Analysis of Simple Planar Linkages

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1   Introduction

The goal of this section is to introduce the basic terminology and notation used for describing, designing and analyzing mechanisms and robots. The scope of this discussion will be limited, for the most part, to mechanisms with planar geometry

We will use the term mechanical system to describe a system or a collection of rigid or flexible bodies that may be connected together by joints. A *mechanism* is a mechanical system that has the main purpose of transferring motion and/or forces from one or more sources to one or more outputs. The rigid bodies are called *links*. A *linkage* is a mechanical system consisting of links connected by either pin joints (also called revolute joints) or sliding joints (also called prismatic joints). In this section, we will also briefly consider mechanical systems with other types of joints.

Definition 1. Degree of freedom
The number of independent variables (or coordinates) required to completely specify the configuration of the mechanical system.

For example, a point on a plane has two degrees of freedom. A point in space has three degrees of freedom. A pendulum restricted to swing in a plane has one degree of freedom. A planar rigid body (or a lamina) has three degrees of freedom (two if you consider translations and an additional one when you include rotations). The mechanical system consisting of two planar rigid bodies connected by a pin joint has four degrees of freedom. Specifying the position and orientation of the first rigid body requires three variables. Since the second one rotates relative to the first one, we need an additional variable to describe its motion. Thus, the total number of independent variables or the number of degrees of freedom is four.

As we go to three dimensions, one needs to appreciate the fact that there are three independent ways to rotate a single rigid body. Thus, a rigid body in three dimensions has six degrees of freedom — three translatory degrees of freedom and three degrees of freedom associated with rotation. For example, consider rotations about the x, y, and z axes. A result due to Euler says that any rigid body rotation

can be accomplished by successive rotations about the x, y, and z axes. If we consider the three angles of rotation to be coordinates and use these coordinates to parameterize the rotation of the rigid body, it is evident there are three rotational degrees of freedom.

Thus, two rigid bodies in three dimensions connected by a pin joint have seven degrees of freedom. Specifying the position and orientation of the first rigid body requires six variables. Since the second one rotates relative to the first one, we need an additional variable to describe its motion. Thus, the total number of independent variables or the number of degrees of freedom is seven.

While the definition of the number of degrees of freedom is motivated by the need to describe or analyze a mechanical system, this concept is also very important for controlling or driving a mechanical system. It tells us the number of independent inputs required to drive all the rigid bodies in the mechanical system.

Definition 2. Kinematic chain *A system of rigid bodies connected together by joints is called a kinematic chain. A kinematic chain is called closed if it forms a closed loop. A chain that is not closed is called an open chain. If each link of an open kinematic chain except the first and the last link is connected to two other links it is called a serial kinematic chain.*

There are mainly four types of joints that are found in mechanisms:

- Revolute, rotary or pin joint (R)

- Prismatic or sliding joint (P)

- Spherical or ball joint (S)

- Helical or screw joint (H)

A revolute joint allows a rotation between the two connecting links. The best example of this is the hinge used to attach a door to the frame. The prismatic joint allows a pure translation between the two connecting links. The connection between a piston and a cylinder in an internal combustion engine or a compressor is via a prismatic joint. The spherical joint between two links allows the first link to rotate in all possible ways with respect to the second. The best example of this is seen in the human body. The shoulder and hip joints, called ball and socket joints, are spherical joints. The helical joint allows a helical motion between the two connecting bodies. A good example of this is the relative motion between a bolt and a nut.

All linkages are kinematic chains. However, they are a special class of kinematic chains with only prismatic or revolute joints.

Definition 3. Planar chain

A planar chain is a kinematic chain in which all links of the chain are constrained to move in or parallel to the same plane.

Because planar chains can only admit prismatic and revolute joints (why?), all planar chains are linkages. In fact, the axes of the revolute joints must be perpendicular to the plane of the chain while the axes of the prismatic joints must be parallel to or lie in the plane of the chain. Two of the most common examples of planar closed kinematic chains are shown in Figure 1.
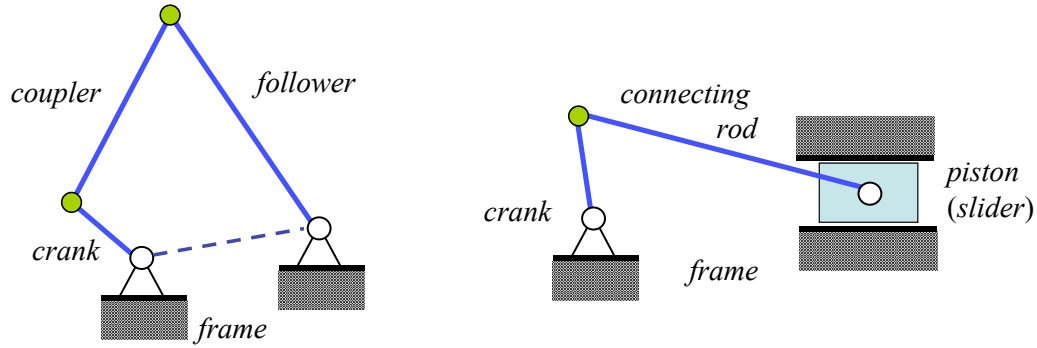

Figure 1: Two examples of planar, kinematic chains. A four-bar linkage (left) and a slider-crank linkage (right). The four-bar linkage has four R-joints while the slider-crank has three R-joints and one P-joint.

Definition 4. Connectivity of a joint

*The number of degrees of freedom of a rigid body connected to a **fixed** rigid body through the joint is called the connectivity of the joint.*

Revolute, prismatic and helical joints have a connectivity 1. The spherical joint has a connectivity of 3. Sometimes one uses the term "degrees of freedom of a joint" instead of the connectivity of a joint.

Definition 5. Mobility of a chain

The mobility of a chain is the number of degrees of freedom of the chain.

