Micro 2 notes

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Notes for Micro 2 Exam

1. Not covered in exam: Bargaining (Lectures 23/24), Incomplete Markets, Balanced Games
2. Focus on: Problem sets, main theorems
3. E.g., GND lemma, Existence of equilibria in GE; questions will cover both GE and Game Theory.

Lectures 1-2: Introduction to Extensive-Form Games

1. Consider sequential games, where Player 1 moves first, then Player 2 observes Player 1's action and moves second.
   (a) Player 1 chooses $a_1 \in A_1$
   (b) Player 2 observes a_1 and chooses $a_2 \in A_2$.
   (c) Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.
   (d) Player 2 maximises $\max_{a_2 \in A_2} u_2(a_1, a_2)$, where a_1 is given.
   The FOC is where $\frac{\partial u_2}{\partial a_2} = 0$.
   (e) Let $R_2(a_1) = arg \max_{a_2 \in A_2} u_2(a_1, a_2)$.
   (f) Knowing this, Player 1 maximises $\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$, where $R_2(.)$ is known.

   The FOC is where $\frac{\partial u_1}{\partial a_1} + \frac{\partial u_1}{\partial a_2} \cdot \frac{\partial R_2}{\partial a_2} = 0$.
   (g) The backward induction outcome is $(a_1^*, R_2(a_1^*))$
   (See graphical notes; note that payoffs are reported as a column vector).

   (h) The above depends on both players fully knowing the payoffs and the game tree, i.e., $S = (s_1, s_2, u_1, u_2; \text{tree})$.

2. Example 1: Stackleberg duopoly
   (a) There are two firms, and the strategy is the quantity to produce. Each firm's payoff is their profit.
   (b) The inverse demand function is $P(Q) = a - Q$.
   $\pi_i(q_i, q_j) = q_i[a - q_i - q_j - c]$.
   (c) Firm 1 chooses quantity q_1, then Firm 2 observes q_1 and chooses q_2.
   (d) Firm 2's problem is $R_2(q_1) = arg \max_{q_2} q_2[P(q_1 + q_2) - c]$.
   (e) Firm 1's problem is $\max_{q_1} q_1[P(q_1 + R_2(q_1)) - c]$.
   (f) Firm 2 takes q_1 as given, so its problem is $\max_{q_2} q_2[(a - q_1 - q_2) - c] \rightarrow R_2(q_1) = \frac{a-q_1-c}{2}$
   (g) Hence, Firm 1's problem is $\max_{q_1} q_1[a - q_1 - \frac{a-q_1-c}{2} - c]$
   $\rightarrow q_1^* = \frac{a-c}{2}$
   $\rightarrow q_2^* = R_2(q_1^*) = \frac{a-c}{4}$
   (h) Compare the strategies to the quantities that would be produced under Cournot duopoly: $q_1^* = q_2^* = \frac{a-c}{3}$.

3. Example 2: Leontief model

(a) There are two players: The firm and the union.

(b) The union maximises $u(w, L)$; the union's strategies are the wages.

(c) The firm maximises $\pi(w, L) = R(L) - wL$.

(d) Timing: the union makes a wage demand, then the firm observes the wage and sets the employment level.

(e) The firm solves $\max_{L\geq 0}[R(L) - wL]$, which implies $\frac{dR(L)}{dL} = w$. This gives $L^* = L^*(w)$ with $\frac{dL^*(w)}{dw} < 0$.

(f) The union solves the problem $\max_{w\geq 0} U(w, L^*(w))$; the solution is

$$\frac{dU(w, L^*(w))}{dw} = 0$$

i.e.,

$$\frac{\partial U(w, L^*(w))}{\partial w} + \frac{\partial U(w, L^*(w))}{\partial L}\frac{L^*(w)}{w} = 0$$

By implicit function theorem,

$$\frac{dL^*(w)}{dw} = -\frac{\frac{\partial u(w, L^*(w))}{\partial w}}{\frac{\partial u(w, L^*(w)}{\partial L}}$$

(g) See graphical notes: The backward induction solution is worse than any solution in the shaded area; the mechanism of wage setting then setting the size of the workforce is not optimal. The leadership position does not make the firm better-off.

4. Example 3: Rubinstein's bargaining model.

(a) Player 1 and 2 are bargaining over a dollar.
At the beginning of the first period, Player 1 proposes to take a share s_1 of the dollar, leaving $1 - s_1$ to Player 2.
If Player 2 rejects the offer, then the game continues. In the second stage, Player 2 proposes that Player 1 take a share s_2, leaving $1 - s_2$ for Player 2.
If Player 1 rejects the offer, then the game continues. In the third stage, Player 1 receives a share s of the dollar, leaving $1 - s$ for Player 2.

(b) Players are impatient and discount payoffs in later periods by a factor of δ per period, $\delta < 1$.

(c) By backward induction, Player 1 accepts the final offer in the third stage if $s_2 \geq \delta s$.
So in the second stage, Player 2 offers $(\delta s, 1 - \delta s)$.
So in the first stage, Player 2 accepts the offer if $1 - s_1 \geq \delta - \delta^2 s$.
So in the first stage, Player 1 offers $s_1 = 1 - \delta - \delta^2 s$.

(d) So the final game is for Player 1 to offer $(1 - \delta - \delta^2 s, \delta - \delta^2 s)$, and for Player 2 to accept the first offer.

Lecture 3: Extensive-Form Games and Subgames

1. An extensive form game consists of:

(a) Players $\{0, 1, ..., N\}$. Player 0 is usually called Nature, and has no payoff

(b) Game tree

(c) Payoffs $\{u_1, ..., u_N\}$

(d) Strategies

2. The game tree consists of nodes and branches.

(a) Each branch b connects two nodes, a predecessor b^p and a successor b^s.

(b) A path connecting two different nodes a and a' is a sequence of branches. The number of branches is the length of the path.

(c) The tree has a node called the root that has no parent. There is a set of nodes without successors, called terminal nodes.

(d) From the root node to any node there is one and only one path. Such a path is called a **history**.

(e) A **strategy** for a player is the list of what actions that player will choose at **every** node that he must play. Note that strategies include what the player should do even on off-equilibrium paths.

(f) (See graphical notes):
Player 1's strategies are W and C;
Player 2's strategies are $(C, C), (C, W), (W, C), (W, W)$.
Backward induction serves as a selection mechanism to select the BI outcome among multiple equilibria.

3. The players relate to the tree since

(a) Each nonterminal node is assigned to a player

(b) Each branch from the node represents an action that the player assigned to that node can take there

(c) If a stochastic event occurs at a node a then Nature is the player in that node and for each branch we assign a probability.

(d) A player may not be aware of the node that they are at. The set of possible nodes that the player faces the same situation/ has the same possibilities is an **information set**.

(e) If at each information set there is only one node, the game has **perfect information**.

(f) In an extensive game, there is **perfect recall** if we require that if two nodes a and a' are in the same information set for a player then the moves made up to a and a' are the same (i.e., the players know what actions they themselves took (if any) to end up at that node). We focus on games with perfect recall.

(g) The extensive game is **finite** if the tree is finite.

4. Formally:

(a) For each player i we denote using H_i the information sets and with $\mathbb{A}(h_i)$ the actions available at $h_i \in H_i$.

(b) The set of all actions allocated to the specific player i is

$$\mathbb{A}_i = \bigcup_{h_k \in H_i} \mathbb{A}(h_k)$$

(c) For each player i a pure strategy is a map

$$s_i : H_i \to \mathbb{A}_i$$

$\forall\ k$ with $s_i(h_k) \in \mathbb{A}_i(h_k)$.

(d) The set of strategies is equivalent to the Cartesian product

$$\prod_{h_k \in H_i} \mathbb{A}_i(h_k)$$

Note that for an extensive game G with a finite set of players N, we can describe the players' strategy spaces (i.e., for each player $i \in N$ we can give a nonempty set S_i and for each player $i \in N$ a payoff function u_i on the set $S = \prod_{j \in N} S_i$.

(e) For any $s_{-i} \in S_{-i}$, define $\hat{B}_i(s_{-i})$ to be the set of player i's best actions given s_{-i}

$$\hat{B}_i(s_{-i}) = arg \max_{s \in S_i} u_i(s, s_{-i})$$

The set-valued function $\hat{B}_i$ is the **best-response correspondence** of player i.

(f) In an extensive game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, the strategies $(s_1^*, ..., s_n^*)$ are a Nash equilibrium if, for each player i, the strategy s_i^* is player i's best response to $(s_1^*, ..., \hat{s_i^*}, ..., s_n^*)$.
This means that

$$u_i(s_1^*, ..., s_i^*, ..., s_n^*) \geq u_i(s_1^*, ..., s_i, ..., s_n^*)$$

for every feasible $s_i \in S_i$.

5. Subgames:

(a) Given the history h^k we denote with $G(h^k)$ the game from stage k up to stage K (possibly ∞) as the game starting with h^k.
We assume that the payoff for player i for history $(a^k, ..., a^{K+1})$ is $u_i(h^k, a^k, ..., a^{K+1})$.
Note that the node to be used as the new root of the subgame must not break an information set (intuitively, the subgame must retain the information from the original game, and if we were to break an information set, this implies we "know where we are", which means we have more information than we did in the original game).

(b) Any strategy profile s of the whole game induces a strategy profile $s|h^k$. For each player i, $s|h_k$ is simply the restriction of s_i to the histories consistent with h^k.

(c) A strategy profile s of a multi-stage game with observed actions is a **subgame-perfect equilibrium** if for each history h^k, the restriction $s|h^k$ to the subgame $G(h^k)$ if a Nash equilibrium for $G(h^k)$.

(d) The definition of SPNE reduces to backward induction in a finite game of perfect information.

Lecture 4: Zermelo-Kuhn Theorem; Behaviour Strategies

1. Theorem 1 : Given any finite game G of perfect information, if we apply backward induction then we obtain a strategy profile which is a Nash equilibrium.
Proof: Let $l(G)$ be the length of G.
If $l(G) = 1$, then there is only one player. Since there is only a finite number of terminal nodes, there exists a Nash equilibrium which is the node with the highest payoff.
Now, we assume that for $l(G) = n - 1$, the statement is true.

   (a) Since the game has perfect information, a finite number of players, and a finite number of terminal nodes, for a game where $l(G) = n$, we can construct a game G' of length $n - 1$ whcih has a NE in pure strategies. We can do this by simply removing the final layer of nodes in the game and replacing them with the highest payoffs from the terminal nodes in each branch.

   (b) Now, we can add back the final layer of the game, choosing the actions that give the highest payoffs in each branch, and we have a game G of length n which has a NE in pure strategies.

2. Note that there can be NE in a game which are not BI (backward induction) solutions.
For example, in the Stackleberg game, the BI solution is

$$q_1^* = \frac{a-c}{2}, \qquad q_2^* = \frac{a-c}{4}$$

However, one possible NE is the Cournot NE, where $q_1^c = q_2^c = \frac{a-c}{3}$.

3. Theorem 2 (Zermelo 1913 - Kuhn 1953): A finite game of perfect information always has a pure strategy Nash equilibrium.
Note the difference from Theorem 1: this does not involve backward induction.

4. **Behaviour Strategies vs Mixed Strategies**

Given a set X, $\Delta(X)$ is the set of all probability measures on X. If $X = \{x_1, ..., x_n\}$, then

$$\Delta(X) = \{(p_1, ..., p_n)| \sum_{j=1}^{n} p_j = 1 \text{ and } p_j \geq 0, \forall j\}$$

(i.e., choosing x_i with probability p_i).

5. Definition 3: For each player i, a behaviour strategy is an element of

$$\prod_{h_i \in H_i} \Delta(\mathbb{A}_i(h_i))$$

Note: The space of behaviour strategies is smaller than the space of mixed strategies; we are "shrinking" the space of strategies.

Consider a two-stage game where Player 1 chooses R or L. If Player 1 chooses R, Player 2 chooses a or b. If Player 1 chooses L, Player 2 chooses A or B.

For the space of **mixed strategies**, the information sets (nodes) assigned to a player are taken together. (e.g., consider how Player 2's pure strategies are $(a, A),\ (a, B),\ (b, A),\ (b, B)$)
The space of **mixed strategies** for Player 1 is

$$sR + (1-s)L \ s.t.\ s \in [0,1]$$

The space of mixed strategies for Player 2 is

$$\begin{aligned} & t(a, A) + u(a, B) + v(b, A) + z(b, B) \\ & t, u, v, z \in [0,1] \text{ and } t + u + v + z = 1 \end{aligned}$$

Note that there are 3 parameters for Player 2 (the fourth parameter is $1 - t - u - v = z$).

For the space of **behaviour strategies**, the information sets (nodes) assigned to a player (e.g., Player 2) are taken separately.
The space of **behaviour strategies** for Player 1 is

$$sR + (1-s)L \ s.t.\ s \in [0,1]$$

(this is identical to Player 1's mixed strategies).
The space of behaviour strategies for Player 2 is

$$\begin{aligned} & pa + (1-p)b \ \&\ rA + (1-r)B \\ & p, r \in [0,1] \end{aligned}$$

Note that there are only two parameters, p and r.
The shrinking of the space of strategies in this example (i.e., moving from 3 parameters to 2 parameters) is analogous to shrinking a 3-dimensional space to a 2-dimensional space.
A behaviour strategy is much less complex to describe than a mixed strategy.

6. Given a behaviour strategy, we can always associate to it a mixed strategy.
For example, Player 2's behaviour strategy is equivalent to the following mixed strategy:

$$pr(a, A) + p(1-r)(a, B) + (1-p)r(b, A) + (1-p)(1-r)(b, B)$$

However, there are mixed strategies that cannot be constructed from behaviour strategies.
For example, the mixed strategy

$$\frac{1}{2}(a, A) + \frac{1}{2}(b, B)$$

cannot be constructed starting from a behaviour strategy as it is inconsistent with the form

$$pr(a, A) + p(1-r)(a, B) + (1-p)r(b, A) + (1-p)(1-r)(b, B)$$

7. **Kuhn's Theorem**: In a game of perfect recall the set of behaviour strategies and mixed strategies are realisation equivalent.
In other words, a player can always replace a mixed strategy with an equivalent behaviour strategy.

Definition: Two randomised strategies m and r of a player are equivalent if for any fixed strategies of the other players, every node of the game tree is reached with the same probability when the player uses m as when he uses r. Two equivalent strategies m and r always give every player the same expected payoff because the nodes of the game tree are reached with the same probability no matter whether m or r is used.

Idea of the proof:

(a) Assume we have a finite extensive game G with n players.

(b) We can construct the associated strategic game that we call G_Γ.

(c) Since G_Γ is finite we can construct the associated mixed extension game $\Delta(G_\Gamma)$.

(d) By Nash's theorem we can claim that there exists a Nash equilibrium in mixed strategies in $\Delta(G_\Gamma)$). Let this be $(s_1^*, ..., s_n^*)$.

(e) By Kuhn's theorem we can replace each s_i^* with a realisation equivalent behaviour strategy b_i^*. Therefore we get the profile $(b_1^*, ..., b_n^*)$.

(f) Finally, we can show that $(b_1^*, ...b_n^*)$ is a Nash equilibrium.

Summary: Every finite extensive game G with perfect recall has an equilibrium in behaviour strategies.

Lecture 5: SPNE and One-Shot Deviation Principle; BI for K-stage and ∞-horizon extensive games

1. We saw earlier (e.g., Week 3 with example of $(W, (C, C))$ vs $(C, (W, W)))$ that there are equilibria (i.e., NE) that are SPNE and also equilibria that are not SPNE.
   For certain classes of games, we can characterise equilibria that are SPNE.
   We restrict ourselves to cases with perfect information.

   We need the following:
   Definition: Given a gave of perfect information, and a Nash equilibrium $(s_1, ..., s_N)$, a strategy s_i' of player i is a **one-shot deviation** at x if $s_i'(y) = s_i(y)$ at each node $y \neq x$ and $s_i'(x) \neq s_i(x)$.
   In other words, the two strategies s_i' and s_i differ at x but are the same everywhere else.

2. **Theorem 1: (one-shot-deviation principle for finite-horizon games)** In a finite game with perfect information (i.e., every node is the beginning of a subgame), a strategy profile s is subgame perfect iff it satisfies the one-shot deviation condition that no player i can gain by deviating from s in a single shot and conforming to s thereafter.
   (Note: "thereafter" does not have a sequential meaning but simply means "at every other node").

3. Proof of SPNE $\rightarrow$ no gain from one-shot deviation:
   This follows directly from the definition of SPNE.

   Proof of no gain from one-shot deviation $\rightarrow$ SPNE:
   Assume that the strategies $(s_1, ..., s_N)$ satisfy the one-shot-deviation principle for finite-horizon games but is not SPNE.
   Then, there is a shot t and a history h^t such that some player i has a strategy $\hat{s}_i$ which is a better response than s_i for the subgame $G(h^t)$ starting at h^t; i.e., the player can profitably deviate.
   Among all deviations, let $\hat{s}_i$ be the deviation such that

   (a) the player i can profitably deviate by using $\hat{s}_i$

   (b) the strategy $\hat{s}_i$ differs from s_i in the minimum number of nodes.

   Among all the histories that come after h^t, let $h^{\hat{t}}$ be the one such that

   (a) the number of nodes in $h^{\hat{t}}$ that do not belong to h^t is the biggest possible

   (b) $\hat{s}_i(h^{\hat{t}}) \neq s_i(h^{\hat{t}})$

   We can claim that the strategy $\hat{s}_i$ is also a profitable deviation at $h^{\hat{t}}$. If this were false, we could take the strategy $\tilde{s}_i$:

   $$\begin{cases} \tilde{s}_i = \hat{s}_i & \hat{t} < t \\ \tilde{s}_i = s_i & \hat{t} \geq t \end{cases}$$

   and this strategy would have the following properties:

   (a) $\tilde{s}_i$ is as profitable as $\hat{s}_i$

(b) the strategy $\tilde{s}_i$ differs from s_i on a smaller number of nodes than $\hat{s}_i$

This is not possible, therefore the strategy s_i^* defined by

$$\begin{cases} s_i^* = s_i(h^s) & h^s \neq h^{\hat{t}} \\ s_i^*(h^{\hat{t}} = \hat{s}_i(h^{\hat{t}} \end{cases}$$

is a profitable one-shot deviation from s_i but this is absurd since we already assumed the one-shot deviation principle.

4. Backward induction can be applied to any finite game of perfect information.
Assume the game has k stages:

   (a) Step 1: Determine the optimal choice in the final stage for each history $h^K \in H^K$

   (b) Step 2: Work back at stage $K-1$ and determine the optimal choice for the player moving there, given that the player moving at stage K with history h^K will play the action determined previously

   (c) ...

   (d) Step K: Work back from stage 2 to stage 1 and determine the optimal choice for the player moving there, given that the player at stage 2 will play the action determined previously.

Recall Theorem 1 from Week 4: Given any finite game G of perfect information, if we apply backward induction then we obtain a strategy profile which is a Nash equilibrium.
By construction of backward induction, all the actions chosen are part of the strategy profiles which are SPNE.
(we can't use backward induction unless there is perfect information).

5. Extension of K-stage extensive games to ∞-horizon:
Assume a game has ∞ stages.
In the first stage of a multi-stage game (i.e., stage 0), all players $i = 1, ..., I$ choose an action

   (a) Player 1 chooses from $\mathbb{A}_1(h^0)$

   (b) ...

   (c) Player I chooses from $\mathbb{A}_I(h^0)$

Let $a^0 = (a_1^0, ..., a_I^0)$ be the stage-0 action profile.
At stage 1 the actions available for each player depend on history:

   (a) Player 1 chooses from $\mathbb{A}_1(h^1)$

   (b) ...

   (c) Player I chooses from $\mathbb{A}_I(h^1)$

Continuing iteratively, we define h^{k+1}, the history at the end of stage k, to be the sequence of actions in previous periods:

$$h^{k+1} = (a^0, ..., a^k)$$

Definition 1: The set of all stage-k histories is H^k
Definition 2: The set of all ∞ histories is H^∞
Definition 3: For each player i we denote with $\mathbb{A}_i(H^k) = \bigcup_{h^k \in H^k} \mathbb{A}_i(h^k)$ the set of all possible moves available to player i at stage k. **Definition 4**: For each player i a pure strategy is a collection of maps

$$(s_i^k) = (s_i^0, ..., s_i^k, ...)$$

such that

$$s_i^k : H^k \to \mathbb{A}_i(H^k)$$

for all $k \in \mathbb{N}$ and $s_i^k(h^k) \in \mathbb{A}_i(h^k)$.

Since the terminal stories represent a sequence of the play, we have:
Definition 4: Player i's payoff is a function $u_i : H^\infty \to \mathbb{R}$
Definition: A Nash Equilibrium is a strategy profile $s = (s_1, ..., s_I)$ such that for each player i,

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

for all possible s'_i.

Definition 5: A multi-stage game has perfect information if, for each stage k, exactly one player has a non-trivial choice set. This means that for each $j \in \mathbb{N}$ there exists i such that

$$\mathbb{A}_j(H^k) = \bigcup_{h^k \in H^k} \mathbb{A}_j(h^k) = \emptyset$$

if $j \neq i$

6. Backward induction in ∞-horizon:
If the horizon is not finite, it is not possible to apply the backward induction approach directly.
However, there are situations where we can extend the BI approach. Typically this is where agents have utility of the following type:

$$\sum_{t=0}^{\infty} U_t^i(s_t^1, ..., s_t^I)$$

where the contribution of U_t^i gets smaller and smaller as t increases (e.g., due to a discount rate $0 < \beta < 1$.

The idea is to stop the game at K_n, and try to apply the following:

(a) Step 1: Determine the optimal choice in the final stage for each history $h^{K_n} \in H^{K_n}$
(b) Step 2: Work back at stage $K_n - 1$ and determine the optimal choice for the player moving there, given that the player moving at stage K with history h^k will play the action determined previously
(c) ...
(d) Step $K_n - 1$: Work back from stage 2 to stage 1 and determine the optimal choice for the player moving there, given that the player at stage 2 will play the action determined previously.

Note that in a game with ∞-horizon, there are no payoffs, and we cannot directly know the optimal choice at each $h^{K_n} \in H_{K_n}$.
One way is to put a payoff for the game $G(h^{K_n})$, i.e., the subgame starting at the history h^{K_n} up to the end at ∞. To do this, we need to know that the new value does not depend on the choice we make.

There is a class of games where this is possible:
Definition: A game is **continuous at infinity** if for each player i the utility function satisfies

$$\sup_{h, \tilde{h} \text{ s.t. } h^t = \tilde{h}_t} |u_i(h) - u_i(\tilde{h})| \to 0 \text{ as } t \to \infty$$

(i.e., for two histories that are identical up to the first t steps, the difference in utility for diverging histories after that point approaches 0 as $t \to \infty$).
In other words, events in the far future are not very important, since the payoff is almost the same as the one you get out of a (very long) finite history.

Going back to the issue of determining the optimal choice in the final stage for each history $h^{K_n} \in H_{K_n}$:

(a) We replace the last node of each history $h^{K_n} \in H_{K_n}$ with the payoff with one $h \in H^\infty$ such that h up to the stage K_n is the same as h^{K_n}.
(note that we can take the payoff of *any* such node with the same history for the first t stages, as our premise is that the payoffs do not change much after getting out of a very long finite history).

(b) We now have a well-defined game with K_n stages, G_{K_n}.

(c) For this game we can find a BI solution where

$$\begin{pmatrix} (s_1^0, ..., s_I^0) \\ ... \\ (s_1^{K_n}, ..., s_1^{K_n}) \end{pmatrix}$$

(d) Repeating this process, we get

$$..., \begin{pmatrix} (s_1^0, ..., s_I^0) \\ ... \\ (s_1^{K_n}, ..., s_1^{K_n}) \end{pmatrix}, \begin{pmatrix} (s_1^0, ..., s_I^0) \\ ... \\ (s_1^{K_{n+1}}, ..., s_I^{K_{n+1}}) \end{pmatrix}, ...$$

Since the game satisfies

$$\sup_{h,\tilde{h} \text{ s.t. } h^t=\tilde{h}_t} |u_i(h) - u_i(\tilde{h})| \to 0 \text{ as } t \to \infty$$

it is possible to show that this process has a limit. This limit is the BI solution for the original game.

Theorem 3 (one-deviation principle for infinite-horizon): In a infinite-horizon game with perfect information that is continuous at infinity, a strategy profile s is a SPNE iff there exists no player i and no strategy $\hat{s}_i$ that agrees with s_i except at a single t and h^t and such that $\hat{s}_i$ is a better response to s_{-i} than s_i conditional on history h^t being reached.

