Micro 1 notes

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June 17, 2024

Lecture 1

1. A resource allocation is Pareto efficient if it is not possible to reallocate these resources without making someone worse off.

   In the following examples, we look at different scenarios and (1) determine efficient allocations (this is conceptually independent of prices), and (2) find Walrasian prices at the efficient allocation.

   Common to these models is that there exist market clearing prices that implement the efficient allocation, assuming complete information about agents' values and costs. These prices depend on values and costs (i.e., agents' "types") but not their identities. Prices with these properties are customarily called **Walrasian prices**.

   A Walrasian price p is such that it clears the market, i.e., $D(p) = S(p)$ if p is a Walrasian price. q_w is the Walrasian (or efficient) quantity traded.

2. **Two-sided homogenous goods model**:

   (a) M sellers, each owning and having demand for one unit of a homogenous good (sellers own the good, buyers don't).
   N buyers, each with demand for one unit of the good.
   A buyer with value v who obtains a unit at price p has a payoff of $v - p$. A seller with cost C who sells at price p has payoff of $p - c$.
   M goods and $M + N$ agents. The solution should be for agents who have the M highest payoffs to be the agents finally holding the good.
   All agents' payoffs of not participating are 0.
   Assume all costs and values are different (this leads all subsequent inequalities to be strict).

   (b) Let $c = (c_1, ..., c_M)$ denote the vector of the sellers' costs, and $v = (v_1, ..., v_N)$ denote the vector of valuations of the buyers.

   (c) We use order statistics as follows: Let θ be a $(M + N \times 1)$ vector $= (v, c)$. $\theta_{(i)}$ denotes the i^{th} highest element and $\theta_{[i]}$ denotes the i^{th} lowest element.

   (d) The efficient allocation is for q_w to be traded, where $c_{[q_w]} < v_{(q_w)}$, i.e., the seller with the last unit sold has a cost lower than the valuation of the buyer with the last unit bought.
   The market does not sell beyond this unit, i.e., $v_{(q_w+1)} < c_{[q_w+1]}$, so for the next unit, the seller's cost is higher than the buyer's valuation.

   (e) Relationship with prices: The upper bound for Walrasian prices is $p\bar{w} = min\{(v_{(q_w)}, c_{[}q_{w+1]}\}$ (i.e., the buyer at this valuation is willing to buy; the next seller is willing NOT to sell), and the lower bound for Walrasian prices is $p\underline{w} = max\{c_{[q_w]}, v_{(q_w+1)}\}$ (i.e., the seller at this cost is willing to sell; the next buyer is willing NOT to buy).
   $p\bar{w} = \theta_{(M)}$ and $p\underline{w} = \theta_{(M+1)}$.
   q_w is equal to the number of values v_i among the M highest elements in θ, which is also equal to the number of costs c_j among the N lowest elements in θ.
   Because all types are different, we have $p\bar{w} < p\underline{w}$, which is a continuum of Walrasian prices.

These prices are independent of any seller of buyer who trades.

(f) The efficient quantity traded q_w is where the number of buyer valuations (v_i) in the top M elements of θ is equal to the number of seller costs (c_j) in the bottom M elements of θ.

3. **Asset market with homogenous goods**

(a) N agents each endowed with r_i units of a homogenous good and with a maximum demand for k_i units, with $r_i \leq k_i$ (i.e., each person's endowment is less than or equal to their demand).
All agents have constant marginal values up to their own demand k_i and thereafter their willingness to pay is 0 for any additional units.
For there to be scarcity and yet something to allocate, assume

$$0 < \sum_{i=1}^{N} r_i \equiv R < \sum_{i=1}^{N} k_i$$

i.e., the total endowment in the economy is less than the maximum demand in the economy.

(b) Let $\theta = (\theta_1, ...\theta_N)$ be the vector of agents' types. This denotes their willingness to pay, and also their reservation value for selling (both values are identical for each agent).

(c) To find the efficient allocation, we rank all the agents from highest θ to lowest θ, then allocate as much as the agent with the highest θ demands, then after satisfying his demand, allocate as much as possible to the agent with the next highest θ, etc.
There is no rationing at the efficient allocation if each agent $i, i = 1, ..., N$ either consumes 0 or k_i. Otherwise, there is rationing.

(d) The set of Walrasian prices is a continuum if there is no rationing, with the upper bound equal to the value of the marginal agent who consumes a positive amount (i.e., the agent with the lowest value among the agents who are allocated their maximum demand) and the lower bound equal to the value of the marginal agent who consumes 0 (i.e., the agent with the highest value among the agents who do not consume).
In contrast, if there is rationing, then the Walrasian price is a singleton and given by the value of the agent who is rationed.

(e) Simple symmetric example: Assume $N = 3, r_i = 1, k_i = 2 \ \forall i$. We rank the three agents from highest to lowest θ, then the $\theta_{(3)}$ agent sells his unit to the $\theta_{(1)}$ agent, and the $\theta_{(2)}$ agent consumes his own unit.
The Walrasian price in this example is $p\bar{W} = p\underline{W} = \theta_{(2)}$, which is also the price at which agent 3 sells to agent 1 (if any lower, agent 2 will buy; if any higher, agent 2 will sell).
In this example, there is rationing (since agent 2 would want to consume 2 but only consumes 1), and the Walrasian price is a singleton at θ_2.

4. **Assignment game (Shapley and Shubik, 1972)**

(a) All sellers have single-unit supplies and all buyers have single-unit demands. Buyers are not sellers and vice-versa.
Sellers' supplied objects are allowed to be heterogenous.
With M sellers, each buyer i's type is an M-dimensional vector $v^i = v^i_1, ...v^i_m$ where v^i_j is i's valuation for j's object.
The social surplus generated if i is matched to j (i.e., "buys" from j or chooses not to trade) is

$$a_{ji} = max\{v^i_j - c_j, 0\}$$

where the max accounts for the possibility that i and j optimally do not trade (which happens if $v^i_j < c_j$).
The efficient allocation or matching maximises the sum of pairwise surpluses $max\{v^i_j - c_j, 0\}$ subject to the constraints that each seller can only sell one object and each buyer has only demand for one object.

(b) A Walrasian price p is an M-dimensional vector, satisfying for each $i \in \{1, ..., M\}, D_i(p) = S_i(p)$, where D_i is the aggregate quantity demanded for good i and S_i is the quantity supplied of good i (which is either 0 or 1 because each seller is the sole owner of that good).
For a seller j to be willing to sell at p, we need $p_j \geq c_j$.

(c) Simple two by two example: Assume $N = 2 = M, v^1 = (10, 8), v^2 = (6, 3), c_1 = 1, c_2 = 2$.
In this example, $a_{11} = 9, a_{12} = 6, a_{21} = 5, a_{22} = 1$. Since each good can only be sold to one buyer, the efficient assignment is for buyer 1 to buy from seller 2, and buyer 2 to buy from seller 1 (this leads to a social surplus of $5+6 = 11$; the alternative would lead to a social surplus of $9+1 = 10$).

To find the set of Walrasian prices, we need to set up both buyers' incentive compatibility constraint such that they are incentivised to buy the specific unit that maximises total social surplus in the economy.

i. For buyer 1, we require $8 - P_2 \geq 10 - P_1$.
ii. For buyer 2, we require $6 - P_1 \geq 3 - P_2$.

We also need to ensure that both sellers are incentivised to sell (to anyone), and both buyers are incentivised to buy (from anyone), so we also need:

i. $1 \leq P_1 \leq 6$
ii. $2 \leq P_2 \leq 8$

i.e., the set of Walrasian prices P^w is the vector of prices $P = (P_1, P_2)$ where:

$$P_w := \{P : 4 \leq P_1 \leq 6, P_2 \geq 2, P_1 - 3 \leq P_2 \leq P_1 - 2\}$$

5. Now, we consider examples that do not permit Walrasian prices that implement the efficient allocation.

6. **Pure public good**

(a) Production fixed cost is $K > 0$.
If good is produced, agents cannot be excluded from consuming it.
There are N agents with values $v = (v_1, ..., v_N)$.
Efficiency dictates that production occurs iff $\sum_{i=1}^{N} v_i > K$, i.e., sum of agents' valuations is greater than the fixed cost.

There are no Walrasian prices that implement the efficient allocation: because every agent benefits from the good whether it pays or not, for any positive price each buyer's demand is 0.
Thus, the only candidate market-clearing price is 0. But at that price, the producer makes a loss and therefore prefers not to supply the public good.

Alternative: Lindahl prices $\lambda = (\lambda_1, ..., \lambda_N)$, which are individualised prices such that at each buyer's individual price, every buyer demands the efficient quantity, i.e., $D_i(\lambda_i) = 1$, and the profit-maximising seller is willing to sell.
If an agent demands 0, then no production occurs, to circumvent the free-rider problem (i.e., if $D_i(\lambda_i) = 0$ for any i, then there is no production.
Lindahl pricing induces efficient production and consumption iff $\sum_{i=1}^{N} \lambda_i \geq K$ and $\forall i : \lambda_i \leq v_i$.

7. **Club good**

(a) Same basic properties as a public good but permits excluding agents from consumption.
If there are marginal costs of serving agent i, where $c_i > 0$ on top of K, the fixed cost of production, then excluding agents can be efficient even if production takes place.

(b) We can find Lindahl prices that implement the efficient allocation:
Let the vector of valuations for each agent be $v = (v_1, ..., v_m)$.
We identify the set of agents for which their valuation is higher than the marginal cost of providing the good to them: $A(v, c) = \{i : v_i > 0\}$.
Production is efficient if $\sum_{i \in A(v,c)} (v_i - c_i) \geq K$. Assuming this holds, we want to set up Lindahl prices such that only agents whose valuations are higher than the marginal costs are incentivised to consume, i.e.,:

i. Lindahl prices for $i \in A(v,c) : c_i \leq \lambda_i \leq v_i$
ii. Lindahl prices for $i \notin A(v,c) : v_i \leq \lambda_i \leq c_i$

We also need total prices paid to be higher than total costs, i.e.,:

$$\sum_{i \in A(v,c)} \lambda_i \geq K + \sum_{i \in A(v,c)} c_i$$

Lecture 2

1. **Precursor: Quasilinear utility**

(a) Recall that typically for a two-goods economy, a decrease in the price of one good x_1 affects the quantity of x_1 through both income and substitution effects:

i. Income effect: change quantity of x_1 consumed because of generalised change in purchasing power
ii. Substitution effect: change quantity of x_1 consumed because of changes in relative price with respect to x_2

To find the overall effect, we do Hicksian decomposition to income and substitution effects: first allow the price to change to the new level while allowing the consumer to stay on the original indifference curve (i.e., substitution effect), then change the consumer's real income (i.e., income effect).

(b) Now consider a two-goods economy with quantities denoted x_1 and x_2.
Consumer i's utility function $u_i(x)$ is quasilinear if it is of the form

$$u_i(x) = v_i(x_1) + x_2$$

where $v_i(.)$ is some well-behaved function (e.g., $v_i' > 0 > v_i''$).

Letting p_i and p_2 be the prices of these two goods and y_i be consumer i's income (and normalising $p_2 = 1$ as the numeraire good), i's utility maximisation problem is

$$\max_x v_i(x_1) + x_2 \text{ st } p_1 x_1 + x_2 \leq y_i$$

For y_i large enough, i will consume both goods in positive amounts, and as the budget constraint holds with equality, one can substitute x_2 with $y_i - p_1 x_1$ and solve

$$\max_{x_1} v_i(x_1) + y_i - p_1 x_1$$

The FOC is

$$v_i'(x_1^{*,i}) - p_1 = 0$$

If $v_i'' < 0$ (and $p_1 x_1^{*,i} < y_i$), the FOC is sufficient for a maximum.
Note that $x_1^{*,i}$ depends on p_1 but not on y_i, provided y_i is large enough (i.e., no income effects; but if income is small, consumer may spend all income on good x_1 and demand for good x_1 is constrained by income at low income).

$v_i'(x)$ is i's inverse demand function

$$P_i(x) = v_i'(x)$$

(where we drop the subscript 1 because the market of interest is the one for good 1).
$P_i(x)$ gives the i_{th} willingness to pay/ marginal utility for the x^{th} unit.

If i consumes q units, its gross surplus is $\int_0^q P_i(x)dx$ and its net surplus (i.e., consumer surplus) is utility minus payment made:

$$CS_i(q) = \int_0^q P_i(x)dx - p_1 q$$

(c) Now assume there are $n \geq 1$ individuals, each of whom has a utility function of the form $u_i(x) = v_i(x_1) + x_2$ (although valuation can differ across individuals), and the amount of good 1 that can be allocated among all of them is $\bar{X} > 0$.

Pareto efficiency requires that all individuals have the same marginal rate of substitution between goods 1 and 2.
Because the marginal utility of good 2 is 1 for all i, this means that all i must have the same marginal utility $v_i'(x_1^i)$.
Thus, the allocation $x^p = (x_1^p, ..., x_n^p)$ being Pareto efficient is equivalent to:

$$x^p = \arg\max_x \sum_{i=1}^n v_i(x_i) \text{ s.t. } \sum_{i=1}^n x_i \leq \bar{X}$$

Letting $\lambda^* > 0$ denote the solution value of the Lagrange multiplier associated with this problem (i.e., associated with the optimisation constraint), the Pareto efficient allocation is such that $v_i'(x_i^P) = \lambda^*$ for all i. In other words, everyone consumes at a different level until all their marginal utilities are identical.
Note: this is similar to problems where we equalise MRS across goods. We have a vector of FOCs where $v_i'(x_i^P) = \lambda^*, \ \forall i$.

Even though it is equivalent to maximising the sum of utilities, Pareto efficiency involves no interpersonal utility comparisons.

2. **Monopoly pricing (textbook approach)**

(a) Assume there is a continuum of consumers, each with a single-unit demand for a homogenous good. $P(Q)$ is the market-clearing price for selling quantity Q, which is assumed to be decreasing, i.e., $P'(Q) < 0$.
$R(Q) = P(Q)Q$ at market-clearing price.
The monopoly's cost is $C(Q)$ and hence its profit is $\Pi^M(Q) = R(Q) - C(Q) - P(Q)Q - C(Q)$.
We assume that $P(Q), C(Q)$ are twice differentiable, $C'(Q) \geq 0, C''(Q) \geq 0$.
Assume $R(Q)$ is concave, i.e., marginal revenue is $R'(Q) = QP'(Q) + P(Q)$ is decreasing, i.e., $R''(Q) = QP''(Q) + 2P'(Q) \leq 0$
Assume $P(0) \geq C'(0)$ (i.e., the monopolist will produce a positive quantity), and $P(\bar{Q}) = 0$ for some finite $\bar{Q}$ (this is a consequence of $P(Q)$ being downward sloping, i.e., $P'(Q) < 0$).
The monopoly sells all units it sells at a single uniform price; no price discrimination.

(b) The monopoly maximises Π^M via the choice of Q. The FOC is:

$$P(Q^*) + Q^* P'(Q^*) = C'(Q^*)$$

that is, $MR = MC$ at optimal quantity.

Proposition 1: the solution Q^* exists, is unique, and characterises the profit maximising quantity.
Proof:

i. MR is decreasing in Q by assumption and MC is non-decreasing by assumption. Hence, at most one point of intersection exists.
ii. Because $P(0) > C'(0) \geq P(\bar{Q})$, $\exists Q^* \in (0, \bar{Q}$ that satisfies the FOC.
iii. Because $R(Q)$ is concave and $C(Q)$ is convex, $\Pi^M(Q) = R(Q) - C(Q)$ is convex.
iv. Hence, the FOC is sufficient for a maximum.

(c) Example: Linear demand and constant MC
Assume: $P(Q) = a - Q$ (this is the inverse demand function) and $C(Q) = c \cdot Q$
Thus, $\Pi^M(Q) = (a - Q)q - cQ$ and $P'(Q) = -1 \implies P(Q) + P'(Q)Q = a - 2q$
So, $a - 2Q = c \Leftrightarrow Q^* = \frac{a-c}{2}$.
Plugging Q^* into $P(Q)$ yields $P(Q^*) = \frac{a+c}{2}$.
Hence, the maximum profit is $\Pi^M(Q^*) = (\frac{a-c}{2})^2$.

(d) Graphical notes for the same example:
Inverse demand function is $P(q) = a - q$.
Marginal revenue is $P(q) + P'(q)q = a - 2q$.
Optimal quantity for the monopolist, q^*, is where $MR = MC$. The market clearing price is $p(q^*)$.
Producer surplus is Π
Consumer surplus is $\int_0^{q*} p(x)dx - p(q^*)q^*$ (i.e., total willingness to pay minus total actual price paid).

By contrast, the optimal quantity for society, Q^e, is where $P(Q^e) = MC$.
Deadweightloss is the difference between social surplus in the efficient quantity scenario (i.e., society optimal) vs. social surplus in the monopoly optimal quantity scenario.

$$\begin{aligned}
DWL &= \int_{q^*}^{q^e} P(x) - c \, dx \\
&= SS^e - SS^m \\
&= \Pi^m(q^e) + CS(q^e) - Pi^m(q*) + CS(q*)
\end{aligned}$$

Note that $\Pi^m(q^e) = 0$, i.e., no profit at q^e.

(e) The uniform-pricing monopoly sells inefficiently few units because to increase q, it will have to lower the price for all previous units (since uniform pricing).
E.g., assuming zero costs, increasing quantity from Q_0 to $Q_0 + \Delta$ also means that price has to fall from $P(Q_0)$ to $P(Q_0 + \Delta)$.
An intuitive optimality condition for the monopoly (assuming zero costs) is to produce until the point where marginal gain (from increasing quantity) = inframarginal loss (from decreasing price). This is equivalent to:

$$\begin{aligned}
\Delta \cdot P(Q_0 + \Delta) &= -(P(Q_0 + \Delta) - P(Q_0)) \cdot Q_0 \\
P(Q_0 + \Delta) &= -\frac{(P(Q_0 + \Delta) - P(Q_0))}{\Delta} Q_0
\end{aligned}$$

As $\Delta \to 0$, this becomes the definition of differentiation, so:

$$P(Q_0) = -P'(Q_0)(Q_0) \Leftrightarrow Q_0 = \frac{P(Q_0)}{-P'(Q_0)}$$

i.e., marginal gain = -inframarginal loss.

The monopoly's optimal quantity given $MC = 0$ is $Q* = \frac{P(Q^*)}{-P'(Q^*)}$

(f) Discussion:
Usual justifications for imposing uniform prices (vs. price discrimination) include transaction costs and resale, but imposing uniform prices is an ad-hoc restriction on the contracting space and not necessarily optimal for the monopolist (see Lecture 3).
At the same time, decreasing marginal revenue may be an unwarranted restriction (e.g., inverse demand may be kinked, leading to non monotone marginal revenue and hence multiple quantities where MC = MR, and both are local maxima. See Lecture 3).

3. **Monopsony pricing (textbook approach)**

(a) Assume the monopsony employer faces an inverse labour supply function $W(Q)$, which gives the market-clearing wage for hiring Q workers.
Workers supply labour at the price $W(Q)$, and the monopsony employer "buys" labour from them.
$W(Q)$ arises from a continuum of workers each supplying one unit of labour inelastically, with $W(Q)$ representing the opportunity cost of working for the worker with the Q^{th} lowest opportunity cost. Each worker has private information about its opportunity cost.

$W(Q)$ is strictly increasing and continuously differentiable.
The cost of procurement

$$C(Q) := W(Q)Q$$

of hiring Q workers at the market-clearing wage is strictly increasing and differentiable, satisfying

$$C'(Q) = W(Q) + W'(Q)Q \geq W(Q)$$

since $W'(Q)Q \geq 0$ given that $W(Q)$ is strictly increasing.

The monopsony's willingness to pay (or marginal profit) for the Q^{th} unit of input is $V(Q)$, which is assumed to be weakly decreasing and continuous (this is the monopsony's valuation for the labour inputs).
We assume $V(0) > W(0)$ (otherwise the monopsonist won't hire at all) and $V(Q) < W(Q)$ for sufficiently large Q (since $V(Q)$ is weakly decreasing).
The efficient (or price taking, i.e, the quantity that the monopolist would choose if it was a price taker) quantity Q^p satisfies $V(Q^p) = W(Q^P)$. This is the efficient level of employment.

The monopsony is profit-maximising.
Assume $C(Q)$ is convex, i.e., $C''(Q) = 2W'(Q) + W''(Q)Q \geq 0$.
Assume that the monopsony is required to set a uniform wage across all workers it employs. The monopsony's profit maximisation problem becomes

$$\max_Q \int_0^Q V(x)dx - C(Q)$$

(i.e., marginal benefit of procurement - cost of procurement).
The FOC is

$$V(Q^*) - C'(Q^*) = 0$$

(note that this is because $\frac{\partial \int_0^Q V(x)dx}{\partial x} = V(Q^*)$).
Because V is non-increasing and $C'' > 0$, the second-order condition for a maximum is satisfied.

Because $C'(Q) > W(Q)$ for any positive Q, we have $Q^* < Q^p$, that is, the monopsony hires a (socially) inefficiently small number of workers.
Intuition is the same as for monopoly - the monopsony optimally trades off marginal gains vs. inframarginal losses.
Because $W(.)$ is increasing, we have $W(Q^*) < W(Q^p)$, i.e., fewer workers are employed and those who are employed get lower wages than under efficiency.

(b) Example: Assume $V(Q) = 1,\ \forall Q, W(Q) = Q$
The monopsony's costs are $C(Q) = W(Q)Q = Q^2$, and optimal quantity is when $V(Q) = C'(Q) \implies 1 = 2Q \implies Q^* = \frac{1}{2}$.
At the same time, the socially optimial quantity Q^p is where $V(Q^p) = W(Q^p) \implies Q^p = 1$. So we have:

i. $Q^p = 1 = W(Q^p)$, and zero profits at the socially optimal quantity; and
ii. $Q^* = \frac{1}{2} = W(Q^p)$, and profits of $\frac{1}{4}$ at the monopsony's profit maximising quantity.

$Q^* < Q^p$ and hence there is inefficiently low employment at the profit maximising quantity (but note that there is no involuntary employment because everyone who works has wage higher than the opp cost of working $W(Q)$, and everyone whose cost of working is higher than their wage do not work).

We define worker surplus (WS) as total wage earnings minus aggregate costs of working, and social surplus (SS) as the sum of WS and the firm's profit.

$$SS^p = WS^p + 0 = 1 - \int_0^1 QdQ = \frac{1}{2}$$
$$SS^* = WS^* + \frac{1}{4} = \frac{1}{4} - \int_0^{1/2} QdQ + \frac{1}{4} = \frac{3}{8}$$

4. **Curbing market power: Price and wage regulation**

(a) Price caps:
A uniform pricing monopoly trades off losses on inframarginal units against marginal gains.
By imposing a price cap $\bar{p}$, one can eliminate the inframarginal losses on all $D(\bar{p})$ units, where $D(p)$ is the demand function (i.e., $D(p) = P^{-1}(p)$).
Any $\bar{p} \in (P(Q^p), P(Q^*))$ will induce the monopoly to increase quantity to $D(\bar{p})$ and decrease price.
Consequently, CS and SS increase, and the monopoly's profit decreases.

  i. Since $SS = CS + \Pi$, it suffices to show that profits decrease and SS increases.
  ii. We can see that profits decrease because $\bar{p}$ and $D(\bar{p})$ were available when $P(Q^*)$ and Q^* were chosen, so the fact that $P(Q*)$ and Q^* were revealed preferred by the monopoly shows that that was the profit maximising quantity (and hence profits decrease with a price cap).
  iii. Meanwhile, $SS(Q) = \int_0^Q P(x)dx - C(Q)$, so $SS'(Q) = P(Q) - C'(Q)$, which is positive for any $Q < Q^p$. In other words, increasing quantity increases SS.

(b) Graphical notes on price caps:
Assume costs = 0.
The demand function is $P^{-1}(p)$ (i.e., inverse of the inverse demand function).
Revenue $R(Q) = P(Q)Q$
Assume a price ceiling $\bar{p} \in (0, P(0))$, where $P(0)$ is the high price where 0 quantity is sold.

$$R_c(Q, \bar{P}) = \begin{cases} \bar{P} \cdot Q, & Q \leq D(\bar{P}) \text{ (i.e., price ceiling is binding)} \\ R(Q), & Q > D(\bar{P}) \text{ (i.e., price ceiling is not binding)} \end{cases}$$

then,

$$R'_c(Q, \bar{P}) = \frac{\partial R_c(Q, \bar{P})}{\partial Q} = \begin{cases} \bar{P}, & Q \leq D(\bar{P}) \\ R'(Q), & Q > D(\bar{P}) \end{cases}$$

(c) Minimum wages:
As first observed by Robinson (1933) and Stigler (1946), in the presence of monopsony power, any minimum wage $\underline{w} \in (W(Q^*), W(Q^p))$ increases total employment, without inducing involuntary unemployment.
It increases WS and SS and decreases the firm's profits.
The proof and intuition are the same as for price caps and monopoly power (minimum wages are a "price ceiling" for the monopsony).

(d) Network infrastructure and natural monopoly:
Assumptions: Large fixed cost K (e.g., network infrastructure).
Variable cost of production $C(Q)$ with $C'(Q) \geq 0, C''(Q) \geq 0$.
Average variable cost $AVC(Q) = \frac{K+C(Q)}{Q}$
AVC'(Q) $= \frac{QC'+C-K}{Q^2} < 0$ for small Q.

Let $P(Q)$ be the inverse demand function for q.
Consumer surplus $CS(Q)$ given quantity Q is:

$$CS(Q) = \int_0^Q P(x)dx - P(Q)Q$$

i.e., the inverse demand (willingness to pay) minus the price paid.

Producer surplus $PS(Q)$ (i.e. profit) given quantity Q is:

$$PS(Q) = P(Q)Q - C(Q) - K$$

Social surplus is:

$$SS(Q) = CS(Q) + PS(Q)$$

$SS(Q)$ is maximised at the efficient (or price-taking) quantity Q^p satisfying $P(Q^p) = C'(Q^p)$ (i.e., inverse demand function = marginal cost).
Problem: If $AVC'(Q^p) < 0, PS(Q^p) < 0$, so the firm would not break even.
Following Walras (1875), this is referred to as natural monopoly.

(e) Ramsey pricing:
This addresses the tradeoff between the firm's profit maximisation and the pricing distortion away from $P = MC$. The solution to the problem of maximising

$$(1-\alpha)(CS(Q) + PS(Q)) + \alpha PS(Q)$$

is called Ramsey pricing (Ramsey, 1927), where $\alpha \in [0, 1]$ is the additional weight given to producer surplus.
When $\alpha = 1$, PS is maximised. When $\alpha = 0$, SS is maximised.

Taking FOC for the equation above, the solution Q_α satisfies

$$P(Q_\alpha) - C'(Q_\alpha) = -\alpha P'(Q_\alpha) Q_\alpha$$

This is equivalent to

$$\frac{P(Q_\alpha) - C'(Q_\alpha)}{P(Q_\alpha} = \frac{\alpha}{\epsilon(Q_\alpha)}$$

where $\epsilon(Q) = -\frac{P(Q)}{P'(Q)Q}$ is the price elasticity of demand at $p = P(Q)$.

Note that at $\alpha = 0, Q_0 = Q^p$, while at $\alpha = 1, Q_1 = Q^m$, where Q^m is the monopoly quantity satisfying $R'(Q^m) = C'(Q^m)$, i.e., $MR = MC$. So as a function of the weight α we get anything between social surplus maximisation and profit maximisation.

We can derive the same results in another way: For $\pi \in [PS(Q^p), PS(Q^m]$, consider the problem of

$$\max_Q SS(Q) \text{ s.t. } PS(Q) \geq \pi$$

(note that PS need not be exactly $= \pi$; e.g., in cases of price ceiling, there is a difference between the producer's profit = price - cost, and PS = price - willingness to sell.
Let $\lambda_p^* i$ be the solution value of the Lagrange multiplier for the Lagrangian

$$\max_{Q \in [Q^m, Q^p]} \mathcal{L}(Q, \lambda) = SS(Q) - \lambda[\pi - PS(Q)]$$

The FOCs are:

$$\frac{\partial \mathcal{L}(Q, \lambda)}{\partial Q} = SS'(Q) + \lambda_\pi^* PS'(Q) = 0$$
$$\frac{\partial \mathcal{L}(Q, \lambda)}{\partial \lambda} = PS(Q) - \pi = 0$$

Since SS = CS + PS, the optimal quantity maximises

$$CS(Q) + PS(Q) + \lambda_\pi^* PS(Q)$$

For $\alpha = \frac{\lambda_\pi^*}{1+\lambda_\pi^*}$, this is equivalent to maximising

$$(1-\alpha)(CS(Q) + PS(Q)) + \alpha PS(Q)$$

i.e., Ramsey pricing.

Observe that for $Q \in (Q^m, Q^p)$ we have

$$SS'(Q) = P(Q) - C'(Q) > 0 > P'(Q) - C''(Q) = SS''(Q)$$
$$PS'(Q) = R'(Q) - C'(Q) < 0$$
$$PS''(Q) = R''(Q) - C''(Q) < 0$$

Because $PS' < 0$, Q decreases as π increases.
From the first FOC, we have

$$\lambda_\pi^* = -\frac{SS'(Q)}{PS'(Q)}$$

Given the aforementioned properties, it follows that $\lambda_p^* i$ increases in π, implying $\alpha = \frac{\lambda_\pi^*}{1+\lambda_\pi^*}$ increases in π.
Moreover, $\lambda_\pi^* = 0$ at $\pi = PS(Q^p)$ and $\lambda_\pi^* \to \infty$ as $\pi \to PS(Q^m)$.
These imply that $\alpha = 0$ is equivalent to $\pi = PS(Q^p)$ and $\alpha = 1$ to $\pi = PS(Q^m)$.

Lecture 3

1. **Monopoly pricing revisited (Loertscher and Muir 2022): Monopoly pricing, optimal randomisation, and resale**

   (a) We no longer assume concave objective functions and uniform pricing.
   Setup: Assume there is a continuum of consumers, each with single-unit demand for a homogenous good.
   Let $P(Q)$ denote the market-clearing price for selling quantity Q, which is assumed to be decreasing, i.e., $P'(Q) < 0$.
   Revenue $R(Q)$ given a market-clearing price is $R(Q) = P(Q)Q$.
   Consumers are privately informed about their values v. Utility for each consumer is their value v_i minus the price they pay to receive the good.
   The seller wants to introduce a mechanism such that it can earn more than $R(Q)$ when selling Q units, whilst assuming that consumers act rationally.

