Comphys Homework 1

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I. WRITING ASSIGNMENTS

1. Evaluate the general solutions of the below ODEs:

   (a) $m\ddot{x} + kx = 0$
   Substitute $x = e^{bt}$ into the equation:

$$mb^2 e^{bt} + ke^{bt} = 0 \tag{1}$$

Divide it by e^{bt}:

$$mb^2 + k = 0 \Rightarrow b = \pm i\sqrt{k/m} = \pm i\omega_0, \tag{2}$$

$$x = A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t},\ A_+, A_- \in \mathbb{C}. \tag{3}$$

Now, since we consider real x,

$$x = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t,\ A_1, A_2 \in \mathbb{R}, \tag{4}$$

or

(5)

$$x = A \cos(\omega_0 t + \phi),\ A, \phi \in \mathbb{R}. \tag{6}$$

   (b) $m\ddot{x} + \lambda\dot{x} + kx = 0$
   Rearrange this equation first:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \text{ where } \gamma = \lambda/2m,\ \omega_0 = \sqrt{k/m}, \tag{7}$$

Substitute $x = e^{bt}$ into the equation and divide it by e^{bt}:

$$b^2 + 2\gamma b + \omega_0^2 = 0 \Rightarrow b = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \tag{8}$$

Consider real solutions, we have three conditions:

   i. $\gamma > \omega_0$

$$x = A_1 e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + A_2 e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t},\ A_1, A_2 \in \mathbb{R}. \tag{9}$$

   ii. $\gamma = \omega_0$

$$x = (A_1 + A_2 t)e^{-\gamma t},\ A_1, A_2 \in \mathbb{R}. \tag{10}$$

   iii. $\gamma < \omega_0$
   Let $\omega = \sqrt{\omega_0^2 - \gamma^2}$, consider part (a):

$$x = Ae^{-\gamma t} \cos(\omega t + \phi),\ A, \phi \in \mathbb{R}. \tag{11}$$

Thus,

$$x = \begin{cases} A_1 e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + A_2 e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t} & \text{for } \gamma > \omega_0 \text{ (overdamping)} \\ (A_1 + A_2 t)e^{-\gamma t} & \text{for } \gamma = \omega_0 \text{ (critical damping)} \\ Ae^{-\gamma t} \cos(\omega t + \phi) & \text{for } \gamma < \omega_0 \text{ (underdamping)} \end{cases} \tag{12}$$

(c) $m\ddot{x} + \lambda\dot{x} + kx = F_0 \cos\omega_f t$

Notice that the solution for part (b) is also the general solution of this problem. Let's call it x_h. Now we need to find the particular solution x_p Rearrange this equation first:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega_f t},\ \text{take the real part.} \tag{13}$$

Substitute $x = Ae^{i\omega_f t}$ into this equation and divide it by $e^{i\omega_f t}$:

$$A(-\omega_f^2 + 2i\gamma\omega_f + \omega_0^2) = \frac{F_0}{m}. \tag{14}$$

$$A = \frac{F_0}{m(-\omega_f^2 + 2i\gamma\omega_f + \omega_0^2)} = \frac{F_0}{m} \frac{\omega_0^2 - \omega_f^2 - 2i\gamma\omega_f}{(\omega_0^2 - \omega_f^2)^2 + (2\gamma\omega_f)^2}, \tag{15}$$

consider the real part,

$$x_p = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega_f^2)^2 + (2\gamma\omega_f)^2}} \cos(\omega_f t + \theta_0),\ \theta_0 = \arctan\left(-\frac{2\gamma\omega_f}{\omega_0^2 - \omega_f^2}\right), \tag{16}$$

and the final solution is

$$x = x_h + x_p \tag{17}$$

Notice that as time pass, x becomes closer to x_p (stable state).

II. PROGRAMMING ASSIGNMENTS

1. The initial conditions of (b), (c) is weird. From part (b) of the writing assignments, (b) is critical damping, and (c) is overdamping. However, all of them use the underdamping case (equation (11)) to provide $x(0)$ and $\dot{x}(0)$ in the problem. Use equation (11) to get $x(0)$ and $\dot{x}(0)$, with $A = 1$ cm, $\phi = -\pi/2$ rad, $\omega_0 = 1$ rad s^{-1}. Then:

   (a) $\gamma = 0.2$ s^{-1} (underdamping)

   $x(0) = A\cos\phi = 0$ cm, $\dot{x}(0) = A(-\gamma\cos\phi - \omega\sin\phi) = \sqrt{0.96}$ cm s^{-1}