We are in this "environment" if the players are of the type

$$\sum_{t=0}^{\infty} \delta^t u_t^i(s_t^1, ..., s_t^i)$$

and there exists a constant M such that

$$|u_t^i(s_t^1, ..., s_t^I)| \leq M$$

Lecture 6: Two-stage games of complete but imperfect information; Repeated games

1. The general model is as follows:

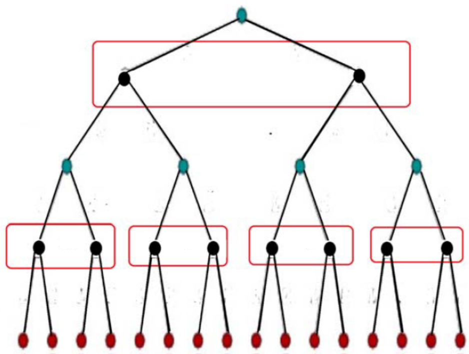
   (a) Player 1 and 2 simultaneously choose actions a_1 and a_2 from the feasible sets A_1 and A_2

   (b) Player 3 and 4 observe the outcome of the first stage and choose actions a_3 and a_4

   (c) The payoffs are:

$$u_1(a_1, a_2, a_3, a_4)$$
$$u_2(a_1, a_2, a_3, a_4)$$
$$u_3(a_1, a_2, a_3, a_4)$$
$$u_4(a_1, a_2, a_3, a_4)$$

2. Example game tree:
   (P1 chooses in the first row, P2 in the second row, P3 in the third row, P4 in the fourth row. Each player can choose either high or low, i.e., $\underline{a}_i$ or $\bar{a}_i$ at each node).



The set of strategies for each player is:

$$
\begin{aligned}
&S_1 = \{\underline{a}_1, \bar{a}_1\} \\
&S_2 = \{\{\underline{a}_2, \underline{a}_2\}, \{\bar{a}_2, \bar{a}_2\}\} \qquad \text{(since info set)} \\
&S_3 = \{\{\underline{a}_3, \underline{a}_3, \underline{a}_3, \underline{a}_3\}, \{\underline{a}_3, \underline{a}_3, \underline{a}_3, \bar{a}_3\}...\} \qquad \text{(16 different permutations)} \\
&S_4 = \{\{\{\underline{a}_4, \underline{a}_4\}, \{\underline{a}_4, \underline{a}_4\}, \{\underline{a}_4, \underline{a}_4\}, \{\underline{a}_4, \underline{a}_4\}\}...\} \qquad \text{(16 different permutations)}
\end{aligned}
$$

Notes:

(a) There are 4 subgames (excluding the main game itself).

(b) SPNE can be in mixed strategies too; here we will focus on pure strategies.

(c) Since this is a game of perfect recall, by Kuhn's theorem, there is an equivalent behaviour strategy for each mixed strategy.

3. To find the SPNE, we **assume** that for any choice of (a_1, a_2) the game $G(a_1, a_2)$ has a Nash equilibrium.

   (a) Assume that Player 1 and 2 choose (a_1, a_2)

   (b) Given the choice above, Player 3 and 4 play the subgame $G(a_1, a_2)$

   (c) By assumption $G(a_1, a_2)$ has a Nash equilibrium that we call

   $$a_3^*(a_1, a_2), \qquad a_4^*(a_1, a_2)$$

   (d) Since Player 1 and 2 know this, they choose a_1^* and a_2^* sych that

   $$a_3^*(a_1^*, a_2^*), \qquad a_4^*(a_1^*, a_2^*)$$

   solve

   $$\max_{a_1 \in A_1} u_1(a_1, a_2^*, a_3^*(a_1, a_2^*), a_4^*(a_1, a_2^*))$$

   This means that

   $$u_1(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*)) = \max_{a_1 \in A_1} u_1(a_1, a_2^*, a_3^*(a_1, a_2^*), a_4^*(a_1, a_2^*))$$

   and the same is true for

   $$\max_{a_2 \in A_2} u_2(a_1^*, a_2, a_3^*(a_1^*, a_2), a_4^*(a_1^*, a_2))$$

   This is like "backward induction": Player 1 and 2 assume that Player 3 and 4 will choose the best responses, and then they choose the best responses to that.

   (e) Example 1: Bank runs (Diamond and Dybvig (1983):
   Two investors have each deposited D with a bank; the bank has invested these deposits in a long-term project.
   If the bank is forced to liquidate its investment before it matrues, a total of $2r$ can be recoved, where $D > r > \frac{D}{2}$.
   If the project reaches maturity, the project will pay out a total of $2R$, where $R > D$.
   There are two dates at which investors can get the money out of the bank:

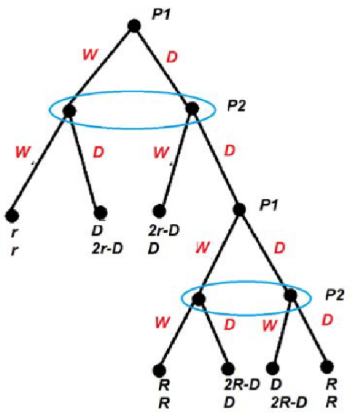
      i. Date 1: Before the investment matures

      ii. Date 2: After the investment matures

   The payoffs and game tree are:

Stage 1

	withdraw	Don't
withdraw	r, r	$D, 2r - D$
Don't	$2r - D, D$	next stage



Stage 2

	withdraw	Don't
withdraw	R, R	$2R - D, D$
Don't	$D, 2R - D$	R, R

The players' strategy sets are:

$$S_1 = \{\{W, W\}, \{D, D\}, \{W, D\}, \{D, W\}\}$$
$$S_2 = \{\{\{W, W\}, \{W, W\}\}...\} \qquad \text{(4 different permutations, because of info set)}$$

Since we are looking for SPNE, by backward induction, we look at the second stage first.
Since $R > D$, W strictly dominates D for both players, implying that the NE for the second stage is for both players to W.
This implies that, if we assume SPNE, we can consider the game

	withdraw	Don't
withdraw	r, r	$D, 2r - D$
Don't	$2r - D, D$	R, R

and since $r < D$, this game has two pure-strategy NE: (W, W) and (D, D).
Note that the (W, W) equilibrium can be interpreted as a bank run where both players withdraw. This equilibrium is socially inefficient.

(f) Example 2: Tariffs and imperfect international competition
There are two identical countries where prices are $P_i(Q_i) = a - Q_i$

i. In each country, the firm produces h_i for home consumption and e_i for export, so total quantity available in a country is the sum of domestic production and imports from the other country:

$$Q_i = h_i + e_j$$

ii. Both countries have identical marginal cost c but pay a tariff t_j on exports, so the total cost borne by firms in each country is

$$c(h_i + e_i) + t_j e_i$$

iii. Payoffs for each firm are total revenue minus total costs:

$$\pi_i(t_i, t_j, e_i, h_i, e_j) = [a - (h_i + e_j)]h_i + [a - (h_j + e_i)]e_i - c(h_i + e_i) - t_j e_i$$

Meanwhile, "payoffs" for each country's governent is based on the government's welfare fnction, which is a function of total quantity available in the country, firms' profits, and tariff revenue

$$W_i(t_i, t_j, e_i, h_i, e_j) = \frac{1}{2}Q_i^2 + \pi_i(t_i, t_j, e_i, h_i, e_j) + t_i e_j$$

iv. The game is played sequentially. First, governments set tariffs, then firms observe tariffs and then decide how much to produce.

v. By backward induction, we look at the firms' problem first. The firms solve

$$\max_{h_i, e_i \geq 0} \pi_i(t_i, t_j, e_i, h_i, e_j)$$

We can solve

$$\max_{h_i \geq 0}[a - (h_i + e_j) - c]h_i$$

and

$$\max_{h_j, e_i \geq 0}[a - (h_j + e_i) - c]e_i - t_j e_i$$

and get the solutions (note that all solutions are symmetric for the two countries):

$$h_i = \frac{1}{2}(a - e_j - c)$$

and

$$e_i = \frac{1}{2}(a - h_j - c - t_j)$$

Writing the system of four equations and four unknowns (h_i, e_i, h_j, h_i), we have:

$$h_i = \frac{1}{3}(a + t_i - c)$$
$$e_i = \frac{1}{2}(a - c - 2t_j)$$

Now, we use these solutions for the first stage of the game.
The government sets t_i to maximise W_i, and the solution is

$$t_i = \frac{a - c}{3}$$

Substituting this into the second stage, we get

$$h_i = \frac{4(a - c)}{9}$$
$$e_i = \frac{a - c}{9}$$

and the output in each market is

$$\frac{5(a - c)}{9}$$

Note that if tariffs were 0, quantity would be the same as that in Cournot duopoly, i.e., $\frac{2(a-c)}{3}$. So tariffs reduce welfare.

4. Introduction to repeated games:
   Consider the Prisoners' Dilemma but with two stages:

   (a) Date 1: Players play Prisoners' dilemma
   (b) Date 2: Players play Prisoners' dilemma
   (c) Date 3: Players sum up the payoffs for each stage

   The game has the following structure:

Stage 1

	L	R
L	1, 1	5, 0
R	0, 5	4, 4

Stage 2

	L	R
L	1, 1	5, 0
R	0, 5	4, 4

The situation is like the standard PD but with the following extra assumptions:

(a) Players 1 and 2 simultaneously choose actions a_1 and a_2 from feasible sets A_1 and A_2

(b) Players 1 and 2 observe the outcome of the first stage (a_1, a_2) before choosing actions $a_3 \in A_1$ and $a_4 \in A_2$.

Notes:

(a) Games of this form (repetitions of some base game) are called **stage games**.

(b) If this was a one-off game, the NE would be (L, L) but the Pareto optimal choice would be (R, R).

To find the SPNE, we assume that for any choice of (a_1, a_2) the game $G(a_1, a_2)$ has a Nash Equilibrium.
Assume that Player 1 and Player 2 choose (a_1, a_2).
Given the choice above, then Players 1 and 2 play the game $G(a_1, a_2)$.
By assumption, $G(a_1, a_2)$ has a NE that we call

$$a_3^*(a_1, a_2) \qquad a_4^*(a_1, a_2)$$

Since Player 1 and 2 know this, then they choose a_1^* and a_2^* such that

$$a_3^*(a_1^*, a_2^*) \qquad a_4^*(a_1^*, a_2^*)$$

Payoffs are

$$u_1(a_1, a_2, a_3, a_4) \qquad u_2(a_1, a_2, a_3, a_4)$$

Note that if the two players get to the second stage, they will play the Nash (L, L), which means the payoffs considering both stages is an affine transformation of the Stage 1 game (i.e., just add $(1, 1)$ to every payoff):

	L	R
L	2, 2	6, 1
R	1, 6	5, 5

This game has only one Nash Equilibrium, so the two players will play (L, L) in both stages, with a payoff of $(2, 2)$.

5. Consider a separate example of a repeated game:

Example 2

	L_2	M_2	R_2
L_1	1, 1	5, 0	0, 0
M_1	0, 5	4, 4	0, 0
R_1	0, 0	0, 0	3, 3

There are two NE: (L_1, L_2) and (R_1, R_2).
Claim: there is a subgame perfect outcome when the strategy (M_1, M_2) is played in the first stage.
At the second stage the players are forced to play a NE, but not in the first stage.

We know that the players anticipate a Nash in the second stage (this is the definition of SPNE), so let's assume that they anticipate the following "dynamics":

(a) Play (R_1, R_2) in the second stage if (M_1, M_2) is played in the first stage
(b) Play (L_1, L_2) in the second stage if $(X, Y) \neq (M_1, M_2)$ is played in the first stage

This means that we get the following game:

	L_2	M_2	R_2
L_1	2, 2	6, 1	1, 1
M_1	1, 6	7, 7	1, 1
R_1	1, 1	1, 1	4, 4

This game has the following NE:

(a) (L_1, L_2): Players play two NE: at the first stage and at the second stage (i.e., always (L_1, L_2)).
(b) (R_1, R_2): Players play two NE: (R_1, R_2) at the first stage, and (L_1, L_2) at the second stage.
(c) (M_1, M_2): This is more interesting - the players do not play a NE in the first stage.

Overall, there are several possible cases:

(a) The case where there is only one NE (e.g., the PD stage game);
(b) The case when we repeat the game many times and we want SPNE (e.g., Example 2 but if we restrict ourselves to only SPNE for both stages);
(c) The case where we repeat the game many times and we are not rigid about SPNE (e.g., Example 2 but if we do not require the players to choose a NE in the first stage);
(d) If it is possible to get close to interesting outcomes and which type of strategies we need to use to achieve this goal.

Lecture 7: Repeated Games and Folk Theorems - Introduction

1. A **normal-form representation** of a n-player game specifies:

(a) A finite set of players N

(b) Players' strategy spaces, i.e., for each player $i \in N$ a nonempty set S_i

(c) For each player $i \in N$ a payoff function u_i on the set $S = \prod_{j \in N} S_i$

(d) We use the symbol $G = \{S_1, ..., S_n; u_1, ..., u_n\}$.

Definition: Given a stage game G,let $G(T)$ denote the finitely repeated game in which G is played T times, with the outcomes of all preceding plays observed before the next play begins. The payoffs are just sums of single payoffs with a discount factor δ where $0 < \delta \geq 1$. So, if the stage game is $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, the payoff for player i is

$$U_i(s_i, s_{-i}) = \sum_{t=0}^{T} \delta_i^t u_i(s_i^t, s_{-i}^t)$$

2. **Theorem 1:** If the stage game has a unique Nash equilibrium then for any finite T, the repeated game has a unique SPNE: The Nash equilibrium is played at each stage.
Proof: Note that the claim is true if $T = 1$. We assume that the claim is true for each repeated game $G(T-1)$ and we prove that it has to be true for $G(T)$. In fact, using the definition of SPNE, we know that the strategy at the last stage has to be a NE, and by assumption, this is unique so the strategy at the last stage has to use the NE. Now, we are left with a situation as in the case $G(T-1)$. For this case, the theorem is true, so it is true in general and we are done.
The idea is the same as in backward induction.

3. Assume we play the following game:

$$\mathcal{G} = \begin{array}{ccc} & C_{ooperate} & D_{efect} \\ C_{ooperate} & 1, 1 & -0.01, 1.01 \\ D_{efect} & 1.01, -0.01 & 0, 0 \end{array}$$

and repeat this game 100 times (so, in our notation, $G(100)$).

This is clearly a PD. There are many pure strategies; at time $T = 100$, the number of pure strategies is larger than 2^{99}, but according to the theorem earlier, the players always just play D.

However, according to simulations, agents with different levels of cooperation, incentive to defect and mutation rate do not keep playing D (see graphics in lecture slides).

Lecture 8: Repeated Games and Folk Theorems - Model and Examples

1. Recap: we are considering **normal-form representations** of n-player games, which have:

   (a) A finite set of players N

   (b) Players' strategy spaces, i.e., for each player $i \in N$ a nonempty set S_i

   (c) For each player $i \in N$ a payoff function u_i on the set

   $$u_i : S = \prod_{j \in N} S_i \to \mathbb{R}$$

   (d) We use the symbol $G = \{S_1, ..., S_n; u_1, ..., u_n\}$.

2. Given a stage game G,let $G(\infty, \delta)$ denote the infinitely repeated game in which G is played at each time, with the outcomes of all preceding plays observed before the next play begins. The payoffs are just sums of

single payoffs with a discount factor δ where $0 < \delta \geq 1$. So, if the stage game is $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, the payoff for player i is

$$U_i(s_i, s_{-i}) \stackrel{def}{=} \sum_{t=0}^{\infty} \delta_i^t u_i(s_i^t, s_{-i}^t)$$

Notes:

(a) We can use the factor $(1 - \delta_i)$ to normalise the sum $\sum_{t=0}^{\infty} \delta_i^t$ to 1.

(b) For the moment we assume that every player has the same discount rate. This is not necessary and the following discussion will still work (with some additions) if the players have different discount factors.

The central result in the theory of ∞-repeated games is the **Folk Theorem** (by Friedman).

(a) This result is essentially a negative result in classical game theory that says that repeated games have too many equilibria.

(b) To make it possible to achieve a Pareto optimal outcome, we need to make an infinite number of equilibria possible.

Definition: The Folk Theorem for repeated games asserts that if the players are sufficiently patient (i.e., δ close enough to 1), then any feasible, individually rational payoff can be enforced by an equilibrium. Therefore, in the limit of extreme patience, repeated games allow for virtually every payoff to be an equilibrium outcome.
Note: Contrast this with Lecture 7, where the NE is played at each stage. The difference is that here we have an infinite game (Lecture 7 was for finite games).

(see graphical notes):
These are relevant in cases where the payoff for the Pareto optimal outcome is higher than the payoff for the NE. By the Folk Theorem, with sufficient patience, if the game is repeated infinitely many times, the players can construct a strategy that is a SPNE and gives a payoff that is better than the NE (up to the Pareto optimal outcome).

3. Example 1: Bertrand duopoly with two players
Consider a duopoly over an infinite number of periods $t = 1, 2, ...$. Assume the duopolists are price setters (Bertrand), so a pure strategy for firm 1 and 2 at period t is just (p_1^t, p_2^t) respectively, conditioned on the history of prices in previous periods.
The payoffs are

$$\pi_1 = \sum_{t=0}^{\infty} \delta^t \pi(p_1^t, p_2^t)$$
$$\pi_2 = \sum_{t=0}^{\infty} \delta^t \pi(p_1^t, p_2^t)$$

We also assume that they have the same marginal cost c and there is downward-sloping demand.

A strategy that dictates one course of action until a certain condition is satisfied and then follows a different course of action for the rest of the game is called a **trigger strategy**.

Bertrand paradox: There is a SPNE of the game where $p_1^t = p_2^t = c$ for each t.

Tacit collusion:
Suppose $\delta > \frac{1}{2}$. Denote with p^m the monopoly price, and $c \leq p \leq p^m$.
Note: Depending on how patient the firms are (δ), they can potentially choose any p within this range.
Let s be a strategy profile such that Firm 1 sets $p_1^t = p$ and Firm 2 sets $p_2^t = p$ and each firm responds to a deviation from this behaviour on the part of the other by setting the price equal to c forever. Then, this strategy profile s is a SPNE.

Proof: Choose p satisfying the conditions in the theorem, and let $\pi(p)$ be the industry total profit.
Since $\pi(p) \geq 0$, the payoff for setting price p throughout is $\frac{\pi(p)}{2(1-\delta)}$ (infinite geometric series).
The payoff for defecting on the first round by charging less than p is $\pi(p-\epsilon) \approx \pi(p)$.
Therefore, for the players to optimally follow this strategy, it must be that

$$\frac{\pi(p)}{2(1-\delta)} > \pi(p)$$
$$\implies \delta > \frac{1}{2}$$

Note: The same holds even if the firms deviate in period t instead of period 1, as we would have the same inequality but with both sides discounted by δ^t.

4. Example 2: Bertrand with n-players
The setup is the same, except that the strategy is that each firm responds to any deviation from s on the part of any other player by setting their price equal to c forever.
Similarly, in this case we require

$$\frac{\pi(p)}{n(1-\delta)} > \pi(p)$$
$$\implies \delta > 1 - \frac{1}{n}$$

In other words, as n increases, δ has to be closer to 1, i.e., firms have to be "even more patient" for this strategy to be a SPNE.

5. Example 3: Bertrand with n-players and delay in trigger strategy
Suppose there are N firms but a firm that has been defected upon cannot implement the trigger strategy until $k > 1$ periods have passed.
In this case we require

$$\frac{\pi(p)}{n(1-\delta)} > \pi(p)(\frac{1-\delta^k}{1-\delta})$$
$$\implies \delta^k > 1 - \frac{1}{n}$$

(the RHS has a finite geometric series)

6. Example 4: Symbiotic relationship between fish
Mutual trust is involved. The situation is modelled as follows:

	C	D
C	$5, 5$	$-3, 8$
D	$8, -3$	$0, 0$

where the payoff is the sum of the payoffs weighted by a discount factor δ.
Note: A player using a trigger strategy initially cooperates but punishes the opponent if a certain level of defection (i.e., the trigger) is observed. The level of punishment and the sensitivity of the trigger differ with different trigger strategies.

The cooperative solution $(5, 5)$ can be achieved as a SPNE of the repeated game if:

(a) δ is sufficiently close to 1 and
(b) each player uses the trigger strategy of cooperating as long as the other player cooperates, and defecting forever if each other player defects.

Proof: If both players cooperate they get

$$(\frac{5}{1-\delta}, \frac{5}{1-\delta})$$

If instead one defects then it will play D and get 8 but the fish gets nothing thereafter. So, it will cooperate iff

$$\frac{5}{1-\delta} \geq 8$$

Consider the following strategy for the little fish: alternating C, D, C, D as long as the big fish alternates D, C, D, C. If the big fish deviates from this pattern then the little fish will defect forever. Suppose the big fish plays the complementary strategy. Then, these two strategies form a SPNE Nash.
Proof: The payoff is

$$\frac{8\delta - 3}{1-\delta^2}$$

if the little fish decides to defect at some point (the most advantageous time is when it gets -3). But if it defects it will get 0 thereafter so it does so only if

$$\frac{8\delta - 3}{1-\delta^2} \geq 0$$

this means that

$$8\delta - 3 \geq 0$$

Overall, in many examples it is possible to cooperate to achieve a certain goal but the players must be able to punish if there is a defection.

Lecture 9: Folk Theorem(s) for $G(\infty, \delta)$

1. A normal-form representation of an n-player game specifies:

   (a) a finite set of players N

   (b) players' strategy spaces, i.e., for each player $i \in N$ a nonempty set S_i

   (c) for each player $i \in N$ a payoff function u_i on the set $S = \prod_{j \in N} S_i$

   (d) We use the symbol $G = \{S_1, ..., S_n; u_1, ..., u_n\}$

   Given a stage game G, let $G(\infty, \delta)$ denote the infinitely repeated game in which G is played at each time, with the outcomes of all preceding plays observed before the next play begins. The payoffs are just sums of single payoffs with a discount factor δ, where $0 < \delta < 1$.
   So if the stage game is

   $$G = \{S_1, .., S_n; u_1, ..., u_n\}$$

   then the payoff for player i is

   $$U_i(s_i, s_{-i}) \overset{def}{=} (1-\delta_i) \sum_{t=0}^{\infty} \delta_i^t u_i(s_i^t, s_{-i}^t)$$

   The Folk Theorem for repeated games assets that if the players are sufficiently patient, then any feasible, individually rational payoffs can be enforced by an equilibrium. Therefore, in the limit of extreme patience, repeated games allow for virtually every payoff to be an equilibrium outcome.

2. Definition: Given a stage game G (which must be finite) and the game $G(\infty, \delta)$, the payoffs $(x_1, ..., x_n)$ are feasible in the game G if they are convex combinations of the pure strategy payoffs of G.

   Consider the example of a bi-matrix game played by two players, where the first player has the set of n pure strategies $S_1 = \{a_1, ..., a_n\}$ and the second player has m pure strategies $S_2 = \{b_1, ..., b_m\}$. The set of payoffs is:

	b_1	b_2	$\cdots$	b_m
a_1	$\alpha_{1,1}, \beta_{1,1}$	$\alpha_{n,1}, \beta_{n,1}$	$\cdots$	$\alpha_{1,m}, \beta_{1,m}$
a_2	$\alpha_{2,1}, \beta_{2,1}$	$\alpha_{2,2}, \beta_{2,2}$	$\cdots$	$\alpha_{2,m}, \beta_{2,m}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
a_n	$\alpha_{n,1}, \beta_{n,1}$	$\alpha_{n,2}, \beta_{n,2}$	$\cdots$	$\alpha_{n,m}, \beta_{n,m}$

Graphing the payoffs in $\mathbb{R}^2$ (because there are 2 players), we have the collection of points

$$\{(\alpha_{i,j}, \beta_{i,j})\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$$

The set of feasible payoffs is the smallest convex set which contains these nm points in $\mathbb{R}^2$. Specifically, the smallest convex set is the set which contains all $(x_1, x_2) \in \mathbb{R}^2$ such that

$$(x_1, x_2) = \sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} t_{i,j}(\alpha_{i,j}, \beta_{i,j})$$

with $t_{i,j} \geq 0$ and

$$\sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} t_{i,j} = 1$$

In other words, the coefficient $\{t_{i,j}\}$ can be thought of as probabilities (i.e., mixed strategies). (see graphical notes).

3. Given the game $G(\infty, \delta)$, the **average payoff** of the infinite sequence of payoffs $\pi_1, ..., \pi_t$ is

$$(1-\delta)\sum_{t=0}^{\infty} \delta^t \pi_t$$

4. Setup for proof of Friedman's Folk Theorem:
   (a) We have a desirable payoff in the one-stage game that we can achieve with a (possibly mixed) strategy profile, which we call $(C_1, ..., C_N)$ (these are payoffs).
   (b) The game has a Nash Equilibrium with payoffs $(E_1, ..., E_N)$.
   (c) For each player i,
   $$u_i(C_1, ..., C_N) > u_i(E_1, ..., E_N)$$
   (d) The strategy to get the payoff $u_i(C_1, ..., C_N)$ is simply to play C_i as long as nobody deviates; if somebody deviates play E_i foever.
   (e) The deviation is the trigger; $u_i(C_1, ..., C_N)$ is the carrot for player i and E_i is player i's stick.
   (f) E_i works well as a punishment device because it is a Nash equilibrium and
   $$u_i(C_1, ..., C_N) > u_i(E_1, ..., E_N)$$
   (g) Using E_i prevents deviation, since it is a Nash equilibrium, and makes the other player suffer for the deviation since there is a drop on payoff, namely the quantity
   $$\delta_i u_i(C_1, ..., C_N) - u_i(E_1, ..., E_N)$$

5. Formally, Friedman (1971)'s theorem is:
   Let $G = \{S_1, ..., S_n; u_1, ..., u_n\}$ be a static game of complete information. Let $(e_1, ..., e_n)$ denote the payoffs from a Nash equilibrium of the stage game and let $(x_1, ..., x_n)$ be any feasible payoff (i.e., achievable using mixed strategies).
   If $x_i > e_i$ for any player i (i.e., for every player i), and if δ is sufficiently close to 1 then there exists a SPNE of $G(\infty, \delta)$ that achieves $(x_1, ..., x_n)$ as the average payoff.

