

Homework 2

Due: October 12, 2022

Note: No late homework will be accepted. You may discuss with your classmates but **you may not plagiarize.** You need to turn in **your analysis and also your code** written in Octave or Matlab.

Part A. (20%)

We will consider three finite difference schemes for the first derivative, i.e.

forward difference

$$f'(x_j) = \frac{f(x_{j+1}) - f(x_j)}{h},$$

second-order central difference

$$f'(x_j) = \frac{f(x_{j+1}) - f(x_{j-1}))}{2h},$$

fourth-order central difference

$$f'(x_j) = \frac{f(x_{j-2}) - 8f(x_{j-1}) + 8f(x_{j+1}) - f(x_{j+2}))}{12h}.$$

A.1 Please use Taylor series to show that these schemes are first-order $O(h)$, second-order $O(h^2)$ and fourth-order $O(h^4)$, respectively. Here we use uniform grid points and h is the spacing between two consecutive grid points, i.e. $h = x_{j+1} - x_j$.

A.2 Now we take the function to be

$$f(x) = \frac{\sin x}{x^3},$$

so that the exact first derivative is known. Please use the above three finite difference schemes to numerically evaluate $f'(x_j)$ at $x_j = 4$ and compute the absolute values of error. You may choose $h = 1$, $h = 0.5$, $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$ and plot the absolute values of error versus grid spacing on a log-log plot.

Part B. (20%)

The fourth-order Padé scheme for the first derivative is

$$f'(x_{j-1}) + 4f'(x_j) + f'(x_{j+1}) = \frac{3}{h}(f(x_{j+1}) - f(x_{j-1})).$$

B.1 Please use Taylor series to show this Padé scheme is fourth-order $O(h^4)$.

B.2 Please derive the modified wavenumber k' for the second-order central difference (given in Part A), fourth-order central difference (given in Part A) and fourth-order Padé scheme for the first derivative. Plot $k'h$ versus kh for the three modified wavenumbers for $0 \leq kh \leq \pi$, where h is the grid spacing. In this plot, please also include $k'h = kh$, which is the exact wavenumber. (Note that $k = 2\pi n/L$, $n = 0, 1, 2, \dots, N/2$ where L is

the period and $h = L/N$ is the grid spacing. The grid points are $x_j = hj$, $j = 0, 1, 2, \dots, N-1$.)

Part C. (20%)

We will use the fourth-order Padé scheme (given in Part B) to numerically evaluate the first derivative of a known function

$$f(x) = \sin(5x) \quad \text{for } 0 \leq x \leq 3.$$

Fifteen uniformly spaced points are used here. Please note that $x_0 = 0$ and $x_N = 3$ while $N = 14$. For the left boundary ($x_0 = 0$) and right boundary ($x_N = 3$), use the following schemes

$$f'_0 + 2f'_1 = \frac{1}{h}(-\frac{5}{2}f_0 + 2f_1 + \frac{1}{2}f_2),$$

and

$$f'_N + 2f'_{N-1} = \frac{1}{h}(\frac{5}{2}f_N - 2f_{N-1} - \frac{1}{2}f_{N-2}).$$

C.1 Please use Taylor series to show the above two boundary schemes are third-order $O(h^3)$.

C.2 Using the fourth-order Padé scheme with the boundary schemes, you may derive fifteen equations for the first derivative at fifteen grid points. What are your solutions of $f'(x_j)$ for $j = 0, 1, 2, \dots, 14$? Plot your solutions at the fifteen grid points with big dots and also plot the exact first derivative $f'(x) = 5\cos 5x$ as a continuous line for $0 \leq x \leq 3$.

Part D. (20%)

For a periodic function

$$f(x) = e^{ikx},$$

its second derivative is $-k^2 f$. A finite difference scheme for second derivative of $f(x)$ would lead to $-k'^2 f$, where k'^2 is the ‘modified wavenumber’ for the second-derivative.

D.1 Derive the ‘modified wavenumber’ for the central difference formula

$$f''(x_j) = \frac{f(x_{j+1}) - 2f(x_j) + f(x_{j-1}))}{h^2}.$$

D.2 Use Taylor series to show that the following Padé scheme

$$\frac{1}{12}f''(x_{j-1}) + \frac{10}{12}f''(x_j) + \frac{1}{12}f''(x_{j+1}) = \frac{f(x_{j+1}) - 2f(x_j) + f(x_{j-1}))}{h^2}$$

is fourth-order accurate.

D.3 Derive the ‘modified wavenumber’ for the Padé scheme given in D.2.

D.4 Plot the ‘modified wavenumber’ in D.1 and D.3 in terms of $k'^2 h^2$ versus kh for $0 \leq kh \leq \pi$. In this plot, please also include $k'^2 h^2 = k^2 h^2$, which is the exact ‘wavenumber’.

Part E. (20%)

Consider the central finite difference scheme

$$\frac{\delta u_n}{\delta x} = \frac{u_{n+1} - u_{n-1}}{2h}.$$

In calculus we have

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}.$$

E.1 Does the following finite difference expression hold?

$$\frac{\delta(u_n v_n)}{\delta x} = u_n \frac{\delta v_n}{\delta x} + v_n \frac{\delta u_n}{\delta x}$$

E.2 Please show that

$$\frac{\delta(u_n v_n)}{\delta x} = \bar{u}_n \frac{\delta v_n}{\delta x} + \bar{v}_n \frac{\delta u_n}{\delta x},$$

where

$$\bar{u}_n = \frac{1}{2}(u_{n+1} + u_{n-1}) \quad \text{and} \quad \bar{v}_n = \frac{1}{2}(v_{n+1} + v_{n-1}).$$