MFE notes

Nicholas Chiang

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Math camp notes

1. **Basic notation:**
   $\to$ To
   $\implies$ Implies
   $\iff$ If and only if
   $\in$ In/ is a member of
   | Such that
   st such that, or subject to
   := Defined equal to
   $\mathbb{R}^+$ Non-negative reals
   $\mathbb{R}^++$ Strictly positive reals

   Set notation: Denote using { }, e.g., {p $\in \mathbb{R}, p > 0$}
   Intervals: Denote using [ ]
   Functions: e.g., $f : x \to y$ has domain x and codomain y. The inputs x $\in$ X are sometimes referred to as the argument of x.
   Inverse functions: $f^{-1}(f(x)) = x$ if $f : x \to y$ is invertible.

2. Tuples: Finite ordered list of n elements $(x_1, ...x_n)$ is a n-tuple.
   Cartesian product $A * B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
   e.g., Cartesian product of $\{3, 4\} * \{0, 1\}$ is $\{\{3, 0\}, \{3, 1\}, \{4, 0\}, \{4, 1\}$
   $\mathbb{R}^n$ is the Cartesian product of n copies of $\mathbb{R}$.

3. Linear combinations: e.g., $f(x, y) = ax + by,\ \forall a, b \in \mathbb{R}$ maps $\mathbb{R}^2 \to \mathbb{R}$.
   Convex combinations are linear combinations where all coefficients are non-negative and sum to 1.
   Convex sets are sets where all convex combinations of any two points in the set are also in the set itself.
   i.e., $\forall x, y \in D, \forall \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in D$.

4. Limits:
   e.g., $\lim_{x\to a} f(x) = L$ or $\lim_{x\to\infty} 1/x$ where evaluating function at ∞ is not possible.
   L'Hopital's rule: for functions f, g differentiable near $x = a$, and with $g'(x) \neq 0\ \forall x$ close to a,
   Then, if $\lim_{x\to a} \frac{f(x)}{g(x)}$ is either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided $\lim f'(x)$ and $\lim g'(x)$ exist.
   One-sided limits: Approaching a from specific directions, e.g.,

   $$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   The limit above 0 is 1, and the limit below 0 is 0, so the limit at $x = 0$ doesn't exist.
   Continuity: f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

5. **Convexity and concavity:**
   f is concave on $[x, y]$ if $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y), \forall \lambda \in [0, 1]$
   Sum of two concave functions is concave. Any local max of a concave function is also a global max.
   If the second derivative $f''(x) < 0$ on an interval, then f is concave on that interval.

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Sum of two convex functions is convex. Any local min of a concave function is also a global min.
If the second derivative $f''(x) > 0$ on an interval, then f is convex on that interval.
Concavity and convexity can also be defined using the same definitions for functions $f : \mathbb{R}^n \to \mathbb{R}$. However, the points x and y are now arbitrary points in $\mathbb{R}^2$, although f is still in $\mathbb{R}$.

6. **Open ball:** An open ϵ-ball with centre x and radius ϵ is the subset of points in $\mathbb{R}^n$ satisfying:

$$B_\epsilon(x) = \{y \in \mathbb{R}^n : d(x, y) < \epsilon\}$$

Closed ball: A closed ϵ-ball with centre x and radius ϵ is the subset of points in $\mathbb{R}^n$ satisfying:

$$B_\epsilon[x] = \{y \in \mathbb{R}^n : d(x, y) \leq \epsilon\}$$

A set $S \subset \mathbb{R}^n$ is called an **open set** if:

$$\forall x \in S, \exists \epsilon > 0 : B_\epsilon(x) \subset S$$

A set S is a **closed set** if its complement S^c is open.
A set S is bounded if:

$$\exists \epsilon > 0 : S \in B_\epsilon(x)$$

for some $x \in \mathbb{R}^n$ (i.e., we can draw an ϵ ball around S)
(Heine–Borel definition) A set $S \in \mathbb{R}^n$ is a compact set if it is closed and bounded.

7. Proofs:
A statement/ proposition is either true or false.
Basic notation for proofs:
$\wedge, \cap$ and
$\vee, \cup$ or (inclusive)
$\neg$ not
p is sufficient for q if $p \to q$
p is necessary for q if $q \to p$
If p is both necessary and sufficient for q, then $p \Leftrightarrow q$. Need to prove both directions if it is a proof.
$(A \cap B)^c = A^c \cup B^c$
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Axioms and definitions: We can assume that these are true for proofs.
Proof structure: Premise (i.e., the statement assumed true), steps, then the conclusion (the statement you wanted to prove).

8. Methods of proof:
Direct proof (generally, disprove a "for all" statement with a counterexample, and prove a "there exists" with an example)
Proof by contradiction: Assume the negation is true, then arrive at two statements r and $\neg r$ that cannot both be true, so the premise (i.e., negation of the statement) must be wrong.
Proof by cases: For statements that have a finite number of cases (e.g., positive and negative signs).
Proof by induction: if $p(n), n \in \mathbb{N}$ are statements such that $p(1)$ is true and $\forall k \in \mathbb{N}, p(k) \implies p(k+1)$ is true, then $\forall n \in \mathbb{N}, p(n)$ is true.

9. Uniqueness: Sometimes we need to show that there is only one solution. We show that a solution exists, and show that there is at most one solution (e.g., we can show that if there are two solutions, they must be identical).

10. Fixed points: Suppose that $f : \mathbb{R} \to \mathbb{R}^n$. A fixed point is an element $x \in \mathbb{R}$ such that $f(x) = x$.

11. A field is a set F with two operations called addition and multiplication, which satisfy the following field axioms:

(a) Axioms for addition:
   i. If $x \in F$ and $y \in F$, then their sum $x + y \in F$. (closedness for addition)
   ii. Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
   iii. Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$
   iv. F contains an element 0 such that $0 + x = x$ for every $x \in F$. (existence of identity elements)
   v. To every $x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$. (existence of inverse elements)

(b) Axioms for multiplication:
   i. If $x \in F$ and $y \in F$, then their product $x \cdot y \in F$. (closedness for multiplication)
   ii. Multiplication is commutative: $x \cdot y = y \cdot x$ for all $x, y \in F$.
   iii. Multiplication is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in F$
   iv. F contains an element $1 \neq 0$ such that $1 \cdot x = x$ for every $x \in F$. (existence of identity elements)
   v. If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that $x \cdot (1/x) = 1$. (existence of inverse elements)

(c) Distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ holds for all $x, y, z \in F$.

12. Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

    (a) Completeness: If $x \in S$ and $y \in S$ then one and only one of the statements $x < y, x = y, x > y$ is true.

    (b) Transitivity: If $x, y, z \in S$, then $x < y$ and $y < z$ implies $x < z$.

    An ordered set is a set S in which an order is defined.
    Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $\beta \geq x$ for every $x \in E$, then E is bounded above, and β is an upper bound of E.
    Suppose S is an ordered set, $E \subset S$, and E is bounded above. We call $\alpha \in S$ the least upper bound of E or supremum of E if α is an upper bound of E, and if $\gamma < \alpha$ then γ is not an upper bound of E. In other words, $\alpha = \sup(E)$ if $\forall \gamma < \alpha, \exists x \in E$ such that $x > \gamma$.

13. A set V is a metric space if for any two elements x and y of V there is an associated real number $d(x, y) \in \mathbb{R}$ such that:

    (a) Positivity: $d(x, y) \geq 0$, with equality iff $x = y$

    (b) Symmetry: $d(x, y) = d(y, x)$

    (c) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

    Any function with these three properties is called a distance function, or a metric.

    Let $V = \mathbb{R}^n$. Define the metric d_p on V by:

$$d_p(x, y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p} \text{ for } 1 \leq p \leq \infty$$

    d_p is the p-metric on $\mathbb{R}^n$.
    When $p = 1$, x and y are both scalars and $d_p(x, y) = |x - y|$.
    When $p = 2$, the p-metric d_p on $\mathbb{R}^n$ is called the Euclidean metric.
    Accordingly, $(\mathbb{R}^2, d_2)$ is called the n-dimensional Euclidean space. The Euclidean metric is related to the Euclidean norm $||.||$ through the identity $d(x, y) = ||x - y||$.

14. A sequence in a set S is the specification of a point $x_k \in S$ for each integer $k \in \{1, 2, ...\}$.
    Convergence: A sequence of points x_k in metric space (V, d) converges to a limit x, denoted $lim_{k \to \infty} x_k = x$, if the distance $d(x_k, x)$ tends to zero as k goes to infinity.
    i.e., if $\forall \epsilon > 0,\ \exists \delta(\epsilon) \in \mathbb{Z}$ such that $d(x, x_k) < \epsilon, \forall k > \delta(\epsilon)$

15. A sequence can have at most one limit. The proof is as follows:
    If $x_n \to x$, then $\forall \epsilon_x > 0, \exists N_{\epsilon_x}$ such that $|| \, x_n, x \, || < \epsilon_x, \forall n \geq N_{\epsilon_x}$.
    Suppose that $x_n \to y$. Then $\forall \epsilon_y > 0, \exists N_{\epsilon_y}$ such that $|| \, x_n, y \, || < \epsilon_y, \forall n \geq N_{\epsilon_y}$.
    We choose $\epsilon_x = \epsilon_y = \epsilon/2$. Then $|| \, x_n, x \, || < \epsilon/2$ and $|| \, x_n, y \, || < \epsilon/2, \forall n \geq N = max\{N_{\epsilon_x}, N_{\epsilon_y}\}$.

By triangle inequality, $|| x, y || \leq || x_n, x || + || x_n, y || < \epsilon$.
This implies $|| x, y || = 0$ and hence $x = y$ based on the definition of a metric. Hence the limit is unique if it exists.

16. A sequence x_k in $\mathbb{R}^n$ is bounded if there exists a real number M such that $||x_k|| \leq M, \ \forall k$.
Every convergent sequence in $\mathbb{R}^n$ is bounded. The proof is as follows:
If $\{x_n\}$ is convergent, then $\forall \epsilon > 0, \exists N$ such that $|| x_n, x || < \epsilon, \forall n \geq N$.
Let $\epsilon = 1$ and $n = N + 1$.
This means that $max\{|| x_0, ||, || x_1 ||, ... || x_N ||, 1+ || x_{N+1} ||\}$ bounds all the terms of the sequence and the sequence is bounded.

17. Let x_k be a sequence in $\mathbb{R}$ converging to a limit x.
Suppose that for every k, we have $a \leq x_k \leq b$. Then, we have $a \leq x \leq b$.
The proof is as follows:
If $\{x_n\}$ is convergent, then $\forall \epsilon > 0, \exists N$ such that $|| x_n, x || < \epsilon, \forall n \geq N$.
Suppose the statement is false, i.e., x lies outside $[a, b]$, e.g., without loss of generality, $x < a$.
We take ϵ such that $\epsilon < ||x, a||$.
This will mean that $\exists N$ such that $|| x_n, x || < \epsilon, \forall n \geq N$, and the points lie outside $[a, b]$. But this contradicts the fact that for every k, we have $a \leq x_k \leq b$. So, the statement must be true.

18. A sequence $\{x_k\}$ in a metric space V is called a Cauchy sequence if for any $\epsilon > 0$, there exists a $M \in \mathbb{R}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq M$.
If $\{x_k\}$ is convergent, it is Cauchy. The proof is as follows:
If $\{x_n\}$ is convergent, then $\forall \epsilon_x > 0, \exists N$ such that $|| x_n, x || < \epsilon, \forall n \geq N$.
We choose N' such that $|| x_n, x || < \epsilon/2, \forall n \geq N'$.
Then, by triangle inequality, $|| x_n, x_m || \leq || x_n, x || + || x_m, x || < \epsilon, \forall m, n \geq N'$.
Hence, $|| x_n, x_m || < \epsilon, \forall n, m \geq N$, and $\{x_n\}$ is a Cauchy sequence.

19. If $\{x_k\}$ is Cauchy, then it is bounded. The proof is as follows:
If $\{x_n\}$ is a Cauchy sequence, then $\forall \epsilon > 0, \exists N$ such that $|| x_m, x_n || < \epsilon, \forall n, m \geq N$.
By triangle inequality, $|| x_n || \leq || x_n, x_m || + || x_m ||$.
Let $\epsilon = 1$ and $m = N + 1$.
Combined with triangle inequality, this gives $|| x_n || \leq || x_n, x_m || + || x_m || < 1+ || x_{N+1} ||$. This holds $\forall n > N$, and means that all terms after x_N are bounded.
For all terms up to x_N, $|| x_n || \leq max\{|| x_0, ||, || x_1 ||, ... || x_N ||\}$. This means that all terms up to x_N are bounded.
Hence, $max\{|| x_0, ||, || x_1 ||, ... || x_N ||, 1+ || x_{N+1} ||\}$ bounds all the terms of the sequence and the sequence is bounded.
However, the sequence need not converge in V. Consider the example where $V = (0, 1]$ and the sequence is $\{1, 1/2, 1/3...\}$. The sequence is a Cauchy sequence but does not converge in the space V.

20. **A metric space is complete** if every Cauchy sequence in V converges to a point in M.
E.g., the sequence $\{1, 1/2, 1/3...\}$ makes $V = (0, 1]$ incomplete.

21. Continuity: Let $S \in \mathbb{R}^n$ and $T \in \mathbb{R}^l$. Then, $f : S \to T$ is continuous at $x \in S$ if for all sequences $x_k \in S$ such that $\lim_{k\to\infty} x_k = x$, we have $lim_{x\to\infty} f(x_k) = f(x)$.
A function $f : S \to T$ is continuous on S if it is continuous at all points in S.

22. Differentiability: Let $S \in \mathbb{R}^n$ and $T \in \mathbb{R}^l$.
Then, $f : S \to T$ is differentiable at a point $x \in S$ if there exists an $l \times n$ matrix A such that $\forall \epsilon > 0, \exists \delta > 0$ such that $t \in S$ and $||t - x|| < \delta$ implies $||f(t) - f(x) - A(t - x)|| < \epsilon ||t - x||$.
Equivalently, f is differentiable at $x \in S$ if

$$lim_{t\to x}\left(\frac{||f(t) - f(x) - A(t - x)||}{||t - x||}\right) = 0$$

The matrix A in this case is called the derivative of of f at x, and is denoted $Df(x)$. If f is differentiable at all points in S, then f is differentiable on S. If $Df : S \to \mathbb{R}^{n\times l}$ is a continuous function, then f is continuously differentiable on S, denoted as f is C^1.
Not all functions that are differentiable everywhere are continuously differentiable. Consider the following example for $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} 0 & x = 0 \\ x^2 sin(1/x^2) & x \neq 0 \end{cases}$$

The limit of $f'(x)$ as $x \to 0$ is not well defined, but $f'(0) = 0$ still exists. So the function is differentiable everywhere, but not continuously differentiable (as the derivative is not continuous).

23. Intermediate value theorem: Let $D = [a, b]$ be an interval in $\mathbb{R}$ and let $f : D\ to\mathbb{R}$ be a continuous function. If $f(a) < f(b)$, and if c is a real number such that $f(a) < c < f(b)$, then $\exists x \in (a, b)$ such that $f(x) = c$. A similar statement holds if $f(a) > f(b)$.

24. Intermediate value theorem for the derivative: Let $D = [a, b]$ be an interval in $\mathbb{R}$ and let $f : D\ to\mathbb{R}$ be a function that is differentiable everywhere on D. If $f'(a) < f'(b)$, and if c is a real number such that $f'(a) < c < f'(b)$, then $\exists x \in (a, b)$ such that $f'(x) = c$. A similar statement holds if $f'(a) > f'(b)$.

25. Intermediate value theorem in $\mathbb{R}^n$: Let $D \subset \mathbb{R}^n$ be a convex set, and let $f : D \to \mathbb{R}$ be continuous on D. Suppose that a and b are points in D such that $f(a) < f(b)$. Then, for any c such that $f(a) < c < f(b)$, there is λ such that $f((1 - \lambda)a + \lambda b) = c$

26. Weierstrass' Extreme Value Theorem: Let $D = \mathbb{R}^n$ be compact, and let $f : D \to \mathbb{R}$ be a continuous function on D. Then, f attains a maximum and minimum on D.
Proof: If $f : D \to \mathbb{R}$ is continuous on D, and D is compact, then $f(D)$ is also compact. Further, if $A \subset \mathbb{R}$ is compact, then $\sup A \in A$ and $\inf A \in A$, so the maximum and minimum of A are well-defined.

27. The **dot product** of two vectors $a = (a_1, a_2..., a_n)$ and $b = (b_1, b_2..., b_n)$ is

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + ... + a_n b_n$$

28. Exercise: Show that it is possible for two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ to be discontinuous, but for their product to be continuous. What about their composition?

Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = x, x \neq 0$. Both functions are discontinuous at $x = 0$, but their product $h(x) = 1$ which is continuous for all $x \in \mathbb{R}$.

