1 Definitions

1.1 Notation

We will use $\delta(v)$ to refer to the set of edges adjacent to v. We identify half-edges in a graph $G = (V, E)$ with a pair (e, v) where $e \in \delta(v)$, referring to the half of e adjacent to v.

We use $\{\{\star\}\}$ to refer to multisets, and the following notation for multisubsets of a set A:

- $A^{\{\{k\}\}}$ refers to the set of all multisubsets of A of size $k \in \mathbb{N}$,
- $A^{\{\{\mathbb{N}\}\}}$ refers to the set of all finite multisubsets of A,
- $A^{\{\{\star\}\}}$ refers to the set of all multisubsets of A.

1.2 Common and uncommon assumptions

T-hop neighbourhoods: we define a *T-hop centered neighbourhood* as a *centered graph of radius T* using the definitions from page 4 of [NS93]. Note that, in the case of subgraphs, the T-hop neighbourhood of v may be different from the subgraph induced by all nodes that have distance at most T from v.

Local model: we work in the *deterministic* LOCAL model with input labels. No assumptions are made about the number of input or output labels. We assume the nodes are aware of the exact number n of nodes in the input graph, but not of any information about the maximum degree Δ. Because of this information, we equivalently describe a LOCAL algorithm as a $T(n)$-round communication algorithm with unbounded message size, or as a *possibly uncomputable* function from $T(n)$-hop centered neighbourhoods to output labels. Additionally, we assume there is a finite integer c such that every node is assigned a unique ID in the set $\{1, \ldots, n^c\}$, but the nodes are not aware of the value of c.

Solvability: we say that a problem is *weakly solvable* if there are finitely many unsolvable instances for it (this includes problems that are always solvable). We can, and often will, treat weakly solvable problems as solvable problems by implictly adding one output label U and requiring the problem to output U on all nodes if the instance is unsolvable; this requires only constant time, so it does not affect asymptotic complexity. We call the maximum diameter of an unsolvable instance the *solvability horizon* of a weakly solvable problem. We say that a problem is *strongly unsolvable* if there are infinitely many unsolvable instances for it.

LCL problems: we define LCL problems as tuples $\Pi = (\Delta, \Sigma_{\text{in}}, \Sigma_{\text{out}}, r, \mathcal{C})$ where:

- Δ is a natural number, representing the maximum degree of the graph class considered,
- Σ_{in} is a set of input labels,
- Σ_{out} is a set of output labels,
- r is a finite integer, called the *radius* of Π, and
- $\mathcal{C}$ is a *finite* set of r-hop centered neighbourhoods of maximum degree Δ, where each node is labelled by a pair in $\Sigma_{\text{in}} \times \Sigma_{\text{out}}$.

Though we haven't explicitly stated this in the definition, WLOG we can and will also assume that Σ_{in} is finite (else the problem would be trivially strongly unsolvable, since only finitely many input labels appear in $\mathcal{C}$) and that Σ_{out} is finite (any labels that don't appear in $\mathcal{C}$ will not appear in any valid solution and can be ignored).

Node-edge checkable problems: we define node-edge checkable problems as tuples $\Pi = (\Delta, \Sigma_{\text{in}}, \Sigma_{\text{out}}, \mathcal{C})$ where:

- Δ is a natural number, representing the maximum degree of the graph class considered,
- Σ_{in} is a set of input labels,
- Σ_{out} is a set of output labels,
- r is a finite integer, called the *radius* of Π, and
- $\mathcal{C}$ is an indexed set $\{\mathcal{C}_{d,\chi}\}_{d\leq\Delta,\chi\in\Sigma_{\text{in}}}$ where each $\mathcal{C}_{d,\chi}$ is a subset of $\Sigma_{\text{out}}^{\{\{d\}\}}$ (representing the set of allowed configurations for a node of degree d with input label χ).

Similar reasoning to the above as to why we can implictly assume that Σ_{in} and Σ_{out} are finite sets.

1.3 New-ish definitions

Hanging trees: *connected* tree graphs containing a finite number of incomplete half-edges, called *hooks*. Specifically, we will call hanging trees with one hook *ornaments* and hanging trees with two hooks *tinsels*.

Class: given an ornament T and a node-edge checkable problem Π, we define the *class* of T to be the set $\mathfrak{Cl}(T) \subseteq \Sigma_{\text{out}}$ containing exactly the labellings of the hook that can be completed to a valid labelling for T according to Π. Up to the changed role of nodes and edges, this is the *class* definition from Section 3 of [CP17]. Note that since Σ_{out} is finite, so is the set of all possible classes.

