Master SRI & Master SAR

Game Theory and Applications

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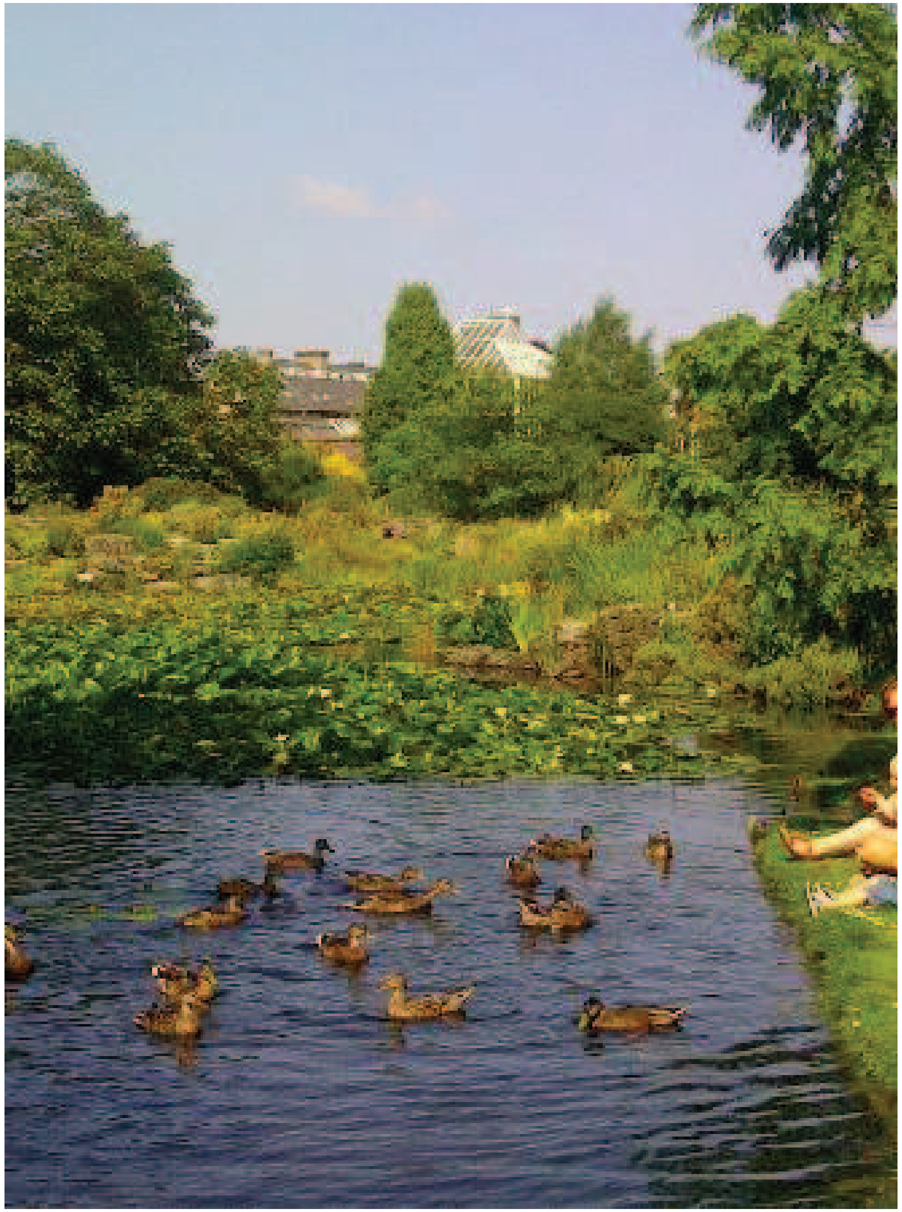
March 12 2021 – April 2 2021

0. Introduction

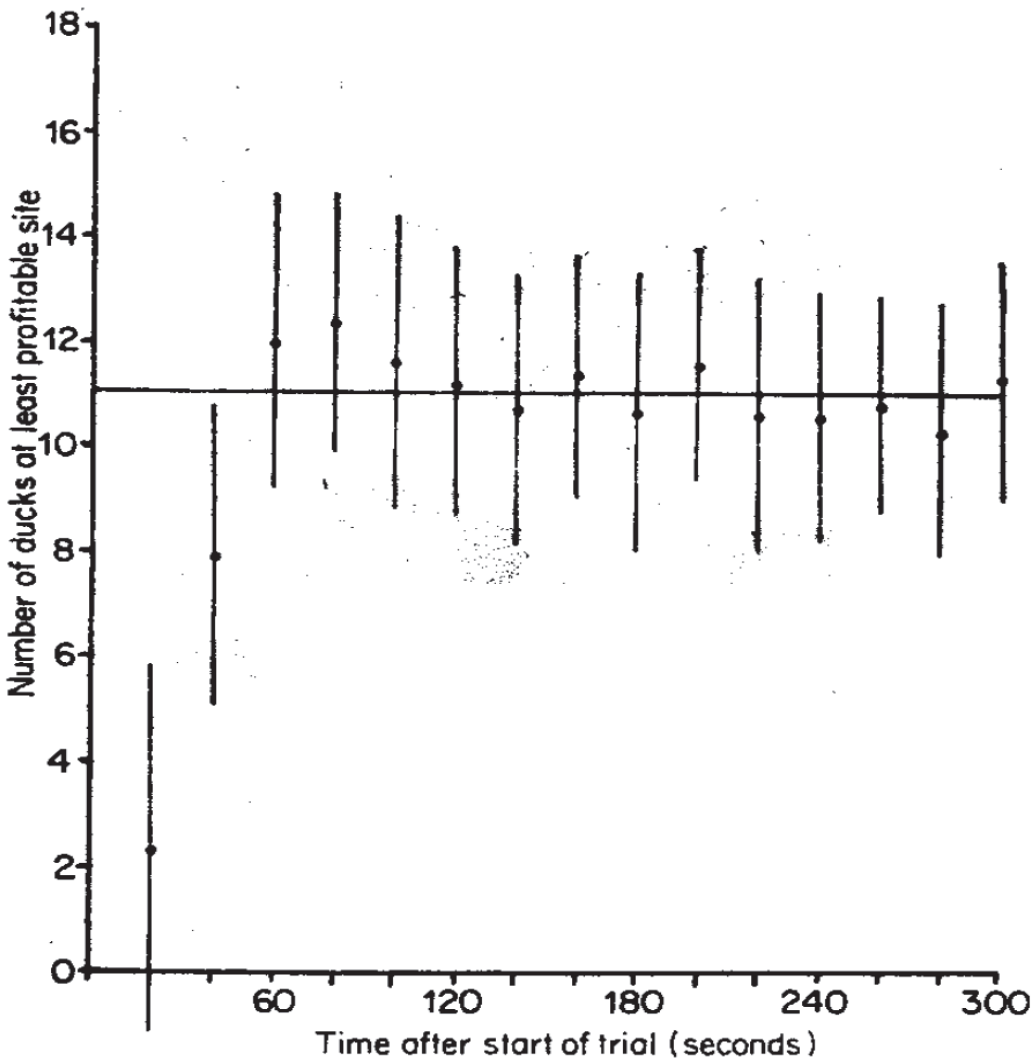
Cambridge University, UK, winter 1979



A couple of details about one of the experiments

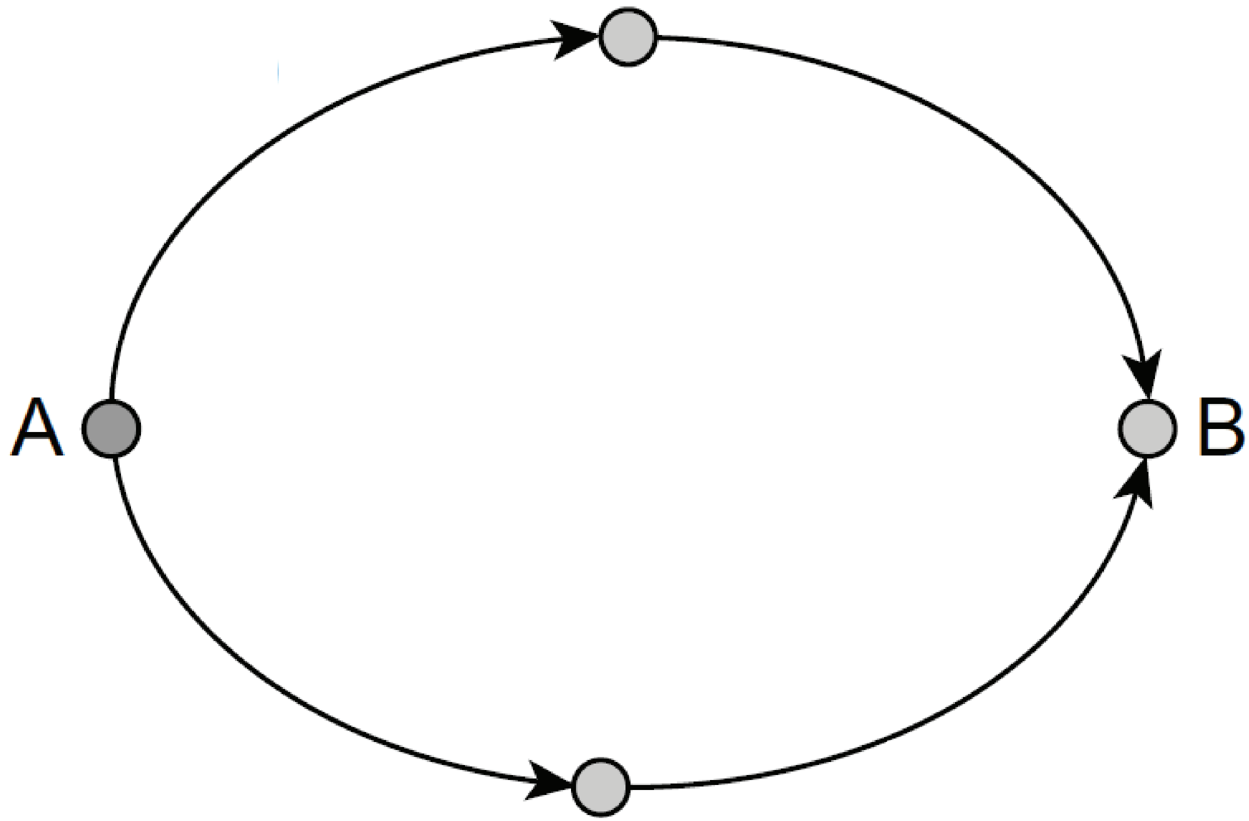
- 33 ducks.
- Two observers/sites 20 m apart.
- Site 1: 12 items/min.
- Site 2: 24 items/min.

Observations

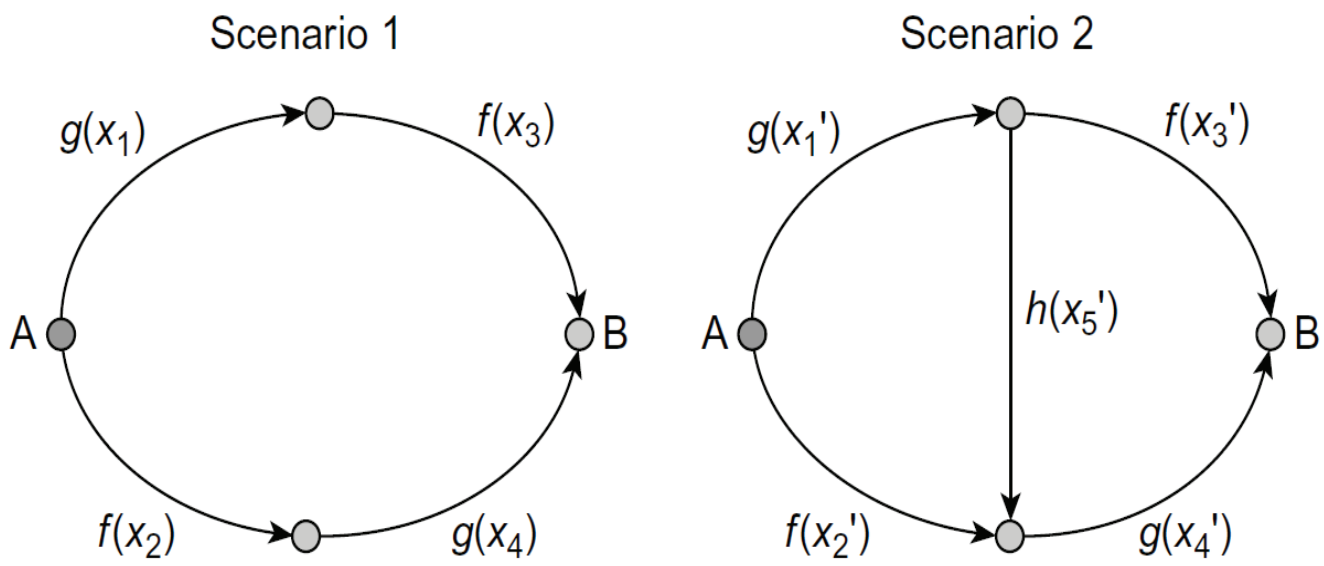


D. G. Harper, "Competitive foraging in mallards: Ideal free ducks", *Anim. Behav.*, 1982, 30, 575-585.

Ducks become drivers

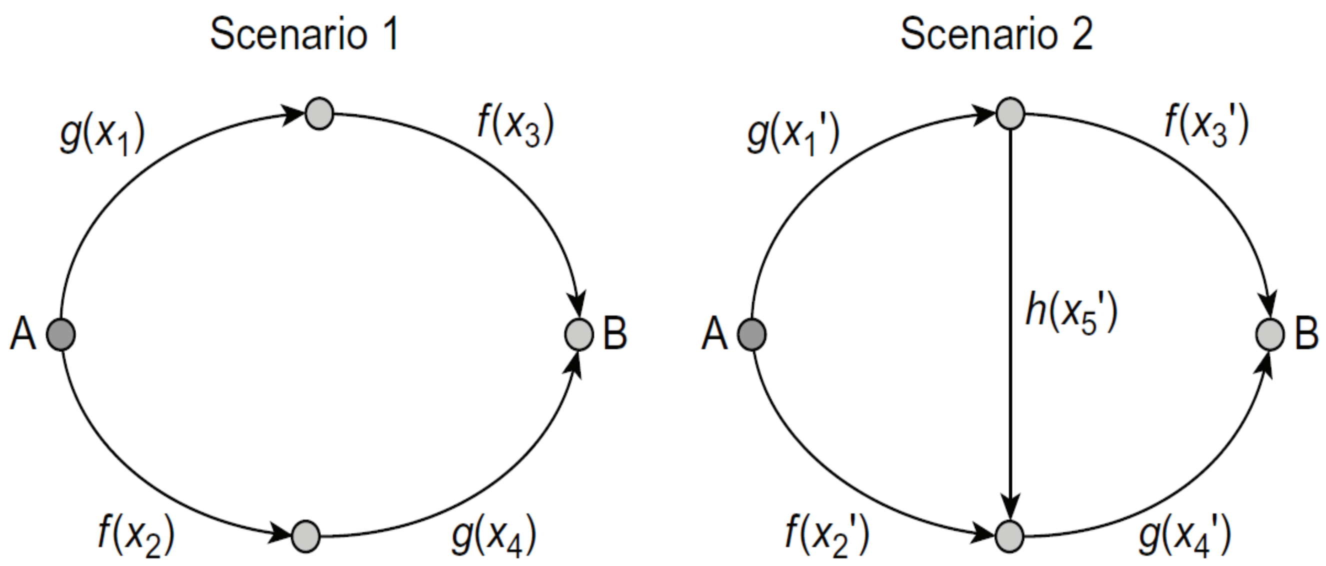


Stuttgart, Germany, 1969 [Braess 1969]



Input flow = 6	
$f(x) = x + 50$, $g(x) = 10x$	
$h(x) = +\infty$	$h(x) = x + 10$

Stuttgart, Germany, 1969 [Braess 1969]



Input flow = 6	
$f(x) = x + 50$, $g(x) = 10x$	
$h(x) = +\infty$	$h(x) = x + 10$
83 min	92 min
$(x_1, x_2) = (3, 3)$	$(x_1', x_2', x_3') = (4, 2, 2)$

Observing Braess-type instances

In the real life

- Stuttgart 1969: investments into the road network $\Rightarrow$ traffic $\searrow$. Section of newly-built road closed $\Rightarrow$ traffic $\nearrow$ [Knödel 1969].
- NYC 1990: closing of 42nd street in New York City $\Rightarrow$ amount of congestion in the area $\nearrow$ [New York Times 1990].
- Seoul 2003: one of the three tunnels shut down to restore a river and a park $\Rightarrow$ traffic flow improved.

About the paradox

Trivial inequality in standard optimization

$$\max_{x \in \mathcal{A}} f(x) \leq \max_{x \in \mathcal{B}} f(x)$$

when $\mathcal{A} \subseteq \mathcal{B}$.

About the paradox

Trivial inequality in standard optimization

$$\max_{x \in \mathcal{A}} f(x) \leq \max_{x \in \mathcal{B}} f(x)$$

when $\mathcal{A} \subseteq \mathcal{B}$.

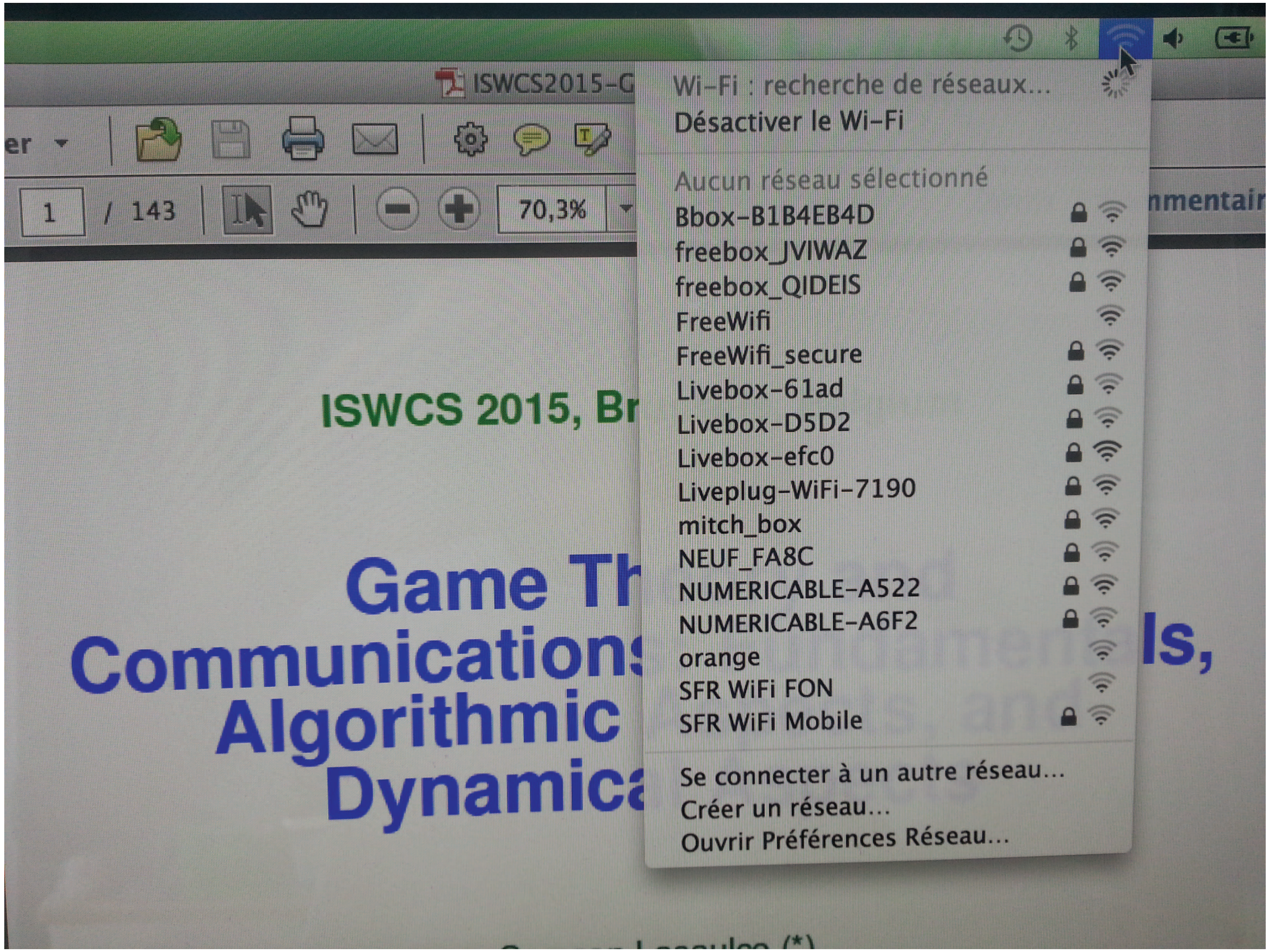
This inequality does not hold anymore under partial control of

$$x = (x_1, ..., x_K).$$

and in presence of multiple performance metrics

Why only partial control?

① **Complexity** issues. Example: channel selection with 16 channels and 16 users $16^{16} = 2^{64} > 10^{18}$.



Why only partial control? Continued

② There are **several effective decision-makers** (DMs).

Let's recap. DO-GT-MOO

► Distributed optimization (DO): typically about partial control with one DM.

► Multi-objective optimization (MOO): typically about one DM with full control + several objectives [Björnson et al 2015].

► ”Non-cooperative” game theory (GT): typically about several (virtual/real) DMs with partial control + several objectives.
What is the meaning of optimality then?

Typical issues in scenarios with partial control and multiple objectives

► Which solution concept to consider as a possible game outcome?

► Does it exist for the game of interest? Is it unique?

► Is it efficient? How do we measure efficiency? How do we improve it?