The composition $f(g(x)) = \frac{1}{x}$ is still discontinuous at $x = 0$.

Lecture notes

1. A **metric space** (X, d) is a set X, together with a metric $d : X \times X \to \mathbb{R}_+$, such that for all $x, y, z \in X$:

   (a) $d(x, y) \geq 0$, with equality iff $x = y$ (nonnegativity);
   (b) $d(x, y) = d(y, x)$ (symmetry);
   (c) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

   Examples of metrics on sequences: Consider two elements of $\mathbb{R}_n$, x and y.

   (a) $d_2(x, y) := (\sum_{i=1}^{n} |x_i - y_i|^2)^{\frac{1}{2}}$
   (b) $d_\infty(x, y) := \sup\{|x_i - y_i\}$
   (c) $d_p((x_m, y_m)) := (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{\frac{1}{p}}$

   Examples of metrics on continuous functions $f : [0, 1] \to \mathbb{R}$:

   (a) $d^1(f, g) := \int_0^1 |f(t) - g(t)|dt$
   (b) $d^\infty = \sup_{t\in[0,1]} |f(t) - g(t)|$

   What if we want to have a metric that maps $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ to $[-1, 1]$?
   Consider the function

   $$\phi \begin{cases} \phi(x \in \mathbb{R}) = \frac{x}{1+|x|} \\ \phi(\infty) = 1 \\ \phi(-\infty) = -1 \end{cases}$$

   This allows us the distance $d(x, y) = |\phi(x) - \phi(y)|$ to satisfy all the necessary properties:

   (a) $d(x, y) = 0$ iff $x = y$
   (b) $d(x, y) = d(y, x)$
   (c) $|\frac{x}{1+|x|} - \frac{z}{1+|z|}| \leq |\frac{x}{1+|x|} - \frac{y}{1+|y|}| + |\frac{y}{1+|y|} - \frac{z}{1+|z|}|$

   Another metric on the space of functions: Consider the set X of functions $f : [a, b] \to \mathbb{R}$ that are r times continuously differentiable. For $i \leq r$, the C^i metric on X is defined as:

   $$d_i(f, g) = \sup_{x\in[a,b]} \{|f(x) - g(x)|, |f'(x) - g'(x)|, ..., |f^i(x) - g^i(x)|\}$$

   i.e., for f and g to be close, their values and also their derivatives up to order i must be close.

2. A vector space is a set of elements that is closed under addition and scalar multiplication and that has a "zero" element denoted by θ.

   A **normed vector space** $(V, ||\cdot||)$ is a vector space V, together with a nonnegative real-valued function $||\cdot|| : X \to \mathbb{R}_+$, such that for all $x, y \in V$ and $\alpha \in \mathbb{R}$:

   (a) $||x|| \geq 0$, with equality iff $x = \theta$;
   (b) $||\alpha x|| = |\alpha| \cdot ||x||$; and
   (c) $||x + y|| \leq ||x|| + ||y||$ (the triangle inequality).

   **Note the reverse triangle inequality:* $|\ ||x|| - ||y||\ | \leq ||x - y||$

   It is standard to view any normed vector space as a metric space by defining $d(x, y) = ||x - y||$. Note that the reverse is not true as we cannot always go from distance $\to$ norm. For example, the discrete metric:

   $$D(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

   is not a norm, as it does not satisfy $||\alpha X|| = |\alpha|||x||$.

   Examples of norms:

(a) Euclidean norm $||x|| := (x_1^2 + ... + x_n^2)^{\frac{1}{2}}$

(b) Sup norm on $R^n := \{|x_i| x = (x_1, x_2, ...x_n)\}$

(c) L^1 norm on the space of functions: $||f||| := \int_0^1 ||f(t)||dt$

(d) L^∞ norm on the space of functions: $||f|| := \sup_{t\in[0,1]} |f(t) - g(t)|$

3. **Cauchy-Schwarz Inequality**: For $x, y \in \mathbb{R}^n$, we have

$$|x \cdot y| \leq ||x|| \cdot ||y||$$

Another way to write it in the Euclidean norm: $\sqrt{\sum x_i y_i} \leq \sqrt{\sum x_i{}^2}\sqrt{\sum y_i{}^2}$

Proof of CS inequality: For $\lambda \in \mathbb{R}$, consider:

$$0 \leq \sum_{i=1}^{n}(x_i - \lambda y_i)^2 = \sum_{i=1}^{n} x_i^2 - 2\lambda \sum_{i=1}^{n} x_i y_i + \lambda^2 \sum_{i=1}^{n} y_i^2$$

This is a quadratic in λ, with the form $a - 2\lambda ab + b\lambda^2$. The discriminant ("$b^2 - 4ac$") is

$$4(\sum x_i y_i)^2 - 4\sum x_i^2 \sum y_i^2 \leq 0$$

Since the quadratic term is positive and the function is always positive, the function has at most one real root, i.e., the discriminant must be negative, and hence:

$$(\sum x_i y_i)^2 \leq \sum x_i^2 \sum y_i^2$$

Another proof of CS inequality by induction: We want to prove that

$$(x_1y_1 + x_2y_2 + ... + x_ny_n)^2 \leq (x_1^2 + ... + x_n^2)(y_1^2 + ... + y_n^2)$$

We first examine the $n = 1$ case, and can see that $(x_1y_1)^2 \leq (x_1^2)(y_1^2)$.
We assume that the inequality holds for arbitrary k and want to show that it then holds for $k + 1$. For the $k + 1$ case, we have:

$$\begin{aligned}
\sum_{i=1}^{k} x_i y_i + x_{k+1} &\leq \sqrt{\sum_{i=1}^{k} x_i^2}\sqrt{\sum_{i=1}^{k} y_i^2} + x_{k+1}y_{k+1} \\
&\leq \sqrt{\sum_{i=1}^{k} x_i^2 + x_{k+1}^2}\sqrt{\sum_{i=1}^{k} y_i^2 + y_{k+1}^2} && \text{by CS inequality} \\
&\leq \sqrt{\sum_{i=1}^{k+1} x_i^2}\sqrt{\sum_{i=1}^{k+1} y_i^2}
\end{aligned}$$

Cauchy-Schwarz-Bunyakovsky Inequality: Let $f, g : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be continuous functions. Then,

$$(\int_a^b f(x)g(x)dx)^2 \leq (\int_a^b (f(x))^2 dx)(\int_a^b (g(x))^2 dx)$$

Proof. The analogy with the Cauchy-Schwarz argument is as follows. A vector $x = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ is a finite list of real numbers, indexed by 1, 2, 3,..., n. A function f is a list of numbers, indexed by $\mathbb{R}$, where $f(x)$ is the number indexed by the real number x. The inner product of the two functions f and g is

$$f \cdot g = \int_a^b f(x)g(x)dx,$$

analogous to

$$x \cdot y = \sum_{i=1}^n x_i y_i.$$

In each case, we multiply the entries of the two vectors and then sum the products over the index set, with an integral replacing the summation as part of the move to a continuous index set.

The analogue of (1) is then

$$0 \leq \int_a^b (f(x) - \lambda g(x))^2 \, dx = \int_a^b (f(x))^2 \, dx - 2\lambda \int_a^b f(x)g(x)dx + \lambda^2 \int_a^b (g(x))^2 \, dx.$$

Looking at (2), we set

$$\lambda = \frac{\int_a^b f(x)g(x)dx}{\int_a^b (g(x))^2 \, dx}.$$

Substituting, we have

$$\begin{aligned} 0 &\leq \int_a^b (f(x) - \lambda g(x))^2 \, dx \\ &= \int_a^b (f(x))^2 \, dx - 2\frac{\left(\int_a^b f(x)g(x)dx\right)^2}{\int_a^b (g(x))^2 \, dx} + \frac{\left(\int_a^b f(x)g(x)dx\right)^2}{\int_a^b (g(x))^2 \, dx} \\ &= \int_a^b (f(x))^2 \, dx - \frac{\left(\int_a^b f(x)g(x)dx\right)^2}{\int_a^b (g(x))^2 \, dx} \end{aligned}$$

or

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b (f(x))^2 \, dx \cdot \int_a^b (g(x))^2 \, dx$$

4. A function f such that $|f(x) - f(y)| \leq C|x - y|$ for all x and y, where C is a constant independent of x and y, is called a **Lipschitz function** or Lipschitz continuous.
   Example of a Lipschitz function: $f(x) = x$.
   Example of a function that is not Lipschitz: $f(x) = x^2$.
   e.g., if we want to prove that $f(x) = \sqrt{x}$ is not Lipschitz continuous:
   (a) We pick $y = 0$ and $x \geq (\frac{1}{c})^2$, e.g., $x = 4(\frac{1}{c})^2$. We fix c.
   (b) We can then see that

   $$|f(x) - f(y)| = |4\sqrt{(\frac{1}{c})^2} - \sqrt{0}| = \frac{2}{c} > c|x - y| = c|4(\frac{1}{c})^2|$$

   Two norms $||\cdot||_1$ and $||\cdot||_2$ are Lipschitz-equivalent if there exist two numbers $m, M > 0$ such that

   $$m||x||_1 \leq ||x||_2 \leq M||x||_1$$

5. Let (X, d_1) and (Y, d_2) be metric spaces. We define the product metric space $(X \times Y, d_\pi)$ with the metric d_π defined by

   $$d_\pi[(x, y), (x', y')] = \sqrt{(d_1(x, x'))^2 + (d_2(y, y'))^2}$$

   Note: there are different ways we could have defined the distance but it must still satisfy the conditions for a metric.

6. **Open and Closed Balls:** Given a metric space (X, d),
   (a) The open ball centered at x with radius ϵ is the set $B_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}$
   (b) The closed ball centered at x with radius ϵ is the set $B_\epsilon[x] = \{y \in X | d(x, y) \leq \epsilon\}$
   (c) A set S is bounded if we can find a closed ball of finite radius that contains it.
   (d) A function $f : Y \to X$ is bounded if $f(Y)$ is a bounded set.

   Some examples of balls:
   (a) $B_1(0, 0)$ with the Euclidean distance is a circle of radius 1 around $(0, 0)$.
   (b) $B_1(0, 0)$ with sup distance ≤ 1 (i.e., maximum distance along 1 dimension ≤ 1) is a square of radius 1 around $(0, 0)$.

(c) Consider the distance:

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Using this distance:

i. $B_{0.5}(0,0)$ is just the origin since it contains no other points.
ii. $B_2(0,0)$ is the entire $\mathbb{R}^2$ since it includes both the origin and everything not in the origin.
iii. $B_1(0,0)$ would be just the origin if it is an open ball, and the entire $\mathbb{R}^2$ if it is a closed ball.

We can define the distance between a point x and some subset $A \subset X$ as $d(x,A) = \inf_{a \in A} d(x,A)$. We can see that $\{d(x,a)|a \in A\}$ is bounded below by 0, so this set has a greatest lower bound.

If A and B are two subsets of X, the distance between them is $d(A,B) = \inf\{d(a,b)|a \in A, b \in B\}$.

7. **Convergence of sequences in metric spaces**: Let (X,d) be a metric space and $\{x_n\}$ a sequence in X. $\{x_n\}$ converges to x if for any $\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n > N(\epsilon)$. A sequence that does not converge is said to diverge.

A sequence $\{x_n\}$ is bounded if there exists $M \in \mathbb{R}$ such that $d(x_n, x_m) < M$ for all $n \in \mathbb{N}, m \in N$. Other equivalent definitions are:

(a) A sequence $\{x_n\}$ is bounded if there exists $M \in R_+$ such that $d(x_n, x) < M$ for all $n \in \mathbb{N}$.
(b) A sequence $\{x_n\}$ is bounded if there exists $M \in R_+$ such that $d(x_n, 0) < M$ for all $n \in \mathbb{N}$.
(c) A sequence $\{x_n\}$ is bounded if there exists $N \in \mathbb{N}$ and $M \in R_+$ such that $d(x_n, x_m) < M$ for all $n > N$ and $m > N$.

Every convergent sequence in a metric space is bounded.
Proof: Suppose $\{x_n\} \to x$. There exists a number $N(1)$ such that $x_n \in B_1(x)$ for $n > N(1)$. Now, let

$$k = \sup\{\{d(x, x_n)|n \leq N(1)\} \cup \{1\}\}$$

It is clear that all of the terms in the sequence lie within $B_k(x)$. In other words, every term before term $N(1)$ is bounded by the sup ball; every term after is bounded by the ball of radius 1.

A point c is a **cluster point** of a sequence $\{x_n\}$ if any open ball centered at c contains an infinite number of terms in the sequence. Formally, for $\epsilon > 0$ and any $N \in \mathbb{N}$, there exists $n > N$ such that $x_n \in B_\epsilon(c)$.

Note: the limit of a sequence is a cluster point, but not necessary vice-versa. Consider the sequence $\{x_n\}$ which $= 0$ when n is odd and $= 1$ when n is even - there are two cluster points but no limit.

If c is a cluster point of a sequence $\{x_n\}$, then there exists some subsequence of $\{x_n\}$ that converges to c.
Proof: Choose $z_1 = x_{n_1}$ from the sequence so that $z_1 \in B_1(c)$. Next, choose $z_2 = x_{n_2}$ from the sequence such that $z_2 \in B_{1/2}(c)$ and $n_2 > n_1$. The k^{th} stage is defined as follows: choose $z_k = x_{n_k}$ from the sequence so that $z_k \in B_{1/k}(c)$ and $n_k > n_{k-1}$. This clearly defines a subsequence of $\{x_n\}$ that converges to c.

Consider the sequence $\frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$ in $\mathbb{Q}$, the set of rational numbers $\frac{p}{q}$, where the notation $\lfloor\ \rfloor$ means rounding to the closest integer.

(a) $n = 1, x_1 = 1$
(b) $n = 2, x_2 = 1.4$
(c) ...

The sequence does not converge to $\sqrt{2}$ in the space of $\mathbb{Q}$ as the limit $\sqrt{2}$ is not in the space.

Consider the discrete metric again. Under the discrete metric, does $\frac{1}{n}$ converge to 0 as $n \to \infty$? No, as the distance between $\frac{1}{n}$ and 0 is always 1 and cannot be contained in an ϵ ball around 0. Under this metric, only sequences where all the elements are identical from some point are convergent.

Now consider the distance $d^2(x,y) = |f(x) - f(y)|$ and define $f(0) = 1, f(1) = 0, f(x) = x$.
Under this metric, $\frac{1}{n}$ converges to 1 as for any $\epsilon > 0$, there exists N such that $D^2(\frac{1}{m}, 1) - |\frac{1}{m} - 0| < \epsilon$, for all $n > N$.

8. **Sequences in $\mathbb{R}$ and $\mathbb{R}^m$**
Note: Since the sequence consists of real numbers, we can apply the Axiom of Completeness/ the Supremum property, i.e., there is no gap between two sets on the real line and hence any set X of real numbers that has an upper bound has a $\sup X$.

Every increasing sequence $\{x_n\}$ that is bounded above converges to its supremum, and every decreasing sequence that is bounded below converges to its infimum.
Proof: Let $x = \sup\{x_n | n \in \mathbb{N}\}$ and consider any $\epsilon > 0$. If $x_{n^*} \in B_\epsilon(x)$ for some value of $n^* \in \mathbb{N}$, then the fact that the sequence is increasing and bounded above by x implies that $x_n \in B_\epsilon(x)$ for all $n \geq n^*$. We can therefore complete the proof by showing that some term in the sequence must be in $B_\epsilon(x)$. If this were not true, then $x - \frac{\epsilon}{2}$ is an upper bound of the sequence, contradicting the assumption that $x = \sup\{x_n | n \in \mathbb{N}|\}$.

A sequence is bounded if it is bounded above and below.

Every sequence of real numbers $\{x_n\}$ contains either an increasing subsequence or a decreasing subsequence, or both.
Note: we allow for the subsequence to be weakly increasing or decreasing.
Proof: See 2024 Assignment 1 Q7.

Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers contains at least one convergent subsequence.
Proof: The sequence as a monotonic subsequence that is also bounded. If it is increasing, then it converges to its supremum; if it is decreasing, then it converges to its infimum.

Let $\{x_n\} \to x$ and $\{y_n\} \to y$ be convergent sequences.

(a) If $x_n \leq y_n$ $\forall n \in \mathbb{N}$, then $x \leq y$.
(b) $\{x_n + y_n\} \to x + y$.
(c) $\{x_n y_n\} \to xy$.
(d) $\{\frac{x_n}{y_n}\} \to \frac{x}{y}$ if $y \neq 0$ and each $y_n \neq 0$.