   Consider a selling mechanism involving two prices: p_1, p_2, with $p_1 > p_2$. Consumers can pay the high price p_1 to receive the good with certainty, while alternatively there is rationing at low price p_2 (note that consumers only pay if they are selected to receive the good).
   The mechanism is characterised by two parameters, Q_1, Q_2. Q_1 is the mass of units sold at the high price, and Q_2 is the mass of consumers that participate in the mechanism.
   Since $Q_2 > Q_1$, $\alpha = \frac{Q-Q_1}{Q_2-Q_1} \in [0,1]$ denotes the probability of service at price p_2.

   For individual rationality and incentive compatibility, we require consumers to be indifferent between lottery or not participating at the low price, and indifferent between lottery or receiving the good with certainty at the high price, i.e.:

   $$\begin{aligned} &p_2 = P(Q_2) \\ &p(Q_1) - p_1 = \alpha(P(Q_1) - p_2) \Leftrightarrow p_1 = (1-\alpha)P(Q_1) + \alpha P(Q_2) \end{aligned}$$

   Because the slope of $u^{certainty}(v) = v - p_1$ with respect to v is steeper than the slope of $u^{lottery}(v) = \alpha(v - P(Q_2))$, we have single-crossing of the two curves.
   Hence, consumers with valuations between $P(Q_2)$ and $P(Q_1)$ prefer the lottery, while consumers with valuations above $P(Q_1)$ prefer the market with certainty. Consumers with valuations below $P(Q_2)$ prefer not to participate at all.

   Revenue R^L of the lottery mechanism is:

   $$\begin{aligned} R^L(Q, Q_1, Q_2) &= Q_1 p_1 + (Q - Q_1)p_2 \\ &= Q_1[(1-\alpha)P(Q_1) + \alpha P(Q_2)] + (Q - Q_1)P(Q_2) \\ &= (1-\alpha)R(Q_1) + \alpha R(Q_2) \end{aligned}$$

   Thus, $R^L > R(Q)$ (i.e., the two-price revenue is greater than the the single-price revenue) iff R is not concave at Q, i.e., the lottery is not optimal if R is concave.
   Maximising $R^L(Q, Q_1, Q_2)$ over Q_1, Q_2 yields $\bar{R}(Q) = R(Q, Q_1^*, Q_2^*)$, where $\bar{R}$ is the concavification of the revenue function, and Q_1^*, Q_2^* denote optimal parameters that are chosen to maximise $\bar{R}$.

   (b) Graphical notes:
   Suppose revenue is non-concave, with two local maxima Q_l and Q_h, where $Q_l < Q_h$ and $R(Q_l) < R(Q_h)$.
   Under uniform pricing and assuming zero costs, the monopolist will sell at Q_h if quantity is not constrained, and sell at Q_l if quantity is constrained by k such that $Q_l < k < Q_h$ and $R(k) < R(Q_l)$.

   Under a two-price mechanism, the monopolist concavifies its revenue:

   $$\bar{R}(Q) = \begin{cases} R(Q), & Q \notin (Q_1^*, Q_2^*) \\ R(Q_1^*) + (Q - Q_1^*)R'(Q_1^*), & Q \in (Q_1^*, Q_2^*) \end{cases}$$

   The optimal quantities (Q_1^*, Q_2^*) are chosen by the slope-secant-slope condition such that the marginal revenues along the slope are equal, i.e.:

   $$R'(Q_1^*) = R'(Q_2^*) = \frac{R(Q_2^*) - R(Q_1^*)}{Q_2^* - Q_1^*}$$

Note that underpricing/ the two-price mechanism pays of iff revenue is not concave.
To see this, let $< q, t >$ be a mechanism that allocates the good to a buyer who reports to be of type v, with probability $q(v)$ and asks the buyer to pay $t(v)$.
$< q, t >$ satisfies IC if no agent/type benefits from misreporting, i.e.,

$$q(v)v - t(v) \geq q(\hat{v})v - t(\hat{v})$$
$$q(v)\hat{v} - t(v) \leq q(\hat{v})\hat{v} - t(\hat{v})$$

i.e., the utility for each individual in the case where they report their identity correctly should be higher than the utility where they report as being of a different identity.
Subtracting the second line from the first line gives

$$q(v)(v - \hat{v}) \geq q(\hat{v})(v - \hat{v})$$

That is, $< q, t >$ being IC implies that q is non-decreasing.
In other words, non-concave revenue induces a tension between rent extraction and consumers' incentive compatibility and the mechanism requires that no agent/ type of consumer benefits from misreporting (e.g., if the high price has a lower probability of getting the good than the low price, all consumers would just report as being the low value consumers).

Lottery is equivalent to "ironing" of marginal revenue (Myerson 1981) since in the case of non-monotone marginal revenue, we have the non-concave revenue function as discussed.
The monopoly wants to sell to consumers with the highest MR, but consumers' IC constraints prohibit this and thus the uniform probability offered by rationing and the two-price mechanism is the best it can do.

The prior discussions assumed zero costs.
In the case of constant marginal costs or increasing marginal costs, the marginal cost curve must intersect the ironed marginal revenue curve in the ironed component of the MR curve in order for the lottery to be optimal. Otherwise, it is optimal for the monopoly to just carry out uniform pricing (i.e., no lottery).

(c) Discussion:
Integrating two markets into one will induce non-concave revenue and hence there is empirical support for this phenomenon.
The inefficiency (in welfare terms) due to random rationing is deliberate and by design, and cause of scope for resale (see Lecture 4).
Private information and consumers' IC also immediately rule out perfect price discrimination, because if the probability of allocation is the same for consumers of two different values, then the amount they pay must be is the same. Otherwise, both consumers would choose the allocation that induces the smaller payment, which violates IC for the consumer who lies.

(d) Example:
Consumer a monopoly with a mass $K > 0$ of homogenous goods, and an inverse demand function of

$$P(Q) = \begin{cases} 1 - \frac{1-a}{\underline{Q}}Q, & Q \in [0, \underline{Q})] \\ \frac{a}{1-\underline{Q}}(1 - Q), & Q \in [\underline{Q}, 1] \end{cases}$$

We can consider this intuitively as an integrated market with a mass $\frac{1-a-\underline{Q}}{1-a}$ of ordinary consumers with values uniform on $[0, a]$, and a mass $\frac{\underline{Q}}{1-a}$ of rich consumers with values uniform on $[0, 1]$.
(refer to Tutorial 3 for details).

2. **Monopsony pricing revisited (Loertscher and Muir 2024: Wage dispersion, minimum wages and involuntary unemployment**

(a) Involuntary unemployment (which is induced by efficiency wage) is optimal iff cost of procurement is not convex.
Minimum wage can decrease or eliminate involuntary unemployment and increase employment.
Whether there is wage dispersion provides guidance as to the effects of minimum wages.

(b) Setup: monopsony employer faces an inverse labour supply function $W(Q)$, which gives the market-clearing wage for hiring Q workers.
$W(Q)$ arises from a continuum of workers each supplying one unit of labour inelastically, with $W(Q)$ representing the opportunity cost of working for the worker with Q^{th} lowwest opportunity cost.
Each worker has private information about its opportunity cost.

$W(Q)$ is strictly increasing and continuously differentiable, and hence the cost of procurement $C(Q) := W(Q)Q$ of hiring Q workers at the market-clearing wage is strictly increasing and differentiable, satisfying

$$C'(Q) = W(Q) + W'(Q)Q \geq W(Q)$$

where $W(Q)$ is the market-clearing wage.

Consider a two-wage mechanism where the monoposony sets two wages w_1 and w_2 satisfying $w_1 < w_2$.
In equilibrium, the mass Q_1 of workers with the lowest opportunity cost are hired at wage w_1 with certainty, while the mass $Q - Q_1$ of workers with opportunity costs between $W(Q_1)$ and $W(Q_2)$ are hired at w_2 with probability $\alpha = \frac{Q-Q_1}{Q_2-Q_1}$.
Similar to the lottery case for monopsony, each worker can only participate in one market: either the w_1 market or the lottery market.

For exactly Q_2 workers to participate, we need $w_2 = W(Q_2)$.
Meanwhile, workers with opportunity cost $W(Q_1)$ need to be indifferent between being employed for sure at w_1 and being employed at wage w_2 with probability α, so:

$$\begin{aligned} w_1 - W(Q_1) = \alpha(W(Q_2) - W(Q_1)) \implies \\ w_1(Q) = (1-\alpha)W(Q_1) + \alpha W(Q_2) \\ = w(Q_1) + (Q - Q_1)\frac{W(Q_2) - W(Q_1)}{Q_2 - Q_1} \end{aligned}$$

Similar to the case of monopoly, because the slope of $w_1 - c$ with respect to c is steeper than the slope of $\alpha(W(Q_2) - c)$, we have single-crossing of the two curves.
Hence, workers with opportunity costs between $W(Q_2)$ and $W(Q_1)$ prefer the lottery, while workers with valuations below $W(Q_1)$ prefer to be hired at the low wage with certainty (workers with opportunity costs above $W(Q_2)$ prefer not to participate at all).

The total wage payment to the low-wage workers is:

$$Q_1 w_1 = (1-\alpha)C(Q_1) + \alpha W(Q_2)Q_1$$

The total wage payment to the high-wage workers is:

$$(Q - Q_1)w_2 = W(Q_2)(Q - Q_1) = W(Q_2)\alpha(Q_2 - Q_1)$$

Hence, procurement cost C^L with efficiency wage and involuntary unemployment is

$$C^L(Q, Q_1, Q_2) = Q_1 W_1 + (Q - Q_1)w_2 = (1-\alpha)C(Q_1) + \alpha C(Q_2)$$

Consequently, $C^L(Q, Q_1, Q_2) < C(Q)$ iff $C(Q)$ is not convex at Q.

(c) Graphical notes:
If procurement costs are not convex, the monopsonist "irons" the marginal cost and implements a two-wage mechanism:

$$\underline{C}(Q) = \begin{cases} C(Q), & Q \notin (Q_1^*, Q_2^*) \\ C(Q_1^*) + (Q - Q_1^*)C'(Q_1^*), & Q \in (Q_1^*, Q_2^*) \end{cases}$$

The optimal quantities (Q_1^*, Q_2^*) are chosen by the slope-secant-slope condition such that the marginal costs along the slope are equal, i.e.:

$$C'(Q_1^*) = C'(Q_2^*) = \frac{C(Q_2^*) - C(Q_1^*)}{C_2^* - C_1^*}$$

The low wage

$$w_1(Q) = W(Q_1) + (Q - Q_1)\frac{W(Q_2) - W(Q_1)}{Q_2 - Q_1}$$

is linear on $[Q_1, Q_2]$.
For $Q \in [Q_1, Q_2]$, $\underline{C}'(Q)$ is constant and above $W(Q)$.

(d) Discussion: The intuition is that the monopsony wants to procure Q units of labour at the lowest marginal cost.
Since the cost curve C is not convex, the marginal cost C' is not monotone before ironing.
This conflicts with workers' incentive compatibility as workers with opportunity costs $W(Q') < W(Q'')$ cannot be hired with lower probability than workers with opportunity costs $W(Q'')$.
Random rationing, achieved with efficiency wage and involuntary unemployment, maximises the probability that low marginal cost (and high opportunity cost) workers are hired, subject to hiring the higher marginal cost (and lower opportunity cost) workers with at least that probability.

The monopsony's willingness to pay (or marginal profit) for the Q^{th} unit of input is $V(Q)$, which is assumed to be strictly decreasing and continuous.
Assume $V(0) > W(0)$ (so the monopoly hires nonzero labour) and $V(Q) < W(Q)$ for sufficiently large Q.
Efficient (price-taking) quantity Q^p satisfies $V(Q^p) = W(Q^p)$.
The monopsony is said to use an efficiency wage if it hires Q workers and its optimal wage schedule involves a wage $w_i > W(Q)$ at which it hires a positive mass of workers.
Optimal procurement involves efficiency wage and involuntary unemployment (i.e., the random rationing/ two-wage mechanism) iff $V(Q^*)$ intersects the ironed marginal cost $\underline{C}'(Q)$ in the "flat" range, i.e., if $Q_1 < Q^* < Q^2$, where the Q_i are the optimal ironing parameters.

3. **Minimum wage effects**

(a) Minimum wages in price-taking scenario:
Efficient (price-taking) wage and quantity of labour $W(Q^p), Q^p$ are where $V(Q)$ and $W(Q)$ curves intersect.
If minimum wages are below the equilibrium wage $W(Q^p)$, then the minimum wage is non-binding.
If minimum wages are above the equilibrium wage $W(Q^p)$, then the minimum wage induces involuntary unemployment as the demand for labour falls while the supply of labour rises.

(b) Minimum wages under monopsony power:
Efficient (price-taking) wage and quantity of labour $W(Q^p), Q^p$ are where $V(Q)$ and $W(Q)$ curves intersect.
However, because monopsonist has market power, it has a marginal cost curve $C'(Q)$ which lies above the $W(Q)$ curve (as seen in Lecture 2). The monopsony's optimal wage and quantity $W(Q^*), Q^*$ are where $C'(Q)$ and $V(Q)$ intersect, and thus $W(Q^p) > W(Q^*)$ and $Q^p > Q^*$, i.e., the monopsony hires a socially inefficiently small number of workers and at lower wages than under efficiency.
If minimum wages are below $W(Q^*)$, they are non-binding.
If minimum wages are between $W(Q^*)$ and $W(Q^p)$, minimum wages will increase employment.
If minimum wages are above $W(Q^p)$, they will decrease employment and increase involuntary unemployment (similar to case of price-taking scenario).

(c) Minimum wages under two-wage mechanism and monopsony power:
$W(Q)$ and hence $C'(Q)$ are both non convex (they are piecewise linear).
Monopsonist has market power and its marginal cost curve $C'(Q)$ lies above the $W(Q)$ curve, but now from some region Q_1 to Q_2 the monopsonist "irons" the marginal cost curve.
Consequently, the ironed wage $W_1(Q)$ is lower than $W(Q)$ over the region (Q_1, Q_2) (visualise as a straight line between the two points $W(Q_1)$ and $W(Q_2)$).
Efficient (price-taking) wage and quantity of labour $W(Q^p), Q^p$ are where $V(Q)$ and $W(Q)$ curves intersect.
Similar to the previous scenario, the monopsony's optimal wage and quantity $W(Q^*), Q^*$ are where $C'(Q)$ and $V(Q)$ intersect, and thus $W(Q^p) > W(Q^*)$ and $Q^p > Q^*$, i.e., the monopsony hires a socially inefficiently small number of workers and at lower wages than under efficiency.

If minimum wages are below $W_1(Q^*)$, they are non-binding. Note that there is involuntary unemployment iff $Q^* \in (Q_1, Q_2)$ as there are more individuals willing to work than there are workers.
If minimum wages are between $W(Q^*)$ and $W(\hat{Q})$ (*where* $W(\hat{Q})$ *is defined as the smallest quantity* $Q \leq Q^p$ *such that for all* $\underline{w} \in [W(Q), W(Q^p)]$, *the monopsony optimally hires* $S(\underline{w})$ *workers at the minimum wage* $\underline{w}$), minimum wages will increase employment and decrease involuntary unemployment.
If minimum wages are between $W(\hat{Q})$ and $W(Q^p)$, minimum wages will increase employment.
If minimum wages are above $W(Q^p)$, they will decrease employment and increase involuntary unemployment (similar to case of price-taking scenario).

(d) Minimising procurement cost:
Suppose $C(Q)$ is convex. $C_F(Q, \underline{w})$ (i.e., cost of procurement with a wage floor) is:

$$C_F(Q, \underline{w}) = \begin{cases} \underline{w} \cdot Q, & Q \leq S(\underline{w}) \\ C(w), & Q > S(\underline{w}) \end{cases}$$

However, now with non-convex marginal costs, the monopolist has to consider three different regions:

$$C_F(Q, \underline{w}) = \begin{cases} \underline{w} \cdot Q, & Q \leq S(\underline{w}) \\ \text{(nontrivial)} & Q \in (S(\underline{w}, w_1^{-1}(\underline{w})) \\ C(w), & Q > w_1^{-1}(\underline{w}) \end{cases}$$

where $w_1^{-1}(\underline{w})$ is the point on the ironed wage W_1 where wages $= \underline{w}$.

For $Q \in (S(\underline{w}, w_1^{-1}(\underline{w}))$, the minimal cost of procurement $\underline{C}(Q, \underline{w})$ is convex in Q, increasing in both arguments, and satisfies

$$\frac{\partial \underline{C}'(Q, \underline{w})}{\partial Q} > 0 > \frac{\partial \underline{C}'(Q, \underline{w})}{\partial \underline{w}}$$

Optimal quantity given by intersection of $V(Q)$ and $\underline{C}'(Q, \underline{w})$ increases in $\underline{w}$.

(e) Discussion:
When there is involuntary unemployment and wage dispersion at a given minimum wage, a sufficiently small increase in the minimum wage increases employment and decreases involuntary unemployment.
If there is involuntary unemployment and no wage dispersion at a given minimum wage, increasing the minimum wage decreases employment and increases involuntary unemployment.
Moreover, provided the minimum wage $\underline{w} \neq W(Q^p)$, if there is no involuntary unemployment at a given minimum wage, a sufficiently small increase in the minimum wage increases employment.

Lecture 4

1. **Resale (Loertscher and Muir 2022, 2024a)**

(a) Graphical notes and recap:
Suppose revenue is non-concave, with two local maxima Q_l and Q_h, where $Q_l < Q_h$ and $R(Q_l) < R(Q_h)$. Also suppose that quantity is constrained by k such that $Q_l < k < Q_h$ and $R(k) < R(Q_l)$.
Without randomisation, the firm will sell at the first local maxima Q_l.
With randomisation, the firm concavifies its revenue:

$$\bar{R}(Q) = \begin{cases} R(Q), & Q \notin (Q_1^*, Q_2^*) \\ R(Q_1^*) + (Q - Q_1^*)R'(Q_1^*), & Q \in (Q_1^*, Q_2^*) \end{cases}$$

The optimal quantities (Q_1^*, Q_2^*) are chosen by the slope-secant-slope condition such that the marginal revenues along the slope are equal, i.e.:

$$R'(Q_1^*) = R'(Q_2^*) = \frac{R(Q_2^*) - R(Q_1^*)}{Q_2^* - Q_1^*}$$

(b) When we introduce resale, the focus is on the goods and monopolist in the goods market.
There is no clear analogue for the labour market context - the closest is subcontracting.
Note that besides what is discussed here, there may be other reasons for the optimal mechanism/ randomisation (e.g., demand uncertainty) or for resale (e.g., preference shocks).

Setup: Private information implies that perfect first-degree price discrimination is not possible.
Uniform pricing (i.e., no price discrimination) is optimal if revenue is concave.
Randomisation is in the form of conflation (where differentiated goods are sold at a single opaque price, and buyers learn product purchased only after payment), with rationing as a special case, is part of the optimal selling mechanism only if revenue is non-concave.
Randomisation in the form of conflation/ rationing is also a form of price discrimination.
Randomisation leads to larger quantities sold, relative to when market clearing pricing is imposed.
If effect is sufficiently strong, randomisation increases consumer surplus.
Resale ensuring from randomisation harms the monopoly without necessarily eliminating all gains from randomisation.
Resale prohibition can increase consumer surplus, depending on the size of the quantity effect due to resale.

We saw in Lecture 3 that a two-price mechanism with underpricing and random rationing outperforms uniform pricing when revenue is not concave.
Because random allocations are inefficient, such two-price mechanism open scope for resale among the consumers.
Resale potentially harms the monopoly seller.

We model the resale market by assuming that a competitive resale market operates with probability $\rho \in [0, 1]$.
In the resale market, all resale transactions occur at the competitive price.
Three-step procedure to set up ρ-competitive resale market:

i. If monopoly sells Q_1 to the highest value consumers and randomly allocates $Q - Q_1$ units among the $Q_2 - Q_1$ consumers with the next highest value, what is the market clearing price in the resale market (provided it operates)?
ii. Given this price, what is the revenue of the monopoly anticipating that a competitive resale market will operate with probability ρ?
iii. What is the optimal quantity sold by monopoly given ρ?

Note that the resale market is confined to the consumers with values $v \in [Q_1^*, Q_2^*]$. Consumers with higher values would already have purchased the good with certainty from the monopoly, and consumers with lower values would not participate in either market.
In the resale market, since only α share of the consumers have the good, α will be sellers and $1 - \alpha$ will be buyers.

Step 1: Market clearing price in resale market.
Let $\alpha = \frac{Q - Q_1}{Q_2 - Q_1}$ and denote by $D(p)$ the demand function, i.e., $D(p) = P^{-1}(p)$.
The demand function $D^R(p)$ in the resale market for $p \in [P(Q_2), P(Q_1)]$ is

$$D^R(p) = (1 - \alpha)[D(p) - Q_1]$$

since Q_1 units have already been allocated and the fraction α of consumers obtained a unit in the lottery. The demand function thus represents the remaining individuals who did not receive any goods.
$Q \in [Q_1, Q_2]$, and $D^R(P(Q_2)) = (1 - \alpha)(Q_2 - Q_1) = (Q_2 - Q)$ (i.e., if resale is at the low price, all consumers from Q_2 to Q demand the good), and $D_R(P(Q_1)) = (1 - \alpha)(Q_1 - Q_1) = 0$ (i.e., if resale is at the high price, there will be no demand in the resale market).

The supply function $S^R(p)$ in the resale market for $p \in [P(Q_2), P(Q_1)]$ is

$$S^R(p) = \alpha[Q_2 - D(p)]$$

since $Q_2 - D(p)$ is the mass of individuals with values less than p who participated in the lottery, out of which the fraction α obtained the good.

Similar to the case for demand, $Q \in [Q_1, Q_2]$, and $S^R(P(Q_2)) = \alpha(Q_2 - Q_2) = 0$, and $S_R(P(Q_1)) = \alpha(Q_2 - Q_1) = Q - Q_1$.

The market clearing condition is $D^R(p^*) = S^R(p^*)$, which is equivalent to

$$
\begin{aligned}
(1-\alpha)[D(p^*) - Q_1] &= \alpha[Q_2 - D(p^*)] \\
D(p^*) - (1-\alpha)Q_1 &= \alpha Q_2 \\
D(p^*) &= Q
\end{aligned}
$$

that is, $p^* = P(Q)$ (note that this means that it is independent of Q_1^* and Q_2^*).

Step 2: Revenue anticipating ρ-competitive resale.
Incentive compatibility and individual rationality for consumers means that they must be indifferent as to whether there is a resale market or not.

For consumers with value $P(Q_2)$, this means that

$$p_2^\rho = (1-\rho)P(Q_2) + \rho P(Q)$$

(i.e., the price they are willing to pay is the probability of no resale market x the outcome of buying in the lottery, plus the probability of having a resale market x the outcome of buying in the resale market).

For consumers with value $P(Q_1)$,

$$P(Q_1) - p_1^\rho = \alpha(P(Q_1) - p_2^\rho) + (1-\alpha)\rho(P(Q_1) - P(Q))$$

Combining both equations yields

$$p_1^\rho = (1-\rho)[(1-\alpha)P(Q_1) + \alpha P(Q_2)] + \rho P(Q)$$

Revenue with lottery and ρ-competitive resale is

$$
\begin{aligned}
R^{L,\rho}(Q, Q_1, Q_2) &= Q_1 p_1^\rho + (Q - Q_1)p_2^\rho \\
&= (1-\rho)R^L(Q, Q_1, Q_2) + \rho R(Q)
\end{aligned}
$$

which is maximised at the same Q_1^* and Q_2^* that maximise $R^L(Q, Q_1, Q_2)$.
Letting

$$\bar{R}(Q) = R^L(Q, Q_1^*, Q_2^*) \text{ and } \bar{R}^\rho(Q) = R^{L,\rho}(Q, Q_1^*, Q_2^*)$$

we thus have

$$\bar{R}^\rho(Q) = (1-\rho)\bar{R}(Q) + \rho R(Q)$$

In other words, resale undermines concavification. When $\rho = 0$, the concavified revenue is identical to the lottery case without resale. When $\rho = 1$, revenue is identical to the case with no lottery (i.e., the original nonconcavified revenue).

Step 3: Effects on quantity sold.
For a fixed Q, consumers are better off under resale because allocation is more efficient and $\bar{R}^\rho(Q)$ decreases in ρ.
But resale can decrease Q in equilibrium. If the monopolist is capacity constrained, a sufficiently large ρ may cause it to reduce its quantity sold in order to maximise its revenue, moving it closer to the quantity it would have sold under uniform pricing (i.e., the local maxima of the non-concave revenue curve).
This is why consumer surplus may also be larger if resale is prohibited altogether.

2. **Consumer surplus effects of resale**

(a) Keeping Q fixed, ρ-competitive resale increases CS because it reduces revenue and increases the efficiency of the allocation.

Assume $R(Q)$ has two local maxima, Q_L and Q_H satisfying $R(Q_H) > R(Q_L)$.
Let $\bar{Q} \in (Q_L, Q_H)$ be such that $R(\bar{Q}) = R(Q_L)$

Denote by $CS^R(Q)$ the CS with randomisation (i.e., the two-price mechanism) and by $CS^P(Q)$ the CS with the "pure" mechanism without randomisation.
With ρ-competitive resale, CS is

$$(1-\rho)CS^R(Q) + \rho CS^P(Q)$$

where

$$CS^R(Q) = \int_0^{Q_1^*} P(x)dx + \frac{Q - Q_1^*}{Q_2^* - Q_1^*} \int_{Q_1^*}^{Q_2^*} P(x)dx - \bar{R}(Q)$$
$$CS^P(Q) = \int_0^{Q} P(x) - P(Q)dx = \int_0^{Q} P(x)dx - R(Q)$$

where $\bar{R}(Q)$ is the total payment made by consumers for the goods in the lottery scenario and $R(Q)$ is the total payment made for the goods in the scenario without randomisation.

Proposition: Suppose the monopoly faces a ρ-competitive resale market. Then:

i. For any $k_1 \in (Q_L, \bar{Q})$, there exists $\hat{\rho}(k_1) \in (0,1)$ such that the monopoly optimally sells all k_1 units of quality 1 iff $\rho \in [0, \hat{\rho}(k_1)]$ and $CS^\rho(k_1)$ is strictly increasing in ρ for $\rho \in [0, \hat{\rho}(k_1)$.
ii. If $k_1 \in (\underline{Q}, \bar{Q})$ then $CS^\rho(k_1, \theta)$ decreases discontinuously at $\rho = \hat{\rho}(k_1)$.
iii. If $\hat{Q} < \bar{Q}$ and $k_1 \in (\hat{Q}, \bar{Q})$, then there exists $\check{\rho}(k_1) \in [\check{\rho}(k_1), 1)$ such that expected equilibrium consumer surplus is strictly higher under resale prohibition for any $\rho \in (\check{\rho}(k_1), 1]$.

(b) Graphical illustration: When ρ is low, CS under randomisation is higher than CS without randomisation.
However, when ρ is high enough, CS falls sharply because the monopolist makes a sharp fall in its quantity to maximise revenue.

(c) ρ-competitive resale vs. random matching and take-it-or-leave-it (TIOLI) offers:
Similar to the scenario of ρ-competitive resale, resale with random matching and TIOLI offers decreases the monopolist's revenue in comparison to the scenario with randomisation but without resale. Because trade between buyers and sellers only takes place when the buyer's valuation exceeds the seller's costs, i.e., $v \geq c$, if v and c are both uniformly distributed on $[0,1]$, trade only occurs in a subset of all efficient trades.

3. **Price regulation revisited (Loertscher and Muir 2023 a,b)**

(a) **Price floors with two-price mechanism:**
Consider a non-concave revenue function $R(Q)$ with an unique ironing interval Q_1, Q_2 satisfying $R(Q_1) < R(Q_2)$ with R being twice continuously differentiable.
Suppose the monopoly can sell up to K units at 0 marginal cost (and cannot sell additional units), with $Q_1 < K < Q_2$.
Assume also (to simplify) that $R(K) > \sup_{Q \in [0,K)} R(Q)$.

Consider then a regulator imposing a price floor $\underline{p}$ satisfying $\underline{p} \in (P(Q_2), P(K)]$, i.e., the price floor is between the lower bound (so it is binding; $\underline{p} > P(Q_2^*) \implies D(\underline{p}) < Q_2^*$) and the price at the quantity K (which is the capacity constraint).

The monopoly's optimum in the presence of the price floor $\underline{p}$ will be the two-price mechanism parameterised by quantities $q_2^F(\underline{p}) = D(\underline{p})$ and $q_1^F(\underline{p})$, where

$$R'(q_1^F(\underline{p})) = \frac{R(D(\underline{p})) - R(q_1^F(\underline{p}))}{D(\underline{p}) - q_1^F(\underline{p})}$$

As $\underline{p}$ increases, $q_2^F(\underline{p})$ decreases and $q_1^F(\underline{p})$ increases.
The maximum revenue given $\underline{p}$ is

$$\bar{R}_F(Q,\underline{p}) = \begin{cases} R(Q), & Q \le q_1^F(\underline{P}) \\ R(q_1^F(\underline{p})) + (Q - q_1^F(\underline{p}))R'(q_1^F(\underline{p})), & Q \in (q_1^F(\underline{p}), D(\underline{p})] \end{cases}$$

Note that $\bar{R}_F(Q,\underline{p})$ decreases in $\underline{p}$ since $q_1^F(\underline{p})$ increases in $\underline{p}$.