   Proof of Friedman's theorem:

(a) Let $(a_{e1}, ...a_{en})$ be the Nash equilibrium of G that yields the payoffs $(e_1, ..., e_n)$.

(b) Let $(a_{x1}, ..., a_{xn})$ be the collection of actions of G that yield the better payoff $(x_1, ..., x_n)$.

(c) Consider the following strategy: Play a_{xi} in the first stage. In the t^{th} stage, if the outcome of all $t-1$ preceding stages has been $(a_{x1}, ..., a_{xn})$, then play a_{xi}. Otherwise, play a_{ei}.

(d) Suppose that all the players other than player i are using this trigger strategy. If player i is defecting, after his defection he must play a_{ei} since the other players will play

$$(a_{e1}, ..., a_{e,i-1}, a_{e,i+1}, ..., a_{en})$$

because $(a_{e1}, ..., a_{en})$ is a Nash equilibrium.

(e) When defecting, player i solves the problem

$$arg \max_{a_i \in A_i} u_i(a_{x1}, ..., a_{x,i-1}, a_i, a_{x,i+1}, ..., a_{xn})$$

since after that the other players are going to stop him (Note: this assumes that the utilities are separable across players). We assume that a_{di} solves this problem and gives a payoff of d_i, so we have

$$u_i(a_{x1}, ..., a_{x,i-1}, a_{di}, a_{x,i+1}, ..., a_{xn}) = d_i$$

Overall, $d_i > x_i > e_i$, where:

$$\begin{aligned} x_i &= u_i(a_{x1}, ..., a_{x,i-1}, a_{xi}, a_{x,i+1}, ..., a_{xn}) \\ e_i &= u_i(a_{e1}, ..., a_{en}) \end{aligned}$$

This gives, as present value, the payoff

$$d_i = \delta e_i + \delta^2 e_i + ... = d_i + \frac{\delta}{1-\delta} e_i$$

Alternatively, if player i plays a_{xi} each time, the payoff is

$$V_i = x_i + \delta V_i = \frac{x_i}{1-\delta}$$

Therefore, playing a_{xi} is optimal iff

$$\begin{aligned} \frac{x_i}{1-\delta} &\geq d_i + \frac{\delta}{1-\delta} e_i \\ \implies \delta &\geq \frac{d_i - x_i}{d_i - e_i} \end{aligned}$$

Therefore the profile is a NE iff

$$\delta \geq \frac{d_i - x_i}{d_i - e_i}$$

holds for every player, i.e.,

$$\delta \geq \max_{i=1,...,n} \frac{d_i - x_i}{d_i - e_i}$$

Since $d_i > x_i > e_i$, we have that

$$1 > \max_{i=1,...,n} \frac{d_i - x_i}{d_i - e_i}$$

(f) Now, we need to check that this is also SPNE, i.e., the trigger strategy is a Nash equilibrium in any subgame of $G(\infty, \delta)$.
We consider two possible classes of subgames:

i. Case 1: The subgame that starts with a history where all the previous played strategies are the same strategies $(a_{x1}, ..., a_{xn})$.
In this case, the strategy of "Play a_{xi} in the first stage. In the t^{th} stage, if the outcome of all $t-1$ preceding stages has been $(a_{x1}, ..., a_{xn})$, then play a_{xi}. Otherwise, play a_{ei}" is still a Nash equilibrium for the subgames since it is the same as the original game.

ii. Case 2: The subgame with a history where at one node along the same history the played strategy was "$(a_{e1}, ..., a_{en})$, the Nash equilibrium of G that yields the payoffs $(e_1, ..., e_n)$". Then, by definition of trigger strategy, the played strategies are always the same (i.e., $(a_{e1}, ..., a_{en})$). Since this is a Nash equilibrium in the static game then it is also a Nash equilibrium in the repeated game if played at every stage.

6. Theorem (Folk Theorem for 2 player games): Consider any two-player stage game with a Nash equilibrium with payoffs (a, b). Suppose that there exists a pair of strategies for the two players that gives the players (c, d). If $c > a$ and $d > b$, and the discount factors of the players are sufficiently close to unity, then there is a subgame perfect Nash equilibrium of the repeated game with expected payoff (c, d) in each period.
7. In summary, it is possible, by repeating a game, to obtain a rich set of Nash equilibria which are subgame perfect.
   The way to achieve this goal is to use trigger strategies by using:
   (a) $u_i(C_1, ..., C_N)$, the payoff for player i;
   (b) E_i, the stick for player i; and
   (c) δ_i, the discount for player i.

Lecture 10: More on $G(\infty, \delta)$ and new look at $G(T, \delta)$

1. A normal-form representation of an N-player game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$ specifies:
   (a) a finite set of players N
   (b) players' strategy spaces, i.e., for each player $i \in N$ a nonempty set S_i
   (c) for each player $i \in N$ a payoff function
   $$u_i : S = \prod_{j \in N} S_i \to \mathbb{R}$$
   For this game, a strategy profile
   $$m^j = (m_j^j, m_{-j}^j)$$
   is a **maximum punishment payoff** for player j if the other players' chosen strategy profile m_{-j}^j is such that the best response of player j, i.e., m_j^j, gives him the lowest possible payoff (among all payoffs for plaer j).
   This implies that $\pi_j(m^j) = \pi_j^*$ is the payoff for player j when everyone else gangs up on him.

   We call $\pi^* = (\pi_1^*, ..., \pi_n^*)$ the **minimax point** of the game and we denote
   $$\prod = \{(\pi_1(s) ..., \pi_n(s)) \text{ s.t. } \pi_j(s) \geq \pi^*(j), \forall j\}$$
   In other words, the set $\prod$ is the set of strategy profiles in the stage game with payoffs at least as good as the maximum punishment payoff for each player.

2. Using this, we can construct a repeated game $G(\infty, \delta)$ in the standard way.

   Folk theorem: For any $(\pi_1, ..., \pi_n)$ in $\prod$, if δ is sufficiently close to unity, there is a Nash equilbrium of the repeated game such that π_j is j's payoff for $j = 1, 2, ..., n$ in each period.
   Note that this Folk theorem says nothing about SPNE because it contains threats that are not credible.

   We can also prove a more plausible version:
   Folk theorem (with subgame perfection): Suppose that $(y_1, ..., y_n)$ is the payoff of a Nash equilibrium of the underlying one-shot game, and
   $$(\pi_1, ..., \pi_n) \in \prod$$

If $\pi_1 \geq y_i$ for each i, and if δ is sufficiently close to unity, then there is a SPNE of the repeated game such that π_j is j's payoff for $j = 1, 2, ..., n$ in each period.

Observe that in this form we need to dominate a Nash equilibrium in any one-shot game. This is not strictly necessary since we can do without it as proved by Fudenberg and Maskin (1986) but we need to impose a very high level of coordination among players.

3. Note that if the stage game has only **one** Nash equilibria, the SPNE for the $G(T, \delta)$ finite game is that the NE is played at every stage.
Specifically: If the stage game has a unique (i.e., one) Nash equilibrium, then for any finite T, the repeated game has a unique SPNE: the Nash equilibrium is played at each stage.

4. "Folk Theorem" for finitely repeated games:
Our starting point normal-form representation of an n-player game specifies:
   (a) A finite set of players N
   (b) Players' strategy spaces, i.e., for each player $i \in N$ a nonempty set A_i
   (c) For each player $i \in N$ a payoff function u_i on the set $A = \prod_{j \in N} A_i$
   (d) We use the symbol $G = \{A_1, ..., A_n; u_1, ..., u_n\}$.

   For each player i and each

$$a_{-i} \in \prod_{\substack{j \in N \\ j \neq i}} A_i$$

   we set

$$M_i(a_{-i}) = \max_{\hat{a}_i \in A_i} u_i(a_{-i}, \hat{a}_i)$$

   (i.e., the maximum payoff if others play a_{-i})

   and

$$\bar{M}_i(a_{-i}) = \max_{a \in A} u_i(a)$$

   for each

$$\underline{v}_i = \min_{a \in A} M_i(a_{-i})$$

   Let

$$a^{\underline{v}_i} = arg \min_{a \in A} M_i(a_{-i})$$

   Note that $u_i(a_i^{\underline{v}}) = \underline{v}_i$.

   The vector $(\underline{v}_i, ..., \underline{v}_n)$ is the **minimax payoff vector**. In other words, $\underline{v}_i$ is the maximum payoff for player i in the stage game provided that all other players are trying to minimise player i's payoff ("the best among the worst").

   Definition: Given a set $S \subset \mathbb{R}^n$, S is convex if $\forall x, y \in S$ and for any $t \in [0, 1]$ we have

$$xt + (1 - t)y \in S$$

   Definition: Given a set $S \subset \mathbb{R}^n$, the convex hull of S is the minimal convex set which contains S, i.e.,

$$conv(S) = \bigcap_{C \text{ is convex, } S \subset C} C$$

Definition: Given a stage game G and the game $G(T, \delta)$, the payoffs $(x_1, ..., x_n)$ are feasible in the stage game G if they are convex combinations of the pure strategy payoffs of G. The set of feasible payoffs is

$$F = conv\{(u_1(a), ..., u_n(a))|a \in A = \prod_{j \in N} A_i\}$$

Definition: Given a stage game G and the game $G(T, \delta)$, the set of feasible payoffs

$$F = conv\{(u_1(a), ..., u_n(a))|a \in A = \prod_{j \in N} A_i\}$$

has **full dimension** (i.e., the dimension is the no. of players) if there exists $a \in A$ and $\epsilon > 0$ such that the set

$$\{x \in \mathbb{R}^n ||x - u(a)|| < \epsilon\} \subset F$$

(i.e., there is some ϵ such that an epsilon ball around any of the payoffs is contained in the set of feasible payoffs).

Example : Given the bimatrix game

	L	R
L	x_1, y_1	x_2, y_2
R	x_3, y_3	x_4, y_4

the set F for this game has full dimension (i.e., is "2-dimensional") iff it is not possible to find $\alpha, \beta, \gamma \in \mathbb{R}$ (not identically 0) such that for every $i = 1, 2, 3, 4$ we have

$$\alpha x_i + \beta y_i = \gamma$$

i.e., the points are not all in a line (See graphical notes).

5. For the previous discussion to hold, we needed the following assumptions:
   (a) Compact set of actions: This is a standard assumption for strategic games in order to apply Fixed Point theory.
   (b) Continuous payoff functions: This is a standard assumption for strategic games in order to apply Fixed Point theory (it is possible to use a weaker version).
   (c) Average discounted payoff: This means that players must have

   $$\frac{1-\delta}{1-\delta^T} \sum_{t=0}^{T-1} \delta_i^t u_i(s_i^t, s_{-i}^t)$$

   There are other possibilities for the case $T = \infty$.
   (d) Perfect monitoring: Each player, at the end of each period, observes the actions chosen by other players. This is the closest we can get to perfect information in a repeated game. The only source of imperfect information comes from the fact that they choose at the same time.
   (e) Public randomisation: The players can condition their actions on the realisation of an exogenous variable, and can use correlated strategies.
   (f) Observability of mixed actions: This allows us to work with the mixed extension of some strategic game. Basically, we can think of mixed strategies as a pure strategy with a pattern of repetition (e.g., A A B B). Note that perfect monitoring implies that the players can observe not only the pure actions resulting from randomisation but also the random process itself, which makes analysis easier.

Lecture 11: Folk Theorem for $G(T, \delta)$ when $\delta = 1$

Note: In this setting, we have no discounting of payoffs, and the game is finite.

1. Recall the situation for the Prisoner's Dilemma:

	$Cheat$	$Cooperate$
$Cheat$	$0, 0$	$3, -1$
$Cooperate$	$-1, 3$	$2, 2$

Note that there is only one Nash equilibrium.

Considering a repeated game, it is clear that in the last stage you must play the Nash equilibrium. However, players can afford to deviate at the beginning, as long as they have something to retaliate that is not the unique Nash equilibrium.

Considering the minmax we have for each player i:

$$a_{-i} \in \prod_{j \in N, j \neq i} A_i$$

we set

$$M_i(a_{-i}) = \max_{\hat{a}_i \in A_i} u_i(a_{-i}, \hat{a}_i)$$

and

$$\bar{M}_i = \max_{a \in A} u_i(a)$$

for each

$$\underline{v}_i = \min_{a \in A} M_i(a_{-i})$$

and let

$$a^{\underline{v}_i} = arg \min_{a \in A} M_i(a_{-i})$$

and note that

$$u_i(a^{\underline{v}_i}) = \underline{v}_i$$

Applying this to the Prisoner's Dilemma, we have

$$M_1(Cheat) = \max\{0, -1\} = 0$$
$$M_1(Cooperate) = \max\{3, 2\} = 3$$

for each

$$\underline{v}_1 = \min\{0, 3\} = 0$$

and let

$$a^{\underline{v}_1} = arg \min_{a \in A} M_i(a_{-i}) = Cheat$$

Note that in this game the Nash equilibrium is the minmax. This seems to tell us that in this game, we cannot use the minmax.

Notice that:

(a) In the period before the final period you do not need to play the Nash, as long as you can threaten with future retaliation.

(b) **If the Nash equilibrium is different from the minmax** there is the possibility to construct Nash equilibria in the repeated games which are not simple repetitions of the Nash in the static game.

For cases where the Nash equilibrium e has a payoff higher than the minmax, if we choose a rational payoff (higher than the minmax), i.e., a, we can play that strategy at the beginning and the Nash equilibrium at the end. In order to threaten the player (so he does not deviate), we can use the minmax that would lower his payoff.

2. The technical complications above are present since we need to calibrate the number of times we play e and the number of times we play a to have a best response for each player (i.e., the Nash equilibrium). To simplify, we start with the case where we take the average payoff, i.e., $\delta = 1$.
(Note: The discussion above still holds for $\delta < 1$, just that there will be discounting of future payoffs and the periods separating when to play the initial strategy and when to play the Nash equilibrium will be different).

Theorem: Let $G(T)$ be a finitely repeated game with

$$U^i = \frac{1}{T}\sum_{t=0}^{T-1} u_i(s_i^t, s_{-i}^t)$$

(this is the average payoff).

Suppose that, for each player $i \in N$, there is a Nash equilibrium e^i of G such that

$$u_i(e^i) > \underline{v}_i$$

(i.e., a Nash equilibrium exists for every player i where their utility playing the Nash is higher than the minmax for player i).
Then, for each v in $\bar{F}$ and $\epsilon > 0$, there is $T^0 \in \mathbb{N}$ such that, for each $T \geq T^0$, there is a Nash equilibrium whose payoff vector $\hat{v}$ satisfies

$$||\hat{v} - v|| < \epsilon \tag{Eq 1}$$

(i.e., the payoff from this NE strategy is $\hat{v}$; the longer the game, the closer $\hat{v}$ gets to v, the pure cooperation payoff).

Proof:
Let a be a strategy profile such that

$$u(a) = v = (\tilde{v}_1, \tilde{v}_2, ...\tilde{v}_n)$$

The most important claim we need is that there are integers

$$T^0, ..., T^n$$

with $T^0 > \sum_{i=1}^n T^i$, such that there is a Nash equilibrium of $G(T)$ with the following path:

$$\overline{\pi} = \left(\underbrace{a, ..., a}_{T^0 - \sum\limits_{i=1}^n T^i}, \underbrace{e^1, ..., e^1}_{T^1}, \underbrace{e^2, ..., e^2}_{T^2}, ..., \underbrace{e^n, ..., e^n}_{T^n} \right)$$

where e^i is the NE that satisfies the condition in (Eq 1) for player i. Note that multiple players could share the same NE, or the same player could have multiple NE that satisfies (Eq 1).

Note that the players cannot play a forever - they must move to the NE actions at some point otherwise there is not enough time to "punish" a player if they potentially defect.

We consider the following strategy σ for each player:

(a) On the main path, play according to $\overline{\pi}$. If one of the players i deviates from it, then go to the minmax phase.
(b) On the minmax phase, use the minmax, i.e., play $a^{\underline{v}_i} = arg\min_{a \in A} M_i(a_{-i})$.

Players will play according to the main path, and play the minmax strategy if they observe a player deviating (i.e., play the minmax of the player that defects first).

To show that this is a Nash equilibrium in the repeated game we need to show that the strategy designed is a best response.
Assume a possible deviation for player i at time t:

(a) The deviation cannot be for the later stages
$$t > T^0 - \sum_{i=1}^{n} T^i$$
since we play Nash at each stage.
(b) So, the deviation must be for the earlier stages (when a is played):
$$t \leq T^0 - \sum_{i=1}^{n} T^i$$
then we must have
$$u_i(b_i(a^t), a_{-i}^t) + \sum_{s=t+1}^{T^0} \underline{v}_i \leq \sum_{s=t}^{\tilde{T}} u_i(a^s) + \sum_{k=1}^{n} T^k u_i(e^k) \tag{Eq 1}$$
(Note: we set $T^0 - \sum_{i=1}^{n} T^i = \tilde{T}$ to simplify notation.)

$(b_i(a^t))$ is the best response of player i at the deviation point t (i.e., the action that gives player i the highest utility if all other players play a^t). This inequality is true if
$$(u_i(b_i(a^t), a_{-i}^t - u_i(a^t) + \sum_{s=t+1}^{\tilde{T}} (v_i - u_i(a^s)) \leq T^i(u_i(e^i) - v_i)$$
holds, which is possible if T^i is big enough since
$$u_i(e^i) - \underline{v}_i > 0$$
Taking the largest, i.e., $\max_{i \in N} T^i$, we get that it is a Nash equilibrium. We now establish the number of periods that the players will play a.

3. We now prove that if k is an arbitrary integer, then
$$\overline{\pi} = \underbrace{\underbrace{a, \ldots\ldots\ldots, a}_{\left(T^0 - \sum_{i=1}^{n} T^i\right) \times k}, \underbrace{e^1, \ldots, e^1}_{T^1}, \underbrace{e^2, \ldots, e^2}_{T^2}, \ldots, \underbrace{e^n, \ldots, e^n}_{T^n}}_{\left(T^0 - \sum_{i=1}^{n} T^i\right) \times k + \sum_{i=1}^{n} T^i}$$
is a Nash equilibrium.
Note that we are only considering a subset of dates, namely the one of the form
$$(T^0 - \sum_{i=1}^{n} T^i) \times k - \sum_{i=1}^{n} T^i$$

As before, the deviation cannot happen for

$$t > T^0 - \sum_{i=1}^{n} T^i$$

so the deviation must be for

$$t \leq T^0 - \sum_{i=1}^{n} T^i$$

To simplify notation, we set

$$(T^0 - \sum_{i=1}^{n} T^i)k + \sum_{i=1}^{n} T^i = \hat{T}$$
$$(T^0 - \sum_{i=1}^{n} T^i) = \hat{\hat{T}}$$

then we have

$$u_i(b_i(a^t), a^t_{-i}) + \sum_{s=t+1}^{\hat{T}} v_i \leq \sum_{s=t}^{k\hat{\hat{T}}} u_i(a^s) + \sum_{k=1}^{n} T^k u_i(e^k)$$

since it must be that

$$\sum_{s=\hat{\hat{T}}+1}^{k\hat{\hat{T}}} u_i(a^s) = (k-1)(T^0 - \sum_{i=1}^{n} T^i)u_i \geq (k-1)(T^0 - \sum_{i=1}^{n} T^i)\underline{v}_i$$

then, for it to be a best response, the following is sufficient:

$$u_i(b_i(a^t), a^t_{-i}) + (T^0 - t)v_i \leq u_i(a^t) + \sum_{s=t+1}^{\hat{\hat{T}}} u_i(a^s) + \sum_{l=1}^{n} T^l u_i(e^l)$$

but we already verified this in (Eq 1) so the claim must be true.

To complete, observe that the average payoff for player i is

$$\frac{k\hat{\hat{T}}\tilde{v}_i + \sum_{k=1}^{n} T^k u_k(e^k)}{\hat{\hat{T}}k + \sum_{i=1}^{n} T^i}$$

and we have

$$\lim_{k \to \infty} \frac{k\hat{\hat{T}}\tilde{v}_i + \sum_{k=1}^{n} T^k u_k(e^k)}{\hat{\hat{T}}k + \sum_{i=1}^{n} T^i} = \tilde{v}_i$$

Therefore, if k is big enough, we get as close as we wish to $\tilde{v}_i$ (i.e., for every ϵ we can choose a k such that we will get within ϵ of the ideal payoff) if we set, for an opportune big $\hat{k}_i$,

$$\hat{\hat{v}}_i = \frac{\hat{k}_i\hat{\hat{T}}v_i + \sum_{k=1}^{n} T^k u_k(e^k)}{\hat{\hat{T}}\hat{k}_i + \sum_{i=1}^{n} T^i}$$

we can make

$$|\hat{\hat{v}}_i - \tilde{v}_i| < \frac{\epsilon}{n}$$

We can improve by using $k^* = \max \hat{k}_i$ if we note that

$$\begin{aligned}
||\hat{v} - v|| &= \sum_{i}^{n} |\hat{v}_i - \tilde{v}_i| \\
&= \sum_{i}^{n} |\frac{k^*\hat{\hat{T}}v_i + \sum_{k=1}^{n} T^k u_k(e^k)}{\hat{\hat{T}}k^* + \sum_{i=1}^{n} T^i} - \tilde{v}_i| \\
&= \sum_{i}^{n} |\hat{v}_i - \tilde{v}_i| = n\frac{\epsilon}{n} = \epsilon
\end{aligned}$$

where

$$\hat{v}_i = \frac{\hat{k}_i \hat{\hat{T}} v_i + \sum_{k=1}^{n} T^k u_k(e^k)}{\hat{\hat{T}} \hat{k}_i + \sum_{i=1}^{n} T^i}$$

At this point, for dates T of the form

$$(T^0 - \sum_{i=1}^{n} T^i) \times k - \sum_{i=1}^{n} T^i$$

we are done since we use the strategy above. If instead we have

$$(T^0 - \sum_{i=1}^{n} T^i)k + \sum_{i=1}^{n} T^i < T < (T^0 - \sum_{i=1}^{n} T^i)(k+1) + \sum_{i=1}^{n} T^i$$

then we use a different strategy

$$\overline{\overline{\pi}} = \overline{\pi} \text{ and we add } \underbrace{e^n, \ldots\ldots\ldots\ldots\ldots\ldots, e^n}_{T - \left(T^0 - \sum_{i=1}^{n} T^i\right) \times k - \sum_{i=1}^{n} T^i} .$$

which means

$$\underbrace{\underbrace{a, \ldots\ldots, a}_{\left(T^0 - \sum_{i=1}^{n} T^i\right) \times k}, \underbrace{e^1, ., e^1}_{T^1}, \underbrace{e^2, ., e^2}_{T^2}, .., \underbrace{e^n, \ldots\ldots\ldots\ldots, e^n}_{T - \left(T^0 - \sum_{i=1}^{n} T^i\right) \times k - \sum_{i=1}^{n-1} T^i}}_{T}$$

Lecture 12: Review of Folk Theorems for $G(T)$ and $G(\infty, \delta)$ and Introduction to GE

1. **Theorem (Friedman 1971)**: Let $G = \{S_1, ..., S_N; u_1, ..., u_n\}$ be a static game of complete information. Let $(e_1, ..., e_n)$ denote the payoffs from a Nash equilibrium of the stage game and let $(x_1, ..., x_n)$ be any feasible payoff.
   If $x_i > e_i$ for any player i and if δ is sufficiently close to 1 then there exists a SPNE of $G(\infty, \delta)$ that achieves $(x_1, ..., x_n)$ as average payoff.