Proof. We will just prove 3, as an exercise in proving a result about a limit. Consider

$$|x_n y_n - xy|,$$

which we need to show becomes small as n gets big: given any $\varepsilon > 0$, we need to find an $N(\varepsilon)$ such that $|x_n y_n - xy| < \varepsilon$ for all $n > N(\varepsilon)$. We have

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \\ &= |(x_n - x) y_n + x (y_n - y)| \\ &\leq |(x_n - x) y_n| + |x(y_n - y)| \\ &= |(x_n - x)| \cdot |y_n| + |x| \cdot |(y_n - y)| \end{aligned}$$

We'll now find $N(\varepsilon)$ to make each of these last two terms smaller than $\varepsilon/2$ for all $n > N(\varepsilon)$. Consider first the second term, $|x| \cdot |(y_n - y)|$. If $x = 0$, this term is zero and therefore less than $\varepsilon/2$. If $x \neq 0$, then $\{y_n\} \to y$ implies that for

$$\varepsilon' = \frac{\varepsilon}{2|x|}$$

there exists a $N(\varepsilon')$ such that

$$|(y_n - y)| < \frac{\varepsilon}{2|x|} \Leftrightarrow |x| \cdot |(y_n - y)| < \frac{\varepsilon}{2}$$

for all $n > N(\varepsilon')$.
Turning to $|(x_n - x)| \cdot |y_n|$, the sequence $\{y_n\}$ is bounded because it converges. There must exist a number $B > 0$ so that $|y_n| < B$ for all $n \in \mathbb{N}$. The fact that $\{x_n\} \to x$ implies that for

$$\varepsilon'' = \frac{\varepsilon}{2B}$$

there exists a $N(\varepsilon'')$ such that

$$|(x_n - x)| < \frac{\varepsilon}{2B} \Leftrightarrow |(x_n - x)| \cdot |y_n| \leq |(x_n - x)| \cdot B < \frac{\varepsilon}{2}$$

for all $n > N(\varepsilon'')$.
The proof is completed by defining $N(\varepsilon) = \max\{N(\varepsilon'), N(\varepsilon'')\}$. ■

A sequence $\{x_n\}$ in $\mathbb{R}^m$ converges to a vector $x = (x^1, ..., x^m)$ iff $\{x_n^i\} \to x^i$ for each $i = 1, 2, ..., m$.

Proof. $\Rightarrow$: Suppose that $\{x_n\} \to x$. We need to show that for each j and each $\varepsilon > 0$, there exists an $N(\varepsilon) \in \mathbf{N}$ such that

$$|x_n^j - x^j| < \varepsilon$$

for $n > N(\varepsilon)$. $\{x_n\} \to x$ implies that there exists an $N(\varepsilon) \in \mathbf{N}$ such that

$$\sqrt{\sum_{i=1}^m (x_n^i - x^i)} = \|x_n - x\| < \varepsilon$$

for $n > N(\varepsilon)$. We have

$$|x_n^j - x^j| = \sqrt{\left(x_n^j - x^j\right)^2} \leq \sqrt{\sum_{i=1}^m (x_n^i - x^i)^2} = \|x_n - x\| < \varepsilon,$$

which is the desired result.

$\Leftarrow$: Conversely, suppose that $\{x_n^i\} \to x^i$. We need to show that for each $\varepsilon > 0$, there exists an $N(\varepsilon) \in \mathbf{N}$ such that

$$\|x_n - x\| < \varepsilon$$

for $n > N(\varepsilon)$. For each i, $\{x_n^i\} \to x^i$ implies there exists an $N_i\left(\frac{\varepsilon}{\sqrt{m}}\right) \in \mathbf{N}$ such that

$$|x_n^i - x^i| < \frac{\varepsilon}{\sqrt{m}}$$

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for $n > N_i\left(\frac{\varepsilon}{\sqrt{m}}\right)$. Let

$$N(\varepsilon) = \max\left\{N_1\left(\frac{\varepsilon}{\sqrt{m}}\right), N_2\left(\sqrt{\frac{\varepsilon}{m}}\right), ..., N_m\left(\sqrt{\frac{\varepsilon}{m}}\right)\right\}.$$

For $n > N(\varepsilon)$, we have

$$|x_n^i - x^i| < \frac{\varepsilon}{\sqrt{m}} \Leftrightarrow (x_n^i - x^i)^2 < \frac{\varepsilon^2}{m}$$

for all i, and therefore

$$\sum_{i=1}^m (x_n^i - x^i)^2 < m \cdot \frac{\varepsilon^2}{m} = \varepsilon^2.$$

Thus,

$$\|x_n - x\| = \sqrt{\sum_{i=1}^m (x_n^i - x^i)^2} < \varepsilon^2,$$

which completes the proof. ■

$\{x_n\} \to \infty$ if for any $K \in \mathbb{R}$ there exists an integer $N(K)$ such that $x_n > K$ for $n > N(K)$. In other words, there is always a term larger than K for any arbitrarily large K.

9. Open and Closed Sets

Let (X, d) be a metric space. $A \subset X$ is an open set if for every $x \in A, \exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq A$. A set C is closed if its complement $\sim C$ in X is open.

Properties of open sets: Let (X, d) be a metric space. Then:

(a) $\emptyset$ and X are both open.

(b) The union of an arbitrary number of open sets is open.

(c) The intersection of a finite number of open sets is open.

Proof.

1. In the case of $\varnothing$, the definition holds vacuously. It clearly holds in the case of X.
2. Consider any $x \in \cup_{i\in I} A_i$, where each A_i is open. There must exist a value $j \in I$ such that $x \in A_j$. Because A_j is open, there exists an ε_j such that
$$x \in B_{\varepsilon_j}(x) \subset A_j \subset \cup_{i\in I} A_i,$$
which verifies that $\cup_{i\in I} A_i$ is open.
3. Consider any $x \in \cap_{i\in I} A_i$, where each A_i is open and the set I is finite. We have $x \in A_i$ for each $i \in I$. The openness of A_i implies that there exists $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(x) \subset A_i$. Define $\varepsilon = \min\{\varepsilon_i \,|\, i \in I\}$. ε exists and positive because I is finite. We have
$$B_\varepsilon(x) \subset B_{\varepsilon_i}(x) \subset A_i$$
for each i, and so $B_\varepsilon(x) \subset \cap_{i\in I} A_i$. The set $\cap_{i\in I} A_i$ is therefore open.

Notes:

(a) If $X = \mathbb{R}$, $[0, 1)$ is neither open nor closed because its complement is neither open nor closed.

(b) If $X = \mathbb{R}^{++}$, $(0, 1]$ is closed because $(1, \infty)$ is open.

(c) The intersection of an infinite number of open sets may not be open. Consider the example $\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}$ where the intersection is 0 (note: Not the empty set, but the point 0), which is a singleton and thus closed.

(d) If A is open in X, and $\tilde{X} \subset X$, A is open in $\tilde{X}$.
Proof: Since A is open in X, there exists $\epsilon > 0$ where $\forall x \in A, B_\epsilon(x) \subseteq A$. Since $\tilde{X} \subset X, B_\epsilon(x) \cap \tilde{X}$ is an open ball in $\tilde{X}$. Moreover, $B_\epsilon(x) \cap \tilde{X} \subseteq A \cap \tilde{X}$. Hence if A is open in X, and $\tilde{X} \subset X$, A is open in $\tilde{X}$.

(e) If A is open in $\tilde{X}$, and $\tilde{X} \subset X$, A is not necessarily open in X. Consider the above example where $[0, 1)$ is open in $\mathbb{R}^+$ but not open in $\mathbb{R}$.

(f) An ϵ-ball is an open set.

Properties of closed sets: Let (X, d) be a metric space. Then:

(a) $\emptyset$ and X are both closed.

(b) The intersection of an arbitrary number of closed sets is closed.

(c) The union of a finite number of closed sets is closed.

Notes:

(a) Singleton sets are closed (and their complement is open).

(b) The empty set is a subset of every set.

10. **Interior, Boundary and Closure of a Set**

Let (X, d) be a metric space and A a subset (not necessarily open). We define:

(a) An interior point of A ($intA$): $x \in intA \Leftrightarrow \exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq A$

(b) An exterior point of A ($extA$): $x \in extA \Leftrightarrow \exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq \sim A$

(c) A boundary point of A ($bdyA, \delta A$): $x \in bdyA \Leftrightarrow \forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$ and $B_\epsilon(x) \cap \sim A \neq \emptyset$

(d) A closure point of A ($clA, \bar{A}$): $x \in clA \Leftrightarrow \forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$.

Note that:

(a) $intA \subset A \subset clA$.

(b) $A \cup extA \cup bdyA$ is a partition of X.

(c) $clA = A \cup bdyA$.

(d) $extA = int \sim A$.

Other points:

(a) $intA$ is the largest open set contained in A. Proof: suppose $intA \subset A' \subset A$, where A' is an open set. For $x \in A'$, the fact that A' is open implies that there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset A$. By the definition of $intA$, this means $x \in intA$, and so $A = A'$, i.e., there cannot be a strictly larger open subset of A that contains $intA$. (Note that we still need to show that $intA$ is an open set, which can be done using the definition.

(b) A is open iff $A = intA$ (this follows from the previous point).

(c) clA is the smallest closed set that contains A (see 2023 Assignment 1, Q4 for proof).

(d) A is closed iff $A = clA$ (this follows from the previous point).

11. **Limit Points and Characterisation of Closed Sets in terms of Sequences**

Let $A \subset X$ and (X, d) be a metric space. A point x_L is a **limit point or cluster point** of A if every open ball centered at x_L contains at least one point of A other than x_L. The set of all limit points of A is its derived set, $D(A)$, where:

$$x_L \in D(A) \Leftrightarrow \forall \epsilon > 0, B_\epsilon(x_L) \cap (A \backslash \{x_L\}) \neq \emptyset$$

Recall that x is a closure point of A if

$$\forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$$

Every point in A is a closure point of A. An isolated point of A is a point $y \in A$ with the property that there exists a $B_\epsilon(y)$ such that $B_\epsilon(y) \cap (A \backslash \{y\}) = \emptyset$. The set of closure points of A is the disjoint union of its derived set $D(A)$ of limit points and the set of A's isolated points.

A point x_L is a limit point of A iff there exists a sequence in $A \backslash \{x_L\}$ that converges to x_L.

Proof. $\Rightarrow$: For each $n \in \mathbb{N}$, choose $x_n \in B_{1/n}(x_L) \cap (A \setminus \{x_L\})$, which is nonempty because x_L is a limit point of A.
$\Leftarrow$: We have a sequence $\{x_n\} \subset A \setminus \{x_L\}$ such that $\{x_n\} \to x_L$. For any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $x_n \in B_\varepsilon(x_L)$ for all $n > N(\varepsilon)$. It is therefore clear that $B_\varepsilon(x_L) \cap (A \setminus \{x_L\}) \neq \varnothing$, as required for x_L to be a limit point of A. ■

A is a closed set iff it contains all its limit points.

Proof.
$\Rightarrow$: Suppose x_L is a limit point of A and $x_L \notin A$. Because A is closed, $\sim A$ is open. There exists an open ball $B_\varepsilon(x_L)$ such that

$$B_\varepsilon(x_L) \subset \sim A \Longrightarrow B_\varepsilon(x_L) \cap A = \varnothing,$$

which contradicts x_L being a limit point of A.
$\Leftarrow$: Consider $x \in \sim A$, which by assumption is not a limit point of A. Because it is not a limit point of A, there must exists $B_\varepsilon(x)$ such that

$$B_\varepsilon(x) \cap (A \setminus \{x\}) = \varnothing.$$

Because $x \notin A$, $A \setminus \{x\} = A$ and so

$$B_\varepsilon(x) \cap A = \varnothing.$$

We have shown that an open ball exists around each x outside A that does not intersect A. The set $\sim A$ is therefore open, and so A is closed. ■

A is closed iff every convergent sequence in A has its limit in A.

Proof. $\Rightarrow$: Suppose $\{x_n\} \subset A$ and $\{x_n\} \to x \in \sim A$. Choose $B_\varepsilon(x) \subset \sim A$, which is possible because $\sim A$ is open. We have $\{x_n\} \cap B_\varepsilon(x) = \varnothing$, contradicting $\{x_n\} \to x \in \sim A$.
$\Leftarrow$: By the previous theorem, if A is not closed, then it must have a limit point $x_L \in \sim A$. By the theorem that precedes the previous theorem, there exists a sequence $\{x_n\} \subset A \setminus \{x_L\}$ such that $\{x_n\} \to x_L$. The hypothesis of this part of the proof, however, implies that $x_L \in A$, a contradiction. ■

Consider the function $f(x, y) = \frac{xy}{x^2+y^2}$.
The function $f : A \to Y$ is $\mathbb{R}^2 \to \mathbb{R}$ and is defined in $\{\mathbb{R}^2 \backslash (0, 0)\}$ (i.e., we don't allow both x and y to both $= 0$ simultaneously).
$(0, 0)$ is a limit point of A because a δ-ball around $(0, 0)$ has infinitely many (x, y) points.
We can think of a sequence $(\frac{1}{n}, \frac{1}{n}) \to (0, 0)$ where f converges to $\frac{1}{2}$ and a different sequence $(\frac{1}{n}, 0)$ where f converges to 0, hence the function has more than one limit limit point in Y.

12. **Limits of Functions**

Let (X, d) and (Y, ρ) be metric spaces, $A \subset X$, and x^0 be a limit point of A, and $f : A \to Y$.
f has a limit y^0 as x approaches x^0 if

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \text{ s.t. } x \in A \cap B_\delta(x^0), x \neq x^0 \implies f(x) \in B_\epsilon(y^0).$$

We can write this as $f(x) \to y^0$ as $x \to x^0$, or $lim_{x \to x^0} f(x) = y^0$.
We define this only for limit points x^0 of A such that we can always find points in A arbitrarily close to x^0. Note that f need not even be defined at x^0, but y^0 must be defined.

Let $f : X \to Y$ and x^0 be a limit point of X. f has limit y^0 as $x \to x^0$ iff for every $\{x_n\} \to x^0$ with $x_n \neq x^0$, the sequence $\{f(x_n)\} \to y^0$.
Proof:

(a) $\Rightarrow$: Consider $B_\epsilon(y^0)$. For this ϵ, there exists $\delta > 0$ such that $x \in B_\delta(x^0), x \neq x^0$ implies $f(x) \in B_\epsilon(y^0)$. The assumption that $\{x_n\} \to x^0$ means that there exists $N(\delta)$ such that $x_n \in B_\delta(x^0)$ for $n > N(\delta)$. Because $x_n \neq x$ by assumption we therefore have $f(x_n) \in B_\epsilon(y^0)$ for all $n > N(\delta)$. Setting $N(\epsilon) = N(\delta)$ completes the argument.

(b) $\Leftarrow$: Suppose $\lim_{x\to x_0} f(x) \neq y^0$ (or alternatively, that the limit does not exist). There must exist a $B_\epsilon(y^0)$ such that, for any $\delta \geq 0$, there exists $x \in B_\delta(x^0), x \neq x^0$ such that $f(x) \notin B_\epsilon(y^0)$. We consider $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$ and choose x_n with this property. In this way, we define a sequence $\{x_n\} \to x^0$ with the property that $\{f(x_n)\} \not\to y^0$, therefore contradicting the hypothesis of this part of the proof.

$\lim_{x\to\infty} f(x) = y^0$ if for every $\epsilon > 0$ there exists a $B > 0$ such that $f(x) \in B_\epsilon(y^0)$ for all $x > B$.
$\lim_{x\to-\infty}$ is defined similarly.

Let $f : X \to \mathbb{R}$, (X, d) be a metric space. $\lim_{x\to x^0} f(x) = \infty$ if for every $B > 0$ (note: B is a large positive number) there exists a $\delta > 0$ such that $f(x) > B$ for all $x \in B_\delta(x^0)\backslash\{x^0\}$. $\lim_{x\to x^0} f(x) = \infty$ is defined similarly.

If $\lim_{x\to x^0} f(x) = \infty, \lim_{x\to x^0} g(x) = -\infty$, we generally cannot say anything about $\lim_{x\to x^0} f(x) + g(x)$ as it depends on the relative rates which these two functions approach their infinite limits as $x \to x^0$.

13. Continuity in Metric Spaces

Let $(X, d), (Y, \rho)$ be metric spaces, and $f : X \to Y$. f is **continuous** at $x^0 \in X$ if $\lim_{x\to x^0} f(x) = f(x^0)$. Equivalently, for any $\epsilon > 0, \exists\delta > 0$ such that $f(B_\delta(x^0)) \subseteq B_\epsilon(f(x^0))$. A function f is discontinuous at x^0 if either $lim_{x\to x^0} f(x) \neq f(x^0)$ or $lim_{x\to x^0} f(x)$ fails to exist.

Consider the function:

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$$

The function is continuous at $x = 2$ but not continuous at $x = 0$ (if we use the usual distance metrics). However, if we use the discrete metric, all distances are 1 unless $x = x_0$, so the function is continuous, i.e., for a small enough ϵ, the ϵ-ball and the δ-ball each only have one element and $f(B_\delta(x_0)) = B_\epsilon(f(x_0))$.