Note that the term number of degrees of freedom is also sometimes used for the mobility. In a serial chain, the mobility of the chain is easily calculated. If there are n joints and joint i has a connectivity f_i, the mobility of the serial chain is given by:

$$M = \sum_{i=1}^{n} f_i \tag{1}$$

For example, most industrial robots are serial chains with either revolute or prismatic joints ($f_i = 1$) and therefore the mobility or the number of degrees of freedom of the robot arm is also equal to the number of joints. Sometimes, an n degree-of-freedom robot or a robot with mobility n is also called an n-axis robot.

When closed loops are present in the kinematic chain (that is, the chain is no longer serial, or even open), it is more difficult to determine the number of degrees of freedom or the mobility of the robot. But there is a simple formula that one can derive for this purpose.

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Let n be the number of links and let j be the number of joints, with f_i being the connectivity of joint i, with $i = 1, 2, \ldots j$. Each planar rigid body has three degrees of freedom. If there were no joints, since there are $n - 1$ moving rigid bodies, the system would have $3(n - 1)$ degrees of freedom. The effect of each joint is to constrain the relative motion of the two connecting bodies. If a joint has a connectivity f_i, it imposes $(3 - f_i)$ constraints on the relative motion. In other words, since there are f_i different ways for one body to move relative to another, there $(3 - f_i)$ different ways in which the body is constrained from moving relative to another. Therefore, the number of degrees of freedom or the mobility of a chain (including the special case of a serial chain) is given by:

$$M = 3(n - 1) - \sum_{i=1}^{j} (3 - f_i) = 3(n - j - 1) + \sum_{i=1}^{j} f_i \tag{2}$$

A straightforward application of this formula for the two planar robot arm designs in Figure 2 tells us that both designs have three degrees of freedom and therefore require three actuators (rotary in one case and linear in the other case) for full control. Similarly, this formula can be applied to the linkages in Figure 3. For the figure on the left, $n = 7$ and $j = 8$, while all joints have connectivity 1 ($f_i = 1$). This gives us a mobility of 2. The figure on the right can be similarly analyzed with one exception. You will see that the joint between links 3, 4, and 5 is different from all other joints. All the other joints we have looked at thus far are *binary joints*. They connect one link to another and therefore serve to join a single pair of links. The joint between links 3, 4, and 5 in Figure 3 (right) connects two pairs of links. It connects link 3 to link 4, and link 4 to link 5 (alternatively, link 3 to link 5 and link 4 to link 5, or link 3 to link 4 and link 3 to link 5). Thus it is really a complex joint consisting of two *binary joints*. The calculation with Equation (2) proceeds smoothly if one counts this joint as two joints. Taking this into account, $n = 6$, $j = 7$ and $f_i = 1$ gives us $M = 1$.

2 Introduction to Linkage Analysis

This section presents analytical techniques for position, velocity and acceleration analysis of planar linkages. We will consider four-bar linkages and slider-crank linkages (Figure 1), but the basic techniques here can be applied to most planar linkages.

The essence of each linkage is captured by the rigid links and the joints connecting the rigid links. Recall that we can only have pin joints (R) and sliding joints (P) in a planar linkage. Each pin (revolute) joint can be characterized by a center of rotation and therefore a point on the linkage. Therefore the geometry of a linkage with only revolute joints is described by the lengths between the pin joints and the angles between the lines joining pairs of adjacent pin joints. See Figure 4. On the other hand, each sliding joint can be characterized by an axis associated with the sliding direction. The relative motion between
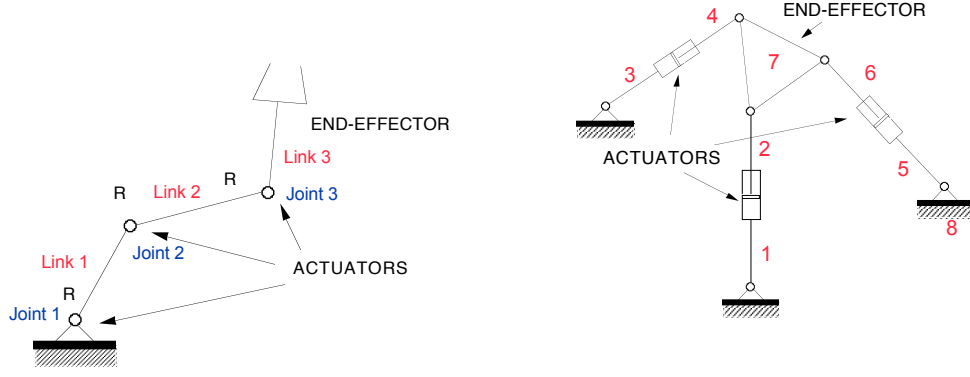
Figure 2: Two designs for planar, three degree-of-freedom robot arms. (a) The mobility can be calculated from Equation (1) or Equation (2) to be 3; (b) The mobility can be calculated from Equation (2) to be 3.
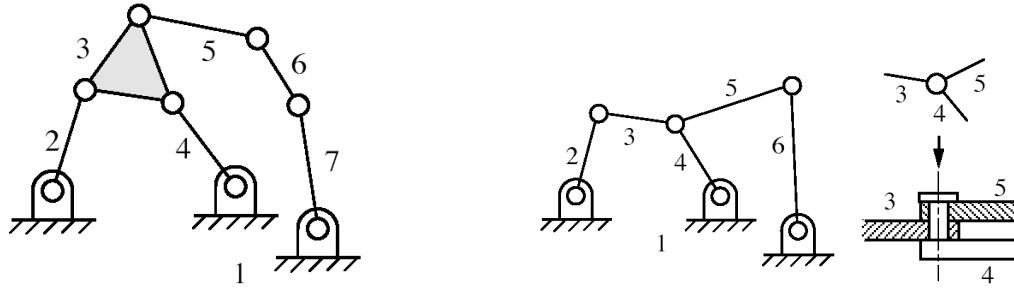

Figure 3: Examples of planar linkages with closed chains. Equation (2) reveals the mobility to be 2 (left) and 1 (right).

the links connected by the sliding joint is determined only by the direction of the axis. The shapes of the links are irrelevant for kinematic analysis.