1.4 Actually new definitions

Finitely Represented Configuration: given an alphabet of symbols Σ, we call a *finitely represented configuration* of Σ a pair of a *finite* multisubset of Σ (the *requirement*) and a *finite* subset of Σ (the *filler*). We denote the set of finitely represented configurations in Σ as

$$\Sigma^{FSL} := \Sigma^{\{\{\mathbb{N}\}\}} \times [\Sigma]^{<\omega}$$

Extended Finite State Labelling problems: we define Extended FSL problems as tuples $\Pi = (\Sigma_{\text{in}}, \Sigma_{\text{out}}, f, \mathcal{N}, \mathcal{E})$ where:

- Σ_{in} is a *finite* set of input labels,
- Σ_{out} is a *finite* set of output labels,
- $f : \mathbb{N} \to 2^{\Sigma_{\text{in}}}$ is a *computable* function describing which input labels are allowed to appear for a node of degree $d \in \mathbb{N}$ which is surjective on some cover of Σ_{in} (in other words, each label in Σ_{in} appears in some set in $f(\mathbb{N})$),
- $\mathcal{N} : \Sigma_{\text{in}} \to 2^{(\Sigma_{\text{out}})^{FSL}}$ is a function assigning to each input label a set of *finitely represented configurations* of Σ_{out}, and
- $\mathcal{E}$ is a set of pairs of elements of Σ_{out}.

An *instance* for Π is a graph $G = (V, E)$ together with a labelling function $l : V \to \Sigma_{\text{in}}$ such that $l(v) \in f(\deg(v))$, that is, the labelling is coherent with the "allowed" labels for each degree.
A *solution* for this instance is a labelling s of the half-edges of G such that:

- $\forall v \in V$, let $S(v) := \{\{s(e,v) \mid e \in \delta(v)\}\}$; then there exists $(R, F) \in \mathcal{N}(l(v))$ such that $R \subseteq S(v)$ and $(S(v) \setminus R) \subseteq F^{\{\{\star\}\}}$, that is to say, every symbol in the requirement appears exactly once in $S(v)$, and every other symbol of $S(v)$ is a filler symbol, and
- $\forall e = \{u, v\} \in E$ we have $\{s(e,u), s(e,v)\} \in \mathcal{E}$.

Finite State Labelling problems: we define FSL problems as tuples $\Pi = (\Sigma_{\text{out}}, \mathcal{N}, \mathcal{E})$ where:

- Σ_{out} is a *finite* set of output labels,
- $\mathcal{N} \subseteq 2^{(\Sigma_{\text{out}})^{FSL}}$ is a *finite* set of finitely represented configurations of Σ_{out},
- $\mathcal{E}$ is a set of pairs of elements of Σ_{out}.

An *instance* for Π is any graph; a solution for this instance is a labelling s of the half-edges of G such that:

- $\forall v \in V$, let $S(v) := \{\{s(e,v) \mid e \in \delta(v)\}\}$; then there exists $(R, F) \in \mathcal{N}$ such that $R \subseteq S(v)$ and $(S(v) \setminus R) \subseteq F^{\{\{\star\}\}}$, that is to say, every symbol in the requirement appears exactly once in $S(v)$, and every other symbol of $S(v)$ is a filler symbol, and
- $\forall e = \{u, v\} \in E$ we have $\{\{s(e,u), s(e,v)\}\} \in \mathcal{E}$.

Minimum degree of a label: we define the *minimum degree* of an input label $\chi \in \Sigma_{\text{in}}$ as $\deg_{\min}(\chi) := \min\{d \in \mathbb{N} \mid \chi \in f(d)\}$ (the minimum degree for which that input label is allowed). This set is never empty. Generally, we can assume that $\forall (R, F) \in \mathcal{N}(\chi)$ we have $|R| = \deg_{\min}(\chi)$, since a configuration with a longer requirement would be unattainable by a finite number of degrees (which can be encoded by putting them in separate input classes) and any configuration with a shorter requirement implicitly requires using a number of filler symbols, which can be encoded by adding all combinations of filler symbols to the requirements multiset.

Width: we define the *width* of a EFSL problem Π as the maximum length of one of its requirements: accounting for hidden requirements encoded in the degree, we get

$$\max\left\{\max_{\chi \in \Sigma_{\text{in}}} \deg_{\min}(\chi), \max\{|R| \mid \exists \chi \in \Sigma_{\text{in}}, \exists F \subset \Sigma_{\text{out}} : (R, F) \in \mathcal{N}(\chi)\}\right\}$$

For FSL problems we define it as $\max\{|R| \mid \exists F(\subset \Sigma_{\text{out}} : R, F) \in \mathcal{N}\}$.