Sequential characterisation of continuity: f is continuous at x^0 iff for every $\{x^n\} \to x^0$ we have $\{f(x^n)\} \to f(x^0)$.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational, } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x \in \mathbb{Q},\ x = \frac{p}{q}, \text{ with } p, q \text{ integers with no common factors other than one} \end{cases}$$

f is continuous at every irrational number and discontinuous at every rational number.
Proof: Consider first $x^0 \in \mathbb{Q}$ with $f(x^0) = \frac{1}{q}$. We choose $0 < \epsilon < \frac{1}{q}$. The ball $B_\epsilon f(x^0)) = B_\epsilon(\frac{1}{q})$ is an interval that does not contain 0. There are irrational numbers that are arbitrarily close to x^0; consequently no matter how small $\delta > 0$ is, the interval $B_\delta(x^0)$ contains numbers x at which $f(x) = 0 \notin B_\epsilon(\frac{1}{q})$. The function f is clearly not continuous at $x^0 \in \mathbb{Q}$.
Now consider $x^0 \neq \mathbb{Q}$ and any $\epsilon > 0$. We have $f(x^0) = 0$. We need to show that there exists $\delta > 0$ such that, if $|x - x^0| < \delta$, then

$$|f(x) - f(x^0)| = f(x) < \delta$$

Consider first the interval (x^0-1, x^0+1). There are an infinite number of rational numbers in this interval, but at most a finite number of rational numbers whose denominators (common factors removed) are less than or equal to $\frac{1}{\epsilon}$. Because this set is finite and x^0 is irrational and therefore outside it, there exists a $B_\delta(x^0) \subset (x^0 - 1, x^0 + 1)$ that does not intersect this set. Therefore, if $x \in B_\delta(x^0) \cap \mathbb{Q}$, it must be the case that the denominator q of x is greater than $\frac{1}{\epsilon}$, and so $f(x) = \frac{1}{q} < \epsilon$. We thus have

$$f(B_\delta(x^0)) \subset B_\epsilon(f(x^0) = (-\epsilon, \epsilon)$$

$f : X \to Y$ is continuous on X (i.e., at all points in X) iff $f^{-1}(C)$ is closed $\forall$ closed subsets $C \subset Y$. Equivalently, f is continuous iff $f^{-1}(O)$ is open $\forall$ open subsets $O \subset Y$.

(a) Note that this is a definition of continuity that doesn't require distances.

(b) The equivalence of the two statements follows because

$$\begin{aligned} f^{-1}(O) &= \{x \in X | f(x) \in O\} \\ &=\sim \{x \in X | f(x) \notin O\} \\ &=\sim \{x \in X | f(x) \in \sim O\} \end{aligned}$$

since a closed set is the complement of an open set.

(c) Proof of the theorem above:
$\Rightarrow$: Assume f is continuous and consider $x^0 \in f^{-1}(O)$ for an open set $O \subset Y$. For any $B_\epsilon(f(x^0)) \subset O$, continuity implies that there exists a $B_\delta(x^0)$ such that

$$f(B_\delta(x^0)) \subset B_\epsilon(f(x^0)) \subset O$$

It follows that $B_\delta(x^0) \subset f^{-1}(O)$ (since $f(B_\delta(x^0)) \subset O$) and hence we have verified that $f^{-1}(O)$ is open.
$\Leftarrow$: We prove continuity at x^0. Consider $B_\epsilon(f(x^0))$, which is an open set. By our hypothesis in this part of the proof, we know that $f^{-1}(B_\epsilon(f(x^0)))$ is an open set. Consequently, there exists $B_\delta(x^0) \subset f^{-1}(B_\epsilon(f(x^0)))$ and so $f(B_\delta(x^0)) \subset B_\epsilon(f(x^0))$, which verifies continuity at x^0.

Consider the function

$$f = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

then $f^{-1}(\{1\}) = \mathbb{R}_{++} \equiv (0, \infty)$.
Then using the definition immediately above, f is not continuous since f^{-1} for the closed set $[0, 2]$ is an open set $\mathbb{R}_{++}$.

Define the metric d' as $min(d, 1)$.
If a function is continuous in d', it is also continuous in d.
Intuitively, we only look at continuity when ϵ is small, and here the metric d' is $= d$ for small ϵ, and hence the function is continuous.

14. Uniform Continuity

$f : X \to Y$ is **uniformly continuous** on $A \subset X$, if for any $\epsilon > 0, \exists \delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$ for all $x \in A$ (this is the same as continuity but we look at all $x_0 \in A$; for each $\epsilon > 0$, the same $\delta > 0$ works at every $x_0 \in A$.

Uniform continuity always implies continuity.
In a closed interval, continuity implies uniform continuity. We can show this using convergent subsequences in a bounded space.

Is $f(x) = \frac{1}{x}$ uniformly continuous on $(0, 1)$?

(a) We first check if $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$.

If the function is continuous, the ϵ-ball around x_0 must satisfy

$$(\frac{1}{x_0 + \delta}, \frac{1}{x_0 - \delta}) \subseteq (\frac{1}{x_0} - \epsilon, \frac{1}{x_0} + \epsilon)$$

After some algebra, this simplifies to

$$\delta \leq \frac{\epsilon (x_0)^2}{1 + \epsilon x_0}$$

So the function is continuous on $(0, 1)$ because we can just pick a δ that satisfies this inequality.

(b) Now we check whether the function is uniformly continuous on $(0, 1)$.

We want to show that $\exists\epsilon, \forall\delta, \exists x_0$ such that $f(B_\epsilon(x_0)) \not\subset B_\epsilon(f(x_0))$.
We fix ϵ and δ, and are trying to find a x_0 that satisfies this.
In other words, we want to show that

$$\frac{1}{x_0+\delta} > \frac{1}{x_0} + \epsilon$$

After some algebra, we get

$$\frac{x_0 - (x_0-\delta)}{(x_0-\delta)x_0} > \frac{\delta}{(x_0)^2} > \epsilon$$

and hence we can show that the function is not uniformly continuous as we can just pick an x_0 for which

$$\sqrt{\frac{\delta}{\epsilon}} > x_0$$

15. Lipschitz Functions

A function $f : X \to Y$ is **Lipschitz** on $E \subset X$, if there exists a $K > 0$ such that $|f(x) - f(x^0)| \leq K \cdot |x - x^0|$ for all $x, x^0 \in E$.

A function that is Lipschitz on E is necessarily uniformly continuous on E, and hence also continuous.

16. Properties of Continuous Real Functions

If $f : [a, b] \to \mathbb{R}$ is continuous, then it is bounded above and below ($[a, b]$ is a closed set).
Proof: Suppose f is not bounded above on $[a, b]$. FOr every $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $f(x_n) > n$. Applying Bolzano-Weierstass, the bounded sequence $\{x_n\}$ has a convergent subsequence $\{z_n\}$ that converges to some x_0. We must have $x_0 \in [a, b]$ because this interval is closed.
Continuity of f implies $\lim_{m\to\infty} f(z_n) = f(x_0)$. But we also have $\lim_{m\to\infty} f(z_n) = \infty$, which is a contradiction.
We can use the same argument to show that f is bounded below.

Extreme Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $x_m, x_M \in [a, b]$ such that

$$f(x_m) \leq f(x) \leq f(x_M)$$

for all $x \in [a, b]$.
In other words, a continuous function on a closed interval achieves a maximum and a minimum somewhere on that interval.
Proof: We prove the existence of a maximum. We know from the previous theorem that $\mu = \sup_{x\in[a,b]} f(x)$ exits. We show that there exists $x_M \in [a, b]$ such that $f(x_M) = \mu$.
Applying the definition of supremum, for each $n \in \mathbb{N}$ we can find $x_n \in [a, b]$ such that $|f(x_n) - \mu| < \frac{1}{n}$.
The sequence $\{x_n\}$ is bounded and therefore has a subsequence $\{z_n\}$ that converges to some $x_m \in [a, b]$.
We complete the proof by showing that $f(x_M) = \mu$. Continuity of f implies that $\lim_{n\to\infty} f(z_n) = f(x_M)$ and by construction of the sequence $\{x_n\}$, $\lim_{n\to\infty} f(z_n) = \mu$. Because a sequence can have at most one limit, we must have $f(x_M) = \mu$.

Intermediate Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, then for every y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = y$.
Proof: Define $c = \sup\{x \in [a, b] | f(x) \leq y\}$. The set over which the sup is taken is nonempty (since one of a or b must be in it because y lies between $f(a)$ and $f(b)$ and it is bounded and so the sup exists. We have $f(x) > y$ for $x > c$.
It is clear that $c \in [a, b]$ because this interval is closed. We derive contradictions from assuming either $f(c) > y$ or $f(c) < y$, and therefore $f(c) = y$.
If $f(c) > y$, set $\epsilon = \frac{f(c)-y}{2}$ and apply the definition of continuity to ensure the existence of $B_\delta(c)$ such that

$$f(B_\delta(c)) \subset B_\epsilon(f(c))$$

Every number in $(c-\delta, c)$ is therefore not in $\{x \in [a,b] | f(x) \leq y\}$, which mean that the sup of this set is strictly less than c, a contradiction.
If $f(c) < y$, set $\epsilon = \frac{y-f(c)}{2}$ and apply continuity to ensure the existence of $B_\delta(c)$ such that

$$f(B_\delta(c)) \subset B_\epsilon(f(c))$$

Every number in $(c, c+\delta)$ is therefore in $\{x \in [a,b] | f(x) \leq y\}$, which again contradicts the definition of c as the sup of this set. Hence $f(c) = y$.

If $f : [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is a closed interval.
Proof: It follows from the last two results that

$$f([a,b]) = [f(x_m), f(x_M)]$$

(i.e., the interval is bounded by the maximum and the minimum).

17. Cauchy sequences

A sequence $\{x_n\}$ is **Cauchy** if

$$\forall \epsilon > 0, \exists N(\epsilon) \text{ s.t. } \forall m, n > N(\epsilon), d(x_m, x_n) < \epsilon$$

Note that a sequence can be Cauchy without having a limit, if the metric space X has "holes" in it (e.g., for $\mathbb{Q}$).

A metric space (X, d) is **complete** if every Cauchy sequence in X converges. A normed vector space that is complete is a **Banach space**.

Let $\{x_n\}$ be Cauchy in a metric space. If $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with limit x^0, then $\{x_n\} \to x^0$.
Proof: Consider any $\epsilon > 0$. We need to show that there exists $M \in \mathbb{N}$ such that $d(x_m, x^0) < \epsilon$ for all $m > M$.
$\{x_{n_k}\} \to x^0$ implies that there exists $N_1 \in \mathbb{N}$ such that $\forall n_k > N_1, d(x_{n_k}, x^0) < \frac{\epsilon}{2}$. The fact that $\{x_n\}$ is Cauchy implies that there exists N_2 such that $\forall m, n > N_2, d(x_m, x_n) < \frac{\epsilon}{2}$. Let

$$M = \max\{N_1, N_2\}$$

For $m > M$, we have by the triangle inequality,

$$\begin{aligned} d(x_m, x^0) &\leq d(x_m, x_{n_m}) + d(x_{n_m}, x^0) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Notice that $n_m \geq m > M$. The first inequality follows because $m, n_m > M < N_2$. The second inequality follows because $n_m > M > N_1$.

A subset $Y \subset X$ of a complete metric space X is complete iff it is closed.
Proof: $\Rightarrow$: Suppose Y is complete. It is sufficient to show that if $\{y_n\} \to x_0$ for some $\{y_n\} \in Y$, then $x^0 \in Y$ (i.e., Y contains all of its limit points and is therefore closed). The sequence $\{y_n\}$ converges and so it must be a Cauchy sequence. A Cauchy sequence in Y converges to a point in Y because Y is assumed to be complete. Therefore, $x^0 \in Y$.
$\Leftarrow$: Suppose $\{y_n\}$ is some Cauchy sequence in Y. Because X is complete, this sequence converges to some $x^0 \in X$. Because Y is closed, the limit of $\{y_n\}$ must be in Y. Existence of the limit is therefore given by the completeness of X, and the fact that it must be in Y follows because Y is closed.

$\mathbb{R}$ is complete with its usual metric (i.e., absolute difference).
Proof: Suppose $\{x_n\}$ is Cauchy. It must therefore be bounded, and so the Bolzano-Weierstrass Theorem implies that it has a convergent subsequence. The limit of this convergent subsequence must also be the limit of $\{x_n\}$.

Any finite-dimensional Euclidean space $\mathbb{R}^m$ is complete.
Proof: We first show that a sequence $\{x_n\}$ in $\mathbb{R}^m$ also defines n Cauchy sequences $\{x_n^j\}$ for each of its component vectors. We have

$$\forall \epsilon > 0, \exists N(\epsilon) \text{ s.t. } \forall m, n > N(\epsilon), d(x_m, x_n) < \epsilon$$

We have for each j,

$$|x_m^j - x_n^j| \leq \sqrt{\sum_{i=1}^{n}(x_m^i - x_n^i)^2} = d(x_m, x_n) < \epsilon$$

and so the $N(\epsilon)$ that establishes the Cauchy property for $\{x_n\}$ also establishes it for each sequence $\{x_n^j\}$ of real numbers. The completeness of $\mathbb{R}$ then implies $\{x_n^j\} \to x_0^j$ for some x_0^j. We have proven the equivalence of convergence in sequences in $\mathbb{R}^m$ to the convergence of the sequences of m components, and thus $\{x_m^j\} \to x_0) = (x_0^1, ..., x_0^m)$.

Given a set $X \subset \mathbb{R}^m$, the set $C(X)$ of bounded continuous functions $f : X \to \mathbb{R}$ equipped with the sup norm is a complete normed vector space.
Proof: Consider a Cauchy sequence $\{f_n\} \in C(X)$. We need to show that there exists a continuous and bounded function $f : X \to \mathbb{R}$ such that, for any $\epsilon > 0$, there exists $N(\epsilon)$ such that

$$\sup\{|f_n(x) - f(x)| \; |x \in X\} < \epsilon$$

for all $n > N(\epsilon)$
For $\{f_n\}$ to be Cauchy, it must be that for any $\epsilon > 0$, there exists $N_1(\epsilon)$ such that

$$\forall m, n > N_1(\epsilon), sup\{|f_m(x) - f_n(x)| \; |x \in X\} < \epsilon$$

(i.e., the difference between functions $< \epsilon$).
For each particular value of $x \in X$, the sequence $\{f_n(x)\} \subset \mathbb{R}$ is therefore Cauchy and thus converges to some number $f(x)$ by the completeness of $\mathbb{R}$.
We need to show that $\{f_n\}$ converges to f not just at each point but also in the sup norm, which is the norm given here for $C(X)$. We also need to show that this function f is bounded and continuous in order to show that the sequence $\{f_n(x)\}$ converges to an element of $C(X)$. We show each of these points in separate steps.

(a) First, we want to prove that $\{f_n\} \to f$ in the sup norm on $C(X)$. For every $\epsilon > 0$, we need to find $N(\epsilon)$ such that

$$\sup\{|f_n(x) - f(x)| \; |x \in X\} < \epsilon$$

holds for all $m > N(\epsilon)$. Because $\{f_n\}$ is Cauchy, there exists $N_3(\epsilon/2)$ such that

$$\forall m, n > N_3(\frac{\epsilon}{2}), \sup\{|f_n(x) - f(x)| \; |x \in X\} < \frac{\epsilon}{2}$$

Convergence of $\{f_n(x)\}$ to $f(x)$ at each value of x implies

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2}$$

for n sufficiently large. By triangle inequality, we have

$$|f_m(x) - f(x)| \leq |f_m(x) - f_n(x)| \text{ (since Cauchy) } + |f_n(x) - f(x)| \text{ (since pointwise convergence)}$$

For $m > N_3)\frac{\epsilon}{2}$, we can pick $n > N_3(\frac{\epsilon}{2})$ sufficiently large to conclude

$$|f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and thus $N(\epsilon) = N_3(\frac{\epsilon}{2})$ suffices to complete the argument.

(b) Now we show that $f(x)$ is a bounded function. First, there exists a $N(1) \in \mathbb{N}$ such that

$$\forall m, n > N(1), sup\{|f_m(x) - f_n(x)| \; |x \in X\} < 1$$

The function $f_{N(1)+1}(x)$ is a bounded function and so there exists $B \in \mathbb{R}$ such that

$$\sup_{x\in X} f_{N(1)+1}(x) \leq B$$

For all $n > N(1)$ and all $x \in X$, we have

$$\begin{aligned} |f_n(x)| &\leq |f_n(x) - f_{N(1)+1}(x)| + |f_{N(1)+1}(x)| \\ &\leq 1 + B \end{aligned}$$

It follows that

$$|f(x)| = \lim_{n\to\infty} |f_n(x)| 1 + B$$

and so $f(x)$ is a bounded function.

