COMP2804 – Winter 2020 – Assignment 4 – Andy Chia – 101111058

Coin Flips

1.		You flip a fair coin, independently, six times. Consider the events A = “the coin comes up heads at least four times”, B = “the number of heads is equal to the number of tails”, C = “there are at least four consecutive heads”. Compute the following probabilities:
	1.	Pr(A) At least four times: $\Pr(A) = \Pr(4) + \Pr(5) + \Pr(6)$ $= C_4^6 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 + C_5^6 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^1 + C_6^6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0$ $= \frac{15}{64} + \frac{6}{64} + \frac{1}{64}$ $= \frac{22}{64}$ $= \frac{11}{32}$
	2.	Pr(B) # Heads = # Tails $\Pr(\mathrm{B}) = \Pr(3)$ $= C_3^6 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3$ $= \frac{20}{64}$ $= \frac{5}{16}$
	3.	Pr(C) Four Consecutive Heads {(HHHHT_), (THHHHT), (_THHHH), (HHHHHT), (THHHHH), (HHHHHH)} $\Pr(C) = \frac{2}{64} + \frac{1}{64} + \frac{2}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64}$ $= \frac{8}{64}$ $= \frac{1}{8}$
	4.	Pr(A|B) This is impossible because for B to be true the number of tails to be equal to heads, it has to be 6/2 that is 3, but for A to be true it can’t be 3. $\Pr(A\|B) = 0$

	5.	Pr(C|A) $\Pr(C\|A) = \Pr(C \cap A)\ /\Pr\ (A)$ $\Pr(C \cap A)$ $= \Pr\ (C)$ $= \frac{1}{8}$ $\Pr(C\|A)$ $= \Pr(C \cap A)\ /\ \Pr(A)$ $= (\frac{1}{8})/(\frac{11}{32})$ $= \frac{4}{11}$

Probability of 5 before 7

2.	Consider the following model, which corresponds to repeatedly rolling two dice d_1 and d_2 and stopping the first time $d_1 + d_2 \in \{5, 7\}$. You roll two 6-sided dice d_1 and d_2. Consider the events A = "$d_1 + d_2 = 7$" B = "$d_1 + d_2 = 5$" $C = A \cup B$
	What is Pr(A|C)? A: {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)} |A| = 6 B: {(1,4), (2,3), (3,2), (4,1)} |B| = 4 C: A ∪ B |C| = 6 + 4 = 10 The probability of A happening given C 6/10 Pr(A|C) = 3/5

Four-Door Monte Hall

3.		Consider the following version of the Monte Hall Problem: • There are four doors numbered $1, \ldots, 4$. Three of these doors have goats behind them. One door has a sports car. You want to win the sports car. • You pick one door, i uniformly at random and put a chalk mark on it. That door stays closed for now. • Monte Hall opens a door j, with $j \neq i$, showing you a goat. • You get to pick any of the three unopened doors in $\{1, 2, 3, 4\} \setminus \{j\}$ and keep whatever is behind it.
	1.	Suppose you decide to stick with your first choice, i . What is the probability that you win the sports car? The probability of picking the car on the first choice is ¼ because there are 3 doors with no car and one with a car
	2.	Suppose you decide to choose one of three unopened doors {1,2,3,4}\{j} uniformly at random. What is the probability that you win the sports car? 1/3 because since the first door is not the sports car it has to be one of the other 3
	3.	Suppose you decide to choose one of the two unopened and unmarked doors {1,2,3,4}\{i,j} uniformly at random. What is the probability that you win the sports car? To find the probability of winning: Let S be the sports car Let A be the sports car in the first door picked $\Pr(S\|A) * Pr(A) + \Pr(W\|A') * \Pr(A') = 0 * \frac{1}{4} + \frac{1}{2} * \frac{3}{4} = \frac{3}{8}$

Estimating Genetic Diseases

<table>
<tr><td>4.</td><td></td><td>One out of 25 healthy people carries a single gene for <u>cystic fibrosis</u> (CF), these people are called carriers and healthy people without a CF gene are called non-carriers. A uniformly chosen random healthy person has probability 1/25 of being a carrier.
A person with two CF genes is not healthy; they are sick (with cystic fibrosis). The child of a carrier has probability 1/2 of inheriting a CF gene from that parent. The child of two carriers inherits each of their parents CF genes independently, so the child has probability 1/4 of having CF.</td></tr>
<tr><td></td><td>1.</td><td>Two uniformly-chosen random healthy people have a child together. What is the probability that the child has CF?