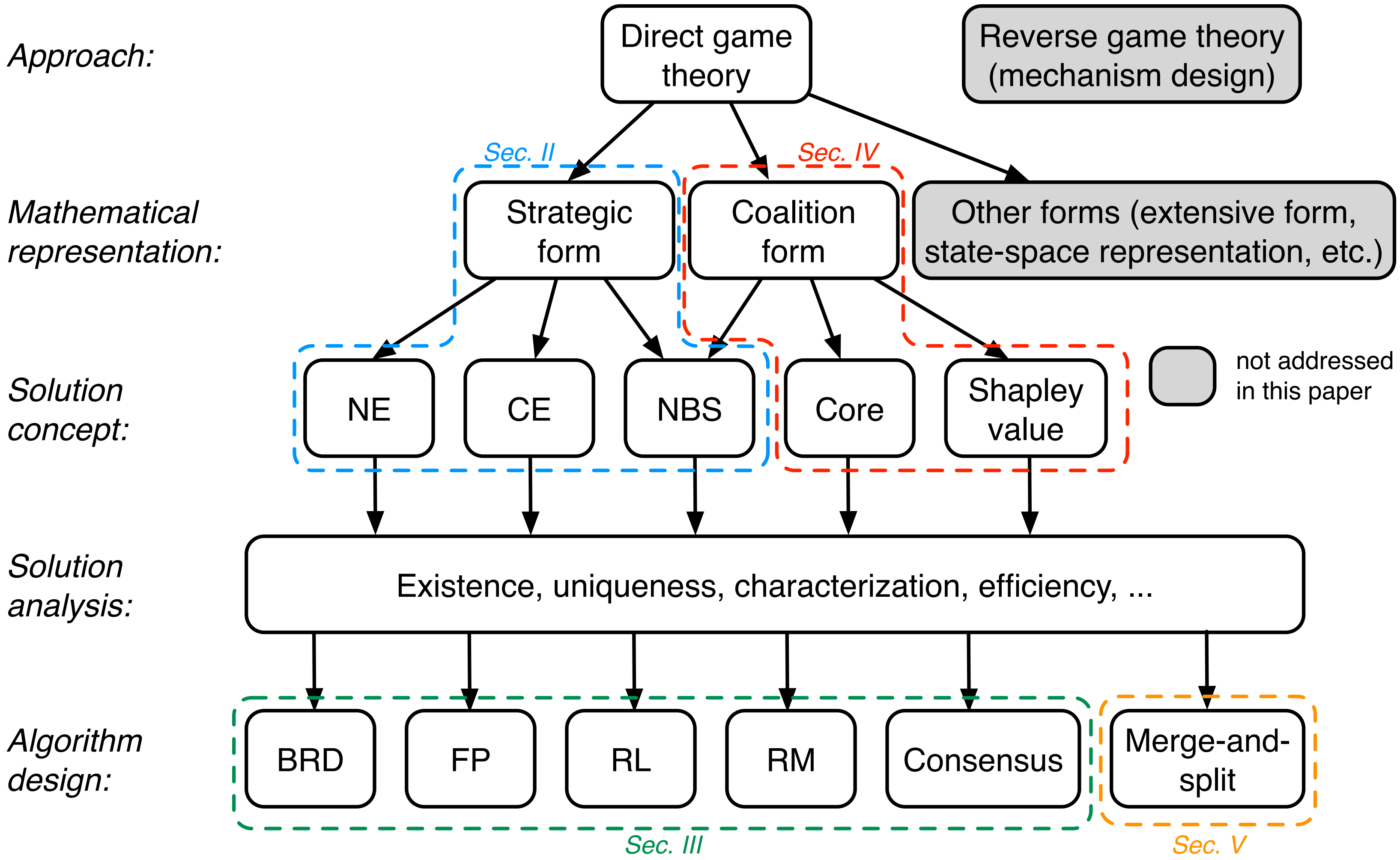
► NE: What is it? Existence? Uniqueness? Efficiency? Existence of a convergent and implementable algorithm?

What is a game exactly?

Main mathematical representations

- Strategic or normal form games
- Extensive form games
- Coalitional form games

Very cheap map of the game theory jungle



[Bacci et al 2015]

Outline

- ☐ 0. Introduction (01–18).
- ☐ 1. Strategic form games (19–71).
- ☐ 2. Dynamic games (72–120).
- ☐ 3. Learning algorithms and strategic form games (121–134).
- ☐ 4. Coalitional form games (135–152).
- ☐ 5. Extensive form games (153–158).

Outline. Continued

☐ 6. Conclusion (159–167).

☐ 7. References (168–176).

☐ 8. Tutorial work (177–202).

☐ 9. Short research project (203–204).

1. Strategic form games

Strategic form definition

Game $\equiv$ triplet:

$$\mathcal{G} = \left(\mathcal{K}, \{\mathcal{S}_i\}_{i\in\mathcal{K}}, \{u_i\}_{i\in\mathcal{K}}\right).$$

- $\mathcal{K} = \{1, ..., K\}$ is the set of players.
- $\mathcal{S}_i$ is the set of strategies for Player i.
- Player i's payoff/utility function:

$$\begin{array}{rccc} u_i: & \mathcal{S}_1 \times ... \times \mathcal{S}_i \times ... \times \mathcal{S}_K & \to & \mathbb{R} \\ & \underbrace{(s_1, ..., s_i, ... s_K)}_{s\ :\ \text{strategy profile}} & \mapsto & u_i(s_i, s_{-i}) \end{array}.$$

Strategic form. Continued

General form

$$\mathcal{G} = \left(\mathcal{K}, \{\mathcal{S}_i\}_{i\in\mathcal{K}}, \{\succeq_i\}_{i\in\mathcal{K}}\right).$$

Utility function definition

The function $u : \mathcal{X} \to \mathbb{R}$ is a utility function for the binary relation $\succeq$ if

$$\forall (x, y) \in \mathcal{X}^2,\ x \succeq y \ \Leftrightarrow\ u(x) \geq u(y)$$

Existence of a utility function

Proposition (lexicographic preferences): there exist no utility functions for lexicographic ordering on $\mathbb{R}^2$ [Debreu 1954].

Proposition (countable preferences): there exists a utility function for every transitive and complete ordering on any **countable** set:

- completeness: $x \succeq y$ or $y \succeq x$ or both;
- transitivity: "$x \succeq y$ and $y \succeq z$" $\Rightarrow x \succeq z$.

Proposition (continuous preferences): there exists a utility function for every transitive, complete, and continuous ordering on a **continuous** set $\mathcal{X} \subset \mathbb{R}^N$ provided $\mathcal{X}$ is non-empty, closed, and connected:

- continuity: $\mathcal{B}(x) = \{y \in \mathcal{X} : x \succeq y\}$ and $\mathcal{W}(x) = \{y \in \mathcal{X} : y \succeq x\}$ are closed [Debreu 1954].

Security dilemma under strategic form

- Players $\mathcal{K} = \{\text{Country 1}, \text{Country 2}\}$.

- Strategies are merely actions
$\mathcal{S}_1 = \mathcal{S}_2 = \{\text{No Weapon}, \text{Weapon}\}$.

- Utility function for Player 1:

$$u_1(s_1, s_2) = \left| \begin{array}{llcll} 0 & \text{if} & (s_1, s_2) & = & (\text{N}, \text{W}) \\ 1 & \text{if} & (s_1, s_2) & = & (\text{W}, \text{W}) \\ 3 & \text{if} & (s_1, s_2) & = & (\text{N}, \text{N}) \\ 4 & \text{if} & (s_1, s_2) & = & (\text{W}, \text{N}) \end{array} \right. .$$

Security dilemma under matrix form

C1, C2	N	W
N	(3, 3)	(0, 4)
W	(4, 0)	(1, 1)

A fundamental solution concept: The Nash equilibrium (NE)

Pure Nash equilibrium. Strategy vector/profile such that

$$\forall i \in \mathcal{K}, \forall s_i \in \mathcal{S}_i, \; u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

Mixed Nash equilibrium ...

Mixed strategies

Mixed strategies

Mixed strategies and mixed NE

- **Mixed strategies** $\sigma_i \in \Delta(\mathcal{S}_i)$ with

$$\Delta(\mathcal{S}_i) = \left\{ x \in \mathbb{R}^{|\mathcal{S}_i|} : x_j \geq 0, \sum_j x_j = 1 \right\}$$

- **Expected utility**

$$\widetilde{u}_i(\sigma_1, ..., \sigma_K) = \mathbb{E}_{\sigma_1 \otimes ... \otimes \sigma_K} \left[u_i(s_1, ..., s_K) \right].$$

- **Mixed Nash equilibrium**

$$\forall i \in \mathcal{K}, \forall \sigma_i \in \Delta(\mathcal{S}_i),\ \widetilde{u}_i(\sigma_i^*, \sigma_{-i}^*) \geq \widetilde{u}_i(\sigma_i, \sigma_{-i}^*).$$

Examples

- **Prisoner's dilemma**

	N	W
N	(3,3)	(0, 4)
W	(4,0)	(1,1)

- **Coordination game**

	Low	High
High	(5 Mbit/s, 1 Mbit/s)	(0, 0)
Low	(4 Mbit/s, 4 Mbit/s)	(1 Mbit/s, 5 Mbit/s)

Free rider game

C1, C2	Provide	Free ride
Provide	$\left(1-\frac{c}{2}, 1-\frac{c}{2}\right)$	$\left(1-c, 1\right)$
Free ride	$\left(1, 1-c\right)$	(0, 0)

where $1 < c < 2$ (resp. $\frac{c}{2}$) represents the cost of contributing alone (resp. at two)

Solution concepts for strategic/extensive form games

- **Pure/mixed Nash equilibrium, Wardrop equilibrium**,
- **correlated equilibrium, coarse correlated equilibrium,**
- $K-$strong equilibrium,
- Nash equilibrium refinements : trembling hand perfect equilibrium, proper equilibrium,
- $\epsilon-$Nash equilibrium,
- logit equilibrium,

Solution concepts for strategic/extensive form games. Continued

- maxmin strategy profiles,
- Bayesian equilibrium,
- evolutionary stable solution,
- satisfaction equilibrium, generalized Nash equilibrium,
- Stackelberg equilibrium,
- **Pareto optimum, social optimum**,
- bargaining solutions (Nash, egalitarian, Kalai-Smorodinsky, etc.),...

Solution concepts for coalition form games

- **core, nucleolus,**
- $\epsilon-$ core,
- least core,
- kernel,
- bargaining set,
- **Shapley value**, Harsanyi value, Banzhaf index,...

Three strengths of the Nash equilibrium

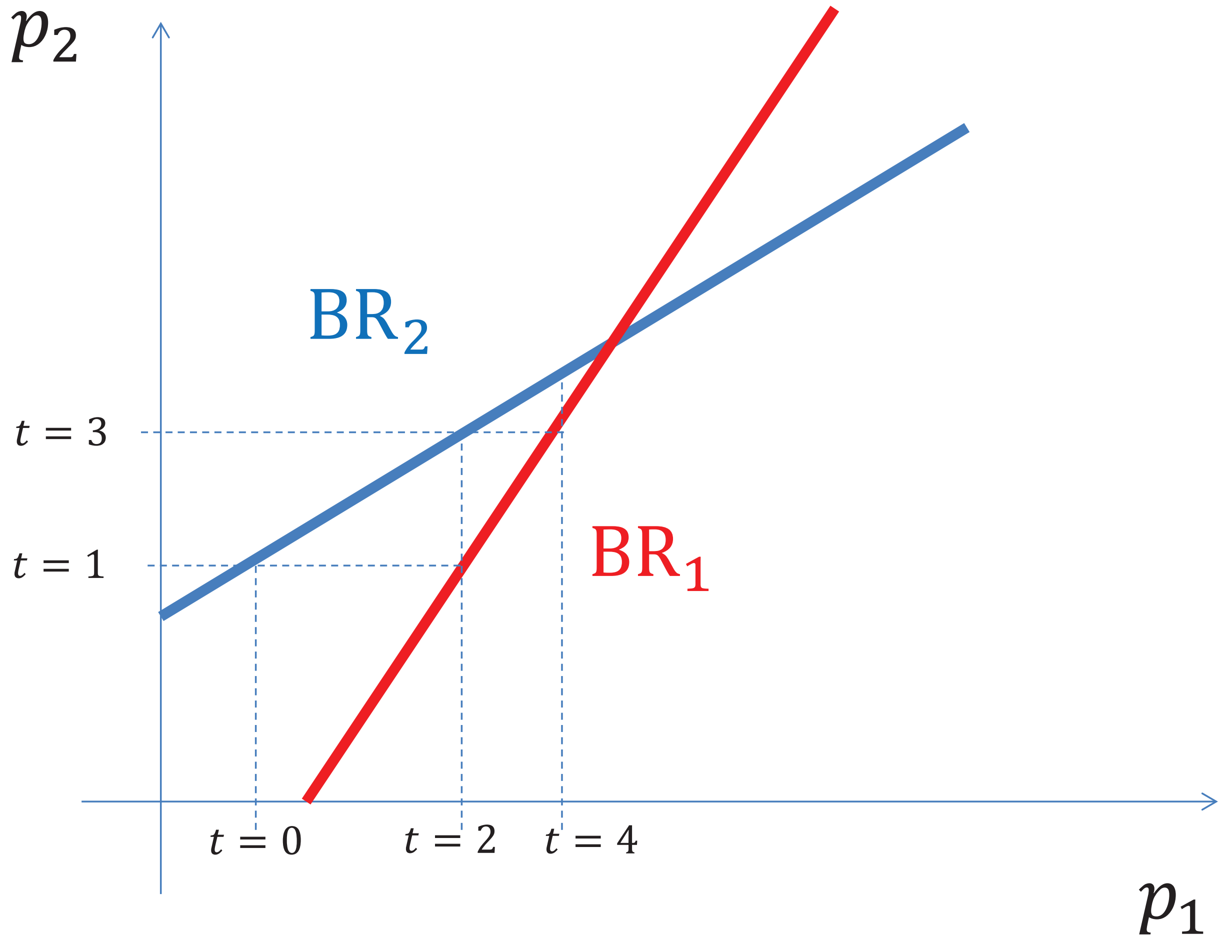
- It almost "always" exists
- Stability property (once you are there)
- Dynamical property (to get there)

Dynamical property: Special case

Best-response

$$\mathrm{BR}_i(s_{-i}) = \text{arg}\max_{s_i \in \mathcal{S}_i} u_i(s_i, s_{-i}).$$

Illustration for continuous sets [Cournot 1838]



The sequential best-response dynamics (1/3)

C1, C2	N	W
N	(3, 3)	(0, 4)
W	(4, 0)	(1, 1)

The sequential best-response dynamics (2/3)

C1, C2	N	W
N	(3, 3)	(0, 4)
W	(4, 0)	(1, 1)

The sequential best-response dynamics (3/3)

C1, C2	N	W
N	(3, 3)	(0, 4)
W	(4, 0)	(1, 1)

Nash equilibrium characterization

A strategy profile s^* is an NE of $\mathcal{G}$ iff:

$$s_i^* \in \mathrm{BR}_i(s_{-i}^*) \;\Leftrightarrow\; s^* \in \mathrm{BR}(s^*)$$

where

$$\begin{aligned} \mathrm{BR}: \;\; \mathcal{S} \;&\rightarrow\; 2^{\mathcal{S}} \\ s \;&\mapsto\; \mathrm{BR}_1(s_{-1}) \times \mathrm{BR}_2(s_{-2}) \times \ldots \times \mathrm{BR}_K(s_{-K}) \end{aligned}$$

Nash existence theorem [Nash 1950]. $\mathcal{S} = \mathcal{S}_1 \times ... \times \mathcal{S}_K$ is finite. Then, there is a mixed NE.

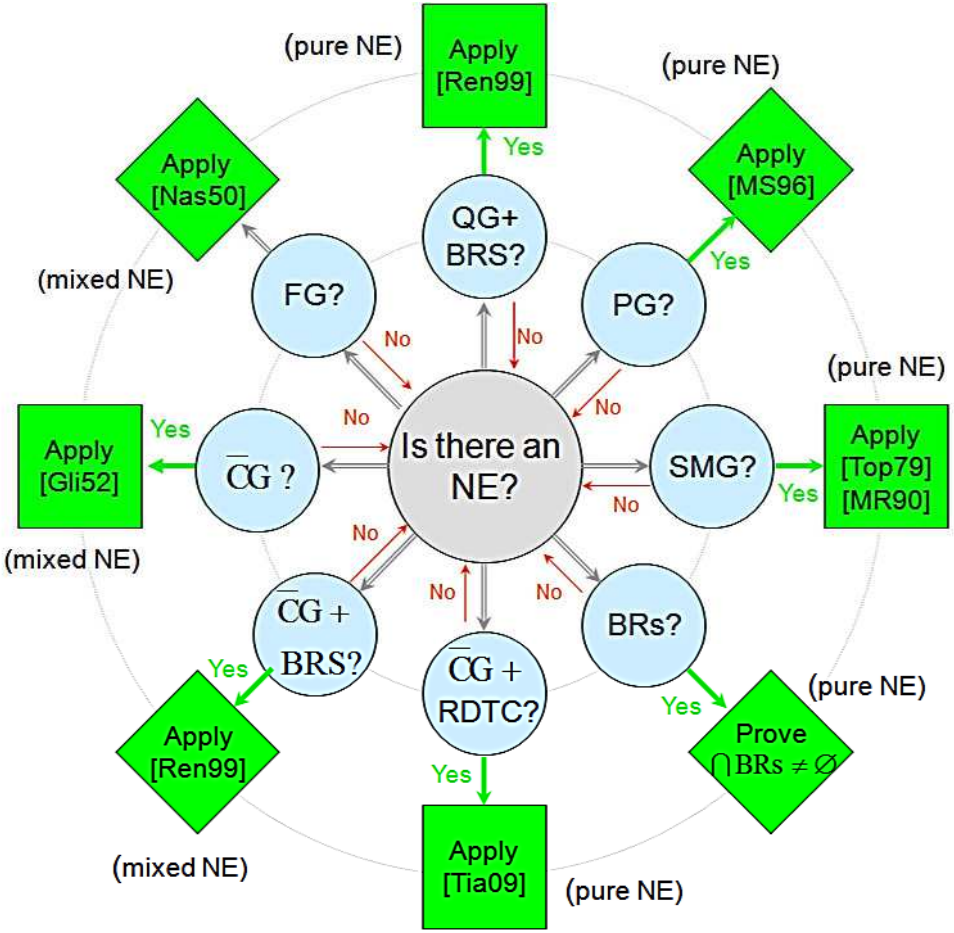
Kuhn existence theorem [Kuhn 1953]. Every finite game of perfect information has at least one pure NE.

Nash equilibrium existence (continuous strategy sets)

Glicksberg theorem [Glicksberg 1952]. $\mathcal{S}_i$ compact, u_i continuous in s. Then, there is a <u>mixed</u> NE.

Debreu-Fan-Glicksberg theorem [Debreu, Fan, Glicksberg 1952]. Above assumptions & u_i quasiconcave in s_i. Then, there exists a <u>pure</u> NE.

More about the existence of NE



[Lasaulce & Tembine 2011]

Simplified methodology for studying NE

Static games	**Dynamic games (repeated g., stochastic g.,**
Existence	Existence
Uniqueness	Characterization of equilibrium utilities
Efficiency	Design of strategies

Uniqueness (concave games)

Rosen theorem [Rosen 1965]

- $\mathcal{S}_i = \{s_i \in \mathbb{R}^{n_i} : h_{ij}(s_i) \geq 0\}$ with h_{ij} concave differentiable
- u_i continuous in s
- u_i concave in s_i + differentiable
- Diagonally strict concavity:

$$\exists r > 0,\ \forall s \neq s',\ \ [s - s']^T \left[\gamma(s;r) - \gamma(s';r)\right] < 0$$

where

$$\gamma(s;r) = \left[r_1 \nabla_{s_1} u_1(s), ..., r_K \nabla_{s_K} u_K(s)\right].$$

Then, there is a unique pure NE.

Uniqueness (concave games)

Remark: If $\mathbf{G}(s;r)+\mathbf{G}^{\mathrm{T}}(s;r)<0$ then the DSC is met (where $\mathbf{G}$ is the Jacobian matrix of $\gamma(s;r)$.

Uniqueness (standard games)

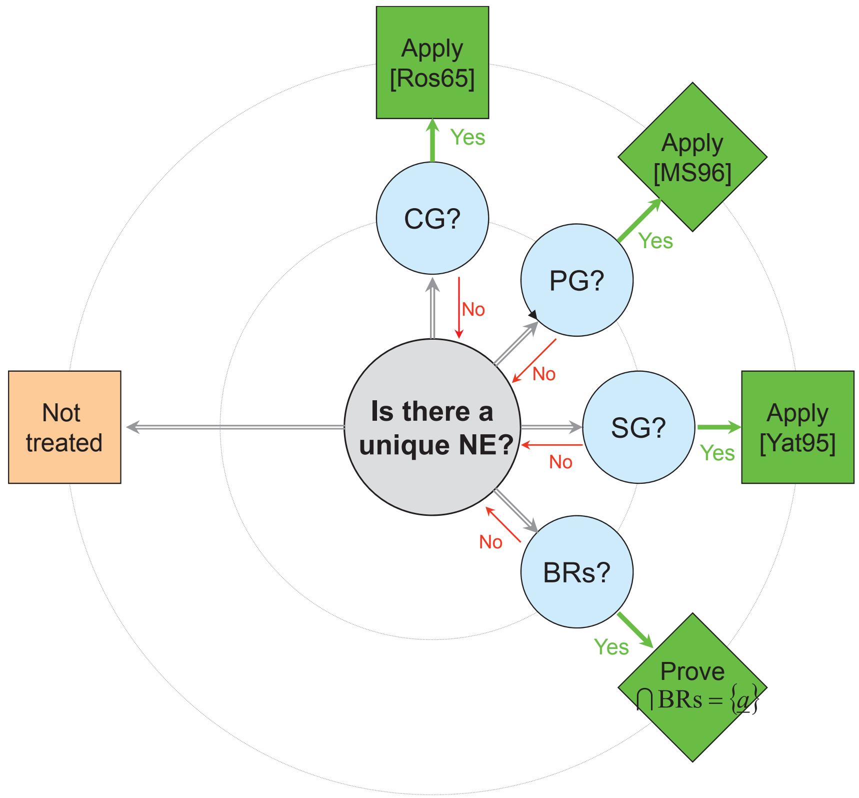
Definition (standard functions) A vector function $g : \mathbb{R}_+^K \to \mathbb{R}_+^K$ is standard if we have:

- Monotonicity: $\forall (x, x') \in \mathbb{R}_+^{2K}$, $x \leq x' \Rightarrow g(x) \leq g(x')$.
- Scalability: $\forall \alpha > 1$, $\forall x \in \mathbb{R}_+^K$, $g(\alpha x) < \alpha g(x)$.