(c) Now we show that $f(x)$ is a continuous function. Given any $x_0 \in X$ and $\epsilon > 0$, we need to show that there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. We proved above that $\{f_n\} \to f$ in the sup norm on $C(X)$. Therefore, there exists $N_2(\frac{\epsilon}{3})$ such that

$$\sup\{|f_n(x) - f(x)| \; |x \in X\} < \frac{\epsilon}{3}$$

for all $n > N_2(\frac{\epsilon}{3})$. For any $n > N_2(\frac{\epsilon}{3})$ we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + |f_n(x) - f_n(x_0)| + \frac{\epsilon}{3} \end{aligned}$$

Now we use the continuity of each f_n to choose a $\delta > 0$ so that

$$|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$$

whenever $|x - x_0| < \delta$. We then have

$$|f(x) - f(x_0)| \leq \epsilon$$

whenever $|x - x_0| < \delta$, which proves the continuity of f.

Consider the sequence $x_i = \sum 1 + \frac{1}{2} + \frac{1}{3} + ...$, such that

$$|x_{i+1} - x_i| = \frac{1}{1+i}$$

For this sequence, we also have that

$$\forall \epsilon, \exists N \text{ s.t. } \frac{1}{i+1} < \epsilon, \forall i > N$$

The sequence is not Cauchy because even if N is large, as long as m and $n > N$ are far enough away from each other, their difference is still $\to \infty$.
The sequence $\{x_i\}$ also does not converge.

18. Complete Metric Spaces and Contraction Mapping Theorem

A function $T : X \to X$ from a metric space into itself is an **operator**.
Let $\beta \in (0, 1)$. An operator $T : X \to X$ is a **contraction of modulus** β if for any $x, y \in X$, we have $d(Tx, Ty) \leq \beta d(x, y)$.

Every contraction is continuous.
Proof: We have $T(B_\epsilon(x^0)) \subset B_\epsilon(Tx_0)$ for every $x^0 \in X$ and $\epsilon > 0$; given $\epsilon > 0$, one can simply use $\delta = \epsilon$ to prove continuity at any $x^0 \in X$.

Consider $f : [0, 1] \to [0, 1]$ with the property that $0 < f(x) \leq \beta < 1$ at all $x \in [0, 1]$. For $y \geq x$ we have

$$f(y) - f(x) = \int_x^y f'(t)dt \leq \beta(y - x)$$

and so f is a contraction of modulus β. A **fixed point** of f is a solution to the equation $f(x) = x$. It appears that any graph of f must cross the 45-degree line $y = x$ once and only once, i.e., any function f with the assumed properties has a unique fixed point.
Consider also the sequence starting from any x_0,

$$x_0, x_1 = f(x_0), x_2 = f(x_1), x_3 = f(x_2), ...$$

i.e., $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$. This iterative process can be depicted graphically using the 45-degree line $y = x$. It appears that the sequence constructed in this way converges to the unique fixed point of f.

Contraction Mapping Theorem: Let X be a complete metric space.

(a) A contraction $T : X \to X$ of modulus β has precisely one fixed point $x^* \in X$.

(b) For any $x_0 \in X$ the sequence $\{x_n\}$ defined with the formula $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ converges to x^*.

Proof: We first show that $\{x_n\}$ is Cauchy. Note first that

$$\begin{aligned} d(x_{n+1}), x_n) &= d(Tx_n, dTx_{n-1}) \\ &\leq \beta d(x_n, x_{n-1}) = \beta d(Tx_{n-1}, Tx_{n-2}) \\ &\leq \beta^2 d(x_{n-1}, x_{n-2}) = ... \\ &\leq \beta^n d(x_1, x_0) \end{aligned}$$

Consequently, for $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n) \\ &\leq \beta^{m-1} d(x_1, x_0) + \beta^{m-2} d(x_1, x_0) + ... + \beta^n d(x_1, x_0) \\ &= (\sum_{t=n}^{m-1} \beta^t) \cdot d(x_1, x_0) \\ &= \beta^n \cdot (\sum_{t=0}^{m-1-n} \beta^t) \cdot d(x_1, x_0) \end{aligned}$$

The series

$$\sum_{t=0}^{\infty} \beta^t$$

converges to $\frac{1}{1-\beta}$, since $\beta \in (0, 1)$. We therefore have

$$d(x_m, x_n) \leq \beta^n d(x_1, x_0) \frac{1}{1 - \beta}$$

To prove that $\{x_n\}$ is Cauchy, we select for any $\epsilon > 0$ a number K such that

$$\beta^K \frac{d(x_1, x_0)}{1 - \beta} \leq \epsilon$$

Because $\{x_n\}$ is Cauchy and X is complete, there exists an x^* such that $\{x_n\} \to x^*$. We now verify that x^* is a fixed point of T and there can be no other fixed point of T.
As a contraction, T is continuous. Therefore

$$Tx^* = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*$$

As to uniqueness, suppose T has two fixed points, x' and x''. We have

$$d(x', x'') = d(Tx', Tx'') \leq \beta d(x', x'')$$

The equality follows because x' and x'' are fixed points and the inequality follows because T is a contraction. Because $\beta \in (0,1)$, it follows that $d(x', x'') \leq \beta d(x', x'')$ is possible only if $d(x', x'') = 0$, i.e., $x' = x''$.

Consider a function $f \in C[0,1]$ with the sup norm. Let the transformation T be defined as $Tf = kf, k < 1$, e.g., if $f = x^2, Tf = kx^2$.
Let $d(f,g) = \delta$. We want to show that $\exists \beta < 1 : d(Tf, Tg) < \beta d(f,g)$.
We have that $\sup |kf, kg| = k \sup |f - g| = k\delta = \beta$, so T is a contraction mapping.

Now consider $f \in C[0,1]$ where $Tf(x) = \int_0^x f(t)dt$. e.g., if $f(x) = x^2, Tf = \int_0^x t^2 dt = \frac{x^3}{3}$.
Let $d(f,g) = \delta = \sup |f - g|$.
Then, $d(Tf, Tg) = |\int_0^x (f-g)(t)dt|_{sup} \leq |\int_0^x \delta dt|_{sup} = \sup_{x\in[0,1]} |\delta x| = \delta$.
So, T is not a contraction mapping since $\beta = 1$ in this case.

Now consider $TTf = T^2 f$ where $Tf(x) = \int_0^x f(t)dt$.
e.g., if $f(x) = x, Tf = \frac{x^2}{2}, T^2 f = \frac{x^3}{6}$.
We have

$$\begin{aligned}
d(T^2 f, T^2 g) &= || \int_0^x (Tf - Tf)(t)dt|| \\
&\leq \int_0^x (\int_0^t f(y)dy - \int_0^t g(y)dy)dt \\
&= \int_0^x \delta t dt \\
&= \frac{\delta x^2}{2} \leq \sup_{x\in[0,1]} |\delta \frac{x^2}{2}| = \frac{\delta}{2}
\end{aligned}$$

so T^2 is a contraction mapping (even though T is not a contraction mapping).

19. Continuity of Correspondences in $\mathbb{E}^n$

Recall that $\mathbb{E}^n$ denotes $\mathbb{R}^n$ equipped with the Euclidean metric.

A correspondence $\Psi : X \rightrightarrows Y$ is **closed-valued** at $x \in X$ if $\Psi(x)$ is a closed subset of Y, and Ψ is **compact-valued** at $x \in X$ if $\Psi(x)$ is a compact subset of Y.

$\Psi : X \rightrightarrows Y$, where X and Y are finite-dimensional Euclidean spaces.

(a) Ψ is **upper-hemicontinuous (uhc)** at $x_0 \in X$ if for every open set V containing $\Psi(x_0)$ there exists an open set U containing x_0 such that $\Psi(x) \subset V$ for every $x \in U$.
In other words, uhc guarantees that every point in $\Psi(x)$ is close to a point in $\Psi(x_0)$.

(b) Ψ is **lower-hemicontinuous (lhc)** at $x_0 \in X$ if for every open set V in Y such that $\Psi(x_0) \cap V \neq \emptyset$ there exists an open set U containing x_0 such that $\Psi(x) \cap V \neq \emptyset$ for every $x \in U$.
In other words, lhc means that every point in $\Psi(x_0)$ is close to a point in $\Psi(x)$.

(c) Ψ is **continuous** at $x_0 \in X$ if it is both uhc and lhc at x_0.
In other words, continuity at x_0 means that every point in $\Psi(x)$ is close to a point in $\Psi(x_0)$ and every point in $\Psi(x_0)$ is close to a point in $\Psi(x)$.

Suppose $\Psi : X \rightrightarrows Y$ is uhc at x_0, $\Psi(x_0)$ is a closed set, and $\{x_n\} \to x_0$. If $\{y_n\} \subset Y$ such that $y_n \in \Psi(x_n)$ and $\{y_n\} \to y_0$, then $y_0 \in \Psi(x_0)$ (i.e., the sequence converges to an element in the correspondence).
Proof: Suppose that $y_0 \neq \Psi(x_0)$. Because $\Psi(x_0)$ is closed, there exists $B_\epsilon(y_0)$ such that $B_\epsilon(y_0) \cap \Psi(x_0) = \emptyset$.
Consider

$$V = \bigcup_{y \in \Psi(x_0)} B_{\frac{\epsilon}{3}}(y)$$

The set V is an open subset of Y such that $\Psi(x_0) \subset v$ and $V \cap B_{\frac{\epsilon}{3}}(y_0) = \emptyset$. Applying uhc at x_0, there exists a $B_\delta(x_0)$ such that for $x \in B_\delta(x_0)$ we have $\Psi(x) \subset V$. The convergence $\{x_n\} \to x_0$ implies that

there is an $N(\delta)$ such that $x_n \in B_\delta(x_0)$ for $n > N(\delta)$.
As a consequence, $y_n \in \Psi(x_n) \subset V$ for $n > N(\delta)$, and so $y_n \notin B_{\epsilon/3}(y_0)$ for $n > N(\delta)$. This contradicts $\{y_n\} \to y_0$ and completes the proof.

Suppose $\Psi : X \rightrightarrows Y$ is lhc at x_0. For every $y_0 \subset \Psi(x_0)$, there exists a sequence $\{x_n\} \to x_0$ and $\{y_n\} \to y_0$ such that $y_n \in \Psi(x_n)$ for each $n \in \mathbb{N}$.
Proof:

Step 1. Consider $B_1(y_0)$. There exists $0 < \delta(1) < 1$ such that

$$x \in B_{\delta(1)}(x_0) \Rightarrow \Psi(x) \cap B_1(y_0) \neq \varnothing.$$

Choose $x_1 \in B_{\delta(1)}(x_0)$ and $y_1 \in \Psi(x_1)$.
Step 2. Consider $B_{1/2}(y_0)$. There exists $0 < \delta(2) < \frac{1}{2}$ such that

$$x \in B_{\delta(2)}(x_0) \Rightarrow \Psi(x) \cap B_{\frac{1}{2}}(y_0) \neq \varnothing.$$

Choose $x_2 \in B_{\delta(2)}(x_0)$ and $y_2 \in \Psi(x_2)$.

$\vdots$

Step n. Consider $B_{1/n}(y_0)$. There exists $0 < \delta(n) < \frac{1}{n}$ such that

$$x \in B_{\delta(n)}(x_0) \Rightarrow \Psi(x) \cap B_{\frac{1}{n}}(y_0) \neq \varnothing.$$

Choose $x_n \in B_{\delta(n)}(x_0)$ and $y_n \in \Psi(x_n)$.

$\vdots$

In this way, we construct sequences $\{x_n\}$ and $\{y_n\}$ such that

1. $y_n \in \Psi(x_n)$ for each $n \in \mathbb{N}$
2. $|x_n - x_0| < \delta(n) < \frac{1}{n}$
3. $|y_n - y_0| < \frac{1}{n}$.

It is clear that $\{x_n\} \to x_0$ and $\{y_n\} \to y_0$, which completes the proof. ■

Consider the correspondence:

$$\begin{cases} \Psi(x) = 2 & x < 1 \\ \Psi(1) = [2,3] \cup \{1\} \\ \Psi(x > 1) = [1,3] \end{cases}$$

We want to show that the sequence is not uhc at $x_0 = 1$.
Let $v = (1.9, 3.1) \cup (0.9, 1.1)$, and u be an arbitrary open interval that includes 1, i.e., $u = (1-\epsilon, 1+\epsilon)$.
For $1 + \frac{\epsilon}{2} \in u$, $\Psi(1 + \frac{\epsilon}{2}) = [1,3] \not\subseteq V$.
And hence the correspondence is not uhc at $x_0 = 1$.
Now we want to show that the correspondence is uhc at $x_0 = 2$.
Since $\Psi(2) = [1,3]$, we pick any arbitrary v that includes $[1,3]$, i.e., $[1,3] \subseteq V$.
We pick u to be the interval $u = [1.9, 2.1]$, then for any x in u, we have $\Psi(x) = [1,3] \subseteq V$.

de la Fuente questions

1. Problem 1.15: Show that the set of bounded real sequences is a metric space, with the norm defined by $d(x,y) = \sup_n |x_n - y_n|$

   To show that this is a metric space, we need to verify the following properties:

   (a) Non-negative function $d : X \times X \to \mathbb{R}_+$
   (b) $d(x,y) = 0$ iff $x = y$
   (c) $d(x,y) = d(y,x)$
   (d) $d(x,z) \leq d(x,y) + d(y,z)$

   Property (a) holds because of the absolute function in the distance.
   Property (b) holds because if $x = y$ for all elements of the sequence, $d(x,y) = 0$, and if at least one element is different then $d(x,y) \neq 0$.
   Property (c) holds because of the absolute sign, i.e., $sup_n|x_n - y_n| = sup_n|y_n - x_n|$.
   Property (d) holds because

   $$\begin{aligned} sup_n|x_n - z_n| &= sup_n|x_n - y_n + y_n - z_n| \\ &\leq sup_n|x_n - y_n| + sup_n|y_n - z_n| \end{aligned}$$

2. Problem 1.16: Let (X_2, d_2) be a metric space, X_1 a set, and $f : X_1 \to X_2$ a one-to-one function. Define a function $d_1()$ by

   $$d_1(x,y) = d_2[f(x), f(y)] \; \forall x, y \in X_1$$

   Show that (X_1, d_1) is a metric space.

   Refer to the same four properties in the preceding question.
   Property (a) holds because $d_2(x,y)$ is a metric and hence all values that d_2 can take are non-negative.
   Property (b) holds because d_2 is a metric and hence $d_1(x,y) = 0 \equiv d_2(f(x), f(y)) = 0$ which only happens iff $f(x) = f(y)$. Since f is one-to-one, this only happens iff $x = y$.
   Property (c) holds because d_2 is a metric and hence symmetric, and thus $d_1(x,y) = d_2(f(x), f(y)) = d_2(f(y), f(x)) = d_1(y,x)$.
   Property (d) holds because

   $$\begin{aligned} d_1(x,z) &= d_2(f(x), f(z)) \\ &\leq d_2(f(x), f(y)) + d_2(f(y), f(z)) \\ &= d_1(x,y) + d_1(y,z) \end{aligned}$$

3. Problem 1.17: Give an example of two sets A and B in a metric space such that $A \cap B = \emptyset$, but $d(A,B) = 0$.

   Consider the sets $A = \{0\}$ and $B = \{\frac{1}{n} | n \in \mathbb{N}\}$, and the distance $d(x,y) = |x - y|$. The distance between both sets is 0 but both sets do not intersect.

4. Problem 1.18: Prove that the set $C[a,b]$ of continuous real functions defined on the interval $[a,b]$ is a metric space when the distance between two functions f and g is defined by

   $$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

   We have to verify the properties of a metric. Use the same (a) to (d) as in the earlier questions.
   Property (a) holds because the sup function on an absolute function is nonnegative.
   Property (b) holds because the distance is 0 when the functions are identical for all $x \in [a,b]$, and if $f(x) \neq g(x)$ for $x \in [a,b]$, the distance is greater than zero.
   Property (c) holds because the absolute function is symmetric.

Property (d) holds because

$$\begin{aligned} d(f,g) &= \sup_{x\in[a,b]} |f(x)-h(x)| \\ &= \sup_{x\in[a,b]} |f(x)-g(x)+g(x)-h(x)| \\ &\le \sup_{x\in[a,b]} |f(x)-g(x)| + \sup_{x\in[a,b]} |g(x)-h(x)| \\ &= d(f,g)+d(g,h) \end{aligned}$$

5. Problem 1.19: Show that the following inequality holds for any $x \in \mathbb{R}^n$:

$$||x||_E \le \sum_{i=1}^{n} |x_i|$$

Assume that $n = 2$. Then we have

$$(|x_1|+|x_2|)^2 = |x_1|^2 + |x_2|^2 + 2|x_1||x_2| \ge |x_1|^2 + |x_2|^2 = x_1^2 + x_2^2$$

and taking the square root of this expression,

$$|x_1|+|x_2| \ge \sqrt{x_1^2+x_2^2} = \|x\|_E$$

as desired.