Social surplus given the price floor $\underline{p}$ is:

$$SS_F(\underline{p}) = \int_0^{q_1^F(\underline{p})} P(x)dx + \frac{K - q_1^F(\underline{p})}{D(\underline{p}) - q_1^F(\underline{p})} \int_{q_1^F(\underline{p})}^{D(\underline{p})} P(x)dx$$

$SS_F(\underline{p})$ is increasing in $(\underline{p})$ on $(P(Q_2), P(K))$. To see this, consider for $a < K < b$ and $\beta = \frac{K-a}{b-a}$, the function $H(a,b) = \int_0^a P(x)dx + \frac{K-a}{b-a}\int_a^b P(x)dx$ and notice:

$$\frac{\partial H}{\partial a} = (1-\beta)[P(a) - \frac{\int_a^b P(x)dx}{b-a}] \ge 0$$
$$\frac{\partial H}{\partial b} = -\beta[\frac{\int_a^b P(x)dx}{b-a} - P(b)] < 0$$

where the inequalities hold because P is decreasing.
Because consumer surplus given $\underline{p}$ is $CS_F(\underline{p}) = SS_F(\underline{p}) - \bar{R}_F(K,\underline{p})$, we have the following result:
Assume $R(K) > \sup_{Q\in[0,K)} R(Q)$. Then for $\underline{p} \in (P(Q_2), P(K))$, the firm's revenue $\bar{R}_F(K,\underline{p})$ decreases as $\underline{p}$ increases while both social surplus and cosumer surplus, i.e., $SS_F(\underline{p})$ and $CS_F(\underline{p})$, increase in $\underline{p}$.
In other words, with non-concave revenue, price floors can increase social and consumer surplus.

(b) **Recap: Price ceilings without two-price mechanism:**
Assume revenue is concave and there is a price ceiling $\bar{P}$.
The monopolist's revenue in the presence of the price ceiling is:

$$R_c(Q,\bar{P}) = \begin{cases} \bar{P}\cdot Q, & Q \le D(\bar{P}) \\ R(Q), & \text{otherwise} \end{cases}$$

and marginal revenue is:

$$R_c'(Q,\bar{P}) = \begin{cases} \bar{P}, & Q \le D(\bar{P}) \\ R'(Q), & \text{otherwise (note the discontinuous drop in MR).} \end{cases}$$

where $R'(Q) = P(Q) + P'(Q)Q < P(Q)$.

(c) **Price ceiling with two-price mechanism:**
Now assume that R exhibits a unique ironing interval (Q_1, Q_2) satisfying $R(Q_2) > R(Q_1)$.
For $Q \in (Q_1, Q_2)$, let

$$p_1(Q) = \frac{Q_2 - Q}{Q_2 - Q_1}P(Q_1) + \frac{Q - Q_1}{Q_2 - Q_1}P(Q_2)$$

be the high price set by the monopoly under laissez faire (this is the high price in the two-price mechanism).

Let $\bar{R}_c(Q,\bar{p})$ be the maximum revenue of the monopoly when selling the quantity Q facing the price ceiling $\bar{p}$.

i. If $\bar{p} > p_1(Q)$, then $\bar{R}_c(Q,\bar{p}) = \bar{R}(Q)$, i.e., the price ceiling is not binding.
ii. If $\bar{p} < P(Q)$, then $\bar{R}_c(Q,\bar{p}) = \bar{p}Q$, i.e., the monopolist just sells all its quantity at the price ceiling.

iii. Our interest is what is the optimal selling mechanism for Q when

$$P(Q) < \bar{p} < p_1(Q)$$

and here we will focus on two-price mechanisms, parameterised by q_1 and q_2.

If q_1 and q_2 are optimal given Q and $\bar{p}$, then

$$(\frac{P(q_2) - P(q_1)}{q_2 - q_1})^2 = P'(q_2)P'(q_1) \tag{Lemma 1}$$

In other words, the square of the slope of the high price function is the product of the slopes at q_1 and q_2.
For P piecewise linear, this implies that $\frac{P(q_2)-P(q_1)}{q_2-q_1} = \frac{P(Q_2)-P(Q_1)}{Q_2-Q_1}$ (i.e., the slope of $P_1(Q)$) since P' is constant on each segment.
Only the vertical intercept of the high-price function adjusts, in such a way that

$$P(q_1) + (Q - q_1)\frac{P(Q_2) - P(Q_1)}{Q_2 - Q_1} = \bar{p}$$

For piecewise linear demand and $\bar{p} \in (P(Q), p_1(Q))$, Lemma 1 implies that the optimal values of q_1 and q_2, denoted $q_1^C(\bar{p})$ and $q_2^C(\bar{p})$, are such that

$$P(q_1^C(\bar{p})) + (Q - q_1^C(\bar{p}))\frac{P(Q_2) - P(Q_1)}{Q_2 - Q_1} = \bar{p}$$

Totally differentiating, we obtain:

$$\frac{dq_1^c}{d\bar{p}} = -\frac{1}{B - P_1'}$$
$$\frac{dq_1^c}{dQ} = BB - P_1'$$

where B is the slope of the high price function.
Both derivatives are constants, i.e., $q_1^c(\bar{p})$ is linear in $\bar{p}$ and Q.
Moreover, the cross-partials are zero, i.e.,

$$\frac{\partial \frac{dq_1^c}{dp}}{\partial Q} = \frac{\partial \frac{dq_1^c}{dQ}}{\partial p} = 0$$

Given $q_1^c(\bar{p})$, $q_2^c(\bar{p})$ is pinned down by the equality

$$\frac{P(q_2^c(\bar{p})) - P(q_1^c(\bar{p}))}{q_2^c(\bar{p}) - q_1^c(\bar{p})} = \frac{P(Q_2) - P(Q_1)}{Q_2 - Q_1}$$

implying that $q_2^c(\bar{p})$ is also linear in $\bar{p}$ and Q and has cross partials $= 0$.
Accordingly, for piecewise linear demand, maximised revenue given $\bar{p} \in (P(Q), p_1(Q))$ is

$$\begin{aligned}\bar{R}_C(Q, \bar{p}) &= \tilde{R}(q_1^C(\bar{p}), q_2^C(\bar{p})) \\ &= \bar{p}q_1^C(\bar{p}) + (Q - q_1^c(\bar{p}))\frac{P(Q_2) - P(Q_1)}{Q_2 - Q_1} \\ &= \bar{p}Q + (Q - q_1^C(\bar{p}))(q_2^C(\bar{p}) - Q)\frac{P(Q_2) - P(Q_1)}{Q_2 - Q_1}\end{aligned}$$

Because $q_1^c(\bar{p})$ and $q_2^c(\bar{p})$ are linear in $\bar{p}$ and Q with zero cross partials, and given the functional form of $\bar{R}_c(Q, \bar{p})$, it follows that $\bar{R}_c(Q, \bar{p})$ is quadratic in Q.
This implies $\bar{R}_c'(Q, \bar{p}) := \frac{\partial R_c(Q,\bar{p})}{\partial Q}$ is linear in Q.
Moreover, $\bar{R}_c'(Q, \bar{p})$ is decreasing in $\bar{p}$.

Proof of Lemma 1:
Revenue under a two-price mechanism given $\bar{p}$ and Q is

$$q_1\bar{p} + (Q - q_1)P(q_2) = Q(\bar{p}) + (Q - q_1)(P(q_2) - \bar{p})$$

The high-price constraint is

$$\frac{q_2 - Q}{q_2 - q_1}P(q_1) + \frac{Q - q_1}{q_2 - q_1}P(q_2) = \bar{p}$$

which must be binding, otherwise $q_i = Q_i$ is the solution, which contradicts the ceiling price since it will make it such that $\bar{p} < p_1(Q)$.
Subtracting the high-price constraint from $P(q_2)$ gives $P(q_2) - \bar{p} = (q_2 - Q)\frac{P(q_2)-P(q_1)}{q_2-q_1}$.
Thus, revenue can be written as

$$\tilde{R}(q_1, q_2) = Q\bar{p} + (Q - q_1)(q_2 - Q)\frac{P(q_2) - P(q_1)}{q_2 - q_1}$$

Note that based on this notation, revenue is quadratic in Q and marginal revenue is linear in Q and linear and increasing in $\bar{p}$.
We will maximise $\tilde{R}(q_1, q_2)$ over q_1 and q_2 subject to the high price constraint, and then use the implicit function theorem to express q_1 as a function of q_2 in the constraint and then use $\tilde{r}'(q_2) = 0$ where $\tilde{r}'(q_2) := \tilde{R}(q_1(q_2), q_2$.
Note that the high price constraint is equivalent to

$$P(q_1) + (Q - q_1)\frac{P(q_2) - P(q_1)}{q_2 - q_1} = \bar{p}$$

where $\frac{P(q_2)-P(q_1)}{q_2-q_1} := B$ is the slope of the high price function.
Observing that $\frac{\partial B}{\partial q_1} = \frac{B-P'(q_1)}{q_2-q_1}$ and $\frac{\partial B}{\partial q_2} = \frac{P'(q_2)-B}{q_2-q_1}$, totally differentiating this form of the high price constraint gives:

$$-(1 - \beta)[B - P'(q_1)]dq_1 - \beta[B - P'(q_2)]dq2 = 0$$

where $\beta = \frac{Q-q_1}{q_2-q_1}$
Thus, $\frac{dq_1}{dq_2} = -\frac{\beta}{1-\beta}\frac{B-P'(q_2)}{B-P'(q_1)}$.

$$\begin{aligned}
\tilde{r}'(q_2) &= (Q - q_1)[B - (q_2 - Q)\frac{B - P'(q_2)}{q_2 - q_1}] + (q_2 - Q)[(Q - q_1)\frac{B - P'(q_1)}{q_2 - q_1} - B]\frac{dq1}{dq2} \\
&= (Q - q_1)[\beta B + (1 - \beta)P'(q_2)] - (q_2 - Q)[(1 - \beta)B + \beta P'(q_1)]\frac{dq_1}{dq_2}
\end{aligned}$$

So, $\tilde{r}'(q_2) = 0$ is equivalent to

$$(Q - q_1)[\beta B + (1 - \beta)P'(q_2)] = (q_2 - Q)[(1 - \beta)B + \beta P'(q_1)]\frac{dq1}{dq2}$$

which is equivalent to

$$\begin{aligned}
\frac{Q - q_1}{q_2 - Q}[\beta B + (1 - \beta)P'(q_2)] &= [(1 - \beta)B + \beta P'(q_1)]\frac{dq1}{dq2} \\
\frac{\beta}{1 - \beta} &= [(1 - \beta)B + \beta P'(q_1)]\frac{dq1}{dq2}
\end{aligned}$$

Substituting in the expression for $\frac{dq_1}{dq_2}$ and writing P_i' in lieu of $P'(q_i)$, this is equivalent to

$$\beta B(B - P_i')) + (1 - \beta)P_2'(B - P_1') = -(1 - \beta)B(B - P_2') - \beta(P_1')(B - P_2')$$

which after some algebra is equivalent to

$$B^2 = P_1'P_2' = 0$$

which was to be shown.

(d) Graphical notes and recap of optimal randomisation with various cost scenarios:
Consider the two-price mechanism with constant marginal costs (i.e., horizontal MC) and concavified MR:

i. If MC intersects MR before the concavified region (i.e., at lower quantity then the concavified quantity), it is optimal for the monopoly to just carry out uniform pricing.
ii. If MC intersects MR after the concavified region, it is optimal for the monopoly to just carry out uniform pricing.
iii. If MC intersects MR exactly at the concavified region, and MC is horizontal, this means that any point of intersection between Q_1 and Q_2 is profit-maximising for the monopolist. The social planner would prefer a higher Q (at Q_2 in this case.

Now consider the two-price mechanism with increasing marginal costs. The two-price mechanism is optimal if MC intersects MR in the concavified region.

If a price ceiling is imposed, the tighter the price ceiling (i.e., the lower the binding price ceiling), the higher the quantity produced by the monopolist at equilibrium.

The price ceiling behaves as an inward parallel shift of the $P_1(Q)$ curve (this is what was shown algebraically earlier, in relation to the slope of the high-price function and the slope of $p_1(Q)$).

Marginal revenue with a price ceiling is:

$$\bar{R}_c'(Q,\bar{p}) = \begin{cases} \bar{p}, & Q \leq D(\bar{p}) \\ \bar{R}_c'(Q,\bar{p}) \text{ (see equation for max revenue above)}, & D(\bar{p}) < Q < P^{-1}(\bar{p}) \\ \bar{R}'(Q), & Q \geq P^{-1}(\bar{p}) \end{cases}$$

4. **Conflation and opaque pricing**

(a) Premise: Vertically differentiated goods are often conflated and sold at a single opaque price. Buyers learn product they purchased only after payment.

(b) Setup:
The monopoly seller has a mass $K \in (0,1]$ of units available in N different qualities $\{\theta_n\}_{n=1}^N$, with quality $\theta_n > \theta_{n+1}$ for $n \in \{1,...,N-1\}$ and $\theta_1 = 1$.
k_n is the mass of units of quality θ_n so that $K = \sum_{n=1}^N k_n$

There is a unit mass of buyers with private values $v \in [0,1]$.
The payoff of buyer with value v from paying p to consume a unit of quality θ with probability q is $q\theta v - p$ (i.e., the buyer's personal value of the good $\times$ the probability of receiving the good - price of the good).
$P(Q) = F^{-1}(1-Q)$ where F denotes the cdf of buyers' values. This admits density f with full support on $[0,1]$.
Note: The support of a real-valued function f is the subset of the function domain containing the elements which are not mapped to zero.

Revenue from selling Q units of quality θ at the market clearing price is

$$\theta R(Q) = \theta P(Q) Q$$

(c) Market clearing prices:
Let $\theta = (\theta_1, ..., \theta_N)$ and $k = (k_1, ..., k_N)$.
Let the quality difference between goods $\Delta_n = \theta_n - \theta_{n+1}$ and $K_{(n)} := \sum_{i=1}^n k_i$
Assume monopoly sells the Q highest quality units at market clearing prices, let $N(Q)$ denote the associated index of lowest quality units.
By IR, the market clearing price for units of quality $\theta_{N(Q)}$ is $p_{N(Q)} = \theta_{n(Q)} P(Q)$

The market clearing price p_n for units of quality θ_n with $n < N(Q)$ satisfies, by IC,

$$p_n = p_{n+1} + \Delta_n P(K_{(n)})$$

Iterative substitution yields

$$p_n = \theta_{N(Q)} P(Q) + \sum_{i=n}^{N(Q)-1} \Delta_i P(K_{(i)})$$

When selling the Q highest quality units at market clearing prices, the monopoly's revenue $R_{\theta,k}(Q)$ is

$$R_{\theta,k}(Q) = \theta_{N(Q)}R(Q) + \sum_{n=1}^{N(Q)-1} \Delta_n R(K_{(n)})$$

Contrast "pure" selling mechanisms, which have market clearing prices, with "categorical" selling mechanisms, where seller can create and price arbitrary (conflated) categories.

i. Category l is conflated if it contains units of multiple quality levels.
ii. Conflated categories are said to be opaquely priced.
iii. Rationing is a special case of conflation, which occurs if any category contains units of quality $\theta_{N+1} = 0$ (i.e., where "nothing" is conflated with the lowest quality good so that not everyone gets a good).
iv. The optimal selling mechanism may involve randomisation in the form of conflation and rationing.

Revenue under the optimal mechanism for selling a given quantity Q is

$$\bar{R}_{\theta,k}(Q) = \bar{R}(Q)\theta_{N(Q)} + \sum_{i=1}^{N(Q)-1} \bar{R}(K_{(i)})\Delta_i$$

where $\bar{R}$ denotes the concavification of R.
Moreover, there is a categorical selling mechanism that achieves $\bar{R}_{\theta,k}(Q)$.

Optimal selling mechanisms now involve randomisation in the form of conflation, impemented using opaque pricing, and rationing, implemented using underpricing.

Categories are formed top-down (i.e., from highest to lowest quality), and then prices are constructed bottom-up (i.e, from lowest to highest price).
At the highest price, there is uniform pricing for (a share of) the highest-quality good.

(d) Revenue maximisation: The revenue maximising monopoly sells Q_K units in total and all k_n units of quality θ_n for all $n < N(Q_k^*)$.
Intuition: For any k_n with $n < N(Q_k^*)$, the maximiser of $\theta_n\bar{R}(Q)$ over $Q \in [0, k_n]$ is k_n because $\bar{R}$ is concave.

By contrast, if the monopoly is restricted to market clearing prices, it may be optimal to sell $q_n < k_n$ high-quality units because $\theta_n R(Q)$ has the same curvature properties as $R(Q)$, implying that non-concavity carries over to these units.

Hence, in the general setting, randomisation has positive quantity and composition effects.

Lecture 5

1. **Model of uncertainty**

(a) Setup: The outcome space is $A \equiv \{a_1, ..., a_i, ..., a_n\}$
The probability that outcome a_i occurs is p_i, where

$$p_i \geq 0 \ \forall i = 1, ..., n \text{ and } \sum_{i=1}^{n} p_i = 1$$

The gamble that yields outcome $a_i \in A$ with probability p_i is

$$(p_1 \circ a_1, ...p_n \circ a_n)$$

and is called a simple gamble.

(b) The **set of simple gambles**, G_S, is

$$G_S = \{(p_1 \circ a_1, ...p_n \circ a_n) | p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$$

The notation $\alpha \circ a_i$ means that a_i occurs with probability α.
When $p_i = 0$ for some i, we just drop this outcome.
The gamble that yields a_i with probability 1 is just denoted as a_i (since only one outcome).

(c) **Compound gambles** are gambles where one or more outcomes is another gamble.
For simplicity, focus on compound gambles that yield an outcome after finitely many randomisations.
Let G be the set of all gambles (compound and simple). If $g \in G$, then $g = (p_1 \circ g^1, ..., p_k \circ g^k)$ for some $k \geq 1$ and $g^i = G$, where g^i can be a compound gamble, simple gamble, or an outcome.

(d) Axioms:
G1: Completeness. For any two g and $g' \in G$, either $g \succcurlyeq g'$ or $g' \preccurlyeq g$.
G2: Transitivity. For any three gambles $g, g', g'' \in G$, $g \succcurlyeq g'$ and $g' \succcurlyeq g''$ implies $g \succcurlyeq g''$.
Axioms G1 and G2 imply that the finitely many elements of A are completely ordered.

G3: Continuity. For any gamble $g \in G$, there is some probability $\alpha \in [0, 1]$ such that $g \sim (\alpha \circ a_1, (1 - \alpha) \circ a_n)$. In other words, the individual is indifferent between gamble g or the new outcome with only two choices (i.e., no inherent aversion to uncertainty).
Note: $\sim$ means "indifferent to".

G4: Monotonicity. For all probabilities $\alpha, \beta \in [0, 1]$,

$$(\alpha \circ a_1, (1 - \alpha) \circ a_n) \succcurlyeq (\beta \circ a_1, (1 - \beta) \circ a_n)$$

iff $\alpha \geq \beta$.
Note that if $a_1 \sim a_n$, then $(\alpha \circ a_1, (1-\alpha) \circ a_n) \sim (\beta \circ a_1, (1-\beta) \circ a_n)$ for any $\alpha, \beta \in [0, 1]$. Since this is ruled out by G4, G4 implies $a_1 \succ a_n$.

Let $g = (p_1 \circ g^1, ..., p_k \circ g^k)$ and $h = (p_1 \circ h^1, ..., p_k \circ h^k)$ be in G.
G5: Substitution. If $h_i \sim g_i$ for every i, then $h \sim g$.
Note: G5 with G1 implies that a decision maker who is indifferent between two gambles must be indifferent between any convex combinations of the two. In particular, if $g \sim h$, then G5 implies

$$(\alpha \circ g, (1 - \alpha) \circ h) \sim (\alpha \circ g, (1 - \alpha) \circ g) = g$$

G6: Reduction to simple gambles. For any $g \in G, (p_1 \circ a_1, ...p_n \circ a_n) \sim g$.
G5 and G6 imply the following more widely known axiom:
Independence Axiom: Let $p = (p_1 \circ a_1, ..., p_n \circ a_n), q = (q_1 \circ a_1, ..., q_n \circ a_n), r = (r_1 \circ a_1, ..., r_n \circ a_n)$ be three simple gambles.
Then, for any $\alpha \in [0, 1]$,

$$p \sim q$$

implies

$$((\alpha p_1 + (1 - \alpha) r_1) \circ a_1, ..., (\alpha p_n + (1 - \alpha) r_n) \circ a_n)$$
$$\sim$$
$$((\alpha q_1 + (1 - \alpha) r_1) \circ a_1, ..., (\alpha q_n + (1 - \alpha) r_n) \circ a_n)$$

i.e., the individual is indifferent to either p or q combined in the same way with a third gamble r.

Proof that G5 and G6 together imply IA: Denote the simple gamble that yields outcome a_i with probability $\alpha p_i + (1 - \alpha) r_i$ by pr and the one that yields a_i with probability $\alpha q_i + (1 - \alpha) q_i$ by qr.
We need to show that if $p \sim q$, then $pr \sim qr$.
Observe that pr and qr are simple gambles induced by the compound gambles $g \equiv (\alpha \circ p, (1-\alpha) \circ r)$ and $h \equiv (\alpha \circ q, (1 - \alpha) \circ r)$. By G6, $pr \sim g$ and $qr \sim h$.
So we only need to show that that $g \sim h$. But G5 implies that $g = (\alpha \circ g^1, (1 - \alpha) \circ g^2) \sim (\alpha \circ h^1, (1 - \alpha) \circ h^2) = h$ if $g^1 \sim h_1$ and $g_2 \sim h_2$. But $g_1 = p$ and $h_1 = q$, so $g_1 \sim h_1$ holds and $g^2 = r = h_2$, so $g^2 \sim h_2$ holds and we are done.

(e) **Expected utility property**:
Let $u : G \to \mathbb{R}$ be a utility function representing $\succcurlyeq$ on G (i.e., it maps from the choice set to a real-valued utility).
The utility function $u : G \to R$ has the expected utility property if for every gamble $g \in G$,

$$u(g) = \sum_{i=1}^{n} p_i u(a_i)$$

where $(p_1 \circ a_1, ..., p_n \circ a_n)$ is the simple gamble induced by g.

2. **Expected utility, non-uniqueness, FOSD**

(a) Expected utility
Utility functions with the expected utility property are called **Von-Neumann-Morgenstern (VNM)** utility functions.
Theorem 2.7: Existence of a VNM utility function on G. Assume $\succcurlyeq$ on G satisfies axioms G1 to G6. Then, there exists a utility function $u : G \to \mathbb{R}$ representing $\succcurlyeq$ that has the expected utility property.

(b) Proof of Theorem 2.7:
The idea is to take some gamble g and find a number $u(g)$ that makes the decision maker indifferent between g and another gamble over the best and the worst outcome a_1 and a_n. This number $u(g)$ will be the utility associated with gamble g.

We take any $g \in G$ and construct $u(g)$ as follows:

$$g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n)$$

$u(g) \in [0, 1]$, i.e., it is a probability. By G3, a number $u(g)$ such that the above equation holds exists. By G4, it is unique. Hence, for every g we can construct a unique number $u(g)$ in this way.
We now show that (i) $u(g)$ represents $\succcurlyeq$ on G, i.e., $g \succcurlyeq g' \Leftrightarrow u(g) \geq u(g')$ and (ii) that $u(g)$ has the expected utility property.

Part (i): Observe that $g \succcurlyeq g'$ iff $(u(g) \circ a_1, (1 - u(g)) \circ a_n) \succcurlyeq (u(g') \circ a_1, (1 - u(g')) \circ a_n)$ because of transitivity (these are the simple gambles for g and g').
But by G4, $(u(g) \circ a_1, (1 - u(g)) \circ a_n) \succcurlyeq (u(g') \circ a_1, (1 - u(g')) \circ a_n)$ iff $u(g) \geq u(g')$ (i.e., there exists a utility function that represents the preferences).

Part (ii): Let g be an arbitrary gamble and let $g_s = (p_1 \circ a_1, ..., p_n \circ a_n)$ be the simple gamble it induces. We need to show that

$$u(g) = \sum_{i=1}^{n} p_i u(a_i)$$

i.e., that $u(g)$ has the expected utility property.
Since $u(g) = u(g_s)$ by G6, it suffices to show that $u(g_s) = \sum_{i=1}^{n} p_i u(a_i)$.
First, observe that $u(a_i)$ is defined as

$$a_i \sim (u(a_i) \circ a_1, (1 - u(a_i) \circ a_n)) \equiv q^i$$

By G5, $g' \equiv (p_1 \circ q^1, ..., p_n \circ q^n) \sim (p_1 \circ a_1, ..., p_n \circ a_n) = g_s$.
The simple gamble g'_s induced by g' is $g'_s = (\sum_{i=1}^{n} p_i u_i(a_i) \circ a_i, 1 - \sum_{i=1}^{m} p_i u_i(a_i) \circ a_n)$.
In other words, a_1 occurs with probability $\sum_{i=1}^{n} p_i u_i(a_i)$ and a_n occurs with probability $1 - \sum_{i=1}^{m} p_i u_i(a_i)$.
Hence,

$$g'_s = (\sum_{i=1}^{n} p_i u(a_i) \circ a_1, (1 - \sum_{i=1}^{n} p_i u(a_i)) \circ a_n)$$

By G6, $g' \sim g'_s$, and hence

$$g_s \sim (\sum_{i=1}^{n} p_i u(a_i) \circ a_1, (1 - \sum_{i=1}^{n} p_i u(a_i)) \circ a_n)$$

But $u(g_s)$ is such that

$$g_s \sim (u(g_s) \circ a_1, (1 - u(g_s)) \circ a_n)$$

where $u(g_s) \sum_{i=1}^{n} p_i u(a_i)$ follows.

Summary of the argument of the proof of part (ii):
For any compound gamble g, $g \sim g_s$ by $G6$, where g_s is the simple gamble induced by $G6$. In general, g_s will yield many outcomes in A with positive probability.
We construct a compound gamble g' whose outcomes are only a_1 and a_n, such that $g' \sim g_s$.
Find the simple gamble induced by g', g'_s, and show that $g'_s = (\sum_{i=1}^{n} p_i u_i(a_i) \circ a_1, (1 - \sum_{i=1}^{n} p_i u_i(a_i)) \circ a_n)$.
By G6, $g'_s \sim g_s \sim g' \sim g$. Observe that $u(g'_s)$ is such that $g'_s \sim (u(g'_s) \circ a_1, (1 - u(g'_s)) \circ a_n)$. Together with 4, this implies $u(g'_s) = \sum_{i=1}^{n} p_i u_i(a_i) = u(g)$ (i.e., the expected utility property).

(c) Uniqueness of VNM utility
Theorem 2.8: VNM utility functions are unique up to positive affine transformations.
Suppose $u(g)$ represents $\succsim$. Then $v(g)$ represents the same preferences iff $v(g) = \alpha + \beta u(g)$, where $\beta > 0$.

(d) Proof of Theorem 2.8:
We only consider simple gambles. Since $u(a_1) \geq u(a_2) \geq ... \geq u(a_n)$ with $u(a_1) > u(a_n)$, there is a unique $\alpha_i \in [0, 1]$ for every $i = 1, ..., n$ such that

$$u(a_i) = \alpha_i u(a_1) + (1 - \alpha_i) u(a_n)$$

(i.e., we reduce the simple gamble to this gamble which has the same utility).
Note that $\alpha_i > 0$ iff $u(a_i) > u(a_n)$.
Because u has the expected utility property, the equation above implies that $u(a_i) = u(\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n)$. Because u represents $\succsim$ this implies

$$a_i \sim (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n)$$

Because v also represents $\succsim$, we must have $v(a_i) = v(\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n)$ and because v has the expected utility property, this implies

$$v(a_i) = \alpha_i v(a_1) + (1 - \alpha_i) v(a_n)$$

Taken together with the equation for $u(a_i)$, we have

$$\frac{u(a_1) - u(a_i)}{u(a_i) - u(a_n)} = \frac{1 - \alpha_i}{\alpha_i} = \frac{v(a_1) - v(a_i)}{v(a_i) - v(a_n)}$$

for every i with $\alpha_i > 0$. Rearranging this yields

$$[u(a_1) - u(a_i)][v(a_i) - v(a_n)] = [v(a_1) - v(a_i)][u(a_i) - u(a_n)]$$

which holds for all i. Rearranging, we can express

$$v(a_i) = \alpha + \beta u(a_i)$$

for all $i = 1, ..., n$ where

$$\alpha \equiv \frac{u(a_1) v(a_n) - v(a_1) u(a_n)}{u(a_1) - u(a_n)} \text{ and } \beta \equiv \frac{v(a_1) - v(a_n)}{u(a_1) - u(a_n)}$$

And now

$$\begin{aligned} v(g) = \sum_{i=1}^{n} p_i v(a_i) &= \sum_{i=1}^{n} p_i (\alpha + \beta u(a_i)) = \alpha + \beta \sum_{i=1}^{n} p_u u(a_i) \\ &= \alpha + \beta u(g) \end{aligned}$$

(e) Preferences over FOSD-ranked gambles

We can show that any decision maker satisfying axioms G1 to G6 (whose preferences can therefore be represented by a VNM utility function) prefers gamble p to gamble q if p **first-order stochastically dominates (FOSD)** q.

We relabel outcomes such that a_1 is the worst and a_n is the best, and also rule out indifferences by assuming that $\forall i < n, a_{i+1} \succ a_i$.

p is said to FOSD q if $\forall k \in \{1, ..., n\}$

$$\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i$$

with strict inequality for at least one k.

Proposition 1: p FOSD q implies $u(p) > u(q)$.

Proof: Let $u_i \equiv u(a_i)$ and notice that

$$\begin{aligned} u(p) = \sum_{i=1}^{n} p_i u_i = &- p_1(u_2 - u_1) - (p_1 + p_2)(u_3 - u_2) - \\ &(p_1 + p2 + p3)(u_4 - u_3) ... - \sum_{i=1}^{n=1} p_i(u_n - u_{n-1}) + u_n \end{aligned}$$

and analogously for q.

Hence,

$$\begin{aligned} u(p) - u(q) =& (q_1 - p_1)(u_2 - u_1) + (q_1 + q_2 - (p_1 + p_2))(u_3 - u_2) + \\ & ... + (\sum_{i=1}^{n-1} q_i - \sum_{i=1}^{n-1} p_i)(u_n - u_{n-1}) \\ & > 0 \end{aligned}$$

where the inequality follows from FOSD and the fact that $u_{i+1} - u_i > 0$ for all i.