Theorem [Yates 1995] If $\mathrm{BR} = (\mathrm{BR}_1, ..., \mathrm{BR}_K)$ is standard, then there is a unique pure NE.

Remark: BR intersection.

More about the uniqueness of NE



[Lasaulce & Tembine 2011]

Another important special class of games: Supermodular games

Definition (supermodularity): $\mathcal{S}_i$ compact subset of $\mathbb{R}$, u_i upper semi-continuous in s, $\forall s_{-i} \geq s'_{-i}$, $u_i(s) - u_i(s_i, s'_{-i})$ is non-decreasing in s_i.

Characterization:

$$\forall i \neq j, \frac{\partial^2 u_i}{\partial s_i \partial s_j} \geq 0.$$

Supermodular games. Continued

Properties

- Convergence of important dynamics ✓
- Existence of a pure NE ✓

Examples. Queueing problems [Yao 1995], power control problems [Saraydar et al 2002].

An important class of games: Potential games

Exact potential games [Monderer and Shapley 1996].
$\exists \Phi, \forall i, \forall s, \forall s'_i,$

$$u_i(s) - u_i(s'_i, s_{-i}) = \Phi(s) - \Phi(s'_i, s_{-i}).$$

Ordinal potential games [Monderer and Shapley 1996].
$\exists \Phi, \forall i, \forall s, \forall s'_i,$

$$u_i(s) - u_i(s'_i, s_{-i}) > 0 \Leftrightarrow \Phi(s) - \Phi(s'_i, s_{-i}) > 0$$

Remark: weighted EPG ($w_i \Phi$), generalized OPG ($\Rightarrow$), ...

An important class of games: Potential games

Characterization (special case). $\mathcal{S}_i = I_i \subset \mathbb{R}$. A game is an exact PG iff:

$$\forall (i,j) \in \mathcal{K}^2, \ \frac{\partial^2 (u_i - u_j)}{\partial s_i \partial s_j} = 0.$$

Potential games. Continued

Properties

- Convergence (sequential BRD, Lyapunov function, FIP, etc). ✓
- Existence of a pure NE ✓
- Class of **exact** potential games $\equiv$ vector subspace

Examples. Team games ($u_i = u$), dummy games ($u_i(s_{-i})$), self-motivated games ($u_i(s_i)$), congestion games.

Congestion games

Definition: A (strategic form) symmetric congestion game is given by $\mathcal{C} = (\mathcal{K}, \mathcal{P}, (C_i)_{i \in \mathcal{K}})$ where

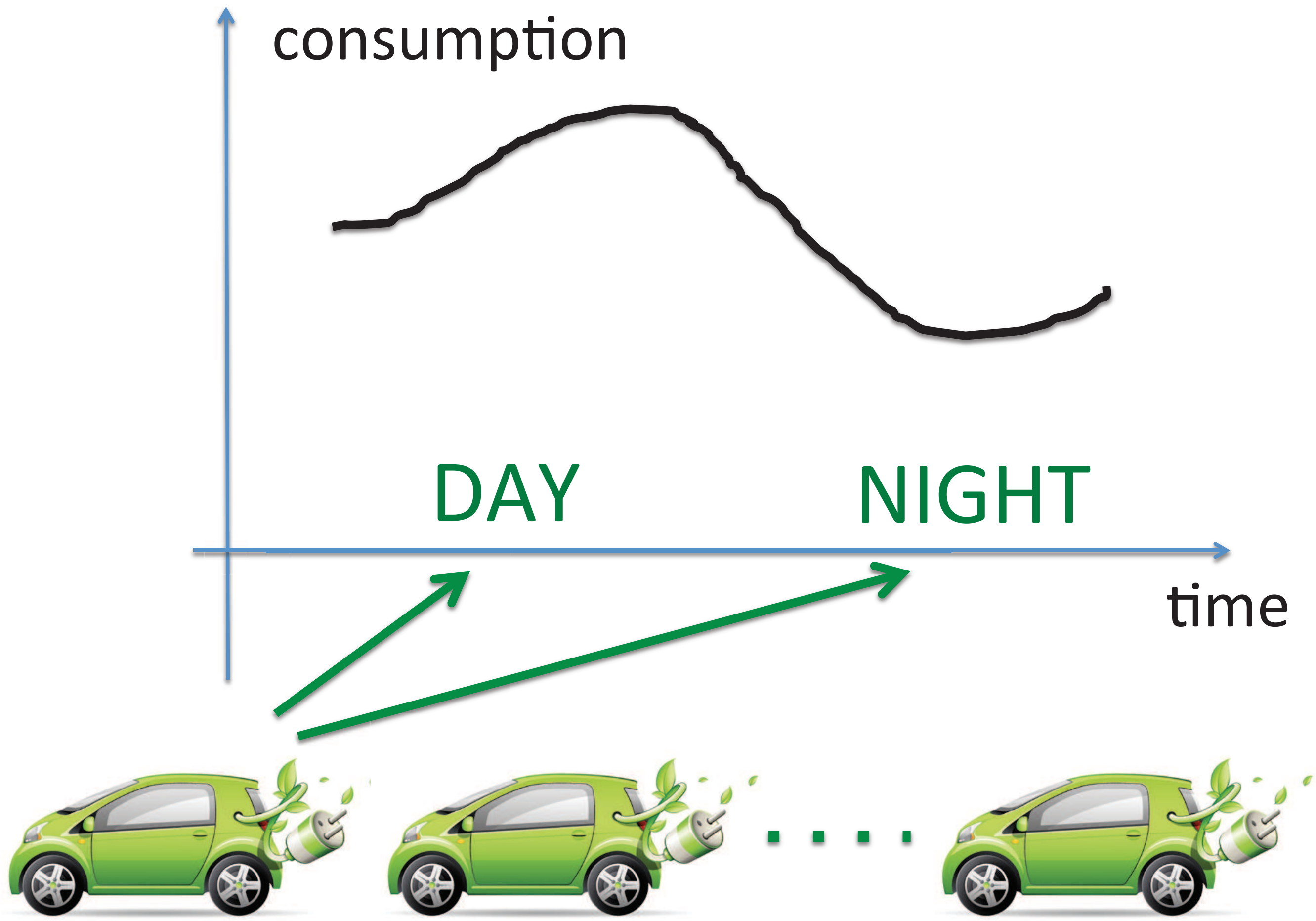
- $\mathcal{P} = 2^{\mathcal{E}}$ is the set of possible strategies (paths) for each player
- $\mathcal{E}$ is the set of facilities or resources (edges)
- $C_i(p_1, ..., p_K) = \sum_{e \in p_i} c_e(x_e)$
- $x_e = \sum_{p : e \in p} |\{i \in \mathcal{K} : p_i = p\}|$

Congestion games (continued)

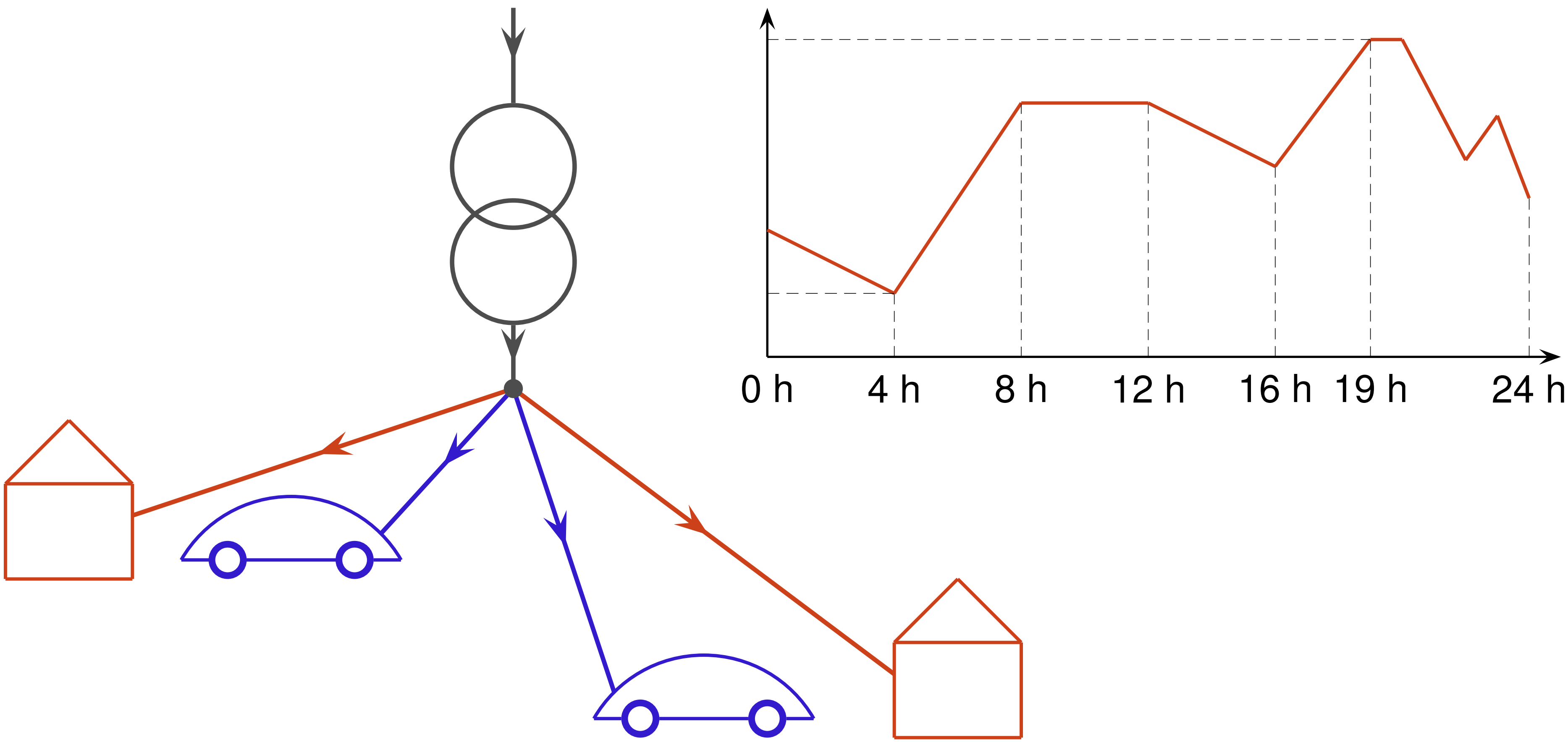
Result (Rosenthal 1973): The function

$\phi(p_1, ..., p_K) = \sum_{e \in \mathcal{E}} \sum_{i=1}^{x_e} c_e(i)$ is an exact potential for $\mathcal{C}$

Ducks → vehicles



The scheduling problem in a nutshell



Obviously the Nash equilibrium has also drawbacks

- Efficiency: typical consequence of partial control and non-alignement
- Correlation: mixed NE assume independent lotteries
- Strategic stability: only stable to single deviations
- Not fully adapted to constraints

For more drawbacks see [Perlaza & Lasaulce 2014]

How to measure efficiency: Pareto efficiency

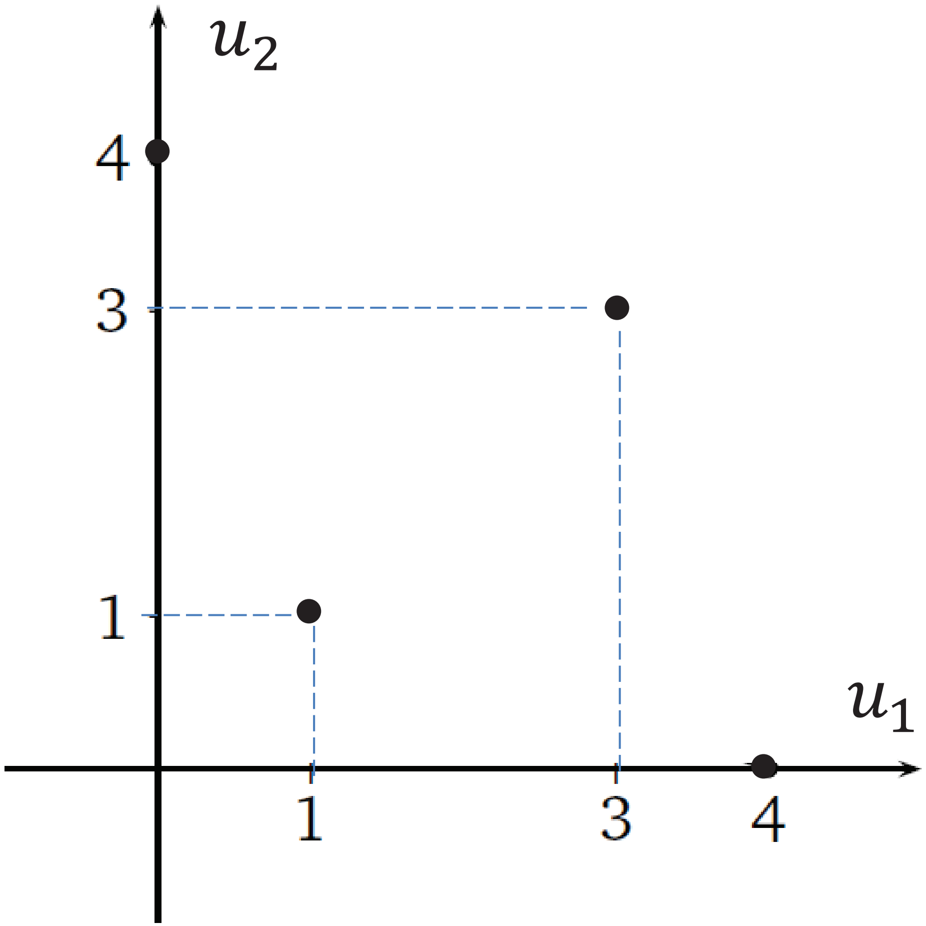
Definition (Pareto-dominance): s Pareto-dominates s' if:

$$\forall i \in \mathcal{K},\ u_i(s) \geq u_i(s'),$$

with strict inequality for at least one player.

Definition (Pareto-optimum): s^* is Pareto-optimal (-efficient) if it is dominated by no other profile.

Illustration of Pareto optimality



How to measure efficiency : Social welfare

Definition (social welfare): the social welfare of a game is defined as:

$$w = \sum_{i=1}^{K} u_i.$$

Definition (social optimum): an SO is a strategy profile which maximizes w.

Remark: An SO is a PO.

How to measure efficiency : PoA and PoS

Definition (price of anarchy):

$$\text{PoA} = \frac{\max\limits_{s \in \mathcal{S}} w(s)}{\min\limits_{s^* \in \mathcal{S}^{\text{NE}}} w(s^*)}$$

where $\mathcal{S}^{\text{NE}}$ is the set of NE of the game.

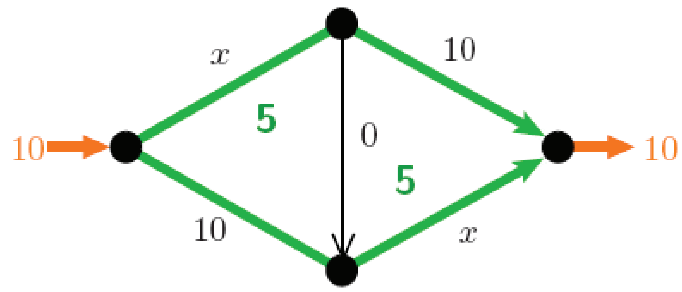
Definition (price of stability):

$$\text{PoS} = \frac{\max\limits_{s \in \mathcal{S}} w(s)}{\max\limits_{s^* \in \mathcal{S}^{\text{NE}}} w(s^*)}.$$

[Papadimitriou 2001] [Anshelevich et al 2004].

Example: PoA in non-atomic routing games

The network cost is defined by:



$$C(x) = \sum_{e \in \mathcal{E}} x_e c_e(x_e)$$

Theorem. For polynomials costs of maximum degree d, the PoA is bounded as:

degree	1	2	3	4	...	d
PoA	$\frac{4}{3}$	1.626	1.896	2.151	...	$\Omega(\frac{d}{\ln(d)})$

[Correa et al 2005].

How to improve efficiency

Possible approaches (non-exhaustive list)

► Introduce pricing.

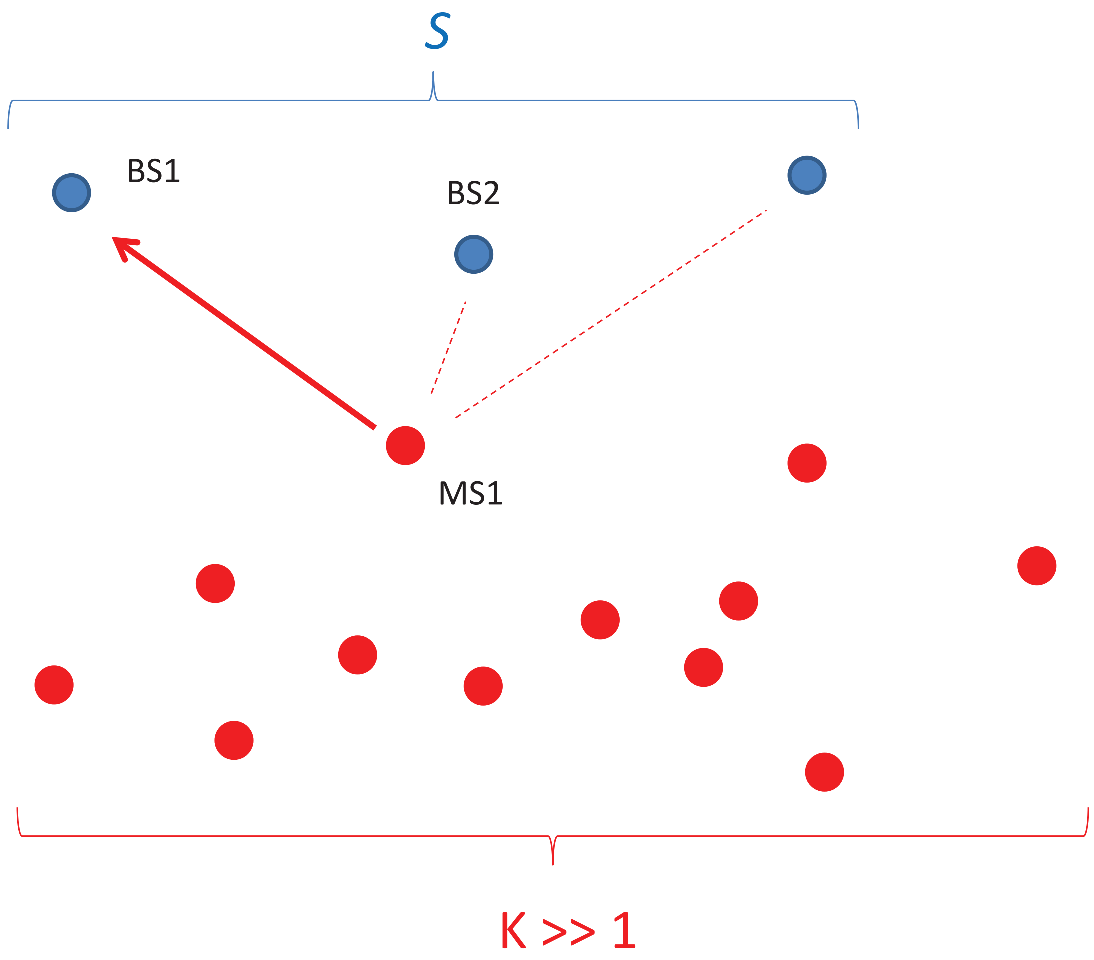
► Introduce hierarchy.

► Introduce coordination (e.g., correlated equilibrium).

► Introduce cooperation (bargaining, cooperation plan in dynamic games, agreement/contract in coalitional games, ...).

