Comphys Homework 3

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I. WRITING ASSIGNMENTS

The 2D Poisson equation is

$$u_{xx} + u_{yy} = \rho(x, y) \tag{1}$$

Use finite difference method to make it discrete:

$$\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{\Delta x^2} + \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{\Delta y^2} = \rho_{i,j} \tag{2}$$

If $\Delta x = \Delta y$,

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = -\rho_{i,j}\Delta x^2 \tag{3}$$

Consider the boundary condition, the final result of $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ is:

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{pmatrix} = -\Delta x^2 \begin{pmatrix} \rho_{11} \\ \rho_{21} \\ \rho_{31} \\ \rho_{12} \\ \rho_{22} \\ \rho_{32} \\ \rho_{13} \\ \rho_{23} \\ \rho_{33} \end{pmatrix} \tag{4}$$

II. PROGRAMMING ASSIGNMENTS

1. The source is $\rho_{22} = 1$ and the others are zero. Outside the grid, the potential is zero:

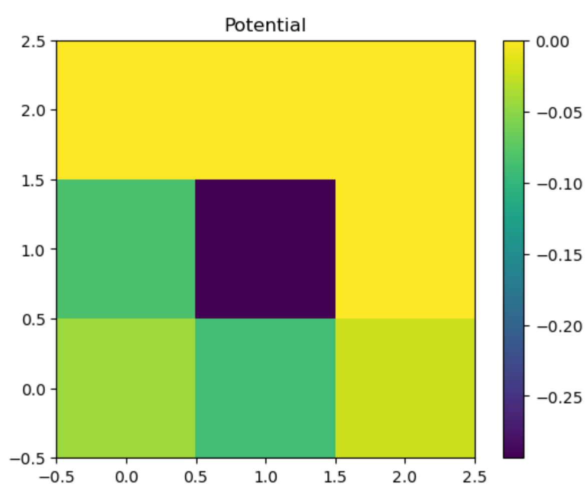
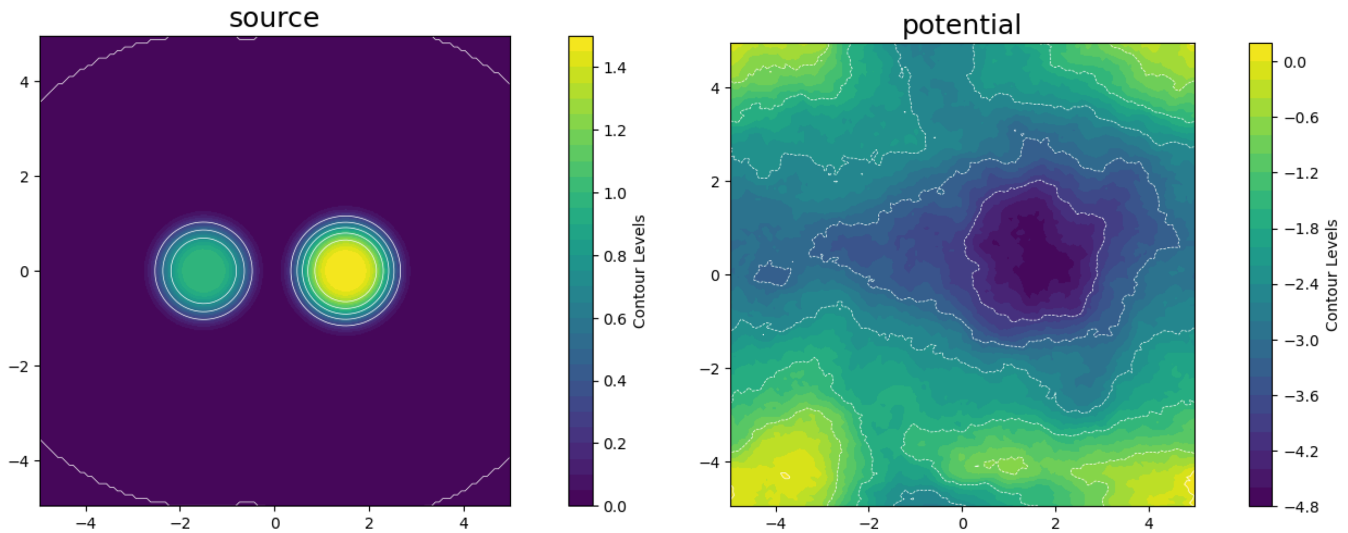


FIG. 1: Solution for problem 1.

2. Periodic boundary condition: the finite difference at the edge of grid need to consider the grid elements "on the other side". So matrix $\mathbf{A}$ from laplace operator will change.

