COMP2804 – Winter 2020 – Assignment 3 – Andy Chia – 101111058

Probability of scrabble words

<table>
<tr><td>1.</td><td></td><td>A scrabble hand is a set of 7 tiles, each having one of the English uppercase letters on them, drawn uniformly at random from a bag of 100 tiles. The number of tiles of each letter are as follows:

E×12, A×9, I×9, O×8, N×6, R×6, T×6, L×4, S×4, U×4, D×4, G×3, B×2, C×2, M×2, P×2, F×2, H×2, V×2, W×2, Y×2, K×1, J×1, X×1, Q×1, Z×1</td></tr>
<tr><td></td><td>1.</td><td>What is the probability that a scrabble hand contains the word “HEXAGON”?

<table>
<tr><td></td><td>Amount Needed</td><td>Amount Of the letter</td><td># of subsets</td></tr>
<tr><td>H</td><td>1</td><td>2</td><td>$\binom{2}{1} = 2$</td></tr>
<tr><td>E</td><td>1</td><td>12</td><td>$\binom{12}{1} = 12$</td></tr>
<tr><td>X</td><td>1</td><td>1</td><td>$\binom{1}{1} = 1$</td></tr>
<tr><td>A</td><td>1</td><td>9</td><td>$\binom{9}{1} = 9$</td></tr>
<tr><td>G</td><td>1</td><td>3</td><td>$\binom{3}{1} = 3$</td></tr>
<tr><td>O</td><td>1</td><td>8</td><td>$\binom{8}{1} = 8$</td></tr>
<tr><td>N</td><td>1</td><td>6</td><td>$\binom{6}{1} = 6$</td></tr>
</table>
Since the word is 7 characters long, we don’t have to add the chance of getting missing characters that wasn’t picked
*After removing a letter from the bag, we must lower the count of tiles in the bag
*Starting tile pulled can be any of the pool (HEXAGON)
Number of subsets for the word HEXAGON:
$$2 * 12 * 1 * 9 * 3 * 8 * 6 = 31104$$
The 7 tiles can be arranged in 7! ways: $31104 * 7!$
7-permutatuons of 100
$$P(100,7) = \frac{100!}{(100-7)!} = 100 * 99 * 98 * 97 * 96 * 95 * 94$$
P(HEXAGON) =
$$\frac{31104 * 7!}{P(100,7)} = \frac{31104}{P(100,7)/7!}$$
Note: $\frac{100!}{(100-7)! * 7!} = \binom{100}{7}$
$$\frac{31104}{\binom{100}{7}} = \frac{31104}{16007560800}$$
</td></tr>
</table>

2. *What is the probability that a scrabble hand contains the word GARBAGE?*

	Amount Needed	Amount Of the letter	# of subsets
G	2	3	$\binom{3}{2} = 3$
A	2	9	$\binom{9}{2} = 36$
R	1	6	$\binom{6}{1} = 6$
B	1	2	$\binom{2}{1} = 2$
E	1	12	$\binom{12}{1} = 12$

Number of subsets for the word HEXAGON:

$$3 * 36 * 6 * 2 * 12 = 15552$$

The 7 tiles can be arranged in 7! ways: $15552 * 7!$

7-permutatuons of 100

$$P(100,7) = \frac{100!}{(100-7)!} = 100 * 99 * 98 * 97 * 96 * 95 * 94$$

P(GARBAGE) =

$$\frac{15552 * 7!}{P(100,7)} = \frac{15552}{P(100,7)/7!}$$

Note: $\frac{100!}{(100-7)! * 7!} = \binom{100}{7}$

$$\frac{15552}{\binom{100}{7}} = \frac{15552}{16007560800}$$

3. *What is the probability that a scrabble hand contains the word APPLE?*

	Amount Needed	Amount Of the letter	# of subsets
A	1	9	$\binom{9}{1} = 9$
P	2	2	$\binom{2}{2} = 1$
L	1	4	$\binom{4}{1} = 4$
E	1	12	$\binom{12}{1} = 12$