Example of FOSD over wealth: Let

$$\begin{aligned} &G(W) = Pr(x \leq w), F(W) = Pr(x \leq w) \\ &g(w) = G'(w), f(w) = F'(w) \end{aligned}$$

Then,

$$\begin{aligned} &\forall w \in [a, b], F(w) \leq G(w) \implies u(F) > u(G) \\ &u(F) = \int_a^b u(w) f(w) dw > \int_a^b u(w) g(w) dw = u(G) \end{aligned}$$

Using integration by parts,

$$[u(w)F(w)]_a^b - \int_a^b u'(w)F(w)dw > [u(w)g(w)]_a^b - \int_a^b u'(w)G(w)dw$$

And hence

$$\begin{aligned} - \int_a^b u'(w)F(w) &> - \int_a^b u'(w)G(w) \\ &\implies F(w) \leq G(w) \end{aligned}$$

(refer to Tutorial 5 for details)

(f) Risk preferences

Consider gambles over wealth w.

Let $A = \mathbb{R}_{++}$. A contains infinitely many elements.

We focus on gambles that yield finitely many outcomes with strictly positive probability p_i : $(p_1 \circ$

$w_1, ..., p_n \circ w_n)$, where n is a finite positive integer.
Assume that $u'(w) > 0$ for all wealth levels w, i.e., utility is differentiable and higher w is preferred.

The expected value of the simple gamble g is

$$\mathbb{E}(g) = \sum_{i=1}^{n} p_i w_i$$

Suppose we are given the choice between getting the gamble g or getting $\mathbb{E}(g)$ with certainty, i.e., comparing

$$u(g) = \sum_{i=1}^{n} p_i u(w_i) \text{ and } u(\mathbb{E}(g)) = u(\sum_{i=1}^{n} p_i w_i)$$

Let $u(.)$ be an individual's VNM utility function over nonnegative levels of wealth.
For the simple gamble $g = (p_1 \circ w_1, ..., p_n \circ w_n)$ the individual is

i. risk averse at g if $u(\mathbb{E}(g)) > u(g)$ (i.e., $u(w)$ is concave in w)
ii. risk neutral if $u(\mathbb{E}(g)) = u(g)$ (i.e., $u(w)$ is linear in w)
iii. risk loving if $u(\mathbb{E}(g)) < u(g)$ (i.e., $u(w)$ is convex in w)

if an individual is risk averse at every simple non-degenerate gamble g, then they a are risk averse.
Note: the concavity/ convexity of $u(w)$ is an implication of Jensen's inequality.

The **certainty equivalent (CE)** of a simple gamble g is an amount of wealth such that $u(g) = u(CE)$, i.e., such that a decision maker is indifferent between the gamble and taking CE with certainty.

The **risk premium (P)** is an amount of wealth such that $u(g) = u(\mathbb{E}(g) - P)$.
We can show that $P \equiv \mathbb{E}(g) - CE$:

$$\begin{aligned} CE &= u^{-1}(u(g)) \\ &= u^{-1}(u(\mathbb{E}(g) - P)) \\ &= \mathbb{E}(g) - P \end{aligned}$$

The **Arrow-Pratt measure of Absolute Risk Aversion** is

$$R_a \equiv -\frac{u''(w)}{u'(w)}$$

Note the negative sign, which means a positive risk aversion measure means an individual is risk averse.
Decreasing (DARA) and Constant (CARA) measures of absolute risk aversion are often used.
$u(w) = -e^{-\alpha w}$ satisfies CARA with $R_a = \alpha$.
$u'''(w) > 0$ is necessary but not sufficient for DARA. Consider:

$$\frac{\partial R_a}{\partial w} = \frac{-(u''')(u') - (u'')^2}{(u')^2}$$

and we want this to be < 0, which requires (u''') but is not sufficient as we also need that $u''' > \frac{(u'')^2}{u'} > 0$.

The **Arrow-Pratt measure of Relative Risk Aversion** is

$$R_r \equiv R_a(w)$$

where w is wealth.
R_r is an elasticity in that it gives the percentage change in marginal utility as wealth changes marginally.
$u(w) = ln(w)$ satisfies constant relative risk aversion, so do other functions with the form $u(w) = \frac{w^{1-\rho}}{1-\rho}, \rho \neq 1$.

(g) **Generalised independence axiom**: Let p, q, r be as defined in the Independence Axiom but assume now that $p \succcurlyeq q$.
Axioms G4, G5 and G6 imply that for any $\alpha \in [0,1]$,

$$(\alpha \circ p, (1-\alpha) \circ r) \succcurlyeq (\alpha \circ q, (1-\alpha) \circ r)$$

Proof: Let pr be the simple gamble induced by $(\alpha \circ p, (1-\alpha) \circ r)$ and qr be the simple gamble induced by $(\alpha \circ q, (1-\alpha) \circ r)$. By G6,

$$(\alpha \circ p, (1-\alpha) \circ r) \succcurlyeq (\alpha \circ q, (1-\alpha) \circ r) \equiv pr \succcurlyeq qr$$

Recall that $u(a_i)$ is such that

$$a_i \sim (u(a_i) \circ a_1, (1 - u(a_i) \circ a_n)) \equiv \gamma_i$$

By G5,

$$pr' = ((\alpha p_1 + (1-\alpha) r_1) \circ \gamma^1, ..., (\alpha p_n + (1-\alpha) r_n) \circ \gamma^n) \sim pr$$
$$qr' = ((\alpha q_1 + (1-\alpha) r_1) \circ \gamma^1, ..., (\alpha q_n + (1-\alpha) r_n) \circ \gamma^n) \sim qr$$

Let pr'_S and qr'_s be the simple gambles induced by pr' and qr'. By G6, it suffices to show that $pr'_S \succcurlyeq pr'_S$.
Arguing as in the proof of Theorem 2.7,

$$u(pr'_S) = \sum_{i=1}^{n} (\alpha p_i + (1-\alpha) r_i) u(a_i) = \alpha u(p) + (1-\alpha) u(r)$$
$$u(qr'_S) = \sum_{i=1}^{n} (\alpha q_i + (1-\alpha) r_i) u(a_i) = \alpha u(q) + (1-\alpha) u(r)$$

By G4, $p \succcurlyeq q$ iff $u(p) > u(q)$. Thus, $p \succcurlyeq q$ imples $pr \succcurlyeq qr$, which implies that

$$(\alpha \circ p, (1-\alpha) \circ r) \succcurlyeq (\alpha \circ q, (1-\alpha) \circ r)$$

(h) Jensen's inequality
Let $p_i \geq 0$ be the probability that outcome $x_i \in \mathbb{R}$ satisfying $\sum_{i=1}^n p_i = 1$ with $x_1 < ... < x_n$ (i.e., x_1 is the worst outcome, x_n is the best) and $p_i < 1 \ \forall i$.
Denote by $\mathbb{E}[.]$ the expectations operator with respect to the distribution $p = (p_1, ..., p_n)$.
Let $f(x)$ be a real-valued function.
Then, **Jensen's inequality** states that if f is concave, then

$$\mathbb{E}[f(x)] \leq f(\mathbb{E}[x])$$

and if f is convex, then

$$\mathbb{E}[f(x)] \geq f(\mathbb{E}[x])$$

where the inequalities are strict if concavity/ convexity is strict.
Note: Jensen's inequality implies $\mathbb{E}[f(x)] = f(\mathbb{E}[x])$ if f is linear because then it is both weakly concave and weakly convex.

Proof of Jensen's inequality:
We focus on the case where f is concave, and prove by induction.
For $n = 2$, we have

$$\mathbb{E}[f(x)] = p_1 f(x_1) + (1-p_1) f(x_2) \leq f(p_1 x_1 + (1-p_1) x_2)) = f(\mathbb{E}[x])$$

by concavity.
We need to show that if the inequality holds for n, it holds for $n+1$.
We have:

$$\begin{aligned}
f(\mathbb{E}[x]) &= f(\sum_{i=1}^{n+1} p_i x_i) = f(p_1 x_1 + (1-p_1) \frac{\sum_{i=2}^{n+1} p_i x_i}{1-p_1} \\
&\geq p_1 f(x_1) + (1-p_1) f(y) \\
&\geq p_1 f(x_1) + (1-p_1) \frac{\sum_{i=2}^{n+1} p_i f(x_i)}{1-p_1} \\
&= \sum_{i=1}^{n+1} p_i f(x_i) = \mathbb{E}[f(x)]
\end{aligned}$$

Notes: the first inequality follows from concavity with $y = \frac{\sum_{i=2}^{n+1} p_i x_i}{1-p_1}$.
The second inequality follows from applying the induction hypothesis and noting that $\frac{\sum_{i=2}^{n+1} p_i}{1-p_1} = 1$: $f(y) \geq \frac{\sum_{i=2}^{n+1} p_i f(x_i)}{1-p_1}$
The second-to-last equality follows from cancelling and collecting terms, and the last from the definition of $\mathbb{E}[f(x)]$.

Lecture 6

1. **Strategic games**

   (a) Definitions:
   A strategic game of complete information consists of
   i. a finite set N of players
   ii. for each player $i \in N$ a non-empty set A_i of actions (or pure strategies) available to player i
   iii. for each player $i \in N$ a payoff (or utility) function $u_i : A \to \mathbb{R}$, where $A := \times_{j \in N} A_j$ is the set of action profiles or outcomes

   If the set A_i of every player i is finite then the game is finite.

   An action $a \in A_i$ is a **best response** to a_{-i} if

   $$u_i(a, a_{-i}) = \max_{a' \in A_i} u_i(a', a_{-i})$$

   i.e., $a \in arg \max_{a' \in A_i} u_i(a', a_{-i})$

   A **mixed strategy** α_i for player i is a probability distribution over his set of (pure) strategies A_i. If A_i is finite, then $\alpha_i(a)$ is the probability that player i chooses action $a \in A_i$.

   (b) An action $a_i \in A_i$ of player i is a **never best response (NBR)** in the strategic game $< N, (A_i), (u_i) >$ if it is not a best response to any belief of player i.

   The action $a_i \in A_i$ of player i in the strategic game $< N, (A_i), (u_i) >$ is **strictly dominated** if there is a mixed strategy α_i of player i such that $u_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$
   Note: $u_i(\alpha_i, a_{-i})$ is the expected payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is a_{-i}. If A_i is finite, then $u_i(\alpha_i, a_{-i}) = \sum_{a \in A_i} \alpha_i(a) u_i(a, a_{-i})$.

   Lemma 1: An action of a player in a finite strategic game is a never best response iff it is strictly dominated.
   Proof: (Strictly dominated $\to$ NBR:) An action that is strictly dominated cannot be a best response to any belief.
   Proof: (NBR $\to$ strictly dominated:) Suppose an action a_i is NBR but not strictly dominated by any other mixture α_i for some a_{-i}, i.e., $u_i(a_i, a_{-i}) \geq u_i(\alpha_i, a_{-i})$. But then a_i would be a BR against a_{-i}, which is a contradiction.

   (c) The set $X \subset A$ of outcomes of a finite strategic game $< N, (A_i), (u_i) >$ **survives iterated elimination of strictly dominated actions** if $X = \times_{j \in N} X_j$ and there is a collection $((X_j^t)_{j \in N})_{t=0}^T$ of sets that satisfies the following conditions for each $j \in N$:
   i. $X_k^0 = A_j$ and $X_j^T = X_j$
   ii. $X_j^{t+1} \subset X_j^t$ for each $t = 0, ..., T-1$
   iii. For each $t = 0, ..., T-1$ every action of player j in $X_j^t \backslash X_j^{t+1}$ is strictly dominated in the game $< N, (X_i^t), (u_i^t) >$ where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j \in N} X_j^t$
   iv. No action in X_j^T is strictly dominated in the game $< N, (X_i^T), (u_i^T) >$

   Example 1:

	L	R
T	3,0	0,1
M	0,0	3,1
B	1,1	1,0

No action is strictly dominated by a pure strategy.
For the row player, choosing $T(0.5), B(0.5)$ strictly dominates choosing B.
The payoff for choosing $T(0.5), B(0.5)$, assuming that the column player chooses L with probability λ and R with probability $1-\lambda$, is:

$$0.5(3\lambda + 0) + 0.5(3(1-\lambda) + 0) = 1.5$$

While the payoff for choosing B is:

$$1(1(\lambda) + 1(1-\lambda)) = 1$$

After we eliminate B which is strictly dominated, we can eliminate L which is strictly dominated since the row player only chooses T or M.
After that, we can eliminate T since it is strictly dominated.
The only outcome that survives iterated elimination of strictly dominated strategies is (M, R).

The action $a_i \in A_i$ is **weakly dominated** if there is a mixed strategy α_i of player i such that $u_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})$ for some $a_{-i} \in A_{-i}$ and $u_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$.
(i.e., at least one of the inequalities must be $>$).
An action that is weakly but not strictly dominated is a best response to some belief (by Lemma 1).
In analogy to the case of strictly dominated actions, the notion of weak domination leads to the procedure of iterative elimination of weakly dominated actions. However the set of actions that survive iterated elimination of weakly dominated actions may depend on the order of elimination (e.g., try this for the example above).

2. **Nash equilibrium**

(a) In many games the procedure of iterative elimination of strictly or weakly dominated strategies does not help in predicting the outcome of the game.
A **pure strategy Nash equilibrium** of a strategic game $< N, (A_i), (u_i) >$ is a profile $a^* \in A$ of actions with the property that for every player $i \in N$ we have

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \ \forall a_i \in A_i$$

(i.e., no player can make themselves better off by unilaterally deviating to any other action.
For any $a_{-i} \in A_{-i}$ define the set of player i's best responses given a_{-i}:

$$B_i(a_{-i} := \{a_i \in A_i : u_i(a_i, a_{_i}) \geq u_i(a_i', a_{-i}) \ \forall a_i' \in A_i\}$$

We call the set B_i the best-response correspondence of player i (there could be multiple such actions).
A Nash equilibrium $a*$ is a collection of mutually best responses.
An alternative (and equivalent) definition to the definition above: A Nash equilibrium of a strategic game $< N, (A_i), (u_i) >$ is a profile $a^* \in A$ such that

$$a_i^* \in B_i(a_{-i}^*) \ \forall i \in N$$

(b) **Existence of a Nash Equilibrium**:
Kakutani's Fixed Point Theorem: Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a correspondence for which

i. for all $x \in X$ the set $f(x)$ is nonempty and convex
ii. the graph of f is closed; that is, for all sequences x_n and y_n such that $y_n \in f(x_n)$ for all n, $x_n \to x$, and $y_n \to y$, we have $y \in f(x)$.

Then, there exists $x^* \in X$ such that $x^* \in f(x^*)$.

Theorem (Debreu-Glicksberg-Fan): The strategic game $< N, (A_i), (u_i) >$ has a Nash equilibrium if for all $i \in N$

i. the set A_i of actions of player i is a nonempty convex subset of a Euclidean space (this is analogous to the $\lambda \in [0,1]$ for mixed strategies)

ii. the payoff function u_i is continuous and quasi-concave on A_i (this is analogous to $f : X \to X$ for the fixed point theorem).

Proof: Define $B : A \to A$ by $B(\alpha) := \times_{i \in N} B_i(a_{-i})$ where B_i is the best response correspondence of player i. For every $i \in N$, the set $B_i(a_{-i}$ is nonempty since u_i is continuous and A_i is compact, and ix convex since u_i is quasiconcave on A_i. B has a closed graph since each u_i is continuous. Thus, by Kakutani's fixed point theorem B has a fixed point. any fixed point is a Nash equilibrium of the game.

(c) The **mixed extension** of the strategic game $< N, (A_i), (u_i) >$ is the strategic game $< N, (\Delta(A_i)), (u_i) >$ in which $\Delta(A_i)$ is the set of probability distributions over A_i and $u_i : \times_{j \in N} \Delta(A_j) \to \mathbb{R}$ assigns to each $\alpha \in \times_{j \in N} \Delta(A_j)$ the expected value under u_i of the lottery over A that is induced by α (so that $u_i(\alpha) = \sum_{a \in A} (\times_{j \in N} \alpha_j(a_j)) u_i(a)$ if A is finite).
The functions u_i are linear in α: for all mixed strategy profiles α, β and all $\lambda \in [0,1]$,

$$u_i(\lambda\alpha + (1-\lambda)\beta) = \lambda u_i(\alpha) + (1-\lambda) u_i(\beta)$$

which follows from the linearity of VNM-utility functions in probabilities (i.e., VNM utility comes in when where is uncertainty/ mixed strategies).

A **mixed strategy Nash equilibrium (MSE)** of a strategic game is a Nash equilibrium of its mixed extension.
The set of (pure strategy) Nash equilibria of a strategic game is a subset of its set of mixed strategy equilibria.

(d) **Theorem (Nash): Every finite strategic game has a mixed strategy Nash equilibrium**.
Proof: Let m_i be the number of actions in A_i. Then $\delta(A_i)$ is the set of vectors $(p_i, ..., p_{m_i})$ with $p_k \geq 0$ for all k and $\sum_{k=1}^{m_i} pk = 1$. This set is nonempty, compact and convex. Since the expected payoff is linear in $(p_1, ..., p_{m_i})$, it is both continuous and quasi-concave on $\Delta(A_i)$. Hence the previous existence theorem applies.
Note: it is crucial to the proof that the set of actions of each player is finite.

Lemma 2: Let $G =< N, (A_i), (u_i) >$ be a finite strategic game. Then $\alpha^* \in \times_{j \in N} \Delta(A_j)$ is a mixed strategy Nash equilibrium of G iff for every player $i \in N$ every pure strategy in the support of α_i^* (i.e., having positive probability) is a best response to α_{-i}^*.
Note: The support of a real-valued function f is the subset of the function domain containing the elements which are not mapped to zero.
Proof: If a_i is in the support of α_i^* and is not a best response to α_{-i}^* then player i would improve their payoff by switching probability from a_i to a best response to α_{-i}^*. Hence a_i in the support of an equilibrium implies a_i is a best response.
To show that all a_i in the support of α_i^* are best responses for all i when α^* is an equilibrium, suppose to the contrary that α_i' yields a higher payoff than α_i^* against α_{-i}^* for some i. Then, at least one action in the support of α_i' must yied a higher payoff against α_{-i}^* than some action in the support of α_i^*, but then not all actions in a_i^* could be best response against α_{-i}^*, a contradiction.

Corollary: Let $G =< N, (A_i), (u_i) >$ be a finite strategic game and $\alpha^* \in \times_{j \in N} \Delta(A_i)$ be a mixed strategy Nash equilibrium of G. Then, every action in the support of any player's equilibrium mixed strategy yields that player the same payoff.

(e) Example: Battle of the Sexes

	B	F
B	2,1	0,0
F	0,0	1,2

There are two pure strategy Nash equilibria $(B, B), (F, F)$.
There is also a mixed strategy Nash equilibrium where for the row player, $\alpha_1^*(B) = \frac{2}{3}, \alpha_1^*(F) = \frac{1}{3}$ and for the column player, $\alpha_2^*(B) = \frac{1}{3}, \alpha_2^*(F) = \frac{2}{3}$.

Note that

$$u_i(B, \alpha_2^*) = 2\alpha_2^*(B) + 0\alpha_2^*(F)$$
$$u_1(F, \alpha_2^*) = 0\alpha_x^*(B) + 1\alpha_2^*(F)$$

Hence $u_1(B, \alpha_2^*) = u_1(F, \alpha_2^*)$ implies
$2\alpha_2^*(B) = \alpha_2^*(F) = 1 - \alpha_2^*(B)$, or $\alpha_2^*(B) = \frac{1}{3}$.

Similarly,

$$u_2(\alpha_1^*, B) = 1\alpha_1^*(B) + 0\alpha_1^*(F)$$
$$u_2(\alpha_1^*, F) = 0\alpha_1^*(B) + 2\alpha_1^*(F)$$

Hence $u_2(\alpha_1^*, B) = u_2(\alpha_1^*, F)$ implies
$\alpha_1^*(B) = 2\alpha_1^*(F) = 2 - 2\alpha_1^*(B)$, or $\alpha_1^*(B) = \frac{2}{3}$.

Note that if there is a small change to one player's payoffs (e.g., if (B, B) is now $(2 + \epsilon, 1)$, it is the other player's strategy that changes.

Lecture 7

1. **Cournot** (based on Cournot 1838)

   (a) Assumptions:
      i. $i = 1, 2$ firms (findings generalise to $n \geq 2$)
      ii. q_1 and q_2 denote the quantity of firms 1 and 2
      iii. $Q = q_1 + q_2$ denotes aggregate quantity of the two firms
      iv. $C(q_i)$ is firm i's total cost function
      v. $P(Q)$ is the inverse demand function
      vi. Both firms choose $q_i \in [0, \infty)$ simultaneously. Then, the market-clearing price "emerges" (Note: firms do not set prices).

   Profit for firm 2 is

   $$\Pi_2(q_2, q_1) = P(q_1 + q_2)q_2 - C(q_2)$$

   This is firm 2's **reaction (or best response) function**: it gives the optimal q_2 for any q_1.

   (b) Example: Assume
      i. $P(Q) = 1 - Q$
      ii. $C(q_i) = cq_i$, with $0 \leq c < 1$ for $i = 1, 2$

   Firm 2's profit is thus

   $$\Pi_2(q_2, q_1) = (1 - q_1 - q_2)q_2 - cq_2$$

   Firm 2's profit maximisation problem is

   $$\max_{q_2} \Pi_2(q_2, q_1) = (1 - q_1 - q_2)q_2 - cq_2$$

   The FOC is

   $$0 = 1 - c - q_1 - 2q_2$$
   $$q_2^*(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1$$

   The SOC is negative and hence profit is maximised at this quantity.
   We can also denote firm 2's best response function using $r(.)$, i.e.,

   $$r(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1$$

Note that if firm 1 produces nothing, firm 2 will be a monopoly and produce $\frac{1-c}{2}$.
In a scenario with n firms, the reaction is

$$r(Q_{-i}) = \frac{1-c}{2} - \frac{1}{2}Q_{-i}$$

(c) Nash equilibrium of the Cournot model:
Some quantity q^C is a symmetric Nash equilibrium strategy if $q^C = r(q^C)$.
In the example above, this is $q^C = \frac{1-c}{2} - \frac{1}{2}q^C$, and $q^C = \frac{1-c}{3}$.
At this equilibrium:

$$q_1^C + q_2^C = \frac{2(1-c)}{3} > \frac{1-c}{2} = q^*$$
$$\Pi_1^c = \Pi_2^C = \frac{1}{9}(1-c)^2 < \frac{\Pi^M}{2} \text{ i.e., each firm's profit is less than half the monopoly profit}$$
$$P(q_1^C + q_2^C) = \frac{1+2c}{3} < \frac{1+c}{2} = p^* \text{ i.e., price is lower than monopoly price}$$

Quantity supplied is higher than in the monpoly case because when one firm increases their quantity, they don't bear the full inframarginal loss since quantity is shared across firms.
As $n \to \infty$, the inframarginal loss $\to 0$ and the total quantity supplied is where $P = MC$.

(d) Symmetric Cournot Oligopoly with n firms:

i. n identical firms, $i = 1, ..., n$
ii. Cost is $C(q_i) = cq_i$
iii. Inverse demand is $P(Q) = 1 - Q$, where $Q = \sum_{i=1}^n q_i$

Firm i's objective is

$$\max_{q_i} \Pi_i(q_i, q_{-i}) = (1-Q)q_i - cq_i = (1 - \sum_{j\neq i}^n q_j - q_i)q_i - cq_i$$

The FOC is

$$0 = 1 - \sum_{j\neq i}^n q_j - 2q_i - c$$

Invoking symmetry, we can replace all q_i and q_j with q^* to get

$$0 = 1 - (n+1)q^* - c \Leftrightarrow q^* = \frac{1-c}{n+1}$$

Hence,

$$Q \equiv nq^* = \frac{n}{n+1}(1-c) \qquad \text{and} \qquad P(Q^*) = \frac{1}{n} + \frac{n}{n+1}c$$

and

$$\Pi_i^* = (\frac{1-c}{n+1})^2$$

For $n = 1$, we get the monopoly situation.
As n increases, $\Pi_i^*, q^*, P(Q^*)$ decrease and Q^* increases.
As $n \to \infty$, $\Pi_i^* = 0, q^* = 0, Q^* = 1 - c, P(Q^*) = c$ (i.e. P=MC).

(e) Graphical notes:
The Nash equilibrium in the Cournot model is shown by the intersection of both firms' best response functions.
The Nash equilibrium in a Cournot duopoly exists and is unique.

2. **Bertrand**

(a) Assumptions:

i. Two identical firms $i = 1, 2$ know each other's identical constant marginal cost $c_i = c \geq 0$.
ii. Demand function is $D(p)$.
iii. Firms compete for customers by making price offers; they produce after they have competed for their customers.
iv. Customers go to lowest price firm. In case of a tie, the market is split evenly.
v. Firms maximise profits.

The profit function of firm 1 is:

$$\Pi_1(p_1, p_2) = \begin{cases} D(p_1)(p_1 - c), & p_1 < p_2 \\ \frac{D(p_1)}{2}(p_1 - c), & p_1 = p_2 \text{ i.e., identical prices} \\ 0, & p_1 > p_2 \end{cases}$$

Nash equilibrium is for both firms to set price at marginal cost.

i. Any higher than MC, and the other firm will price between the firm and marginal cost, and this continues until both firms are pricing at marginal cost.

ii. Any lower than MC and the firm will make a loss.

iii. The unique equilibrium is where $\Pi_1^* = \Pi_2^* = 0$ and $p_1^* = p_2^* = c$.

(b) Discussion:

i. Bertrand paradox: "two is perfect competition".
ii. What if there are different (but constant) marginal costs, e.g., $c_1 < c_2$? Then one potential equilibrium is $p_1 = c_2 - \epsilon$ and $p_2 = c_2$.
iii. Other extensions: Heterogenous products (see Hotelling-Bertrand).

3. **Hotelling-Bertrand**

(a) Assumptions:

i. Two firms located along the interval $[0, 1]$: Firm 1 is located at 0 and Firm 2 is located at 1.
ii. Firms have constant marginal costs $c \geq 0$.
iii. Consumers have valuation $v > 0$ for one unit of the good and do not buy more than one unit.
iv. Consumers are uniformly distributed along $[0, 1]$.
v. Consumers incur a cost $t > 0$ per unit they have to travel to their provider.

Assume v is "large". More specifically:

$$P(x) = v - t$$
$$xP(x) = R(x) = (v - tx)x$$

The firm's objective function is

$$\max_x x(P(x) - c) = x(v - tx - c)$$

where the derivative is

$$v - 2tx - c$$

If one firm gets all the demand, i.e., $x = 1$, then

$$v - 2t - c > 0$$
$$\frac{v - c}{2} > t$$

Given that v is large,

i. For any (p_1, p_2), there is a critical customer $\tilde{x}$ that is indifferent between buying from Firm 1 and Firm 2.

ii. All consumers with $x < \tilde{x}$ prefer to buy from Firm 1 and all consumers with $x > \tilde{x}$ prefer to buy from Firm 2.

Formally,

$$v - p_1 - t\tilde{x} = v - p_2 - t(1 - \tilde{x})$$

Solving for $\tilde{x}$ yields

$$\tilde{x} = \frac{1}{2} + \frac{p_2 - p_1}{2t}$$

Note that:

i. $\tilde{x}$ is independent of the valuation v.
ii. Demand for seller 1 is $D_1(p_1, p_2) = \tilde{x} = \frac{1}{2} + \frac{p_2 - p_1}{2t}$
iii. Demand for seller 2 is $D_2(p_1, p_2) = 1 - \tilde{x} = \frac{1}{2} + \frac{p_1 - p_2}{2t}$
iv. Profit for seller 1 is

$$\Pi_1(p_1, p_2) = (p_1 - c)D_1(p_1, P_2) = (p_1 - c)(\frac{1}{2} + \frac{p_2 - p_1}{2t})$$

v. Profit for seller 2 is

$$\Pi_2(p_2, p_1) = (p_2 - c)D_2(p_2, p_1) = (p_2 - c)(\frac{1}{2} + \frac{p_1 - p_2}{2t})$$

vi. The FOC for seller 1 is

$$0 = \frac{1}{2} + \frac{p_2 + c}{2t} - \frac{p_1}{t}$$

yielding

$$p_1(p_2) = \frac{t + c}{2} + \frac{1}{2}p_2$$

as a reaction function for firm 1 (and the SOC is < 0, so this is profit maximising).

vii. Similarly, for firm 2, we get:

$$p_2(p_1) = \frac{t + c}{2} + \frac{1}{2}p_1$$

(b) Equilibrium:
The equilibrium can be found by intersecting the two firms' reaction functions (e.g., on a graph of p_2 on p_1).
Alternatively, assume that the two firms will set the same price in equilibrium, i.e., $p_1^* = p_2^* = p^*$. That is,

$$p^* = \frac{t + c}{2} + \frac{1}{2}p^*$$

yielding $p^* = t + c$. Firms will get the same profit in equilibrium, which is $\Pi_1^* = \Pi_2^* = \Pi^* = (p^* - c)\frac{1}{2} = \frac{t}{2}$
Note: In homogenous goods Bertrand, we have $p^* = c$. Thus, with $t = 0$, the products are undifferentiated, whereas with $t > 0$, each seller gets a positive mark-up, and hence makes a positive profit.

(c) Graphical notes:

i. The critical (or indifferent) consumer $\tilde{x}$ in the case $v > \frac{3}{2}t + c$
ii. Reaction functions and equilibrium for the case $v > \frac{3}{2}t + c$
iii. Note that for this analysis to be correct, it must be that for $\tilde{x} = \frac{1}{2}$, it is optimal to buy, i.e.,

$$v - p^* - \frac{1}{2}t = v - t - c - \frac{1}{2}t \geq 0$$

which gives us the condition $\frac{3}{2}t + c < v$.

(d) Discussion on size of v:
Assume $c + t < v < \frac{3}{2}t + c$.
In this case, $v < \frac{3}{2}t + c$, but full market coverage can still occur, such that buyer $\tilde{x} = \frac{1}{2}$ is still indifferent between seller 1 and seller 2.
Equilibrium prices will be smaller than $p^* = c + t$ in this case.
For $\tilde{x} = \frac{1}{2}$ to be indifferent while still willing to buy, equilibrium prices have to satisfy

$$v - \bar{p} - \frac{1}{2}t \geq 0$$

As firms want to set prices as high as possible, they will set $\bar{p} = v - t\frac{1}{2}$. Each firm gets $\bar{\Pi} = (\bar{p} - c)\frac{1}{2} = (v - c)\frac{1}{2} - \frac{1}{4}t$ in equilibrium.