2. **Theorem (Benoit-Krishna 1985)**: Let $G(T)$ be a finitely repeated game with

   $$U^i = \frac{1}{T} \sum_{t=0}^{T-1} u_i(s_i^t, s_{-i}^t)$$

   Suppose that, for each player $i \in N$, there is a Nash equilibrium e^i of G such that

   $$u_i(e^i) > v_i$$

   Then, for each $v \in \bar{F}$ and each $\epsilon > 0$, there is a $T^0 \in \mathbb{N}$ such that, for each $T \geq T^0$, there is a Nash equilibrium whose payoff vector $\hat{v}$ satisfies

   $$||\hat{v} - v|| < \epsilon$$

   It is possible to prove a similar result with discounting for payoffs (as long as utilities are separable across time):

Theorem (Benoit-Krishna 1985): Let $G(T, \sigma)$ be a finitely repeated game. Suppose that, for each player $i \in N$, there is a Nash equilibrium e^i of G such that

$$u_i(e^i) > \underline{v}_i$$

Then, for each $v \in \bar{F}$ and each $\epsilon > 0$, there are $\delta^0 \in (0,1)$ and $T^0 \in \mathbb{N}$ such that, for each $\delta \in (\delta^0, 1)$ and $T \geq T^0$, there is a Nash equilibrium whose payoff vector $\hat{v}$ satisfies

$$||\hat{v} - v|| < \epsilon$$

3. It is possible to get a strong result if we have more than one Nash equilibrium: **Theorem:** Let $G(T, \delta)$ be a finitely repeated game. Suppose that, for each player $i \in N$, there are two Nash equilibria e^i and $\bar{e}^i$ such that

$$u_i(e^i) > u_i * \bar{e}^i)$$

Then, for each $v \in \bar{F}$ and each $\epsilon > 0$ there are $\delta^0 \in (0,1)$ and $T^0 \in \mathbb{N}$ such that, for each $\delta \in (\delta^0, 1)$ and for each $T \geq T^0$, there is a Subgame Perfect Nash Equilibrium of $G(T, \delta)$ whose payoff vector $\hat{v}$ satisfies

$$||\hat{v} - v|| < \epsilon$$

4. It is also possible to get a stronger result in the infinitely repeated games:
Theorem: Let $G(\infty, \delta)$ be a infinitely repeated game. If F is full dimensional, then, for each $v \in \bar{F}$ there is a $\delta^0 \in (0,1)$ such that, for each $\delta \in [\delta^0, 1)$ there is a Subgame Perfect Nash Equilibrium of $G(\infty, \delta)$ with payoff v.

Introduction to General Equilibrium

1. GE and Game Theory share a few aspects:
   (a) The notion of equilibrium
   (b) Use of fixed point theory
   (c) Both give results about pricing (in Game Theory, these are for auctions).

2. Specific to GE, we have:
   (a) The benchmark for competitive behaviour with many agents
   (b) The equilibrium exists under well-known conditions (Walras-Arrow-Debreu) - i.e., monotonicity, quasiconcavity, continuity.
   (c) The equilibrium is always Pareto Optimal (very different from Game Theory)
   (d) There are two important welfare theorems (Arrow-Debreu)
   (e) Negative result: Dynamics is not present in GE (Scarf theorem)
   (f) Aggregation is very problematic.
   e.g., let demand be

$$D^i(p) = arg \max u_i(x_1, ..., x_n)$$

   The idea of aggregation is that there exists some representative D_R where

$$\sum D_i = D_R = arg \max u_R(x_1, ..., x_n)$$

   but this representative agent rarely exists.

   (g) Negative result: Debreu-Mantel-Sonnenschein Theorem
   (h) Gives a foundation for pricing in a competitive setting

3. Roadmap:
   (a) The standard model:
   Existence of equilibria $\exists p^*$ s.t. excess demand $z(p^*) = 0$

(b) The problem of existence
(c) Two welfare theorems
(d) Structure of excess demand: $z(p) = \sum D_i(p) - \sum w_i$ where w_i are endowments.

(e) Problem of aggregation
(f) Problem of uniqueness of equilibria (i.e., there are too many equilibria)

(g) Tantonnement process (i.e., process of finding equilibrium price where $\lim_{t\to\infty} p(t) = p^*$).

(h) Comparative statics
(i) The core:
The core is the set of feasible allocations or imputations where no coalition of agents can benefit by breaking away from the grand coalition.
"The core is not empty"

(j) The intertemporal model (arbitrage and equilibrium)
(k) Back to game theory (Bargaining and Nash solution)

Lecture 13: General Equilibrium Theory 1: Tools

Notation and results

1. The set
$$\mathbb{R}^l = \{x = (x_1, x_2, ..., x_l) : x_i \in \mathbb{R}\}$$
is known as the Euclidean space. We denote
$$\mathbb{R}^l_+ = \{x \in \mathbb{R}^l : x_i \geq 0 \ \forall i\}\mathbb{R}^l_{++} = \{x \in \mathbb{R}^l : x_i > 0 \ \forall i\}$$
We sometimes refer to $\mathbb{R}^l_+$ as the *positive orthant.*

2. For vectors x and y in $\mathbb{R}^l$, $x \geq y$ if $x_i \geq y_i \ \forall i$, and $x > y$ if $x_i \geq y_i \ \forall i$ and $x \neq y$.

$x \gg y$ if $x_i > y_i \ \forall i$.

3. Definition: If $x = (x_1, x_2, ..., x_l)$ and $y = (y_1, y_2, ...y_l)$, the distance between x and y is
$$||x - y|| = \sqrt{\sum_{j=1}^{l}(x_j - y_j)^2}$$
Note that
(a) $||x + y|| \leq ||x|| + ||y||$
(b) If $x = (x_1, ..., x_l)$ then
$$\max_{j=1,...,l} |(x_j - y_j)| \leq ||x - y|| leq \sum_{j=1}^{l} |(x_j - y_j)|$$

4. Definition: A function $f : \mathbb{R}^l \to R^l$ is continuous at $(x_1^0, x_2^0, ..., x_l^0)$ if
$$\lim_{||y - x^0|| \to 0} ||f(y) - f(x^0)|| = 0$$
Note that we can write
$$f(x_1, ..., x_l) = (f_1(x_1, ..., x_l), ..., f_l(x_1, ...x_l))$$

and f is continuous if and only if $\forall j = 1, ..., n$,

$$f_j : R^l \to \mathbb{R}$$

is continuous (i.e., each component is continuous).

5. Definition: A set K in $\mathbb{R}^l$ is convex if $\forall x, y \in K$ we have

$$\{xt + (1-t)y | 0 \leq t \leq 1\} \subset K$$

6. Definition: Given a point x^0 and a positive number r, we define

$$B(x^0, r) = \{y \in \mathbb{R}^l | \; ||y - x^0|| < r\}$$

(i.e., the ball of radius r centered around x).

7. A point x is an accumulation point for the set K in $\mathbb{R}^l$ if for any $r > 0$ we have

$$B(x, r) \cap K \neq \emptyset$$

8. Definition: A set K in $\mathbb{R}^l$ is closed if it contains all its accumulation points.

9. Definition: A set K in $\mathbb{R}^l$ is compact iff it is closed and there exists an $r > 0$ for which $K \subset B(0, r)$.

10. Theorem: If $K \subset \mathbb{R}^l$ is compact and $f : K \to \mathbb{R}^m$ is a continuous function then the set $f(K) \subset \mathbb{R}^m$ is compact. In particular, if $m = 1$, there exists

$$\max_{x \in K} f(X) \qquad \text{and} \qquad \min_{x \in K} f(x)$$

(i.e., the maximisation/ minimisation problem has a solution).

11. Theorem (Brouwer): If K in $\mathbb{R}^l$ is compact and convex and $f : K \to K$ is a continuous function then there exists $x^* \in K$ such that $f(x^*) = x*$.

Looking at the result in another way, we can consider

$$f(x_1, ..., x_l) = (f_1(x_1, ..., x_l)), ..., f_l(x_1, ..., x_l))$$

then the theorem says that for the following system of equations

$$\begin{aligned} f_1(x_1, ..., x_l) &= x_1 \\ f_2(x_1, ..., x_l) &= x_2 \\ &... \\ f_l(x_1, ..., x_l) &= x_l \end{aligned}$$

there exists $x^* = (x_1^*, ..., x_l^*)$ such that

$$\begin{aligned} f_1(x_1, ..., x_l) &= x_1^* \\ f_2(x_1, ..., x_l) &= x_2^* \\ &... \\ f_l(x_1, ..., x_l) &= x_l^* \end{aligned}$$

12. Definition: A correspondence φ defined on $X \subset \mathbb{R}^l$ with values in $Y \subset \mathbb{R}^m$

$$\varphi : X \rightrightarrows Y$$

is a rule that associates to each x a set $\varphi(x) \subset Y$.

13. Theorem (Kakutani): If K in $\mathbb{R}^l$ is compact and convex and

$$\varphi : K \rightrightarrows K$$

is a convex non-empty valued correspondence such that

$$Graph(\varphi) = \{(x, y) | y \in \varphi(x) \forall x \in K\}$$

is closed, then φ has a fixed point, i.e., there exists $x^* \in K$ such that $x^* \in \varphi(x^*)$.

Brouwer's Theorem and Sperner's Lemma

1. Brouwer's theorem in one dimension:
   Let $f : [0, 1] \to [0, 1]$ be a continuous function. Then, there exists a fixed point, i.e., there is $x^* \in [0, 1]$ such that

   $$f(x^*) = x^*$$

   Proof: There are two essential possibilities:

   (a) If $f(0) = 0$ or if $f(1) = 1$, then we are done.
   (b) If $f(0) \neq 0$ and $f(1) \neq 1$, then define

   $$F(x) = f(x) - x$$

   In this case, $F(0) = f(0) - 0 = f(0) > 0$ and $F(1) = f(1) - 1 < 0$.
   Then, by using the Intermediate Value Theorem, there is a $x^* \in [0, 1]$ such that $F(x^*) = 0$.
   By the definition of $F(.)$, then $F(x^*) = f(x^*) - x^* = 0$ and thus $f(x^*) = x^*$.

   For Brouwer's theorem to hold, it is necessary for the following to hold: Any continuous function from a closed n-dimensional ball into itself must have at least one fixed point.

   (a) Continuity of the function is essential.
   (b) The closure of the ball is also essential.
   (c) The round shape of the ball is not essential, but we cannot replace the ball with something with "holes".

2. Sperner's Lemma:

   1. Given: A triangle and a'triangulation' of it into an arbitrary number of baby triangles .
   2. Mark the vertices of the original triangle by 0,1,2.
   3. Mark all other vertices with one of the labels 0,1,2, according to the following rule:
      - if a vertex lies on an edge of the original triangle, label it by any one of the numbers at the endpoints of that edge;
      - if a vertex is inside the original triangle, label it **any way you want.**
   4. **Conclusion** (very surprising!): There always exists a baby triangle which is completely labeled (i.e. by 0 &1 & 2). In fact, there is an odd number of them.

Sperner implies Brouwer.
We will prove the theorem in the two-dimensional case.
We will describe the points of a triangle by its barycentric coordinates:

(a) $p = x_0A + x_1B + x_2C$; $x_0 + x_1 + x_2 = 1$; $p = (x_0, x_1, x_2)$.

(b) x_0, x_1, x_2 represent the weight fractions one should place at the vertices in order to have the center of the mass of the triangle at the desired point.

(c) Label all points of the triangle as follows: If $p = (x_0, x_1, x_2)$ and $f(p) = (x'_0, x'_1, x'_2)$, inspect the coordinates of p and $f(p)$ until you find the first index i (from $0, 1, 2$) for which

$$x'_i < x_i$$

Label the point p with the label "i".

(d) If it happens that for some point p there is no strict inequality like that, it must be that $x_0 = x'_0$, $x_1 = x'_1$, $x_2 = x'_2$, in which case p is a fixed point and we are done.

(e) According to this rule, the vertices A, B, C are labelled $0, 1, 2$.

(f) Each point of the edge 0 to 1 is marked with either 0 or 1; similar statements hold for the other edges.

(g) Divide the triangle into smaller and smaller baby triangles, with diameters $\to 0$.

(h) At each step, label all vertices by the rule described above.

(i) The labelling is as in Sperner's Lemma. For each subdivision, there must exist at least one triangle labelled with 0, 1 and 2. The vertices of the completely labelled baby triangles must accumulate at some point $q = (y_0, y_1, y_2)$.

(j) Due to continuity, for q and $f(q)$ we must have

$$y'_0 \leq y_0;$$
$$y'_1 \leq y_1;$$
$$y'_2 \leq y_2$$

Since the barycentric coordinates of a point add up to 1, we must have equalities in the above list.

(k) Therefore, $f(q) = q$, i.e., q is a fixed point.

It is possible to mimic this construction in every dimension so this is a strategy to prove Brouwer's theorem in its full generality.

Lecture 14: General Equilibrium Theory 2

Theorem of the Maximum

1. Definition: A correspondence $\Gamma : X \rightrightarrows Y$ is lower semi-continuous at x if for all $y \in \Gamma(x)$ and any sequence $x_n \to x$ there exists a sequence $\{y_n\}$ with $y_n \in \Gamma(x_n)$ such that $y_n \to y$.

2. Definition: A non-empty compact-valued correspondence $\Gamma : X \rightrightarrows Y$ is upper semi-continuous at x if for any sequence $x_n \to x$ and for any sequence $\{y_n\}$, with $y_n \in \Gamma(x_n)$ the sequence $\{y_n\}$ has a limit point $y \in \Gamma(x)$.

3. Definition: A correspondence $\Gamma : X \rightrightarrows Y$ is continuous at x if it is upper semi-continuous and lower semi-continuous at x.

4. Theorem: Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$, let $\Gamma : X \rightrightarrows Y$ be a non-empty compact valued continuous correspondence and let $f : X \times Y \to \mathbb{R}$ be a continuous function.
   Then, the following hold:

(a) The function

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

is continuous;

(b) The correspondence

$$\begin{aligned} G(x) &= \{y \in \Gamma(x) : f(x, y) = h(x)\} \\ &= arg \max_{y \in \Gamma(x)} f(x, y) \end{aligned}$$

is non empty, compact-valued and upper semi-continuous.

5. **Berge's Theorem of the Maximum**:
Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$. If $f : X \times Y \to \mathbb{R}$ is a continuous function such that for every x, $f(x, \cdot)$ is strictly concave and if $\Gamma : X \vec{\to} Y$ is a non empty, compact, convex-valued continuous correspondence, then

$$G(x) = arg \max_{y \in \Gamma(x)} f(x, y)$$

is a continuous function.

Agents

1. Assume that there are l commodities and that the consumption space is $\mathbb{R}^l_+$ (the positive orthant).
2. Each agent has the utility function $U : \mathbb{R}^l_+ \to \mathbb{R}$. Furthermore:
   (a) U is increasing if $x' \gg x$ implies $U(x') > U(x)$.
   (b) U is strictly increasing if $x' > x$ implies $U(x') > U(x)$.
   (c) U is quasiconcave if, for any α, the set

   $$\{x \in \mathbb{R}^\infty_+ : U(x) \geq \alpha\}$$

   is a convex set. Equivalently, whenever $U(x') \geq \alpha$ and $U(x) \geq \alpha$, then

   $$U(tx' + (1 - t)x) \geq \alpha$$

   for any $t \in [0, 1]$.
   U is strictly quasiconcave if, whenever $U(x') \geq \alpha$ and $U(x) \geq \alpha$, then

   $$U(tx' + (1 - t)x) \geq \alpha$$

   for any $t \in (0, 1)$.

3. At price (vector) $p = (p_1, ..., p_l)$ in $\mathbb{R}^l_{++}$ and income $w > 0$, the agent's budget set is

$$B(p, w) = \{x \in \mathbb{R}^l_+ : p \cdot x \leq w\}$$

4. A commodity bundle x^* is a demand bundle of the agent at (p, w) if x^* maximises utility in $B(p, w)$; formally,

$$x^* \in arg \max_{x \in B(p,w)} U(x)$$

Note that $B(tp, tw) = B(p, w)$ for any $t > 0$, so

$$arg \max_{x \in B(p,w)} U(x) = arg \max_{x \in B(tp,tw)} U(x)$$

In other words, the budget set is invariant if we multiply both wealth and prices by the same constant; the levels of prices are not important and what matters are relative prices.

5. Let $x^* \in arg\max_{x\in B(p,w)} U(x)$. We can see that:
   (a) If U is increasing, then the inner product $p \cdot x^* = w$. In this case, we say that the agent's demand obeys the budget identity.
   (b) If U is strictly quasiconcave, then
   $$arg \max_{x\in B(p,w)} U(x)$$
   has at most one element.
   (c) $arg\max_{x\in B(p,w)} U(x)$ is nonempty if U is a continuous function (this is a consequence of the Weierstrass Theorem).

6. The results below summarise (a) to (c) above:
   Proposition: Suppose that the utility function $U : \mathbb{R}^l_+ \to \mathbb{R}$ is continuous, strictly increasing and strictly quasiconcave.
   Then, for any (p, w) in $R^l_{++} \times \mathbb{R}_{++}$ (the Cartesian product), there exists a unique element x^* (a commodity bundle) in
   $$\max_{x\in B(p,w)} U^a(x)$$
   Additionally, the function $\bar{x} : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+$ mapping (p, w) to
   $$\bar{x}(p, w) = arg \max_{x\in B(p,w)} U^a(x)$$
   has the following properties:
   (a) It is continuous;
   (b) It obeys the budget identity $p \cdot \bar{x}(p, w) = w$ (i.e., it is on the budget line);
   (c) It is zero-homogeneous, i.e.,
   $$\bar{x}(tp, tw) = \bar{x}(p, w)$$
   for $t > 0$ (i.e., absolute levels of price do not matter);
   (d) It obeys the boundary condition: If $\{(p^n, w^n)\} \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and the set $I : \{i : \bar{p}_i = 0\}$ is nonempty, then
   $$\sum_{i=1}^{l} \bar{x}^a_i(p^n, w^n) \to \infty$$
   In other words, if one of the prices approaches zero, then the demand of at least one good will approach infinity. So, prices must be strictly positive.

Economy

1. Assume that there is a finite set A of agents.
   Agent $a \in A$ has the utility function $U^a : \mathbb{R}^l_+ \to \mathbb{R}$ and endowment $w^\alpha = (w^\alpha_1, ..., w^\alpha_l)$ in $\mathbb{R}^l_+$.
   The economy is thus described by the sequence $\epsilon = \{(U^a, w^a)\}, a \in A$.

2. We require U^a to be continuous, strictly increasing and strictly quasiconcave, and that aggregate endowment
   $$\bar{w} = \sum_{a\in A} w^a \gg 0$$
   (i.e., all endowments are positive).

3. Agent a's demand function is $\bar{x}^a$.
   In an exchange economy, income is determined by the prevailing price p. With an endowment of w^a, agent a's income $w^a = p \cdot w^a$.
   His demand at price p is $\bar{x^a}(p, p \cdot w^a$.

4. Define

$$\hat{x}^a : \mathbb{R}^l_{++} \to \mathbb{R}^l_+$$

by

$$\hat{x}^a(p) = \bar{x}^a(p, p \cdot w^a)$$

(i.e., this is the demand at price p given endowment $p \cdot w^a$).
Note the distinction between $\hat{x}$ and $\bar{x}$. $\bar{x}$ is the demand function; $\hat{x}$ is the demand at price p given the endowment $p \cdot w^a$.

5. Agent a's excess demand is $z^a(p) = \hat{x}^a(p) - w^a$.
Claim: z^a is zero-homogeneous, i.e., $z^a(\lambda p) = z^a(p)$ for any scalar $\lambda > 0$, and $p \cdot z^a(p) = 0$ (i.e., orthogonal) for all p.
Proof:
$\bar{x}^a(p, w) = \max_{x \in B(p,w)} U^a(x)$ is zero-homogeneous, so

$$\hat{x}^a(\lambda p) = \bar{x}^a(\lambda p, (\lambda p) \cdot w^a) = \bar{x}^a(p, p \cdot w^a) = \hat{x}^a(p)$$

which in turn guarantees that $z^a(\lambda p) = z^a(p)$.
Since $p \cdot \bar{x}^a(p, p \cdot w^a) = p \cdot w^a$, we have

$$p \cdot [\bar{x}^a(p, p \cdot w^a) - w^a] = 0$$

so $p \cdot z^a(p) = 0$.

6. Aggregate (or market) demand at price p is

$$X(p) = \sum_{a \in A} \hat{x}^a(p)$$

7. The aggregate excess demand function $Z : \mathbb{R}^l_{++} \to \mathbb{R}^l$ is

$$Z(p) = X(p) - \bar{w}$$

Z is zero-homogeneous and obeys Walras' law: $p \cdot Z(p) = 0 \ \forall p$ (these properties are inherited from z^a).

8. The fundamental question is: **what conditions guarantee that there is $p^* \gg 0$ such that $Z(p^*) = 0$?** (i.e., no excess demand; the economy is in equilibrium).

Note that since Z is zero-homogeneous, if p^* is an equilibrium price, so is λp^* for any $\lambda > 0$.

Recall the boundary property satisfied by $\bar{x}_a$: if $(p^n, w^n) \to (\bar{p}, \bar{w})$ such that $\bar{w} > 0$ and $I = \{i : \bar{p}_i = 0\}$ is non-empty, then $\sum_{i \in I} \bar{x}^a_i(p^n, w^n) \to \infty$.

9. Corollary: In economy ϵ, suppose p^n tends to $\bar{p} \neq 0$, such that $I = \{i : \bar{p}_i = 0\}$ is nonempty. Then $\sum_{i \in I} Z_i(p^n) \to \infty$.
Proof: Suppose that for good k, $\bar{p}_k > 0$. Since

$$\bar{w} = \sum_{a \in A} w^a \gg 0$$

there is $\bar{a}$ with $w^{\bar{a}}_k > 0$. In other words, there is some agent $\bar{a}$ who has a strictly positive endowment of good k. Then, $p^n \cdot w^{\bar{a}}$ tends to $\bar{w}^{\bar{a}} = \bar{p} \cdot w^{\bar{a}} > 0$.
So,

$$\sum_{i \in I} \hat{x}^{\bar{a}}_i(p^n) = \sum_{i \in I} \bar{x}^{\bar{a}}_i(p^n, p^n \cdot w^{\bar{a}}) \to \infty$$

which implies that

$$\begin{aligned}\sum_{i \in I} Z_i(p^n) &= \sum_{i \in I} X_i(p^n) - \sum_{i \in I} \bar{w}_i \\ &\geq \sum_{i \in I} \bar{x}^{\bar{a}}_i(p^n, p^n \cdot w^{\bar{a}}) - \sum_{i \in I} \bar{w}_i \to \infty\end{aligned}$$

Lecture 15: General Equilibrium Theory 3

Excess Demand Function

1. **Theorem**: The excess demand function

$$Z : \mathbb{R}^l_{++} \to \mathbb{R}^l$$

of the economy $\epsilon = (U^1, ...U^n; w^1, ..., w^n)$ satisfies the following properties:

   (a) It is zero-homogenous
   (b) It obeys Walras' law (i.e., perpendicularity)
   (c) It is continuous
   (d) It satsfies the boundary condition (i.e., closer to the boundary, excess demand, i.e., "the perpendicular line" becomes longer until it is arbitrarily large)
   (e) It is bounded below

   Notes:

   (a) We are focusing on the case where the agents are $a \in A$ and the number of agents A is finite.
   (b) Z is bounded from below since we have

$$\bar{w} = \sum_{i=1}^{M} w_I$$
$$Z(p) = X(p) - \bar{w} > -\bar{w}$$

   since $X(p)$ is always positive.

2. **Theorem:** (Arrow and Debreu; McKenzie) Suppose Z satisfies properties (1) to (5). Then there is p^* such that

$$Z(p^*) = 0$$

   The proof uses Kakutani's fixed point theorem, which generalises Brouwer's fixed point theorem to correspondences.

   Note that for $\Delta = \{(P_1, ..., P_n) | \sum_{i=1}^{n} P_i = 1, P_i > 0\}$, Δ is not closed and is hence not compact (since it does not include the end-points where $P_i = 0$), and hence we cannot apply the Brouwer fixed point theorem.

Proof of equilibrium existence in a special case

1. We present a simple proof of the existence of an equilibrium in the case where $w_a \gg 0$ for all a (i.e., every agent has some of every good).

   Define

$$\tilde{B}(p, a) = B(p, p \cdot w_a) \cap \{x \leq 2\bar{w}\}$$

   This is a truncated budget set for agent a (See graphical notes).

   Assuming the utility function is (P1) continuous, (P2) strictly increasing and (P3) strictly quasiconcave, then

$$arg \max_{x \in \tilde{B}(p,a)} U^a(x)$$

   exists and is unique for all

$$p \in \delta^{l-1} = \{p \in \mathbb{R}^l | p = (p_1, ..., p_l) > 0, \sum_{i=1}^{l} p_i = 1\}$$

   Notes:

(a) The unit simplex Δ for l goods is of dimension $l-1$ because if all elements sum to 1, the effective degrees of freedom is reduced by 1.
For $l=2$ goods, the unit simplex is a line segment (1-dimensional).
For $l=3$ goods, the unit simplex is a triangle (2-dimensional) because the condition leaves 2 degrees of freedom.

(b) Now if we take a price $p=0$, it is still bounded by the truncated budget, so demand is still well-defined.

2. We define the modified demand by $\tilde{x}^a(p)$.
The crucial feature of $\tilde{x}^a$ that makes it useful is that it is also defined on the boundary of Δ (the unit simplex) unlike $\hat{x}^a$ which is defined only in the interior of Δ.