How to improve efficiency (pricing). Example

Scenario



Example (utilities)

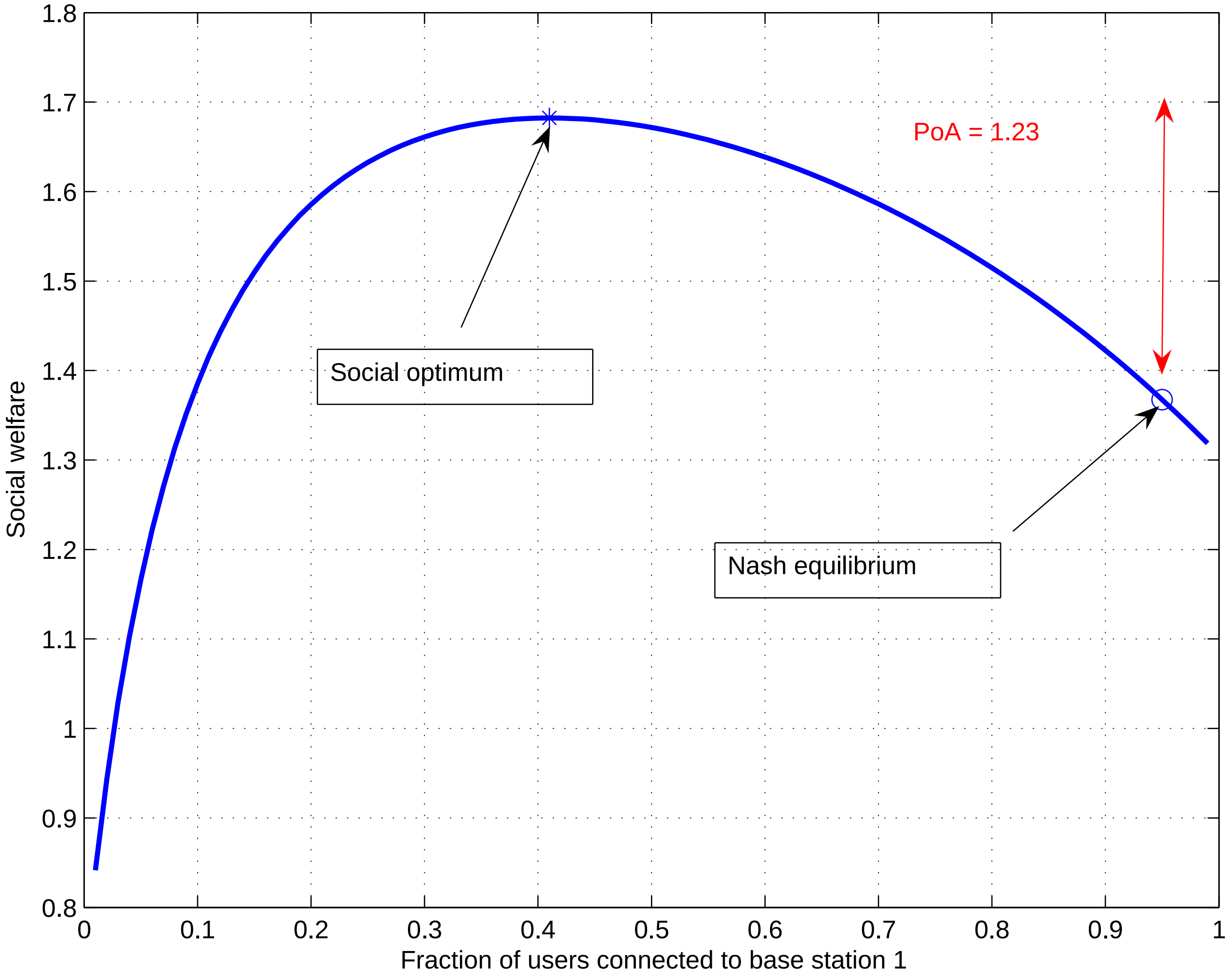
Utility of Player k when connecting to base station s

$$v_{k,s}(x) = \log\left[1 + \frac{1}{a_s + bx_s}\right],$$

$a_s > 0, b > 0$.

Example (illustration)

Social welfare for $S = 2$



Example (pricing and modified game)

Let n_k be the data volume to be transferred:

$$\tau_{k,s}(x) = \frac{n_k}{\upsilon_{k,s}(x)}.$$

Cost function of the new game:

$$c_{k,s}(x) = p(\tau_{k,s}(x)) + \beta_s.$$

Parameter adjustment $\rightarrow$ desired solution.

How to improve efficiency: introduce coordination through correlated equilibria

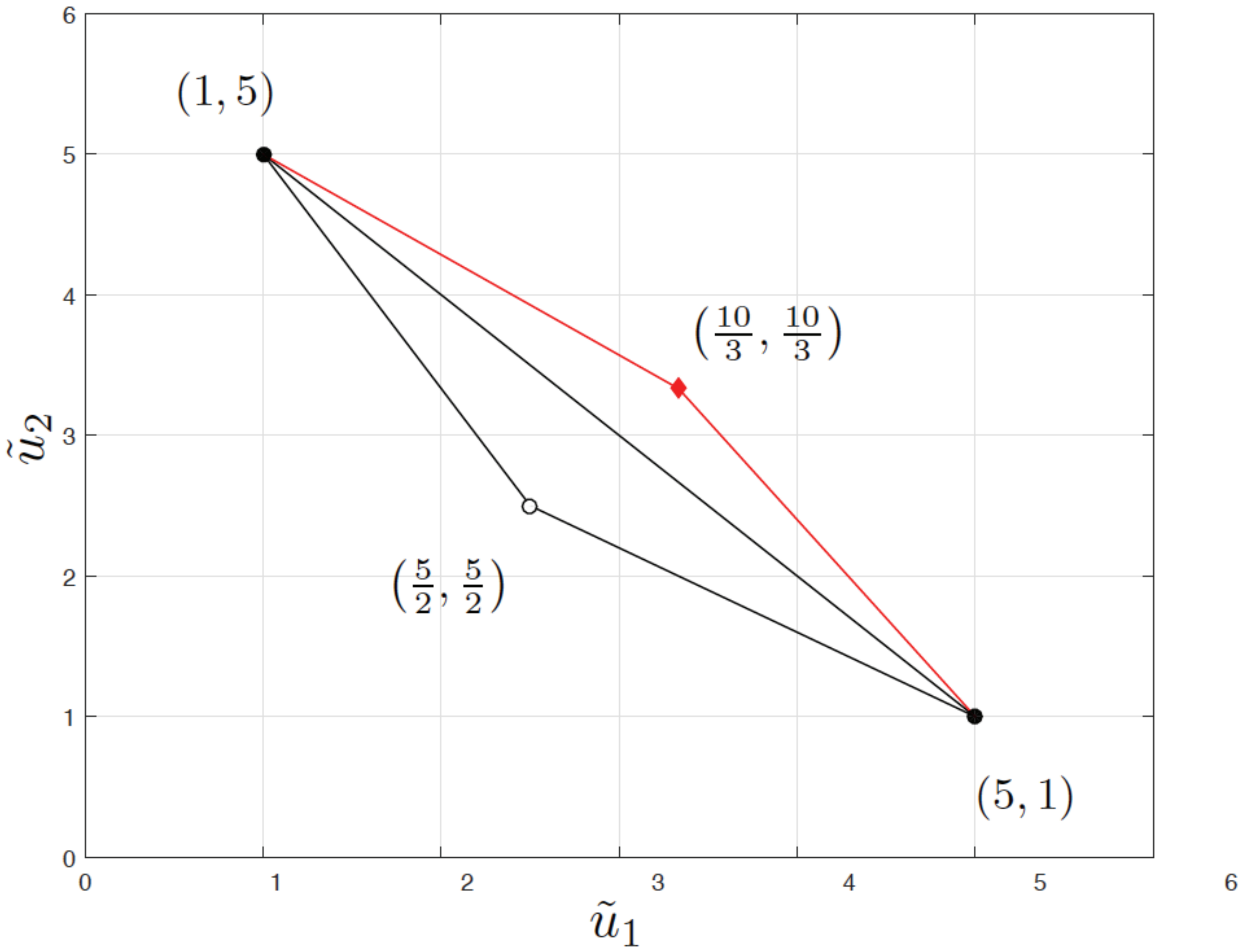
Definition (correlated equilibrium) Let $\sigma_k : \mathcal{A}_k \rightarrow \mathcal{A}_k$ be a mapping. Then q^{CE} is a CE if

$$\forall k, \forall \sigma_k,$$
$$\sum_{a \in \mathcal{A}} q^{\mathrm{CE}}(a_k, a_{-k}) u_k(a_k, a_{-k}) \geq \sum_{a \in \mathcal{A}} q^{\mathrm{CE}}(a_k, a_{-k}) u_k(\sigma_k(a_k), a_{-k}),$$

Example (CR coordination game)

	Low	High
High	(5 Mbit/s, 1 Mbit/s)	(0, 0)
Low	(4 Mbit/s, 4 Mbit/s)	(1 Mbit/s, 5 Mbit/s)

Set of correlated equilibria



General properties of the set of CE

The set of CE is compact and convex. It is a polytope whose vertices can be computed efficiently (e.g., simplex method).

Nash bargaining solution

Definition A bargaining problem is a pair $(\mathcal{S}, d)$ with $\mathcal{S}$ compact, convex, and with $x >> d$ ($(x, d) \in \mathcal{S}^2$)

Definition A bargaining solution concept is an application from the set of bargaining problems to $\mathcal{S}$

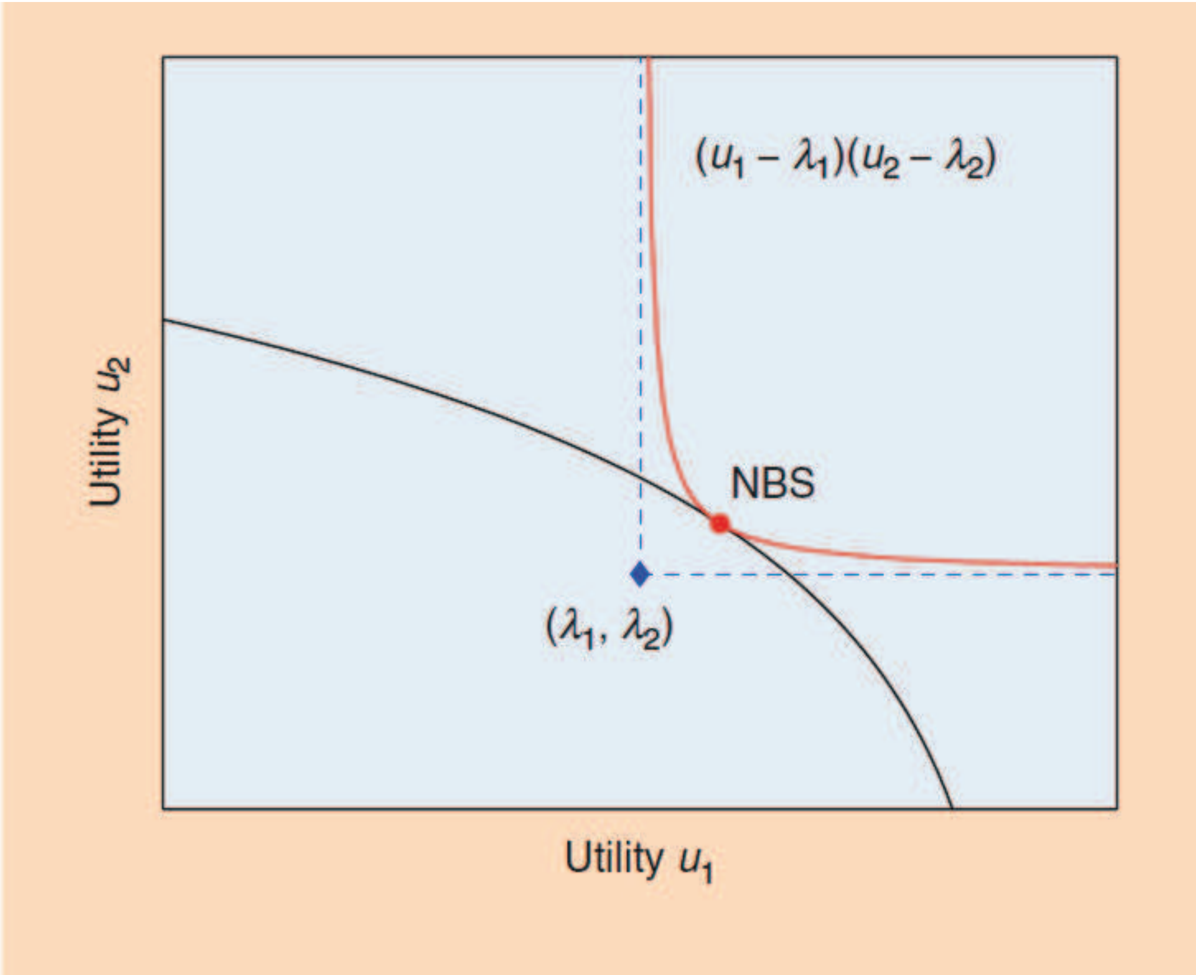
Desired properties of a BP Symmetry, Efficiency, IPAT, IIA.

Definition The NBS is the unique solution of

$$\begin{array}{cc} \max\limits_{(u_1, u_2) \in \mathcal{U}} & (u_1 - \lambda_1)(u_2 - \lambda_2) \\ \text{subject to} & u_1 \geq \lambda_1, \; u_2 \geq \lambda_2 \end{array}$$

where $\mathcal{U}$ is the game feasible utility set.

Illustration of the NBS



[FIG7] The graphical interpretation of the NBS point (red circle) as the intersection between the Pareto boundary of $\mathcal{U}$ and the hyperbola $(u_1 - \lambda_1)(u_2 - \lambda_2) = \kappa$, where the status quo $\lambda = (\lambda_1, \lambda_2)$ is represented by the blue diamond.

2. Dynamic games

Dynamic games

Informal definition. A game in which at least one player can use a strategy depending on previously played actions. No universal definition, only special classes.

Typical ingredients

- Several stages
- The stage utility is state-dependent ($u_i(a, \omega_i)$)
- Average/long-term utility
- Notions of game history, action plans

Important classes of dynamic games

- **Repeated games**
- **Stochastic games**
- Differential/difference games
- Mean-field games

Differential games (linear-quadratic + common state + finite horizon)

- Control functions: $u_i : t \mapsto u_i(t)$, $i \in \{1, ..., K\}$
- State law:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \mathbf{A}(t)x(t) + \sum_{i=1}^{K} \mathbf{B}_i(t)u_i(t)$$

- Cumulative utility:

$$J_i(u_1, ..., u_K) = \int_{t\in[0,T]} x^{\mathrm{T}}(t)\mathbf{Q}_i x(t)\mathrm{d}t + \sum_{j=1}^{K} \int_{t\in[0,T]} u_j^{\mathrm{T}}(t)\mathbf{R}_{ij}u_j(t)\mathrm{d}t + q_i(x_T)$$

More general differential games

- More general control law:

$$u_i(t, y_i(t))$$

- More general state law:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(t, x(t), u_1(t, y_1(t)), \ldots, u_K(t, y_K(t)))$$

- More general observation structures. Closed-loop perfect state example: $y_i(t) = \{x(t') : 0 \leq t' \leq t\}$. Memoryless perfect state example: $y_i(t) = \{x(0), x(t)\}$.

- Remark (stochastic differential game):

$$\mathrm{d}x(t) = f(t, x(t), u_1(t, y_1(t)), \ldots, u_K(t, y_K(t)))\mathrm{d}t + \mathrm{d}w(t)$$

$\rightarrow$ One path to mean field games.

Equilibrium analysis for differential/difference games

Basar, T., Olsder, G.J., 1982. Dynamic noncooperative game theory. In: Classics in Applied Mathematics, first ed. SIAM, Philadelphia.

T. Eisele, ”Nonexistence and Nonuniqueness of Open-Loop Equilibria in Linear-Quadratic Differential Games”, Journal of optimization theory and applications, Vol. 37, No. 4, Aug. 1982.

G. Jank, ”Introduction to Non-cooperative Dynamical Game Theory”, Coimbra, March 2001.

G. Jank and H. Abou-Kandil, ”Existence and Uniqueness of Open-Loop Nash Equilibria in Linear-Quadratic Discrete Time Games”, IEEE Trans. on Automatic Control, Vol. 48, No 2, Feb. 2003.

Engwerda, J.C., 2005. LQ Dynamic Optimization and Differential Games. Wiley.

P. Cardaliaguet, ”Introduction to differential games”, Lecture Notes, 2010.

M. Quincampoix, ”Differential games”, Computational complexity. Vols. 16, 854861, Springer, New York, 2012.

Mean field games

► The general utility function $u_i(a_1, ..., a_K, x_1, ..., x_K)$ boils down to $u_i(a_i, x_i, m_i)$

► The evolution of x_i and m_i are governed by some dynamical equations (e.g., $\mathrm{d}x_i(t) = a_i(t)\mathrm{d}t + \mathrm{d}w(t)$)

Repeated games with perfect monitoring

Definition (game history): $\forall t \geq 1$,
$h_t = (a(1), ..., a(t-1)) \in \mathcal{H}_t$ where $\mathcal{H}_t = \mathcal{A}^{t-1}$.

▶ **Definition (pure strategy)**: A pure strategy for player $i \in \mathcal{K}$ is a sequence $(\tau_{i,t})_{t \geq 1}$ with

$$\begin{array}{rccc} \tau_{i,t}: & \mathcal{H}_t & \rightarrow & \mathcal{A}_i \\ & h_t & \mapsto & a_i(t) \end{array}$$

Other types of strategies

- **Mixed strategies**

- **Definition (behavior strategy)**: A behavior strategy for player $i \in \mathcal{K}$ is a sequence $(\widetilde{\tau}_{i,t})_{t\geq 1}$ with

$$\begin{array}{rrcl} \widetilde{\tau}_{i,t}: & \mathcal{H}_t & \rightarrow & \Delta(\mathcal{A}_i) \\ & h_t & \mapsto & \pi_i(t). \end{array}$$

- **General strategies**

Repeated game utility models

▶ **General model**

▶ **Infinitely repeated games:**

$$v_i^{\infty}(\tau) = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} u_i(a(t)).$$

▶ **Discounted repeated games.** Let $0 < \lambda < 1$ be the discount factor:

$$v_i^{\lambda}(\tau) = \sum_{t=1}^{+\infty} \lambda(1-\lambda)^{t-1} u_i(a(t)).$$

Repeated game utility models

- **Finitely repeated games.** Let $\tau = (\tau_1, ..., \tau_K)$ and $T \geq 1$:

$$v_i^T(\tau) = \frac{1}{T} \sum_{t=1}^{T} u_i(a(t)).$$

Equilibria in repeated games

Definition (equilibrium strategies). A joint strategy τ^* supports an equilibrium of the repeated game $\big(\mathcal{K}, \{\mathcal{T}_i\}_{i\in\mathcal{K}}, \{v_i^y\}_{i\in\mathcal{K}}\big)$, $y \in \{T, \infty, \lambda\}$, if:

$$\forall i \in \mathcal{K}, \forall \tau_i',\ v_i^y(\tau^*) \geq v_i^y(\tau_i', \tau_{-i}^*).$$

Remark (equilibrium analysis): Existence for finite games, compact games, static games with a Nash equilibrium. In contrast with static games, there can be many equilibria.

Equilibrium characterization for infinitely repeated games with perfect monitoring

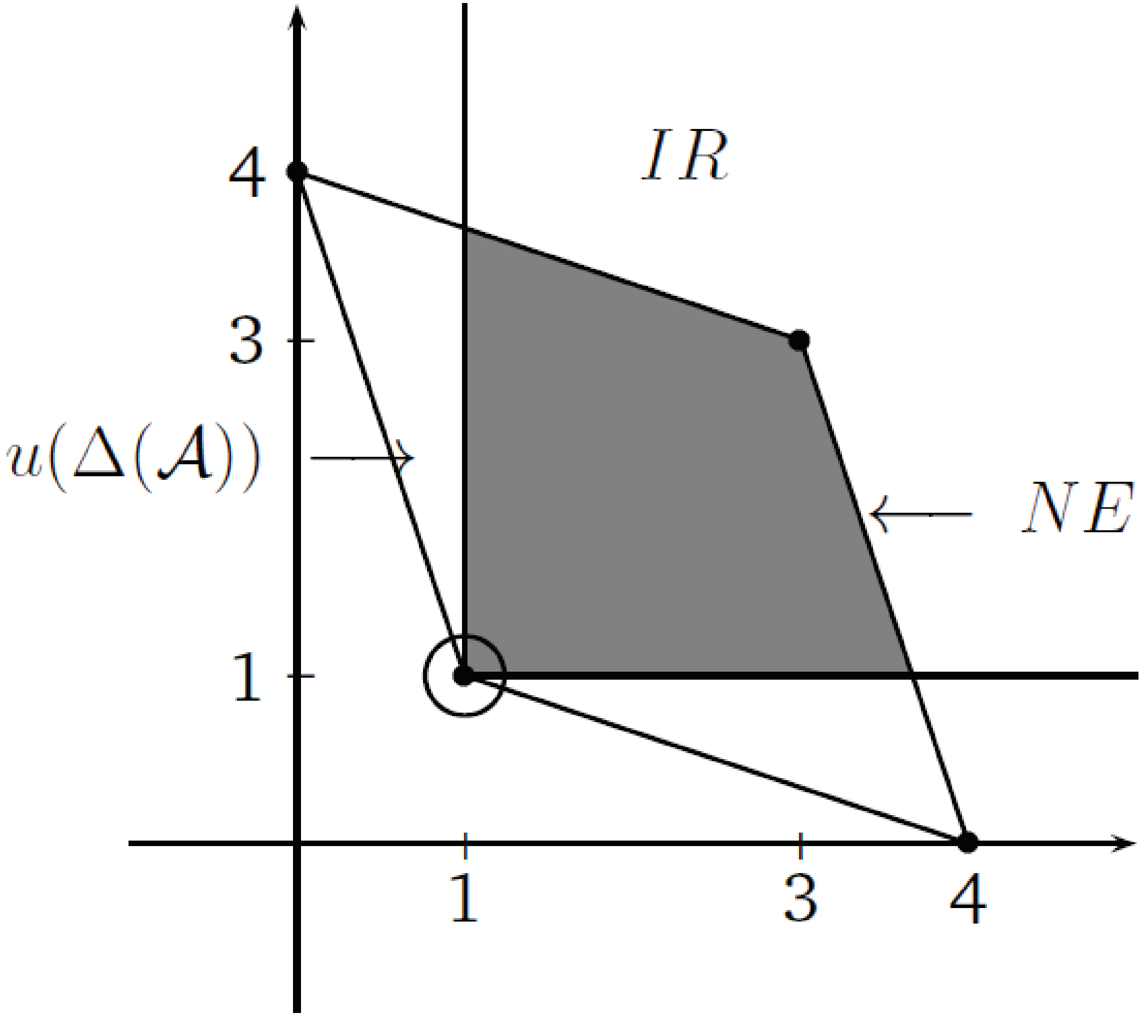
Folk theorem. The set of equilibrium utilities for Γ^∞ is given by

$$E^\infty = \mathrm{IR}(\mathcal{G}) \cap \mathrm{co}\,(\mathcal{U}(\mathcal{G}))$$

where:

- $\mathrm{IR}(\mathcal{G}) = \{u \in \mathbb{R}^K : \forall i \in \mathcal{K}, u_i \geq \min_{\pi_{-i}} \max_{\pi_i} \widetilde{u}_i(\pi)\}$;
- $\mathcal{U}(\mathcal{G}) = \{u' \in \mathbb{R}^K : \exists a, u(a) = u'\}$.

Repeated prisoner's dilemma



Relaxing the perfect monitoring assumption: $2-$connected graphs

Definition (strongly connected graph) A graph Γ is said to be strongly connected if for each pair of vertices (i, j), there is a directed path from i to j.