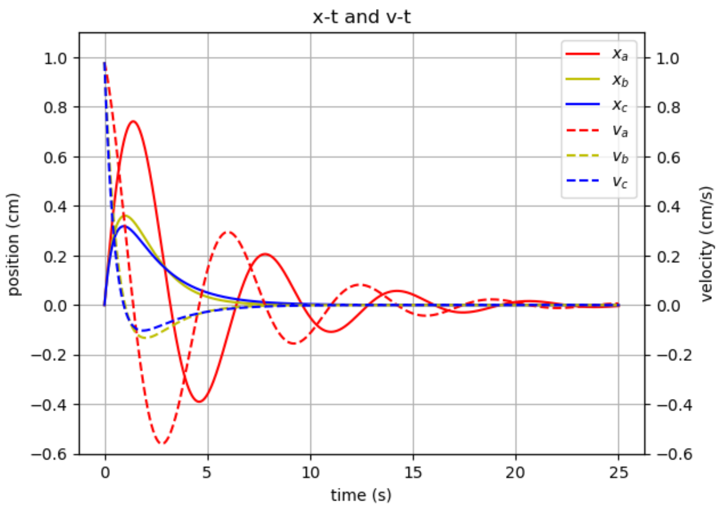
   (b) $\gamma = 1.0$ s^{-1} (critical damping)

   $x(0) = A\cos\phi = 0$ cm, $\dot{x}(0) = A(-\gamma\cos\phi - \omega\sin\phi) = 0$ cm s^{-1}

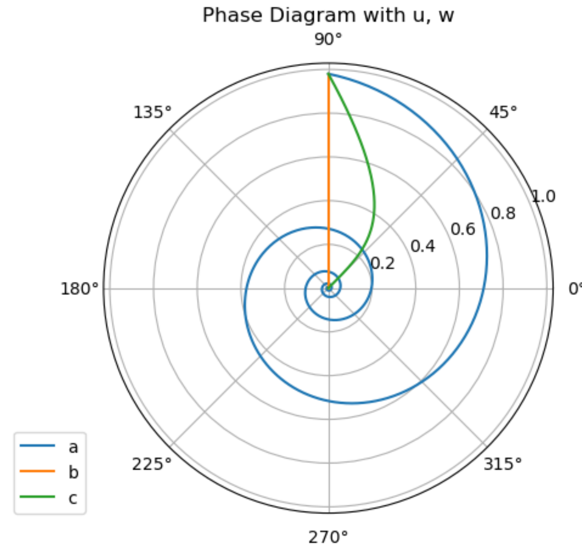
   (c) $\gamma = 1.2$ s^{-1} (overdamping, $\omega = \sqrt{\gamma^2 - \omega_0^2}$ to get initial condition.)

   $x(0) = A\cos\phi = 0$ cm, $\dot{x}(0) = A(-\gamma\cos\phi - \omega\sin\phi) = \sqrt{0.44}$ cm s^{-1}

   Notice that the case of critical damping will not move, and there is no need to simulate this case. This is because the problem uses wrong case of $x(t)$. Instead, I will use $x(0) = 0$ cm and $\dot{x}(0) = \sqrt{0.96}$ cm s^{-1} as initial conditions for all cases.