Height: we define the *height* of an EFSL problem as $h(\Pi) := \max\{|\mathcal{N}(\chi)| \mid \chi \in \Sigma_{\text{in}}\}$ (the maximum number of configurations any node can be in). For FSL problems we define it as $|\mathcal{N}|$.

Restricted (E)FSL: we define the *restriction* of an (E)FSL the *node-edge checkable* problem with input labels obtained by replacing every finitely represented configuration (R, F) with the configurations $\{R \cup \{\{x\}\} \mid x \in F\}$.

2 List of results

1. Extended FSL = FSL (up to changing the number of labels) input/degree info in the structure and output labels
2. Node-edge checkable problems $\subseteq$ FSL
3. Weak and strong solvability are decidable on FSL

3 List of possible results

1. If a FSL problem is known to be $\Omega(\log n)$ on trees, its complexity on trees is decidable and an efficient algorithm can be found. (This holds because of Gustav's proof which should be stronger, it's in here because I still haven't written out all the details of the FSL based one)
2. (Belief I haven't been able to fully prove) For each FSL problem Π there is a value d that only depends on the problem description (specifically, number of labels, height and width) such that there is a worst-case family of graphs for Π of maximum degree d. My theory is either width+1 or width·height (also possible: width+height, using one of each filler). Latter is required for solvability.
3. (Investigating) A form of the Round Elimination procedure applies to FSL problems.

4 Proofs

4.1 Old(er) proofs that I finally formalised

4.1.1 Solvability

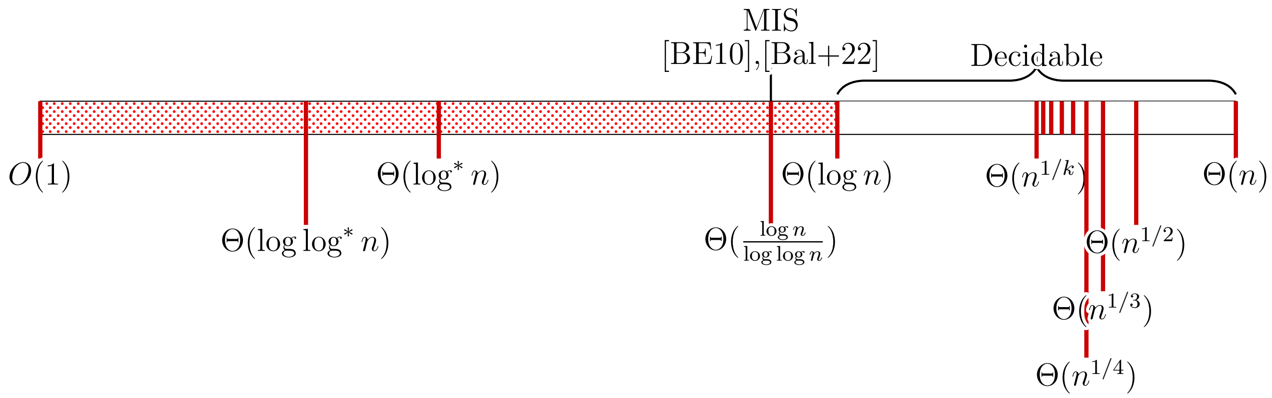
Theorem 4.1. *Let Π be a node-edge checkable problem. Then it is possible to decide whether Π is strongly unsolvable or weakly solvable on trees and compute the solvability horizon.*

Proof. Let $\Pi = (\Delta, \Sigma_{\text{in}}, \Sigma_{\text{out}}, \mathcal{C})$. Let $\mathcal{T}^1_\Delta$ be the set of all ornaments of maximum degree Δ: this is a countably infinite set, but $\mathfrak{Cl}(\mathcal{T}^1_\Delta)$ is a finite set since it is a subset of $2^{\Sigma_{\text{out}}}$. We want to classify the possible classes of ornaments for the problem Π in three sets: classes that appear infinitely many times in the image of $\mathfrak{Cl}$, classes that appear finitely many times, and classes that never appear at all. $\square$

Theorem 4.2. *Let Π be a FSL problem. Then it is possible to decide whether Π is strongly unsolvable or weakly solvable and compute the solvability horizon (if it exists).*

5 What do we know about the complexity landscape?

5.1 Trees



5.2 General graphs

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