Definition ($2-$connected graph) The graph Γ is $2-$connected if, for any vertex i, $\Gamma\backslash\{i\}$ is strongly connected.

Theorem The following two assertions are equivalent:

(i) the observation graph of the infinitely repeated games is $2-$connected;

(ii) $E_\infty = \mathrm{IR}(\mathcal{G}) \cap \mathrm{co}\,(\mathcal{U}(\mathcal{G}))$.

[Renault and Tomala 1998]

Stochastic games with i.i.d. states. General problem statement

- Set of players: $\mathcal{K} = \{1, ..., K\}$

- Stage/instantaneous perf. criteria (utilities):

$$u_k(a_0, a_1, ..., a_K)$$

- Action sets: $\mathcal{A}_k$ with $|\mathcal{A}_k| < \infty$

- T iterations/samples/stages: $t \in \{1, ..., T\}$

Stochastic games with i.i.d. states. Continued

- Observation: $(a_0(t), a_1(t), ..., a_K(t)) \to o_k(t)$

Ultimate Goal: characterize the feasible long-term utility region

- Long-term utilities:

$$v_k^\infty = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} u_k(a_0(t), a_1(t), ..., a_K(t))$$

Ultimate goal: characterize the feasible long-term utility region

- Long-term utilities:

$$v_k^\infty = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[u_k(a_0(t), a_1(t), ..., a_K(t))\right]$$

Ultimate goal: characterize the feasible long-term utility region

► Long-term utilities:

$$v_k^{\infty}(\tau_1, ..., \tau_K) = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[u_k(a_0(t), a_1(t), ..., a_K(t))\right]$$

Ultimate goal: characterize the feasible long-term utility region

- Long-term utilities:

$$v_k^{\infty}(\tau_1, ..., \tau_K) = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[u_k(a_0(t), a_1(t), ..., a_K(t))\right]$$

- $\tau_1, ..., \tau_K =$???

Strategies

► Strategies informal definition: HISTORY $\mapsto$ ACTION

► Observation structure: $o_k = (s_k, y_k)$ where s_k is given by $\daleth_k(s_k|a_0)$ and y_k by $\Gamma_k(y_k|a_0, a_1, ..., a_K)$

- Definition (causal scenario):

$$a_k(t) = \tau_{k,t}(s_k(1), ..., s_k(t), y_k(1), ..., y_k(t-1))$$

Strategies. Continued

- Definition (causal scenario):

$$a_k(t) = \tau_{k,t}(s_k(1), ..., s_k(t), y_k(1), ..., y_k(t-1))$$

- Definition (noncausal scenario):

$$a_k(t) = \tau_{k,t}(s_k(1), ..., s_k(T), y_k(1), ..., y_k(t-1))$$

Problem statement: recap

Find the feasible region $(v_1^\infty, ..., v_K^\infty)$

- for the causal/**noncausal** scenario

- when $a_0(t)$ is i.i.d. and $\sim \rho_0$

- with the memoryless observation structure given

by $\daleth_k$ and Γ_k

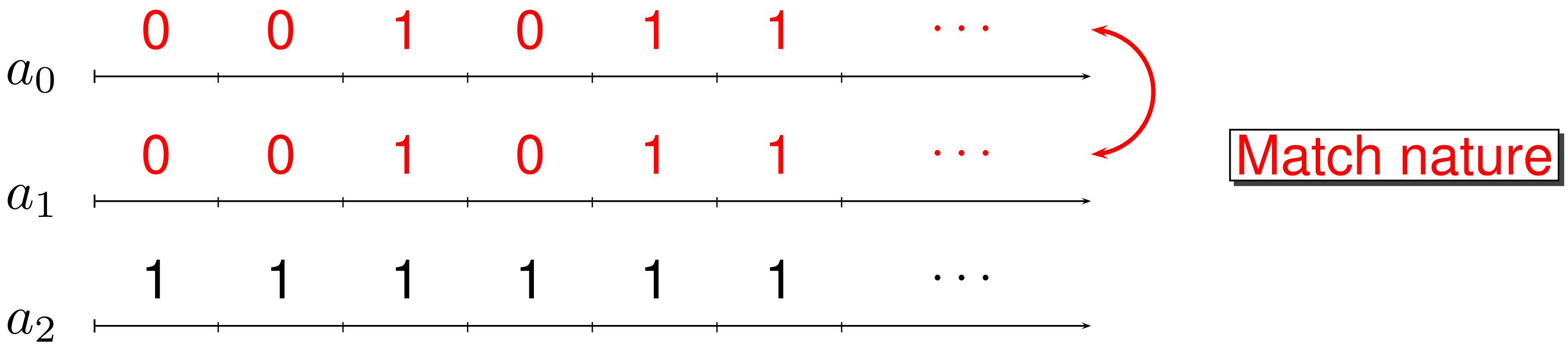
Example (noncausal+asymmetrical scenario)

- Agents: $\{1,2\}$; $0 \equiv$ nature.
- Action sets: $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{0,1\}$
- Stage utility function:

$$u(a_0, a_1, a_2) = \left| \begin{array}{ll} 1 & \text{if } a_0 = a_1 = a_2 \\ 0 & \quad \text{otherwise} \end{array} \right. .$$

Long-term utility

- Scheme 1:

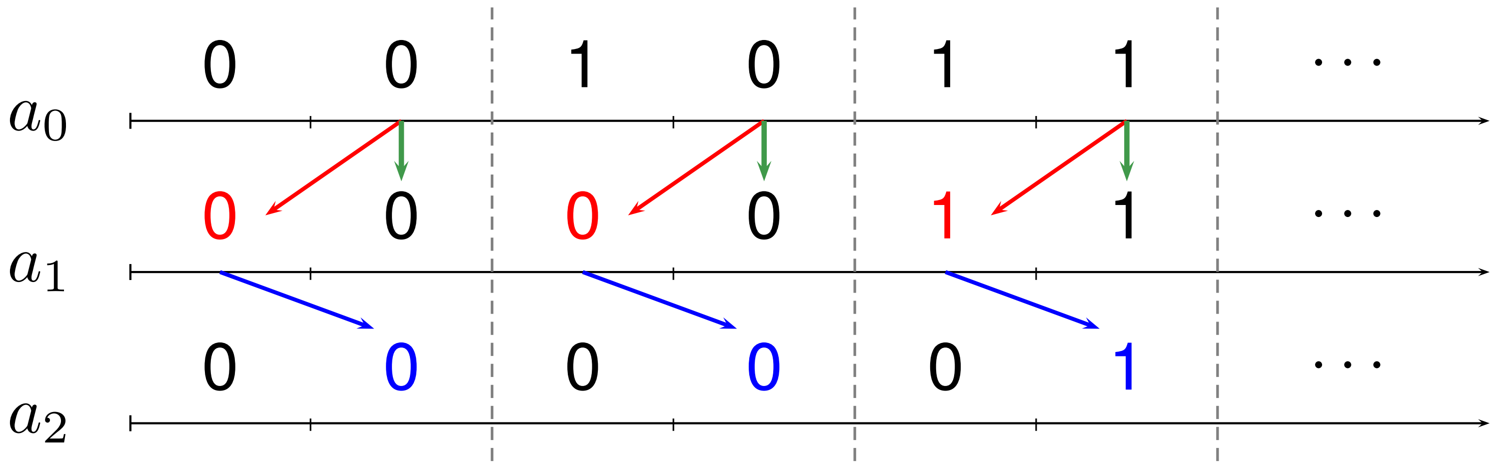


- Long-term utility

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u(a(t))\right] \to \frac{1}{2} = 0.5. \quad \text{for } A_0 \sim \mathcal{B}\left(\frac{1}{2}\right)$$

Long-term utility

- Scheme 2:



- Long term utility:

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u(a(t))\right] \to \frac{5}{8} = 0.625.$$

Maximal long-term utility

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u(a(t))\right] \to \mu^\star \simeq 0.81$$

where

$$\mu^\star \quad \text{is the solution of} \quad \frac{h(x)-1}{x-1} = \log_2 3$$

where h is the entropy function.

Review of basic information-theoretic notions

- Entropy, conditional entropy, chain rule
- Mutual information, conditional mutual information
- Log-sum inequality
- Memoryless conditional probability
- Typical sequence

Key observation. Say $\mathcal{A} = \{0, 1\}$, $K = 1$

$$\frac{1}{T}\sum_{t=1}^{T} u(a(t))$$

$$= \tfrac{1}{T}\left[u(0) + u(1) + u(1) + u(0) + \ldots + u(1)\right]$$

$$= \tfrac{N_0}{T}u(0) + \tfrac{N_1}{T}u(1)$$

$$= q_0 u(0) + q_1 u(1) \qquad (q_0 \geq 0, q_1 \geq 0, q_0 + q_1 = 1)$$

$$= \sum_{a \in \mathcal{A}} q_a u(a) = \mathbb{E}[u(a)]$$

More formally

$$
\begin{aligned}
&v_i^{\infty}(f_1, \ldots, f_K) \\
&= \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[u_k(a_0(t), a_1(t), \ldots, a_K(t))\right] \\
&= \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{a_0, \ldots, a_K} P_t(a_0, \ldots, a_K) u_k(a_0, \ldots a_K) \\
&= \sum_{a_0, \ldots, a_K} u_k(a_0, \ldots, a_K) \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} P_t(a_0, \ldots, a_K)
\end{aligned}
$$

Implementable coordination

Definition $Q(a_0, a_1, ..., a_K)$ is implementable if $\exists$ $(\tau_1, ..., \tau_K)$ such that

$$\frac{1}{T}\sum_{t=1}^{T} P_t(a_0, ..., a_K) \to Q(a_0, a_1, ..., a_K)$$

Reminder:

$$a_k(t) = \tau_{k,t}(s_k(1), ..., s_k(T), y_k(1), ..., y_k(t-1))$$

Extreme cases of distribution

- Team game $u_k = u$

- Zero correlation (lower bound):

$$Q(a_0, a_1, ..., a_K) = \rho_0(a_0) \prod_{k=1}^{K} P_{A_k}(a_k)$$

- Full correlation (upper bound):

$$(a_1(t), ..., a_K(t)) \in \text{arg} \max_{a_1,...,a_K} u(a_0(t), a_1, ..., a_K)$$

$$\rightarrow Q(a_0, \underbrace{a_1, ..., a_K}_{a}) = \rho_0(a_0)\delta(a - m(a_0))$$

Implementable distribution set characterization

Proposition Solving the problem is at least as hard as solving the two-way channel.

Theorem [Larrousse and Lasaulce ISIT 2013]

– $K = 2$

– $a_1(t) = \tau_{1,t}(a_0(1), ..., a_0(T), a_1(1), ..., a_1(t-1))$

– $a_2(t) = \tau_{2,t}(a_0(1), ..., a_0(t-1), y(1), ..., y(t-1), a_2(1), ..., a_2(t-1))$

► Then $Q(a_0, a_1, a_2)$ is implementable iff it verifies:

$$I_Q(A_0; A_2) \leq I_Q(A_1; Y | A_0, A_2).$$

Proof

- Converse
- Achievability

Proof

▶ [Larrousse and Lasaulce ISIT 2013]

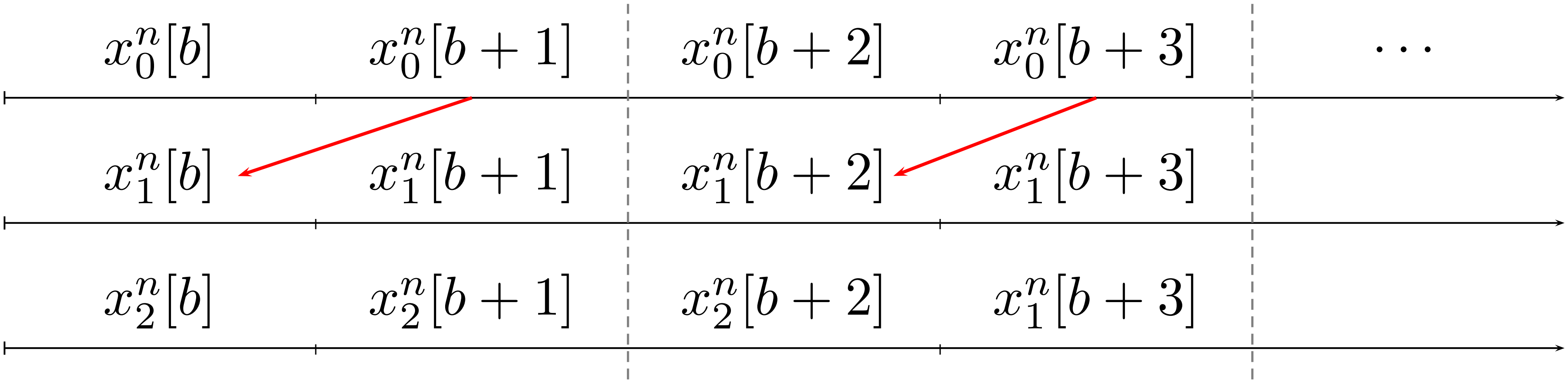
▶ Main ingredients for the **achievability** part

- Block coding
- Markov coding
- Separate source-channel coding
- Strong typicality

▶ Main ingredient for the **converse** part: chain rules which exploit the three assumptions.

About the achievability (player 1 observes player 2)

- $T = nB$, $b \in \{1, 2, ..., B\}$.



- Block coding, Markov decoding:

$$\begin{cases} x_1^n[b] & = & f_1\left(x_0^n[b+1], x_0^n[b], x_2^n[b]\right) \\ x_2^n[b+1] & = & f_2\left(x_0^n[b], y^n[b], x_2^n[b]\right). \end{cases}$$

About the achievability. Continued

- Separate coding: $f_i = f_{i,C} \circ f_{i,S}$, $\mathcal{M} = \{1, 2, ..., M\}$.

$$f_{1,S} : \left| \begin{array}{ccc} \mathcal{X}_0^n & \to & \mathcal{M} \\ x_0^n[b+1] & \mapsto & m_b \end{array} \right.$$

$$f_{1,C} : \left| \begin{array}{ccc} \mathcal{M} \times \mathcal{X}_0^n \times \mathcal{X}_2^n & \to & \mathcal{X}_1^n \\ (m_b, x_0^n[b], x_2^n[b]) & \mapsto & x_1^n[b] \end{array} \right.$$

$$f_{2,C} : \left| \begin{array}{ccc} \mathcal{X}_0^n \times \mathcal{Y}^n \times \mathcal{X}_2^n & \to & \mathcal{M} \\ (x_0^n[b], y^n[b], x_2^n[b]) & \mapsto & \widehat{m}_b \end{array} \right.$$

$$f_{2,S} : \left| \begin{array}{ccc} \mathcal{M} & \to & \mathcal{X}_2^n \\ \widehat{m}_b & \mapsto & x_2^n[b+1] \end{array} \right.$$

About the achievability. Continued

- Source coding part: a special case

$$f_{1,S} : \left| \begin{array}{ccc} \mathcal{X}_0^n & \rightarrow & \mathcal{M} \\ x_0^n & \mapsto & m \end{array} \right.$$

$$f_{2,S} : \left| \begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{X}_0^n \\ m & \mapsto & \widehat{x}_0^n \end{array} \right.$$

$$\begin{array}{rcl} \log M & \geq & n \log |\mathcal{X}_0| \\ \log M & \geq & n \left[H(X_0) + \delta_n \right] \end{array} \qquad \Leftrightarrow \qquad \begin{array}{c} M \geq |\mathcal{X}_0|^n \\ n \text{ large + lossless case} \end{array}$$

About the achievability. Continued

- Source coding part: our case

$$f_{1,S} : \left| \begin{array}{ccc} \mathcal{X}_0^n & \rightarrow & \mathcal{M} \\ x_0^n & \mapsto & m \end{array} \right.$$

$$f_{2,S} : \left| \begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{X}_2^n \\ m & \mapsto & \widehat{x}_2^n \end{array} \right.$$

$$\begin{array}{rcll} \log M & \geq & n \log |\mathcal{X}_0| & \Leftrightarrow \quad M \geq |\mathcal{X}_0|^n \\ \log M & \geq & n \left[H(X_0) + \delta_n \right] & n \text{ large + lossless case} \\ \log M & \geq & n \left[H(X_0) - H(X_0|X_2) + \delta'_n \right] & n \text{ large + lossy case} \end{array}$$

About the achievability. Continued

- Channel coding part: a special case

$$f_{1,C}: \left| \begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{X}_1^n \\ m & \mapsto & x_1^n \end{array} \right.$$

$$f_{2,C}: \left| \begin{array}{ccc} \mathcal{X}_1^n & \rightarrow & \mathcal{M} \\ x_1^n & \mapsto & \widehat{m} \end{array} \right.$$

$$\begin{array}{lll} \log M & \leq & n \log |\mathcal{X}_1| \\ \log M & \leq & n \left[H(X_1) - \delta_n'' \right] \end{array} \qquad \Leftrightarrow \qquad \begin{array}{c} M \leq |\mathcal{X}_1|^n \\ n \text{ large} \end{array}$$

About the achievability. Continued

- Channel coding part: (almost) our case

$$f_{1,C} : \left| \begin{array}{ccc} \mathcal{M} \times \mathcal{X}_0^n \times \mathcal{X}_2^n & \to & \mathcal{X}_1^n \\ (m, x_0^n, x_2^n) & \mapsto & x_1^n \end{array} \right.$$

$$f_{2,C} : \left| \begin{array}{ccc} \mathcal{X}_0^n \times \mathcal{X}_1^n \times \mathcal{X}_2^n & \to & \mathcal{M} \\ (x_0^n, x_1^n, x_2^n) & \mapsto & \widehat{m} \end{array} \right.$$

$$\begin{array}{lcll} \log M & \leq & n \log |\mathcal{X}_1| \quad \Leftrightarrow & M \leq |\mathcal{X}_1|^n \\ \log M & \leq & n\left[H(X_1) - \delta_n''\right] & n \text{ large + partial correlation} \\ \log M & \leq & n\left[H(X_1|X_0, X_2) - \delta_n'''\right] & n \text{ large + full correlation} \end{array}$$

About the achievability. Continued

▶ Channel coding part: (exactly) our case (imperfect monitoring)

$$f_{1,C} : \left| \begin{array}{ccc} \mathcal{M} \times \mathcal{X}_0^n \times \mathcal{X}_2^n & \rightarrow & \mathcal{X}_1^n \\ (m, x_0^n, x_2^n) & \mapsto & x_1^n \end{array} \right.$$

$$f_{2,C} : \left| \begin{array}{ccc} \mathcal{X}_0^n \times \mathcal{Y}^n \times \mathcal{X}_2^n & \rightarrow & \mathcal{M} \\ (x_0^n, y^n, x_2^n) & \mapsto & \widehat{m} \end{array} \right.$$

$$\begin{array}{lcl} \log M & \leq & n \log |\mathcal{X}_1| \qquad \Leftrightarrow \quad M \leq |\mathcal{X}_1|^n \\ \log M & \leq & n\left[H(X_1) - \delta_n''\right] \\ \log M & \leq & n\left[H(X_1|X_0, X_2) - H(X_1|X_0, X_2, Y) - \delta_n''''\right] \end{array}$$

Link with Q: offline operations

- Fix $\overline{Q}(x_0, x_1, x_2) = Q_{X_1|X_0 X_2}(x_1|x_0, x_2) Q_{X_2|X_0}(x_2|x_0) Q_{X_0}(x_0)$.
- Generate the codebook $\mathcal{C}_S = \{x_2^n(1), x_2^n(2), ..., x_2^n(M)\}$ with $X_2 \sim \sum_{x_0} Q_{X_2|X_0}(x_2|x_0) Q_{X_0}(x_0)$.
- Generate the codebook $\mathcal{C}_C = \{x_1^n(1), x_1^n(2), ..., x_1^n(M)\}$ with $X_1 \sim \sum_{x_0, x_2} Q_{X_1|X_0 X_2}(x_1|x_0, x_2) Q_{X_0 X_2}(x_0, x_2)$.

Link with Q: online operations

• Player 1. Find m_b such that $(x_0^n[b+1], x_2^n(m_b)) \in \mathcal{T}_\epsilon^n(X_0 X_2)$ with

$$\mathcal{T}_\epsilon^n(X_0X_2) = \left\{ (x_0^n, x_2^n) \,:\, \forall (x_0, x_2),\ \left| \frac{1}{n}\mathcal{N}(x_0, x_2 | x_0^n, x_2^n) - Q_{X_0X_2}(x_0, x_2) \right| \leq \epsilon \right\}.$$

Condition $\frac{\log M}{n} \geq I_Q(X_0; X_2) + \delta_n' \Rightarrow$ uniqueness.

• Player 1. Send $x_1^n(m_b)$.

• Player 2. Find m_b such that $(x_0^n[b], y^n[b], x_2^n[b], x_1^n(m_b)) \in \mathcal{T}_\epsilon^n(X_0X_1X_2Y)$. Condition $\frac{\log M}{n} \leq I_Q(X_1; Y | X_0, X_2) + \delta_n'''' \Rightarrow$ uniqueness.