Note that $\tilde{x}^a$ satisfies $p \cdot \tilde{x}^a(p) = p \cdot w^a$. Furthermore, $\tilde{x}^a$ is continuous in p (this property relies crucially on $w^a \gg 0$, so everyone has some demand even if prices of one or more goods go to 0. Otherwise if they only hold assets of the 0-valued goods, they will have 0 wealth and thus 0 demand in this case).

Therefore, the mapping

$$\tilde{Z}(p) = \sum_{a \in A} [\tilde{x}^a(p) - w^a]$$

is continuous and obeys Walras' law.

3. **Lemma 1**: There is $p^* \gg 0$ such that $\tilde{Z}(p^*) = 0$.

Proof: Define $\psi : \Delta^{l-1} \to \Delta^{l-1}$ by

$$\psi_j(p) = \frac{p_j + \max\{\tilde{Z}_j(p), 0\}}{1 + \sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\}}, \qquad \forall j$$

Notes:

(a) $\tilde{Z}$ is continuous (by theorem of the maximum)
$\implies \forall j, \tilde{Z}_j$ is continuous
$\implies \forall j, \max\{\tilde{Z}_j, 0\}$ is continuous.

(b) $\tilde{Z} = (\tilde{Z}_i, ..., \tilde{Z}_l)$.
$\tilde{\psi} = (\psi_1, ..., \psi_l)$

(c) We need to show that

$$\sum_{j=1}^{l} \psi_j = 1 \qquad \text{and} \qquad \psi_j \geq 0, \qquad j = 1, ..., l$$

$$\sum_{j=1}^{l} \psi_j = \sum_{j=1}^{l} \frac{p_j + \max\{\tilde{Z}_j(p), 0\}}{1 + \sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\}} = 1 \qquad \text{(shown)}$$

$$\psi_j \geq 0 \qquad \text{because both the numerator and denominator are positive (shown)}$$

Δ^{l-1} is a compact and convex set and this map is continuous.
Brouwer's theorem guarantees there is p^* such that $\psi(p^*) = p^*$ (these are our candidate equilibria).

If $p_k^* = 0$ for some k, then $\max\{\tilde{Z}_k(p^*), 0\} > 0$ and so $\psi_k(p^*) \neq p_k^* = 0$, which is a contradiction.
So, it must be that $p^* \geq 0$, i.e., each component > 0. By Walras' law, there is h such that $\max\{\tilde{Z}_h(p^*), 0\} = 0$.
Note: This is since $\tilde{Z}(p) \cdot p = 0$ and $P_j \geq 0$ so it must be that some excess demands $\tilde{Z}(p)$ have to be negative or 0 for the inner product to be $= 0$.

Since $\psi(p^*) = p^*$, in particular,

$$p_h^* = \psi_h(p^*) = \frac{p_h^* + 0}{1 + \sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\}}$$

This gives

$$\sum_{i=1}^{l} \max\{\tilde{Z}_i(p), 0\} = 0$$

so $\tilde{Z}_i(p^*) \leq 0$ for all i. By Walras' law and the fact that $p^* \gg 0$, we have $\tilde{Z}_i^*(p^*) = 0$ for all i (i.e., no excess demands).

4. **Lemma 2**: If there is $p^* \gg 0$ such that $\tilde{Z}(p^*) = 0$, then $Z(p^*) = 0$.

   Proof:
   We claim that $\hat{x}^a(p^*) = \tilde{x}^a(p^*)$ for all agents (i.e., at p^*, the demand on the truncated simplex = demand on the full simplex).
   Clearly, this implies that $Z(p^*) = \tilde{Z}(p^*) = 0$.
   Suppose, to the contrary, that for some agent b, $\hat{x}^b(p^*) \neq \tilde{x}^b(p^*)$, which means that $\hat{x}^b(p^*)$ is not less than $2\bar{w}$ and $U^b(\hat{x}^b(p^*)) > U^b(\tilde{x}^b(p^*))$ (since the original budget set is greater than the truncated budget set).
   Since $\tilde{Z}(p^*) = 0$, it must be the case that $\tilde{x}^b(p^*) \ll 2\bar{w}$.
   Choose $t \in (0, 1)$ such that $x = t\tilde{x}^b(p^*) + (1 - t)\hat{x}^b(p^*)$ satisfies $x \ll 2\bar{w}$.
   Note that $U^b(x) > U^b(\tilde{x}^b(p^*))$ by the strict quasiconcavity of U^b, and that $x \in \tilde{B}(p^*, b)$.
   This contradicts the optimality of $\tilde{x}^b(p^*)$ in $\tilde{B}(p^*, b)$. QED.

5. Summary of what we have done so far:
   If we have an economy $\epsilon = \{(U^a, w^a)\}, a \in A$, with

   (a) $w^a \gg 0, \forall a$, and
   (b) U^a satisfies (P1) continuity, (P2) strictly increasing, and (P3) strictly quasiconcave

   then

   $$\implies \exists p^* \text{ s.t. } Z(p^*) = 0$$

   But what we want to do is replace the requirement that $w_a \gg 0$ with a less strict requirement that $\sum w_a \gg 0$ (i.e., agents do not need to have some of every good, but just need to have non zero wealth).

A new tool

1. We proved the existence of the equilibrium in a special case (where $w_a \gg 0$) which is not too satisfactory from an economic point of view.

   To give the general proof we need to use Kakutani's result. The application of this theorem is more effective if we prove the following:

   Lemma (Gale-Nikaido-Debreu): Let S be a compact and convex subset of the simplex $\Delta^{l-1} = \{p > 0 : \sum_{i=1}^{l} p_i = 1\}$, let $f : S \to \mathbb{R}^l$ be a continuous map such that

   $$p \cdot f(p) = 0$$

   (i.e., they are perpendicular) for all $p \in S$. Then there exists a $p^* \in S$ such that

   $$p \cdot f(p^*) \leq 0, \forall p \in S$$

   Proof (sketch):
   Consider the correspondence $\mu : f(S) \to S$ defined by

   $$\mu(z) = \{p \in S | p \cdot z = \max_{q \in S} q \cdot z\}$$

It is simple to verify that this correspondence is convex and compact valued and that

$$Graph(\mu) = \{(z, y) | y \in \mu(z), \forall z \in f(S)\}$$

is closed. We consider

$$\Gamma : S \times f(S) \rightrightarrows S \times f(S)$$

where $\Gamma(p, z) = (\mu(z), f(p))$.
It is simple to verify that:

(a) Γ is convex-valued
(b) Γ is compact valued
(c) $Graph(\Gamma)$

$$\{((p, z), y) | y \in (\mu(z), f(p)), \forall (p, z) \in S \times f(S)\}$$

is closed.

We can apply Kakutani's theorem and this implies that there exists $(p^*, z^*) \in S \times f(S)$ such that

$$p^* \in \mu(z^*) \text{ and } z^* = f(p^*)$$

Hence for all $p \in S$ we have

$$\begin{aligned} p \cdot f(p^*) &= p \cdot z^* \\ &\leq p^* \cdot z^* \qquad (\text{because } p \cdot z = \max_{q \in S} q \cdot z) \\ &= p^* \cdot f(p^*) \\ &= 0 \end{aligned}$$

and hence $p \cdot f(p^*) \leq 0$.

Lecture 16: General Equilibrium Theory 4 - Existence

Existence of Equilibria

1. Recap: We almost proved, as a direct consequence of Kakutani's theorem, the following result:

   Lemma (Gale-Nikaido-Debreu): Let S be a compact and convex subset of the simplex $\Delta^{l-1} = \{p > 0 : \sum_{i=1}^{l} p_i = 1\}$, let $f : S \to \mathbb{R}^l$ be a continuous map such that

   $$p \cdot f(p) = 0$$

   (i.e., they are perpendicular) for all $p \in S$. Then there exists a $p^* \in S$ such that

   $$p \cdot f(p^*) \leq 0, \forall p \in S$$

2. The main ingredients for the application of Kakutani Theorem are:

   (a) The correspondence $\mu : f(S) \to S$ defined by

   $$\mu(z) = \{p \in S | p \cdot z = \max_{q \in S} q \cdot z\}$$

   is convex and compact valued and $Graph(\mu)$ is closed.

   (b) We consider $\Gamma : S \times f(S) \rightrightarrows S \times f(S)$ and defined $\Gamma(p, z) = (\mu(z), f(p))$, where it is simple to verify that Γ is convex valued, compact valued and $Graph(\Gamma)$ is closed.

   (c) Apply Kakutani's theorem and get $(p^*, z^*) \in S \times f(S)$ such that $p^* \in \mu(z^*)$ and $z^* = f(p^*)$

3. **Theorem:** Let $f : \mathbb{R}^l_{++} \to \mathbb{R}^l$ be a function which is zero-homogenous, obeys Walras' law, is continuous, satisfies the boundary condition and is bounded below.
Notes:

   (a) Zero-homogeneous: $f(tx) = t^0 f(x) = f(x)$ so we can restrict ourselves to focus on the simplex.

   (b) the inputs of f are in R^l_{++}, i.e., each component > 0 (i.e., all prices are positive).

   Then, there exists p^* such that

$$f(p^*) = 0$$

   Proof: For any $n \in \mathbb{N}$ we define

$$\Delta^{l-1}_n = \{p \in \Delta^{l-1} | \sum_{i=1}^{l} p_i = 1 \text{ such that } p_i \geq \frac{1}{n} \forall i\}$$

   (See graphical notes. This is basically different simplexes within one another, where the simplexes of smaller n are subsets of the simplexes of larger n. Note that n is just a parameter; the dimension of the simplex is still $l-1$.)

   If we apply Gale-Nikaido-Debreu to each of the simplexes, we get, for $n \in \mathbb{N}$, $p^*_n \in \Delta^{l-1}_n$ such that

$$p \cdot f(p_n) \leq 0, \forall p \in \Delta^{l-1}_n$$

   now, since

$$\Delta^{l-1}_1 \subset ... \subset \Delta^{l-1}_n \subset ... \subset \Delta^{l-1}$$

   we have a sequence $\{p_n\} \subset \Delta^{l-1}$. Since Δ^{l-1} is compact there exists a limit point p^* for the sequence $\{p^*_n\}$.
   We still need to show that $p^* \gg 0$; the limit point is not on the boundary.

4. Since we know that f satisfies the boundary condition by definition, it is enough to show that the sequence

$$f(p^*_1)...f(p^*_n)....$$

   stays bounded (otherwise the sequence will converge to $f(p^*)$ because f is continuous).

   We observe that:

   (a) the sequence is bounded below by construction

   (b) The sequence is bounded above, since if $f(p^*_n) = (f(p^*_n)_1, ..., f(p^*_n)_l) \in \Delta^{l-1}$ we have, if n is large enough,

$$f(p^*_n)1\frac{1}{l} \leq -\sum_{k=2}^{l} f(p^*_n)k\frac{1}{l} \leq \sum_{k=2}^{l} w_k\frac{1}{l}$$

   (Notes: In the first inequality, we are comparing the first component against the 2nd to l^{th} components. The second inequality holds because of boundedness from below, i.e., $Z(p) = f(p) > -\bar{w}$).

   Therefore we have

$$f(p^*_n)_1 \leq \sum_{k=2}^{l} w_k$$

5. In general we have

$$f(p^*_n)_j \leq \sum_{\substack{k=1 \\ k \neq j}}^{l} w_k \leq \sum_{k=1}^{l} w_k$$

Therefore

$$f(p^*) = \lim_{n\to\infty} f(p_n^*) \qquad \text{(because } f \text{ is continuous)}$$

is bounded and

$$p \cdot f(p^*) \leq 0, \forall p \in \Delta^{l-1}$$

but this implies that

$$f(p^*) \leq 0, \forall p \in \Delta^{l-1}$$

(i.e., the components of the vector are nonpositive)

and since $p^* \gg 0$ and $p^* \cdot f(p^*) = 0$, we must have $f(p^*) = 0$ (otherwise $p^* \cdot f(p^*) \neq 0$, which is a contradiction).

6. Corollary: **Theorem (Arrow and Debreu, McKenzie)**: Suppose Z satisfies properties (a) to (e) from earlier (i.e., zero-homogeneity, Walras' law, continuity, boundary condition, bounded below). Then there is p^* such that $Z(p^*) = 0$.

   Now that we have established existence of the equilibrium, it is important to notice that:

   (a) It is an existence proof, but not constructive (it does not say how to get the equilibrium).
   (b) It uses very few assumptions on the utility functions, and does not rely on special forms.
   (c) It does not say anything about the equilibrium itself.
   (d) It is based entirely on the use of the excess demand function.

   Therefore, after existence, we need to analyse the equilibrium more deeply (e.g., the optimality of the equilibrium).

Lecture 17: General Equilibrium Theory 5: Indeterminacy and Weak Axiom

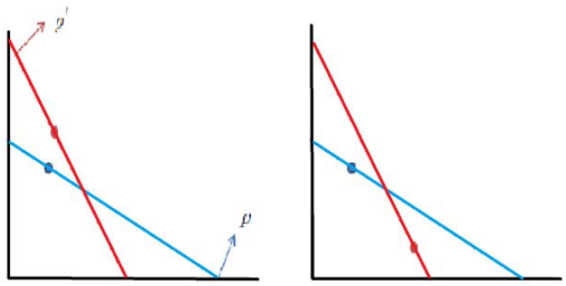
Weak Axiom of Revealed Preference

1. Agent a's demand function $\bar{x}^a$ (this is just demand; see lecture 14) obeys the **weak axiom of revealed preference** if at any pair (p, w) and (p', w') (i.e., prices and endowments), with $\bar{x}^a(p, w) \neq \bar{x}(p', w')$, the following holds:

$$p' \cdot \bar{x}^a(p, w) \leq w' \implies p \cdot \bar{x}^a(p', w') > w$$

   Equivalently, $\bar{x}^a(p, w) \in B(p', w') \implies \bar{x}^a(p', w') \notin B(p, w)$.

   Graphical illustration:



(a) On the LHS chart, the dots are the choices under different pricing vectors p and p'. The choices here are consistent with the weak axiom of revealed preference.

(b) On the RHS chart, the choices are **not** consistent with the weak axiom of revealed preference. Consider the red dot under price p'. The agent should not be choosing this consumption bundle when prices are p', because the agent could already afford this bundle at price p.

2. **Proposition:** Agent a's demand function obeys the weak axiom of revealed preference if a is utility-maximising.

Proof: If $p' \cdot \bar{x}(p,w) \leq w'$ then $x^a(p,w)$ is in $B(p',w')$. But $\bar{x}^a(p,w)$ is not the demand at (p',w'), so

$$U(\bar{x}^a(p',w')) > U(\bar{x}^a(p,w))$$

Thus $\bar{x}^a(p',w') \notin B(p,w)$, otherwise it would be chosen over $\bar{x}^a(p,w)$. In other words, $p \cdot \bar{x}^a(p',w') > w$.

Corollary: Agent a's excess demand function z^a obeys the weak axiom of revealed preference: at prices p and p', if $z^a(p) \neq z^a(p')$, then

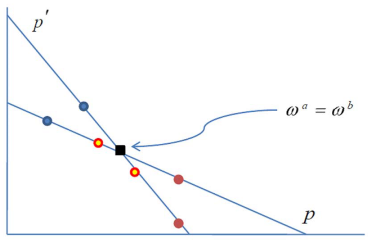
$$p' \cdot z^a(p) \leq 0 \implies p \cdot z^a(p') > 0$$

Note: what we are doing here is swapping the demand function $\bar{x}$ with the excess demand function z, using the fact that $z = \bar{x} - \bar{w}$.

Proof: Since $z^a(p) = \bar{x}^a(p, p \cdot w^a)$, we may rewrite $p'^a(p) \leq 0$ as $p' \cdot \bar{x}^a(p, p \cdot w^a) \leq p' \cdot w^a$. Set $p \cdot w^a = w$ and $p' \cdot w^a = w'$. By the previous result, $p \cdot \bar{x}^a(p',w') > w = p \cdot w^a$. Rewrite this as $p \cdot z^a(p') > 0$.

What is the structure of $Z(p)$?

1. We can use a simple example from Hicks to show that the aggregate excess demand function Z need not obey the weak axiom.



Hicks' counterexample:

(a) Let the blue dots be the consumption bundles chosen by agent 1 and the red dots be the consumption bundles chosen by agent 2.

(b) The red circled dots are the aggregate demands across both agents.

(c) The illustration shows that even if the individual demands obey the weak axiom of revealed preference, the aggregate demand (and hence aggregate excess demand, Z) may not obey the weak axiom.

2. **Indeterminacy Theorem (Sonnenschein-Mantel-Debreu):** Let K be a compact set in $\mathbb{R}^l_{++}$ and let $S : K \to \mathbb{R}^l$ be any function with the following properties: it is zero-homogeneous, it obeys Walras' Law, and it is continuous, then there is an exchange economy of agents with utility functions obeying the standard assumptions such that the excess demand function $Z : \mathbb{R}^l_{++} \to \mathbb{R}^l$ satisfies

$$Z(p) = S(p), \forall p \in K$$

Practically, this means that multiple equilibria and unstable equilibria (i.e., a small change would cause the system to move far from equilibrium) are possible.

3. **Theorem:** Suppose that all agents in E have the same utility function and thus the same demand function, so

$$\bar{x}^a(p, w) = \bar{x}(p, w), \forall a \in A$$

Suppose also that $\bar{x}(p, w) = \bar{x}(p, 1)w$, i.e., demand is linear in income. Then, Z obeys the weak axiom. Note: Demand is linear if preferences are homothetic. Such a preference is representable by a 1-homogeneous utility function, i.e., $U(tz) = tU(x)$ for any $t > 0$ and $x \in \mathbb{R}^l_+$.

Proof: If we denote the economy's aggregate endowment $\sum_{a\in A} w^a$ by $\bar{w}$, then

$$\begin{aligned} X(p) &= \sum_{a\in A} \bar{x}(p, p \cdot w^a) \\ &= \sum_{a\in A} \bar{x}(p, 1)(p \cdot w^a) \\ &= \bar{x}(p, 1)[\sum_{a\in A} (p \cdot w^a)] \\ &= \bar{x}(p, 1)[p \cdot (\sum_{a\in A} w^a)] \\ &= \bar{x}(p, p \cdot (\sum_{a\in A} w^a)) \\ &= \bar{x}(p, p \cdot \bar{w}) \end{aligned}$$

so an economy's aggregate demand behaves like the demand of an agent with endowment $\bar{w}$. Therefore, its excess demand function

$$Z(p) = \bar{x}(p, p \cdot \bar{w}) - \bar{w}$$

obeys the weak axiom.
Note: This result is not very useful because it is almost impossible to see a real economy with such homogeneity, and the aggregation here is quite artificial.

Weak Axiom at equilibrium

1. **Proposition:** Suppose Z obeys the weak axiom and let p^* be an equilibrium. Then, for any p such that $Z(p) \neq 0$, we have

$$(p - p^*) \cdot Z(p) < 0$$

Proof: Note that $Z(p) \neq Z(p^*) = 0$. Furthermore, $p \cdot Z(p^*) = 0$. By the weak axiom, $p^* \cdot Z(p) > 0$, so

$$(p - p^*) \cdot Z(p) = -p^* \cdot Z(p) < 0$$

2. Let p^* be an equilibrium of the economy ϵ. We say that its excess demand function Z obeys the weak axiom at equilibrium if $(p - p^*) \cdot Z(p) < 0$ for all p not collinear with p^*.

By proposition, Z obeys the weak axiom at equilibrium (WAE) if it obeys the weak axiom (WA) and has a unique equilibrium price.

If Z obeys the weak axiom, then the set of equilibrium prices is convex, i.e., if p^* and p^{**} are equilibrium prices, so is

$$tp^* + (1 - t)p^{**}$$

for $t \in [0, 1]$.

Note: Non singleton convex equilibrium sets are non-generic (in general equilibrium, a property is generic if it holds with probability 1).

So when Z obeys the WA, generically, the price equilibrium is unique and Z obeys the WAE.

Interpretation of WAE: If $p = (p_1, p_2^*, ..., p_l^*)$, then

$$(p - p^*) \cdot Z(p) = (p_1 - p_1^*)Z_1(p) < 0$$

i.e., when the price of good 1 is higher than its equilibrium price, then there is excess supply of good 1, etc.

Walras Tatonnement

1. What is the solution to the differential equation

$$\frac{dp_i}{dt} = Z_i(p) : \forall i$$

along with the condition $p(0) = \bar{p}$?

   (a) If $Z_i(p) < 0$ then we are asking too little and the price should decrease.
   (b) If $Z_i(p) > 0$ then we are asking too much and the price should increase.

Ideally a dynamic model of General equilibrium should be such that for any initial price $\bar{p}$, the solution converges to an equilibrium price as $t \to \infty$.
However, this is not generally true:

Lemma: Let $p(t)$ be the solution to Walras Tantonnement (i.e., the Walrasian price after the Walrasian auction price discovery process) at the initial condition $p(0) = \bar{p}$. Then,

$$\sum_{i=1}^{l} p_i^2(t) = \sum_{i=1}^{l} \bar{p}_i^2, \forall t$$

Proof: Note that

$$\begin{aligned}\frac{d}{dt}(\sum_{i=1}^{l} p_i^2(t)) &= 2(\sum_{i=1}^{l} p_i(t)\frac{dp_i}{dt}(t)) \\ &= 2(\sum_{i=1}^{l} p_i(t)Z_i(p(t)))\end{aligned}$$

By Walras' Law, $\frac{d}{dt}(\sum_{i=1}^{l} p_i^2(t)) = 0, \forall t$, QED.

Thus, the solution $p(t)$ lies on the surface of a higher dimensional sphere with radius $\sqrt{\sum_{i=1}^{l} \bar{p}_i^2}$.

2. **Theorem:** Suppose Z obeys the WAE. Then, at any initial condition, $p(t)$ converges to the equilibrium price.

Sketch of proof: Suppose $p(0) = \bar{p}$. By the previous result, we know that $p(t)$ lies on a sphere. Scale the equilibrium price p^* such that $\sum_{i=1}^{l} p_i^{*2} = \sum_{i=1}^{l} \bar{p}_i^2$, so p^* also lies on the sphere.

Consider the Lyapunov function $L(p) = \sum_{i=1}^{l}(p_i - p_i^*)^*$. Then,

$$\frac{dL}{dt} = 2\sum_{i=1}^{l}(p_i - p_i^*)\frac{dp_i}{dt}$$

The latter equals

$$2(p - p^*) \cdot Z(p)$$

and hence

$$\frac{dL}{dt} < 0$$

Lecture 18: General Equilibrium Theory 6: Complete and Incomplete Markets

General Equilibrium and Time

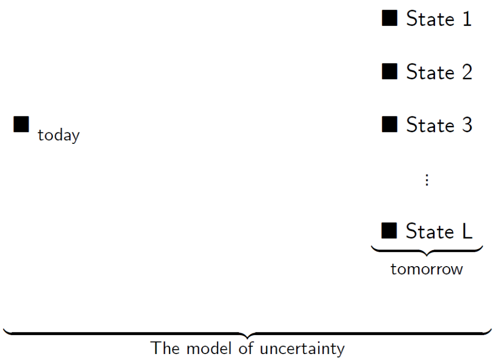
1. We study the simple two-period model in the General equilibrium framework.

   We assume that goods at different dates and contingencies (i.e., states of the world) are considered to be different, i.e., bread today is different from bread tomorrow.

   We need to account for the possibility to transfer wealth from today to tomorrow. The extent of this transfer is going to be the main characteristic (i.e., completeness vs incompleteness of markets).

Financial Assets

1. Assume that there are two dates, date 0 and date 1. There are L states of the world tomorrow.



2. A security/ asset is a promise of payment (positive or negative), conditional on the realisation of the state. We write the payoff of security s as a $L \times 1$ column vector (for the L possible states). The payoff vector of security s is

$$\begin{pmatrix} d_{1s} \\ d_{2s} \\ d_{3s} \\ . \\ . \\ d_{Ls} \end{pmatrix}$$

   If an economy has S securities (called $1, 2, ..., S$; note that this has nothing to do with the number of goods), then the payoff matrix D is

$$D = \begin{pmatrix} d_{11} & d_{12} & . & . & d_{1S} \\ d_{21} & d_{22} & . & . & d_{2S} \\ d_{31} & d_{32} & . & . & d_{3S} \\ . & . & . & . & . \\ . & . & . & . & . \\ d_{L1} & d_{L2} & . & . & d_{LS} \end{pmatrix}$$

   Trades in securities/ the agent's portfolio is represented by the column vector $z \in \mathbb{R}^S$.
   The portfolio can account for both buying and selling. If $z_s > 0$ then the agent is buying asset s; if $z_s < 0$ then the agent is selling asset s.

3. A portfolio $z \in \mathbb{R}^S$ gives a payoff in state i of

$$z_1 d_{i1} + ... + z_S d_{iS}$$

This is the i^{th} entry in the column vector Dz in $\mathbb{R}^L$ since

$$Dz = z_1 \begin{pmatrix} d_{11} \\ d_{21} \\ . \\ . \\ d_{L1} \end{pmatrix} + z_2 \begin{pmatrix} d_{12} \\ d_{22} \\ . \\ . \\ d_{L2} \end{pmatrix} + ... z_S \begin{pmatrix} d_{1S} \\ d_{2S} \\ . \\ . \\ d_{LS} \end{pmatrix}$$

So Dz is the payoff vector of the portfolio z.