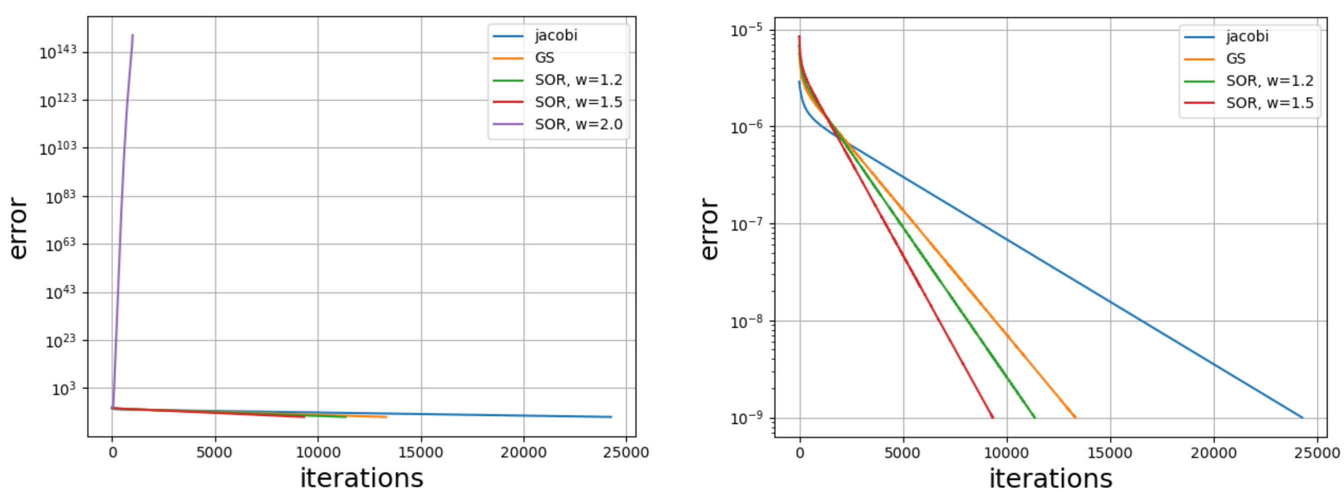


(a) (b)

FIG. 2: Color plots of source and potential solution in problem 2.

For problem 1 and problem 2, we can see that positive source actually means "sink": the divergence is positive. Notice that this problem uses periodic boundary condition without specific numbers, so we can shift potential.

3. Since the behavior of SOR method with $w = 2.0$ is weird, I make another plots without it.



(a) Number of iterations for SOR method with $w = 2.0$ is $2 \cdot 10^5$

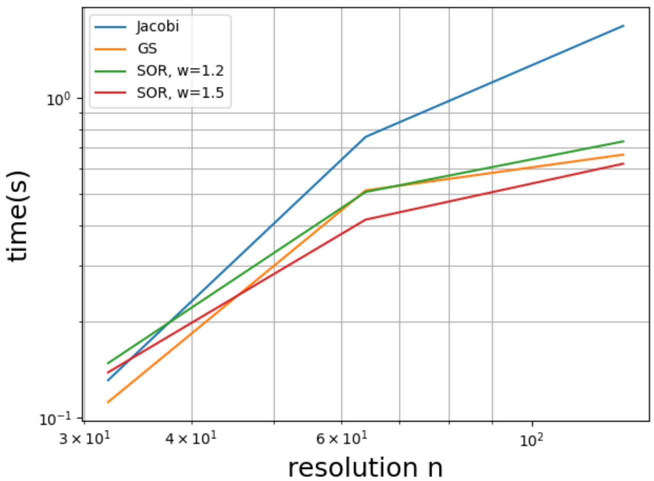
(b)

FIG. 3: Color plots of source and potential solution in problem 2.

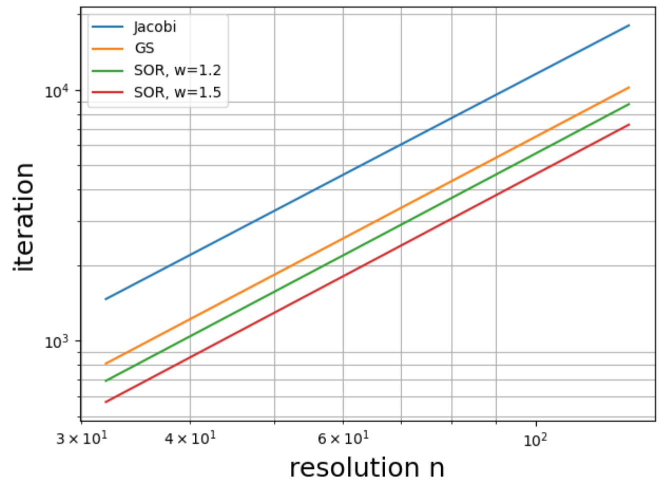
From Fig. 3b, if we only run these methods till error$> 10^{-6}$, Jacobi method has the smallest error. But when we require smaller error such as 10^{-9}, SOR behaves better. Since Jacobi method converges slow, the change in the beginning of simulation is small, which reduces error. As for SOR with $w = 2.0$, the strange behaviors might comes from discarding information from past completely. Remember $x^{k+1}_{SOR} = (1-w)x^k + wx^{k+1}_{GS}$, and just half of information in Gauss-Seidel model is from past, so when $w = 2.0$, it differs itself from past a lot.

4. iteration for certain error seams to be proportional to n^m. If we fit it, then we can get $n \sim 1.82$. This means the error is larger when we use higher resolution, meaning hard to convergence. Also, from Fig. 3b, log of error seams to be linear with iteration, so error is proportional to exp(-A $\times$ iterations), which means increasing iterations really helps convergence.

Now we look at Fig. 4b again, we can know error is proportional to exp(B $\times$ n^C) and confirm the idea we mentioned before.



(a) Computing times vs. resolution



(b) Iteration vs. resolution