We will use position vectors to describe the positions and orientations of links. The *ith* link connecting points A and B is mathematically modeled by the vector $\overrightarrow{AB}$. In the notation of [1], this is the position vector $\mathbf{r}_{B/A}$. When we have position vectors that connect two points on link i, and when there is no ambiguity, we will call this link position vector $\mathbf{r}_i$. The vector has magnitude r_i and the angle[1] it makes with the positive x axis will be called θ_i as shown in Figure 5. Each such position vector can be written in terms of components in a Cartesian coordinate system:

$$\mathbf{r}_i = r_i cos\theta_i \mathbf{i} + r_i sin\theta_i \mathbf{j}$$

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[1]All angles are measured positive counter-clockwise, usually from the positive x axis.
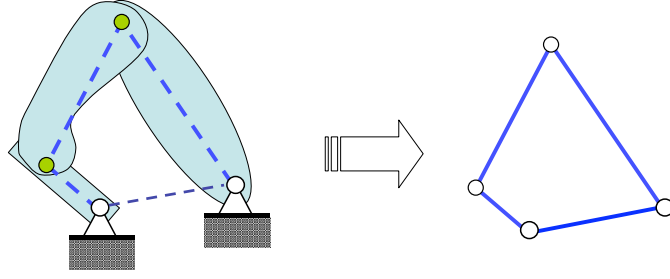
Figure 4: The kinematic model of a linkage (left) is simply a series of concatenated line segments with joints (right).
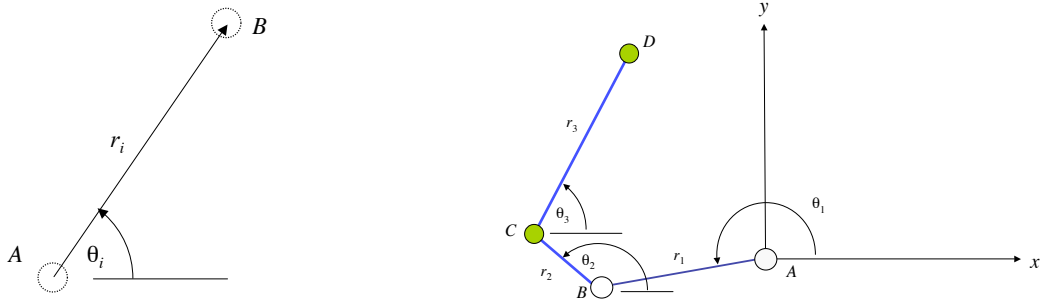

Figure 5: A position vector describing link i connecting two points A and B (left) and the concatenation of three position vectors describing three links connected by two revolute joints at B and C (right).

or as a 2×1 column vector:

$$\mathbf{r}_i = \left[\begin{array}{c} r_i cos\theta_i \\ r_i sin\theta_i \end{array} \right] \tag{3}$$

In Figure 5 (right), the position vector of the point D can be obtained by summing up the vectors $\mathbf{r}_{B/A} = \overrightarrow{AB}$, $\mathbf{r}_{C/B} = \overrightarrow{BC}$, and $\mathbf{r}_{D/C} = \overrightarrow{CD}$. Thus we have the vector equation:

$$\mathbf{r}_{D/A} = \mathbf{r}_{B/A} + \mathbf{r}_{C/B} + \mathbf{r}_{D/C} \tag{4}$$

$$= \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 \tag{5}$$

It is important to remember that each such vector equation represents two scalar equations.

One can also differentiate these position vectors to get equations involving velocities. The derivative of $\mathbf{r}_i$ is obtained by differentiating Equation (3), recognizing that r_i, the magnitude, is a link length and therefore constant:

$$\frac{d\mathbf{r}_i}{dt} = -r_i sin\theta_i \dot{\theta}_i \mathbf{i} + r_i cos\theta_i \dot{\theta}_i \mathbf{j}$$

$$= r_i \dot{\theta}_i (-sin\theta_i \mathbf{i} + cos\theta_i \mathbf{j}) \tag{6}$$

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Similarly, one can differentiate the above equation to get a vector equation involving not only the position θ_i and the velocity $\dot{\theta}_i$, but also the acceleration $\ddot{\theta}_i$.

Alternatively, we could have differentiated Equation (5) to obtain first an equation involving the velocities $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$, and then (after differentiating again) an equation involving the velocities $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$, and the accelerations $\ddot{\theta}_1$, $\ddot{\theta}_2$, and $\ddot{\theta}_3$.

In most problems involving kinematic analysis, there are typically three steps or subproblems.

1. First, you will need to determine the values for all the joint positions (angles for revolute joints and linear displacements for prismatic joints). Typically, the values of some of the joint positions are known and the **position analysis** step involves writing down vector equations like (5) to solve for the unknown joint positions.

2. The next step involves **velocity analysis**, which involves differentiating Equation (5) to obtain an equation which includes the velocities $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in addition to the positions. The goal is to use the position information (all known from the position analysis step) and some known velocities to solve for the unknown velocities.

3. The third step involves differentiating the velocity equations to get equations involving the accelerations $\ddot{\theta}_1$, $\ddot{\theta}_2$, and $\ddot{\theta}_3$ and lower order derivatives. In this step, **acceleration analysis**, one solves for unknown accelerations given all positions, all velocities and some known accelerations.

We will illustrate these three steps using the example of four bar linkages next.

3 Position analysis for four-bar linkages

3.1 Notation

A schematic of a four-bar linkage is shown in Figure 6. The fixed link or the base is called a frame. There are two fixed pivots or revolute joints that connect the base to two other links. One of these links is an input and usually the other is an output. If the input link is able to rotate 360 degrees, this input link is usually called a crank. The output link is called the follower while the intermediate link is called the coupler. The pivots attached to the coupler move in circles whose centers are the fixed pivots.

