

# Assignment 04

## PUBH 8878

### Requirements

- Show complete mathematical work for any derivations.
- Submit well-documented **R code** with clear comments and a fixed seed (`set.seed(8878)`).
- Interpret results in a **genetic/biological** context where applicable.
- **Submit the rendered PDF only** (do not submit the source `.qmd`).

A helpful vignette on using `cmdstanr` is available at <https://mc-stan.org/cmdstanr/articles/cmdstanr.html>.

### Problem 1: Population Substructure and Allele Frequency Estimation (30 pts)

Let there be  $n$  diploid individuals sampled at random from a population made of subpopulations  $k = 1, \dots, K$ . In subpopulation  $k$ , the allele  $A$  frequency is  $p_k$ , and let  $w_k$  be the fraction of sampled individuals from subpopulation  $k$  ( $\sum_k w_k = 1$ ). The “global” allele frequency we want to estimate is the mixture average

$$p = \sum_k w_k p_k.$$

Let  $n_{AA}, n_{Aa}, n_{aa}$  be the counts of genotypes  $AA, Aa, aa$  in the sample of size  $n$ . The standard estimator of the allele frequency is

$$\hat{p} = \frac{2n_{AA} + n_{Aa}}{2n}.$$

(a) Show that in the presence of population substructure,  $\hat{p}$  is unbiased.

*Hint:* Let  $D_i$  be the  $A$ -dosage for individual  $i$  ( $D_i \in \{0, 1, 2\}$ ). Under HWE within  $k$ ,  $D_i \mid Z_i = k \sim \text{Bin}(2, p_k)$ . Use LOTUS:  $\mathbb{E}[D_i] = \sum_k w_k \mathbb{E}[D_i \mid Z_i = k]$ .

(b) What is  $\text{Var}(\hat{p})$  under *random mixture sampling* (i.i.d. individuals from the mixture)? Assume each subpopulation is in HWE. Compare to the case with no substructure ( $K = 1$ ).

*Hint:* Use the law of total variance on  $D_i$ :

$\text{Var}(D_i) = \mathbb{E}\{\text{Var}(D_i \mid Z)\} + \text{Var}\{\mathbb{E}(D_i \mid Z)\}$ , with  $\text{Var}(D_i \mid Z = k) = 2p_k(1 - p_k)$  and  $\mathbb{E}(D_i \mid Z = k) = 2p_k$ .

Define

$$\bar{p} = \sum_k w_k p_k \quad (1)$$

$$\text{Var}_w(p_k) = \sum_k w_k (p_k - \bar{p})^2, \quad (2)$$

and note  $\mathbb{E}[p_k(1 - p_k)] = \bar{p}(1 - \bar{p}) - \text{Var}_w(p_k)$ .

(c) Now consider a *stratified* sample: take exactly  $n_k$  individuals from subpopulation  $k$  ( $\sum_k n_k = n$ ; write  $w_k = n_k/n$ ). Maintain HWE within subpopulations. Derive  $\text{Var}(\hat{p})$  under this design and compare it to your answer in (b). State which design yields the larger variance, and by how much, in terms of  $\text{Var}_w(p_k)$ .

*Hint:* Write  $\hat{p} = \sum_k w_k \hat{p}_k$  with  $\hat{p}_k = \frac{1}{2n_k} \sum_{i: Z_i=k} D_i$ . Use independence across strata and  $\text{Var}(\hat{p}_k) = \frac{p_k(1-p_k)}{2n_k}$ .

## Problem 2: Population Substructure, LD, and Association Testing (40 pts)

Let  $X$  be the genotype dosage (0/1/2 copies of the effect allele) at a tag SNP and  $X_c$  at a causal SNP. The causal SNP has effect size  $\beta_c$  on quantitative trait  $Y$ . The observed effect from simple regression of  $Y$  on  $X$  is  $\beta_{\text{obs}}$ .

**Convenience (scaling):** Work with **standardized** genotypes

$$\tilde{X} = \frac{X - \mathbb{E}[X]}{\sqrt{\text{Var}(X)}}, \quad \tilde{X}_c = \frac{X_c - \mathbb{E}[X_c]}{\sqrt{\text{Var}(X_c)}}.$$

With this scaling,  $\beta_{\text{obs}} = r \beta_c$  exactly, where  $r = \text{Corr}(\tilde{X}, \tilde{X}_c) = \text{Corr}(X, X_c)$ .