Number of subsets for the word APPLE:

$$9 * 1 * 4 * 12 = 432$$

The 4 tiles can be arranged in 4! ways: $432 * 4!$

4-permutatuons of 100

$$P(100,4) = \frac{100!}{(100-4)!} = 100 * 99 * 98 * 97$$

P(GARBAGE) =

$$\frac{432 * 4!}{P(100,4)} = \frac{432}{P(100,4)/4!}$$

Note: $\frac{100!}{(100-4)! * 4!} = \binom{100}{4}$

$$\frac{432}{\binom{100}{4}} = \frac{432}{3921225}$$

Feeding Your Rat

2.		A rat feeder is essentially a straw whose diameter is just large enough for 1 (medicine) pill or 1 (food) pellet but is long enough to hold many pills and pellets. The pill and pellets are put in at one of the feeders and come out the other end (when the rat presses a pedal) in the same order they were put in. Suppose we place 25 identical pellets and 4 identical pills uniformly at random into a rat feeder. The rat then comes and consumes one item x1 from the feeder and then consumes another item x2 from the feeder.
	1.	Let A be the event "x1 is a pellet" and let B be the event "x2 is a pill".
	2.	What is $\Pr(A \cap B)$? Pr(A∩B) = Pr(A) * Pr(A|B) - First item consumed must be a pellet - Second item consumed must be a pill Pr(A) = 25/29 If A is true, then we can pass to test B Pr(A|B) = 4/28 Pr(A∩B = Pr(A) * Pr(A|B) = 25/29 * 4/28 = 100/812 = 25/203
	3.	What is $\Pr(A \cup B)$? - At least one of the two items consumed must follow A and B Pr(A∪B) = Pr(A) + Pr(B) – Pr(A∩B) = 25/29 + 4/28 – 25/203 ≈ 0.88177339
	4.	Are the events A and B independent? In other words, is $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$? No Events A and B are not independent because after removing one of the items, the probability changes because there is one less item in the feeder

A Coin-Flipping Game

3.		*Consider the following coin tossing games. For each one, compute the probability that you win the game. For each question, the answer is a rational number so you should give this number exactly and give a decimal approximation of it as well.*
	1.	You toss a fair coin twice and win if it comes up heads at least once. 2 Possibilities | 2 actions | Heads at least once A = If the first coin flips lands on head B = If the second coin flipped lands on head Pr(A∩B) = Pr(A) * Pr(B) = 0.5 * 0.5 = 0.25 Pr(A∪B) = $\Pr(A)$ + $\Pr(B)$ – $\Pr(A \cap B)$ = 0.5 + 0.5 – 0.25 = 0.75

	2.	You toss a fair coin 10 times and win if comes up heads at least five times. 2 Possibilities | 10 Actions | Heads at least 5 times $$\Pr(\geq 5) = \Pr(5) + \Pr(6) + \Pr(7) + \Pr(8) + \Pr(9) + \Pr(10)$$ 1. $Pr(5) = \binom{10}{5} 0.5^5 0.5^5 \approx 0.246$ 2. $Pr(6) = \binom{10}{6} 0.5^6 0.5^4 \approx 0.205$ 3. $Pr(7) = \binom{10}{7} 0.5^7 0.5^3 \approx 0.117$ 4. $Pr(8) = \binom{10}{8} 0.5^8 0.5^2 \approx 0.044$ 5. $Pr(9) = \binom{10}{9} 0.5^9 0.5^1 \approx 0.010$ 6. $Pr(10) = \binom{10}{10} 0.5^{10} 0.5^0 \approx 0.001$ $$0.246 + 0.205 + 0.117 + 0.044 + 0.010 + 0.001 = 0.623$$ $$\Pr(\geq 5) = 0.623$$
	3.	You toss a fair coin twice and win if it comes up heads exactly once. 2 Possibilities | 2 Actions | Heads once $$Pr(1) = \binom{2}{1} 0.5^1 0.5^1 = 0.5$$ [{H, H}, {H, T}, {T, H}, {T, T}] {H, T} and {T,H} are the only one that holds true 2/4 = 1/2 = 0.5
	4.	You toss a fair coin 10 times and win if comes up heads exactly five times. 2 Possibilities | 10 Actions | Heads exactly 5 times $$Pr(5) = \binom{10}{5} 0.5^5 0.5^5 \approx 0.246$$

Blindfolded Musical Chairs

4.		We are playing a game of blindfolded musical chairs with 20 blindfolded people and 40 chairs. When the music stops each person picks a chair uniformly at random and sits on it.
	1.	What is the probability that some chair has at least two people sitting on it? - c = 40 (chairs) - n = 20 (people) - $\|S\| = c^n$ - A = "At least two people picked the same chair" - A = "$b_i = b_j$ for some i != j" - A' = "everyone has a different chair" - A' = "b_i != b_j for any i != j" - |A'| = Number of one-to-one functions n to c *It is easier finding the probability of one-to-one functions and then substracting it by 1 to find the non-one-to-one functions (Pr(A)) $$\Pr(A') = \frac{A'}{\|S\|} = \frac{c!}{(c-n)!} / c^n = \frac{c!}{(c-n)! * c^n}$$ $$\Pr(A') = \frac{40!}{(40-20)! * 40^{20}} \approx 0.00305$$ $$\Pr(A) = 1 - \Pr(A') = 1 - 0.00305 = 0.99695$$