We will adopt the notation in Figure 7. We number the links 1 through 4, with the base being called link 1, the crank link 2, the coupler link 3 and the follower link 4. Each link has a length r_i associated
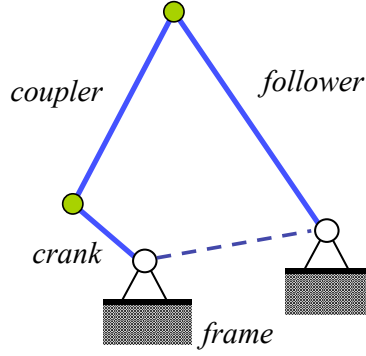
Figure 6: A four-bar linkage. The fixed base link is called the frame. The driving link, usually called the crank, is pivoted to the frame. The other link that is pivoted to the frame is usually the driven link and called the follower.

with it. Define

$$\mathbf{r}_1 = \overrightarrow{OR}$$
$$\mathbf{r}_2 = \overrightarrow{OQ}$$
$$\mathbf{r}_3 = \overrightarrow{QP}$$
$$\mathbf{r}_4 = \overrightarrow{RP}$$

The angles the links make with the horizontal (positive) x axis are denoted by θ_i. Note that θ_1 is fixed, while θ_2, θ_3, and θ_4 are variables.

3.2 Closure equations

We will now write the position vectors as in Equation (5), but in a form that captures the closed-chain geometry of the four-bar linkage. Consider the position vector of point P, $\mathbf{r}_{P/O}$. This position vector can be written in two ways:

$$\mathbf{r}_{P/O} = \mathbf{r}_2 + \mathbf{r}_3 \tag{7}$$
$$\mathbf{r}_{P/O} = \mathbf{r}_1 + \mathbf{r}_4 \tag{8}$$

This leads us to the vector closure equation:

$$\mathbf{r}_2 + \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_4 \tag{9}$$

Or the two scalar equations:

$$r_2 cos\theta_2 + r_3 cos\theta_3 = r_1 cos\theta_1 + r_4 cos\theta_4 \tag{10}$$
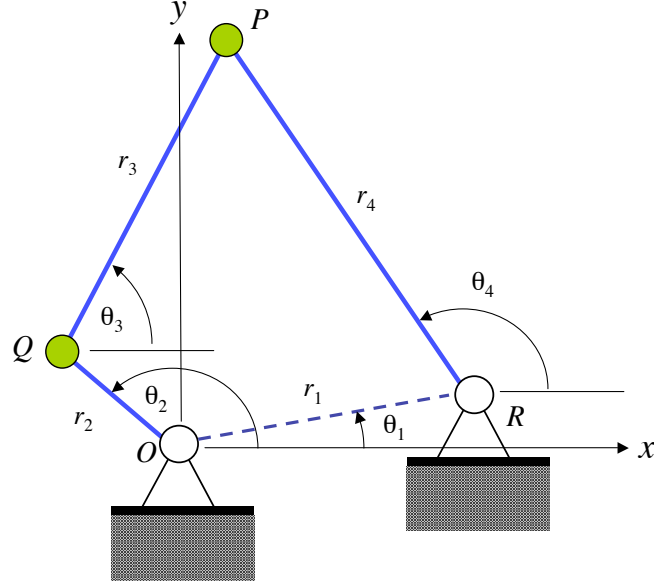$$r_2 sin\theta_2 + r_3 sin\theta_3 = r_1 sin\theta_1 + r_4 sin\theta_4 \tag{11}$$

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Figure 7: Definitions of key variables and constants for modeling a four-bar linkage.

If one of the angles, for example, the crank angle θ_2 is known (presumably θ_1 is known from the geometry since it is a constant), it is, in principle, possible to solve Equations (10)-(11). Let us see how to do this next.

3.3 Solving the closure equations when the crank angle is known

Eliminating θ_3 Consider the case where the crank angle is stepped through at some known rate and therefore θ_2 is known. θ_1 is known to be a fixed constant. Rearrange Equations (10-11) so only θ_3 is on one side:

$$r_3 cos\theta_3 = r_1 cos\theta_1 + r_4 cos\theta_4 - r_2 cos\theta_2 \tag{12}$$

$$r_3 sin\theta_3 = r_1 sin\theta_1 + r_4 sin\theta_4 - r_2 sin\theta_2 \tag{13}$$

Squaring both sides of each equation and adding, we get one equation:

$$[r_3 cos\theta_3]^2 + [r_3 sin\theta_3]^2 = [r_1 cos\theta_1 + r_4 cos\theta_4 - r_2 cos\theta_2]^2$$
$$+ [r_1 sin\theta_1 + r_4 sin\theta_4 - r_2 sin\theta_2]^2$$

After rearranging, we get:

$$(2r_1 r_4 cos\theta_1 - 2r_2 r_4 cos\theta_2)cos\theta_4 + (2r_1 r_4 sin\theta_1 - 2r_2 r_4 sin\theta_2)sin\theta_4$$
$$-2r_1 r_2(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2) + r_1^2 + r_2^2 + r_4^2 - r_3^2 = 0 \tag{14}$$

We have eliminated θ_3 from Equations (10)-(11) to arrive at one equation in one unknown. Unfortunately this equation is nonlinear. There is a trick to solving a nonlinear trigonometric equation of this type which is discussed next.