Assume individuals are sampled i.i.d. from a mixture with  $\Pr(Z = k) = w_k$ ,  $\sum_k w_k = 1$ . Within each subpopulation  $k$ : - HWE holds at each locus, - LE (no within- $k$  LD) holds between  $X$  and  $X_c$ .

Define

$$\bar{p} = \sum_k w_k p_k, \quad \bar{p}_c = \sum_k w_k p_{c,k} \quad (3)$$

$$\text{Var}_w(p_k) = \sum_k w_k (p_k - \bar{p})^2 \quad (4)$$

$$\text{Cov}_w(p_k, p_{c,k}) = \sum_k w_k (p_k - \bar{p})(p_{c,k} - \bar{p}_c) \quad (5)$$

### Part A (10 pts): Correlation induced by population structure

Let the allele frequencies at the tag and causal SNPs be  $p_k$  and  $p_{c,k}$  in subpopulation  $k$ . Show that

$$r = \frac{\text{Cov}(X, X_c)}{\sqrt{\text{Var}(X) \text{Var}(X_c)}},$$

and express  $\text{Cov}(X, X_c)$ ,  $\text{Var}(X)$ , and  $\text{Var}(X_c)$  in terms of  $\{w_k, p_k, p_{c,k}\}$ .

*Hint:* Law of total covariance:

$\text{Cov}(X, X_c) = \mathbb{E}[\text{Cov}(X, X_c \mid Z)] + \text{Cov}(\mathbb{E}[X \mid Z], \mathbb{E}[X_c \mid Z])$ . Under LE, the first term is 0. Use  $\mathbb{E}[X \mid Z = k] = 2p_k$ .

### Part B (10 pts): Bias from ignoring structure in the trait

Suppose population structure also affects the trait mean:  $\mathbb{E}[Y \mid Z = k] = \mu_k$ . Consider the model  $Y = \mu_Z + \beta_c X_c + \varepsilon$  with  $\mathbb{E}[\varepsilon] = 0$ . Show that the naïve regression of  $Y$  on  $X$  (without structure covariates) is biased:

$$\hat{\beta}_{\text{naïve}} \approx \frac{\beta_c \text{Cov}(X, X_c) + \text{Cov}(X, \mu_Z)}{\text{Var}(X)} = r \beta_c + \underbrace{\frac{\text{Cov}(X, \mu_Z)}{\text{Var}(X)}}_{\text{bias}}.$$

Under the assumptions above, prove that  $\text{Cov}(X, \mu_Z) = 2 \text{Cov}_w(p_k, \mu_k)$ , and give the bias in terms of  $w_k, p_k, \mu_k$ .

*Hint:*  $\text{Cov}(X, \mu_Z) = \text{Cov}(\mathbb{E}[X \mid Z], \mu_Z) = \text{Cov}(2p_Z, \mu_Z)$ .

### Part C (20 pts): Brief interpretation

In a few sentences each:

1. Explain why  $r \neq 0$  can arise even if **within** each subpopulation there is no LD. What feature of the mixture induces it?

2. Give a sign-consistent example: if subpopulations with larger  $p_k$  also have larger  $\mu_k$ , what is the expected direction of the naïve bias?
3. Name **two** standard strategies to mitigate both components of bias (structure-induced  $r$  and trait mean differences) in practice.

### Problem 3: Bayesian Analysis (30 pts)

#### Part A (10 pts): Beta–Binomial conjugacy

1. With prior  $\text{Beta}(4, 18)$  and data  $x = 11$  successes out of  $n = 27$ , write the posterior distribution for  $p$ .
2. Compute the posterior mean and a **central 95% credible interval** in R using `qbeta`. Compare to the MLE  $\hat{p} = 11/27$ . Briefly interpret the *shrinkage*.
3. **Sensitivity:** repeat with priors  $\text{Beta}(1, 1)$  and  $\text{Beta}(8, 32)$ . Summarize how the posterior mean and width change across priors, and why.

#### Part B (10 pts): Beta–Binomial in Stan

1. **Write a Stan model** to estimate the allele frequency  $p$  from Binomial data with a  $\text{Beta}(4, 18)$  prior. Use  $x = 11, n = 27$ .
2. **Run the model** in R using `cmdstanr`. **Check convergence** and effective sample size; report  $\hat{R}$  and bulk ESS for  $p$ .
3. **Summarize** the posterior mean and a central 95% credible interval. Compare to your analytical result from Part A.

