Final Exam:
December 23 (1:20pm) - December 26 (1:20pm), 2022

Part A. (50%)

Non-linear differential equations with several degrees of freedom often exhibit chaotic solutions. Chaos is associated with sensitive dependence to initial conditions; however, numerical solutions are often confined to a so-called strange attractor, which attracts solutions resulting from different initial conditions to its vicinity in the phase space. An example of a strange attactor is the Lorenz attractor, which results from the solution of the following equations:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

The values of $\sigma = 10$ and $b = 8/3$ are usually fixed leaving r as the control parameter. For low values of r, the stable solutions are stationary. When r exceeds 24.74, the trajectories in xyz space become irregular orbits about two particular points. **Find your solutions by writing a code using the 4th-order Runge-Kutta method with $\Delta t = 0.001$. Using built-in functions to solve the equations will not be graded.**

A.1 Solve these equations using $r = 20$. Start from point $(x, y, z) = (1, 1, 1)$, and plot the solution trajectory for $0 \le t \le 25$ in the xy, xz, and yz planes. Plot also x, y, and z versus t.

A.2 Observe the change in the solution by repeating A.1 for $r = 28$. In this case, plot also the trajectory of the solution in the three-dimensional xyz space (you can use the command `plot3(z,y,x)` in Matlab/Octave).

A.3 Observe the unpredictability at $r = 28$ by overplotting two solutions versus time starting from two initially nearby points: $(6, 6, 6)$ and $(6, 6.01, 6)$.

Part B. (50%)

We will solve the Blasius equation

$$f''' + ff'' = 0 \quad \text{with} \quad f(0) = f'(0) = 0 \quad \text{and} \quad f'(\eta \to \infty) = 1$$

using the **relaxation method**. Here $f = f(\eta)$ and η is the independent variable.

For numerical purposes we will truncate the upper extent of the computational domain at $\eta_{max} = 6$. Using the following finite difference schemes:

- 3-point central difference schemes at the interior points
- 3-point one-sided schemes at the end points

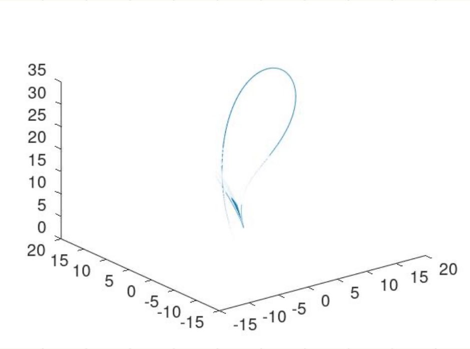
Consider the following four cases: using a uniform grid of spacing $\Delta\eta = 0.4$ (16 points), 0.3 (21 points), 0.2 (31 points), 0.1 (61 points).

Plot your solutions f, f', f'' versus η in the range $0 \le \eta \le \eta_{max}$ and monitor how many iterations are required for the change in solution $f(\eta)$ at all the points is less than 10^{-4}. Compare the numerical value of $f''(0)$ with its exact value of 0.4696.

You may NOT discuss with others about the exam. You may use notes, books and computers. You need to turn in your code in Octave or Matlab. No late exam will be accepted.

You must certify in writing that you take the exam honestly. Please sign and date after your written statement. Your exam will NOT be graded without your written statement.

Part A.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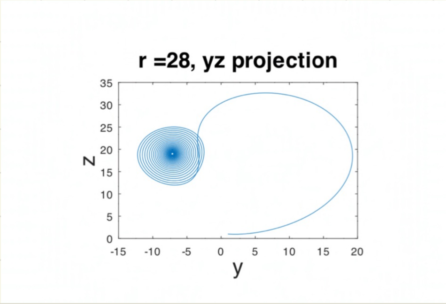
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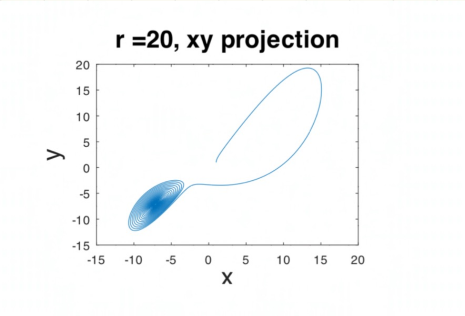
z
y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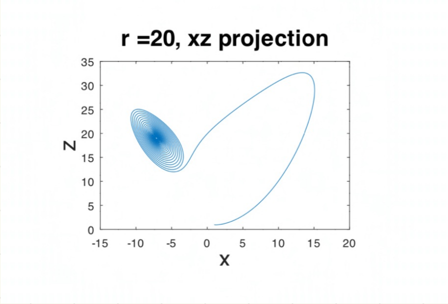
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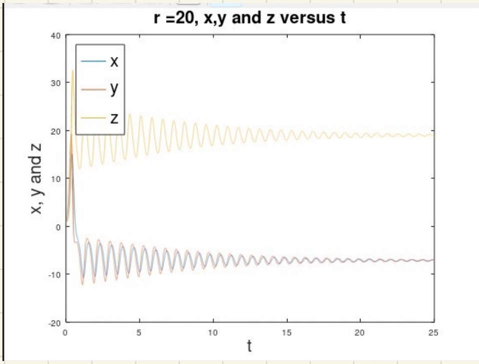
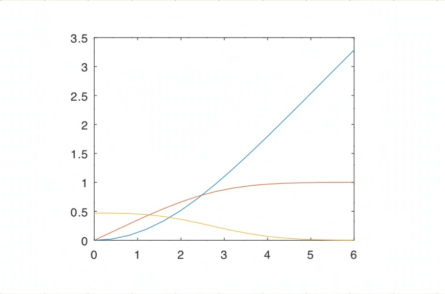
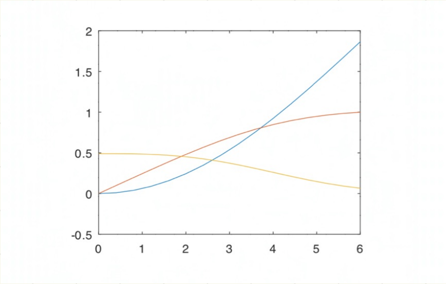
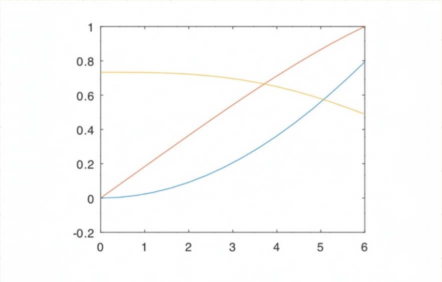
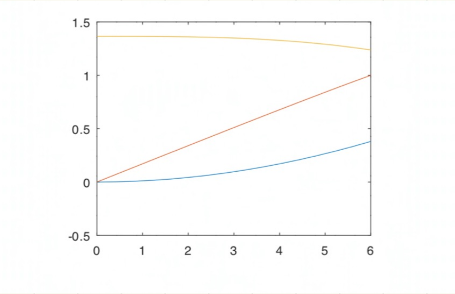

r =20, x,y and z versus t
x
y
z
x, y and z
t

Part B.









誠實作答聲明：蔣依倢.