Solving an equation involving cosine and sine of a variable Consider the equation:

$$A cos\alpha + B sin\alpha + C = 0 \tag{15}$$

Define γ so that

$$cos\gamma = \frac{A}{\sqrt{A^2 + B^2}}; \qquad sin\gamma = \frac{B}{\sqrt{A^2 + B^2}}.$$

Note that this is always possible. γ can be determined from A and B using the $atan2$ function:

$$\gamma = atan2\left(\frac{B}{\sqrt{A^2 + B^2}}, \frac{A}{\sqrt{A^2 + B^2}}\right).$$

Now divide all terms of Equation (15) by $\sqrt{A^2 + B^2}$. The result can be rewritten as:

$$cos\gamma cos\alpha + sin\gamma sin\alpha + \frac{C}{\sqrt{A^2 + B^2}} = 0$$

or,

$$cos(\alpha - \gamma) + \frac{C}{\sqrt{A^2 + B^2}} = 0$$

Knowing the inverse cosine function has two possible solutions in the interval $(-\pi, \pi]$, we get two possible solutions for α in terms of the known angle γ:

$$\alpha = \gamma \pm cos^{-1}\left(\frac{-C}{\sqrt{A^2 + B^2}}\right). \tag{16}$$

Solving for θ_4 and θ_3 We can now apply the solution methodology in Equation (16) to the nonlinear equation in Equation (14) by letting:

$$
\begin{aligned}
A &= (2r_1 r_4 cos\theta_1 - 2r_2 r_4 cos\theta_2), \\
B &= (2r_1 r_4 sin\theta_1 - 2r_2 r_4 sin\theta_2), \\
C &= (r_1^2 + r_2^2 + r_4^2 - r_3^2) - 2r_1 r_2(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2).
\end{aligned}
$$

Define γ:

$$\gamma = atan2\left(\frac{(r_1 r_4 sin\theta_1 - r_2 r_4 sin\theta_2)}{\sqrt{(r_1^2 r_4^2 + r_2^2 r_4^2 - 2r_1 r_2 r_4^2 cos(\theta_2 - \theta_1)}}, \frac{(r_1 r_4 cos\theta_1 - r_2 r_4 cos\theta_2)}{\sqrt{(r_1^2 r_4^2 + r_2^2 r_4^2 - 2r_1 r_2 r_4^2 cos(\theta_2 - \theta_1)}}\right).$$

Then θ_4 is given by:

$$\theta_4 = \gamma \pm cos^{-1}\left(\frac{r_3^2 - r_1^2 - r_2^2 - r_4^2 + 2r_1 r_2(cos(\theta_1 - \theta_2)}{\sqrt{(r_1^2 r_4^2 + r_2^2 r_4^2 - 2r_1 r_2 r_4^2 cos(\theta_2 - \theta_1)}}\right). \tag{17}$$

We can now substitute for θ_4 in (12-13) to get:

$$\theta_3 = atan2\left(\frac{r_1 sin\theta_1 + r_4 sin\theta_4 - r_2 sin\theta_2}{r_3}, \frac{r_1 cos\theta_1 + r_4 cos\theta_4 - r_2 cos\theta_2}{r_3}\right) \tag{18}$$

Note that there are two solutions for θ_4 because of the sign ambiguity in Equation (17) but for each θ_4, there is a unique θ_3.
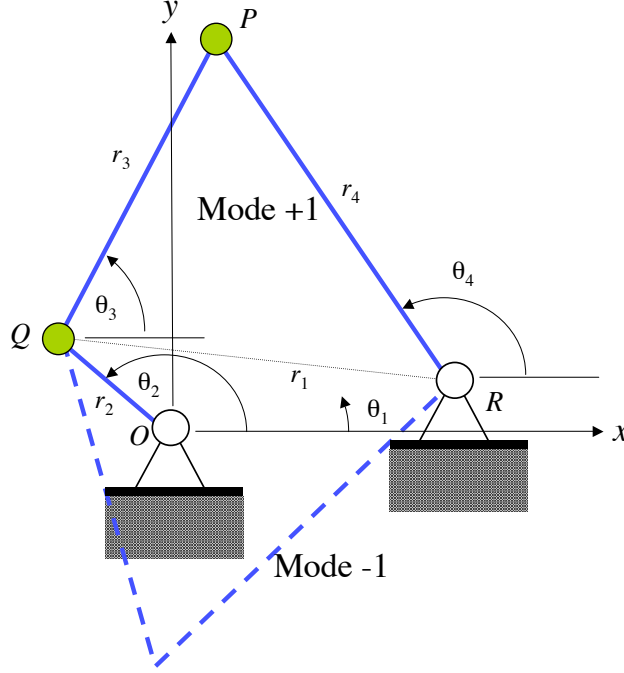

Figure 8: There are two modes of assembly for a four-bar linkage. If the crank angle is given, changing the sign in Equation (17) gives us two assembly configurations which correspond graphically to mirror reflections of the point P about the line QR.

This sign ambiguity can be easily explained as an ambiguity in the assembly of the linkage. As shown in Figure 8, there are two distinct assembly modes labeled $+1$ and -1 corresponding to the sign for the arc cosine in Equation (17). One of them will yield a value of θ_4 that corresponds to the point P being above the line QR while the other sign will yield a value of θ_4 that corresponds to a its mirror reflection about the line QR. Note that depending on the values of the constants in the equation, the plus sign may well yield the configuration below QR, and not the one above QR as shown in the figure.

3.4   Velocity analysis

The velocity equations are obtained by differentiating the closure equations in Equations (10-11):

$$-r_2 sin(\theta_2)\dot{\theta}_2 - r_3 sin(\theta_3)\dot{\theta}_3 = -r_4 sin(\theta_4)\dot{\theta}_4 \tag{19}$$

$$r_2 cos(\theta_2)\dot{\theta}_2 + r_3 cos(\theta_3)\dot{\theta}_3 = r_4 cos(\theta_4)\dot{\theta}_4 \tag{20}$$

Suppose, as in the previous subsection, the crank motion is known. In other words, $\dot{\theta}_2$, the crank angular velocity is given. Assume the first step, position analysis, has been conducted and *all* angles have been solved for. We can move all the terms involving unknowns to the left hand side leaving the known quantities on the right hand side:

$$-r_3 sin(\theta_3)\dot{\theta}_3 + r_4 sin(\theta_4)\dot{\theta}_4 = r_2 sin(\theta_2)\dot{\theta}_2$$

$$r_3 cos(\theta_3)\dot{\theta}_3 - r_4 cos(\theta_4)\dot{\theta}_4 = -r_2 cos(\theta_2)\dot{\theta}_2$$