Now assume an alternative situation where $c < v < t + c$.
Market segmentation occurs: There is some $x_1 < \frac{1}{2}$ and $x_2 > \frac{1}{2}$ such that all consumers with $x < x_1$ buy from seller 1 and all consumers with $x > x_2$ buy from seller two, and the other consumers do not buy.
In this case, each seller is a local monopoly.
x_1 is such that $v - p_1 - tx_1 = 0 \Leftrightarrow x_1(p_1) = \frac{v-p_1}{t}$. Maximising profit $x_1(p_1)(p_1 - c)$ yields the optimal price $\hat{p} = \frac{v+c}{2}$.
Since $x_1 < \frac{1}{2}$ must hold, we must have $x_1(\hat{p}) = \frac{v-c}{2t} < \frac{1}{2} \Leftrightarrow v < c + t$.
Equilibrium profit when this is the case is $\hat{\Pi} = (\hat{p} - c)x_1(\hat{p}) = \frac{(v-c)^2}{4t}$.

4. **Multi-product Monopoly**

(a) Assumptions:

i. There is only 1 firm that can sell products, and it can do so by choosing any two points on the $[0, 1]$ interval, but information about consumers' locations is private.
ii. The firm sells products at location 0 and 1.
iii. v is large ($v > \frac{1}{2}$), zero marginal costs.
Note that $v > \frac{1}{2}$ is required such that full market coverage occurs, and there is some "overlap" between products. Otherwise, similar to the local monopoly case in Hotelling-Bertrand, there is market segmentation.

(b) Comparison of pricing strategies:
If the firm's pricing strategy is to set posted prices for each of the two products, the posted prices are $p_0 = v - \frac{1}{2} = p_1$
Profit is $\Pi^{PP} = v - \frac{1}{2}$.
Revenue is v at points $x = 0$ and $x = 1$, and revenue is $v - 1$ at $x = \frac{1}{2}$, so marginal revenue of good 0, $MR_0 = v - 2x$; marginal revenue of good 1, $MR_1 = v - 2(1 - x)$.
Total consumers' utility is $v - \frac{1}{2}$.
Consumers located at the ends of the $[0, 1]$ interval have the highest utility (as they have the lowest travel distance); consumers in the middle have the lowest utility as they have the highest travel distance.

If the firm's pricing strategy is to have an "opaque product", i.e., a lottery where consumers have a 50% chance of receiving either good, then consumers' utility is the same as in the posted prices case: $U_L = \frac{1}{2}(v - x + v - (1 - x)) = v - \frac{1}{2}$ (i.e., this is consumers' valuation of the lottery product since it is equivalent to a 50% chance of travelling to 0 or travelling to 1)
$\Pi_L = v - \frac{1}{2} = \Pi^{PP}$.

If the firm's pricing strategy is a combination of posted prices and a lottery, where the consumers closest to points 0 and 1 pay the pure prices for the products, and the consumers in between participate in a lottery with a 50% chance of receiving either good:
$MR(\underline{x}) = v - 2\underline{x} = v - \frac{1}{2}, \underline{x} = \frac{1}{4}$
$MR(\bar{x}) = v - 2(1 - \bar{x}) = v - \frac{1}{2}, \bar{x} = \frac{3}{4}$
Posted prices for good 0 and 1 are: $P_0^L = v - \underline{x} = v - \frac{1}{4}$
$P_1^L = v - (1 - \bar{x}) = v - \frac{1}{4}$
Profits under posted prices + lottery is $\Pi^{PL} = \frac{1}{2}(v - \frac{1}{4}) + \frac{1}{2}(v - \frac{1}{2}) = v - \frac{3}{8}$.
Profits are higher than under either pure posted prices or pure lottery.

(see graphical notes)

(c) Impact of mergers v:
When $v < \frac{1}{2}$, and products are positioned at 0 and 1, there is no overlap between the markets for good 0 and good 1. Consumers within $x \in (v, 1-v)$ are simply inactive as it is not worth it for them to purchase either good.

If there are 2 independent sellers selling good 0 and good 1 respectively , then the marginal consumer is where $MR = 0$.
For good 0, this is $\underline{x} = \frac{v}{2}$; for good 1, this is $\bar{x} = 1 - \frac{v}{2}$.
This also means that if $\frac{1}{2} < v < 1$, there is no overlap between the sellers, since $\frac{v}{2} < \frac{1}{2} < 1 - \frac{v}{2}$.
We can show from this that the merger of two independent sellers increases social surplus iff $v < \frac{5}{6}$.

(d) Optimal product placement:
Now assume that the firm can place the products to maximise profit instead of locating them at 0 and 1.
Under the posted prices strategy, if $v > \frac{1}{4}$, the firm can place the products at $\frac{1}{4}$ and $\frac{3}{4}$ and minimise transportation costs.
If $v < \frac{1}{4}$, the firm cannot cover the entire market with two goods.
The profit maximising prices are still $\frac{v}{2}$ for each good.
$\Pi^{PP} = 2v\frac{v}{2} = v^2$. (see graphical notes)

If v is small, and the two products placed at locations a and $1-a$, and a pricing strategy to have a combination of posted prices and lottery,
Consumers' utility is $\frac{1}{2}(v - (x-a) + v - (1-a-x)) = v - \frac{1}{2} + a$. The closer the products are placed to one another, the higher the consumers' utility from this pricing strategy.
Profits are $\Pi^{PL}(a) = v^2 + (v - \frac{1}{2} + a)(1 - 2a - v)$, where v^2 is the profit under posted prices, $(v - \frac{1}{2} + a)$ is the price in the lottery, and $(1 - 2a - v)$ is the mass of lottery consumers.
FOC is $2 - 4a - 3v = 0 \implies a^* = -\frac{3}{4}v + \frac{1}{2}$.
SOC if $-4 < 0$ (profit maximising)
$\Pi^{*PL} = \frac{1}{8}v^2 + v^2$.
(see graphical notes)

If v is large,
$P_0 = v - a = P_1$
$\Pi^{PL}(a) = 4a(v-a) + (v - \frac{1}{2} + a)(1 - 4a)$
FOC is $4v - 8a + 1 - 4a - 4v + 2 - 4a = 0 \implies a^* = \frac{3}{16}$; i.e., instead of placing products at $\frac{1}{4}$ and $\frac{3}{4}$, the firm places the products at $\frac{3}{16}$ and $\frac{13}{16}$ to maximise profits.
SOC is $-16 < 0$ (profit maximising)
(see graphical notes)

Now assume that instead of an interval $[0, 1]$, consumers are located on the edge of a circle, and products are placed at two opposite ends of the circle.
The multi-product monopoly can offer two lotteries (one on each side of the circle).

5. **Generalised Cournot Competition** (based on Loertscher and Muir, 2024)

(a) Assumptions:
Inverse demand function is $P(Q), P' < 0$
Costs are $C(q_i), c' \geq 0, c'' \geq 0$
Total quantity is $Q = \sum_{i=1}^{N} q_i$
The FOC for firm i is

$$0 = P'(Q)q_i + P(Q) - C'(q_i)$$

The SOC is

$$0 > 2P'(Q) + P''(Q)\frac{Q}{n} - C''(\frac{Q}{n})$$

If the SOC is not satisfied, no symmetric equilibrium exists and no pure strategy equilibrium exists. Note that the FOC cannot be satisfied for two firms with quantities $q_i \neq q_j$ because the RHS is strictly smaller for the firm with the larger quantity.

(b) **Aggregative property of Cournot**: We first find a solution to the aggregated equation, then solve the n individual equations.
At symmetric equilibrium $q_i = \frac{Q^*}{n}$.
Since the FOC has to hold for all i, we can sum up over all i to obtain

$$P'(Q)\frac{Q}{n} + P(Q) = C'(\frac{Q}{n})$$

which is a single equation with a single unknown (Q).
The RHS is increasing in Q, and the LHS is decreasing in Q, so there is a unique solution for Q^*. We can solve this single equation first and then use the earlier FOC for each firm i, n times to obtain the individual equilibrium actions q^* to characterise the equilibrium.

Note that while we assumed symmetry, this property does not require symmetry: e.g., if each firm i has constant marginal costs of c_i, then the RHS of the aggregated FOC becomes $\frac{1}{n}\sum_i c_i$ and we can proceed in the same way.
Specifically, let $c_i(q_i) = c_i x q_i$ for each firm i.
Each firm's objective function is $\max_{q_i} P(Q)q_i - c_i q_i$
The FOC is $P'(Q)q_i + P(Q) - c_i = 0, \forall i$.
We can still sum up over all i to get the FOC in the form:

$$P'(Q)\frac{Q}{n} + P(Q) = \frac{1}{n}\sum c_i$$

The RHS is constant in Q, and the LHS is decreasing in Q, so there is a unique solution for Q^*.

(c) Generalised quantity competition in the product market setting:
Inverse demand function $P(Q)$ satisfying $P' < 0$
Revenue under market-clearing pricing is $R(Q) = P(Q)Q$
Each firm's cost of production is $C(x_i)$ satisfying $C' \geq 0$ and $C'' \geq 0$
Aggregate quantity $Q = \sum_{i=1}^{n} x_i$ is sold optimally, generating revenue $\bar{R}(Q)$
A "Cournot auctioneer" sells Q optimally to maximise revenue at $\bar{R}(Q)$

i. If the revenue function is concave, $\bar{R}(Q) = R(Q) = P(Q)Q$
ii. If the revenue function is not concave, $\bar{R}$ is from the optimal selling mechanism.

Each firm obtains the revenue share $\frac{x_i}{Q}\bar{R}(Q)$
Notes:

i. If $\bar{R}(Q) = R(Q)$, this is exactly like the Cournot case since $\frac{x_i}{Q}R(Q) = x_i P(Q)$
ii. Moreover, $\sum_i \frac{x_i}{Q}\bar{R}(Q) = \bar{R}(Q)$, so the "auctioneer's budget" is balanced.

Let Q_n^p denote the quantity under price-taking behaviour given n, i.e.,

$$P(Q_n^p) = C'(\frac{Q_n^P}{n})$$

Q^e denotes the efficient quantity under perfect competition, i.e., $Q^e = \lim_{n\to\infty} Q_n^P$. This means that

$$P(Q^e) = C'(0)$$

i.e., as $n \to \infty, Q \to Q^e$.
Index ironing intervals by $m \in M$, i.e., $(Q_1(m), Q_2(m))$ denotes a typical ironing interval.

Lemma 4: An equilibrium always exists, is unique and symmetric. Each firm's equilibrium quantity is $x^* = \frac{Q_n^*}{n}$ and Q_n^* satisfies

$$\frac{n-1}{n}\frac{\bar{R}(Q_n^*)}{Q_n^*} + \frac{1}{n}\bar{R}'(Q_n^*) = C'(\frac{Q_n^*}{n})$$

By concavity, $\frac{\bar{R}(Q)}{Q} > \bar{R}'(Q)$.
Thus, Lemma 4 says that the firm-level marginal revenue is a convex combination of $\bar{R}'$ and $\frac{\bar{R}(Q)}{Q}$, which is larger.
Moreover, for $Q \in Q_1(m), Q_2(m))$ for some $m \in M$, we have $\frac{\bar{R}(Q)}{Q} > \frac{R(Q)}{Q} = P(Q)$.
This means that for $n > 1$, it is possible to have $Q_n^* > Q_n^p$, i.e., socially excessive production.
(see graphical notes)

(d) Proof of Lemma 4:
Firm i's problem is $\max_{x_i} \bar{R}(Q) - C(x_i)$, yielding the FOC

$$0 = \frac{\bar{R}}{Q} + x_i \frac{\bar{R}'Q - \bar{R}}{Q^2} - C'(x_i)$$

Note that $\bar{R}'(Q) < \bar{R}$ because $f'(y)y \leq f(y)$ for any concave function f. Here the inequality is strict because $\bar{R}$is strictly concave for small Q.
Consequently, $\frac{\bar{R}'Q-\bar{R}}{Q^2} < 0$ and only a symmetric solution exists for the n FOCs.
The SOC is

$$2(1 - \frac{x_i}{Q})\frac{\bar{R}'Q - \bar{R}}{Q^2} + \frac{x_i}{Q}\bar{R}'' - C''(x_i)$$

which is strictly negative for $x_i < Q$, i.e., this is profit maximising.

(e) Proposition 4:
The aggregate equilibrium quantity $Q*_n$ is increasing in n. If $Q_n^p \leq Q_n^*$ holds, then $n > 1$ and $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in M$. As $n \to \infty$, we have
  i. $Q_n^* \to Q_e$ if $R(Q^e) = \bar{R}(Q^e)$, and
  ii. otherwise $Q_n^* \to \tilde{Q}$, where $\tilde{Q}$ satisfies $Q^e < \tilde{Q} < Q_2(m_e)$ for some $m_e \in M$.

Proof that Q_n^* increases in n:
Suppose to the contrary that $Q_{n+1}^* \leq Q_n^*$ for some $n \geq 1$.
Then, $C'(\frac{Q_n^*}{n}) \geq C'(\frac{Q_{n+1}^*}{n+1})$ would hold.
But the LHS of the equation in Lemma 4 is strictly increasing in n (because $\frac{\bar{R}}{Q} > \bar{R}'$) and both $\bar{R}'(Q_{n+1}^* \geq \bar{R}'(Q_n^*)$ and $\frac{\bar{R}(Q_{n+1}^*)}{Q_{n+1}^*} \geq \frac{\bar{R}(Q_n^*)}{Q_n^*}$ would hold (because both $\bar{R}'$ and $\frac{\bar{R}}{Q}$ are decreasing in Q).
Thus, the equation in Lemma 4 would not be satisfied at $n + 1$, which is the desired contradiction.

(f) Generalised quantity competition in the procurement setting:
There are n symmetric firms with decreasing marginal benefit functions $V(x_i)$.
Firms choose quantities x_i simultaneously. The Walrasian auctioneer procures $Q = \sum_{i=1}^n x_i$ at minimum cost $\bar{C}(Q)$.
Given Q and x_i, firm i pays $\frac{x_i}{Q}\bar{C}(Q)$.
The findings are exactly like Cournot competition with ironing except that we do not require the auctioneer to set market-clearing wages since under Cournot, each firm pays $\frac{x_i}{Q}C(Q) = x_i W(Q)$.

If the n firms behaved as price-takers, the aggregate employment Q_n^p would satisfy

$$V(\frac{Q_n^p}{n}) = W(Q_n^p)$$

If market-clearing wages are imposed and a symmetric equilibrium exists, the equilibrium quantity Q_n^C satisfies

$$V(\frac{Q_n^C}{n}) = W(Q_n^C) + \frac{Q_n^C}{n}W'(Q_n^C)$$

Because $W' > 0$, we have $Q_n^C < Q_n^p$.

The equilibrium quantity Q_n^* satisfies

$$V(\frac{Q_n^*}{n}) = \frac{n-1}{n}\frac{\underline{C}(Q_n^*)}{Q_n^*} + \frac{1}{n}\underline{C}'(Q_n^*)$$

Note that $\frac{C}{Q} < \frac{C(Q)}{Q} = W(Q)$ for $Q \in (Q_1(m), Q_2(m))$.
$Q_n^p < Q_n^*$ is thus possible for $n > 1$, i.e., we may have excessive employment in equilibrium with competition.
(see graphical notes)

Analogously to the product market case, denote by $Q^e = \lim_{n\to\infty} Q_n^p$ the Walrasian (efficient) quantity, which satisfies

$$V(0) = W(Q^e)$$

(g) Proposition 5:
The quantity setting game has a unique equilibrium, and this equilibrium is symmetric. The aggregate equilibrium quantity Q_n^* is increasing in n.
If $Q_n^p \leq Q_n^*$, then $n > 1$ and $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in M$.
As $n \to \infty$, we have:

i. $Q_n^*\ toQ^e$ if $C(Q^e) = \underline{C}(Q^e)$, and
ii. otherwise $Q_n^* \to \tilde{Q}$, where $\tilde{Q}$ satisfies $Q^e < \tilde{Q} < Q_2(m_e)$ for some $m_e \in M$.

In case (ii), there is involuntary unemployment in the $n \to \infty$ limit, of size $Q_2(m_e) - \tilde{Q}$.

(h) Sketch of proof of Proposition 5:
Uniqueness, symmetry and monotonicity of Q_n^* in n follow from the convexity of $\underline{C}(Q)$.
Using symmetry, a firm's FOC is

$$V(\frac{Q_n^*}{n}) = \frac{n-1}{n}\frac{\underline{C}(Q_n^*}{Q_n^*} + \frac{1}{n}\underline{C}'(Q_n^*)$$

Letting $n \to \infty$, we get $V(0) = \frac{\underline{C}(Q_\infty^*)}{Q_\infty^*}$.
Because $\underline{C}$ is convex, Q_∞^* is unique.
In case (i), $\underline{C}(Q^e) = C(Q^e)$, implying $\frac{\underline{C}(Q^e)}{Q^e} = W(Q^e)$ and thus $Q_\infty^* = Q^e$ as claimed.
In case (ii), $\frac{\underline{C}(Q^e)}{Q^e} < W(Q^e)$, implying $Q_\infty^* = \tilde{Q} > Q^e$.
Finally, $\frac{\underline{C}(Q_2(m_e))}{Q^2(m_e)} = W(Q_2(m_e)) > W(Q^e) = V(0)$ implies $\tilde{Q} < Q_2(m_e)$.

(i) Competition and involuntary unemployment:
As Q_n^* is monotone in n, increasing competition unambiguously increases employment.
If Q_n^* and Q_{n+1}^* are elements of the same ironing range, then the increase in competition decreases involuntary unemployment.
However, an increase in n can also:

i. induce involuntary unemployment (if Q_n^* is not in an ironing interval but Q_{n+1}^* is)
ii. eliminate involuntary unemployment (if Q_n^* is in an ironing interval but Q_{n+1}^* is not)
iii. decrease or increase involuntary unemployment if Q_n^* and Q_{n+1}^* are both elements of different ironing intervals.

Lecture 9

1. Extensive Form Games

(a) Graphical notes:
Sequential game with three subgames:

i. Note that the whole game is also a subgame.
ii. The subgame perfect Nash equilibrium (SPNE/SPE) is

$$(\sigma_1^*, \sigma_2^*) = ((L, L''), L')$$

this is derived using backward induction.
iii. Note that while we only observe L, the unobserved subgames are still critical because they explain why we observe L (because of the payoffs of the rest of the game).

(b) Definitions:
Subgame: A subgame starts at a singleton information set and consists of all the decision nodes, moves and payoffs from there onward.
SPNE: A subgame perfect Nash equilibrium (SPNE) is a strategy profile that induces an equilibrium at every subgame of the game.

(c) The extensive form representation accounts for the dynamic (or sequential) nature of moves.
A finite extensive form game consists of

i. A set of decision nodes
ii. A set of actions at each decision node
iii. A specification of the player who is given the move at any decision node
iv. Payoffs for every player at the terminal nodes of the game tree

(d) Finite dynamic games of complete information can be solved by **backwards induction**.
One can also represent simultaneous moves games such as the prisoner's dilemma in extensive form. The dashed lens represents a **non-singleton information set**: it contains the two decision nodes and reflects the notion that at the point when player 2 is given the move, he doesn't know what player 1 chooses or has chosen.

(e) Graphical notes:
Prisoner's Dilemma in Extensive Form Game
Player 1 moves first, but Player 2 doesn't know the choice of Player 1 (represented graphically as a dashed circle).

Prisoner's Dilemma in Normal Form

In both representations of the game, the dominant strategy is for both players to defect. The strategy is (D, D), and the payoffs are (P, P).

2. Discussion about specific games:

(a) Entrant-Incumbent game
The entrant first chooses In or Out.
If the entrant chooses Out, the game ends, with payoff (0,2) for the entrant and the incumbent respectively.
If the entrant chooses In, then the incumbent chooses Fight (with payoff (-1,0), or Accommodate (with payoff (1,1)).

There are two Nash equilibria:

i. (Out, Fight)
ii. (In, Accommodate)

However, the first NE rests on the threat by the incumbent to Fight if the entrant plays In.
This threat is not credible because the best response by the incumbent to In is Accommodate.
Thus, equilibrium (Out, Fight) is not subgame perfect; (In, Accomodate) is subgame perfect.

(b) Cournot vs Stackelberg Duopoly
Setup: There are two firms 1 and 2, each with marginal costs of 0.
Inverse market demand is $p(Q) = 1 - Q$ for $Q \in [0, 1]$ with $Q = q_1 + q_2$.

Cournot game:

i. Firms choose q_1 and q_2 simultaneously.
ii. Firm i's problem is

$$\max_{q_i}(1 - q_j - q_i)q_i \tag{1}$$

Taking FOC,

$$q_i^* = \frac{1 - q_j}{2}$$

and hence the best response function is

$$r(q) = \frac{1-q}{2}$$

The best responses both both firms is symmetric:

$$\begin{aligned} q_1^* &= r(q_2^*) = q_2^* = r(q_1^*) \\ q^c &= \frac{1}{3}, \qquad \pi^c = \frac{1}{3} \end{aligned}$$

Stackleberg game:

i. Firm 1 chooses q_1 first, which it cannot change subsequently. Firm 2 observes q_1 and then chooses q_2.
ii. Firm 2's best response is $q_2 = r(q_1)$.
iii. Firm 1's problem is

$$\begin{aligned} &\max_{q_1} (1 - q_1 - r(q_1))q_1 \\ &= \max_{q_1} \frac{1}{2}(1-q_1)q_1 \end{aligned}$$

Taking FOC,

$$q_1^* = \frac{1}{2}$$

The SPE is $(\frac{1}{2}, r(q_1)$; the SPE outcome is $(\frac{1}{2}, \frac{1}{4})$.
Profits are:

$$\pi_1^s = \frac{1}{8}, \qquad \pi_2^s = \frac{1}{16}$$

Note that Firm 1 is able to commit to a quantity that is not optimal ex-post (because $r(\frac{1}{4}) = \frac{3}{8} < \frac{1}{2}$.

(c) Simultaneous vs Sequential Location Games
Hotelling (EJ 1929):

i. Consumers with total mass of 1 are uniformly distributed on $[0, 1]$. Each consumer visits/buys/consumers the good from the closest firm.
ii. The payoffs of the firms are given by their market shares.
iii. If the game is simultaneous, both players $i = 1, 2$ choose locations $x_i \in [0, 1]$ simultaneously. Note that in this case, both players choose to locate at $x = \frac{1}{2}$ because otherwise their payoff is strictly less than $\frac{1}{2}$.
iv. If the game is sequential, by backward induction, since the second player will always choose "adjacent to the first player, on the longer side", the first player chooses to locate at $x = \frac{1}{2}$ and hence both players locate at $x = \frac{1}{2}$.

Prescott and Visscher (BJE 1977):

i. Assumptions for consumers are the same as in the Hotelling setup (mass 1, uniformly distributed, buy from the nearest firm).
ii. There is a large number of firms $i = 1, 2, ...$
iii. Firms choose sequentially whether or not to enter the market, observing the choices of their predecessors.
iv. Upon entering, each firm i chooses a location $x_i \in [0, 1]$ and bears a cost of entry $K > 0$.
v. Firm i's payoff is the market share minus its entry cost K if it enters, and 0 otherwise.

There are several possible scenarios:
(see graphical notes)

i. If $K > 1$, all firms stay out of the market.
ii. If $\frac{1}{2} < K < 1$, the first firm enters and the second firm stays out of the market. Any $x_1 \in [K - 1, K]$ will be an optimal location for Firm 1 (i.e., part of the SPE).

iii. If $K < \frac{1}{2}$, revenue at $x \in (K, 1-K)$ is $\frac{1}{2} - K$. So if $K < \frac{1}{4}$, only 2 firms will enter, and they will locate at K and $1-K$.
iv. If $K < \frac{1}{4}$, one equilibrium is for firm 1 to locate at K, firm 2 to locate at $1-K$, and firm 3 to locate at $\frac{1}{2}$.
v. If K is very small, one equilibrium is for firms to locate at $(0, K, 3K, 5K, ...)$, which deters entry for any further firm within the intervals between two firms.

(d) Capacity Constraints and Price Competition
Background:
i. Cournot (1838): Positive equilibrium profits for quantity competition.
ii. Bertrand (1883): In contrast to Cournot, firms set prices, and when they do so, they earn 0 profits in equilibrium when marginal costs are constant and identical.

Kreps and Scheinkman (BJE 1983)
i. Two-stage game:
Firms $i = 1, 2$ choose capacities $k_i \in [0, \infty)$ simultanously at cost $C(k_i)$ with $C' \geq 0, C'' > 0$.
After observing (k_1, k_2), firms choose prices p_1 and p_2 simultaneously.
ii. Capacities k_i are such that i cannot sell more than k_i units.
iii. In stage 2, consumers go to the lower price firm. Rationing is efficient, i.e., the demand that firm j faces with $p_j > p_i$ is $\max\{D(p_j) - k_i, 0\}$
iv. The inverse demand function $P(Q)$ is downward-sloping, concave (i.e., $P'' \leq 0$) and satisfies $P(0) > C'(0)$.

(e) Hotelling as a two-stage game
i. Stage 1: Two firms simultaneously choose locations $x_i \in [0, 1]$ (at no costs)
ii. Stage 2: upon observing $\mathbf{x} = (x_1, x_2)$, the two firms choose prices p_i simultaneously.

For Stage 2, Hotelling assumed that consumers substitute, as in our treatment of Hotelling-Bertrand. As pointed out by d'Apremont, Gabszewicz and Thisse (ECMA 1979), this is problematic (even for V large).
Assume, instead that $U_1(p_1, x) = V - p_1 - t(x - x_1)^2$ and $U_2(p_2, x) = V - p_2 - t(x_2 - x)^2$, i.e., transport costs are quadratic.
In this case, firms will locate at the extreme points $(0, 1)$ to mitigate price competition.

(f) Loertscher and Muehlheusser (RJE 2011)
Same setup as Prescott-Visscher but consumers are distributed according to non-uniform density $f(x)$
Paper finds subgame perfect equilibrium locations when $f' > 0, f' < 0$, f is trough-shaped and f is hump-shaped.

(g) Selten's Chain Store Paradox
i. A monopoly (chain store) has 20 branches in 20 cities.
ii. 20 entrants (one in each city) move sequentially. Each entrant chooses between In and Out; the chain store chooses between Fight and Accommodate.
iii. Entrants who stay out get a payoff of 1 and the chain store's payoff is 5 in cities without new entrants.
iv. Upon In and Accomodate, the payoffs (in this city) are 2 for both the entrant and the chain store.
v. Upon In and Fight, the payoffs (in this city) are 0 for both the entrant and the chain store.

To find the SPNE, we use backward induction and start from the last city.
i. The last incumbent doesn't fight.
ii. Hence, the last entrant enters.
iii. This repeats for every city, and the chain store accommodates all entrants in every city.

(h) Barro-Gordon model (1983)
i. There are two players: Public and Policymaker.

ii. The public chooses the expected inflation rate $\pi^e \in \mathbb{R}$
iii. The policymaker chooses the actual inflation rate $\pi \in \mathbb{R}$

iv. The utility function for the public is

$$u_p(\pi^e, \pi) = -(\pi^e - \pi)^2 - \pi^2$$

i.e., the public doesn't like inflation and doesn't like inflation to differ from expectations.

v. The utility function for the policymaker is

$$u(\pi, Y) = -\pi^2 - (Y - kY^s)^2, k > 1$$

i.e., the policymaker doesn't like inflation and wants output Y to be above the steady state Y^s.

vi. According to the Phillips curve,

$$Y = Y^s + (\pi - \pi^e)$$

i.e., higher-than-expected inflation makes Y higher than Y^s.

If the game is a simultaneous-move/ static game:
i. In the NE, the public chooses $\pi^e = \pi$ to maximise utility.
ii. The policymaker's objective function is

$$-\pi^2 - (Y^s + (\pi - \pi^e) - kY^s)^2$$

(obtained by substituting the Phillips curve into the utility function)

The FOC is

$$\frac{\partial u}{\partial \pi} = -2\pi - 2(\pi - \pi^e - (k-1)Y^s) = 0$$

Since in equilibrium the public's best response is $\pi - \pi^e = 0$, the policymaker's optimum is

$$\pi^* = (k-1)Y^s > 0$$

Hence, the best responses are

$$BR_p(\pi) : \pi^e = \pi \qquad \text{(BR for public)}$$
$$BR_{pm}(\pi^e) : \pi^* = (k-1)Y^s \qquad \text{(BR for policymaker)}$$

The Nash equilibrium is $(\pi^{e^*}, \pi^*) = ((k-1)Y^s, (k-1)Y^s)$

If the policymaker moves first, the game is sequential.
Intuitively, we can think of this as a case where the policymaker delegates π to an independent central bank that the policymaker cannot influence afterward.
i. In Stage 1, the policymaker/central bank chooses π; in Stage 2, the public chooses π^e after observing π.
ii. By backward induction, the public's best response is

$$BR_p(\pi) : \pi^e = \pi$$

iii. This means that in Stage 1, the policymaker's problem is

$$\max_{\pi}\{-\pi^2 - (-(k-1)Y^s)^2\}$$

and hence from the FOC, $\pi^* = 0$.

Comparing payoffs, utility is greater in the sequential game vs. the simultaneous game as $u(0, Y^s) > u(\pi^0, Y^s),\ \forall \pi^0 > 0$.

Lecture 10

1. Repeated Games

   (a) Preliminaries:

   For feasible action profile $\mathbf{a} \in A = \times_i(A_i)$ (i.e., product over all i; where A_i is each individual's action set), let

   $$\pi(\mathbf{a}) = (\pi_1(\mathbf{a}), ..., \pi_n(\mathbf{a}))$$

   be the payoff vector.