	2.	What is the probability that some chair has at least three people sitting on it? Adding the probability of exactly 2 people sitting to Pr(A') from 4.1: *To find the probability for exactly a pair of people sitting in a spot or multiple, we must keep that in consideration B = "At least 3 people picked the same chair" B' = "At most 2 people picked a single chair" Since there are n = 20 people, there can be 1 to n/2 (10) pairs 1. first pair chooses that cuts 1 chair from the pool 2. if there is a second pair, it takes another chair and cuts 1 from the pool, ect…. $$\sum_{i=0}^{10} \binom{20}{2i} \left(\frac{1}{2^i}\right) \binom{2i}{i} i!$$ Then we need to pick a distinct chair for each of the people alone and the pairs $$\lvert B' \rvert = \sum_{i=0}^{10} \binom{20}{2i} \left(\frac{1}{2^i}\right) \binom{2i}{i} i! * \frac{40!}{(20+i)!}$$ $$\Pr(B') = \frac{\lvert B' \rvert}{40^{20}} \approx$$ $$\Pr(B) = 1 - \Pr(B') = 1 - \frac{\lvert B' \rvert}{40^{20}}$$ *I don't know how to do sums properly on my calculator for a final anwser

Random Number Weirdness

5.		We independently pick two random numbers R1 and R2 from the set {1,...,1000}. (Note: Independence means we may pick R1=R2.)
	1.	What is the probability that R1 and R2 are both even? A = "R1 is Even" B = "R2 is Even" Pr(A∩B) = ? A and B and both either be true or false independently $$Pr(A) = 500/1000 = 1/2$$ $$Pr(B) = 500/1000 = 1/2$$ $$Pr(A \cap B) = \Pr(\mathrm{A}) * \Pr(\mathrm{B}) = 0.5 * 0.5 = 0.25$$
	2.	Suppose I tell you that at least one of R1 or R2 is even. What is the (conditional) probability that R1 and R2 are both even? - {A', B}, {A, B'}, {A, B} - There are only 3 possible permutations - Pr(A∩B) = 1/3

	3.	What is the probability that at least one of R1 or R2 is equal to 1000? Same as 3.3 - {A', B'}, {A', B}, {A, B'}, {A, B} - Pr(A∪B) = Pr(A) + Pr(B) - Pr(A∩B) = 0.5 + 0.5 – 0.25 = 0.75
	4.	Suppose I tell you that at least one of R1 or R2 is even. What is the probability that at least one of R1 or R2 is equal to 1000? If R1 or R2 is even, then 1/500 Else then 0 From 5.3: 0.75 for it being even 0.75 * 1/500 = 0.0015
	5.	Suppose I tell you that at least one of R1 or R2 is even. What is the probability that at least one of R1 or R2 is equal to 999? 1 – 0.75 = 0.25 0.25 * 1/500 = 0.0005

Uniqueness of Maximum and Median

6.		Let n be an odd integer. The \emph{median} of a sequence $x_1, \ldots, x_n$, denoted of numbers, $\text{median}(x_1, \ldots, x_n)$, is the unique value x such that there are at least $\lceil n/2 \rceil$ values less than or equal to x and at least $\lceil n/2 \rceil$ values greater than or equal to x. For example, $$\text{median}(8, 4, 3, 4, 7, 5, 6) = 5$$ since $4, 3, 4, 5 \leq 5$ and $8, 7, 5, 6 \geq 5$; and $$\text{median}(8, 5, 3, 4, 7, 5, 6) = 5$$ since $5, 3, 4, 5 \leq 5$ and $8, 5, 7, 5, 6 \geq 5$. Consider a uniform random sequence $x_1, \ldots, x_7$ of 7 numbers each chosen from the set $\{1, 2, 3, \ldots, 10\}$
	1.	What is the probability that max{x1,…,x7} occurs exactly once in x1,…,x7? |S| = 7^10 = number of possible permutations Varies depending on the max() number: If 10 - 1 * 6^9 If 9 - 1 * 6^8 If 8 - 1 * 6^7 Ect…

		1 has no lower so its impossible, $$\frac{\sum_{i=1}^{10}(6^i)}{7^{10}}$$
	2.	What is the probability that median (x1, …, x7) occurs exactly once in x1, …, x7? Since the median must not repeat more than once, there is no decreasing value $$\frac{6^9}{7^{10}}$$