These two equations are linear in $\dot{\theta}_3$ and $\dot{\theta}_4$ and therefore are easy to solve. Write them in matrix form:

$$\begin{bmatrix} -r_3 sin(\theta_3) & r_4 sin(\theta_4) \\ r_3 cos(\theta_3) & -r_4 cos(\theta_4) \end{bmatrix} \begin{bmatrix} \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix} = \begin{bmatrix} r_2 sin(\theta_2)\dot{\theta}_2 \\ -r_2 cos(\theta_2)\dot{\theta}_2 \end{bmatrix} \quad (21)$$

Solving this matrix equation gives us a unique solution for the angular velocities $\dot{\theta}_3$ and $\dot{\theta}_4$.

Once these equations are solved, the velocity of any point on the linkage can be easily determined. For example, let us suppose we are interested in the velocity of the point P. The position vector for P is given by:

$$\mathbf{r}_{P/O} = \mathbf{r}_1 + \mathbf{r}_4.$$

The derivative of this position vector gives us the velocity:

$$\mathbf{v}_{P/O} = \dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_4 = \dot{\mathbf{r}}_4 = r_4 \dot{\theta}_4 \left(-sin(\theta_4)\mathbf{i} + cos(\theta_4)\mathbf{j} \right).$$

3.5 Acceleration analysis

The acceleration equations are obtained by differentiating the velocity equations in Equations (19-20):

$$-r_2 sin(\theta_2)\ddot{\theta}_2 - r_2 cos(\theta_2)(\dot{\theta}_2)^2 - r_3 sin(\theta_3)\ddot{\theta}_3 - r_3 cos(\theta_3)(\dot{\theta}_3)^2 = -r_4 sin(\theta_4)\ddot{\theta}_4 - r_4 cos(\theta_4)(\dot{\theta}_4)^2 \quad (22)$$

$$r_2 cos(\theta_2)\ddot{\theta}_2 - r_2 sin(\theta_2)(\dot{\theta}_2)^2 + r_3 cos(\theta_3)\ddot{\theta}_3 - r_3 sin(\theta_3)(\dot{\theta}_3)^2 = r_4 cos(\theta_4)\ddot{\theta}_4 - r_4 sin(\theta_4)(\dot{\theta}_4)^2 \quad (23)$$

Once again if the crank motion is known, $\ddot{\theta}_2$, the crank angular acceleration is given. We can also assume the first two steps, position analysis and velocity analysis, have been conducted and *all* angles and angular velocities have been solved for. We can move all the terms involving unknowns to the left hand side leaving only the known quantities on the right hand side:

$$-r_3 sin(\theta_3)\ddot{\theta}_3 + r_4 sin(\theta_4)\ddot{\theta}_4 = r_2 sin(\theta_2)\ddot{\theta}_2 + r_2 cos(\theta_2)(\dot{\theta}_2)^2 - r_4 cos(\theta_4)(\dot{\theta}_4)^2 + r_3 cos(\theta_3)(\dot{\theta}_3)^2$$

$$r_3 cos(\theta_3)\ddot{\theta}_3 - r_4 cos(\theta_4)\ddot{\theta}_4 = -r_2 cos(\theta_2)\ddot{\theta}_2 + r_2 sin(\theta_2)(\dot{\theta}_2)^2 + r_3 sin(\theta_3)(\dot{\theta}_3)^2 - r_4 sin(\theta_4)(\dot{\theta}_4)^2$$

These two equations are linear in $\ddot{\theta}_3$ and $\ddot{\theta}_4$ and therefore can be written in matrix form:

$$\begin{bmatrix} -r_3 sin(\theta_3) & r_4 sin(\theta_4) \\ r_3 cos(\theta_3) & -r_4 cos(\theta_4) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{bmatrix} = \begin{bmatrix} r_2 sin(\theta_2)\ddot{\theta}_2 + r_2 cos(\theta_2)(\dot{\theta}_2)^2 - r_4 cos(\theta_4)(\dot{\theta}_4)^2 + r_3 cos(\theta_3)(\dot{\theta}_3)^2 \\ -r_2 cos(\theta_2)\ddot{\theta}_2 + r_2 sin(\theta_2)(\dot{\theta}_2)^2 + r_3 sin(\theta_3)(\dot{\theta}_3)^2 - r_4 sin(\theta_4)(\dot{\theta}_4)^2 \end{bmatrix}$$
$$(24)$$

Solving this matrix equation yields unique solutions for $\ddot{\theta}_3$ and $\ddot{\theta}_4$.
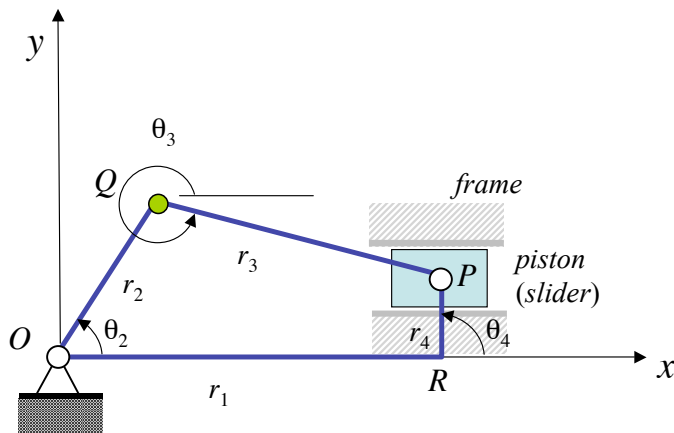
4   The Slider Crank Linkage

Figure 9: Definition of key variables and constants for modeling a slider crank linkage.

