COMP2804 – Winter 2020 – Assignment 2 – Andy Chia – 101111058

Arrangements of PUNKEYDOODLES

<table>
<tr><td>2.</td><td>How may distinct ways are there to rearrange the letters in PUNKEYDOODLES (the name of a town in Ontario)?

P – 1
U – 1
N – 1
K – 1
E – 2
Y – 1
D – 2
O – 2
L – 1
S – 1

1. Chose 1 position for P from 13 positions $\binom{13}{1}$
2. Chose 1 position for U from 12 positions $\binom{12}{1}$
3. Chose 1 position for N from 11 positions $\binom{11}{1}$
4. Chose 1 position for K from 10 positions $\binom{10}{1}$
5. Chose 2 positions for E from 9 positions $\binom{9}{2}$
6. Chose 1 position for Y from 7 positions $\binom{7}{1}$
7. Chose 2 positions for D from 6 positions $\binom{6}{2}$
8. Chose 2 positions for O from 4 positions $\binom{4}{2}$
9. Chose 1 position for L from 2 positions $\binom{2}{1}$
10. Make the last position S $\binom{1}{1}$
$$\binom{13}{1}\binom{12}{1}\binom{11}{1}\binom{10}{1}\binom{9}{2}\binom{7}{1}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 778377600$$</td></tr>
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Modern Coupling

<table>
<tr><td>3.</td><td>1.</td><td>Suppose we have a group of n women and n men, all heterosexual and all strictly monogamous. How many ways are there to make n couples out of these $2n$ people?
- n men
- n women
- Heterosexual And all monogamous
- # men = # women
1. first man chooses out of n women $\binom{n}{1}$
2. second man chooses out of n - 1 women $\binom{n-1}{1}$
3. repeat for each man
$$\binom{n}{1} * \binom{n-1}{1} * \dots * \binom{1}{1} = n * (n-1) * \dots * 1 = !n$$</td></tr>
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<table>
<tr><td></td><td>2.</td><td>Suppose we have 2n people, each of whom is bisexual but strictly monogamous. How many ways are there to make n couples out of these 2n people?
- 2n people
- bisexual
1. First person chooses out of n – 1 people (the one missing being the picker) $\binom{n-1}{1}$
2. Third person chooses out of n – 3 people $\binom{n-3}{1}$
3. Fifth person chooses out of n – 5 people $\binom{n-5}{1}$
4. repeat for every next person
5. Second last person chooses last person (sorry)
$$\prod_{i=1}^{n}\binom{n-1-((i-1)*2)}{1} = \prod_{i=1}^{n}(n-1-((i-1)*2))$$</td></tr>
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Pigeonholing

<table>
<tr><td>4.</td><td>1.</td><td>In an anonymous survey of a group of 20 men and 20 women, the men reported a total of 81 sexual encounters with women in the group, and all women reported having at most 4 sexual encounters with men in the group. Is everyone telling the truth?
- 20 men and 20 women
- 81 total sexual encounters from men with the women
- women having at most 4 sexual encounters with men
Suppose every woman had sexual encounters 4 times being the most amount
$$20 * 4 = 80$$
Since there are more reported sexual encounters from men than the maximum amount of sexual encounter that women can have added together, this is false.
$$81 > 80$$
By PHP there must have been one woman that has had a sexual intercourse $\left\lceil \frac{81}{20} \right\rceil$ times (5) to match the number of reported sexual encounters by the men</td></tr>
<tr><td></td><td>2.</td><td>A group of n agents all start at the same location and each one takes a *≤m-walk* on the line, where a ≤m-walk is a sequence of *at most* m steps and each step moves the agent one unit to the left or one unit to the right. (Different agents might take different walks.) Prove that, if n>2m+1, then some pair of agents finishes their walk at the same location.
- n = # of agents
- m = # of steps
- Each agent travels at most m steps
- Agents either move to the left or the right
- if $n > 2m + 1$ then some pair of agents finish at the same location
n = agents -> pigeons
l = locations -> holes
if n > l then PHP $\left\lceil \frac{n}{l} \right\rceil \geq 2$
show $n > 2m + 1 \geq l$
since you can only take left and right steps [-m, m]
at most means that the agents can stop an infinite distance
|l| = 2m + 1 because m to the left, m to the right and its initial position</td></tr>
</table>