For $n > 2$, we proceed by induction. Let $x \in \mathbb{R}^{n+1}$, partition $x = (y, z)$ with $y \in \mathbb{R}^n$ and $z = x_{n+1} \in \mathbb{R}$, and assume that

$$\|y\|_E = \sqrt{\sum_{i=1}^{n} y_i^2} \le \sum_{i=1}^{n} |y_i| \qquad (1)$$

Then we have

$$\|x\|_E^2 = \sum_{i=1}^{n+1} x_i^2 = \sum_{i=1}^{n} y_i^2 + z^2 = \|y\|_E^2 + |z|^2 \le \|y\|_E^2 + |z|^2 + 2|z|\|y\|_E = (|z| + \|y\|_E)^2$$

Taking the square root of this expression, using (1), and recalling that $z = x_{n+1}$, we have

$$\|x\|_E \le \|y\|_E + |z| \le \sum_{i=1}^{n} |y_i| + |z| = \sum_{i=1}^{n} |x_i| + |x_{n+1}| = \sum_{i=1}^{n+1} |x_i|$$

which is the desired result. □

6. Problem 3.14: (Bernoulli Inequality) For each positive integer n and any $x \ge -1$, $(1+x)^n \ge 1+nx$. Prove this is true by induction. Where in the proof do you need $x \ge -1$?

We prove by induction. First for $n = 1$, we have

$$(1+x) \ge 1+n$$

Now that we have proved that this holds for $n = 1$, we assume that the case $n = k$ holds for arbitrary k, and show that if so, the inequality holds for the $k+1$ case.
For the $k+1$ case, we have:

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x) \\ &\ge (1+kx)(1+x) \\ &= (1+x+kx+kx^2) \\ &= (1+(k+1)x+kx^2) \end{aligned}$$

Since k is positive and x^2 is positive for $x \geq -1$, we have $kx^2 \geq 0$, which completes our conclusion that $(1+x)^{k+1} \geq 1 + (k+1)x$.

7. Problem 3.15: Let a be a real number, and consider the sequence $\{a^n\}$. Prove that as $n \to \infty$, we have:

   (a) If $|a| < 1, \{a^n\} \to 0$

   (b) If $a > 1, \{a^n\} \to \infty$

   (c) If $a \leq -1, \{a^n\}$ diverges

   Hint: Use Problem 3.14.

   For (a), we want to prove that for any $\epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $|a^n - 0| \leq \epsilon$ for all $n > N$.
   We choose N such that $|a^N| \leq \epsilon$. Then, $|a^n - 0| \leq \epsilon \; \forall n > N$ and hence we have shown that $\{a^n\} \to 0$.

   For (b), we want to prove that for any $K \in \mathbb{R}, \exists N(\epsilon) \in N$ such that $a^n > K$ for $n > N$. We choose N such that $N > log_a(K)$. Then $a^n > K$ for all $n > N$ and hence we have shown that $\{a_n\} \to \infty$.
   (Note: We could also have used the answer from 3.14 for this)

   For (c), the sign of $\{a^n\}$ alternates for odd and even n, and the sequence does not approach a single value, hence it diverges.

8. Problem 3.19: Given a sequence of real numbers $\{x_n\}$, the sequence $\{S_N\}$ defined by $S_N = \sum_{n=0}^{N} x_n$, is called the sequence of partial sums of the infinite series $\sum_{n=0}^{\infty} x_n$. If $\{S_N\}$ converges to some finite limit S, then we write $\sum_{n=0}^{\infty} x_n = S$.

   Consider the sequence $\{a^n; n = 0, 1, ...\}$ where $0 < a < 1$, and define S_N as before. Verify that $(1-a)S_N = 1 - a^{N+1}$. Use this to show that $\sum_{n=0}^{\infty} a^n = 1/(1-a)$.

   We prove using induction. The case where $n = 0$ is:

   $$(1-a)S_N = (1-a)(a^0) = (1-a) = 1 - a^1$$

   Hence the equality holds for $n = 0$. Now we assume that the equality holds for some $n = k$, and consider the $k+1$ case:

   $$\begin{aligned}(1-a)S_{k+1} &= (1-a)(S_k + a^{k+1}) \\ &= 1 - a^{k+1} + (1-a)a^{k+1} \\ &= 1 - a^{k+2}\end{aligned}$$

   And hence if the equality holds for the case $n = k$, it holds for $n = k+1$, and we have verified it by induction.

   Since we know that $(1-a)S_n = 1 - a^{n+1}$, and $0 < a < 1$, it follows that:

   $$S_n = \frac{1 - a^{n+1}}{1-a} \to \frac{1}{1-a} \text{ as } n \to \infty$$

9. Problem 3.20: Given the function

   $$f(x) = \frac{x^2 + 2}{2x}$$

   define a sequence $\{x_n\}$ of rational numbers by $x_1 = 1$ and $x_{n+1} = f(x_n)$ for all $n > 1$.

   (a) Prove that if $\{x_n\}$ converges, its limit is $\sqrt{2}$. Hint: Complete the following expression: $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = ...$
   Let the limit of the sequence be x. If the sequence converges, we have:

   $$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x)$$

Substituting x into the limits, we have

$$\begin{aligned} x &= \frac{x^2+2}{2x} \\ 2x^2 &= x^2+2 \\ x^2 &= 2 \\ x &= \sqrt{2} \end{aligned}$$

(b) Prove that for $n \geq 2$, we have $x_n \geq \sqrt{2}$. Hint: Show that $f(x) \geq \sqrt{2}$ using $a^2+b^2 \geq 2ab$.

We can use induction. The base case is $n = 2$, in which case $x_n = \frac{1^2+2}{2(1)} = \frac{3}{2} \geq \sqrt{2}$.

Now we assume that the inequality holds for some $k > 2$ and will verify whether it exists for $k+1$. The $k+1$ case is:

$$\begin{aligned} x_{k+1} = \frac{(x^k)^2+2}{2(x^k)} &\geq \frac{\sqrt{2}^2+2}{2\sqrt{2}} \\ &\geq \sqrt{2} \end{aligned}$$

Hence the $k+1$ case holds and we have proved the inequality by induction.

(note: we could also have used the hint and set $a = x$, $b = \sqrt{2}$).

(c) Calculate the value of $(x_{n+1} - x_n)$ as a function of x_n and x_{n+1}. Use the resulting expression to prove that for $n \geq 2$, $\{x_n\}$ is decreasing (by induction).

Using the definition of $f(x)$, we have

$$\begin{aligned} x_{n+1} - x_n &= \frac{(x_n)^2+2}{2(x^n)} - x_n \\ &= \frac{2-(x_n)^2}{2(x^n)} \end{aligned}$$

When $n = 2$, the difference is

$$\frac{2-(\frac{3}{2})^2}{2 \cdot \frac{3}{2}} < 0$$

We want to show that if for some $k > 2$, $(x_{k+1} - x_k) < 0$, then $(x_{k+2} - x_{k+1}) < 0$. The $k+1$ case is:

$$\begin{aligned} x_{k+2} - x_k &= \frac{(x_{k+1})^2+2}{2(x^{k+1})} - x_{k+1} \\ &= \frac{2-(x_{k+1})^2}{2(x^{k+1})} \end{aligned}$$

Since $x_{k+1} \geq \sqrt{2}$ from part (b), the expression is negative and hence the difference between terms is negative. Hence we have shown that $\{x_n\}$ is decreasing (by induction).

10. Problem 4.4: Prove that:

(a) $\emptyset$ and X are closed in X.

(b) The intersection of an arbitrary collection of closed sets is closed.

(c) The union of a finite family of closed sets is closed.

For (a), these hold because the complements of $\emptyset$ and x, i.e., X and $\emptyset$, are both open.

For (b), the complement of the intersection of an arbitrary collection of closed sets is the union of an arbitrary collection of open sets, which is open, hence the intersection of an arbitrary collection of closed sets is closed.

For (c), the complement of the union of a finite family of closed sets is the intersection of a finite family of open sets, which is open, hence the union of a finite family of closed sets is closed.

11. Problem 4.7: Prove that $bdyA = clA \cap cl(\sim A)$.
    This is straightforward from the definition of boundary and closure. Observe that:
    (a) $x \in bdyA \Leftrightarrow \forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$ and $B_\epsilon(x) \cap \ A \neq \emptyset$
    (b) $x \in clA \Leftrightarrow \forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$
    (c) $x \in cl \sim A \Leftrightarrow \forall \epsilon > 0, B_\epsilon(x) \cap \sim A \neq \emptyset$

    So $x \in clA \cap cl(\sim A) \Leftrightarrow \forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$ and $B_\epsilon(x) \cap \sim A \neq \emptyset$, which is the definition of $bdyA$.

12. Problem 4.14: Show that in a metric space the closed ball $B_r[x]$ is a closed set. Hint: Take a limit point a of $B_r[x]$ and consider an arbitrary sequence $\{x_n\}$ in $B_r[x]$ with limit a. Use the triangle inequality to show that a must be in $B_r[x]$.

    Let a be a limit point of $B_r[x]$ and consider an arbitrary sequence $\{x_n\}$ in $B_r[x]$ with limit a. Since a is a limit point, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $d(x_n, a) < \epsilon,\ \forall n > N$.

    Then, by the triangle inequality, $\forall n \geq N$ we have
    $$d(x_n, x) \leq d(x_n, a) + d(a, x) \leq \epsilon + d(a, x)$$
    At the same time, since $x_n \in B_r[x], d(x_n, x) < r$, which implies
    $$d(a, x) \leq d(a, x_n) + d(x_n, x) < r + \epsilon$$
    Since we can choose N such that ϵ is arbitrarily small, this implies $d(a, x) \leq r$, so $a \in B_r[x]$.

    Therefore, every limit point of $B_r[x]$ is also in $B_r[x]$ which means $B_r[x]$ contains all its limit points and hence is closed.

13. Problem 4.15: Let B be a nonempty set of real numbers bounded above. Let $s = \sup B$. Show that $s \in \bar{B}$ (i.e., the closure of B). Notice that this implies $s \in B$ if B is closed.

    We want to show that $\sup B \in clB$. Since $\sup B \in clB \Leftrightarrow \forall \epsilon > 0, B_\epsilon(\sup B) \cap B \neq \emptyset$, we need to show that every ϵ ball around $\sup B$ contains at least one point of B.

    Let $\epsilon > 0$ be given. Since $\sup B$ is the least upper bound of B, by the definition of the supremum $\exists b \in B$ such that $\sup B - \epsilon < b < \sup B$, and hence $b \in B$ is a point within the ϵ ball around $\sup B$.

14. Problem 4.16: Let A be a set in metric space (X, d). Show that if A is closed and $x \notin A$, then $d(x, A) > 0$. Hint: Prove the contrapositive.

    We prove the contrapositive, i.e., if $d(x, A) = 0$, then either $x \in A$ or A is not closed.

    Let A be a closed set in (X, d). If $d(x, A) = 0$, then by definition of the distance from a point to a set, either $x \in A$ or x is a limit point of A.

    If $x \in A$, then we have shown that if $x \notin A$, then $d(x, A) > 0$.

    If x is a limit point of A, then there exists a sequence $\{a_n\}$ in A such that $\lim_{n\to\infty} a_n = x$. If A is closed, it contains all its limit points and hence $x \in A$. The only case where $x \neq A$ is if A is not closed.

    Hence we have shown the contrapositive, i.e., if $d(x, A) = 0$, either $x \in A$ or A is not closed.

15. Problem 6.2: Let f be a continuous function from a metric space (X, d) to $\mathbb{R}$, with the usual metric. Prove that the set $\{x \in X, f(x) > 0\}$ is open.

    Let the set $\{x \in X, f(x) > 0\}$ be A.

    Let x be such that $f(x) > 0$. By the continuity of f, $\forall \epsilon > 0, \exists \delta > 0$ such that

    $$|f(y) - f(x)| < \epsilon, \forall y \in B_\delta(x)$$

    We choose $\epsilon = f(x)/2$, and continuity ensures that we can find some $\delta > 0$ such that $\forall y \in B_\delta(x)$

    $$f(x) - f(y) < f(x)/2 \implies f(y) > f(x)/2 > 0$$

    and hence $y \in A$. This shows that $B_\delta(x) \subseteq A$ for all $x \in A$ and hence $\{x \in X, f(x) > 0\}$ is open.

16. Problem 6.5: Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 1$ for x rational and $f(x) = 0$ for x irrational. Show that f is discontinuous everywhere.

    Sketch of proof: Show that a δ-ball around any point in $\mathbb{R}$ contains both irrational and rational numbers and hence we cannot pick a δ such that $f(B_\delta(x^0)) \subseteq B_\epsilon(f(x^0))$.

17. Problem 6.6: Given a function $f : \mathbb{R} \to \mathbb{R}$, define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(x) = (x, f(x))$. Use the sequential characterisation of continuity to show that if f is continuous at some point x^0, then so is g.

    (refer to 2023 Assignment 1 Q5)

18. Problem 6.8: Show that in any normed vector space $(X, ||\cdot||)$ the norm is a continuous function from X to $\mathbb{R}$.

    Let $x \in X$ be any arbitrary point in the normed vector space X. We want to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $y \in X$, if $||y - x|| < \delta$, then $||y|| - ||x|| < \epsilon$.

    Since $||\cdot||$ is a norm, it satisfies the reverse triangle inequality: $||y|| - ||x|| \leq ||y - x||$.

    We choose $\delta = \epsilon$. Then, if $||y - x|| < \delta$, we have $||y|| - ||x|| \leq ||y - x|| \leq \delta = \epsilon$.

    This shows that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $||y - x|| < \delta$, then $||y|| - ||x|| < \epsilon$. Therefore, the norm function $||\cdot|| : X \to \mathbb{R}$ is continuous.

19. Problem 7.17: Let (X, d) be a complete metric space and $T : X \to X$ a function whose n^{th} iteration T^n is a contraction. Show that T has a unique fixed point.

    If T^n is a contraction, there exists some $\beta \in (0, 1)$ such that

    $$d(T^n x, T^n y) \leq \beta d(x, y), \forall x, y, \in X$$

    and by the CMT, T^n has one fixed point $x^* \in X$, for which $T^n x^* = x^*$. Substituting this back into the previous equation, we have

    $$d(Tx^*, x^*) = d(T(T^n x^*), T^n x^*) = d(T^n(Tx^*), T^n x^*) \leq \beta d(Tx^*, x^*) = 0$$

    and therefore $Tx^* = x^*$ and x^* is a fixed point of T.

20. Problem 11.5: Show that a closed correspondence is closed-valued.

    Consider a sequence of points $\{y_n\}$ in $\Phi(x)$ converging to some point in Y. We want to show that y lies in $\Phi(x)$.

    Take a sequence $\{x_n\}$ where $x_n = x$ for all n, and notice that $\{y_n\}$ is a companion sequence of $\{x_n\}$. Because $\{x_n\}$ converges to x,and because the correspondence is closed and $\{y_n\} \to y$, we have also that $y \in \Phi(x)$.

21. Problem 1.3: Prove that any intersection of convex sets is convex.

    Let $\{X_i\}$ be a collection of convex sets, and consider two points in their intersection: $x', x'' \in \bigcap_i X_i$. Since all the sets are convex, we have that for any $\lambda \in [0, 1]$,

$$x^\lambda = \lambda x' + (1 - \lambda)x'' \in \bigcap_i X_i$$

22. Problem 1.17: Let X be a convex set with a nonempty interior. Show that $int(clX) = intX$.

    If X is convex, both its closure and interior are convex sets. We pick an arbitrary point $x \in int(clX)$. Since x is not a boundary point of $cl(X)$, it is also not a boundary point of X, and hence it must be an interior point of X, hence $int(clX) = intX$.

23. Problem 2.6: Prove that the function $f : \mathbb{R}^n \supseteq X \to \mathbb{R}$, where X is a convex set, is concave iff the function $\phi(\lambda) = f[(1 - \lambda)x' + \lambda x'']$ is concave for any two points x' and x'' in the domain of f.

- First assume that ϕ is concave, and fix two arbitrary points x' and x'' in X. Then for each $\lambda \in [0,1]$, we have

$$\phi(\lambda) = \phi[\lambda 1 + (1-\lambda)0] \geq \lambda\phi(1) + (1-\lambda)\phi(0)$$

but then

$$\phi(\lambda) = f[\lambda x' + (1-\lambda)x''] \geq \lambda f(x') + (1-\lambda)f(x'') = \lambda\phi(1) + (1-\lambda)\phi(0)$$

for all $\lambda \in [0,1]$. That is, $f(\ )$ is concave.