4.1   Notation

The analysis of a slider crank linkage is very similar to the analysis of the four bar linkage. Indeed if we examine the slider crank linkage in Figure 9, both linkages consist of four links connected by four joints. The frame is link 1, the crank (OQ) is link 2, the connecting rod (PQ) is link 3, and the slider or piston is link 4. The joints at O, Q, and R are revolute joints while the joint between the piston and the frame is a prismatic joint.

We fix the coordinate system of the slider crank linkage with the origin at the crank pivot and the x axis parallel to the axis of the slider as shown in the figure. Define

$$
\begin{aligned}
\mathbf{r}_1 &= \overrightarrow{OR} \\
\mathbf{r}_2 &= \overrightarrow{OQ} \\
\mathbf{r}_3 &= \overrightarrow{QP} \\
\mathbf{r}_4 &= \overrightarrow{RP}
\end{aligned}
$$

As before the angles the links make with the horizontal (positive) x axis are denoted by θ_i. Note that θ_1 is fixed at zero in this figure, while θ_2 and θ_3 are variables. The segment RP is always perpendicular to OR. Therefore θ_4 is fixed at $\frac{\pi}{2}$. However, while r_2, r_3, and r_4 are link lengths and therefore constants (as before in the four bar linkage), the length r_1 varies as the piston translates along the x-axis.

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4.2 Closure equations

Consider the position vector of point P, $\mathbf{r}_{P/O}$. This position vector can be written in two ways as before:

$$\mathbf{r}_{P/O} = \mathbf{r}_2 + \mathbf{r}_3 \tag{25}$$

$$\mathbf{r}_{P/O} = \mathbf{r}_1 + \mathbf{r}_4 \tag{26}$$

This leads us to the vector closure equation:

$$\mathbf{r}_2 + \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_4 \tag{27}$$

Or the two scalar equations:

$$r_2 cos\theta_2 + r_3 cos\theta_3 = r_1 \tag{28}$$

$$r_2 sin\theta_2 + r_3 sin\theta_3 = r_4 \tag{29}$$

where we have used the facts $\theta_1 = 0$ and $\theta_4 = \frac{\pi}{2}$. These two equations are nonlinear and involve three variables: r_1, θ_2 and θ_3.

4.3 Position analysis

Consider the case where the crank angle is stepped through at some known rate and therefore θ_2 is known. Rearrange Equations (28-29) so only θ_3 (one of the unknowns) is on one side:

$$r_3 cos\theta_3 = r_1 - r_2 cos\theta_2$$

$$r_3 sin\theta_3 = r_4 - r_2 sin\theta_2$$

Squaring both sides and adding gives one equation:

$$r_3^2 = r_1^2 + r_4^2 + r_2^2 - 2r_2(r_1 cos\theta_2 + r_4 sin\theta_2)$$

which can be written as a quadratic equation in r_1:

$$r_1^2 - 2r_2 cos(\theta_2)r_1 + [r_4^2 + r_2^2 - 2r_2 r_4 sin\theta_2 - r_3^2] = 0 \tag{30}$$

with two solutions:

$$r_1 = r_2 cos(\theta_2) \pm \sqrt{r_2^2 cos^2(\theta_2) - [r_4^2 + r_2^2 - 2r_2 r_4 sin\theta_2 - r_3^2]} \tag{31}$$

For each value of r_1, there is a distinct solution for θ_3 given by:

$$\theta_3 = atan2\left(\frac{r_4 - r_2 sin\theta_2}{r_3}, \frac{r_1 - r_2 cos\theta_2}{r_3}\right) \tag{32}$$

The ambiguity in the value of r_1 can be easily explained as an ambiguity in the assembly of the linkage. As shown in Figure 10, there are two distinct assembly modes labeled $+1$ and -1 corresponding to the sign outside the square root in Equation (31). The positive sign will yield a value of r_1 that corresponds to the point P being on one side of the y axis (positive value of r_2) while the negative sign will yield a value of r_1 that corresponds to the point P being to the left of the y axis.

4.4   Velocity analysis

The velocity equations are obtained by differentiating the closure equations in Equations (28-29):

$$-r_2 sin(\theta_2)\dot{\theta}_2 - r_3 sin(\theta_3)\dot{\theta}_3 \quad = \quad \dot{r}_1 \tag{33}$$

$$r_2 cos(\theta_2)\dot{\theta}_2 + r_3 cos(\theta_3)\dot{\theta}_3 \quad = \quad 0 \tag{34}$$

or

$$\dot{r}_1 + r_3 sin(\theta_3)\dot{\theta}_3 \quad = \quad -r_2 sin(\theta_2)\dot{\theta}_2$$

$$r_3 cos(\theta_3)\dot{\theta}_3 \quad = \quad -r_2 cos(\theta_2)\dot{\theta}_2$$

If the crank angular velocity, $\dot{\theta}_2$, is known, it is convenient to write these equations in matrix form:

$$\begin{bmatrix} 1 & r_3 sin(\theta_3) \\ 0 & r_3 cos(\theta_3) \end{bmatrix} \begin{bmatrix} \dot{r}_1 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} -r_2 sin(\theta_2)\dot{\theta}_2 \\ -r_2 cos(\theta_2)\dot{\theta}_2 \end{bmatrix} \tag{35}$$

These equations can be solved to get:

$$\dot{r}_1 \quad = \quad r_2 \frac{sin(\theta_3 - \theta_2)}{cos(\theta_3)}\dot{\theta}_2 \tag{36}$$

$$\dot{\theta}_3 \quad = \quad -\frac{r_2 cos(\theta_2)}{r_3 cos(\theta_3)}\dot{\theta}_2 \tag{37}$$