<table>
<tr><td></td><td>3.</td><td>A group of n agents all start at the same location and each one takes an m-walk on the line, where a m-walk is a sequence of exactly m steps and each step moves the agent one unit to the left or one unit to the right. (Different agents might take different walks.) Prove that, if $n>m+1$, then some pair of agents finishes their walk at the same location.
- n = # of agents
- m = # of steps
- All agents travel m steps exactly
- Agents either move to the left or the right
- Prove that, if $n>m+1$, then some pair of agents finishes their walk at the same location.
n = agents -> pigeons
l = locations -> holes
if n > l then PHP $\left\lceil \frac{n}{l} \right\rceil \geq 2$
show $n > m + 1 \geq l$
Since you can only take left and right steps [-m, m]
||l| = m + 1 because all agents walk the same distance and initial starting position</td></tr>
<tr><td></td><td>4.</td><td>Let $V = \{v1, \ldots, vk\}$ be any set of vectors in $R2$. Suppose n agents each start at $(0,0)$ and each takes a mV-walk where a mV-walk consists of a sequence of exactly m steps and each step moves the agent along a vector in V. Prove that, if $n>(m+k-1\ \ k-1)$, then some pair of agents finishes their walk at the same location.
- n = agents -> pigeons
- p = position -> holes
- m = steps
- Prove that, if $n > \binom{m+k-1}{k-1}$, then some pair of agents finishes their walk at the same location.
if n > p then PHP $\left\lceil \frac{n}{p} \right\rceil \geq 2$
show $n > \binom{m+k-1}{k-1}$
M being a step in a direction and k being the possible vectors in the direction that the agent walks
$|p| = n > \binom{m+k-1}{k-1}$because all agents walk the same distance and they go all in the possible k directions</td></tr>
<tr><td></td><td>5.</td><td>Let S be a k-element subset of $\{1,\ldots,n\}$. Prove that, if $k>\lceil n/2\rceil$, then there exists x,y∈S such that $x-y=1$
- S = {1, …, n}
- Prove that, if $k > \left\lceil \frac{n}{2} \right\rceil$, then there exist $x, y \in S$ such that $x - y = 1$
x and y are in s
x > y, x = y + 1
If more than half there exist two numbers following each other
Since there are more than half the amount of numbers, there must be a number between two numbers that will follow each other to have all the numbers
If $\left|\frac{x+y}{2}\right| = 0.5$ PHP</td></tr>
<tr><td></td><td>6.</td><td>Let S be a k-element subset of $\{1,\ldots,n\}$. Prove that, if $(k\ \ 2)>n-1$, then there exists a,b,x,y∈S such that a≠b, {a,b}≠{x,y} and $b-a=y-x$. (Note that it may be the case that $b=x$ or $a=y$.)
- a,b,x,y are in S</td></tr>
</table>

		- a is not equal to b - {a,b} cant have the same numbers as {x,y} - b and a cant have the same gap as x and y - b and be equal to x or a can be equal to y - Prove k choses 2 > n – 1
	7.	

Recurrences

5.	1.	Let $a, b \geq 0$ be two real numbers and consider the function $f : \mathbb{N} \to \mathbb{R}$ given by the recurrence $$f(n) = \begin{cases} a & \text{if } n = 0 \\ f(n-1) + b & \text{if } n \geq 1 \end{cases}$$ Write a closed form formula for $f(n)$ and prove that your formula is correct. N = occurrences a = end (0) b = number to f(n) $$g(n) = n * b + a$$ Proof: A = 5 B = 8 N = 3 $$f(3)$$ $$f(2) + 8$$ $$f(1) + 8 + 8$$ $$f(0) + 8 + 8 + 8$$ $$5 + 8 + 8 + 8$$ $$29$$ $$3 * 8 + 5$$ $$29$$ Base case: N=0 $$f(0)$$ $$a$$ $$g(0)$$ $$0 * b + a$$ $$0 + a$$ $$a$$ Induction: $$f(0) = g(0)$$ $$a = a$$

2. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by the recurrence

$$f(n) = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ \frac{1}{2} \cdot n \cdot f(n-1) & \text{if } n \geq 1. \end{cases}$$

Write a closed form formula for $f(n)$ and prove that your formula is correct.