- Assume that f is concave, and fix two arbitrary points x' and x'' in X. We have to show that for any $\mu_1, \mu_2 \in \mathbb{R}$ and all $\lambda \in [0,1]$,

$$\phi(\lambda\mu_1 + (1-\lambda)\mu_2) \geq \lambda\phi(\mu_1) + (1-\lambda)\phi(\mu_2) \tag{1}$$

To verify that this expression holds by the concavity of f, put

$$\begin{aligned} y^1 &= \mu_1 x' + (1-\mu_1)x'' \\ y^2 &= \mu_2 x' + (1-\mu_2)x'' \\ t &= \lambda\mu_1 + (1-\lambda)\mu_2 \end{aligned}$$

Then

$$\begin{aligned} \phi(\mu_1) &= f[\mu_1 x' + (1-\mu_1)x''] = f(y^1) \\ \phi(\mu_2) &= f[\mu_2 x' + (1-\mu_2)x''] = f(y^2) \\ \phi[\lambda\mu_1 + (1-\lambda)\mu_2] &= \phi(t) = f[tx' + (1-t)x''] \end{aligned}$$

but notice that

$$\begin{aligned} tx' + (1-t)x'' &= [\lambda\mu_1 + (1-\lambda)\mu_2]x' + [1 - \lambda\mu_1 - (1-\lambda)\mu_2]x'' \\ &= \lambda\mu_1 x' + (1-\lambda)\mu_2 x' + (1-\lambda+\lambda)x'' - \lambda\mu_1 x'' - (1-\lambda)\mu_2 x'' \\ &= \lambda[\mu_1 x' + x'' - \mu_1 x''] + (1-\lambda)[\mu_2 x' + x'' - \mu_2 x''] \\ &= \lambda[\mu_1 x' + (1-\mu_1)x''] + (1-\lambda)[\mu_2 x' + (1-\mu_2)x''] \\ &= \lambda y^1 + (1-\lambda)y^2 \end{aligned}$$

Hence, (1) becomes

$$f[\lambda y^1 + (1-\lambda)y^2] \geq \lambda f(y^1) + (1-\lambda)f(y^2)$$

which holds by the concavity of f. □

24. Problem 2.9: Let $f : \mathbb{R}^n \supseteq X \to \mathbb{R}$ be a concave function and $g : \mathbb{R} \to \mathbb{R}$ an increasing and concave function defined on an interval I containing $f(X)$. Prove that the function $g[f(X)]$ is concave.

   (refer to 2023 Assignment 2 Q11)

25. Problem 2.11: Let f and g be concave functions $\mathbb{R}^n \supseteq X \to \mathbb{R}$. Given arbitrary scalars α and β, prove that the function $h = \alpha f + \beta g$ is concave.

   If f is concave, it means that for all $x', x'' \in X$ we have

   $$tf(x') + (1-t)(f(x'') \leq f(tx' + (1-t)x'')\ \forall t \in [0,1]$$

   The same is true for g.

   This means that for $h = \alpha f + \beta g$ we have

   $$\begin{aligned} t(h(x')) + (1-t)(h(x'')) &= t(\alpha f(x') + \beta g(x')) + (1-t)(\alpha f(x'') + \beta g(x'')) \\ &\leq \alpha(f(tx' + (1-t)x'')) + \beta(g(tx' + (1-t)x'') \\ &= (\alpha f + \beta g)tx' + (\alpha f + \beta g)(1-t)x'' \end{aligned}$$

   for all $t \in [0,1]$ and any $x', x'' \in X$, and hence h is concave.

26. Problem 2.13: Let $\{f^s, s \in S\}$ be a (possibly infinite) family of concave functions $\mathbb{R}^n \supseteq X \to \mathbb{R}$, all of which are bounded below. Prove that the function f defined on X by

$$f(X) = \inf\{f^s(X); s \in S\}$$

is concave.

Sketch of proof:
Each f^i is concave, and hence its hypograph is convex.
Since $f(x) = \inf$ of all the f^i, the hypograph of f is the intersection of all the hypographs.
This intersection is also convex since any intersection of convex sets is convex. So $f(X)$ is concave since its hypograph is convex.

Efe Ok questions

1. Let $(x_m), (y_m), (z_m)$ be real sequences such that $x_m \leq y_m \leq z_m$ for each m. Show that if $\lim x_m = \lim z_m = a$, then $y_m \to a$.

   Given that the limits of (x_m) and (z_m) are a, by the definition of a limit, $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that x_m and $z_m \in B_\epsilon(a)$ $\forall n > N(\epsilon)$. Since $x_m \leq y_m \leq z_m$ for each m, it must also be that $x_m \leq a + \epsilon$ and $z_m \geq a - \epsilon$ and hence $a - \epsilon < y_m < a + \epsilon$ for all $m > N(\epsilon)$.

   Thus, $y_m \in B_\epsilon(a)$ $\forall n > N(\epsilon)$ and hence $y_m \to a$.

2. Show that every unbounded real sequence has a subsequence that diverges to either ∞ or $-\infty$.

   Assume $\{x_n\}$ is unbounded above. Then $\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}$ such that $x_n > k$. This defines a subsequence x_{n_k} that diverges to ∞.

   Assume $\{x_n\}$ is unbounded below. Then $\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}$ such that $x_n < k$. This defines a subsequence x_{n_k} that diverges to $-\infty$.

3. Let S be a nonempty bounded subset of $\mathbb{R}$. Show that there is an increasing sequence $(x_m) \in S^\infty$ such that $x_m \nearrow \sup S$, and a decreasing sequence $(y_m) \in S^\infty$ such that $y_m \searrow \inf S$.

   For the increasing sequence, we start with $x_0 = \inf S$, and select terms such that $x_m = \inf\{s \in S : s > x_{m-1}\}$ thereafter.

   For the decreasing sequence, we start with $y_0 = \sup S$, and select terms such that $y_m = \sup\{s \in S : s < y_{m-1}\}$ thereafter.

4. For any real number x and $(x_m) \in \mathbb{R}^\infty$, show that $x_m \to x$ iff every subsequence of (x_m) has itself a subsequence that converges to x.

   We first prove that if $x_m \to x$, then every subsequence of (x_m) has itself a subsequence that converges to x. Using the definition of convergence, if $x_m \to x$, it means that $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $x_m \in B_\epsilon(x)$ $\forall n > N(\epsilon)$. This means that for every subsequence x_{m_k} in (x_m), we can also find a $N(\epsilon)$ such that $x_{m_k} \in B_\epsilon(x)$ $\forall n > N(\epsilon)$. Hence every subsequence of x_m has itself a subsequence that converges to x.

   Now, we prove that if every subsequence of (x_m) has itself a subsequence that converges to x, then $x_m \to x$. We prove by contradiction. Suppose x_m does not converge to x. Then, for some $\epsilon > 0$, there are infinitely many terms of x_m outside the interval $(x - \epsilon, x + \epsilon)$. This implies that there exists a subsequence (x_{m_k}) such that $|x_{m_k} - x| > \epsilon$ for all k. But this contradicts the fact that every subsequence of (x_m) has itself a subsequence that converges to x. So x_m must converge to x.

5. If (X, d) and (X, ρ) are metric spaces, is $(X, max\{d, \rho\})$ a metric space? How about $(X, min\{d, \rho\})$?

   We note that both the max and min metrics satisfy non-negativity, symmetry and equality by their definition, so the crucial criteria to check is the triangle inequality.

   Without loss of generality, assume that we have two points x and y, where $d(x, y) < \rho(x, y)$.

   For $(X, max\{d, \rho\})$, the triangle inequality is:

   $$max\{d, \rho\}(x, y) = \rho(x, y) \leq \rho(x, z) + \rho(z, y) = max\{d, \rho\}(x, z) + max\{d, \rho\}(z, y)$$

   So the triangle inequality holds and $max\{d, \rho\}$ is a metric.

For $(X, min\{d,\rho\})$, the triangle inequality is:

$$min\{d,\rho\}(x,y) = d(x,y) \le d(x,z) + d(z,y) = min\{d,\rho\}(x,z) + min\{d,\rho\}(z,y)$$

So the triangle inequality holds and $min\{d,\rho\}$ is a metric.

6. For any metric space X, show that $|d(x,y) - d(y,z)| \le d(x,z)$ for any $x, y, z \in X$.

This is the reverse triangle inequality. We know that in a metric space, the triangle inequality holds, i.e.,

$$d(x,y) \le d(x,z) + d(z,y)$$

This gives

$$d(x,y) - d(y,z) \le d(x,z) \tag{Eq 1}$$

At the same time, we also have the following triangle inequality:

$$d(y,z) \le d(y,x) + d(x,z)$$

This gives

$$d(y,z) - d(x,y) \le d(x,z) \tag{Eq 2}$$

Since Eq 1 and Eq2 are both cases of the absolute sign on the LHS, we have:

$$|d(x,y) - d(y,z)| \le d(x,z)$$

7. Show that $(C^1[0,1], D_\infty)$ is a metric space.

We want to show that the set of all continuously differentiable functions on the interval $[0,1]$ is a metric space with the sup norm.

We check each property of a metric space:
(a) Nonnegativity holds because for any $f(x), g(x) \in C^1[0,1]$, the sup norm D_∞ is nonnegative.
(b) $D_\infty(f(x), f(y)) = 0$ iff $f(x) = g(x)$ on $[0,1]$, because the distance $= 0$ iff both functions are identical in $[0,1]$ because of the sup norm.
(c) Symmetry holds because of the absolute function in the sup norm.
(d) For the triangle inequality, we can verify that:

$$\begin{aligned} D_\infty(f,g) &= \sup_{x\in[0,1]} |f(x) - g(x)| \\ &= \sup_{x\in[0,1]} |f(x) - h(x) + h(x) - g(x)| \\ &\le \sup_{x\in[0,1]} |f(x) - h(x)| + \sup_{x\in[0,1]} |h(x) - g(x)| \\ &= D_\infty(f,h) + D_\infty(h,g) \end{aligned}$$

Hence (D_∞) satisfies all properties of a metric and $(C^1[0,1], D_\infty)$ is a metric space.

8. Let (X,d) be a metric space and $f : \mathbb{R}_+ \to \mathbb{R}$ a concave and strictly increasing function with $f(0) = 0$. Show that $(X, f \circ d)$ is a metric space.

We check each property of a metric space:
(a) Nonnegativity holds because f is a strictly increasing function where $f(0) = 0$, hence $f \circ d$ must be nonnegative since d is nonnegative.
(b) $(f \circ d)(x,y) = 0$ iff $x = y$ because f is a strictly increasing function where $f(0) = 0$ and $d(x,y)$ is a metric and only $= 0$ when $x = y$.
(c) Symmetry holds because d is a metric and f is one-to-one as it is strictly increasing.
(d) For the triangle inequality, we can verify that:

$$\begin{aligned} (f\circ d)(x,y) &= f(d(x,y)) \\ &\le f(d(x,z) + d(z,y)) \qquad \text{since the function is concave} \\ &= (f\circ d)(x,z) + (f\circ d)(z,y) \end{aligned}$$

Hence $(f \circ d)$ satisfies all the properties of a metric and $(X, f \circ d)$ is a metric space.

9. Given any metric space X, let Y be a metric subspace of X, and take any $S \subseteq Y$. Show that S is open in Y iff $S = O \cap Y$ for some open subset O of X, and it is closed in Y iff $S = C \cap Y$ for some closed subset C of X.

   We first show that S is open in Y if $S = O \cap Y$ for some open subset O of X. Since Y has the properties of a metric space, it is open, and hence S is the intersection of two open sets and thus S is open.

   Now, we show that $S = O \cap Y$ for some open subset O of X if S is open in Y. Since S is open, and S is a subset of Y, then $\forall s \in S, \exists O \in X$ such that $B_X(s, r) = O \cap Y$. Therefore, $S = O \cap Y$ where $O = \cup_{s \in S} B_X(s, r)$ is open in X.

   Now we show that S is closed in Y if $S = O \cap Y$ for some closed subset O of X. Since Y has the properties of a metric space, it is closed, and hence S is the intersection of two closed sets and is thus closed.

   Now, we show that $S = O \cap Y$ for some closed subset O of X if S is closed in Y. Since S is closed, then its complement $Y \backslash S$ is open in Y, and there exists an open set $O^c \in Y$ such that $Y \backslash S = O^c \cap Y$, which implies $S = O \cap Y$.

10. Can you find a metric on $\mathbb{N}$ such that $\emptyset \neq S \subseteq \mathbb{N}$ is open iff $\mathbb{N} \backslash S$ is finite?

    No. Suppose we can find such a metric. Then, the nonempty set S would be closed iff it is finite. But this is a contradiction since $\mathbb{N}$ is infinite, and we cannot have both $\mathbb{N} \backslash S$ and $\emptyset \neq S \subseteq \mathbb{N}$ to be finite.

11. Given a metric space X, let Y be a metric subspace of X, and $S \subseteq X$. Show that

$$int_X(S) \cap Y \subseteq int_Y(S \cap Y) \text{ and } cl_X(S) \cap Y \supseteq cl_Y(S \cap Y)$$

    and give examples to show that the converse containments do not hold in general. Also prove that

$$int_X(S) \cap Y = int_Y(S \cap Y)$$

    provided that Y is open in X. Similarly, $cl_X(S) \cap Y = cl_Y(S \cap Y)$ holds if Y is closed in X.

12. Let S be a closed subset of a metric space X, and $x \in X \backslash S$. Show that there exists an open subset O of X such that $S \subseteq O$ and $x \in X \backslash O$.

    Consider the set $\{x\}$. Since $\{x\}$ is a singleton, it is closed.

    Since S is closed and $\{x\}$ are closed, their complements $X \setminus S$ and $X \setminus \{x\}$ are open sets.

    Since $x \in X \backslash S$, it follows that $S \subseteq X \backslash \{x\}$.

    Hence we have found an open set $X \backslash \{x\} = O$, for which $S \subseteq O$ and $x \in X \backslash O$.

13. For any given metric space (X, d), show that for any $(x^m) \in X^\infty$ and $x \in X$, we have $x^m \to x$ in (X, d) iff $x^m \to x$ in $(X, \frac{d}{1+d})$.

    Sketch of proof:
    We first show that if $x^m \to x$ in (X, d), then $x^m \to x$ in $(X, \frac{d}{1+d})$.
    If $x^m \to x$ in (X, d), this means that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m > N, d(x^m, x) < \epsilon$.

    Now consider the distance $\frac{d}{1+d}(x^m, x)$. $\forall m > N$, we have

$$d'(x^m, x) = \frac{d(x^m, x)}{1 + d(x^m, x)} < \frac{\epsilon}{1 + \epsilon}$$

Since this is true for any arbitraty ϵ, this shows that $x^m \to x$ in $(X, \frac{d}{1+d})$.

The proof in the other direction is similar.

14. Prove that for any subset S of a metric space X, the following are equivalent:

    (a) $x \in cl_X(S)$;

    (b) Every open neighbourhood of $x \in X$ intersects S;

    (c) There exists a sequence in S which converges to x.

    Sketch of proof: We will show that (1) implies (2), (2) implies (3), and (3) implies (1) in a metric space X.

    (a) (1) implies (2): Assume $x \in \mathrm{cl}_X(S)$, the closure of S. Let U be an open neighborhood of x. We want to show that $U \cap S \neq \emptyset$. Since x is in the closure of S, every open neighborhood of x intersects S, so $U \cap S \neq \emptyset$.

    (b) (2) implies (3): Assume every open neighborhood of x intersects S. Consider the open neighborhood $B(x, \frac{1}{n})$ for each positive integer n. Since this neighborhood intersects S, there exists $s_n \in S$ such that $s_n \in B(x, \frac{1}{n})$, i.e., $d(s_n, x) < \frac{1}{n}$. Thus, the sequence (s_n) converges to x.

    (c) (3) implies (1): Assume there exists a sequence (s_n) in S that converges to x. We want to show that x is in the closure of S, i.e., every open neighborhood of x intersects S. Let U be an open neighborhood of x. Since (s_n) converges to x, there exists $N \in \mathbb{N}$ such that $s_n \in U$ for all $n \geq N$. Since $s_n \in S$ for all n, this means that U intersects S, proving that x is in the closure of S.

15. Let (x^m) be a sequence in a metric space X. We say that $x \in X$ is a cluster point of (x^m) if for each $\epsilon > 0, N_{\epsilon,X}(x)$ contains infinitely many terms of (x^m).

    (a) Show that every convergent sequence has exactly one cluster point.

    (b) For any $k \in \mathbb{Z}_+$, give an example of a sequence in some metric space with exactly k many cluster points.

    (c) Show that x is a cluster point of (x^m) iff there is a subsequence of (x^m) that converges to x.

    Sketch of proof:
    (a) every convergent sequence has one unique limit which is the cluster point.
    (b) consider the sequence defined by:

    $$x_n = n \mod k$$

    (c) Prove both directions using definition of cluster point.