Corollary

– $K = 2$

– $a_1(t) = \tau_{1,t}(a_0(1), ..., a_0(T))$

– $a_2(t) = \tau_{2,t}(a_1(1), ..., a_1(t-1))$

► Then $Q(a_0, a_1, a_2)$ is implementable iff its marginal w.r.t (a_1, a_2) is ρ_0 and

$$H_Q(A_0, A_1, A_2) \geq H_Q(A_0) + H_Q(A_2).$$

Utility region characterization

Pareto frontier: use $w_\alpha = \alpha u_1 + (1-\alpha) u_2$

$$\begin{array}{lrcl} \text{minimize} & \displaystyle -\sum_{a_0,a_1,a_2} Q(a_0,a_1,a_2) w_\alpha(a_0,a_1,a_2) & & \\ \text{subject to} & H_Q(A_0) + H_Q(A_2) - H_Q(A_0,A_1,A_2) & \leq & 0 \\ & -Q(a_0,a_1,a_2) & \leq & 0 \\ & \displaystyle -1 + \sum_{a_0,a_1,a_2} Q(a_0,a_1,a_2) & = & 0 \\ & \displaystyle -\rho_0(a_0) + \sum_{a_1,a_2} Q(a_0,a_1,a_2) & = & 0 \end{array}$$

Let us try to interpret by specializing further

- $H_Q(a_0) =$ constant $= -\sum_{a_0} \rho_0(a_0) \log \rho_0(a_0)$
- $H_Q(a_2) \sim$ constant
- Boltzmann-Gibbs is optimal:

$$Q^\star(a_0, a_1, a_2) = \frac{e^{\lambda u(a_0,a_1,a_2)}}{\sum_{a_0,a_1,a_2} e^{\lambda u(a_0,a_1,a_2)}}$$

3. Learning algorithms and strategic-form games

Algorithm 1: The best-response dynamics (BRD)

Updating rule (asynchronous BRD) $K = 2$. Action sequence: $a_1(0)$, $a_2(1) \in \mathrm{BR}_2[a_1(0)]$, $a_1(2) \in \mathrm{BR}_1[a_2(1)]$, etc. More generally:

$$a_i(t+1) \in \mathrm{BR}_i\left[a_1(t+1), ..., a_{i-1}(t+1), a_{i+1}(t), ..., a_K(t)\right].$$

Updating rule (synchronous BRD):

$$a_i(t+1) \in \mathrm{BR}_i\left[a_{-i}(t)\right].$$

[Cournot 1838]

Comments on Algorithm 1

Main features

► Fast convergence.

► Steady state: NE.

► Required knowledge: Action profile and individual utility function (in general).

The iterative water-filling algorithm [Yu et al 2002]

- Actions: $a_i = p_i = (p_{i,1}, ..., p_{i,S})$ with $\sum_s p_{i,s} \leq P^{\max}$ and $p_{i,s} \geq 0$
- BRD:

$$p_i(t+1) \in \text{argmax}_{p_i} \sum_{s=1}^{S} \log\left(1 + \frac{g_{ii,s}p_{i,s}}{\sigma^2 + \sum_{j\neq i} g_{ji,s}p_{j,s}(t)}\right)$$

- The water-filling solution writes as

$$p_{i,s}(t+1) = \left[\frac{1}{\lambda_i} - \frac{p_i(t)}{\mathrm{SINR}_i(t)}\right]^+$$

Algorithm 2: Fictitious play (FP)

Updating rule (synchronous FP): [Brown 1951]

$$a_i(t+1) \in \text{arg}\max_{a_i \in \mathcal{A}_i} \sum_{a_{-i}} f_{-i,t}(a_{-i}) u_i(a_i, a_{-i}).$$

Recursive structure

$$\begin{aligned}
f_{i,t+1}(a_{i,j}) &= \frac{1}{t+1} \sum_{t'=1}^{t+1} \mathbb{1}_{\{a_i(t')=a_{i,j}\}} \\
&= \frac{1}{t+1} \sum_{t'=1}^{t} \mathbb{1}_{\{a_i(t')=a_{i,j}\}} + \frac{1}{t+1} \mathbb{1}_{\{a_i(t+1)=a_{i,j}\}} \\
&= \frac{t}{t+1} f_{i,t}(a_{i,j}) + \frac{1}{t+1} \mathbb{1}_{\{a_i(t+1)=a_{i,j}\}} \\
&= f_{i,t}(a_{i,j}) + \frac{1}{t+1} \left[\mathbb{1}_{\{a_i(t+1)=a_{i,j}\}} - f_{i,t}(a_{i,j}) \right] \\
&= f_{i,t}(a_{i,j}) + \lambda_i(t) \left[\mathbb{1}_{\{a_i(t+1)=a_{i,j}\}} - f_{i,t}(a_{i,j}) \right]
\end{aligned}$$

where $\mathbb{1}$ is the indicator function.

Algorithm 3: Reinforcement learning

A reinforcement learning algorithm. $|\mathcal{A}_i| < +\infty$,
$\forall i \in \mathcal{K}, \forall j \in \{1, ..., |\mathcal{A}_i|\}$,

$$\pi_{i,j}(t+1) = \pi_{i,j}(t) + \lambda_i(t)\mu_i(t)\left[\mathbb{1}_{\{a_i(t)=a_{i,j}\}} - \pi_{i,j}(t)\right],$$

$0 < \lambda_i(t) < 1$, $\mu_i \; in[0,1]$ normalized utility.

[Bush and Mosteller 1955][Sastry et al 1994].

Main features of Algorithm 3

- Required knowledge: individual utility realizations.
- Slow convergence.
- Steady state: NE/boundary points/limit cycle.

Convergence issue

► Convergence depends on: the updating rule + the associated game.

► For algorithms 1, 2, and 3, it is sufficient that the game be:

□ dominance solvable, or

□ potential, or

□ supermodular.

A bit of stochastic approximation

- **Updating rule:** $x(t+1) = x(t) + \lambda(t)Y(t)$, $t \in \mathbb{N}$
- **Model:** $Y(t) = f(x(t)) + Z(t)$
- **Idea:** interpolate $x(t)$ by $\overline{x}(\tau)$, $\tau \geq 0$

A bit of stochastic approximation. Continued

Theorem 1. Let $\overline{x}(\tau) = x(t) + \frac{\tau - \tau_t}{\tau_{t+1} - \tau_t}(x(t+1) - x(t))$ with $\tau_t = \sum_{t'=0}^{t} \lambda(t')$. If: $\lambda(t)$ is well chosen; f is Lipschitz; $x(t)$ is bounded; $Z(t)$ is a square integrable Martingale difference.
Then $\forall T > 0$:

$$\begin{cases} \Pr\left[\lim_{s\to+\infty} \sup_{\tau\in[s,s+T]} \|\overline{x}(\tau) - X^s(\tau)\| = 0\right] & = & 1 \\ \Pr\left[\lim_{s\to+\infty} \sup_{\tau\in[s-T,s]} \|\overline{x}(\tau) - X_s(\tau)\| = 0\right] & = & 1 \end{cases}$$

where $X^s(\tau)$ (resp. $X_s(\tau)$) corresponds to the trajectory given by the ODE $\dot{X}(\tau) = f(X(\tau))$ that starts (resp. ends) at time $s > 0$ in the point $\overline{x}(s)$...

A bit of stochastic approximation. Continued

Conclusion 1. The interpolation process converges a.s. to the solution of the ODE $\dot{X}(\tau) = f(X(\tau))$.

Theorem 2. The discrete process $\{x(t)\}$ converges when $t \to +\infty$ almost surely to a (possibly path dependent) compact connected internally chain transitive set of the ODE.

Reference. [See e.g., Seminar notes ”Dynamics of stochastic approximation algorithms” by Michel Bénaïm (1999).]

Algorithm 4: Regret Matching

Definition (regret) [Hart & Mas-Collel 2000]

$$\forall n,\ r_{k,a_{k,n}}(t{+}1) = \frac{1}{t}\sum_{t'=1}^{t} u_k(a_{k,n}, a_{-k}(t')) - u_k(a_k(t'), a_{-k}(t'))$$

Updating rule

$$\pi_{k,a_{k,n}}(t+1) = \frac{\left[r_{k,a_{k,n}}(t+1)\right]^+}{\sum_{n'=1}^{N_k}\left[r_{k,a_{k,n'}}(t+1)\right]^+}$$

Main features of Algorithm 4

- Required knowledge: action profile

- Convergence: unconditional convergence + intermediate speed

- Steady state: CCE

Remark: ”pure NE $\subseteq$ mixed NE $\subseteq$ CE $\subseteq$ CCE”

Coarse correlated equilibrium

Definition

$$\forall k, \forall a'_k,$$

$$\sum_{a \in \mathcal{A}} q^{\mathrm{CCE}}(a) u_k(a) \geq \sum_{a \in \mathcal{A}} q^{\mathrm{CCE}}(a) u_k(a'_k, a_{-k})$$

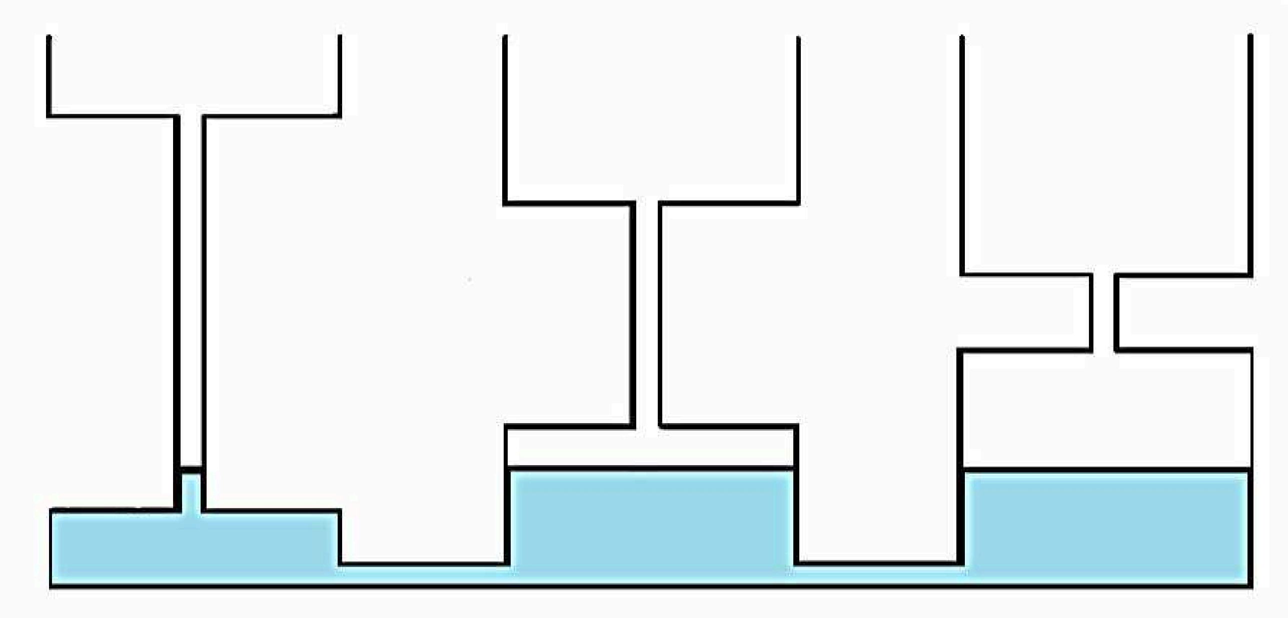
4. Coalitional form games

The bankruptcy problem (Talmud's version)

		Claim		
		100	200	300
Estate	100	$\frac{100}{3}$	$\frac{100}{3}$	$\frac{100}{3}$
	200	50	75	75
	300	50	100	150

Physical interpretation of the (game-theoretic) solution

[Aumann and Maschler 1985].



Messages

Coalition games can be a very powerful tool.

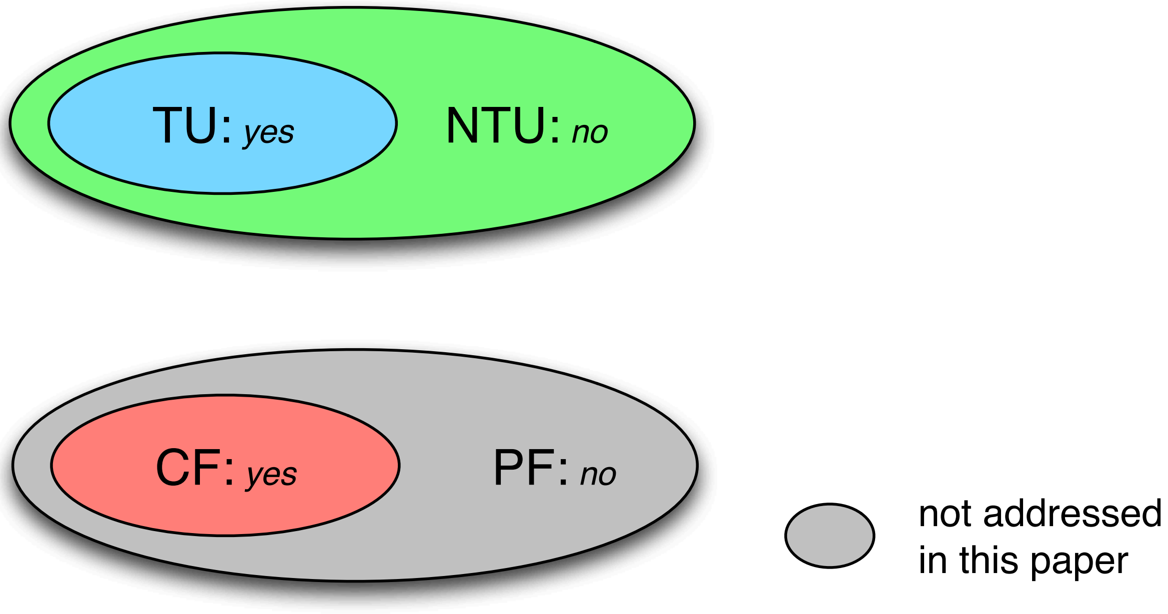
Two important issues in coalition games:

- **utility allocation/division;**
- **coalition formation.**

Classification of coalition-form games

Distribution of utility:
Can the value of any coalition be divided arbitrarily among its members?

Coalition value type:
Does the value function of a coalition depend on its own members only?



Coalition form games with characteristic functions

Definition. Game $\equiv$ pair:

$$\mathcal{G} = (\mathcal{K}, v)\,.$$

Notation (power set). Ex:
if $\mathcal{K} = \{1,2\},\ 2^{\mathcal{K}} = \big\{\emptyset, \{1\}, \{2\}, \{1,2\}\big\}$.

Transferable utility (TU) games:

$$\begin{array}{rccc} v: & 2^{\mathcal{K}} & \to & \mathbb{R} \\ & \mathcal{C} & \mapsto & v(\mathcal{C}) \end{array}.$$

Coalition form games with characteristic functions

Non transferable utility (NTU) games:

$$\begin{aligned} v : 2^{\mathcal{K}} &\to \mathbb{R}^{\mathcal{K}} \\ \mathcal{C} &\mapsto v(\mathcal{C}) = \{(v_1(\mathcal{C}), ..., v_K(\mathcal{C}))\} \end{aligned}.$$

Ice-cream game example (TU game)

 Chris: \$4,  Marvin: \$3,  Terry: \$3

$w = 500$

$p = \$7$

$w = 750$

$p = \$9$

- $v(\emptyset) = v(\{C\}) = v(\{M\}) = v(\{T\}) = 0$
- $v(\{C, M\}) = 500$, $v(\{C, T\}) = 500$, $v(\{M, T\}) = 0$
- $v(\{C, M, T\}) = 750$

Ice-cream game example. General solution concept.

Utility division: $x = (x_{\mathrm{C}}, x_{\mathrm{M}}, x_{\mathrm{T}})$.

- $x = (200, 200, 350)$ not stable ($v(\{\mathrm{C}, \mathrm{M}\}) > x_{\mathrm{C}} + x_{\mathrm{M}}$).
- $x' = (250, 250, 250)$ stable.
- $x'' = (750, 0, 0)$ stable.

Notion of core (TU superadditive games):

$$\mathrm{core}(\mathcal{G}) = \left\{ x \in \mathbb{R}^K : \sum_{i \in \mathcal{K}} x_i = v(\mathcal{K}),\ \forall \mathcal{C} \subseteq \mathcal{K}, \sum_{i \in \mathcal{C}} x_i \geq v(\mathcal{C}) \right\}.$$

Ice-cream game core

$$\begin{cases} x_1 + x_2 + x_3 & = & ??? \\ x_1 & \geq & ??? \\ x_2 & \geq & ??? \\ x_3 & \geq & ??? \\ x_1 + x_2 & \geq & ??? \\ x_1 + x_3 & \geq & ??? \\ x_2 + x_3 & \geq & ??? \\ x_1 + x_2 + x_3 & \geq & ??? \end{cases}$$

Ice-cream game core

$$\left\{\begin{array}{rcl} x_1 + x_2 + x_3 & = & 750 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ x_3 & \geq & 0 \\ x_1 + x_2 & \geq & 500 \\ x_1 + x_3 & \geq & 500 \\ x_2 + x_3 & \geq & 0 \\ x_1 + x_2 + x_3 & \geq & 750 \end{array}\right.$$

Core existence: theorems

Theorem (Bondareva-Shapley) Not treated here. See e.g., [Bacci et al 2015].

Definition (convex TU game)

$$\forall \mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{K}, \;\; v(\mathcal{C}_1) + v(\mathcal{C}_2) \leq v(\mathcal{C}_1 \cup \mathcal{C}_2) + v(\mathcal{C}_1 \cap \mathcal{C}_2)$$

Theorem Convex TU game $\Rightarrow$ non-empty core.

The nucleolus

Core

$$\text{core}(\mathcal{G}) = \Big\{x \in \mathbb{R}^K : \sum_{i \in \mathcal{K}} x_i = v(\mathcal{K}),\ \forall \mathcal{C} \subseteq \mathcal{K}, \underbrace{v(\mathcal{C}) - \sum_{i \in \mathcal{C}} x_i}_{e(\mathcal{C},x)} \leq 0\Big\}.$$

Excess: $e(x) = (e(\mathcal{C}_1, x), ..., e(\mathcal{C}_{2^K}, x))$ (with $e(\mathcal{C}_1, x) \geq e(\mathcal{C}_2, x) \geq ...$).

Nucleolus (relative to $\mathcal{X} \subseteq \mathbb{R}^K$)

$$\text{nucleolus}(\mathcal{G}; \mathcal{X}) = \Big\{x \in \mathcal{X} : \forall x' \in \mathcal{X}, e(x) \preceq_L e(x')\Big\}.$$

Motivation Stability $\rightarrow$ fairness

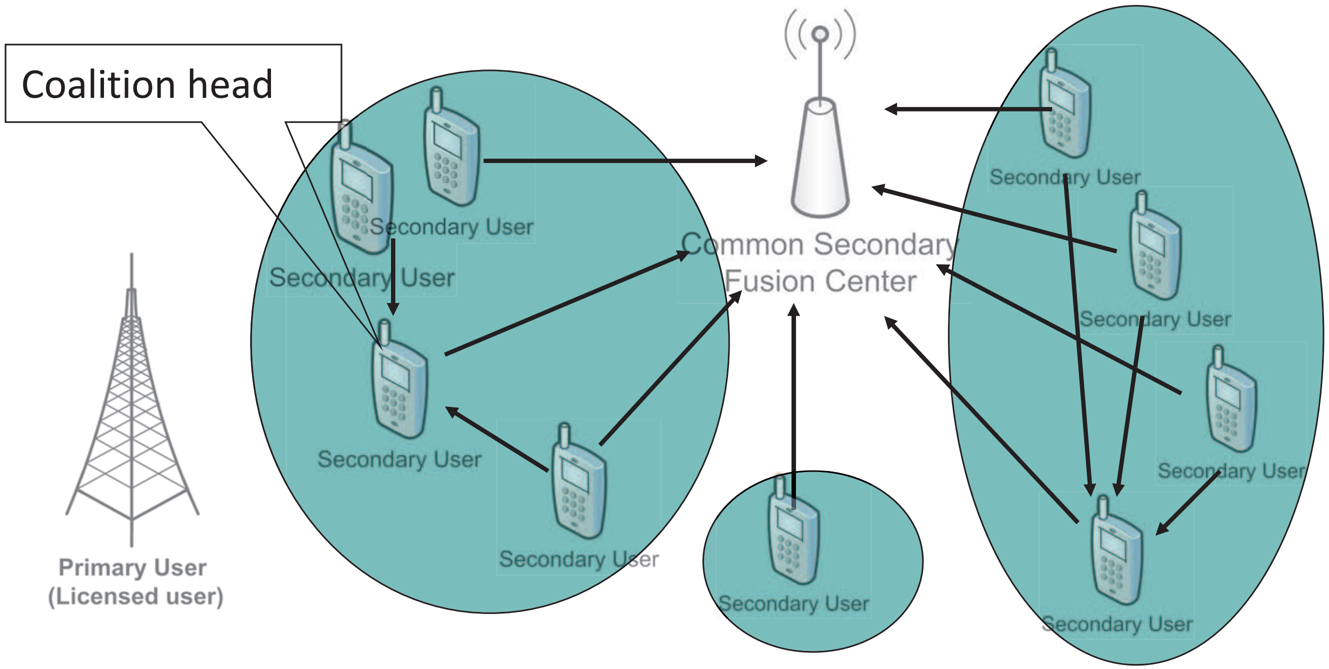
Definition Utility division:

$$x_i = \sum_{\mathcal{C} \subseteq \mathcal{K} \setminus \{i\}} \frac{|\mathcal{C}|!(|\mathcal{K}| - |\mathcal{C}| - 1)!}{|\mathcal{K}|!} \left[v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \right].$$

Axiomatic characterization

- **Efficiency:** $\sum_{i \in \mathcal{K}} x_i = v(\mathcal{K})$
- **Additivity:** $x_i(\mathcal{G}_1 \oplus \mathcal{G}_2) = x_i(\mathcal{G}_1) + x_i(\mathcal{G}_2$ ($\oplus \equiv v = v_1 + v_2$)
- **Dummy:** $\forall \mathcal{C}',\ v(\mathcal{C}') = v(\mathcal{C}' \cup \{i\})$ ($\mathcal{C}$ does not contain i)
- **Symmetry:** $\forall \mathcal{C}'',\ v(\mathcal{C}'' \cup \{i\}) = v(\mathcal{C}'' \cup \{j\})$ ($\mathcal{C}$ does neither contain i nor j)

An NTU coalition game for distributed collaborative spectrum sensing [Saad et al 2011]



Coalitional form (NTU)

▶ **Players:** secondary transmitters $\mathcal{K} = \{1, ..., K\}$.