The subspace $\{Dz; z \in \mathbb{R}^S\}$, denoted $Span(D)$, is called the span of the security payoffs or the asset span (this is just the span of matrix D).

Complete vs. incomplete markets

1. If $Span(D) = \mathbb{R}^L$, then the economy has **complete markets**, i.e., the agent can achieve any consumption bundle by assembling the appropriate portfolio.

   In this case, agents are trading on the full simplex.

2. If $Span(D)$ is a strict subspace of $\mathbb{R}^L$, then the economy has **incomplete markets**.

   Suppose the agent has endowment $w = (w_0, w_1, ..., w_L) \in \mathbb{R}_+^{L+1}$ (where $w_0 \in \mathbb{R}_+$ is its endowment in period $t = 0$, and $w_{-0} = (w_1, ..., w_L)$ is its endowment in states 1 to L in period $t = 1$).

   With a portfolio $\bar{z}$, the agent's contingent consumption at $t = 1$ is $w_{-0} + D\bar{z}$ (i.e., endowment at $t = 1$ plus payoff from portfolio $\bar{z}$).
   With incomplete markets, there are contingent consumption bundles that the agent cannot achieve even if he could assemble any portfolio he likes, i.e., there is $x^* \in \mathbb{R}_+^L$ such that there is no z with $w_{-0} + Dz = x^*$.

   In this case, there are points on the full simplex that the agent cannot reach.

3. Example:
   Suppose there are three states of the world at $t = 1$ and just two securities, 1 and 2, with payoff vectors $(1, 1, 0)$ and $(0, 1, 2)$ respectively.

   An agent's endowment at $t = 1$ is $w_{-0} = (2, 3, 0)$ (for states 1,2 and 3 respectively). The portfolio $z = (z_1, z_2)$ gives the agent contingent consumption of

$$\omega_{-0} + Dz = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

   Note that $M = \{w_{-0} + Dz : z \in R^2\}$ is a plane in R^3 passing through the point w_{-0} (i.e., this does not cover the entire R^3).

   Since $R_+^3 \not\subseteq M$, the market is incomplete.

Agent and its budget

1. The agent has endowment $w = (w_0, w_{-0}) \in \mathbb{R}^{L+1}_+$.
   The agent's utility function is (P1) continuous, (P2) strictly increasing and (P3) strictly quasiconcave, and its utility function $U : \mathbb{R}^{L+1}_+ \to \mathbb{R}$. For example,

   $$U(x_0, x_{-0}) = u(x_0) + \delta[\pi_1 v(x_1) + ... + \pi_L v(x_L)]$$

   Suppose $q \in R^s$ is the asset price vector. Then, the agent's budget set is

   $$B(q, \omega, D) = \left\{ \begin{array}{ll} x \in R^{L+1}_+ : & x_0 \leq \omega_0 - q \cdot z \\ & x_{-0} \leq \omega_{-0} + Dz \\ & \text{for some } z \in R^S \end{array} \right\}$$

   where q is the asset price vector, w is the endowment, and D is the payoff matrix.

   The agent maximises $U(x)$ subject to $x \in B(q, w, D)$.
   Then,

   $$\hat{x} \in arg \max_{x \in B(q,w,D)} U(x)$$

   is the agent's demand for contingent consumption and $\hat{z}$ such that

   $$\begin{aligned} &\hat{x}_0 \leq w_0 - q \cdot \hat{z}, \text{ and} \\ &\hat{x}_{-0} \leq w_{-0} + D\hat{z} \end{aligned}$$

   is the agent's demand for securities.

   Note: If U is strictly monotone (P2), then the inequalities above are equalities, since the agent will consume everything he can consume.

2. Let the agent's utility $U(x_0, x_1, x_2) = \ln x_0 + \frac{1}{2} \ln x_1 + \frac{1}{2} \ln x_2$.
   Let there be two securities with payoff vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively, with the prices $q = (1, 2)$ (for securities 1 and 2 respectively).
   Let the endowment be $w = (3, 0, 0)$ (in state 1, 2, 3 respectively).

   $$\text{Budget Set} = \left\{ \begin{array}{ll} x \in R^3_+ : & x_0 \leq 3 - z_1 - 2z_2 \\ & x_1 \leq z_1 + z_2 \\ & x_2 \leq z_2 \;\text{ for some } z = (z_1, z_2) \end{array} \right.$$

   The agent maximises $\ln(3 - z_1 - 2z_2) + \frac{1}{2} \ln(z_1 + z_2) + \frac{1}{2} \ln z_2$.
   Solution: $\bar{z}_1 = 0, \bar{z}_2 = \frac{3}{4}, \bar{x}_0 = \frac{3}{2}, \bar{x}_1 = \frac{3}{4}, \bar{x}_2 = \frac{3}{4}$.

Financial economy

1. A financial economy $\mathcal{F}$ consists of
   (a) a payoff matrix D, and
   (b) a set A of agents, each of whom has an endowment

   $$w^a = (w^a_0, w^a_{-0}) \in \mathbb{R}^{L+1}_+$$

   and a utility function

   $$U^a : \mathbb{R}^{L+1}_+ \to \mathbb{R}$$

2. A price $q^* \in \mathbb{R}^S$ is an equilibrium price of the financial economy if:

(a) for each agent a, there is $\hat{z}^a$ such that $D\hat{z}^a$ maximises $U^a(x)$ in $B(q^*, w^a, D)$ (i..e, everyone's portfolios maximise their utility), and

(b) $\sum_{a\in A} z^a = 0$ (i.e., buyers and sellers all match; no excess demand).

3. **Proposition:** Suppose that q^* is an equilibrium price of $\mathcal{F}$ and let $\hat{z}^a$ and $\hat{x}^a$ be agent a's asset and consumption demand respectively. Then, provided U^a obeys (P2), we have

$$\sum_{a\in A} \hat{x}^a = \sum_{a\in A} w^a$$

(i.e., total demands = total endowments).

Proof: For agent a, we have $\hat{x}_0^a = w_0^a - q^* \cdot \hat{z}^a$. Summing across a, we obtain

$$\begin{aligned}\sum_{a\in A} \hat{x}_0^a &= \sum_{a\in A} w_0^a - \sum_{a\in A} q^* \cdot \hat{z}^a \\ &= \sum_{a\in A} w_0^a - q^* \sum_{a\in A} \hat{z}^a \\ &= \sum_{a\in A} w_0^a \qquad \text{since } \sum_{a\in A} \hat{z}^a = 0\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{a\in A} \hat{x}_{-0}^a &= \sum_{a\in A} w_{-0}^a + \sum_{a\in A} D\hat{z}^a \\ &= \sum_{a\in A} w_{-0}^a + D\sum_{a\in A} \hat{z}^a \\ &= \sum_{a\in A} w_{-0}^a\end{aligned}$$

Invariance

1. Consider a financial economy and keep endowments and preferences fixed. Then, a change in securities that leaves the span unchanged does not change the equilibrium in an essential way (in other words, we can buy linear combinations of the securities to achieve the same consumption as before).

2. **Lemma:** Let $\{1, ..., S\}$ be a set of securities with no redundant securities and let D be its payoff matrix. Let there be another set of $|S|$ securities, with payoff matrix D' such that $Span(D) = Span(D')$. Then, there is an invertible $S \times S$ matrix K such that $DK = D'$ (i.e., K is all the linear transformations to change D to D').

$$\text{Example: Suppose } D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{aligned}D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= DK\end{aligned}$$

3. **Theorem:** Suppose q^* is an equilibrium price of $F(D)$. At that price, let $\hat{z}^a$ be agent a's equilibrium portfolio, which achieves consumption of $\hat{x}^a$. Then, the economy $F(D')$, where $D' = DK$ for some invertible K, has the equilibrium price q^*K. The equilibrium portfolio of agent a is $K^{-1}\hat{z}^a$ and his consumption is $\hat{x}^a$.

   Proof: Note that $\sum_{a\in A} \hat{z}^a = 0$ implies $\sum_{a\in A} K^{-1}\hat{z}^a = 0$. Furthermore, if $x \in B(q^*, w^a, D)$ and is achieved by z, then $x \in B(q^*, w^a, D')$ and is achieved by $K^{-1}z$, and vice-versa. Hence,

$$B(q^*, w^a, D) = B(q^*K, w^a, D').$$

   It follows that

$$\hat{x}^a = arg \max_{x\in B(q^*K, w^a, D')} U^a(x)$$

   and is achieved by $K^{-1}\hat{z}^a$.

   Since $x = (x_0, x_{-0}) \in B(q^*, w^a, D)$ and is achieved by z, we have that

$$x_0 \leq w_0^a - q^* \cdot z$$
$$x_{-0} \leq w_{-0}^a + Dz$$

   Given that $D' = DK$, we have that

$$x_0 \leq w_0^a - q^*K \cdot (K^{-1}z)$$
$$x_{-0} \leq w_{-0}^a + D'K^{-1}z$$

4. **Definition:** Security prices $q \in \mathbb{R}^S$ admit **arbitrage** if there is $\bar{z} \in \mathbb{R}^S$ such that $q \cdot \bar{z} \leq 0$ and $D\bar{z} \geq 0$, with their inequality strict.

   An equilibrium price **cannot admit arbitrage** because if q^* admits arbitrage then the problem

$$\max_{x\in B(q^*, w^a, D)} U^a(x)$$

   has no solution:

   (a) for any $x = (x_0, x_{-0}) \in B(q^*, w, D)$, the bundle $x + (-q \cdot \bar{z}, D\bar{z})$ is also in $B(q^*, w, D)$, and, by (P2), has a higher utility.

5. **Fundamental theorem:** $q^* \in \mathbb{R}^S$ admits no arbitrage iff there is $p = (p_1, ..., p_L) \gg 0$ (prices in each state), such that

$$q_s = p_1 d_{1s} + ... + p_L d_{Ls}$$

   More succinctly, $q = pD$.

Existence of equilibrium

1. The proof of the equilibrium in this context can be achieved using Cass' trick:

   (a) Let agent 1 use the full budget set $D(q^*, w_1)$ to make his choice.
   (b) Constrain agent i for $i = 2, ..., n$ to trade on the reduced budget set $B(q^*, w_i, D)$, where D is the payoff matrix.
   (c) Construct the excess demand $Z_D(q)$.

   We can verify that the consequent excess demand $Z_D(q)$ satisfies the five conditions, and hence, using Kakutani's theorem, we can get an equilibrium.
   We can also show that at equilibrium, agent 1 is also choosing in $B(q^*, w_1, D)$, hence we get the equilibrium we were looking for.

Lecture 19: General Equilibrium Theory 7: Optimality

General equilibrium and optimality

1. Now, we focus on optimality of equilibria.

   Definition: Given the economy $\mathcal{E} = (U^1, ..., U^n; w^1, ..., w^n)$, where utilities satisfy (P1) (P2) (P3), the n-tuple $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is an allocation is

$$\sum_{i=1}^{n} \mathbf{x}_i \leq \sum_{i=1}^{n} w^i$$

2. **Definition:** Given the economy $\mathcal{E} = (U^1, ..., U^n; w^1, ..., w^n)$, an allocation is:

   (a) Individually rational if for each j we have

$$U^j(\mathbf{x}_j) \geq U^j(w_j)$$

   (b) Weakly Pareto Optimal if there is no allocation $(\mathbf{y}_1, ..., \mathbf{y}_n)$ such that for each j,

$$U^j(\mathbf{y}_j) > U^j(\mathbf{x}_j)$$

   (c) Pareto Optimal if there is no allocation $(\mathbf{y}_1, ..., \mathbf{y}_n)$ such that for each j,

$$U^j(\mathbf{y}_j) \geq U^j(\mathbf{x}_j)$$

   and there exists an i such that

$$U^i(\mathbf{y}_i) > U^i(\mathbf{x}_i)$$

3. **Theorem:** If the consumers of an exchange economy have continuous and strictly monotone utilities, then an allocation is Pareto optimal iff it is weakly Pareto optimal.

   Proof: Assume that preferences are continuous and strictly monotone.
   The result is trivial for $n = 1$, so we assume $n > 1$.
   Let $(\mathbf{x}_1, ..., \mathbf{x}_n)$ be a weakly Pareto optimal allication. Now, suppose than an allocation $(\mathbf{y}_1, ..., \mathbf{y}_n)$ satisfies

$$U^j(\mathbf{y}_j) \geq U^j(\mathbf{x}_j)$$

   and

$$U^k(\mathbf{y}_k) > U^k(\mathbf{x}_k)$$

   for some k. By the continuity of U^k, there exists some $0 < \alpha < 1$ such that

$$U^k((1-\alpha)\mathbf{y}_k) > U^k(\mathbf{x}_k)$$

   Now, if we let

$$\mathbf{z}_i = \mathbf{y}_i + \frac{\alpha}{n-1}\mathbf{y}_k, \forall i \neq k$$

   and

$$\mathbf{z}_k = (1-\alpha)\mathbf{y}_k$$

   then $(\mathbf{z}_1, ..., \mathbf{z}_n)$ is an allocation. Moreover, by the strict monotonicity of the preferences, we see that

$$U^j(\mathbf{z}_j) \geq U^j(\mathbf{x}_j)$$

   holds for all j, contradicting the weak Pareto optimality of $(\mathbf{x}_1, ..., \mathbf{x}_n)$. This implies that $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is a Pareto optimal allocation.

4. Let $\mathcal{A}$ be the set of allocations (i.e., ways to redistribute among agents) where

$$\mathcal{A} = \{(\mathbf{x}_1, ..., \mathbf{x}_n) | \mathbf{x}_i \geq 0, \forall i = 1, ..., n \text{ and } \sum_{i=1}^{n} \mathbf{x}_i = \sum_{i=1}^{n} w^i\}$$

Clearly, we have that

(a) $\mathcal{A}$ is a convex, closed and bounded (and hence convex and compact) subset of $(\mathbb{R}^l)^n$ (recall that we have l goods and n agents).

The **set of all individually rational allocations** is denoted

$$\mathcal{A}_r = \{(\mathbf{x}_1, ..., \mathbf{x}_n) : U^j(\mathbf{x}_j) \geq U^j((w^j), \forall j = 1, ..., n\}$$

(b) Since $(w^1, ..., w^n) \in \mathcal{A}_r$, we have that $\mathcal{A}_r \neq \emptyset$.

(c) If each preference U^j is continuous, then $\mathcal{A}_r$ is a closed subset of $(\mathbb{R}^l)^n$ and hence $\mathcal{A}_r$ is a compact set.

In the case where each preference U^j is convex and continuous, then the set $\mathcal{A}_r$ is a (non-empty) convex and compact subset of $(\mathbb{R}^l)^n$.

(d) Continuity of preferences suffices to guarantee the existence of individually rational Pareto optimal allocations.

5. **Theorem:** If in an exchange economy with a finite number of consumers each consumer has continuous preferences, then individually rational Pareto optimal allocations always exist.

Proof (sketch):

(a) Start by introducing an equivalence relation on $\mathcal{A}_r$:

$$(\mathbf{z}_1, ..., \mathbf{z}_n) \sim (\mathbf{x}_1, ..., x_n)$$

if and only if

$$U^j(\mathbf{z}_j) = U^j(\mathbf{x}_j)$$

holds for all j. On the set

$$\mathcal{A}_r / \sim$$

there is a natural order that we denote by $\succ$.

(b) Recall that a (nonempty) set $\mathcal{C}$ of $\mathcal{A}$ is said to be a **chain** whenever every two elements of $\mathcal{C}$ are comparable. This means that

$$z \succ y \text{ or } y \succ z, \qquad \forall z, y, \in \mathcal{C}$$

(c) Now, let $\mathcal{C}$ be an arbitrary chain of $\mathcal{A}_r$. Then, it is possible to show that $\mathcal{C}$ ois bounded from above in $\mathcal{A}_r$, i.e., there exists some $x \in \mathcal{A}_r$ such that

$$x \succ c \in \mathcal{C}$$

(this uses the fact that $\mathcal{A}_r$ is compact).

(d) Zorn's Lemma tells us that there exists a maximal element $x \in \mathcal{A}_r$. Since there cannot exist a $y \in \mathcal{A}_r$ such that $y \sim x$, it follows that x is an individually rational Pareto optimal allocation.

(e) The proof of the theorem is now complete.

Core

1. **Definition:** Given the economy $\mathcal{E} = (U^1, ..., U^n; w^1, ..., w^n)$, a coalition is a nonempty subset of $A = \{1, ..., n\}$.
   There are $2^n - 1$ possible coalitions, since each member can either be in the subset or not, and we don't allow the coalition to be completely empty.

2. **Definition**: A coalition S improves upon an allocation $(\mathbf{x}_1, ..., \mathbf{x}_n)$ whenever there exists another allocation $(\mathbf{y}_1, ..., \mathbf{y}_n)$ such that
   (a) $\sum_{j\in S} \mathbf{y}_j = \mathbf{j} \in \mathbf{S} w^j$ (i.e., the reallocation is feasible by just reallocatin within the members of the coalition);
   (b) For each $j \in S$ we have
   $$U^j(\mathbf{y}_j) > U^j(\mathbf{x}_j)$$

3. **Definition:** An allocation is a **core allocation** if there does not exist a coalition that can improve upon it. The set of core allocations of the economy $\mathcal{E}$ is called the **core** of the economy and is denoted by $Core(\mathcal{E})$.

   Note: This is much more stringent than Pareto optimality since it means that no subset of the agents can reallocate among themselves and generate an improvement.

4. **Theorem:** Every core allocation is individually rational and weakly Pareto optimal (i.e., either a one-person coalition or the full coalition of all members cannot improve with a reallocation).
   Note: A one-person coalition can still reallocate by reallocating his wealth across the basket of goods in a different way.

   Let $(\mathbf{x}_1, ..., \mathbf{x}_n)$ be a core allocation. To see that $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is individually rational, note that if
   $$U^k(w_k) > U^k(\mathbf{x}_k)$$
   holds for some k, then the coalition S consisting of the k^{th} agent alone (i.e, $S = \{k\}$) can improve on the allocation. Hence it must be that
   $$U^i(\mathbf{x}_i) \geq U^i(w^i)$$
   holds for each i.

   To see that $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is also weakly Pareto optimal, let $(\mathbf{y}_1, ..., \mathbf{y}_n)$ be such that
   $$U^i(\mathbf{y}_i) > U^i(\mathbf{x}_i)$$
   This means that the grand coalition of all agents $A = (1, ..., n)$ can improve upon the allocation, which is impossible.

   Hence, the allocation $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is also weakly Pareto optimal.

Edgeworth box

1. We use the **Edgeworth Box** to illustrate various optimality properties of allocations.

2. Consider a two-consumer exhcange economy with commodity space $\mathbb{R}^2_+$. The total endowment is
   $$w = w^1 + w^2$$

3. We consider the first consumer with commodity space on the xy plane and the second consumer with commodity space on the st plane.
   (a) (see graphical notes)

(b) The point B (the origin of the st plane) corresponds to the point $w = w^1 + w^2$ in the xy plane.

(c) The point B (the origin of the xy plane) corresponds to the point $w = w^1 + w^2$ in the st plane.

(d) An allocation is an arbitrary point in the box determined by the z, y, s, t axes, referred to as the Edgeworth Box.

4. We have that:

   (a) An arbitrary Pareto optimal allocation corresponds to a point Q in the box for which the indifference curves of the two consumers are tangent.

   (b) The set of all Pareto optimal allocations (known as the **contract curve**) is represented by the curve joining the points A and B.
   If P denotes the initial allocation then the points inside the shaded lens correspond to the individually rational allocations. The points on the contract curve inside the darkened lens correspond to the core allocations (which in this case coincide with the set of all individually rational Pareto optimal allocations).

   (c) The Walrasian equilibria lie on the core part of the contract curve.

Cooperative game theory: First step

1. To prove the existence of core allocations, we introduce the notion of an n-person cooperative game with **non-transferable utility** (NTU).

   (a) Every exchange economy defines such a game.

   (b) Each payoff vector in the core of the associated n-person game corresponds to a core allocation in the exchange economy.

   (c) Scarf showed that balanced n-person games have a non-empty core, and concluded from this that neoclassical exchange economies have a non-empty core.

2. To establish Scarf's existence theorem, we need some preliminary discussion.

   Given a finite set of players $N = \{1, ..., n\}$, let

   $$\mathcal{N} = \{S \subseteq N : S \neq \emptyset\}$$

   be the set of all coalitions. A n-person cooperative game with NTU is

   $$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

   where

   $$\mathcal{P}(\mathbb{R}^n) = \{X \subseteq \mathbb{R}^n : X \neq \emptyset\}$$

   (see graphical illustration of example)

   Given the coalition S, the set $V(S) \subset \mathbb{R}^n$ is the set consisting of all payoff vectors that the coalition S can attain for its members.
   Notice that in the definition there is no mention of how the players get the payoffs, just that there is an exact description of what the payoffs are once the coalition is given.
   The most important notion is that the core of the game is the set of all vectors in $V(N)$ that no coalition can improve upon.

3. **Definition:** The **core** of a n-person game

   $$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

   is by definition

   $$Core(V) = \{\mathbf{x} \in V(N) | \not\exists S \in \mathcal{N} \text{ and } \mathbf{y} \in V(S) \text{ such that } y_i > x_i, \forall i \in S\}$$

4. **Definition:** $\forall S \in \mathcal{N}$ the function $\mathbf{1}_S : N \to \{0, 1\}$ is defined as

$$\mathbf{1}_S(i) = \begin{cases} 1 & \forall i \in S \\ 0 & \forall i \notin S \end{cases}.$$

5. **Definition:** A nonempty family $\mathcal{B}$ of $\mathcal{N}$ is **balanced** whenever there exist non-negative weights

$$\{w_S | S \in \mathcal{B}\}$$

such that

$$\sum_{S \in B} w_S \mathbf{1}_S = \mathbf{1}_N$$

6. **Definition (Bondareva):** An n-person game

$$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

is **balanced** whenever every balanced family $\mathcal{B}$ of the coalition satisfies

$$\bigcap_{S \in \mathcal{B}} V(S) \subseteq V(\mathcal{N})$$

Lecture 20: General Equilibrium Theory 8: Scarf's Theorems

Scarf's Theorem about games

1. Recap of key definitions:

Given a finite set of players $N = \{1, 2, ..., n\}$, let

$$\mathcal{N} = \{S \subseteq N : S \neq \emptyset\}$$

be the set of all coalitions. A n-person cooperative game with NTU is

$$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

where

$$\mathcal{P}(\mathbb{R}^n) = \{X \subseteq \mathbb{R}^n : X \neq \emptyset\}$$

2. **Definition:** The **core** of an n-person game

$$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

is by definition

$$Core(V) = \{\mathbf{x} \in V(N) | \nexists S \in \mathcal{N} \text{ and } \mathbf{y} \in V(S) \text{ such that } y_i > x_i, \forall i \in S$$

3. **Definition:** $\forall S \in \mathcal{N}$ the function $\mathbf{1}_S : N \to \{0, 1\}$ is defined as

$$\mathbf{1}_S(i) = \begin{cases} 1 & \forall i \in S \\ 0 & \forall i \notin S \end{cases}$$

4. **Definition:** A nonempty family $\mathcal{B}$ of $\mathcal{N}$ is said to be **balanced** whenever there exist nonnegative weights

$$\{w_S | S \in \mathcal{B}\}$$

such that

$$\sum_{S \mathcal{B}} w_S \mathbf{1}_S = \mathbf{1}_N$$

Example: Let $N = \{1, 2, ..., 10\}$.
We consider two coalitions:

$$S = \{1, ..., 6\} \text{ with weights } W_S$$
$$T = \{5, 6, ..., 10\} \text{ with weights } W_T$$

We require $w_S + w_T = 1$ and $w_S \mathbf{1}_s + w_T \mathbf{1}_T = \mathbf{1}_N$.
Then, we have $T \cap S = \{5, 6\}$.
Since $\{1, 2, 3, 4\} \in S$ but $\notin T$, $w_S = 1$.
Since $\{7, 8, 9, 10\} \in T$ but $\notin S$, $w_T = 1$.
But these weights are not usable for $\{5, 6\}$, so the coalitions are not balanced.

5. **Definition (Bondareva):** An n-person game

$$V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$$

is **balanced** when every balanced family $\mathcal{B}$ of the coalition satisfies

$$\bigcap_{S \in \mathcal{B}} V(S) \subseteq V(N)$$

6. **Theorem (Scarf):** If $V : \mathcal{N} \to \mathcal{P}(\mathbb{R}^n)$ is a balanced n-person game such that
   (a) each $V(S)$ is closed,
   (b) each $V(S)$ is comprehensive from below, i.e., $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \in V(S)$ imply $\mathbf{x} \in V(S)$,
   (c) $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in V(S)$ and $y_i = x_i, \forall i \in S$ imply $\mathbf{x} \in V(S)$, (i.e., agents only care about the values for members in their coalition)
   (d) each $V(S)$ is bounded from above in $\mathbb{R}^S$, i.e., for each coalition S there exists some $M_s > 0$ satisfying $x_i < M_s$ for all $\mathbf{x} \in V(S)$ and $\forall i \in S$ (this means that gains are bounded; "no free lunch").

   then, the n-person game has a nonempty core.