You will need to install `cmdstanr` and `cmdstan` if you haven't already. Please follow installation instructions at <https://mc-stan.org/cmdstanr/>

Boilerplate is provided below. You will need to set `eval` to `TRUE` to run the code when knitting the final document:

```
library(cmdstanr)

stan_beta_binomial <- "
data {
  int<lower=0> n;
  int<lower=0, upper=n> x;
```

```

    real<lower=0> a;
    real<lower=0> b;
  }
  parameters {
    real<lower=0, upper=1> p;
  }
  model {
    // TODO: prior on p
    // Example: p ~ beta(a, b);
    // TODO: likelihood
    // Example: x ~ binomial(n, p);
  }
  generated quantities {
    real logit_p = logit(p);
    int x_rep = binomial_rng(n, p);
  }
  "

library(cmdstanr)
set.seed(8878)

writeLines(stan_beta_binomial, con = "beta_binomial.stan")
mod_bb <- cmdstan_model("beta_binomial.stan")
fit_bb <- mod_bb$sample(
  data = list(n = 27, x = 11, a = 4, b = 18),
  seed = 8878, chains = 4, parallel_chains = 4,
  iter_warmup = 1000, iter_sampling = 1000
)

# TODO: check convergence and summarize posterior

```

### Part C (10 pts): ABO blood group frequencies in Stan (missing AB phenotype)

We observe phenotype counts in a population sample where **AB individuals are not sampled**:

- $n_A = 725$
- $n_B = 258$
- $n_O = 1073$

Under HWE with allele frequencies  $\mathbf{p} = (p_A, p_B, p_O)$ , the **unconditional** phenotype probabilities are

$$\Pr(A) = p_A^2 + 2p_A p_O, \quad \Pr(B) = p_B^2 + 2p_B p_O, \quad \Pr(AB) = 2p_A p_B, \quad \Pr(O) = p_O^2.$$

Because AB is missing, the **observed** category probabilities are the renormalized values

$$q_A = \frac{\Pr(A)}{1 - \Pr(AB)}, \quad q_B = \frac{\Pr(B)}{1 - \Pr(AB)}, \quad q_O = \frac{\Pr(O)}{1 - \Pr(AB)}.$$

1. **Write a Stan model** that estimates  $(p_A, p_B, p_O)$  with prior  $\text{Dirichlet}(1, 1, 1)$  and a **Multinomial** likelihood on  $(n_A, n_B, n_O)$  using  $(q_A, q_B, q_O)$ .
2. **Run the model** and check convergence (report  $\hat{R}$  and ESS).
3. **Prior sensitivity:** re-run with  $\alpha = k(0.26, 0.09, 0.65)$  for  $k \in \{1, 10, 100\}$ . Summarize how posterior means and credible intervals change with  $k$ , and why.

Boilerplate is provided below:

```
stan_abo_missing_ab <- "
data {
  int<lower=0> n_A;
  int<lower=0> n_B;
  int<lower=0> n_O;
  vector<lower=0>[3] alpha;
}
transformed data {
  int N = n_A + n_B + n_O;
  int y[3] = { n_A, n_B, n_O };
}
parameters {
  simplex[3] p;
}
transformed parameters {
  // Unconditional phenotype probabilities under HWE:
  real PrA = square(p[1]) + 2 * p[1] * p[3];
  real PrB = square(p[2]) + 2 * p[2] * p[3];
  real PrAB = 2 * p[1] * p[2];
  real PrO = square(p[3]);

  // Observed (AB excluded): renormalize by (1 - PrAB)
  simplex[3] q;
```

```

{
  real denom = 1 - PrAB;
  q[1] = PrA / denom;
  q[2] = PrB / denom;
  q[3] = Pr0 / denom;
}
}
model {
  // TODO: prior on allele frequencies
  // Example: p ~ dirichlet(alpha);
  // TODO: likelihood for observed counts
  // Example: y ~ multinomial(q);
}
generated quantities {
  // Unconditional phenotype probabilities (optional checks)
  vector[4] phen_prob = [PrA, PrB, PrAB, Pr0]';
}
"

set.seed(8878)
writeLines(stan_abo_missing_ab, con = "abo_missing_ab.stan")
mod_abo <- cmdstan_model("abo_missing_ab.stan")
fit_abo <- mod_abo$sample(
  data = list(n_A = 725, n_B = 258, n_0 = 1073, alpha = c(1,1,1)),
  seed = 8878, chains = 4, parallel_chains = 4,
  iter_warmup = 1000, iter_sampling = 1000
)

```