▶ **Characteristic function:**

$$v(\mathcal{C}) = 1 - P_{\mathrm{m}}(\mathcal{C}) - J(P_{\mathrm{f}}(\mathcal{C}))$$

with

$$J(P_{\mathrm{f}}(\mathcal{C})) = \left| \begin{array}{ll} -q^2 \log \left[1 - \left(\frac{P_{\mathrm{f}}(\mathcal{C})}{q} \right)^2 \right] & \text{if} \quad 0 \leq P_{\mathrm{f}}(\mathcal{C}) < q \\ +\infty & \text{if} \quad q \leq P_{\mathrm{f}}(\mathcal{C}) \leq 1 \end{array} \right. .$$

Coalition formation

Utility division. Not relevant.

Coalition formation. Merge and split coalitions by performing Pareto comparisons.

Results. Converging algorithm. Distributed solution: implementable, good performance in terms of miss and false alarm probabilities [Saad et al 2011].

5. Extensive form games

Extensive form games

Definition: A standard extensive form game is a $6-$uplet

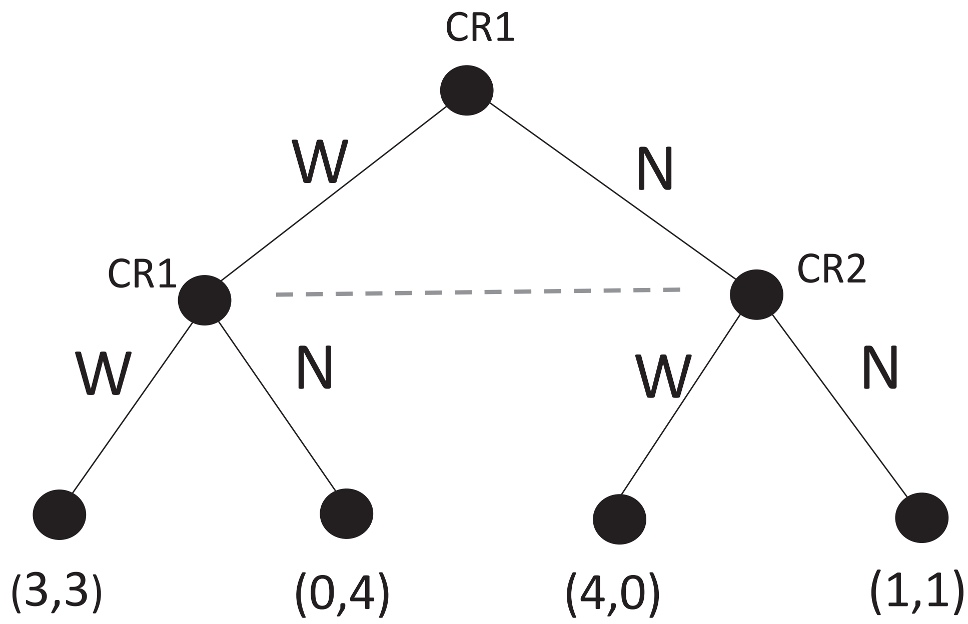
$$\mathcal{G} = (\mathcal{K}, \mathcal{V}, v_{\text{root}}, \pi, \{\mathcal{V}_i\}_{i\in\mathcal{K}}, \{u_i\}_{i\in\mathcal{K}})$$

where:

- $\mathcal{K} = \{1, ..., K\}$ is the set of players;
- $(\mathcal{V}, v_{\text{root}}, \pi)$ is a **tree**;
- $\{\mathcal{V}_i\}_{i\in\mathcal{K}}$ is a partition of $\mathcal{V}$.

Remark: $\forall v \in \mathcal{V}$, $\exists n \geq 1$, $\pi^{(n)} = \pi \circ ... \circ \pi = v_{\text{root}}$.

Representing the forwarder's dilemma under extensive form



Extensive form with imperfect information

Definition: It is a $9-$uplet

$$\mathcal{G} = \Big(\mathcal{K}, \mathcal{V}, v_{\text{root}}, \pi, \mathcal{V}_0, \{q_0^j\}_{j\in\mathcal{V}_0}, \{\mathcal{V}_i\}_{i\in\mathcal{K}}, \{W_i^k\}_{k\in\{1,\dots,k_i\}}, \{u_i\}_{i\in\mathcal{K}}\Big)$$

where:

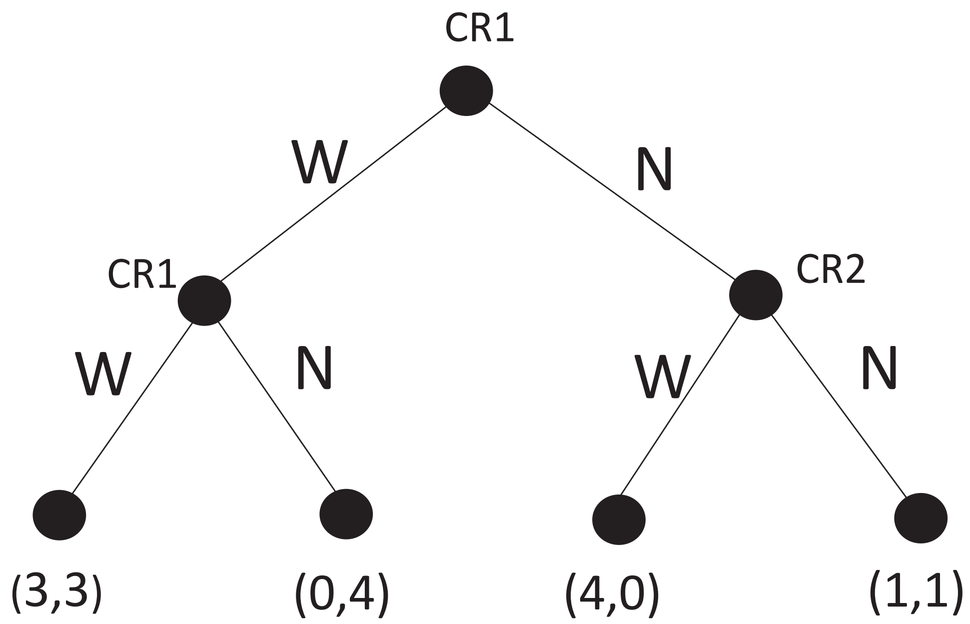
- player 0 is nature;
- $\forall j \in \mathcal{V}_0$ q_j^0 is the transition probability used by player 0 to choose a successor to j;
- W_i^k corresponds to the partition of V_i which defines the information structure for i.

Remark: Games with perfect information $W_i^k = \{w_i^k\}$.

Strategic form and extensive form

- Extensive form **more complete** than strategic form.
- Extensive form usually **less convenient** for mathematical analysis.
- Continuous/discrete action sets.
- Extensive form sometimes more intuitive.
- The tree structure of the extensive form can be useful for computer-based analyses.

Transforming the prisoner's dilemma into a dynamic game



6. Conclusion

Summarizing

► Direct game theory – mechanism design.

► 3 dominant mathematical representations: strategic form, extensive form, coalition form.

► Focus on the Nash equilibrium.

Solution concepts for strategic/extensive form games

- **Pure/mixed Nash equilibrium, Wardrop equilibrium**,
- **correlated equilibrium, coarse correlated equilibrium,**
- $N-$ strong equilibrium,
- Nash equilibrium refinements : trembling hand perfect equilibrium, proper equilibrium,
- $\epsilon-$Nash equilibrium,
- logit equilibrium,

Solution concepts for strategic/extensive form games. Continued

- maxmin strategy profiles,
- Bayesian equilibrium,
- evolutionary stable solution,
- satisfaction equilibrium, generalized Nash equilibrium,
- Stackelberg equilibrium,
- **Pareto optimum, social optimum**,
- bargaining solutions (Nash, egalitarian, Kalai-Smorodinsky, etc.),...

Solution concepts for coalition form games

- **core, nucleolus,**
- $\epsilon-$ core,
- least core,
- kernel,
- bargaining set,
- **Shapley value**, Harsanyi value, Banzhaf index,...

Summarizing

- Static games - dynamic games.
- Relationship between static games and learning.

Algorithms to reach a given solution concepts (strategic case)

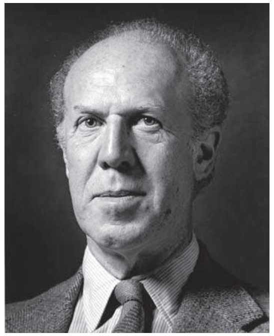
- **Asynchronous/synchronous best response dynamics, fictitious play, a type of reinforcement algorithm, regret matching,**
- Boltzmann-Gibbs learning,
- coupled dynamics learning,
- trial-and-error learning,
- conditional no-regret learning,
- Bayesian learning,...

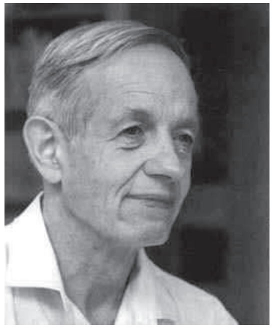
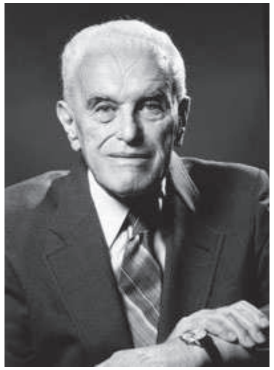
Challenges

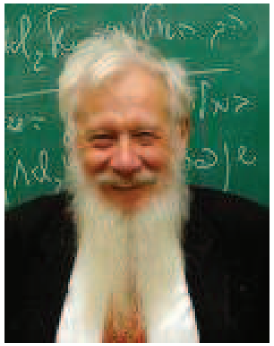
- Tradeoff between efficiency – weak information assumption.
- Bridge the gap between learning and dynamic games.
- Dynamic games with arbitrary observation graphs.
- Mechanism design.

Mechanism design, Nobel prizes,...





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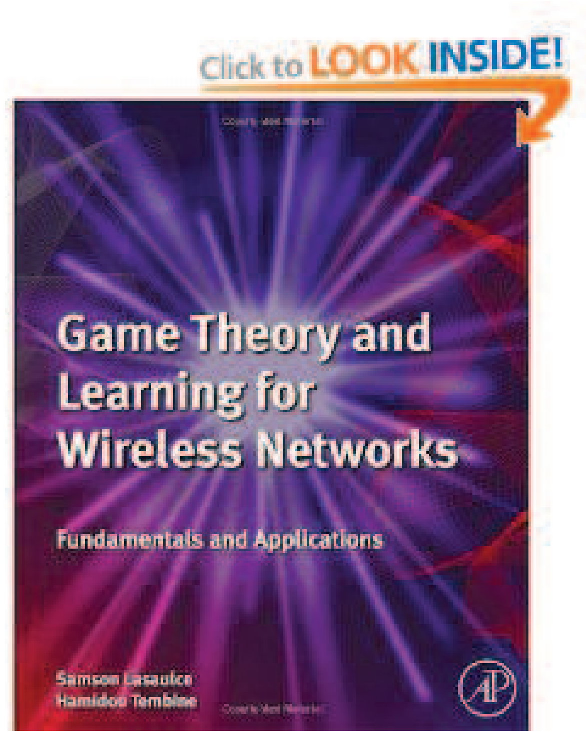

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8. Tutorial work

Global emissions game

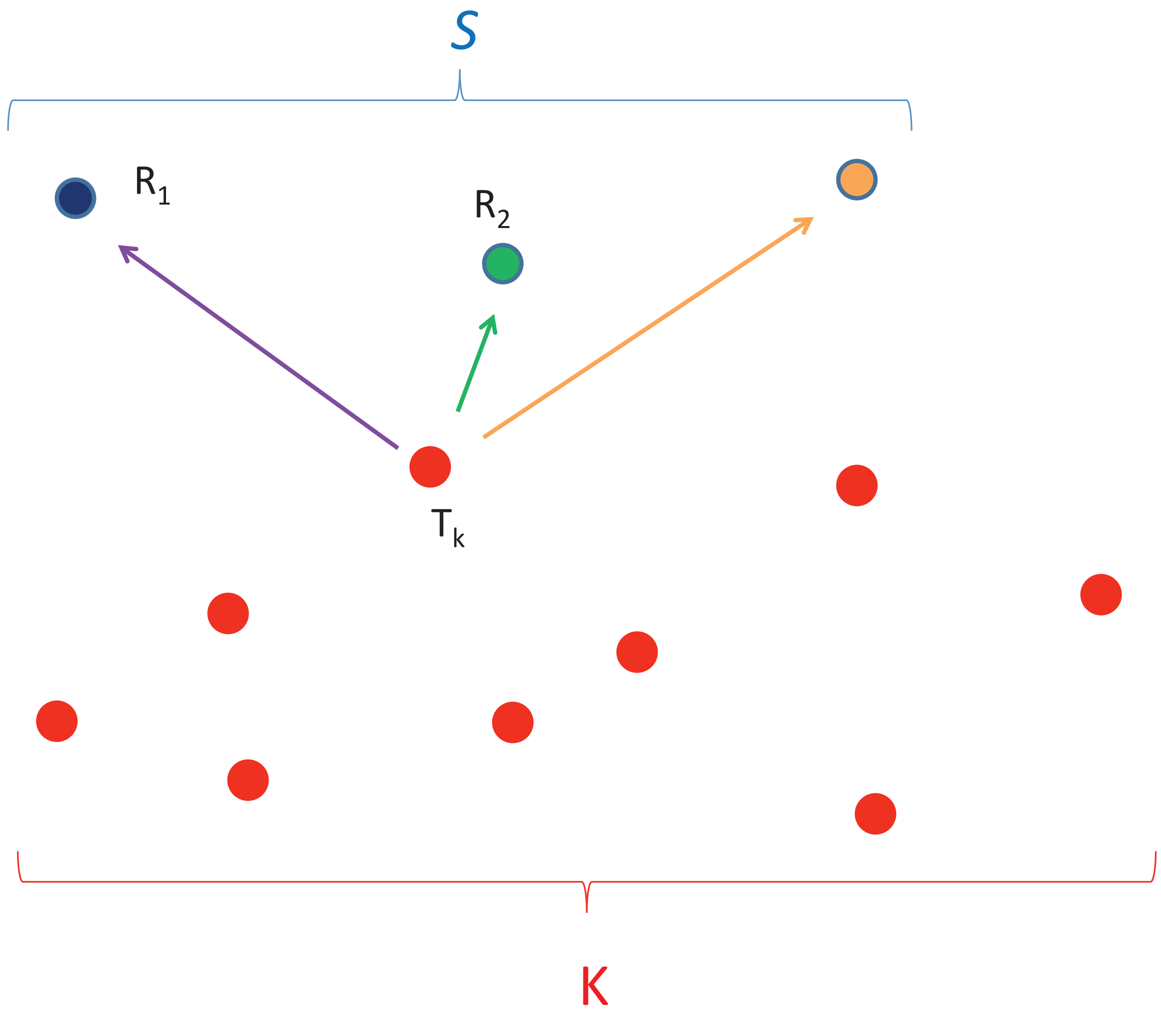
$$u_i(e_i, e_{-i}) = \beta_i(e_i) - \gamma_i\left(\sum_j e_j\right)$$

$$\beta_i(x) = a_i x - b_i x^2 \;\text{ and }\; \gamma_i(x) = c_i x^2$$

where $a_i > 0, b_i > 0$, and $c_i > 0$

Is there a Nash equilibrium? What are the best responses? What is the social optimum?

TW: A simple example of potential game



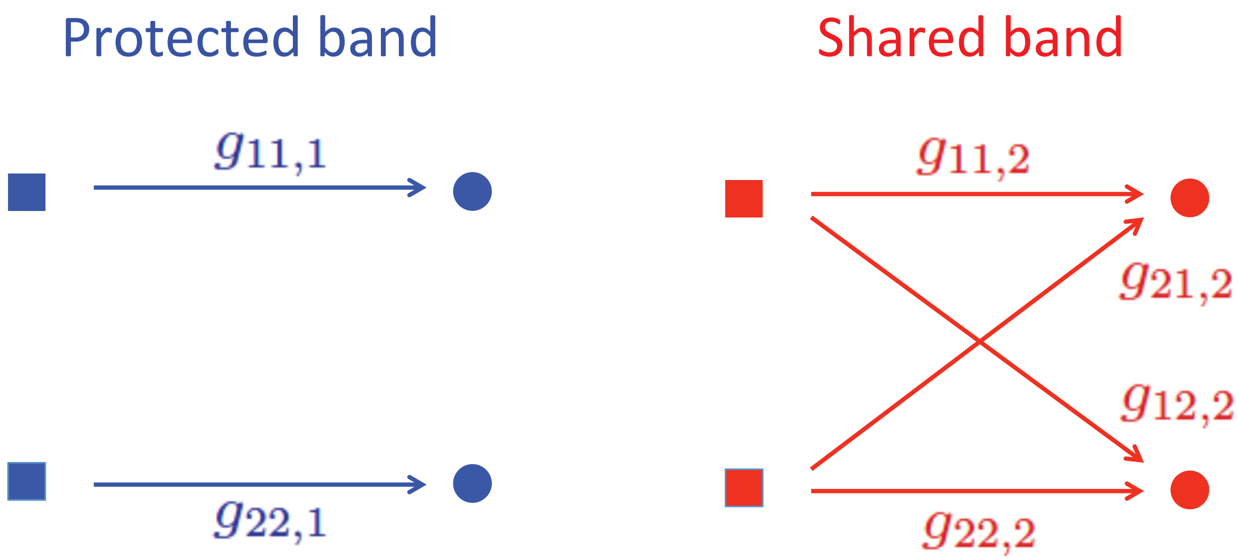
TW: A simple example of potential game [Perlaza et al 2009]

$$u_i(p_1, ..., p_K) \quad = \quad \sum_{s=1}^{S} \log\left(1 + \frac{g_{i,s}p_{i,s}}{\sigma^2 + \sum_{j\neq i} g_{j,s}p_{j,s}}\right)$$

TW: A simple example of potential game [Perlaza et al 2009]

$$
\begin{aligned}
u_i(p_1, ..., p_K) &= \sum_{s=1}^{S} \log \left(1 + \frac{g_{i,s} p_{i,s}}{\sigma^2 + \sum_{j \neq i} g_{j,s} p_{j,s}} \right) \\
&= \sum_{s=1}^{S} \log \left(\frac{\sigma^2 + \sum_{j} g_{j,s} p_{j,s}}{\sigma^2 + \sum_{j \neq i} g_{j,s} p_{j,s}} \right) \\
&= \log \left(\sigma^2 + \sum_{j} g_{j,s} p_{j,s} \right) - \log \left(\sigma^2 + \sum_{j \neq i} g_{j,s} p_{j,s} \right) \\
&= \Phi(p_1, ..., p_K) - \log \left(\sigma^2 + \sum_{j \neq i} g_{j,s} p_{j,s} \right)
\end{aligned}
$$

TW: A simple example of supermodular game [Mochaourab & Jorswieck 2009]



$$u_1(\mu_1, \mu_2) = \log\left(1 + \rho g_{11,1}\mu_1\right) + \log\left(1 + \frac{\rho g_{11,2}\overline{\mu}_1}{1 + \rho g_{21,2}\mu_2}\right)$$

$$u_2(\mu_1, \mu_2) = \log\left(1 + \rho g_{22,1}\overline{\mu}_2\right) + \log\left(1 + \frac{\rho g_{22,2}\mu_2}{1 + \rho g_{12,2}\overline{\mu}_1}\right)$$

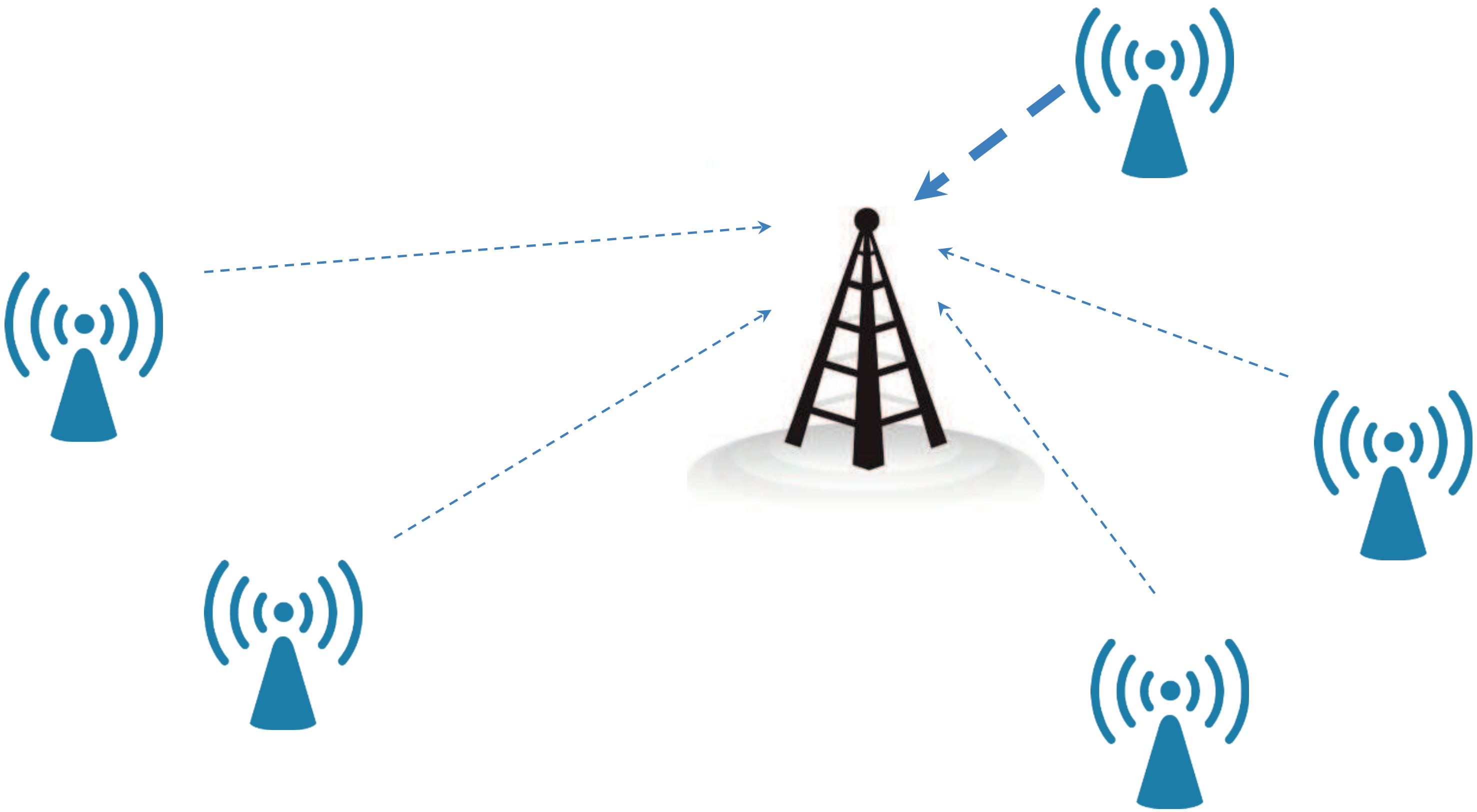
TW: Application of NBS: beamforming game [Larsson & Jorswieck 2008]

- $2-$user MISO interference channel model
- SINR: $\mathrm{SINR}_i = \frac{|h_{ii}^H w_i|^2 \mathbb{E}|x_i|^2}{1+|h_{ji}^H w_j|^2 \mathbb{E}|x_j|^2}$
- Utility: $u_i = \log(1 + \mathrm{SINR}_i)$
- Beamforming vector: $w_i = \alpha_i w_i^{\mathrm{ZF}} + (1 - \alpha_i) w_i^{\mathrm{MRT}}$
- NBS:

$$(\alpha_1^{\mathrm{NBS}}, \alpha_2^{\mathrm{NBS}}) = \arg \max_{(\alpha_1,\alpha_2)\in[0,1]^2} [u_1(\alpha_1, \alpha_2) - u_1(0,0)][u_2(\alpha_1, \alpha_2) - u_2(0,0)]$$

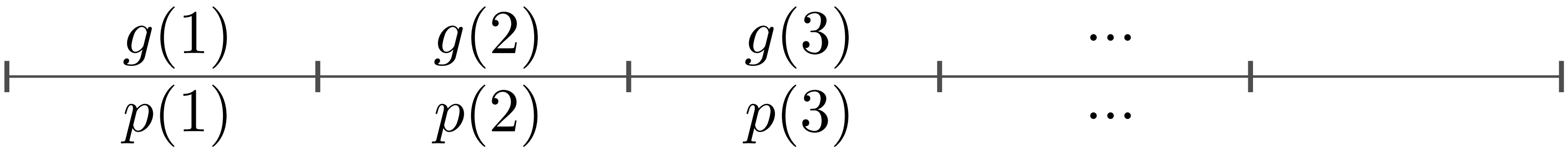
- Static game formulation.
- A repeated game formulation.