   The set of feasible payoffs is

   $$F = \{\tilde{\pi} | \ \exists \mathbf{a} \in A \text{ s.t. } \pi(\mathbf{a}) = \tilde{\pi}\}$$

   (this is the payoff vector such that individual 1 gets payoff $\tilde{\pi}_1$, etc)

   The set of Pareto optimal payoffs is

   $$P = \{\tilde{\pi} \in F | \ \nexists a \in A \text{ s.t. } \pi(\mathbf{a}) \geq \tilde{\pi}\}$$

   (i.e., there is no allocation is available that makes one individual better off without making another worse off when compared to the Pareto optimal payoff).

   (b) Our focus is (stage) games that:
      i. have a unique Nash equilibrium $\mathbf{a}^*$ with associated payoff vector $\pi^* = \pi(\mathbf{a}^*)$, and
      ii. the NE is Pareto dominated, i.e., $\pi^* \notin P$.

   The set of "Pareto optimal and Pareto superior" payoffs is

   $$P^S = \{\tilde{\pi} \in P | \ \nexists \mathbf{a} \in A \text{ s.t. } \pi(\mathbf{a}) > \tilde{\pi}^*\}$$

   (i.e., the set of payoffs that are feasible and also Pareto optimal and also Pareto (weakly) superior to the NE).

   (c) Examples:

   Prisoner's Dilemma

	C	D
C	4,4	0,5
D	5,0	1,1

   Set F of feasible payoffs is $(\pi_1, \pi_2) = (0,5), (1,1), (4,4), (5,0)$
   Set P of Pareto optimal payoffs is $(4,4)$.
   Set P^S is the point $(4,4)$ (compare it to the NE $(1,1)$).

   Cournot competition (see graphical notes)
   We assume constant MC where $c = 0$
   Total quantity is $A = \sum a_i$
   Monopoly quantity is $\pi^M := A^M(P(A^M)$
   Set F is any point on or inside the curve $\pi_2 = -\pi_1 + \pi^M$
   Set P is any point on the line the line $\pi_2 = -\pi_1 + \pi^M$, i.e., firm 1 chooses quantity $x \in [0, A^M]$ and firm 2 chooses quantity $A^M - x$.
   Note that the sets F and P are of payoffs (and hence they are expressed in terms of profits and not quantities).
   Set P^S is any point in the set P (i.e., a point on the line) where either (or both) firms have a higher π than in the NE of (π^*, π^*).

Bertrand competition (see graphical notes)
Set F is anywhere along the x and y-axes from $\pi_2 = \pi^M$ to $(0,0)$ to $\pi_1 = \pi^M$, plus the point $\frac{\pi^M}{2}, \frac{\pi^M}{2}$.
Set P contains the three points $(\pi_1, \pi_2) = (\pi^M, 0), (\frac{\pi^M}{2}, \frac{\pi^M}{2}), (0, \pi^M)$.
Set P^S is the point $(\frac{\pi^M}{2}, \frac{\pi^M}{2})$

2. Finitely Repeated Games

We consider stage games that are played repeatedly in periods $t = 1, ...$

Let $\mathbf{a}(t) \in A$ be an action profile in period t, and suppose the stage game is repeatedly played for $T < \infty$ periods.
We find that the game has a unique SPE:

(a) In this equilibrium, $\mathbf{a}(t) = \mathbf{a}^*,\ \forall t \in \{1, ..., T\}$
(b) In the last period (i.e., last subgame), $\mathbf{a}(T) = \mathbf{a}^*$ is the unique NE.
(c) Consequently, $\mathbf{a}(T-1) = \mathbf{a}^*$ is the unique NE in the second-to-last period, and so on.

So no matter how large is the finite time horizon T, the players cannot escape the dilemma that they obtain Pareto dominated payoffs.

3. Infinitely Repeated Games

(a) Assume now that the time horizon is infinite, and players have a common discount factor $\delta \in [0, 1)$
   i. could be pure time preference
   ii. could be the (constant) probability that the game continues from one period to the next
   iii. or a combination thereof

   Player i's objective is

$$\pi_i(\mathbf{a}(1)) + \delta\pi_i(\mathbf{a}(2)) + ... + \sum_{t=1}^{\infty} \delta^{t-1}\pi_i(\mathbf{a}(t))$$

   Note: The initial SPE, i.e., $\mathbf{a}(t) = \mathbf{a}^*$ for all t remains a SPE of the infinitely repeated game since it induces a NE in every subgame, but there are now other SPE.

(b) **Folk Theorem (Friedman 1971)**, Theorem 1:
   For any $\tilde{\pi} \in P^S$, there exists a $\delta^* < 1$ such that for all $\delta \in [\delta^*, 1)$ there exists a SPE in which, on the equilibrium path of play, every player i obtains $\tilde{\pi}_i$ in every period.
   That is, with sufficient patience, players can "escape" from the dilemma and e.g., perfect collusion becomes possible.

(c) **Proof of Theorem 1**:
   Let $\tilde{\mathbf{a}}$ be such that $\pi(\tilde{\mathbf{a}}) = \tilde{\pi}$.
   i. The way $\tilde{\mathbf{a}}$ is supported as part of the SPE is that as long as every player i plays $\tilde{a}_i$ in every period, all players play $\tilde{\mathbf{a}}$ in the next period.
   ii. After the first deviation from $\tilde{\mathbf{a}}$, all players play $\mathbf{a}^*$ forever after (this is referred to as "Nash reversion" or the "grim trigger" strategy).
   iii. Note that deviations will not occur (i.e., will not be part of play in this SPE).
   iv. Because $\mathbf{a}(t) = \mathbf{a}^*$ for all t is a SPE, the grim trigger threat is capable in the sense that playing $\mathbf{a}^*$ from tomorrow onward if a deviation occurs constitutes a SPE of the subgame that starts tomorrow.
   v. Since any deviation triggers Nash reversion, the strategy profile must be immune to the optimal deviation for every player i.

   For fixed $\tilde{\mathbf{a}}_{-i}$, let $\pi_i^F(\tilde{\mathbf{a}}_{-i}) = \max_{a_i \in A_i} \pi_i(a_i, \tilde{\mathbf{a}}_{-i}$ be i's maximum profit.

i. This is the "best-response" or Stackleberg follower profit.
ii. For Bertrand, strictly speaking, $\pi_i^F(\tilde{\mathbf{a}}_{-i}) = \sup_{a_i \in A_i} \pi_i(a_i, \tilde{\mathbf{a}}_{-i}$ since a best-response typically does not exist.

So, the incentive compatibility (or incentive sustainability) condition is that for every player i, we have

$$\frac{\tilde{\pi}_i}{1-\delta} \geq \pi_i^F(\tilde{\mathbf{a}}_{-i}) + \frac{\delta}{1-\delta}\pi_i^* \tag{Eq 1}$$

(i.e., the discounted infinite geometric sum of payoffs for the Pareto superior action must be greater than the payoff for deviating today plus the discounted infinite geometric sum of payoffs for the static NE payoff starting from next period onward).
Rearranging (Eq 1), we get:

$$\tilde{\pi}_i \geq (1-\delta)\pi_i^F(\tilde{\mathbf{a}}_{-i}) + \delta\pi_i^* \tag{Eq 2}$$

Because $\tilde{\pi}_i > \pi_i^*$, (Eq 2) is satisfied at $\delta = 1$ and by continuity, for values of $\delta < 1$.
For each i, let δ_i be the value of δ such that (Eq 2) hods with equality, i.e.,

$$\delta_i = \frac{\pi_i^F(\tilde{\mathbf{a}}_{-i}) - \tilde{\pi}_i}{\pi_i^F(\tilde{\mathbf{a}}_{-i}) - \pi_i^*}$$

We complete the proof by letting

$$\delta^* = \max \delta_1, ..., \delta_n$$

(d) Examples of games and critical discount factors:

Bertrand:
Consider the homogenous good Bertrand model with equal tie-breaking.
$\tilde{\pi}_i = \frac{\Pi^M}{2}$ (the P^S), $\pi_i^* = 0$ (the NE), and $\pi_i^F(a^M) = \Pi^M$, where a^M is the monopoly price and Pi_M the monopoly profit.
(Eq 2) above becomes

$$\frac{\Pi^M}{2} \geq (1-\delta)\Pi^M$$

implying a critical discount factor of $\delta^* = \frac{1}{2}$.
With $n \geq 2$ firms and equal tie-breaking we have $\delta^* = \frac{n-1}{n}$, with the larger discount factor suggesting that collusion is more difficult to sustain in more competitive environments (since with many firms, if one firm deviates, it earns the entire Π^M that period, compared to $\frac{\Pi_M}{n}$ each period if it does not deviate).

Cournot:
Consider the $n = 2$ Cournot model with linear inverse demand $P(A) = 1 - A$, where $A = \sum_i a_i$, and zero marginal costs.
This implies:

i. $r(a) = \frac{1-a}{2}$ (the best response to the other firm choosing quantity a)
ii. $\pi_F(a) = r(a)P(a + r(a)) = (\frac{1-a}{2})^2$ (the Stackleberg follower profit)
iii. $a^* = \frac{1}{3}$ and $\pi^* = \frac{1}{9}$
iv. $a^M = \frac{1}{2}, P(a^M) = \frac{1}{2}, \Pi^M = \frac{1}{4}$ (the monopoly quantity).

With identical constant marginal costs, for any aggregate quantity $A \in [0, 1]$, firms can transfer the aggregate profits $AP(A)$ by having one firm produce $x \in [0, A]$ and the other firm produce $A - x$.
The monopoly profits $\Pi^M = a^M P(a^M)$ can be shared by firm 1 producing $x \in [0, a^M]$ and firm 2 producing $a^M - x$.
Thus, the Pareto set (or frontier) P consists of $\pi^1 \in [0, \frac{1}{4}]$ and $\pi^2 = \Pi^M - \pi_1$.

i. This can be parameterised by firm 1's quantity $x \in [0, \frac{1}{4}]$.
ii. Firm 1's profits are $\frac{x}{2}$ and firm 2's profits are $\frac{1}{2}(\frac{1}{2} - x)$.
iii. For each firm's profit to be larger than its one-shot NE profit, we need $\frac{x}{2} > \frac{1}{9}$ and $(\frac{1}{2} - x)\frac{1}{2} > \frac{1}{9}$, implying $x \in (\frac{2}{9}, \frac{1}{2} - \frac{2}{9})$.

iv. Thus, any point in P^S corresponds to firm 1 producing some $x \in (\frac{2}{9}, \frac{1}{2} - \frac{2}{9})$ and firm 2 producing $\frac{1}{2} - x$.

For $x \in (\frac{2}{9}, \frac{1}{2} - \frac{2}{9})$, (Eq 2) becomes

$$\frac{x}{2} \geq (1-\delta)(\frac{\frac{1}{2}+x}{2})^2 + \frac{\delta}{9} \tag{Eq 4}$$

Letting $\delta(x)$ be the value of δ for which (Eq 4) holds with equality, we obtain

$$\delta(x) = \frac{9(1-2x)^2}{-7+36x(1+x)}$$

which satisfies

$$\delta(\frac{2}{9}) = 1$$
$$\delta(\frac{1}{4}) = \frac{9}{17}$$
$$\delta(\frac{1}{2} - \frac{2}{9}) = \frac{4}{13}$$

The market is at risk for perfect collusion if the discount factor δ is larger than $\max\{\delta(x), \delta(\frac{1}{2} - x)$. Since

$$\max \delta(x), \delta(\frac{1}{2} - x)$$

is minimised over x at $\tilde{x}$ such that $\delta(\tilde{x}) = \delta(\frac{1}{2} - \tilde{x})$, implying $\tilde{x} = \frac{1}{4}$, the market is at most at risk for perfect collusion that is symmetric.

The critical discount factor is therefore

$$\delta^* = \frac{9}{17}$$

(see graphical notes)

(e) Discussion on critical discount factors
δ^* is the set of discount factors for which perfect and symmetric collusion is sustainable (e.g., being able to earn monopoly profits when aggregating across competing firms).
However, for $\delta > \delta^*$, it is still possible to have imperfect collusion, i.e., collusion with lower payoffs than the perfect collusion payoff.

4. Imperfect Collusion

The focus here is that given some $\delta \in (0, \delta^*)$, we study what is the best the firms can achieve.
The following examples show that for homogenous good Bertrand, the δ^* approach (i.e., focusing on perfect collusion only) is without loss of generality, but for Cournot it is not (nor is it for Hotelling-Bertrand).

Bertrand:
Suppose the firms coordinate on any price $a > 0 = a^*$, generating aggregate profit $\Pi(a) = aD(a)$.
The question is whether $a < a^M$ would be sustainable for some discount factor $\delta < \delta^*$, and the answer (upon revisiting (Eq 3)) is no, since nothing hinges on the magnitude of the profit, or the price, that is sustained. In other words, we will fget the same δ for any π.
Thus, any $a > 0$ and associated $\Pi(a) > 0$ is sustainable iff $\delta \geq \delta^* = \frac{1}{2}$ in the Bertrand model.
Since it seems natural to assume that firms choose the Pareto optimal collusive scheme among all those that are sustainable, the focus on perfect collusion appears to be without loss of generality in the homogenous goods Bertrand model.

Cournot:
Writing (Eq 2) for the linear-demand Cournot model with zero marginal costs when each firm produces a yields

$$a(1-2a) \geq (1-\delta)\pi^F(a) + \delta\frac{1}{9}$$

where the LHS is the revenue for producing a, and the RHS is the deviation payoff.
Given $\delta > 0$, at equality this equation has two solutions:

(a) $a = a^* = \frac{1}{3}$

(b) $a = a^C(\delta) = \frac{1}{3}\frac{9-5\delta}{9-\delta} < a^*$, where C stands for collusion.

We conclude that for Cournot, collusion is always a concern (even if not perfect collusion); i.e., for any $\delta > 0$. The δ^* approach is with loss of generality.
(see graphical notes).

Lecture 11

1. Dynamic Games with Incomplete Information

   Example: Akerlof (1971)

   (a) Setup:
      i. One seller of a used car, and one risk-neutral buyer
      ii. Objective quality of the car is $\theta \in [0, 1]$ (e.g., can assume quality is uniformly distributed on this interval).
      iii. The buyer's willingness to pay for a car of quality θ is $v\theta$.
      iv. The seller's opportunity cost of selling a car of quality θ is $c\theta$.
      v. Assume $0 < c < v$. If so, efficiency dictates that trade should occur.
      vi. θ is private information of the seller, but c is known by the buyer too.
      vii. The buyer's prior belief is that θ is uniformly distributed on $[0, 1]$.

   (b) Analysis:
      i. The seller is willing to sell at price p iff

$$c\theta \leq p \Leftrightarrow \theta \leq \frac{p}{c}$$

      ii. Aware of this, the buyer's estimate of θ given that the seller is willing to sell at price p is

$$\mathbb{E}[\theta|\theta \leq \frac{p}{c}] = \min\{\frac{p}{c}, 1\}\frac{1}{2}$$

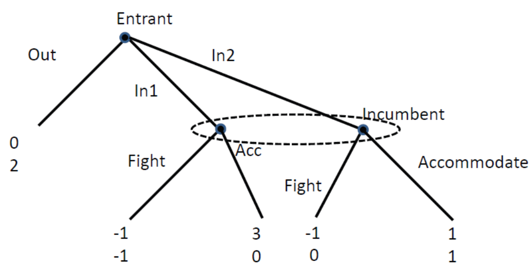
      iii. So the buyer's willingness to pay for a car offered at price p is $v\mathbb{E}[\theta|\theta \leq \frac{p}{c}]$
      iv. Thus, the buyer is willing to buy iff $v\mathbb{E}[\theta|\theta \leq \frac{p}{c}] - p \geq 0$
      v. For $\frac{p}{c} \leq 1$, this is the case iff $v \geq 2c$.
      vi. If $\frac{p}{c} > 1$, it is satisfied iff $v > 2p$.
      vii. Thus, for

$$1 < \frac{v}{2} < 2$$

      there is no trade in equilibrium although there should be under efficiency.

2. Sequential Rationality and Perfect Bayesian Equilibrium

   Motivation: Consider the Entrant-Incumbent game below:



(a) There is uncertainty because the incumbent does not know whether the entrant is entering as "In 1" or "In 2". But note that the incumbent is better of choosing to Accomodate in both cases.

(b) There is only 1 subgame because there is only 1 singleton decision node (i.e., the entire game).

(c) We cannot apply SPNE here. Recall that to apply SPNE, the incumbent has to assign probabilities to each action and then its actions have to be rational given its payoffs and beliefs.

(d) Consider the strategy (Out, Fight). This is a NE, but the probability of reaching this is 0.

System of beliefs: In an extensive form game, a **system of beliefs** μ is a collection of probability distributions over decision nodes x, one distribution for each information set H such that

$$\sum_{x \in H} \mu(x) = 1$$

(i.e., the probabilities sum up to 1).

Sequential rationality: A strategy σ_i for player i in an extensive form game is **sequentially rational** if, given beliefs μ, σ_i maximises i's expected payoff at every information set that player i can reach.
Going back to the Entrant-Incumbent game, we can see that Incumbent's strategy "Fight" is not sequentially rational:

(a) If we assign probabilities μ and $1-\mu$ to In 1 and In 2, we see that regardless of μ, the incumbent will accommodate.

(b) And then, since incumbent accommodates, the entrant chooses In 1, and hence $\mu = 1$.

A strategy profile σ and a system of beliefs μ is a **Perfect Bayesian Equilibrium** (PBE) of an extensive form game if

(a) Each player's strategy σ is sequentially rational given the belief system μ

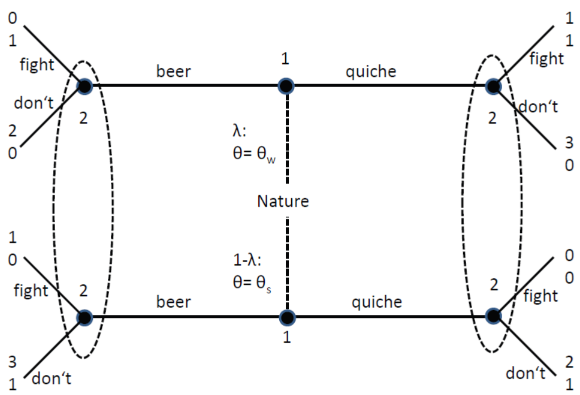
(b) The system of beliefs μ is derived from the strategy profile σ whenever possible. That is, for any information set H such that $Pr(H|\sigma) > 0$,

$$\mu(x) = \frac{Pr(x|\sigma)}{Pr(H|\sigma)} \qquad \forall x \in H$$

Note: This is also referred to as weak PBE.

3. Applications in specific games:

(a) Beer-Quiche Game



i. Nature chooses the "type" for Player 1 first (θ_w or θ_S).
Then, Player 1 chooses Quiche or Beer.
Then, Player 2 chooses Fight or Don't after observing Player 1's choice.

ii. Player 1 always prefers not to fight, but in a case of a fight, Player 2's payoffs are higher if Player 1 is a weak type.

iii. There are no separating equilibria, because if the strong and weak Player 1 types choose different actions, then Player 2 will be able to infer Player 1's type correctly and fight the weak type.

iv. So, there are two pooling equilibria - where both types of player 1 choose quiche, and where both types of player 1 choose beer. In both cases, player 2 decides not to fight.

v. However, the quiche pooling equilibrium is less plausible, as it means that the weak type has to want to fight after choosing beer, but we know that the weak type will not deviate from quiche as it gives the max payoff to the weak type.

(b) Limit Pricing

i. Setup
$T = 2$ periods
$n = 2$ players: Incumbent and Entrant.
The inverse demand function $p(\theta, Q) = \theta - Q$.
In the first period $t = 1$, I observes θ (which is time-invariant, can assume that it is e.g., distributed uniformly on $[0, 1]$. Then, I sets sets price p.
In the second period $t = 2$, E observes p and decides whether to enter or stay out.

ii. Payoffs
For the entrant, we only need to consider its payoffs in the second period:

A. If the entrant enters, its payoff is the Cournot duopoly profit, minus the fixed cost of entry, i.e., $\pi^E = \pi^D(\theta) - K = \frac{\theta}{9} - K$.

B. If the entrant stays out, its payoff is $\pi^E = 0$

It is clear that the entrant wants to enter if θ is large, and will stay out if θ is small.
If $\pi^D = K$, the entrant is indifferent whether to enter or not.

For the incumbent, we have to consider its payoffs in both periods:

A. In period $t = 1$, $\pi^I(p, \theta) = p(\theta - p)$

B. In period $t = 2$, if the entrant stays out, the incumbent's profit is $\pi^I(\theta) = \pi^M(\theta) = \frac{\theta^2}{4}$ if the entrant stays out, and $\pi^I(\theta) = \frac{\theta^2}{9}$ (the Cournot duopoly profit) if the entrant enters.

iii. Analysis

The incumbent's ideal is if it can earn monopoly profits (or as high profits as possible) in the first time period, but still have the entrant stay out.

Is there a perfect Bayesian equilibrium where the incumbent always sets $P^M(\theta)$ and the entrant stays out?

A. (see graphical notes)

B. Since the entrant knows the inverse demand function, if the incumbent sets the monopoly price, the entrant observes p, infers θ and then decides to enter.

C. Hence there is no PBE equilibrium where the incumbent always sets the monopoly price and the entrant always stays out.

D. In this scenario, the equilibrium is "fully separating", since from what the entrant observes, it can fully infer the incumbent's type (i.e., the true value of θ).

If $\theta < \tilde{\theta}$, the entrant always stays out, so the incumbent is incentivised to set the price lower than the monopoly price to make the entrant think that θ is low.

A. The incumbent's tradeoff is

$$p(\theta - p) + \pi^M(\theta) \qquad \text{vs.} \qquad \pi^M(\theta) + \pi^D(\theta)$$

where the LHS is its profits in the entry-deterring case, and the RHS is its profits in the "accommodate" case.

B. $p(\theta - p)$ is the "limit price"; for the strategy of entry deterrence to be profitable, $p(\theta - p) \geq \pi^D(\theta)$.
C. For very small $\theta < \tilde{\theta}$, the entrant doesn't come in regardless, so the incumbent can just charge the monopoly price.
D. If $\theta > \tilde{\theta}$, the incumbent sets the limit price $\bar{p}$ where $\bar{p}(1 - \bar{p}) \geq \pi^D(\theta)$

Scenario 1: "Low" limit pricing PBE
A. (see graphical notes)
B. This is a pooling equilibrium in the sense that for any $\theta \in [\theta_1, \theta_2]$, the monopolist sets the price $\bar{p}$.
C. Upon observing $\bar{p}$, the entrant's belief that $\theta \in [\theta_1, \theta_2]$ is $\mu(\theta, \bar{p}) = \frac{f(\theta)}{F(\theta_2) - F(\theta_1)}$
D. The entrant must stay out upon observing $\bar{p}$, so

$$\int_{\theta_1}^{\theta_2} \frac{\pi^D(\theta) f(\theta)}{F(\theta_2) - F(\theta_1)} d\theta - L \leq 0$$

E. For the incumbent to prefer this PBE over just setting the monopoly price, it must also be the case that

$$\bar{p}(\theta_2 - \bar{p}) \geq \pi^D(\theta_2)$$

F. What if the entrant observes a price slightly above $\bar{p}$?
The entrant "observes something it should not have observed"; the prior probability of observing such a price is zero.
The entrant is hence free to choose its beliefs about what θ is. If the agent still enters, it must be because it put enough weight on its belief that $\theta > \tilde{\theta}$.

Scenario 2: "High" limit pricing PBE
A. (see graphical notes)

Scenario 3: Interior limit pricing PBE
A. (see graphical notes)

(c) Spence's Signalling Modelling

i. Setup
A. There is one worker, who is of productivity type $\theta \in \{\theta_L, \theta_H\}$ with probability $(1 - \lambda)$ and λ respectively, where $\theta_L < \theta_H$
B. θ is the worker's private information
C. The worker can invest in costly but intrinsically useless years of education $e \geq 0$ at cost $c(e, \theta)$, where

$$c(0, \theta) = 0, \qquad \frac{\partial e, \theta}{\partial e} > 0, \qquad \frac{\partial^2 c(e, \theta)}{\partial e^2} > 0, \qquad \frac{\partial^2 c(e, \theta)}{\partial e \partial \theta} > 0$$

i.e., education has an increasing marginal cost, and education is less costly for θ_H type.
e.g., a possible function that satisfies these conditions is $c(e, \theta) = \frac{1}{2\theta} e^2$.
There is single-crossing of the indifference curves of the two types; $w - c(e, \theta_L) = K_0$ and $w - c(e, \theta_H) = K_1$, where both K_0 and K_1 are constants (see graphical notes).

D. Two firms observe education level e, update their beliefs (in the same way) and compete via Bertrand for the worker.

E. The function $c(e, \theta)$, λ, θ_L and θ_H are common knowledge.

F. The payoff of the worker of type θ who invested e when accepting a wage w is $w - c(e, \theta)$.

ii. Analysis

A. Let $\mu(e)$ be the firms' belief that the agent is of high type when observing education level e, in a given PBE.

B. Both firms offer the wage

$$w(e) := \mu(e)\theta_H + (1 - \mu(e))\theta_L \tag{Eq 1}$$

C. Definition of a **separating equilibrium**: The worker chooses different levels of e when of type θ_L and when of type θ_H, i.e., the two worker types take different education levels on the equilibrium path.

D. Claim 1: In any separating equilibrium with education levels $e^*(\theta_L)$ and $e^*(\theta_H)$ the wage $w(e^*(\theta_i)) = \theta_i$ for $i = L, H$.
Proof: Follows from (Eq 1) because $\mu(e^*(\theta_H)) = 1$ and $\mu(e^*(\theta_L)) = 0$ in any separating equilibrium.

E. Claim 2: In any separating equilibrium, $e^*(\theta_L) = 0$.
Proof: The θ_L type will be found out, so there is nothing that can induce this type to depart from the first-best choice of having no education.

F. Let $\tilde{e}$ be such that $c(\tilde{e}, \theta_L) = \theta_H - \theta_L$.
Claim: The following is a separating PBE: The high types choose $\tilde{e}$, low types choose 0, the firms offer $w^*(e) = \theta_L + c(e, \theta_L)$ and have beliefs $\mu^*(e) = \frac{w^*(e) - \theta_L}{\theta_H - \theta_L}$ for any $e \in [0, \tilde{e}]$ (and, say, $\mu^*(e) = 1$ for any $e > \tilde{e}$).
Proof: The wage is constructed such that the low type is indifferent between 0 and any $e \in (0, \tilde{e}]$. Because $\frac{\partial c(e,\theta_H)}{\partial e} < \frac{\partial c(e,\theta_L)}{\partial e}$, the high type prefers $(\tilde{e}, w^*(\tilde{e}))$ to any other effort-wage conbination (it won't choose any higher because it does not pay any better).
Firms' beliefs need only be updated using Bayes' rule for upon observing $e = 0$ and $e = \tilde{e}$, in whcih case Bayes' rule and consistency with the strategy profile imply $\mu^*(0) = 0$ and $\mu^*(\tilde{e}) = 1$, i.e., the equilibrium is fully separating and if $\tilde{e}$ is observed, the type must be θ_H.

G. There are many other separating equilibria because there are some degrees of freedom in choosing beliefs off the equilibrium path.
For example, consider the separating equiibria where $e^*(\theta_L) = 0$ and $e^*(\theta_H) = \tilde{e}$. There is a continuum of other seprating equilibria where the high type chooses $\hat{e} > \tilde{e}$, as long as $\theta_H - c(\hat{e}, \theta_H) \geq \theta_L$.

H. An equilibrium is **pooling** if both (or all) types choose the same action - here, some e^*.
Claim: There is a continuum of pooling PBE in Spence's model. Each is characterised by some education level e^* and $w^*(e^*) = \lambda\theta_H + (1 - \lambda)\theta_L = \mathbb{E}[\theta]$ with e^* such that $\mathbb{E}[\theta] - c(e^*, \theta_L) \geq \theta_L$.
Otherwise, $\mu^*(e^*) = \lambda$ and, say, $\mu^*(e) = 0$.
In other words, both types find it worthwhile to get e^* level of education.

I. There can also be hybrid equilibria where one type always chooses a fixed education level and the other type chooses from a continuum of education levels.

4. **Cho-Kreps Intuitive Criterion**

(a) Motivation and setup

i. Multiplicity of PBE is obviously a problem.
ii. One widely accepted and most uncontroversial refinement is the **intuitive criterion** of Cho and Kreps.

iii. Our definition is confined to two-player, sender-receiver games.
iv. There is one sender, who can be of type $\theta \in \Theta$ and moves first by choosing some action $a_1 \in A_1$.
v. The receiver observes a_1, updates her beliefs μ and chooses an action $a_2 \in A_2$.
vi. This family of games encompasses the beer-quiche game and SPence's model (if we treat the two firms as a single player, the "market").

(b) Analysis

i. Let $\Theta_0 \subset \Theta$ and let $A_2(\Theta_0)$ be the set of sequentially rational (i.e., best responses) of the receiver when her beliefs μ are restricted to Θ_0, i.e., $\sum_{\theta \in \Theta_0} \mu(\theta) = 1$
Note: When we discuss games with only two types, Θ_0 is just a singleton that describes one of the types.

ii. Let $u_1^*(\theta$ be the expected equilibrium payoff of the sender in a given PBE.

iii. Let $u_1(a_1, a_2, \theta)$ be her expected payoff when she is of type θ, chooses a_1 and receiver chooses a_2.

iv. An action a_1 is equilibrium payoff dominated for type θ if

$$u_1^*(\theta) > \max_{a_2 \in A_2(\Theta)} u_1(a_1, a_2, \theta) \tag{Eq 2}$$

i.e., the equilibrium payoff is greater than the maximum payoff after choosing a_1 and the receiver choosing the best a_2 in response.

v. Let $\Theta^*(a_1)$ be the set of types such that $(Eq2)$ does not hold.

vi. Definition: A PBE fails to satisfy the **intuitive criterion** if there is a type θ such that

$$\min_{a_2 \in A_2 \Theta^*(a_1)) u_1(a_1, a_2, \theta) > u_1^*(\theta)}$$

i.e., the minimum payoff from a deviation is greater than the equilibrium payoff.

vii. The intuitive criterion is particularly powerful when there are only two types, because then excluding one type pins down the belief on the other type.
A. In the beer-quiche game, it gets rid of the quiche-quiche equilibrium.
B. In Spence's model, it gets rid of all PBE other than the most efficient separating equilibrium.