   Proof: Omitted from these notes. See Lecture 20 notes.

Scarf's Theorem about exchange economies

1. **Theorem (Scarf)**: Every exchange economy whose consumers' preferences are represented by continuous and quasi-concave utility functions has a non-empty compact core.
   Recall that the economy is $\mathcal{E} = (U^1, ..., U^n, w^1, ..., w^n), w^i \in \mathbb{R}^l$.

   Proof (sketch):
   Consider an exchange economy with n consumers such that the preference of each consumer i is represented by a continuous quasi-concave utility function.

   First, we show that the core is non-empty.
   WE define an n-person game V by

   $$V(S) = \{(x_1, ..., x_n) \in \mathbb{R}^n | \exists \text{ an allocation } (\mathbf{y}_1, ..., \mathbf{y}_n) \text{ such that } \sum_{i \in S} \mathbf{y}_i = \sum_{i \in S} w^i \text{ and } U^j(\mathbf{y}_j) \geq x_j \in S\}$$

   (i.e., the reallocation is feasible within the coalition; $\mathbf{x}_j$ are the payoffs).
   We claim that the n-person game V satisfies the properties listed in Scarf's theorem about games. To see this, note that properties (b) and (c) are trivially true, and (d) follows immediately from the fact that each utility function U^j (as a continuous function) is bounded on the compact set $[0, w]$.
   $V(S)$ is closed.
   Next, we show that the n-person game V is balanced.
   Therefore, we can conclude, by Scarf's theorem, that the Core is not empty.
   The next important thing is to show that the core is compact.

Optimality and Decentralisation

1. The classical intuition, that decentralized competitive markets produce out of the self-interested behavior of economic agents an optimal distribution of resources, dates back at least to Adam Smith's "invisible hand".

2. An interpretation of GE is that his intuition is made precise in the two Welfare Theorems of K. J. Arrow and G. Debreu.

3. Competitive allocations are realized in a decentralised and non-cooperative manner, since each consumer's demand derives his/ her utility maximisation subject only to her budget constraint, without knowledge of the demands or concern for the tastes of other consumers. In this setting, prices serve as signals of scarcity and agents interact with the market rather than with each other, implicit in both the core and Pareto optimality.

4. It is therefore quite surprising that every competitive allocation is Pareto optimal (the First Welfare Theorem) and that every Pareto optimal allocation can be achieved in a decentralised fashion as a competitive allocation subject to income transfers (the Second Welfare Theorem).

5. For the Core the results are even more striking. Every Walrasian allocation is in the core (a stronger version of the first welfare theorem) and "in the limit" only core allocations are Walrasian. The latter theorem is called a Core Equivalence Theorem in the economics literature, and was originally proven by F. Y. Edgeworth for an exchange economy with two goods and identical agents. Edgeworth's model is the first economic model that uses a potential infinity of consumers to express the notion of perfect competition, where each household has a negligible influence in determining equilibrium prices.

6. Edgeworth's construction was elegantly extended by G. Debreu and H. E. Scarf to arbitrary exchange economies, using the notion of replicas of a given economy which formalises the idea of "in the limit" when there are many agents on the market.

7. H. E. Scarf in his paper on the nonemptiness of the core for an n-person game observed that his result together with the Debreu-Scarf core equivalence theorem provides a new proof of the existence of Walrasian equilibria. This proof is independent of the notions of demand and supply functions.

Lecture 21: General Equilibrium Theory 9: Welfare Theorems

Optimality and Decentralisation

1. We are again considering the economy

$$\mathcal{E} = (U^1, ..., U^n; w^1, ..., w^n)$$

Definition: An allocation $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is a Walrasian (or a competitive) equilibrium whenever there exists some price $\mathbf{p} \neq 0$ such that $\mathbf{x}_j \in B(\mathbf{p}, \mathbf{p} \cdot w_j)$ with

$$U^j(\mathbf{x}) > U^j(\mathbf{x}_j) \implies \mathbf{p} \cdot \mathbf{x} > \mathbf{p} \cdot w_j$$

There is a quasiequilibrium whenever there exists some price $\mathbf{p} \neq 0$ such that

$$U^j(\mathbf{x}) \geq U^j(\mathbf{x}_j) \implies \mathbf{p} \cdot \mathbf{x} > \mathbf{p} \cdot w_j$$

A Walrasian equilibrium is always a Pareto optimal allocation (**First Welfare Theorem**).

2. **Theorem (Arrow):** If in an exchange economy preferences are strictly convex, then every Walrasian equilibrium allocation is Pareto optimal.

Proof: Let $(\mathbf{x}_1, ..., \mathbf{x}_n)$ be a Walrasian equilibrium allocation with respect to a price $\mathbf{p}$. Assume that there is another allocation $(\mathbf{y}_1, ..., \mathbf{y}_n)$ such that for each j we have

$$U^j(\mathbf{y}_k) \geq U^j(\mathbf{x}_j)$$

and there exists an i such that

$$U^i(\mathbf{y}_i) > U^i(\mathbf{x}_i)$$

Clearly,

$$U^i(\mathbf{x}) > U^i(\mathbf{x}_i) \implies \mathbf{p} \cdot \mathbf{y}_i > \mathbf{p} \cdot w_i \geq \mathbf{p} \cdot \mathbf{x}_i$$

is true for at least one i. From

$$w = \sum^n \mathbf{x}_i = \sum^n \mathbf{y}_i$$

we have

$$\mathbf{p} \cdot w = \sum^n \mathbf{p} \cdot \mathbf{x}_i = \sum^n \mathbf{p} \cdot \mathbf{y}_i$$

Therefore, $\mathbf{p} \cdot \mathbf{y}_k \geq \mathbf{p} \cdot \mathbf{x}_k$ must hold for at least one k, and thus

$$\mathbf{y}_k \neq \mathbf{x}_k$$

By strict convexity of utility it follows that

$$U^k(\frac{1}{2}\mathbf{y}_k + \frac{1}{2}\mathbf{x}_k) > U^k(\mathbf{x}_k)$$

and therefore we have

$$\frac{1}{2}\mathbf{p} \cdot \mathbf{y}_k + \frac{1}{2}\mathbf{p} \cdot \mathbf{x}_k > \mathbf{p} \cdot w_k \geq \mathbf{p} \cdot \mathbf{x}_k$$

From this, we see

$$\frac{1}{2}\mathbf{p} \cdot \mathbf{y}_k > \frac{1}{2}\mathbf{p} \cdot \mathbf{x}_k$$

so we have

$$\mathbf{p} \cdot \mathbf{y}_k > \mathbf{p} \cdot \mathbf{x}_k$$

which is impossible. Therefore, the allocation $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is Pareto optimal.

3. We also have a stronger theorem:

   Theorem (Arrow): Every Walrasian equilibrium allocation is a core allocation and hence it is weakly Pareto optimal.

   Proof: Let $(\mathbf{x}_1, ..., \mathbf{x}_n)$ be a Walrasian equilibrium allocation with respect to a price $\mathbf{p}$ so that $\mathbf{p} \neq 0$ such that $x_i \in B(\mathbf{p}, \mathbf{p} \cdot w_j)$ with

$$U^i(\mathbf{x}) > U^i(\mathbf{x}_i) \implies \mathbf{p} \cdot \mathbf{x} > \mathbf{p} \cdot w_i$$

   We shall establish that $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is a core allocation. To this end, assume by way of contradiction that there exists an allocation $(\mathbf{y}_1, ..., \mathbf{y}_n)$ and a coalition S such that

$$\sum_{i \in S} \mathbf{y}_i = \sum_{i \in S} w_i \qquad \text{(Eq 1)}$$
$$U^i(\mathbf{y}_i) > U^i(\mathbf{x}_i), \forall i \in S$$

   Then, $\mathbf{p} \cdot \mathbf{y}_i > \mathbf{p} \cdot w_i$ must hold for each $i \in S$, and consequently

$$\mathbf{p} \cdot (\sum_{i \in S} \mathbf{y}_i) = \sum_{i \in S} \mathbf{p} \cdot \mathbf{y}_i > \sum_{i \in S} \mathbf{p} \cdot w_i = \mathbf{p} \cdot (\sum_{i \in S} w_i)$$

   which contradicts (Eq 1). Therefore, $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is in the core.

4. **Definition:** An allocation $(\mathbf{x}_1, ..., \mathbf{x}_n)$ in an exchange economy is said to be **supported** by some price $\mathbf{p} \neq 0$ whenever

$$U^j(\mathbf{x}) \geq U^j(\mathbf{x}_j) \implies \mathbf{p} \cdot \mathbf{x} \geq \mathbf{p} \cdot \mathbf{x}_j$$

It is easy to see that when preferences are monotone, supporting prices are always positive.

5. Notion of supporting price: A linear functional f on a vector space $\mathbb{R}^n$ supports a set $A \subset \mathbb{R}^n$ at some point $a \in A$ whenever

$$f(x) \geq f(a), \forall x \in A$$

Theorem (Separating Hyperplane): Every pair A and B of non-empty and disjoint convex subsets of some $\mathbb{R}^l$ can be separated by a hyperplane. This means that there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^l$ and a constant c such that

$$\sup_{a \in A} \mathbf{p} \cdot \mathbf{a} \leq c \leq \inf_{b \in B} \mathbf{p} \cdot \mathbf{b}$$

Theorem: If $x \in \mathbb{R}^n \backslash A$ and A is convex, then

$$\inf_{y \in A} ||x - y|| = \min_{x \in A} ||x - y||$$
$$\implies y^* = arg \min_{y \in A} ||x - y|| \text{ is unique.}$$

6. **Theorem:** If in an exchange economy preferences are strictly convex and monotone, every weakly Pareto optimal allocation (and hence every Pareto optimal allocation) is supported by a non-zero price (**Second Welfare Theorem**).
(i.e., we can organise a set of transfers to deliver the Pareto optimal as an equilibrium).

Proof: Let the allocation $(\mathbf{x}_1, ..., \mathbf{x}_n)$ be weakly Pareto optimal. For each i, we have that

$$F_i = \{\mathbf{x} \in \mathbb{R}^l_+ | U^i(\mathbf{x}) > U^i(\mathbf{x}_i)\}$$

By non-satiation, each F_i is non-empty and convex. Now, we consider the convex set

$$F = F_1 + F_2 + ... + F_n - w$$

where the total endowment $w = \sum_n w_i$.

Note that $\mathbf{0} \notin F$. In fact, if $\mathbf{0} \in F$ then there exists an allocation $(\mathbf{s}_1, ..., \mathbf{s}_n)$ such that both

$$U^i(\mathbf{s}_i > U^i(\mathbf{x}_i) \text{ and } \mathbf{0} = s_1 + s_2 + ... + s_n - w$$

which contradicts the fact that $(\mathbf{x}_1, ..., \mathbf{x}_n)$ is weakly Pareto optimal.

Therefore, by the Hyperplane Separation Theorem, there exists $\mathbf{p}$ such that

$$\mathbf{p} \cdot \mathbf{y} \geq 0, \forall \mathbf{y} \in F$$

Now, we prove that the price $\mathbf{p}$ supports the allocation $(\mathbf{x}_1, ..., \mathbf{x}_n)$. We assume that $U^r(\mathbf{x}) > U^r(\mathbf{x}_r)$ for some r. We know that for each i there is a $z_i \in \mathbb{R}^n_+$ such that

$$U^i(\mathbf{z}_i) > U^i(\mathbf{x}_i)$$

then we have by strict convexity (i.e., based on assumption (P3)) that for $i \neq r$

$$U^i(t\mathbf{z}_i + (1-t)\mathbf{x}_i) > U^i(\mathbf{x}_i)$$

and

$$U^r(t\mathbf{z}_r + (1-t)\mathbf{x}) > U^r(\mathbf{x}_r)$$

and therefore

$$\sum_{i\neq r}^{n}[t\mathbf{z}_r + (1-t)\mathbf{x}_i] + t\mathbf{z}_r + (1-t)\mathbf{x} - w \in F$$

and this implies that for all $t \in [0,1]$

$$t\sum_{i\neq r}^{n}\mathbf{p}\cdot\mathbf{z}_i + (1-t)\sum_{i\neq r}^{n}\mathbf{p}\cdot\mathbf{x}_i + t\mathbf{p}\cdot\mathbf{z}_r + (1-t)\mathbf{p}\cdot\mathbf{x} \geq \mathbf{p}\cdot w$$

Taking the limit when $t \to 0$, we have

$$\sum_{i\neq r}^{n}\mathbf{p}\cdot\mathbf{x}_i + \mathbf{p}\cdot\mathbf{x} \geq \mathbf{p}\cdot w = \sum_{i\neq r}^{n}\mathbf{p}\cdot\mathbf{x}_i + \mathbf{p}\cdot\mathbf{x}_r$$

or

$$\mathbf{p}\cdot\mathbf{x} \geq \mathbf{p}\cdot\mathbf{x}_r$$

and this means that $\mathbf{p}$ supports the allocation $(\mathbf{x}_1, ..., \mathbf{x}_n)$.

Constrained Pareto Optimality

1. Recap: A financial economy $\mathcal{F}$ consists of
   (a) A payoff matrix D
   (b) A set A of agents, each of whom has an endowment

   $$w^a = (w_0^a, w_{-0}^a) \in R_+^{L+1}$$

   and a utility function

   $$U^a : \mathbb{R}_+^{L+1} \to \mathbb{R}$$

   The agent's budget set is

   $$B(q, w, D) = \{x \in \mathbb{R}_+^{L+1} : x_0 \leq w_0 - q\cdot z,\ x_{-0} \leq w_{-0} + D_z \text{ for some } z \in R^S\}$$

   The agent maximises $U(x)$ subject to x in $B(q, w, D)$.

   A price q^* in R^S is an equilibrium price of this financial economy if:
   (a) for each agent a, there is $\hat{z}^a$ such that $D\hat{z}^a$ maximises $U^a(x)$ in $B(q^*, w^a, D)$ and
   (b) $\sum_{a\in A}\hat{z}^a = 0$. The allocation $\{x^a\}_{a\in A}$ is feasible if $\sum_{a\in A} x^a = \sum_{a\in A} w^a$.

   We know that this type of equilibrium always has an equilibrium, but the question is if the equilibrium is a good one.

2. Definition: An allocation is a constrained feasible allocation if it is feasible and there exists $\{z^a\}_{a\in A}$ such that $\sum_{a\in A} z^a = 0$ and

   $$x_{-0}^a = w_{-0}^a + Dz^a$$

3. : An allocation $\{x^a\}_{a\in A}$ is constrained Pareto optimal if there does not exist a constrained feasible allocation $\{\bar{x}^a\}_{a\in A}$ that is Pareto superior, i.e.,

   $$U^a(\bar{x}^a) \geq U^a(x^a)$$

   for all a, and the equality is strict for at least one agent.

4. **Theorem (First Welfare Theorem for Financial Economies):** Suppose that U^a obeys (P2) for all a. Then, every equilibrium allocation is constrained Pareto optimal.

   Proof: Let q^* be the equilibrium price and $\{\hat{x}^a\}_{a\in A}$ the equilibrium allocation. Assume that $\{\bar{x}^a\}_{a\in A}$ is a Pareto superior constrained feasible allocation. For some agent $\tilde{a}$, we have $U^{\tilde{a}}(\bar{x}^{\tilde{a}}) > U^{\tilde{a}}(\hat{x}^{\tilde{a}})$.
   So, the bundle $\bar{x}^{\tilde{a}}$ cannot be in $\tilde{a}$'s budget set; if it were, agent $\tilde{a}$ would have chosen this bundle instead of $\hat{x}^{\tilde{a}}$. Thus,

   $$\bar{x}_0^{\tilde{a}} > w_0^{\tilde{a}} - q^* \cdot \bar{z}^{\tilde{a}}$$

   Suppose $\bar{z}^a$ is the trade that achieves $\bar{x}^a$. We claim that

   $$\bar{x}_0^a \geq w_0^a - q^* \cdot \bar{z}^a$$

   Suppose not, then there is $\epsilon > 0$ such that $\bar{x}_0^a + \epsilon < w_0^a - q^* \cdot \bar{z}^a$. So, $\tilde{x}^a = \bar{x}^a + (\epsilon, 0, ..., 0)$ is in $B(q^*, w, D)$ and

   $$U^a(\tilde{x}^a) > U^a(\bar{x}^a) \geq U^a(\hat{x}^a)$$

   This cannot happen since $\hat{x}^a$ is a's demand. Summing across a, we obtain

   $$\begin{aligned} sum_{a\in A}\bar{x}_0^a &> \sum_{a\in A} w_0^a - q^* \cdot (\sum_{a\in A} \bar{z}^a) \\ &= \sum_{a\in A} w_0^a \end{aligned}$$

   which is a contradiction.

5. We can extend the framework to model multiple trading dates. In general:
   (a) In financial markets with many dates, the equilibrium exists generically (i.e., with probability 1)
   (b) In financial markets with many dates, the equilibrium is not even constrained Pareto optimal.

Lecture 22: Cooperative Game Theory: Bargaining 1

Cooperative Game Theory

1. In general, from the intuitive point of view, when we discuss cooperative behaviour we consider situations where players can commit to behave in a way that is socially optimal.
   The main issue is how to share the benefits arising from cooperation.
   Important elements in this approach are the different subgroups of players, i.e., coalitions, and the set of outcomes that each coalition can get regardless of what the players outside the coalition do.

   When discussing the equilibria in noncooperative games, we were concerned about whether a given strategy profile was self-enforcing or not, in the sense that no player had incentives to deviate.
   Now, we assume that the players can make binding agreements, and our focus (instead of self-enforceability) is the payoffs that the coalition can achieve, and notions like fairness and equity.

Non-transferable utility games (NTU games)

1. The main source of generality comes from the fact that although binding agreements between the players are implicitly assumed to be possible, utility is not transferable across players.
2. Definition: If $N = \{1, 2, ...n\}$ (i.e., the set of players), then

   $$\mathcal{N} = \{S \subseteq N : S \neq \emptyset\}$$

   is the set of all coalitions (i.e., non-empty subsets of N; there are $2^n - 1$ coalitions).

Given a set $S \in \mathcal{N}$ and a set $A \in \mathbb{R}^S$, it is **comprehensive** if, for each $x, y \in R^S$ such that

$$x \in A \text{ and } y \leq x \implies y \in A$$

where A is the set consisting of all payoff vectors that the coalition S can attain for its members. (see graphical illustration)

Given $A \in \mathbb{R}^S$, its comprehensive hull is the smallest comprehensive set which contains A.

3. **Definition:** An n-player nontransferable utility game (NTU-game) is a pair (N, V), where N is the set of players and V is a function that assigns to each coalition $S \in \mathcal{N} = \{S \subseteq N : S \neq \emptyset\}$, a set $V(S) \subset \mathbb{R}^S$ such that
   (a) $V(S) \subset \mathbb{R}^S$ is a closed nonempty set (i.e., V is the set of all possible payoffs that the coalition can guarantee for its members, and the values of V cannot become arbitrarily large).
   (b) $V(S) \subset \mathbb{R}^S$ is comprehensive (i.e., if the coalition can reach a level of payoffs, they can reach anything below that). Moreover, for each $i \in N = \{1, 2, ..., n\}$, for some $v_i \in \mathbb{R}$,
   $$V(\{i\}) = (-\infty, v_i]$$
   (this is the utility that can be achieved as a single person "coalition")
   (c) The set
   $$V(S) \cap \{\mathbf{y} \in \mathbb{R}^S | \forall i \in S, y_i \geq v_i\} \subset \mathbb{R}^S$$
   is bounded.

4. **Definition:** Let (N, V) be a NTU-game. Then the vectors in $\mathbb{R}^n$ are called allocations. An allocation $x \in \mathbb{R}^n$ is **feasible** if there exists a partition $S_1, ..., S_k$ such that $\forall i \in \{1, 2, ..., k\}$ and for any $y \in V(S_i)$ and for any $l \in S_i$ we have $x_l = y_l$
   (i.e., we can split the aggregate allocation to get the exact levels of allocation required).

5. Notes: The main objective of the theoretical analysis in this field is to find appropriate rules for choosing feasible allocations for the general class of NTU-games. These rules are referred to as *solutions* and aim to select allocations that have desirable properties according to different criteria such as equity, fairness, and stability.
   (a) If a solution selects a single allocation for each game, it is commonly referred to as an *allocation rule*.
   (b) We discuss the two most relevant subclasses of NTU-games: bargaining problems and games with transferable utility (TU-games).

Bargaining

1. In a bargaining problem, there is a set of possible allocations, the feasible set F, and one of them has to be chosen by the players. Importantly, all the players have to agree on the chosen allocation; otherwise, the realised allocation is d_i, the **disagreement point**.

2. **Definition:** An N-player bargaining problem is a pair (F, d), whre
   (a) Feasible set: F is the comprehensive hull of a compact and convex set of $\mathbb{R}^N$
   (b) Disagreement point: d is an allocation in F. We assume that there is some $x \in F$ such that $x > d$ (i.e., d is not the best allocation).
   (see graphical illustration).

3. **Note:** An N-player bargaining problem can be seen as a nontransferrable utility game (N, V), where $V(N) = F \subset \mathbb{R}^N$ and for any set $S \in \mathcal{N} = \{S \subset N : S \neq \emptyset\}$ we set

$$V(S) = \{\mathbf{y} \in \mathbb{R}^S | \forall i \in S, d_i \geq y_i\}$$

(i.e., a smaller coalition than the full grand coalition can only achieve at best something up to the disagreement point, or lower).

4. Hence, given a bargaining problem (F, d):
   (a) The feasible set represents the utilities the players get from the outcomes associated with the available agreements.
   (b) The disagreement point delivers the utilities in the case in which no agreement is reached.
   (c) The assumptions on the feasible set are mathematically convenient, and natural and not too restrictive.
   (d) The convexity assumption can be interpreted as the ability of the players to choose lotteries over the possible agreements, with the utilities over lotteries being derived by means of Von Neumann and Morgenstern utility functions.

5. We use the symbol $\mathcal{B}^N$ to denote the set of all n-player bargaining problems. If $(F, d) \in \mathcal{B}^N$, then we set

$$F_d = \{\mathbf{x} \in F | \forall i \in N, d_i \leq x_i\}$$

(i.e., the set for which there is gain from reaching agreement).

6. We now discuss several solution concepts for n-player bargaining problems. More precisely, we study *allocation rules.*

Definition: An allocation rule for a n-player bargaining problem is a map

$$\varphi : \mathcal{B}^N \to \mathbb{R}^N$$

such that for each $(F, d) \in \mathcal{B}^N$ we have

$$\varphi(F, d) \in F_d$$

(i.e., the vector must be inside F_d).

7. **Theorem (Nash, 1950):**
   (a) Pareto Efficiency (EFF): $\varphi(F, d)$ is a Pareto optimal allocation.
   (b) Symmetry (Sym): The solution is invariant under permutation (basically, we can swap around the players arbitrarily). More formally, if π is a permutation, we set $\mathbf{x} \in \mathbb{R}^N$ where

$$\mathbf{x}_i^\pi \stackrel{def}{=} x_{\pi(i)}$$

   and the bargaining problem is symmetric if, for each permutation π, we have that $d^\pi = d$ and $\mathbf{x} \in F \implies \mathbf{x}^\pi \in F$.
   Note that for symmetry to be satisfied, d must also be symmetric, i.e., it lies on the 45-degree line.

   (c) Invariance (Cat): If $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies $f_i(x)a_i x_i + b_i, \forall i$ (i.e., an affine transformation), then

$$\varphi(f(F), f(d)) = f(\varphi(F, d))$$

   (d) Independence of irrelevant alternatives (IIA): If (F, d) and $(\hat{F}, d) \in \mathcal{B}^N$ with $\hat{F} \subset F$, then we have

$$\varphi(F, d) \in \hat{F} \implies \varphi(F, d) = \varphi(\hat{F}, d)$$

   (see graphical illustration). Basically, if the solution of F is also in the smaller set $\hat{F}$, and $d = \hat{d}$, then both allocation rules are identical.

Note that the four properties above are very appealing; the most controversial is IIA.

8. Given $(F, d) \in \mathcal{B}^N$, we define $g^d : \mathbb{R}^N \to \mathbb{R}$ by

$$g^d(x) = \prod_{i \in N} (x_i - d_i)$$

Note: We don't have to use absolute values because if we take $x \in F_d$, then x_i is always greater than d_i and we can interpret $(x_i - d_i)$ as a distance and g^d as a product of distances.

Theorem: Let $(F, d) \in \mathcal{B}^N$, then there exists a unique $z \in F_d$ that maximises the function g^d over the set F_d (i.e., a unique maximum).