(a) x-t and v-t

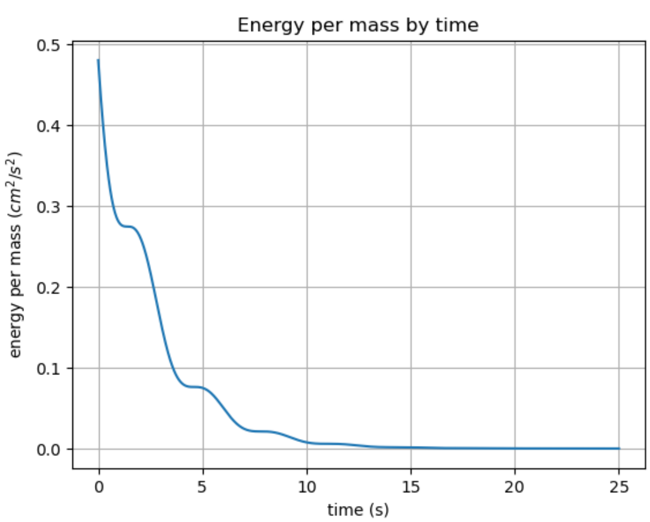


(b) Phase diagram

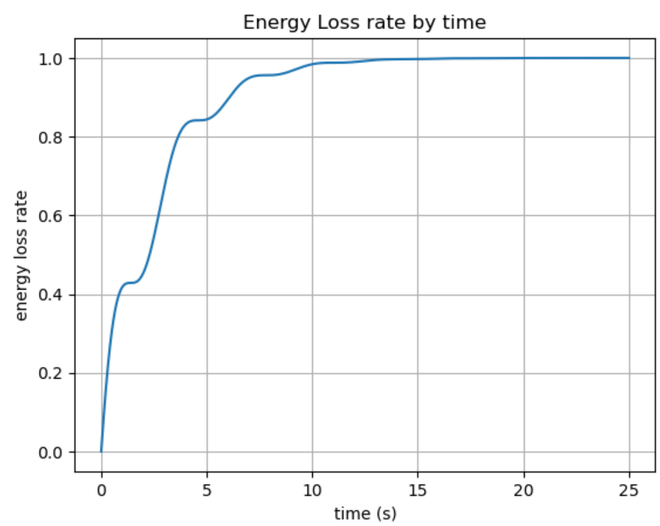
FIG. 1: Time evaluation of damped oscillation. Fig. 1b uses polar coordinate. Its x-axis is $u = \omega x$, and y-axis is $w = \gamma x + \dot{x}$

We can see that in Fig. 1a, only the underdamping case can oscillate back and forth. Also, critical damping case decays faster than overdamping case.

2. Total energy and energy loss of 1(a):



(a) E-t



(b) Energy loss

FIG. 2: Time evaluation of total energy and energy loss with condition of 1(a) (underdamping).

3. In forced oscillation, we can calculate resonance frequency ω_R, which has maximum amplitude, from Eq.(16):

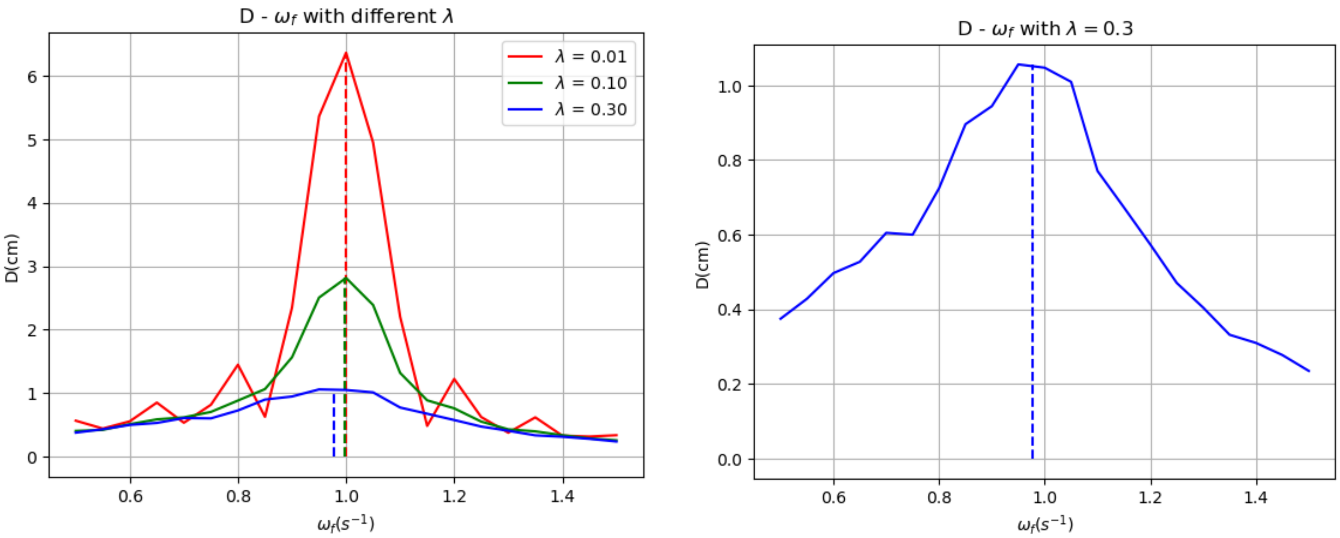
$$|A| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_f^2)^2 + (2\gamma\omega_f)^2}} \tag{18}$$

We only have to consider the minimum of the denominator:

$$h(\omega_f) = (\omega_0^2 - \omega_f^2)^2 + (2\gamma\omega_f)^2 \tag{19}$$

$$h'(\omega_R) = 4\omega_R^3 + (-4\omega_0^2 + 8\gamma^2)\omega_R = 0 \rightarrow \omega_R = \sqrt{\omega_0^2 - 2\gamma^2}$$

Notice that we use $D = \langle |x(t)| \rangle$ for $t = 40$ $50s$ as a parameter to judge magnitude. When amplitude is larger, D will be larger if we begin and end at the same phase (or, adding π). However, because we use time to choose the data for average, the result will be influence larger when λ become larger, producing smaller amplitude.



(a) $D - \omega_f$ with different λ

(b) $\lambda = 0.3$

FIG. 3: $D - \omega_f$ with different λ. The dashed lines are the analytical resonance frequency ω_R of each case.