Lecture 12

1. Introduction to Mechanism Design

(a) Economic policy problems derive their interest and relevance from a tradeoff between profit and social surplus.

(b) Common practice to obtain such a tradeoff is to constrain the contracting space (e.g., monopoly must use uniform pricing), or study arbitrary games, but these constraints may be unsatisfactory because they are arbitrary.

(c) Mechanism design provides a framework in which tradeoffs stem from the primitives of the model, and optimality is well defined.

2. Broad approach to Mechanism Design

(a) Revelation Principle implies that direct mechanisms $< q, t >$ are without loss of generality

(b) If $< q, t >$ satisfies IC, then $q(.)$ is non-decreasing and hence monotone.

(c) Envelope Theorem implies that the Payoff Equivalence Theorem (PET) and Revenue Equivalence Theorem (RET) hold.

(d) Identify the objective/ revenue function, and maximise with respect to $q(.)$ (e.g., maximise profit over direct mechanisms).
Changing the order of integration yields objective

$$\int \Phi(v) q(v) f(v) dv - U(\underline{v})$$

Two key questions for this step:

i. What is the pointwise maximiser of the objective function (i.e., $(q^*(v))$?

ii. Is $q^*(v)$ monotone?
If yes, use $q^*(v)$ and RET to obtain $t^*(v)$, yielding $< q^*, t^* >$.
If no, this case is more complex and requires ironing, etc.

Most of the work typically goes into (c) and (d).
Note: Under ex post efficiency, q is not a choice variable.

3. Example setup

Consider a setup with a single risk-neutral buyer whose value $v \in [\underline{v}, \bar{v}]$ is the buyer's private information.

(a) There is one seller with a single object for sale.
(b) The buyer's payoff when of type v and allocated the good with probability q and paying the (expected) transfer t is

$$vq - t$$

(c) The buyer's payoff when not participating is 0.
(d) The seller's cost is known to be 0.
(e) The seller chooses the mechanism ("game").
Note: Contrast this to game theory where the choice and design of the game is exogenous.

The designer's problem:

(a) Assume the seller believes the buyer's value is distributed according to CDF $F(v)$ with density $f(v) > 0$ on $[\underline{v}, \bar{v}]$.
(b) What is the mechanism that maximises the seller's expected profit, subject to voluntary participation and optimal behaviour by the buyer?
(c) What is the set of admissible mechanisms? (e.g., could make a take-it-or-leave-it offer; could have the buyer make an offer with some probability, etc).

4. Direct Mechanisms/ Revelation Principle

(a) Let $< q, t >$ denote a **direct mechanism**, where

$$q : [\underline{v}, \hat{v}] \to [0, 1]$$

is the **allocation rule** (i.e., that maps buyers' types to a probability of receiving a good, and

$$t : [\underline{v}, \bar{v}] \to \mathbb{R}$$

is the **transfer rule** (i.e., that maps buyers' types to the payment/ transfer they will have to make).
(b) In a direct mechanism, the agent is asked to report its type, and allocations and payments are made as a function of the report.
(c) A direct mechanism is **incentive compatible (IC)** if it is optimal for the buyer to report its true v for all its possible types.
(d) A direct mechanism satisfies **individual rationality (IR)** if the buyer does not want to walk away for any possible type (this is also called the participation constraint), i.e., for all $v \in [\underline{v}, \bar{v}]$,

$$q(v)v - t(v) \geq 0$$

Revelation Principle: The focus on direct mechanisms that satisfy IC and IR is without loss of generality.

(a) That is, whatever expected profit the seller can achieve in the equilibrium of any mechanism can be achieved by a direct IC IR mechanism.
(b) This helps narrow our search for the optimal mechanism to the set of optimal direct mechanisms that satisfy IC and IR.

Proof of Revelation Principle:

(a) Consider any mechanism in which the buyer of type v optimally takes the action $s(v) \in S$, where S is the set of actions (strategies) the alternative mechanism offers the buyer.

(b) Let $\hat{q} : S \to [0,1]$ and $\hat{t} : S \to \mathbb{R}$ be the allocations and transfers the alternative mechanism makes as a function of the action chosen by the buyer.

(c) Now construct the direct mechanism $< q, t >$ by $q(v) := \hat{q}(s(v))$ and $t(v) := \hat{t}(s(v))$

(d) Because $s(v)$ was optimal in the alternative mechanism for type v, reporting v truthfully is optimal in the direct mechanism $< q, t >$ and because participation was voluntary in the alternative mechanism, it is voluntary in $< q, t >$.

5. Envelope Theorem

(a) Traditionally, envelope theorems were discovered in environments in which the decision-maker's objective function $h(x, a)$ is differentiable in its choice variable x and the parameter a.

(b) For example, the choice set could be $X = \mathbb{R}$ and the maximiser $x^*(a)$ is given by the FOC

$$h_x(x^*(a), a) = 0$$

The value function (i.e., the maximum value of the objective function) is

$$V'(a) = \sup_{x \in X} h(x, a)$$

which satisfies

$$V'(a) = \frac{\partial h(x^*(a), a)}{\partial x}\frac{\partial x^*}{\partial a} + \frac{\partial h(x^*(a), a)}{\partial a} = h_a(x^*(a), a)$$

where the first term equals to zero because $\frac{\partial h(x^*(a),a)}{\partial x} = 0$ at the maximum point x^*.

(c) Differentiability in the choice variable x seems key, but this is incorrect.

(d) Consider a buyer with value $a \in [0,1]$ who is given a take-it-or-leave-it offer p by a seller.

i. The buyer's choice is $X = \{0, 1\}$, with $x = 0$ meaning no consumption and no price paid, and $x = 1$ meaning the buyer consumes and pays p.

ii. $h(x, a) = a(a - p)$

iii. Given binary choice, there is no notion of differentiability in x. Nonetheless, for all x, $h_a(x, a) = x$ is well-defined and exists.

iv. $x^*(a) = 0$ for $a \leq p$ and $x^*(a) = 1$ otherwise.

v. We have $V'(a) = h_a(x^*(a), a)$ almost everywhere, and

$$V(a) = V(0) + \int_0^a V'(y)dy$$

vi. **Milgrom-Segal 2002**: Let X denote the choice set, $a \in [0,1]$ be the parameter, and $h : X \times [0,1] \to \mathbb{R}$ the objective function. The value function V and the optimal choice correspondence X^* are given by

$$V(a) = \sup_{x \in X} h(x, a) \tag{Eq 2}$$

$$X^*(a) = \{x \in X : h(x, a) = V(a)\} \tag{Eq 3}$$

Theorem 1:
Take $a \in [0,1]$ and assume that $h_a(x^*, a)$ exists. If $a > 0$ and V is left-hand differentiable at a, then $V'(a^-) \leq h_a(x^*, a)$. If $a < 1$ and V is right-hand differentiable at a, then $V'(a^+) \geq h_a(x^*a, a)$. If $a \in (0,1)$ and V is differentiable at a, then $V'(a) = h_a(x^*, a)$.

vii. Proof of Theorem 1:
Using (Eq 2) and (Eq 3), for any $a' \in [0,1]$, we have

$$h(x^*, a') - h(x^*, a) \leq V(a') - V(a)$$

where the inequality holds because x^* is a maximiser at a but not necessarily at a'.
Taking $a' \in (a, 1)$ and dividing both sides and taking limits yields $h_a(x^*, a) \leq V(a^+)$.
Conversely, taking $a' \in (0, a)$ and dividing both sides and taking limits yields $h_a(x^*, a) \geq V(a^-)$.
When V is differentiable at $a \in (0, 1)$, we have $V'(a) = V(a^-) = V(a^+)$.

viii. **Theorem 2:** Suppose that
   A. $h(x, .)$ is absolutely continuous (i.e., "differentiable almost everywhere") for all $x \in X$, and
   B. there exists an integrable function $b : [0, 1] \to \mathbb{R}_+$ such that $|h_a(x, a)|leqb(a)$ (i.e., bounded by $b(a)$) for all $x \in X$ and almost all $x \in [0, 1]$

   Then V is absolutely continuous.
   If in addition, $h(x, .)$ is differentiable in a for all $x \in X$, and $X^*(a) \neq \emptyset$ (i.e., the set of maximisers is nonempty) almost everywhere on $[0, 1]$, then for any selection $x^*(a) \in X^*(a)$,

$$V(a) = V(0) + \int_0^a h_a(x^*(s), s)ds \qquad \text{(Eq 4)}$$

ix. Proof of Theorem 2: Using (Eq 2), observe that for any a', a'' such that $a' < a''$,

$$\begin{aligned} |V(a'') - V(a')| &\leq \sup_{x \in X} |h(x, a'') - h(x, a')| \\ &= \sup_{x \in X} |\int_{a'}^{a''} h_a(x, a)da| \\ &\leq \int_{a'}^{a''} \sup_{x \in X} |h_a(x, a)|da \leq \int_{a'}^{a''} b(a)da \end{aligned}$$

   where
   A. the first inequality holds because $V(a') = \sup_{x \in X} h(x, a')$ and $V(a'') = \sup_{x \in X} h(x, a'')$, implying $|V(a') - V(a'')| = \sup_{x \in X} h(x, a') - \sup_{x \in X} h(x, a'')|$, which is $\leq sup_{x \in X}|h(x, a') - h(x, a'')|$
   B. the second inequality is strict unless the sign of h_a is constant on $[a', a'']$, and
   C. the last inequality follows from the assumption that $|h_a(x, a)| \leq b(a)$ for all $x \in X$.

   These imply that V is absolutely continuous, which in turn implies that it is differentiable almost everywhere and $V(a) = V(0) + \int_0^a V'(s)ds$.
   If $h(x, a)$ is differentiable in a, $V'(s)$ is given by Theorem 1, i.e., $V'(s) = h_a(x^*, s)$, which gives (Eq 4).

6. Implications of IC; RET and PET

   (a) Monotonicity of q
   Lemma 1: If $< q, t >$ satisfies, IC, then $q(.)$ is non-decreasing.
   Proof: Consider IC for types v and $\hat{v}$:

$$\begin{aligned} q(v)v - t(v) &\geq q(\hat{v})v - t(\hat{v}) \\ q(v)\hat{v} - t(v) &\leq q(\hat{v})\hat{v} - t(\hat{v}) \end{aligned}$$

   Subtracting the latter from the former implies

$$q(v)(v - \hat{v}) \geq q(\hat{v})(v - \hat{v})$$

   That is, $< q, t >$ being IC implies that q is non-decreasing.

   (b) Transfers can't vary if allocation doesn't.
   Note that Lemma 1 only restricts $q(.)$ and is silent about $t(.)$.
   Lemma 2: Suppose $< q, t >$ satisfies IC and is such that $q(v) = q(\hat{v})$ for $v, \hat{v} \in [\underline{v}, \bar{v}]$. Then $t(v) = t(\hat{v})$. (i.e., if the probability of obtaining the good is the same for two different valuations, then the transfer they will pay is the same).
   Proof: Suppose not, e.g., $t(v) > t(\hat{v})$. Then both v and $\hat{v}$ would report $\hat{v}$, which violates IC for v.
   Implication: With private information, perfect price discrimination is not possible.

(c) Revenue Equivalence Theorem (RET)
Let

$$U(v) = q(v)v - t(v) \tag{Eq 5}$$

Lemma 3: Suppose $< q, t >$ satisfies IC. Then

$$t(v) = q(v)v - \int_{\underline{v}}^{v} q(y)dy - U(\underline{v}) \tag{Eq 6}$$

Note: Up to the constant $U(\underline{v})$, $t(v)$ is entirely pinned down by the allocation rule. The result (Eq 6) is called the **revenue equivalence theorem (RET)** because it means that the transfers of the same type under two mechanisms with the same allocation rule will differ at most by a constant.

(d) Payoff Equivalence Theorem (PET)
Lemma 3 is an implication of the following:
Lemma 4: Suppose $< q, t >$ satisfies IC. Then $U'(v) = q(v)$ almost everywhere and

$$U(v) = U(\underline{v}) + \int_{\underline{v}}^{v} q(y)dy \tag{Eq 7}$$

Noting that $\int_{\underline{v}}^{v} q(y)dy \geq 0$, it follows that $U(v) \geq U(\underline{v})$ for all v. Thus, an IC mechanism satisfies IR iff it satisfies IR for $\underline{v}$.

(e) Proof of Lemma 4:
IC implies that for all $v, \hat{v} \in [\underline{v}, \bar{v}]$,

$$U(v) \geq q(\hat{v})v - t(\hat{v})$$

i.e.,

$$v \in arg \max_{\hat{v} \in [\underline{v}, \bar{v}]} q(\hat{v})v - t(\hat{v})$$

Letting

$$X = (q(v), t(v))_{v \in [\underline{v}, \bar{v}]}$$

be the menu of contracts the agent can choose from, and

$$h(x, v) = x \cdot (v, -1) = q_x v - t_x$$

where $x = (q_x, t_x)$, one can equivalently express the agent's optimisation problem as

$$\max_{x \in X} h(x, v)$$

In this formulation, IC requires $x^*(v) = (q(v), t(v))$ for all v.
he agent maximises its utility function $h(x, a)$ over $x \in X$ and the agent's equilibrium utility $V(a)$ in the mechanism is then given by (Eq 2) and the set of outcomes $X^*(a)$ is given by (Eq 3).
Any selection $x^*(a) \in X^*(a)$ is a choice rule implemented by the mechanism.

Theorem 2 above implies:
Corollary 1: Suppose that the agent's utility function $h(x, a)$ is differentiable and absolutely continuous in a for all $x \in X$ and that $\sup_{xinX} |h_a(x, a)|$ is integrable on $[0, 1]$. Then the agent's equilibrium utility V in any mechanism implementing a given choice rule x^* satisfies (Eq 4).

To apply Corollary 1, in our mechanism design setting, a is replaced by v, $a \in [0, 1]$ by $v \in [\underline{v}, \bar{v}]$ and $V(a)$ by $U(v)$.
We have $h_a(x, a) = q_x = |q_x|$ since $q_x \in [0, 1]$.
Hence, $\sup_{x \in X} |q_X| = q(\bar{v})$ since $q(.)$ is increasing. This is a constant, which is integrable.
Moreover, q_x is a constant and hence absolutely continuous.
Thus, Corollary 1 applies directly to our setting, with $h_a(x^*(a), a)|_{a=v} = q(v)$.

(f) Proof of Lemma 3:
Lemma 3 now follows from equating the definition of $U(v)$ in (Eq 5) with $U(v)$ in (Eq 7):

$$q(v)v - t(v) = U(\underline{v}) + \int_{\underline{v}}^{v} q(y)dy$$
$$t(v) = q(v)v - \int_{\underline{v}}^{v} q(y)dy - U(\underline{v}) \qquad \text{(Eq 6)}$$

7. Objective Function Maximisation
   (a) Recall that the goal is to maximise the seller's expected profit (or expected revenue, since zero costs) over IC and IR mechanisms.
   (b) We know that for a given $q(.)$ that satisfies IC, $t(v)$ is given by (Eq 6).

$$\begin{aligned}
\mathbb{E}[t(v)] &= \int_{\underline{v}}^{\bar{v}} t(v)f(v)dv \\
&= \int_{\underline{v}}^{\bar{v}} q(v)vf(v)dv - \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{v} q(y)dyf(v)dv - U(\underline{v}) \\
&= \int_{\underline{v}}^{\bar{v}} q(v)vf(v)dv - \int_{\underline{v}}^{\bar{v}} \int_{y}^{\bar{v}} q(y)f(v)dvdy - U(\underline{v}) \qquad \text{change order of integration} \\
&= \int_{\underline{v}}^{\bar{v}} q(v)vf(v)dv - \int_{\underline{v}}^{\bar{v}} (1-F(y))q(y)dy - U(\underline{v}) \qquad \text{take F from the distribution} \\
&= \int_{\underline{v}}^{\bar{v}} [v - \frac{1-F(v)}{f(v)}]q(v)f(v)dv - U(\underline{v}) \qquad \text{(Eq 8)}
\end{aligned}$$

8. The designer's problem can now be expressed maximising $\mathbb{E}[t(v)]$ in (Eq 8) over $q(.)$ and $U(\underline{v})$, subject to IC and IR.
9. Optimally, $U(\underline{v}) = 0$ as $U(\underline{v})$ will be chosen as small as possible without violating IR.
10. Letting

$$\Phi(v) = v - \frac{1-F(v)}{f(v)}$$

denote the **virtual value**, the optimisation problem is now

$$\max_{q(.)} \int_{\underline{v}}^{\bar{v}} \Phi(v)q(v)f(v)dv$$

Q1: What is the pointwise maximiser $q^*(.)$?
A1:

$$q^*(v) = \begin{cases} 1, & \Phi(v) \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{(Eq 9)}$$

Q2: Is $q^*(.)$ monotone?
A2: Suppose $\Phi(v)$ is increasing (called the "regular case"). Then, $q^*(v)$ is monotone.
Proposition 1: In the regular case, the profit maximising mechanism $< q^*, t^* >$ has the allocation rule $q^*(v)$ given by (Eq 9) and the payment rule $t^*(v) = q^*(v)v - \int_{\underline{v}}^{v} q^*(y)dy$.
The optimal mechanism can be implemented with a take-it-or-leave-it offer p^*. In the regular case, $p^* = \Phi^{-1}(0)$. For example, for F uniform on $[0,1]$, $p^* = \frac{1}{2}$.

11. Intuition that Φ is "marginal revenue":
Suppose $1 - F(p) = Q$
Then, $P(Q) = F^{-1}(1-Q)$.
Revenue is $Q(P(Q) = QF^{-1}(1-Q)$
Marginal revenue is:

$$\begin{aligned}
R'(Q) &= F^{-1}(1-Q) - \frac{Q}{f(F^{-1}(1-Q))} \\
&= p - \frac{1-F(p)}{f(p)} = \Phi(p)
\end{aligned}$$

Lecture 13

1. Mechanism Design with multiple agents

   We extend the principles from Lecture 12 from the case of a single agent (and a designer/ producer) to the case of multiple agents.

   We revisit the examples in Lecture 1 and show that the mechanisms used are indeed the optimal mechanisms.

   (a) Public good

   i. Setup:
   Consider a pure public good.
   Production cost is $K > 0$.
   If the good is produced, agents cannot be excluded from consuming it.
   There are N agents with values $\boldsymbol{v} = (v_1, ..., v_N)$.
   Efficiency dictates that production occurs iff $\sum_{i=1}^{N} v_i > K$.
   Assume, for all $i, v_i \in [0, 1]$ and $1 < K < N$, so that production is sometimes but not always efficient (and no single agent can induce production, which would be the case if $K < 1$).
   The outside option of every agent is 0.

   ii. Direct mechanisms:
   Letting $Q \in [0, 1]$ be the production decision and T_i the transfer, i's payoff when of type v_i is $Qv_i - T_i$.
   A direct mechanism $< Q, \boldsymbol{T} >$ with $Q : [0, 1]^N \to [0, 1]$ and $\boldsymbol{T} : [0, 1]^N \to \mathbb{R}^N$ consists of an allocation rule Q and a payment rule $\boldsymbol{T}$ (note the vector notation as there are N agents).
   The efficient allocation rule $Q^e(\boldsymbol{v})$ is

$$Q^e(\boldsymbol{v}) = \begin{cases} 1, & \sum_{i=1}^{N} v_i > K \\ 0, & \text{otherwise} \end{cases}$$

   i.e., production occurs when the sum of values is $> K$.

   The direct mechanism $< Q, \boldsymbol{T} >$ is **dominant strategy incentive compatible (DIC)** if for all i, all $v_i, \hat{v}_i \in [0, 1]$ and all $\boldsymbol{v}_{-i} \in [0, 1]^{N-1}$,

$$Q(v_i, \boldsymbol{v}_{-i})v_i - T_i(v, \boldsymbol{v}_{-i}) \geq Q(\hat{v}_i, \boldsymbol{v}_{-i})v_i - T_i(\hat{v}_i, \boldsymbol{v}_{-i})$$

   in other words, everyone is better off reporting their true valuation v_i than any other valuation $\hat{v}_i$, for all possible reports by other agents.

   The direct mechanism $< Q, \boldsymbol{T} >$ is **ex post individually rational (EIR)** if for all i, all $v_i \in [0, 1]$ and all $v_{-i} \in [0, 1]^{N-1}$, if $Q = 1$ (i.e., in the case of production),

$$v_i - T_i(\boldsymbol{v}) \geq 0$$

   holds, and if $Q = 0$, $-T_i(\boldsymbol{v}) \geq 0$ holds (i.e. 4edin the case of no production, T_i is nonpositive),
   These are analogous to IC IR in Lecture 12.
   The revelation principle extends in the sense that, among DIC-EIR mechanisms, the focus on direct mechanisms is without loss of generality.

   A DIC-EIR mechanism $< Q, \boldsymbol{T} >$ is efficient if $Q = Q^e$.
   Consider the mechanism $< Q^e, \boldsymbol{T} >$ with

$$T_i(\boldsymbol{v}) = \max\{0, K - \sum_{j \neq i} v_j\}$$

   if $Q^e(\boldsymbol{v}) = 1$ and $T_i(\boldsymbol{v}) = 0$ otherwise (i.e., under this mechanism, the agents must make this transfer).
   This is the **VCG mechanism** named after Vickrey (1961), Clarke (1971), Groves (1973).

iii. Proof that VCG mechanism satisfies DIC and EIR:

$$T_i(\boldsymbol{v}) = \max\{0, K - \sum_{j \neq i} v_j\} \qquad \text{if } Q^e(\boldsymbol{v}) = 1$$

Let $A(v)$ be the set of agents for which $\{i : \sum_{j\neq i} v_j \leq K\}$, i.e., these are agents for which the sum of other agents' values $\leq K$, so these agents are pivotal for production.
If these agents report low enough, no production occurs.

Case 1: $i \notin A(v)$. We focus on the case where $Q^e(\boldsymbol{v}) = 1$.
In this case, $K - \sum_{j\neq i} v_i < 0 \implies T_i(\boldsymbol{v}) = 0$
For such agents, reporting v_i truthfully cannot harm the agents, and hence is optimal.

Case 2: $i \in A(v)$. We focus on the case where $Q^e(\boldsymbol{v}) = 1$.
In this case, $v_i - (K - \sum_{j\neq i} v_j) = \sum_{j\neq i} v_j + v_i - K = \sum_{i=1}^{N} v_i - K > 0$.
$\forall \hat{v}_i \neq v_i$ such that $\hat{v}_i + \sum_{j\neq i} v_j > K$, agents get the same payoff. Hence no such report increases agent i's payoff.
Now suppose $\hat{v}_i + \sum_{j\neq i} v_i \leq K$. Then, there is no production and agent i's payoff is 0, which is strictly less than the payoff from reporting truthfully.

Hence, in both cases, the agent reports truthfully and the mechanism satisfies DIC and EIR.

iv. Impossibility result

Proposition 1: The VCG mechanism runs a deficit whenever production is efficient.

Corollary 1: By RET, any efficient DIC-EIR mechanism runs a deficit whenever production is efficient.
This corollary follows from RET because the VCG mechanism satisfies EIR with equality for the lowest possible type; hence any other efficient DIC-EIR mechanism generates weakly less revenue.

Proof of Proposition 1:

A. Let $\mathcal{A} = \{i | \sum_{j\neq i} v_j < K\}$ with cardinality A be the set of agents who are pivotal (it is possible that $\mathcal{A}$ is empty).
B. The revenue of the VCG mechanism if production occurs is $R(\boldsymbol{v}) = A(K - \sum_{j\in N} v_j) + \sum_{i\in\mathcal{A}} v_i$.
This is because

$$\begin{aligned}\sum_{i\in N} T_i(\boldsymbol{v}) = \sum_{i\in\mathcal{A}} T_i(\boldsymbol{v}) &= \sum_{i\in\mathcal{A}}(K - \sum_{j\neq i} v_j) \\ &= \sum_{i\in\mathcal{A}}(K - \sum_{j\in N} v_j + v_i) \\ &= A(K - \sum_{j\in N} v_j) + \sum_{i\in\mathcal{A}} v_i\end{aligned}$$

C. If $\mathcal{A}$ is empty, then $R(\boldsymbol{v}) = 0$.
D. If $R(\boldsymbol{v}) > 0$, we have

$$R(\boldsymbol{v}) - K = (A-1)(K - \sum_{i\in N} v_i) + \sum_{j\in\mathcal{A}} v_j - \sum_{j\in N} v_j < 0$$

(the inequality is strict because we could have equality iff $A = 1$ and $v_j = 0$ for all $j \neq \mathcal{A}$ but then production is not efficient) because $K - \sum_{i\in N} v_i < 0$ if production is efficient and $\sum_{j\in\mathcal{A}} v_j \leq \sum_{j\in N} v_j$ with strict inequality if $v_j > 0$ for some $j \notin \mathcal{A}$.

v. RET and PET for DIC mechanisms

RET and PET extend from the single-agent case to the multiple agent case for DIC mechanisms.

Let $T_i(v_i, \boldsymbol{v}_{-i})$ be i's transfer when it reports v_i while all others report $\boldsymbol{v}_{-i}$ and $T_i(v_i, \boldsymbol{v}_{-i})$ be i's allocation given these reports in a DIC mechanism.

(with public goods, we have of course $Q_i(v_i, \boldsymbol{v}_{-i}) = Q(v_i, \boldsymbol{v}_{-i})$ for all i).
Then,

$$T_i(v_i, \boldsymbol{v}_{-i}) = Q_i(v_i, \boldsymbol{v}_{-i})v_i = \int_{\underline{v}}^{v_i} Q_i(y, \boldsymbol{v}_{-i})dy - U_i(\underline{v})$$

(b) Club good

i. Setup:
A club good has basically the same properties as a public good but permits excluding agents from consumption.
If there are marginal costs $c > 0$ on top of the fixed cost of production, then excluding agents can be efficient even if production takes place.
Assume $v_i \in [0, 1]$ for all i, $c > 0$ and $0 < K < N(1 - c)$.

ii. Direct mechanism: Letting $q_i \in [0, 1]$ be the production decision and t_i the transfer, i's payoff when of type v_i is $qv_i - t_i$.
A direct mechanism $< \boldsymbol{Q}, \boldsymbol{t} >$ with $\boldsymbol{Q} : [0, 1]^N \to [0, 1]^N$ and $\boldsymbol{T} : [0, 1]^N \to \mathbb{R}^N$ consists of an allocation rule $\boldsymbol{Q} = (Q_1, ..., Q_N)$ and a payment rule $\boldsymbol{T} = (T_1, ..., T_N)$.
Let $A(\boldsymbol{v}) := \{i : v_i > c\}$.
The efficient allocation rule $\boldsymbol{Q}^c(\boldsymbol{v})$ consists for all i of

$$Q_i^e(\boldsymbol{v}) = \begin{cases} 1 & i \in A(\boldsymbol{v}) \text{ and } \sum_{i \in A(\boldsymbol{v})}(v_i - c) > K \\ 0 & \text{otherwise} \end{cases}$$

iii. VCG mechanism:
The VCG mechanism has, by definition, the efficient allocation rule.
The transfers are $T_i(\boldsymbol{v}) = 0$ if $Q_i^e(\boldsymbol{v}) = 0$ and, if $Q_i^e(\boldsymbol{v}) = 1$:

$$T_i(\boldsymbol{v}) = \max\{c, K + c - \sum_{j \in A(\boldsymbol{v}) \setminus \{i\}} (v_j - c)\}$$

This mechanism endows the agents with dominant strategies and respects their EIR.
An analogous argument as the one for public goods shows that the VCG mechanism never runs a budget surplus.

(c) Bilateral trade

i. Consider a setup with one buyer with value $v \in [0, 1]$ and one seller with cost $c \in [0, 1]$.
ii. Under efficiency, the buyer obtains the good iff $v > c$.
iii. No transfers are made under the VCG mechanism if there is no trade; if there is trade, the buyer pays c and the seller obtains v.
iv. EIR binds for the buyer of type 0 and the seller of type 1, whose gains from participating in the mechanism are 0.
v. RET (extended to the seller) implies that any other efficient DIC-EIR mechanism also runs a deficit when there is trade.
vi. Myerson and Satterthwaite (1983) study a version of this setup; Vickrey (1961) analyses a setting with multi-unit demands and supplies and many buyers and sellers.
vii. These impossibility results contrast (but do not contradict) the Coase Theorem (Coase 1960) by showing that private information can be a substantial transaction cost.
viii. With single-unit demands and supplies and multiple buyers and sellers, the VCG mechanism asks $\underline{p}^W(\boldsymbol{v}, \boldsymbol{c})$ from trading buyers and offers $\bar{p}^W(\boldsymbol{v}, \boldsymbol{c})$ to trading sellers.
In other words, if trade is (ex-post) efficient, VCG payments are:

$$T_B(\boldsymbol{v}, \boldsymbol{c}) = c = \underline{p}^W$$
$$T_C(\boldsymbol{v}, \boldsymbol{c}) = v = \bar{p}^W$$

2. Optimal auctions

(a) Profit maximisation with multiple buyers

Setup:

i. $N \geq 1$ buyers, who draw their values v_i independently from distributions F_i with supports $[\underline{v}_i, \bar{v}_i]$ and densities $f_i > 0$ on the supports with $\underline{v}_i \geq 0$.

ii. Buyer i's value v_i is its private information, but the distributions are common knowledge.

iii. The two preceding points taken together means that we have "independent private values", i.e., each buyer's willingness to pay doesn't depend on knowledge of other agents' types.
Note: The opposite of this would be "interdependent values".
Note: Whether or not the values are independent has nothing to do with statistical properties, but rather is related to agents' valuation functions. It is possible for agents to have independent types (which is a statistical property) and interdependent values (which is a property of the valuation function), and vice-versa.

iv. The seller (designer) has one item for sale, which is know to have value 0 for the seller (i.e., the seller has a known cost of 0).