Non-Carrier = 1/25
Carrier = ½
If A = Parent non-Carrier, B = Parent non-Carrier, C = Parent carrier, D = parent carrier.
$$\Pr(A \cap B \cap C \cap D) = \Pr(A) * \Pr(B) * \Pr(C) * \Pr(D)$$ $$= \frac{1}{25} * \frac{1}{25} * \frac{1}{2} * \frac{1}{2}$$ $$= \frac{1}{2500}$$</td></tr>
<tr><td></td><td>2.</td><td>Two uniformly-chosen random healthy people have a child together. What is the probability that the child is a healthy non-carrier?

Pr(child healthy non-carrier)
Parents: {(A, B), (A', B) , (A', B')}
A being a first parent non carrier
B being a second parent non carrier

(A, B):
When both parents are not carriers
$$\Pr(A' \cap B') = \Pr(A') * \Pr(B') = \frac{24}{25} * \frac{24}{25} = 0.9216$$(A',B) (or (A, B')):
When one of the parents is a carrier
$$C_1^2 \left(\frac{1}{25}\right)\left(\frac{1}{2}\right)\left(\frac{24}{25}\right) = 0.0384$$(A', B'):
When both parents are carriers
$$\frac{1}{25} * \frac{1}{25} * \frac{1}{2} * \frac{1}{2} = \frac{1}{2500}$$Together:
Pr(C) = 0.9604</td></tr>
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	3.	A carrier has a child with a uniformly-chosen random healthy person. What is the probability that the child has CF? Similar to 2. but we know that one of the parents is a carrier Parent 2 is a carrier: - Probability that parent 2 is a carrier * gene from parent 2 * gene from parent 1: $\frac{1}{25} * \frac{1}{2} * \frac{1}{2} = 1/100$ Parent 2 is not a carrier: 0 because both parents are not carriers
	4.	A carrier has a child with a uniformly-chosen random healthy person. What is the probability that the child is a (healthy) carrier? Similar to 3. but checking for child being a carrier Parent 2 is a carrier: - Child inherits from parent 1, parent 2 is a carrier * gene from parent 2 * gene from parent 1: $\frac{1}{25} * \frac{1}{2} * \frac{1}{2} = \frac{1}{100}$ - Child inherits from parent 2, parent 2 is a carrier * gene from parent 2 * gene from parent 1: $\frac{1}{25} * \frac{1}{2} * \frac{1}{2} = \frac{1}{100}$ Parent 2 is not a carrier: 1 – 1/25 = 24/25 Child inherits from carrier parents, parent 2 is not a carrier * parent 1 $$\frac{24}{25} * \frac{1}{2} = \frac{24}{50}$$ Total: $\frac{1}{100} + \frac{1}{100} + \frac{24}{50} = \frac{1}{2}$
	5.	Two uniformly-chosen healthy people have a baby. A quick blood test, administered minutes after birth, shows that the baby has at least one CF gene, but gives no other information. What is the probability that the baby has CF? For the child to carry at least one gene, we know that a parent is a carrier. To know if the child has another gene we must calculate the probability that the second parent has the gene and passes it to the child: $$\frac{1}{25} + \frac{1}{2} = \frac{1}{50}$$

Sampling With and Without Replacement

5.		We have a cooler containing 2 cider bottles and $n-2$ beer bottles. We can *sample* from this cooler in two different ways: • *with replacement*: We take a uniformly-random bottle from the cooler, look at it and put it back. • *without replacement*: We take a uniformly-random bottle from the cooler, drink it, and throw the empty bottle into the recyling bin. Suppose we repeatedly sample from the cooler and let X be the number of samples up to and including the first cider bottle.
	1.	What is E(X) if we use sampling with replacement? Grabbed a drink *i* times: Beer(i – 1), cider(1) Probability of grabbing a beer is $\frac{n-2}{n}$ Probability of grabbing a cider is $\frac{2}{n}$ $$\sum_{i=0}^{n}\left(\frac{n-2}{2}\right)^{i-1}\left(\frac{1}{n}\right)^{1}$$
	2.	What is E(X) if we use sampling without replacement? We discard the bottle after consuming it.