4.5   Acceleration Analysis

The acceleration equations are obtained by differentiating the velocity equations in Equations (33-34) and re-arranging the terms:

$$\ddot{r}_1 + r_3 sin(\theta_3)\ddot{\theta}_3 + r_3 cos(\theta_3)(\dot{\theta}_3)^2 \quad = \quad -r_2 sin(\theta_2)\ddot{\theta}_2 - r_2 cos(\theta_2)(\dot{\theta}_2)^2 \tag{38}$$

$$r_3 cos(\theta_3)\ddot{\theta}_3 - r_3 sin(\theta_3)(\dot{\theta}_3)^2 \quad = \quad -r_2 cos(\theta_2)\ddot{\theta}_2 + r_2 sin(\theta_2)(\dot{\theta}_2)^2 \tag{39}$$

Once again if the crank motion is known, $\ddot{\theta}_2$, the crank angular acceleration is given. We can move all the terms involving the unknown accelerations to the left hand side leaving only the known quantities on the
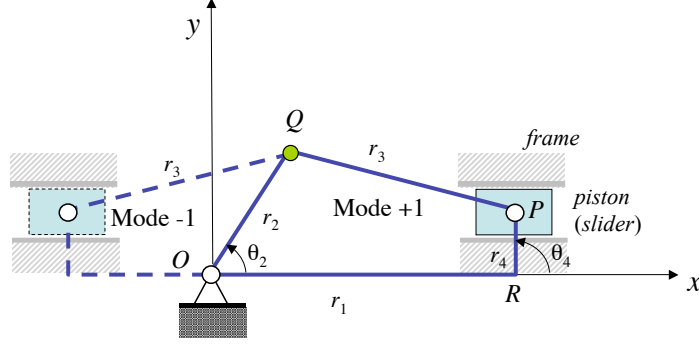
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Figure 10: The two modes of assembly for a slider crank linkage. If the crank angle is given, changing the sign in Equation (31) gives us two assembly configurations which correspond graphically to two intersections between the circle with radius r_3 and center at Q with the axis of translation of the slider (a line parallel to the x axis through P).

right hand side:

$$
\begin{aligned}
\ddot{r}_1 + r_3 sin(\theta_3)\ddot{\theta}_3 &= -r_3 cos(\theta_3)(\dot{\theta}_3)^2 - r_2 sin(\theta_2)\ddot{\theta}_2 - r_2 cos(\theta_2)(\dot{\theta}_2)^2 \\
r_3 cos(\theta_3)\ddot{\theta}_3 &= r_3 sin(\theta_3)(\dot{\theta}_3)^2 - r_2 cos(\theta_2)\ddot{\theta}_2 + r_2 sin(\theta_2)(\dot{\theta}_2)^2
\end{aligned}
$$

These two equations are linear in $\ddot{\theta}_3$ and $\ddot{r}_1$ and therefore can be written in matrix form:

$$
\begin{bmatrix} 1 & r_3 sin(\theta_3) \\ 0 & r_3 cos(\theta_3) \end{bmatrix} \begin{bmatrix} \ddot{r}_1 \\ \ddot{\theta}_3 \end{bmatrix} = \begin{bmatrix} -r_3 cos(\theta_3)(\dot{\theta}_3)^2 - r_2 sin(\theta_2)\ddot{\theta}_2 - r_2 cos(\theta_2)(\dot{\theta}_2)^2 \\ r_3 sin(\theta_3)(\dot{\theta}_3)^2 - r_2 cos(\theta_2)\ddot{\theta}_2 + r_2 sin(\theta_2)(\dot{\theta}_2)^2 \end{bmatrix} \tag{40}
$$

Solving this matrix equation yields unique solutions for $\ddot{\theta}_3$ and $\ddot{r}_1$.

References

[1] Tongue, Benson H. and Sheri D. Sheppard, *Dynamics: Analysis and Design of Systems in Motion,* John Wiley and Sons, Inc., 2005.

[2] Waldron, Kenneth, J., and Gary L. Kinzel, *Kinematics, Dynamics and Design of Machinery,* John Wiley and Sons, Inc., 1999.

A   Appendix: The $atan2$ function

Inverse trigonometric functions have multiple values. Even within a 360 degree range they have two values. For example, if

$$y = sin(x)$$

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the inverse sin function gives two values in a 360 degree interval:

$$sin^{-1}y = x, \pi - x$$

Of course we can add or subtract 2π from either of these solutions and obtain another solution.

The same is true of the inverse cosine and inverse tangent functions as well. If

$$y = cos(x),$$

the inverse cosine function yields:

$$cos^{-1}(y) = x, -x$$

Similarly, for the tangent function

$$y = tan(x),$$

the inverse tangent function yields:

$$tan^{-1}(y) = x, \pi + x.$$

This multiplicity is particularly troublesome in linkage analysis where an ambiguity may mean that there is more than one way of assembling a linkage (two or more angles for a given input). This problem is circumvented by defining the $atan2$ function which requires two arguments and returns a unique answer in a 360 degree range.

The $atan2$ function takes as arguments the sine and cosine of a number and returns the number. Thus if

$$s = sin(x); c = cos(x)$$

the $atan2$ function takes s and c as the argument and returns x:

$$atan2(s, c) = x$$

The main idea is that the additional information provided by the second argument eliminates the ambiguity in solving for x.

To see this consider the simple problem where we are given:

$$s = \frac{1}{2}; c = \frac{\sqrt{3}}{2}$$

and we are required to solve for x. If we use the inverse sine function and restrict the answer to be in the interval $[0, 2\pi)$, we get the result:

$$x = sin^{-1}\left(\frac{1}{2}\right) = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$$

Since we know the cosine to be $\frac{\sqrt{3}}{2}$, we can quickly verify by taking cosines of both candidate solutions that the first solution is correct and the second one is incorrect.

$$cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}; \ \ cos\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}.$$

The atan2 function goes through a similar algorithm to figure out a unique solution in the range $[0, 2\pi)$.

$$atan2\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

The $atan2$ function is a standard function in most C, Pascal and Fortran compilers, and in Matlab.