(b) Direct mechanisms

i. Letting $\mathcal{V} = \times_{i=1}^{N}[\underline{v}_i, \bar{v}_i]$ (every agent's reported values), a direct mechanism $< \boldsymbol{Q}, \boldsymbol{T} >$ consists of the allocation rule

$$\boldsymbol{Q} : \mathcal{V} \to [0,1]^N$$

$$\text{s.t. for all } \boldsymbol{v} \in \mathcal{V}, \sum_{i=1}^{N} Q_i(\boldsymbol{v}) \leq 1 \text{ (this is the budget constraint)}$$

and a payment rule

$$\boldsymbol{T} : \mathcal{V} \to \mathbb{R}^N$$

ii. Letting

$$q_i(v_i) := \mathbb{E}_{v_{-i}}[Q_i(v_i, \boldsymbol{v}_{-i})] \text{ and } t_i(v_i) := \mathbb{E}_{v_{-i}}[T_i(v_i, \boldsymbol{v}_{-i})]$$

a direct mechanism $< \boldsymbol{Q}, \boldsymbol{T} >$ is **Bayesian Nash IC (BIC)** if, for all i, all $v_i, \hat{v}_i \in [\underline{v}_i, \bar{v}_i]$,

$$q_i(v_i)v_i - t_i(v_i) \geq q_i(\hat{v}_i)v_i - t_i(\hat{v}_i) \tag{Eq 6}$$

iii. In other words, BIC (i.e., (Eq 6)) requires truth telling to be a Bayesian Nash Equilibrium of the game.

iv. Meanwhile, **Interim IR** requires that for all i, all $v_i \in [\underline{v}_i, \bar{v}_i]$,

$$q_i(v_i)v_i - t_i(v_i) \geq 0$$

v. By the revelation principle, among BIC-IR mechanisms, the focus on direct BIC-IR mechanisms the focus on direct BIC-IR mechanisms is without loss of generality.

vi. The **designer's problem** is to find the revenue maximising BIC-IR mechanism.

(c) Expected payments

PET, RET extend directly from the single agent case (and are identical).

Consequently, the expected payment from i is

$$\mathbb{E}_{v_i}[\Phi_i(v_i)q_i(v)] - U_i(\underline{v}_i)$$

where $\Phi_i(v) = v - \frac{1-F_i(v)}{f_i(v)}$ and $U_i(v) = q_i(v)v - t_i(v)$ and thus expected revenue is

$$\begin{aligned}
R &= \sum_{i=1}^{N} \mathbb{E}_{v_i}[\Phi_i(v_i)q_i(v)] - \sum_{i=1}^{N} U_i(\underline{v}_i) \\
&= \sum_{i=1}^{N} \int_{\underline{v}_i}^{\bar{v}_i} \Phi_i(v_i)q_i(v_i)f_i(v_i)dv_i - \sum_{i=1}^{N} U_i(\underline{v}_i) \\
&= \sum_{i=1}^{N} \int_{v \in \mathcal{V}} \Phi_i(v_i)Q_i(\boldsymbol{v})f(\boldsymbol{v})d\boldsymbol{v} - \sum_{i=1}^{N} U_i(\underline{v}_i \\
&= \int_{v \in \mathcal{V}} \sum_{i=1}^{N} Q_i(\boldsymbol{v})\Phi_i(v_i)f(\boldsymbol{v})d\boldsymbol{v} - \sum_{i=1}^{N} U_i(\underline{v}_i)
\end{aligned}$$

where $f(\boldsymbol{v}) = f_1(v_1)...f_N(v_N)$, and where the first equality uses the definition of $\mathbb{E}_{v_i}[\Phi_i(v_i)q_i(v)]$; the second the definition of $q_i(v_i)$ and the fact that each buyer draws their values independently, and the third changes the order of integration and summation.

The designer's problem is to choose $\boldsymbol{Q}$ and $U_i(\underline{v}_i)$ to maximise R.
As in the one agent case, $U_i(\underline{v}_i) = 0$ for all i is optimal and satisfies IR with equality.

Two questions:

i. What is the pointwise maximiser $\boldsymbol{Q}^*$ of R?
ii. Is $\boldsymbol{Q}^*$ such that $q_i^*(v_i)$ is monotone?

Under regularity, $q_i(v_i) := \mathbb{E}_{v_{-i}}[Q_i(v_i, \boldsymbol{v}_{-i})]$ is increasing in v_i.
Proposition 2: Assume that for all i, $\Phi_i(v_i)$ is increasing. In other words, as agents report a higher type v_i, their virtual type $\Phi_i(v_i)$ increases.
Then, the optimal auction allocates the good to the agent with the highest value, provided this virtual value is positive. Otherwise, the seller keeps the good.

$$Q_i v = \begin{cases} 1 & \Phi_i(v_i) > 0 \text{ and } \Phi_i(v_i) = \max\{\Phi_i v(i)\} \\ = & \text{otherwise} \end{cases}$$

Given PET, the payment rule is pinned down by $\boldsymbol{Q}^*$.

(d) Two sources of inefficiency

Recall that:

$$\Phi_i(v_i) = v_i - \frac{1 - F_i(v)}{f_i(v)} < v, \ \forall v_i < \bar{v}$$
$$\Phi_i(\bar{v}) = \bar{v}$$

and $\Phi_i(v_i)$ is increasing in v, $\forall i$.

There are in general two sources for why the optimal auction allocates inefficiently in the regular case:

i. If $\Phi_i(\underline{v}_i) < 0$ for some i, where 0 is the seller's cost, the seller will not sell to i when i's value is low even though $\underline{v}_i \geq 0$ and selling would be efficient.
This is analogous to the standard monopoly distortion: $P > MC$.
ii. If $F_i \neq F_j$, then the allocation is based on virtual values, whose ranking is not in general the same as the ranking of values.
This leads to inefficient price discrimination.

If $F_i = F$ for all i, the optimal auction in the regular case can be implemented as a second-price (or English) auction with reserve price $p^* = \Phi^{-1}(0)$.

(e) Example:

i. Assume:

$$F_1(v) = v, f_1(v) = 1$$
$$\Phi_1(v) = v - \frac{1 - F_1(v)}{f_1(v)} = 2v - 1$$
$$F_2(v) = 1 - (1 - v)^a, f_2(v) = a(1 - v)^{a-1}$$
$$\Phi_2(v) = v - \frac{1}{a}(1 - v) = \frac{a+1}{a}v - \frac{1}{a}$$

so that when $a = \frac{1}{2}, \Phi_2(v) = 3v - 2$.
By inspection, we can see that $F_2(v)$ FOSD $F_1(v)$.

ii. The optimal reserve price for bidder 1 is $v_1^* = \frac{1}{2}$ (this is the price that sets $\Phi_1(v) = 0$)

iii. The optimal reserve price for bidder 2 is $v_2^* = \frac{2}{3}$

iv. (see graphical notes for illustration of two types of inefficiency).

v. Without regularity, there is potentially a third source of inefficiency arising from random allocation.

3. Example of BNE: FPA

(a) BNE in FPA

i. Assume $\beta(v)$ is the symmetric increasing, differentiable BNE bidding function (or strategy) satisfying $\beta(\underline{v}) = \underline{v}$.

ii. Consider the problem of a bidder of type v:

$$\max_{\hat{v}}(v - \beta(\hat{v}))F(\hat{v})^{N-1} \tag{Eq 8}$$

iii. If $\beta(.)$ is the equilibrium bidding strategy, the maximiser for (Eq 8) must be $\hat{v} = v$:

$$0 = v(N-1)F(v)^{N-2}f(v) - [\beta(v)F(v)^{N-1}]'$$

iv. Re-arranging we get:

$$[\beta(v)F(v)^{N-1}]' = v(N-1)F(v)^{N-2}f(v)$$

Integrating both sides from $\underline{v}$ to v we get:

$$\beta(v)F(v)^{N-1} = \int_{\underline{v}}^{v} y(N-1)F(y)^{N-2}f(y)dy$$

This is equivalent to

$$\beta(v) = \frac{\int_{\underline{v}}^{v} y(N-1)F(y)^{N-2}f(y)dy}{F(v)^{N-1}}$$

which has all the properties ($\beta(\underline{v}) = \underline{v}, \beta' > 0$ stipulated.

(b) Properties of $\beta(v)$

i. Notice that $\beta(v) = \mathbb{E}[v_{(1:N-1)}|v_{(1:N-1)} \leq v]$, where the expectation of $v_{(1:N-1)}$ is taken wrt the distribution $F_{(1:N-1)}(v_{(1:N-1)})$.

ii. In other words, $\beta(v)$ is the expectation of the highest draw of the competitors, conditional on this draw being less than v.

iii. Therefore, $\beta(v) < v$, i.e., there is bid-shading.

iv. Also, $\beta'(v) = \frac{f_{(1:N-1)}(v)}{F_{(1:N-1)}(v)}(v - \beta(v)) = (N-1)\frac{f(v)}{F(v)}(v - \beta(v))$, which is positive as just observed.

v. By l'Hopital's rule, $\beta(\underline{v}) = \frac{\underline{v}f_{(1:N-1)}(\underline{v})}{f_{(1:N-1)}(\underline{v})} = \underline{v}$

(c) Second-order conditions

i. Denote $u(\hat{v}, v) := (v - \beta(\hat{v}))F_{(1:N-1)}(\hat{v})$.

ii. $\frac{\partial u(\hat{v}),v}{\partial \hat{v}} = vf_{(1:N-1)}(\hat{v}) - [\beta(\hat{v})F_{(1:N-1)}(\hat{v})]'$

iii. Notice that $[\beta(\hat{v})F_{(1:N-1)}(\hat{v})]' = \hat{v}f_{(1:N-1)}(\hat{v})$.

iv. Thus, $\frac{\partial u(\hat{v},v)}{\partial \hat{v}} = f_{(1:N-1)}(\hat{v})(v - \hat{v})$

v. This is evidently positive for $\hat{v} < v$ and negative for $\hat{v} > v$.

vi. Thus we have a maximum and hence a BNE.

(d) IID in FPE

i. Denote by $v_{(1:N-1)}$ the highest of $N-1$ independent draws from the distribution F with density f.

ii. **Proposition 1**: In the IPV model with identical distributions (i.e., $f_i(v) = f(v)$ for all i and all $v \in [\underline{v}, \bar{v}]$), the FPA has a symmetric Bayes Nash equilibrium in which each bidder bids according to

$$\beta^{FPA}(v) = \mathbb{E}[v_{(1:N-1)} | v_{(1:N-1)} \leq v]$$

Proof: The distribution of $v_{(1:N-1)}$ is $F_{(1:N-1)}(v_{(1:N-1)}) := F(v_{(1:N-1)})^{N-1}$
Thus,

$$\mathbb{E}[v_{(1:N-1)} | v_{(1:N-1)} \leq v] = \frac{\int_{\underline{v}}^{v} y f_{(1:N-1)}(y) dy}{F_{(1:N-1)}(v)}$$

and this is a symmetric BNE.

Lecture 14

1. Mechanism design approach to incomplete information bargaining
   (a) Bargaining is central to real world and economic models.
   e.g., Merger reviews, contracts, tariffs, climate policy agreements
   (b) With complete information, bargaining loses "much of its interest" (Fudenberg and Tirole 1991)
   (c) Recent upsurge of empirical and theoretical interest (e.g., Loertscher and Marx 2019; Loertscher and Marx 2022).

2. Myerson and Satterthwaite (1983)
   (a) Setup
      i. One buyer with value v distributed according to F on $[\underline{v}, \bar{v}]$ with density $f > 0$ on the support
      ii. One seller with opportunity cost (value) c distributed according to G with support $[\underline{c}, \bar{c}]$ and density g on the support
      iii. Assume $\underline{v} < \bar{c}$ and $\bar{v} > \underline{c} > 0$, which implies that ex post trade is sometimes but not always efficient
      iv. The value of the buyer's outside option is 0
      v. The value of the seller's outside option is c

   (b) Direct mechanisms
      i. Let $< \boldsymbol{Q}, \boldsymbol{T} >$ with $\boldsymbol{Q} = (Q_B, Q_S)$ (i.e., the probability that the buyer and the seller receive the good respectively) and $\boldsymbol{T} = (T_B, T_S)$ (i.e., the transfers to the buyer and to the seller), such that

$$\boldsymbol{Q} : [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}] \to [0, 1]^2$$
and
$$\boldsymbol{T} : [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}] \to \mathbb{R}^2$$

and $Q_B + Q_S \leq 1$ be a direct mechanism, where $Q_i(v, c)$ is the probability that i is allocated the good and $T_i(v, c)$ is i's payment to the mechanism upon reporting (v, c).

Let

$$q_B(v) = \mathbb{E}_c[Q_B(v, c)] \text{ and } t_B(v) = \mathbb{E}_c[T_B(v, c)]$$
$$q_S(c) = \mathbb{E}_v[Q_s(v, c)] \text{ and } t_S(c) = \mathbb{E}_v[T_S(v, c)]$$

      ii. A direct mechanism $< \boldsymbol{Q}, \boldsymbol{T} >$ satisfies BIC for the buyer if for all $v, \hat{v} \in [\underline{v}, \bar{v}]$,

$$U_B(v) := q_B(v)v - t_B(v) \geq q_B(\hat{v})v - t_B(\hat{v})$$

and satisfies BIC for the seller if for all $c, \hat{c} \in [\underline{c}, \bar{c}]$,

$$U_S(c) := q_S(c)c - t_S(c) \geq q_S(\hat{c})c - t_S(\hat{c})$$

and satisfies interim IR for the buyer if for all $v \in [\underline{v}, \bar{v}]$, and interim IR for the seller if for all $c \in [\underline{c}, \bar{c}]$,

$$U_B(v) \geq 0 \text{ and } U_S(c) \geq c \qquad \text{(in line with Lemma 4 and Eq 7 in Lecture 12)}$$

(c) Implications of BIC for seller

i. By the fundamental theorem of calculus, if H is differentiable almost everywhere on $[\underline{x}, \bar{x}]$, then for any $x, \hat{x} \in [\underline{x}, \bar{x}]$,

$$H(x) = H(\hat{x}) + \int_{\hat{x}}^{x} h'(y)dy$$

Applied to $U(v)$ in (Eq 7) in Lecture 12, this means that for any $v, \hat{v} \in [\underline{v}, \bar{v}]$ we have,

$$U(v) = U(\hat{v}) + \int_{\hat{v}}^{v} q(y)dy$$

This implies that for any $v, \hat{v} \in [\underline{v}, \hat{v}]$,

$$t(v) = q(v)v - \int_{\hat{v}}^{v} q(y)dy - U(\hat{v})$$

ii. Since c is just a value, we can write the seller's interim expected utility under a BIC mechanism as

$$U_S(c) = U_S(\bar{c}) + \int_{\bar{c}}^{c} q_S(y)dy = U_S(\bar{c}) - \int_{c}^{\bar{c}} q_S(y)dy$$

implying

$$t_S(c) = q_S(c)c + \int_{c}^{\bar{c}} q_S(y)dy - U_S(\bar{c})$$

and thus

$$\begin{aligned}
\mathbb{E}_c[t_S(c)] &= \int_{\underline{c}}^{\bar{c}} t_S(c)g(c)dc \\
&= \int_{\underline{c}}^{\bar{c}} q_S(c)cg(c)dc + \int_{\underline{c}}^{\bar{c}} \int_{\underline{c}}^{\bar{c}} q_S(y)dyg(c)dc - U_S(c) \\
&= \int_{\underline{c}}^{\bar{c}} q_S(c)cg(c)dc + \int_{\underline{c}}^{\bar{c}} q_S(c) \int_{\underline{c}}^{y} g(c)dcdy - U_S(c) \\
&= \int_{\underline{c}}^{\bar{c}} [c + \frac{G(c)}{g(c)}]q_S(c)g(c)dc - U_S(c) \\
&= \int_{\underline{c}}^{\bar{c}} \Gamma(c)q_S(c)g(c)dc - U_S(\bar{c}) \qquad \text{(Eq 2)}
\end{aligned}$$

where $\Gamma(c) = c + \frac{G(c)}{g(c)}$ is the seller's **virtual cost**.

(d) IR for and expected payment from the seller

i. Because $U_S'(c) = q_S(c) \leq 1 = U_S^O(c)$, where $U_S^O(c)$ is the seller's value of the outside option, it follows that the IR constraint for the seller is satisfied iff it is satisfied for the seller with cost $c = \bar{c}$.

ii. Making the seller's IR constraint bind by setting $U_S(\bar{c}) = U_S^O(\bar{c}) = \bar{c}$, (Eq 2) yields

$$\begin{aligned}
\mathbb{E}_c[t_S(c)] &= \int_{\underline{c}}^{\bar{c}} \Gamma(c)q_S(c)g(c)dc - \bar{c} \\
&= \int_{\underline{c}}^{\bar{c}} \Gamma(c)(q_S(c) - 1)g(c)dc
\end{aligned}$$

where the second line follows from the fact that $\mathbb{E}_c[\Gamma(c)] = \bar{c}$. Note that $\mathbb{E}_c[t_S(c)]$ is the ex ante expected payment *from* the seller, which is nonpositive since $q_S(c) \leq 1$.

Note: We can show that $\mathbb{E}_c[\Gamma(c)] = \bar{c}$ by:

$$\begin{aligned}
&\mathbb{E}[\Phi(v)|v \geq p] = p \\
&\mathbb{E}[\Phi(c)|c \geq p] = p \qquad \text{(these require integration by parts)} \\
&\mathbb{E}[\Phi(c)] = \bar{c}
\end{aligned}$$

iii. Accordingly, the ex ante expected payment to the seller is

$$-\mathbb{E}_c[t_S(c)] = \int_{\underline{c}}^{\bar{c}} \Gamma(c)(1-q_S(c))g(c)dc = \int_{\underline{c}}^{\bar{c}} \Gamma(c)p(c)g(c)dc$$

where $p(c) := 1 - q_S(c)$ is the probability that the seller produces (sells) the good given report c. $q_S(c)$ is the probability that the seller keeps the good.

As before, the ex ante expected payment from the buyer when its IR constraint binds is

$$\mathbb{E}_v[t_B(v)] = \int_{\underline{v}}^{\bar{v}} \Phi(v)q_B(v)f(v)dv$$

where $\Phi(v) = v - \frac{1-F(v)}{f(v)}$ is the buyer's **virtual value**.

(e) Ex post efficiency

i. Under ex post efficiency, we have $Q_B(v,c) = 1$ iff $v > c$ and $Q_S(v,c) = 1$ otherwise, implying

$$q_B(v) = G(v) \text{ and } p_S(c) = 1 - q_S(c) = 1 - F(c)$$

Note that $q_B(v)$ and $q_S(c)$ are increasing and hence satisfy the monotonicity constraint imposed by BIC.

ii. Let $a = \min\{\underline{v}, \underline{c}\}$ and $b = \max\{\bar{v}, \bar{c}\}$.
The revenue R^0 under ex post efficiency, BIC and interim IR is thus

$$\begin{aligned}
R^0 &= \int_{\underline{v}}^{\bar{v}} \Phi(v)G(v)f(v)dv - \int_{\underline{c}}^{\bar{c}} \Gamma(c)(1-F(c))g(c)dc \qquad \text{(using } G(v) \text{ and } 1-F(c) \text{ from earlier)}\\
&= \int_a^b x[f(x)G(x) - g(x)(1-F(x))] - 2G(x)(1-F(x))dx\\
&= \int_a^b x[-G(x)(1-F(x)]' - 2G(x)(1-F(x))dx\\
&= -\int_a^b G(x)(1-F(x))dx \qquad \text{(using integration by parts)}\\
&< 0
\end{aligned}$$

i.e., the revenue is always negative.
For example, for F and G uniform on $[0,1]$, we get $R^0 = -\frac{1}{6}$.

(f) Impossibility theorem and Second-best problem

i. The result above is the impossibility theorem of Myerson and Satterthwaite (1983): With independently distributed types and overlapping, compact supports, every efficient BIC-IR mechanism for a bilateral trade problem runs a deficit in expectation.

ii. Important result in light of the Coase theorem, showing that the result of Vickrey (1961) generalises to BIC.

iii. The question now is what is the **second-best mechanism**, that is, the mechanism that maximises ex ante expected social surplus subject to BIC, interim IR and no deficit?
This is what Myerson and Satterthwaite (1983) answer for the regular case where we assume Φ and Γ are increasing.

iv. The problem of the second-best is to maximise the value of the allocation (note that now we can choose the allocations Q_B, Q_S):

$$\max_{Q_B,Q_S} \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [vQ_B(v,c) + cQ_S(v,c)]f(v)g(c)dcdv$$
$$\text{s.t.} \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi(v)Q_B(v,c) + \Gamma(c)(Q_S(v,c)-1)]f(v)g(c)dcdv \geq 0$$

subject to:

A. $0 \leq Q_i(v,c)$,
B. $Q_B(v,c) + Q_S(v,c) \leq 1$ for all $(v,c) \in [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}]$,
C. $q_B(v)$ and $q_S(c)$ increasing for all $v \in [\underline{v}, \bar{v}]$ and $c \in [\underline{c}, \bar{c}]$.

We can rewrite the problem as

$$\begin{aligned} &\max_{Q_B, Q_S} \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [vQ_B(v,c) - c(1 - Q_s(v,s))] f(v) g(c) dc dv + \mathbb{E}[c] \\ &\text{s.t.} \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi(v) Q_B(v,c) - \Gamma(c)(1 - Q_S(v,c))] f(v) g(c) dc dv \geq 0 \end{aligned}$$

(g) Weighted virtual types and Lagrangian

i. To solve the problem, we will use $\alpha \in [0,1]$ as a weight on the virtual types (e.g., think of how α is used as a weight in Ramsey pricing).
Previously, the analysis used equal weights for the buyer and the seller ($\frac{1}{2}$ each).

For $\alpha \in [0,1]$, let

$$\Phi_\alpha(v) := (1-\alpha)v + \alpha \Phi(v) \qquad \text{and} \qquad \Gamma_\alpha(c) := (1-\alpha)c + \alpha \Gamma(c)$$

be the $\alpha-$weighted virtual types.

ii. Letting $\rho > 0$ denote the solution value of the Lagrange multiplier associated with the revenue constraint, the problem is now

$$\max_{Q_B, Q_S} (1+\rho) \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi_{\frac{\rho}{1+\rho}}(v) Q_B(v,c) - \Gamma_{\frac{\rho}{1+\rho}}(c)(1 - Q_S(v,c))] f(v) g(c) dc dv$$

subject to:
A. $0 \leq Q_i(v,c)$,
B. $Q_B(v,c) + Q_S(v,c) \leq 1$ for all $(v,c) \in [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}]$,
C. $q_B(v)$ and $q_S(c)$ increasing for all $v \in [\underline{v}, \bar{v}]$ and $c \in [\underline{c}, \bar{c}]$.

iii. Letting $\alpha = \frac{\rho}{1+\rho}$, the pointwise maximisation problem is

$$\max_{Q_B, Q_S} \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi_\alpha(v) Q_B(v,c) - \Gamma_\alpha(c)(1 - Q_S(v,c))] f(v) g(c) dc dv$$

subject to the feasibility constraints.

Claim 1: The pointwise maximiser (Q_B, Q_S) is such that for all (v,c), $Q_B(v,c) = 1 - Q_S(v,c)$.
Proof: Suppose not, e.g., suppose $Q_B(v,c) < 1 - Q_S(v,c)$. Then the objective will increase by decreasing the RHS until $Q_B = 1 - \tilde{Q}_S$ since $\Gamma_\alpha(c) \geq c \geq 0$.

Claim 2: The pointwise maximiser of

$$\max_{Q \in [0,1]} \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi_\alpha(v) - \Gamma_\alpha(c)](Q(v,c) f(v) g(c) dc dv$$

is

$$Q^\alpha(v,c) = \begin{cases} 1 & \Phi_\alpha(v) > \Phi_\alpha(c) \\ 0 & \text{otherwise} \end{cases}$$

Proof: This just means that trade occurs if virtual value is greater than virtual cost, and trade does not occur otherwise.

Claim 3: $Q^\alpha(v,c)$ increases in v and decreases in c and $q_B^\alpha(v) = \mathbb{E}_c[Q^\alpha(v,c)]$ increases in v and $q_S^\alpha(c) = \mathbb{E}_v[1 - Q^\alpha(v,c)]$ increases in c.
Proof: By inspection of the pointwise maximiser.

iv. Letting α^* be the smallest value of $\alpha \in [0,1]$ such that

$$\max_{Q\in[0,1]} \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi(v) - \Gamma(c)](Q(v,c)f(v)g(c)dcdv = 0$$

then the second-best mechanism has the allocation rule $Q^{\alpha^*}(v,c)$.
The LHS is monotone and continuous in α and positive at $\alpha = 1$; hence α is well-defined.
Note: Φ, Γ are no longer functions of α.

v. Remarks on specific numeric examples:
   A. For F, G uniform on $[0,1]$, $\alpha^* = 1/3; Q^{\alpha^*}(v,c) = 1$ iff $v \geq \frac{1}{4} + c$.
   B. For $\alpha = 1$, $Q^1(v,c)$ is the allocation rule of the profit-maximising mechanism for a designer who maximises its profit. For F, G uniform on $[0,1]$, $Q^1(v,c) = 1$ iff $v \geq 1/2 + c$.

3. Incomplete information bargaining with different bargaining weights
   (a) Given Q, the buyer's ex ante expected surplus is

$$\begin{aligned} u_B(Q) &:= \mathbb{E}_v[vq_B(v)] - \mathbb{E}_v[\Phi(v)q_B(v)] + U_B(\underline{v}) \\ &= \mathbb{E}_v[\frac{1-F(v)}{f(v)}q_B(v)] + U_B(\underline{v}) \end{aligned}$$

   and the seller's ex ante expected surplus (gains from trade) is

$$\begin{aligned} u_S(Q) &:= \mathbb{E}^c[\Gamma(c)(1-q_S(c))] - \mathbb{E}_c[c(1-q_S(c))] + U_S(\bar{c}) \\ &= \mathbb{E}_c[\frac{G(c)}{g(c)}(1-q_S(c))] + U_S(\bar{c}) \end{aligned}$$

   (b) Williams (1987): For $w \in [0,1]$, suppose that the designer wants to maximise

$$wu_B(q) + (1-w)u_S(Q)$$

   subject to BIC, IR and no deficit.

   The assumptions are the same as Myerson and Satterthwaite (1983) in their analysis of second-best:

$$\begin{aligned} \max_{Q\in[0,1],U_B(\underline{v}),U_S(\bar{c})} & \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [w\frac{1-F(v)}{f(v)} + (1-w)\frac{G(c)}{g(c)}]Q(v,c)f(v)g(c)dcdv \\ & + wU_B(\underline{v}) + (1-w)U_S(\bar{c}) \\ \text{s.t.} & \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi(v) - \Gamma(c)](Q(v,c)f(v)g(c)dcdv - U_B(\underline{c}) - U_S(\bar{c}) \geq 0 \end{aligned}$$

   s.t. $q_B(v)$ and $q_S(c)$ increasing for all $v \in [\underline{v}, \bar{v}]$ and $c \in [\underline{c}, \bar{c}]$, and $U_B(\underline{v}) \geq 0$, and $U_S(\bar{c}) \geq \bar{c}$.

   Claim: $\rho \geq \max\{w, 1-w\}$, where ρ is the solution value of the Lagrange multiplier on the budget constraint.
   Proof: Suppose not, e.g., $\rho < w$. Then the value of the Lagrangian would be unbounded in $U_B(\underline{v})$, making $U_B(\underline{v}) = \infty$ optimal, which would violate the budget constraint.

   (c) The part of the Lagrangian that depends on the allocation rule is

$$\max_{Q\in[0,1]} \rho \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi_{\frac{\rho-w}{\rho}}(v) - \Gamma_{\frac{\rho-(1-w)}{\rho}}(c)]Q(v,c)f(v)g(c)dcdv$$

   s.t. $q_B(v)$ and $q_S(c)$ increasing for all $v \in [\underline{v}, \bar{v}]$ and $c \in [\underline{c}, \bar{c}]$.

(d) The pointwise maximiser

$$Q^{\rho,w}(v,c) \begin{cases} 1 & \Phi_{\frac{\rho-w}{\rho}}(v) > \Gamma_{\frac{\rho-(1-w)}{\rho}}(c) \\ 0 & \text{otherwise} \end{cases}$$

is such that $q_B^{\rho,w}(v) = \mathbb{E}_v[Q^{\rho,w}(v,c)]$ and $q_S^{\rho,w}(c) = \mathbb{E}_c[1 - Q^{\rho,w}(v,c)]$ are increasing.

(e) The optimal allocation rule is $Q^{\rho,w}(v,c)$, with ρ such that

$$\int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} [\Phi(v) - \Gamma(c)] Q^{\rho,w}(v,c) f(v) g(c) dc dv - \bar{c} = 0$$

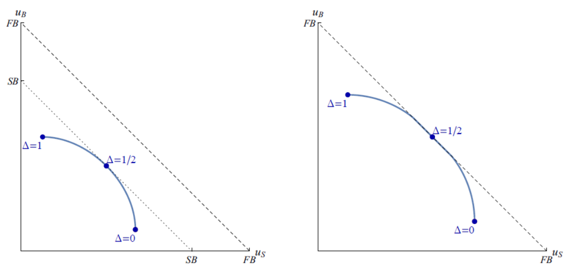
where we made the IR constraints bind (i.e., set $U_B(\underline{v}) = 0$ and $U_S(\bar{c}) = \bar{c}$ since $\rho \geq \max\{w, 1-w\}$). The LHS is monotone and continuous in ρ and positive as $\rho \to \infty$; hence the solution is well-defined.

For F, G uniform on $[0,1]$, $Q^{\rho,w}(v,c)$ is such that there is trade iff

$$v > \frac{2\rho - (1-w)}{2\rho - w} c + \frac{\rho - w}{2\rho - w}$$

(f) Countervailing power:
This setup has the feature that equalising bargaining power (weights) increase aggregate social surplus:

(a) Overlapping supports (b) Nonoverlapping supports



In the figures, $\Delta \equiv w$. Distributions are uniform.

Loertscher and Marx (2022) generalize incomplete information bargaining to multiple buyers and sellers (and multi-unit demands and suppliers).
The left-hand picture is the generalization of the preceding figure settings where agents own share r_1 and $r_2 = 1 - r_1$ of the good.
Larsen and Zhang (2021) provide methodology based on the revelation principle to estimate type distributions.
Larsen (2021) estimates whether bargaining outcomes are on the Pareto frontier (they are not).