Distributed power control in wireless networks



Modeling the problem as a static game [Goodman & Mandayam 2000]

Time-slotted transmissions



Modeling the problem as a static game. Continued

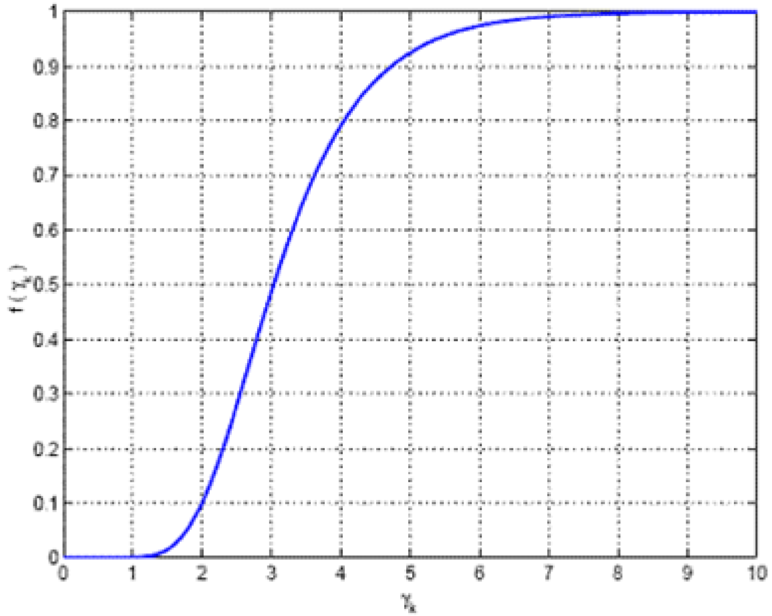
- **Set of players** : $\mathcal{K} = \{1, ..., K\}$.
- **Set of actions** : $\mathcal{A}_i = [0, P^{\max}]$.
- **Utilities** : energy-efficiency;

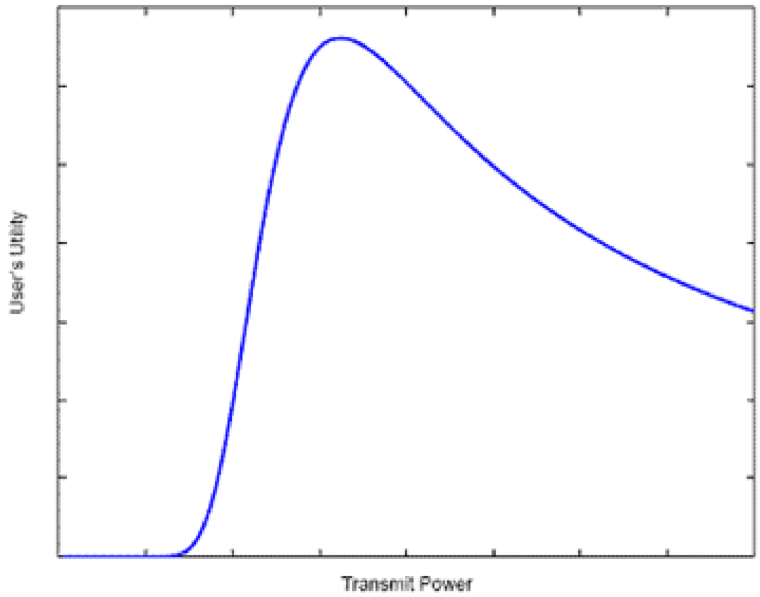
$$u_i(p_i, p_{-i}) = \frac{\text{benefit}}{\text{cost}} = \frac{f(\mathrm{SINR}_i)}{p_i} \quad \text{[bit/J]}.$$

where

$$\mathrm{SINR}_i = \frac{g_i p_i}{1 + \sum_{j \neq i} g_j p_j}.$$

Properties assumed for f





- f non-negative, continuous, and non-decreasing.
- f sigmoidal.
- $\lim_{x \to 0} \frac{f(x)}{x} = 0$, $\lim_{x \to +\infty} f(x) = const \leq 1$, $0 \leq f(x) \leq 1$.

Nash equilibrium analysis (1/3)

Existence

- $\mathcal{A}_i = [0, P_i^{\max}]$: compact, convex.
- u_i is continuous w.r.t. $p = (p_1, ..., p_K)$.
- u_i is quasi-concave w.r.t. p_i ($f(x)$ sigmoidal $\Rightarrow \frac{f(x)}{x}$ is quasi-concave).

Nash equilibrium analysis (2/3)

Uniqueness

The best response is a function and

$$\forall i \in \mathcal{K},\ \mathrm{BR}_i(p_{-i}) = \frac{\beta}{g_i}\left(1 + \sum_{j\neq i} g_j p_j\right)$$

with $\beta f'(\beta) = f(\beta)$.

The game is standard:

- Monotonicity: $p' \leq p \ \Rightarrow \ \mathrm{BR}(p') \leq \mathrm{BR}(p)$.
- Scalability: $\forall \alpha > 1,\ \mathrm{BR}(\alpha p) < \alpha \mathrm{BR}(p)$.

Nash equilibrium analysis (3/3)

Determination (interior point)

Solve the system of equations $\frac{\partial u_i}{\partial p_i}(p) = 0$, which leads to:

$$\forall i \in \{1, ..., K\},\ p_i^\star = \frac{1}{g_i} \frac{\sigma^2 \beta}{1 - (K-1)\beta}.$$

Problem Generally inefficient solution. How to improve efficiency?

Introduce pricing

Main points

- New utility:

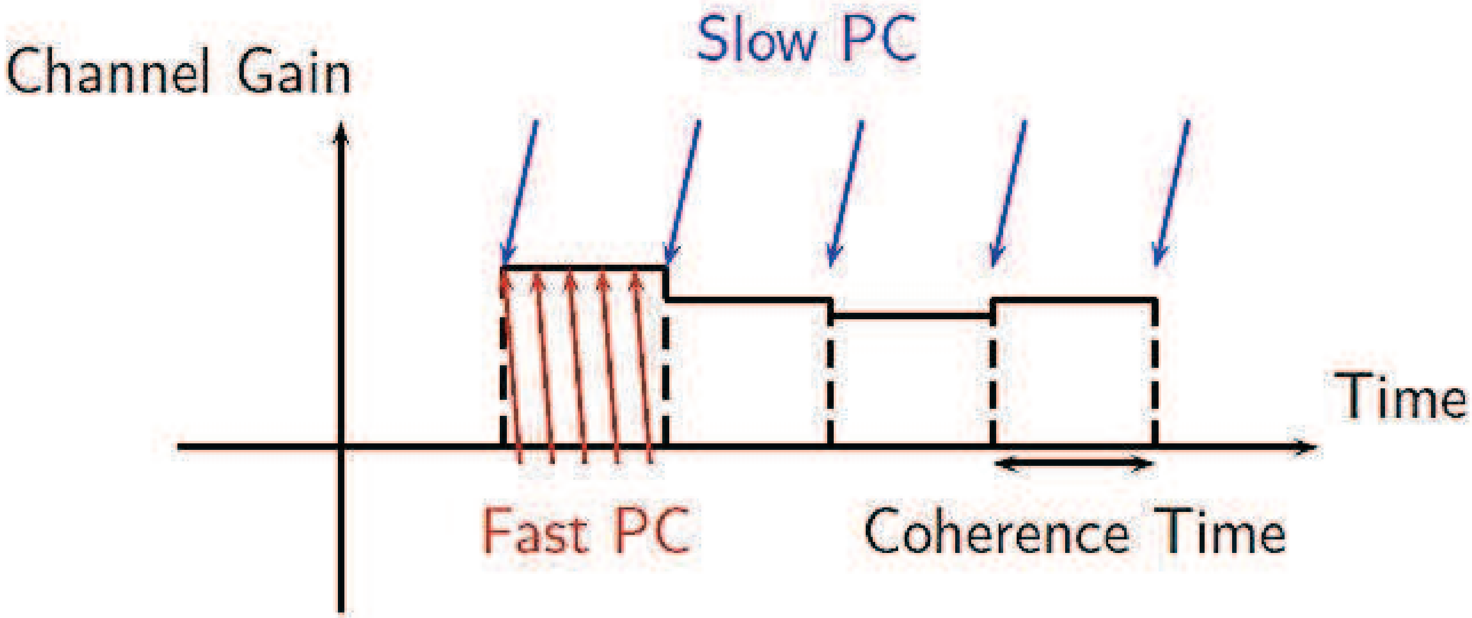
$$\widetilde{u}_i(p) = u_i(p) - \alpha p_i, \quad \alpha \geq 0.$$

- Good news. The new NE profile Pareto-dominates $p^\star$.

- Bad news. Uniqueness not guaranteed, convergence under some specific assumption. Global channel state information is required.

[Saraydar et al 2002].

Modeling the problem as a repeated game: fast/slow power control



Repeated game formulation (fast power control)

Strategic form

$$\mathcal{G}^m = \left(\mathcal{K}, \{\mathcal{T}_i\}_i, \{v_i^m\}_i\right) \text{ with } m \in \{T, \lambda\}.$$

If $m = T$:

$$v_i^T = \frac{1}{T}\sum_{t=1}^{T} u_i(\underline{p}(t)).$$

If $m = \lambda \in (0,1]$:

$$v_i^\lambda = \sum_{t=1}^{+\infty} \lambda(1-\lambda)^{t-1} u_i(\underline{p}(t)).$$

[Le Treust and Lasaulce 2010]

Public signal choice (RSSI/SINR)

$$\omega(t) \triangleq \sigma^2 + \sum_{i=1}^{K} g_i p_i(t) = p_i(t) g_i \times \frac{\mathrm{SINR}_i(t) + 1}{\mathrm{SINR}_i(t)}$$

Strategic form. Continued

Pure strategies

$$\begin{array}{rccc} \tau_{i,t}: & (\mathcal{P}_i \times \Omega)^{t-1} & \rightarrow & [0, P_i^{\max}] \\ & (p_i^{t-1}, \omega^{t-1}) & \mapsto & p_i(t) \end{array}$$

where

- $p_i^{t-1} = (p_i(1), p_i(2), ..., p_i(t-1))$;
- $\omega^{t-1} = (\omega(1), \omega(2), ..., \omega(t-1))$;
- $\Omega = \left[\sigma^2, \sigma^2 + \sum_{i=1}^{K} g_i^{\max} P_i^{\max}\right]$.

An interesting Nash equilibrium of $\mathcal{G}^m$, $m = T$

Proposed equilibrium point

$$\tau_{i,t}^* = \begin{vmatrix} p_i^{\mathrm{LI}} & \text{if } t \in \{1, 2, \ldots, T - t_0\} & \text{and} & \omega(t) = \frac{\sigma^2(1-\gamma)}{1-(K-1)\gamma} \\ p_i^\star & \text{if } t \in \{T - t_0 + 1, \ldots, T\} & \text{and} & \omega(t) = \frac{\sigma^2(1-\gamma)}{1-(K-1)\gamma} \\ P_i^{\max} & \text{if} & & \omega(t) \neq \frac{\sigma^2(1-\gamma)}{1-(K-1)\gamma} \end{vmatrix}$$

where $\gamma[1 - (K-1)\gamma]f'(\gamma) - f(\gamma) = 0$ and

$$\forall i \in \mathcal{K}, \quad p_i^{\mathrm{LI}} = \frac{1}{g_i} \frac{\sigma^2 \gamma}{1 - (K-1)\gamma}.$$

Comments

- To obtain LI, impose $g_j p_j =$ const.
- t_0 comes from the equilibrium condition:

$$\text{Let } \; t_0 = \left\lceil \frac{\frac{f(\alpha)}{\alpha} - \frac{f(\beta)[1-(K-1)\beta]}{\beta}}{\frac{f(\alpha)[1-(K-1)\alpha]}{\alpha} - \frac{f(\alpha)}{\alpha\left(\sigma^2+\sum_{j\neq i} g_j P_j^{\max}\right)}} \right\rceil$$

- Individual CSI property, Pareto dominates the NE of $\mathcal{G}$, good in terms of social welfare.

Slow power control case

▶ Repeated game methodology holds (worst-case scenario).
For instance, t_0 becomes:

$$t_0 = \left\lceil \frac{\frac{g_i^{\max}}{g_i^{\min}} \frac{f(\alpha)}{\alpha} - \frac{f(\beta)[1-(K-1)\beta]}{\beta}}{\frac{f(\alpha)[1-(K-1)\alpha]}{\alpha} - \frac{\frac{g_i^{\max}}{g_i^{\min}} f(\alpha)}{\alpha\left(\sum_{j \neq i} P_j^{\max} g_j^{\min} + \sigma^2\right)}} \right\rceil .$$

▶ **Stochastic game formulation:** i.i.d. state, $\overline{v}_i^T = \mathbb{E}_{\underline{g}}\left[v_i^T(.)\right]$

- good: better performance;
- bad: more information is needed (CDI typically).

Illustration (fast power control)

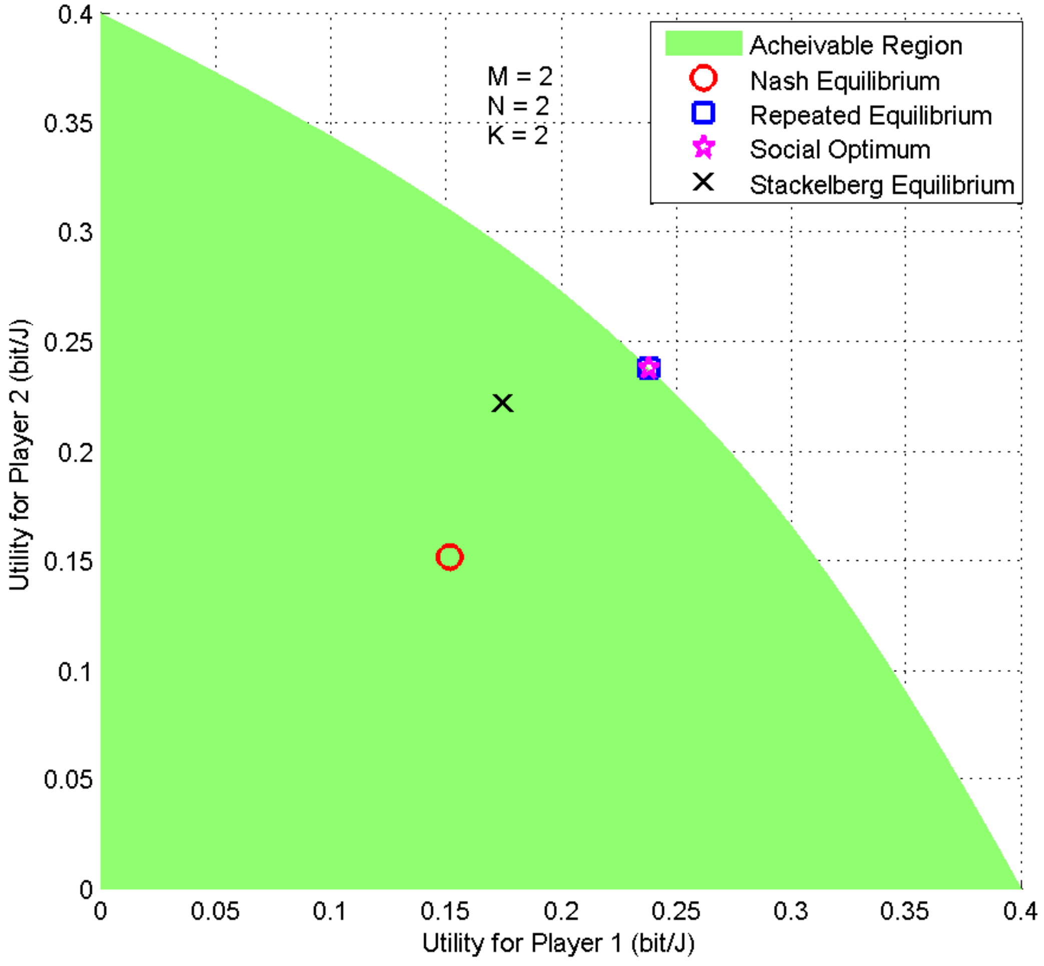
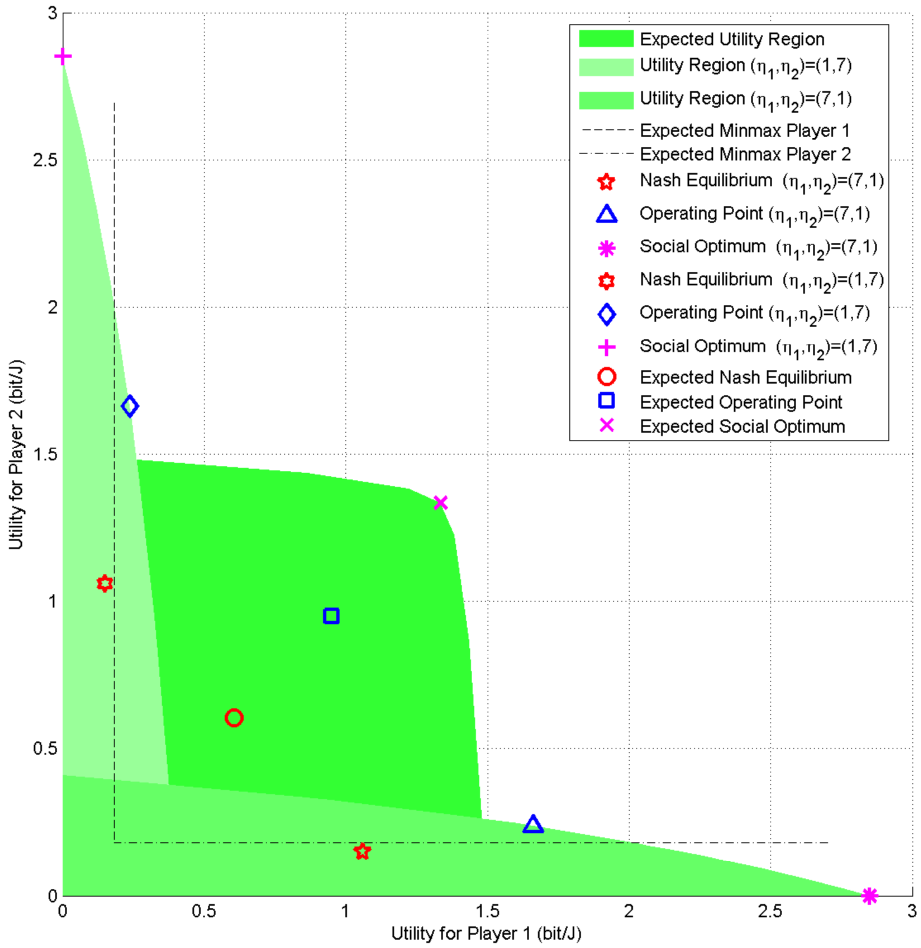


Illustration (slow power control)



[Mériaux et al 2011]

Observations

- ► Stochastic game case = most general case + most efficient policies.
- ► Importance of characterizing equilibrium points.

9. Short research project

Viral marketing

[”Allocating marketing resources over social networks: A long-term analysis”, V.S. Varma, S. Lasaulce, J. Mounthanyvong, I.C. Morarescu, IEEE Control Systems Letters 3 (4), 1002-1007.]

[”Opinion dynamics aware marketing strategies in duopolies”, V.S. Varma, I.C Morarescu, S. Lasaulce, and S. Martin, 2017 IEEE 56th Annual Conference on Decision and Control (CDC), 3859-3864.]