½ = Ending (0)

n = occurrences & to be multiplied

½ = to be multiplied every time

$$g(n) = \left(\prod_{i=1}^{n} \left(\frac{1}{2} * i \right) \right) * \frac{1}{2} = \frac{1^{(n+1)}}{2} * !n$$

Proof

N = 3

$$f(3)$$
$$\frac{1}{2} * 3 * f(2)$$
$$\frac{1}{2} * 3 * \frac{1}{2} * 2 * f(1)$$
$$\frac{1}{2} * 3 * \frac{1}{2} * 2 * \frac{1}{2} * 1 * f(0)$$
$$\frac{1}{2} * 3 * \frac{1}{2} * 2 * \frac{1}{2} * 1 * \frac{1}{2}$$
$$0.375$$

$$\frac{1^{(3+1)}}{2} * !3$$
$$\frac{1^{(4)}}{2} * 3 * 2 * 1$$
$$0.375$$

Base case:

N=0

$$f(0)$$
$$\frac{1}{2}$$

$$g(0)$$
$$\frac{1^{0+1}}{2} * 0!$$
$$\frac{1^{1}}{2} * 1$$
$$\frac{1}{2}$$

Induction:

$$f(0) = g(0)$$
$$\frac{1}{2} = \frac{1}{2}$$

	3.	Consider the function $f : \mathbb{N}^+ \to \mathbb{N}$ defined by: $$f(n) = \begin{cases} 1 & \text{if } n \in \{1,2,3\} \\ 4f(n-3) & \text{if } n > 3. \end{cases}$$ Write a closed form formula for $f(n)$ and prove that your formula is correct. 4 times the occurrences of 3 in n $$g(n) = 4^{\lceil \frac{n}{3} \rceil - 1}$$ Proof: n = 10 $$f(10)$$ $$4 * f(7)$$ $$4 * 4 * f(4)$$ $$4 * 4 * 4 * f(1)$$ $$4 * 4 * 4 * 1$$ $$64$$ $$4^{\lceil \frac{10}{3} \rceil - 1}$$ $$4^{4-1}$$ $$4^3$$ $$64$$ Base case: n = 1 $$f(1)$$ $$1$$ $$g(1)$$ $$4^{\lceil \frac{1}{3} \rceil - 1}$$ $$4^{1-1}$$ $$4^0$$ $$1$$ Induction: $$f(1) = g(1)$$ $$1 = 1$$

4. Consider the set A_n of strings over the 4-character alphabet $\{a, b, c, d\}$ whose length is n and for which cc does not appear as a consecutive substring. For example:

$$A_0 = \{\varepsilon\}$$

$$A_1 = \{a, b, c, d\}$$

$$A_2 = \{aa, ab, ac, ad, ba, bb, bc, bd, ca, cb, \cancel{cc}, cd, da, db, dc, dd\}$$

Write a recurrence for $|A_n|$. Then, using induction, show that this recurrence solves to

$$|A_n| = (1/2 + 5 \cdot \sqrt{21}/42) \cdot \alpha^n + +(1/2 - 5 \cdot \sqrt{21}/42) \cdot \beta^n \ ,$$

where $\alpha = (3 + \sqrt{21})/2$ and $\beta = (3 - \sqrt{21})/2$.

If n = 0 or n = 1 then cc appears 0 times
If n = 2 then cc appears 1 time
If n = 3 then cc appears 7 times (1 * 3) + 4

- 1 time from ac (A_2)
- 1 time from bc (A_2)
- 4 time from cc (A_2)
- 1 time from dc(A_2)

If n = 4 then cc appears 40 times (7 * 3) + 16 + (1 * 3)
If n = 5 then cc appears 205 times (40 * 3) + 64 + (7*3)

$$b(n) = \begin{cases} 0 & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ f(n-1) * 3 + 4^{n-2} + f(n-2) & \text{if } n >= 2 \end{cases}$$

$$f(n) = 4^n - b(n)$$

	5.	Consider the set S_n of binary strings whose length is n and for which 010 does not appear as a consecutive substring. For example, $$S_0 = \{\varepsilon\},$$ $$S_1 = \{0, 1\},$$ $$S_2 = \{00, 01, 10, 11\},$$ $$S_3 = \{000, 001, \cancel{010}, 011, 100, 101, 110, 111\}$$ $$S_4 = \{0000, 0001, \cancel{0010}, 0011, \cancel{0100}, \cancel{0101}, 0110, 0111, 1000, 1001, \cancel{1010}, 1011, 1100, 1101, 1110, 1111\}$$ a. Argue that, for $n \geq 3$, $$\lvert S_n \rvert = \lvert S_{n-1} \rvert + \sum_{k=3}^{n} \lvert S_{n-k} \rvert + 2 \ . \qquad (1)$$ b. Write a program to compute $\lvert S_n \rvert$ for $n = 0, \ldots, 20$ and look up the resulting sequence in the Online Encyclopedia of Integer Sequences. What did you find? a. Base case: n = 3 $$\lvert S_3 \rvert = \lvert S_{3-1} \rvert + \sum_{k=3}^{3} \lvert S_{3-3} \rvert + 2$$ $$= 4 + 1 + 2$$ $$= 7$$ b.