From Fig. 3a, we do find the resonance. Only the resonance frequency with $\lambda = 0.3$ is different from the analytical solution. Apart from the reason we mentioned above, another reason is because the separation of ω_f is too large (0.05 vs $\omega_R = 0.977$).

4. Consider the series RLC circuit shown driven by an alternating emf of value $E_0 \sin \omega t$.

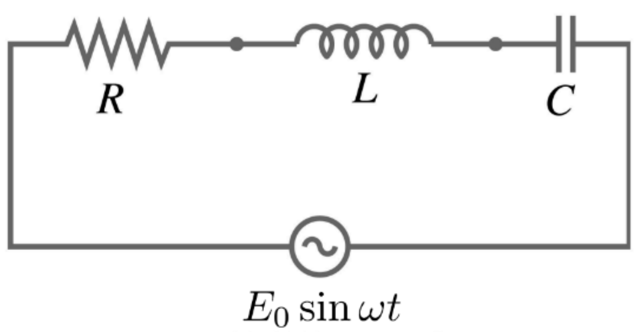


FIG. 4: RLC circuit for problem 4[1].

(a) First, calculate the voltage of every component:

$$V_R = IR,\ V_L = L\dot{I},\ V_C = q/C,\ I = \dot{q} \tag{20}$$

From Kirchoff's equation, the sum of voltage of a closed circuit is zero:

$$E_0 \sin \omega t - V_R - V_L - V_C = 0$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_0 \sin \omega t$$

(b) Current and voltage of the inductor from numerical solution:

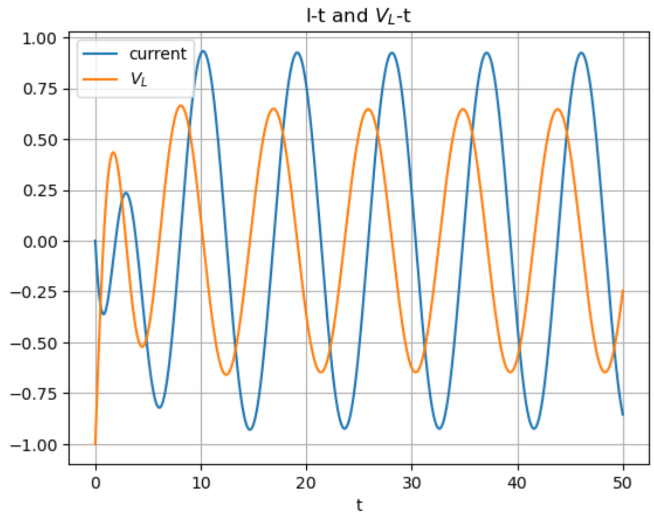


FIG. 5: I-t and V_L-t.

(c) Using similar way in problem 3, we can find resonance frequency for V_C, V_R, V_L:

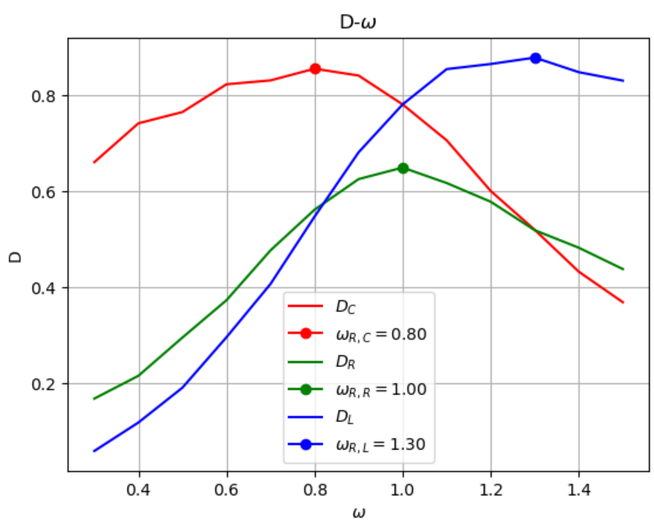


FIG. 6: D-ω for V_C, V_R, V_L. ω_R is the frequency where the amplitude of voltage is maximum.

Fig. 6 shows that the resonances of different components happen at different frequencies, which also means that the resonances of charge and current happen at different frequencies, too. The resonance frequencies from large to small are from inductor, resistor, capacitance. this is because time derivative of charge will multiply amplitude by ω. See Eq. (20), and the form of charge is similar to Eq. 16 with $m \to L$, $\lambda \to R$, $k \to 1/C$, $F_0 \to E_0$.

[1] K.-C. Pan, `https://drive.google.com/file/d/1GOkVetCkKvLAOS5_hORIpOduluZEQzNQ/view`, [Online; accessed 7-April-2024].