Proof: Since g^d is continuous over a compact set, there is a max for g^d. We assume (for proof by contradiction) what there are two $z, \hat{z}$ where

$$g^d(z) = g^d(\hat{z}) = \max_{x \in F_d} g^d(x)$$

We have, for each i,

$$z_i > d_i \text{ and } \hat{z}_i > d_i$$

and

$$\bar{z} = \frac{1}{2}z + \frac{1}{2}\hat{z} \in F_d$$

Notice that since the ln function is strictly concave, we have

$$\begin{array}{rcc} \ln g^d(\overline{z}) & = & \sum_{i \in N} \ln(\overline{z}_i - d_i) \\ & = & \sum_{i \in N} \ln(\frac{z_i - d_i}{2} + \frac{\widehat{z}_i - d_i}{2}) \\ & > & \sum_{i \in N} \left(\frac{1}{2}\ln(z_i - d_i) + \frac{1}{2}\ln(\widehat{z}_i - d_i)\right) \\ & = & \frac{1}{2}\ln(g^d(z)) + \frac{1}{2}\ln(g^d(\widehat{z})) \\ & = & \ln(g^d(z)) \end{array}$$

and therefore

$$g^d(\bar{z}) > g^d(z)$$

which is a contradiction, since we earlier assumed that there are two unique maxima.

9. **Definition:** The Nash solution $\mathcal{NA}((F, d))$ is defined for any $(F, d) \in \mathcal{B}^N$m as it follows

$$\mathcal{NA}((F, d)) = g^d(z) = arg \max_{x \in F_d} g^d(x)$$

Note: We use the notation $\mathcal{NA}((F, d))$ as the input to $\mathcal{NA}$ is a single tuple/ pair of the form (F, d), and not two separate inputs, F and d.

Given $(F, d) \in \mathcal{B}^N$, the Nash solution selects the unique allocation in F_d that maximises the product of gains of players with respect to the disagreement point d (note that this is equivalent to maximising the sum of logs of gains).

10. **Theorem:** Let $(F, d) \in \mathcal{B}^N$, and let $z = \mathcal{NA}((F, d))$. For each $x \in \mathbb{R}^n$, let

$$h(x) = \sum_{i \in N} \prod_{j \neq i} (z_j - d_j)x_i$$

then for each $x \in F$ we have

$$h(x) \leq h(z)$$

Proof: Suppose that there is $x \in F$ with $h(x) > h(z)$. For each $\epsilon \in (0,1)$, we set $x^\epsilon = (\epsilon x + (1-\epsilon)z) \in F$.

We observe that if ϵ is small enough, then $x^\epsilon \in F_d$. We have that

$$\begin{aligned} g^d(x^\epsilon) &= \prod_{i \in N} (z_i - d_i + \epsilon (x_i - z_i)) \\ &= \begin{pmatrix} \prod_{i \in N} (z_i - d_i) + \\ \epsilon \sum_{i \in N} \prod_{j \neq i} (z_j - d_j)(x_i - z_i) + \\ \sum_{i=2} \epsilon^i f(x, z, d) \end{pmatrix} \\ &= g^d(z) + \epsilon (h(x) - h(z)) + \sum_{i=2} \epsilon^i f(x, z, d) \end{aligned}$$

and if ϵ is small enough then

$$g^d(x^\epsilon) > g^d(z) = \max_{x \in F_d} g^d(x)$$

which is absurd (i.e., a contradiction).

Lecture 23: Cooperative Game Theory: Bargaining II

Nash solution

1. **Definition**: The Nash solution $\mathcal{NA}(F,d)$ is defined for any $(F,d) \in \mathcal{B}^N$ as it follows

$$\mathcal{NA}((F,d)) = arg \max_{x \in F_d} g^d(x)$$

Given $(F,d) \in \mathcal{B}^N$, the Nash solution selects the unique allocation in F_d that maximises the product of gains of players with respect to d, the disagreement point.
Notes:

(a) $\mathcal{B}^N$ denotes the set of all n-player bargaining problems.

(b) We assume that F_d is never empty.

2. **Theorem**: Let $(F,d) \in \mathcal{B}^N$, and let $z = \mathcal{NA}((F,d))$. For each $x \in \mathbb{R}^N$, let

$$h(x) = \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) x_i$$

Note: Since z is a specific point that is known, $(z_j - d_j)$ is essentially a constant, so $h(x)$ is a polynomial in x of degree n.

Then, for each $x \in F$,

$$h(x) \leq h(z)$$

Proof: Suppose that there is $x \in F$ with $h(x) > h(z)$. For each $\epsilon \in (0,1)$ we set $x^\epsilon = (\epsilon x + (1-\epsilon)z) \in F$. We observe that if ϵ is small enough then $x^\epsilon \in F_d$. We have

$$g^d(x^\epsilon) = \prod_{i \in N} (z_i - d_i + \epsilon(x_i - z_i))$$

Note that if we perform the product we obtain the sum of 2^N elements. Each element is the product of N factors where each factor is of the form $z_i - d_i$ or $\epsilon(x_i - z_i)$ (and hence we have 2^N elements), and all indices appear once and only once in each summand.

This implies that if we factor out each summand that contains only the first power of ϵ, we get

$$\epsilon(\prod_{j\neq 1}(z_j-d_j)(x_1-z_1)+...+\prod_{j\neq N}(z_j-d_j)x_N-z_N))$$

We also observe that in the sum there are elements that multiply ϵ^i for each $i=2,...,n$. Therefore, we can write

$$\begin{aligned}
g^d(x^\epsilon) &= \prod_{i\in N}(z_i-d_i+\epsilon(x_i-z_i)) \\
&= (\prod_{i\in N}(z_i-d_i)+\epsilon\sum_{i\in N}\prod_{j\neq i}(z_j-d_j)(x_i-z_i)+\sum_{i=2}e^i f_i(x,z,d) \\
&= g^d(z)+\epsilon(h(x)-h(z))+\sum_{i=2}e^i f_i(x,z,d)
\end{aligned}$$

where, if ϵ is small enough, we have

$$g^d(x^\epsilon) > g^d(z) = \max_{x\in F_d} g^d(x)$$

which is absurd.

3. **Theorem:** The Nash solution $\mathcal{NA}((F,d))$ which is defined for any $(F,d)\in\mathcal{B}^N$ as

$$\mathcal{NA}((F,d)) = arg\max_{x\in F_d} g^d(x)$$

satisfies:

(a) Pareto Efficiency (EFF) (i.e., at the frontier)
(b) Symmetry (Sym) (i.e., on the 45-degree line)
(c) Invariance (Cat)
(d) Independence of irrelevant alternatives (IIA).

Proof:

(a) Pareto Efficiency (EFF):

Assume, by way of contradiction, that $\mathcal{NA}((F,d)) = (\hat{x}_1,...\hat{x}_N)$ is not Pareto optimal. Therefore, there exists an allocation $x=(x_1,...,x_N)$ such that $x_i\geq\hat{x}_i$ for any i and there exists a j such that $x_j>\hat{x}_j$.

Since $\mathcal{NA}((F,d))\in F_d$ it follows that such $x=(x_1,...,x_N)\in F_d$. But, for any i,

$$(x_i-d_i)\geq(\hat{x}_i-d_i)$$

and

$$(x_j-d_j)>(\hat{x}_j-d_j)$$

therefore

$$\begin{aligned}
g^d(x) &= \prod_{i\in N}(x_i-d_i) \\
&> \prod_{i\in N}(\hat{x}_i-d_i) \\
&= arg\max_{y\in F_d} g^d(y)
\end{aligned}$$

which is absurd (i.e., a contradiction) since the Nash solution is already the $arg\max$.

(b) Symmetry (Sym): This is trivial.

(c) Invariance (Cat):

We set $f : \mathbb{R}^N \to \mathbb{R}^N$ with

$$f_i(x) = a_i x_i + b_i, \forall i, \text{ and } \forall x \in \mathbb{R}^N \text{ and } a_i > 0$$

Note: We require $a_i > 0$ so that we preserve the signs, but we do not require $b_i > 0$.

Let $\hat{y} = \mathcal{NA}(f(F), f(d))$, then there must exist $\hat{x} \in F_d$ such that $f(\hat{x}) = \hat{y}$.

We claim that $\hat{x} = \mathcal{NA}((F, d))$. Let's assume, by way of contradiction that it is false.

Then, there exists $\tilde{x} \in F_d$ such that $g^d(\tilde{x}) > g^d(\hat{x})$ and we can write (since each $a_i > 0$):

$$\begin{aligned} g^d(\tilde{x}) &= \prod_{i \in N}(\tilde{x}_i - d_i) > \prod_{i \in N}(\hat{x}_i - d_i) = g^d(\hat{x}) \\ \implies & \prod_{i \in N}(a_i\tilde{x}_i - a_i d_i > \prod_{i \in N}(a_i\hat{x}_i - a_i d_i) \\ & = \\ & \prod_{i \in N}(a_i\tilde{x}_i + b_i - a_i d_i - b_i) > \prod_{i \in N} a_i\hat{x}_i + b_i - a_i d_i - b_i) \\ & = \\ & \prod_{i \in N}(f_i(\tilde{x}_i)_i - f_i(d_i)) > \prod_{i \in N}(f_i(\hat{x}_i) - f_i(d_i)) \\ \implies & g^{f(d)}(f(\tilde{x})) > g^{f(d)}(f(\hat{x})) \end{aligned}$$

Note: We retain the signs of these inequalities because we assumed that $a_i > 0$.

but this is absurd because $\hat{y} = \mathcal{NA}(f(F), f(d))$ and therefore for any $\tilde{x} \in F_d$ we have $g^d(\tilde{x}) \leq g^d(\hat{x})$ which implies that $\hat{x} = \mathcal{NA}((F, d))$, and therefore

$$\mathcal{NA}(f(F), f(d)) = f(NA(F, d))$$

(d) Independence of irrelevant alternatives (IIA):

We need to show that if (F, d) and $(\hat{F}, d) \in B^N$ with $\hat{F} \subset F$, then

$$\mathcal{NA}((F, d)) \in \hat{F} \implies \mathcal{NA}((F, d)) = \mathcal{NA}(\hat{F}, d)$$

Since $\hat{F} \subset F$,

$$\max_{x \in \hat{F}_d} g^d(x) \leq \max_{x \in F} g^d(x)$$

(i.e., maximising the function on a smaller set)

and therefore

$$g^d(\mathcal{NA}(\hat{F}, d) \leq g^d(\mathcal{NA}((F, d)))$$

but since $(F, d) \in \hat{F}$,

$$g^d(\mathcal{NA}(\hat{F}, d)) \geq g^d(\mathcal{NA}((F, d))$$

therefore, we have that

$$g^d(\mathcal{NA}(\hat{F}, d) = g^d(\mathcal{NA})F, d)$$

and since the maximum is unique, it must follow that

$$\mathcal{NA}(\hat{F}, d) = \mathcal{NA}((F, d))$$

4. **Theorem:** The Nash solution $\mathcal{NA}((F,d))$ which is defined for any $(F,d) \in \mathcal{B}^N$ as

$$\mathcal{NA}((F,d)) = arg \max_{x \in F_d} g^d(x)$$

is the only allocation rule $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ that satisfies

(a) Pareto efficiency (EFF)
(b) Symmetry (Sym)
(c) Invariance (Cat)
(d) Independence of irrelevant alternatives (IIA).

Proof: We take an allocation rule $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ that satisfies the four assumptions, and the function h such that for each $x \in \mathbb{R}^N$ we have

$$h(x) = \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) x_i$$

Let

$$z = \mathcal{NA}((F,d)) = arg \max_{x \in F_d} g^d(x)$$

we now prove that

$$\varphi((F,d)) = \mathcal{NA}((F,d))$$

We set $\mathcal{NA}((F,d)) = z$ and we define

$$\mathcal{U} = \{x \in \mathbb{R}^N | h(x) \leq h(z)\}$$

where clearly $\mathcal{F} \subset \mathcal{U}$.

We set $f^A : \mathbb{R}^N \to \mathbb{R}^N$ with

$$f_i^A(x) = \frac{1}{z_i - d_i} x_i - \frac{d_i}{z_i - d_i} \forall i, \text{ and } x \in \mathbb{R}^N$$

(this is well-defined because F_d contains at least one point which $\neq$ d).

we now compute $f^A(\mathcal{U})$ and have

$$\begin{aligned} f^A(\mathcal{U}) &= \{y \in \mathbb{R}^N | (f^A)^{-1}(y) \in U\} \\ &= \{y \in \mathbb{R}^N | h((f^A)^{-1}(y)) \leq h(z)\} \\ &= \{y \in \mathbb{R}^N | h((z_i - d_i)x_i + d_i)_i \leq h(z)\} \end{aligned}$$

where

$$h(\frac{1}{z_i - d_i} x_i + d_i)_i$$

denotes

$$h((z_1 - d_1)x_1 + d_1, ..., (z_N - d_N)x_N + d_N)$$

Therefore we have $f^A(\mathcal{U})$ is (we always assume that $y \in \mathbb{R}^N$):

$$f^A(\mathcal{U}) = \left\{ y \,\middle|\, \begin{array}{c} \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) \left((z_i - d_i) y_i + d_i \right) \\ \leq \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) z_i \end{array} \right\}$$

and hence

$$
\begin{aligned}
f^A(\mathcal{U}) &= \left\{ y \,\middle|\, \sum_{i\in N}\prod_{j\in N}(z_j-d_j)y_i \;\le \sum_{i\in N}\prod_{j\in N}(z_j-d_j) \right\} \\
&= \left\{ y \,\middle|\, \sum_{j\in N}(z_j-d_j) \;\sum_{i\in N} y_i \le \sum_{j\in N}(z_j-d_j) \;\sum_{i\in N} 1 \right. \\
&= \left\{ y \,\middle|\; \sum_{i\in N} y_i \le N \right\}
\end{aligned}
$$

(since $\sum_{i\in N} 1 = N$, and we cancel out $\sum_{j\in N}(z_j - d_j)$ on both sides)

Note that $\{y| \sum_{i\in N} y_i \le N\}$ is a symmetric set and

$$f^A(d) = (0, 0, ...0)$$

therefore

$$(f^A(\mathcal{U}, f^A(d))$$

is a symmetric bargaining problem.

Since $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ satisfies Pareto efficiency and Symmetry, we have

$$\varphi(f^A(\mathcal{U}, f^A(d)) = (1, 1, ..., 1)$$

and moreover since $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ satisfies Invariance, we have

$$\varphi(\mathcal{U}, d) = (f^A)^{-1}(1, 1, ..., 1) = z$$

Finally, since $z \in F, F \subset \mathcal{U}$ and φ satisfies Independence of Irrelevant Alternatives, we have

$$\varphi(F, d) = z$$

which means that

$$\varphi(F, d) = \mathcal{NA}((F, d))$$

and hence

$$\mathcal{NA} = \varphi$$

Lecture 24: Cooperative Game Theory: Bargaining III

1. Recap:
   (a) Pareto Efficiency (EFF): $\varphi(F, d)$ is a Pareto optimal allocation.

   (b) Symmetry (Sym): The solution is invariant under permutation (the name of the player does not count).

   More formally, if π is a permutation, we set $\mathbf{x} \in \mathbb{R}^N$ and the permutation

   $$\mathbf{x}_i^\pi \overset{def}{=} x_{\pi(i)}$$

   We say that the bargaining problem is symmetric if, for each permutation π, we have $d^\pi = d$ and $\S \in F \implies \mathbf{x}^\pi \in F$ (i.e., the name of the player does not count for both d and F).

   (c) Invariance (Cat): If $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following: $f_i(x) = a_i x_i + b_i, \forall i$ (i.e., affine transformation), then

   $$\varphi(f(F), f(d)) = f(\varphi(F, d))$$

(d) Independence of irrelevant alternatives (IIA): If (F, d) and $(\hat{F}, d) \in \mathcal{B}^N$ with $\hat{F} \subset F$, we have

$$\varphi(F, d) \in \hat{F} \implies \varphi(F, d) = \varphi(\hat{F}, d)$$

2. **Theorem:** The Nash solution axioms are all necessary for uniqueness. This means that for each axiom there is an allocation rule different from $\mathcal{NA}$ that satisfies the remaining three axioms.

Proof: We show that for each of the axioms there is an allocation rule different from the Nash solution that satisfies the other remaining three axioms.

(a) Pareto efficiency (EFF):

$\varphi(F, d)$ is a Pareto optimal allocation. We just use the rule that for each (F, d) we select

$$\varphi(F, d) = d$$

(i.e., we are just choosing the disagreement point as the allocation since we don't care about efficiency)

and clearly it satisfies (Sym) (Cat) (IIA).

(b) Symmetry (Sym): The solution is invariant under permutation (the name of the player does not count).

More formally, if π is a permutation then we set $\mathbf{x} \in \mathbb{R}^N$ and the permutation

$$\mathbf{x}_i^\pi \overset{def}{=} x_{\pi(i)}$$

We say that the bargaining problem is symmetric if, for each permutation π, we have $d^\pi = d$ and $\S \in F \implies \mathbf{x}^\pi \in F$.

Then, we just use the rule that for each (F, d), for each player $i > 1$ we select

$$\varphi_i(F, d) = \max_{x \in F_d^i} x_i$$

where $F_d^i = \{x \in F_d | \forall j < i, x_j = \varphi_j(F, d)\}$, and for $i = 1$ we select

$$\varphi_i(F, d) = \max_{x \in F_d} x_1$$

(i.e., everyone maximises their allocation in order from the first agent to the last agent, if we don't care about symmetry)

and clearly it satisfies (EFF) (Cat) (IIA). This type of solution is known as *serial dictatorship* because there is an ordering of players and each player chooses among those left when he is given the turn to choose.

(c) Invariance (Cat): If $f : \mathbb{R}^N \to \mathbb{R}^N$ satisfies $f_i(x) = a_i x_i + b_i \forall i$ (i.e., affine transformation), then

$$\varphi(f(F), f(d)) = f(\varphi(F, d))$$

and we can just use the following rule for each (F, d):

$$\varphi(F, d) = d + \bar{t}(1, ..., 1)$$

where

$$\bar{t} = \max\{t \in \mathbb{R} | d + t(1, ..., 1) \in F_d\}$$

(if invariance doesn't matter)

This allocation rule is called the *egalitarian solution* and clearly it satisfies (EFF) (Sym) (IIA).

(d) Independence of Irrelevant Alternatives (IIA): If (F, d) and $(\hat{F}, d) \in \mathcal{B}^N$ with $\hat{F} \subset F$, we have

$$\varphi(F, d) \in \hat{F} \implies \varphi(F, d) = \varphi(\hat{F}, d)$$

We can just use the **KalaiSmorodinsky rule** (discussed next) which satisfies (EFF) (Sym) (Cat).

Kalai-Smorodinsky Solution:

1. **Definition:** The utopia point for a bargaining problem $(F, d) \in \mathcal{B}^N$ is given by the vector

$$b(F, d) = (b_1(F, d), ..., b_N(F, d)) \in \mathbb{R}^N$$

where for each i

$$b_i(F, d) = \max -x \in F_d x_i$$

i.e., we maximise x_i for each agent separately, and take the point where every agent is at their maximum x_i.
(See graphical notes)

2. **Definition:** The Kalai-Smorodinsky solution, KS, is defined for a bargaining problem $(F, d) \in \mathcal{B}^N$

$$KS(F, d) = d + \bar{t}(b(F, d) - d)$$

where

$$\bar{t} = \max t \in \mathbb{R} | d + t(b(F, d) - d) \in F_d$$

(See graphical notes)
Note: Intuitively, we start from d and move to the utopia point; the KS solution is the furthest we can go on the line connecting d and the utopia point. This solution satisfies (Eff) (Sym) (Cat) but is not the Nash solution, hence it does not satisfy (IIA).

The solution is based on the property of Individual Monotonicity (IM):
Let $(F, d), (\hat{F}, d) \in \mathcal{B}^N$ such that

$$\hat{F}_d \subset F_d$$

Let $i \in N$ such that, for each $j \neq i$,

$$b_j(F, d) = b_j(\hat{F}, d)$$

If φ is an allocation rule for N-player bargaining problems that satisfies IM, then

$$\varphi_i(\hat{F}, d) \leq \varphi_i(F, d)$$

3. **Theorem:** The Kalai-Smorodinsky solution is the unique allocation rule for **two-player** bargaining problems that satisfy (EEF) (Sym) (Cat) (IM).
(Note: IIA not satisfied)

Proof:
Let φ be an allocation rule for **two-player** bargaining problems that satisfies the four properties, and let $(F, d) \in B^2$ (we will discuss $n \geq 3$ later). We must prove that

$$KS(F, d) = \varphi(F, d)$$

Since KS and φ satisfy (Cat), we can assume, without loss of generality, that

$$d = (0, 0)$$

and

$$b(F, d) = (1, 1)$$

(we can do this scaling because we assume invariance).

Then, since

$$KS(F,d) = (KS(F,d)_1, KS(F,d)_2) \in \mathbb{R}^2$$

lies in the segment with extreme points

$$d = (0,0) \text{ and } b(F,d) = (1,1)$$

It follows that

$$KS(F,d)_1 = KS(F,d)_2$$

Now, we define

$$\widehat{F} = \left\{ (x_1, x_2) \in \mathbf{R}^2 \middle| \begin{array}{c} \text{there exists } (y_1, y_2) \text{ in} \\ conv \left\{ \begin{array}{c} (0,0), (1,0), \\ (0,1), KS(F,d) \end{array} \right\} \\ \text{with } (x_1, x_2) \leq (y_1, y_2) \end{array} \right\}$$

Since φ satisfies (IM) and $\hat{F} \subset F$ and

$$b(F,d) = b(\hat{F},d)$$

we must have

$$\varphi(\hat{F},d) \leq \varphi(F,d)$$

Moreover, since $\hat{F}$ is symmetric, then (EFF) and (Sym) imply that

$$\varphi(\hat{F},d) = KS(F,d)$$

Then,

$$\varphi(\hat{F},d) = KS(F,d) \leq \varphi(F,d)$$

but since $KS(F,d)$ is Pareto efficient, we have

$$KS(F,d) = \varphi(F,d)$$

4. The previous result provides a nice characterisation of the KS solution. Unfortunately this characterisation cannot be generalised when $n \geq 3$.

   Theorem (Roth): Let $n \geq 3$. Then there is no solution for the n-player bargaining problem that satisfies (EFF) (Sym) (IM).

   Proof: Assume that $n \geq 3$ and φ is a solution for the n-player bargaining problem satisfying (EFF) (Sym) (IM). Let

$$d = (0, ..., 0)$$

   and

$$\widehat{F} = \left\{ x \in \mathbf{R}^N \middle| \begin{array}{c} \text{there exists } y \text{ in} \\ conv \left\{ \begin{array}{c} (0,1,1,....,1), \\ (1,0,1,...,1) \end{array} \right\} \\ \text{with } x \leq y \end{array} \right\}$$

By (EFF), $\varphi(\hat{F}, d)$ belongs to the segment joining $(0, 1, ..., 1)$ and $(1, 0, ..., 1)$, therefore

$$\varphi_3(\hat{F}, d) = 1$$

Let

$$F = \{x \in \mathbb{R}^N | \sum_{i=1}^{N} x_i \leq n - 1 \text{ and } \forall i \in n, x_i \leq 1\}$$

Since φ is a solution for n- player bargaining problems satisfiying (EFF) (Sym), then for each $i \in N$ we have

$$\varphi_i(F, d) = \frac{n-1}{n}$$

However, $\hat{F} \subset F$ and

$$b(F, d) = b(\hat{F}, d) = (1, 1, ..., 1)$$

Then, by (IM)

$$\varphi(\hat{F}, d) < \varphi(F, d)$$

which contradicts the fact that

$$\varphi_3(\hat{F}, d) = 1 > \frac{n-1}{n} = \varphi_3(F, d)$$

5. There are many other possible allocation rules but the Nash solution and the Kalai-Smorodinski solution are the most famous.

6. There are several classes of games cooperative games which are interesting, for example games of *transferable utility.*

   Definition: A TU-game is a pair (N, v) where $N = \{1, 2, ..., n\}$ is the set of players and

   $$v : 2^N \to \mathbb{R}$$

   where $2^N = \{S \subseteq N\}$ and $v(\emptyset) = 0$.

   The standard interpretation is that $v(S)$ is the worth of coalition S, i.e., the benefit that the coalition can generate.

7. The standard example is the **Glove Game**:

   There are three players.
   Player 1 has a left glove and Players 2 and 3 have one right glove each.
   Each left-right pair sells for \$1, so we can formalise the situation with:;

   $$\begin{aligned} &N = \{1, 2, 3\} \\ &v(\emptyset) = 0 \\ &v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0 \\ &v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 1 \end{aligned}$$

   For this class of game we can define the core and prove a Bondareva-type theorem.

   Of course we should also ask ourselves what is the value of each individual player by using the values of the coalitions which has them as a a member.
   The answer is the *Shapley* value (not covered in this course), which is constructed with a similar approach as in the construction of the Nash solution.