16. (see image)

[4] Consider the following real sequences: For each $m \in \mathbb{N}$,

$$x^m := \left(0, ..., 0, \tfrac{1}{m}, 0, ...\right), \quad y^m := (0, ..., 0, 1, 0, ...), \quad z^m := \left(\tfrac{1}{m}, ..., \tfrac{1}{m}, 0, ...\right),$$

where the only nonzero term of the sequences x^m and y^m is the mth one, and all but the first m terms of z^m are zero. Since $d_p((x^m), (0, 0, ...)) = \frac{1}{m}$ for each m, we have $(x^m) \to (0, 0, ...)$ in ℓ^p for any $1 \leq p \leq \infty$. (Don't forget that here (x^m) is a sequence of real sequences.) In contrast, it is easily checked that the sequence $(y^1, y^2, ...)$ is not convergent in any ℓ^p space $(1 \leq p \leq \infty)$. On the other hand, we have $d_\infty(z^m, (0, 0, ...)) = \frac{1}{m} \to 0$, so (z^m) converges to $(0, 0, ...)$ in ℓ^∞. Yet $d_1(z^m, (0, 0, ...)) = 1$ for each m, so (z^m) does not converge to $(0, 0, ...)$ in ℓ^1. Is (z^m) convergent in any ℓ^p, $1 < p < \infty$?

x^m is the following sequence of sequences:

$$x^1 = (1, 0, ...)$$
$$x^2 = (0, \frac{1}{2}, 0, ...)$$
$$x^3 = (0, 0, \frac{1}{3}, 0, ...)$$

$(x^m) \to (0, 0, ...)$ in l^p for any $l \leq p \leq \infty$.

y_m is the following sequence of sequences:

$$y^1 = (1, 0, ...)$$
$$y^2 = (0, 1, 0, ...)$$

z^m is the following sequence of sequences:

$$z^1 = (1, 0, ...)$$
$$z^2 = (\frac{1}{2}, \frac{1}{2}, 0, ...)$$

l^p is the norm

$$||z_m|| = (\sum_{i=1}^{m} |\frac{1}{m}|^p)^{\frac{1}{p}}$$

which approaches 0 as $m \to \infty$. Hence the sequence z_n converges to 0 in the l^p norm for $1 < p < \infty$.

17. Prove that $\mathbb{R}^{n,p}$ is complete for any $(n, p) \in \mathbb{N} \times [1, \infty]$.

$\mathbb{R}^{n,p}$ is $\mathbb{R}^n$ with the p-norm, which is

$$||x||_p = (\sum_{i=1}^{m} |\frac{1}{m}|^p)^{\frac{1}{p}}$$

where $x = (x_1, x_2, ...x_n) \in \mathbb{R}^n$. We want to show that every Cauchy sequence in $\mathbb{R}^{n,p}$ converges to a limit in $\mathbb{R}^{n,p}$.

For $x^k = (x_1^k, ..., x_n^k)$, consider the individual components (which are sequences), i.e., $\{x_i^k\}_{k=1}^{\infty}, i = 1, 2, ..., n$. Let each $\{x_i^k\}$ be a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, it converges to a limit in $\mathbb{R}$.

Let $x = (x_1, x_2, ...x_n)$ where $x_i = \lim_{k\to\infty} x_i^k$. We want to show that $x^k \to x$ in $\mathbb{R}^{n,p}$.

For $1 \leq p < \infty$,

$$||x^k - x||_p = (\sum_{i=1}^{n} |x_i^k - x_i|^p)^{1/p}$$

As $k \to \infty$, each $|x_i^k - x_i| \to 0$. By the continuity of the p-norm, we have $||x^k - x||_p \to 0$.

For $p = \infty$, we have

$$||x^k - x||_\infty = \max_{1 \leq i \leq n} |x_i^k - x_i|$$

As $k \to \infty$, this distance $\to 0$.
Therefore, $x^k \to x$ in $\mathbb{R}^{n,p}$ and $\mathbb{R}^{n,p}$ is complete.

18. Let X be any metric space, and $\phi \in \mathbb{R}^X$.

    (a) Show that if ϕ is continuous, then the sets $\{x : \phi(x) \leq \alpha\}$ and $\{x : \phi(x) \geq \alpha\}$ are closed in X for any real number α. Also show that the continuity of ϕ is necessary for this to hold.

    (b) Prove that if ϕ is continuous, and $\phi(x) > 0$ for some $x \in X$, then there exists an open subset O of X such that $\phi(y) > 0$ for all $y \in O$.

19. Let A and B be two nonempty closed subsets of a metric space X with $A \cap B = \emptyset$. Define $\phi \in \mathbb{R}^X_+$ and $\psi \in \mathbb{R}^X$ by $\phi(x) := d(x, A)$ and $\psi(x) := d(x, A) - d(x, B)$ respectively. Prove:

    (a) $A = \{x \in X : \phi(x) = 0\}$, so we have $d(x, A) > 0$ for all $x \in X \backslash A$.

    (b) ψ is continuous, so $\{x \in X : \psi(x) < 0\}$ and $\{x \in X : \psi(x) > 0\}$ are open.

    (c) There exist disjoint open sets O and U in X such that $A \subseteq O$ and $B \subseteq U$.

20. Consider the self-correspondence Γ on $[0, 1]$ defined as $\Gamma(0) := (0, 1]$ and $\Gamma(t) := (0, t)$ for all $0 < t \leq 1$. Is Γ upper hemicontinuous?

    For Γ to be upper hemicontinuous at all $x_0 \in [0, 1]$, it must be that for every open set V containing $\Gamma(x_0)$, there exists an open set U containing x_0 such that $\Gamma(x) \in V$ for every $x \in U$.

    For $0 < t < 1$, it is clear that $\Gamma(t)$ is upper hemicontinuous because for the open set $V = (0, t)$ containing $\Gamma(t)$, there exists an open set $U = (0, t)$ containing $V = (0, t)$ such that $\Gamma(x) \in V = (0, t)$ for every $x \in U = (0, t)$.

    For $t = 1$, the set $\Gamma(1)$ is $(0, 1)$. For the open set $V = (0, 1)$ containing $\Gamma(1)$, it is not possible to find an open set U such that $\Gamma(x) \in V$ for every $x \in U$, since the only point that maps to $(0, 1)$ is $t = 1$, and 1 is not in $(0, 1)$. Hence Γ is not upper hemicontinuous at $t = 1$.

21. Show that if A and B are λ-convex sets of a linear space, so is $\alpha A + B$ for any $\alpha \in \mathbb{R}$ and $0 < \lambda < 1$.

    Sketch of proof: This is similar to De La Fuente Problem 2.11 but using the definition of convex sets instead of concave functions.

2023 assignments

1. **2023 Assignment 1**

   (a) Let (X, d) be a metric space and $\{x_n\}$ a convergent sequence in (X, d). Let x_{n_i} and x_{n_j} be two subsequences of $\{x_n\}$. Prove that both sequences converge and that they converge to the same limit.

   We first show x_n is a Cauchy sequence.

   If $\{x_n\}$ converges to a limit x, then $\forall \epsilon > 0, \exists N$ such that $d(x_n, x) < \epsilon, \forall n \geq N$.

   We choose N' such that $d(x_n, x) < \epsilon/2, \forall n \geq N'$.

   Then, by triangle inequality, $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon, \forall m, n \geq N'$.

   Hence, $d(x_n, x_m) < \epsilon, \forall n, m \geq N$, and $\{x_n\}$ is a Cauchy sequence.

   Now, we will show that it must be that x_{n_i} and x_{n_j} converge and that they converge to the same limit x. We prove both parts by contradiction.

   First, suppose that either of the subsequences do not converge. Without loss of generality, suppose that x_{n_i} does not converge. That means that $\exists \epsilon > 0$ such that $d(x_{n_i}, x) > \epsilon, \forall n \in \mathbb{N}$. But this contradicts the fact that x_n (the main sequence that both subsequences draw from) is a convergent sequence, since we know that $\forall \epsilon > 0, \exists N$ such that $d(x_n, x) < \epsilon, \forall n \geq N$. So it must be true that both of the subsequences converge.

   Second, suppose that either of the subsequences converge to a limit other than x. Without loss of generality, suppose that x_{n_i} converges to a limit y.

   If the main sequence $x_n \to x$, then $\forall \epsilon_x > 0, \exists N_{\epsilon_x}$ such that $d(x_n, x) < \epsilon_x, \forall n \geq N_{\epsilon_x}$.

   Suppose that $x_{n_i} \to y$. Then $\forall \epsilon_y > 0, \exists N_{\epsilon_y}$ such that $d(x_{n_i}, y) < \epsilon_y, \forall n \geq N_{\epsilon_y}$.

   Finally, since x_n is a Cauchy sequence, it must be that $d(x_n, x_{n_i}) < \epsilon_z, \forall n, m \geq N_{\epsilon_z}$.

   We choose $\epsilon_x = \epsilon_y = \epsilon_z = \epsilon/3$. Then by triangle inequality, $d(x, y) \leq d(x, x_n) + d(x_n, x_{n_i}) + d(x_{n_i}, y)\ \forall n \geq N = max\{N_{\epsilon_x}, N_{\epsilon_y}, N_{\epsilon_z}\}$. This implies $d(x, y) = 0$ and hence $x = y$ based on the definition of a metric. Hence the limit is unique if it exists.

   (b) Let $\{x_n\}$ be a sequence of positive real numbers. Prove that $\{x_n\} \to \infty$ iff $\{1/x_n\} \to 0$.

   (refer to 2024 Assignment 1, Q6)

   (c) Consider the space of all m-dimensional vectors $\mathbb{R}^m$ with the Euclidean metric; de la Fuente calls it $\mathbb{E}^m$. Show that every bounded sequence in $\mathbb{E}^m$ contains at least one convergent subsequence.

   (refer to 2024 Assignment 1, Q7)

   (d) Prove parts (iii) and (iv) of Theorem 33, i.e., (i) cl A is the smallest closed set that contains A; and A is closed iff A = cl A.

   We first show that cl A is a closed set. By definition, $x \in clA \Leftrightarrow \forall \epsilon > 0, B_\epsilon(x) \cap A \neq \emptyset$. This means that for the complement of cl A, $clA^c, \forall \epsilon > 0, B_\epsilon(x) \cap A = \emptyset$. This also means that $\forall x \in clA^c, \exists \epsilon > 0, B_\epsilon(x) \subseteq clA^c$, i.e., clA^c is open. Hence cl A is closed.

   Now, we will show that cl A is the smallest closed set that contains A. Suppose $A \subset A' \subset clA$, where A' is a closed set. Consider any point $x \in clA$, but not in A', i.e., $x \notin A'$. x cannot be in the interior of A' as this would contradict the definition of closure, so it must be in A but not in A'. This leads to a contradiction since $A \subset A'$, so it must be that there cannot be a closed set containing A that is strictly smaller than clA, thus proving that clA is the smallest closed set containing A.

Now, we will show that A is closed if A = cl A. This is trivial since we showed earlier that cl A is a closed set.

Now, we will show that A = cl A if A is closed. Since $clA = A \cup \text{bdy}A$, we need to show that A includes all its boundary points. We show this by contradiction.

Suppose that x is a boundary point of A that lies outside A, i.e., in the complement of A, A^c. Since A is closed, it must be that A^c is open, and hence $\forall x \in A^c, \exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq A^c$. But this contradicts A^c containing the boundary point x since that requires that $\forall \epsilon > 0, B_\epsilon(x) \cap A^c \neq \emptyset$ and $B_\epsilon(x) \cap A \neq \emptyset$. So since assuming that the boundary points of A lie outside A leads to a contradiction, it must be that A contains all of its boundary points.

(e) Given a function $f : \mathbb{R} \to \mathbb{R}$, define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(x) = (x, f(x))$. Use the sequential characterisation of continuity to show that if f is continuous at some point x_0, then so is g.

The sequential characterisation of continuity is that f is continuous at x^0 iff for every $\{x_n\} \to x^0$ we have $\{f(x^n)\} \to f(x^0)$.

Since f is continuous at x^0, it must be that for every $\{x^n\} \to x^0$ we have $\{f(x^n)\} \to f(x^0)$. Since $g(x) = (x, f(x))$, it thus means that both components of g are continuous at x^0 and thus g is continuous at x^0.

(f) Show that in any normed vector space $(X, ||\cdot||)$, the norm is a continuous function from X to $\mathbb{R}$.

We can use the sequential characterisation of continuity, i.e., the norm is a continuous function iff for every sequence converging to a point x in the vector space X, i.e., $\{x_n\} \to x$, we have $||x_n|| \to ||x||$.

Using the reverse triangle inequality, $|||x_n|| - ||x||| \leq ||x_n - x||$. Since we are looking at sequences converging to a point x, it must be that $||x_n - x|| \to 0$, and hence $|||x_n|| - ||x||| \to 0$ and thus $||x_n|| \to ||x||$.

(g) In Theorem 46, we made two claims, for open and for closed sets. We have proved the claim for open sets, but not for closed sets. Using the result we established, prove the theorem for closed sets.

We want to prove that $f : X \to Y$ is continuous on X (i.e., at all points in X) iff $f^{-1}(C)$ is closed for any closed subset $C \subset Y$.

First, we will prove that $f : X \to Y$ is continuous on X (i.e., at all points in X) if $f^{-1}(C)$ is closed for any closed subset $C \subset Y$.

Let $x \in X$ and $\epsilon > 0$. Consider the set $C = (B_\epsilon(f(x)))^c$, where $B_\epsilon(f(x))$ is the open ball of radius ϵ around $f(x)$. Since $B_\epsilon(f(x))$ is open, C is closed.

$f^{-1}(C) = f^{-1}((B_\epsilon(f(x)))^c)$. Since $f^{-1}(B_\epsilon(f(x)))$ is open (by the continuity of f), its complement $f^{-1}(C)$ is closed. This means that $f^{-1}(C)$ is closed.

Since $f^{-1}(C)$ is closed, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(C)$, which means $f(B_\delta(x)) \subseteq C$. This implies that f is continuous at x. Since x was arbitrary, f is continuous on X.

Now, we will prove that $f^{-1}(C)$ is closed for any closed subset $C \subset Y$ if $f : X \to Y$ is continuous on X (i.e., at all points in X).

If C is closed, it must be that its complement C^c is open. Since f is continuous, it must be that $f^{-1}(C^c)$ is open. Thus its complement $f^{-1}(C)$ is closed.

(h) We claimed that a Lipschitz function is uniformly continuous without a proof. Prove that.

Recall that $f : X \to Y$ is Lipschitz on $E \subset X$ if $\exists K > 0$ such that $|f(x)-f(x^0)| \leq K\cdot|x-x^0|$, $\forall x, x^0 \in E$.

Definition of uniform continuity: $f : X \to Y$ is uniformly continuous on $A \subset X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$, $\forall x \in A$.

Let f be Lipschitz on A and let ϵ be given. We choose $\delta = \epsilon/K$. This means that if $|x - x^0| \leq \delta$, then $|f(x) - f(x^0)| \leq K \cdot |x - x^0| \leq K\delta = \epsilon$ and hence f is uniformly continuous.

(i) Consider two infinite sequences, $x_m = (0, ..., 0, 1, 0, ...)$ and $y_m = (\frac{1}{m}, \frac{1}{m}$ (m times) $, 0, 0, ...)$. Do these sequences converge in Euclidean norm? Do they converge in sup norm?

In Euclidean norm, the distance between the two sequences is:

$$\sqrt{(0-\frac{1}{m})^2, (0-\frac{1}{m})^2, ...(1-0^2), (0-0)^2...} = \sqrt{\frac{m}{m^2}+1}$$

As $m \to \infty$, the distance approaches 1, so the two sequences do not converge in Euclidean norm.

In sup norm, the distance between the two sequences is $max\{\frac{1}{m}, 1\}$. As $m \to \infty$, the distance approaches 1, so the two sequences do not converge in sup norm.

(j) Let (X_m, d_m) be an infinite sequence of metric spaces and $X = X_1 \times X_2 \times ....$ Define $\rho : X \times X \to \mathbb{R}+$ as

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{min\{1, d_i(x_i, y_i)\}}{2^i}$$

Is ρ a metric on X? Why is min needed in the above?

We check each criteria of a metric in turn:

i. Non-negativity: This condition holds as $\rho(x, y)$ is a sum of positive terms (minimum between 1 and a distance), divided by a positive term (powers of 2).
ii. $d(x, y) = 0$ iff $x = y$: This condition holds, because the distance between x and $y = x$ is an infinite sum of zeroes, and this only holds if $d(x_i, y_i) = 0$ for all i.
iii. $d(x, y) = d(y, x)$: This condition holds given the structure of $\rho(x, y)$ (i.e., being a sum of terms with minimum of 1 and a distance, which is itself symmetric).
iv. Triangle inequality: Let $x, y, z \in X \times X$. Then, the distance between x and z is:

$$\begin{aligned}
\rho(x, z) &= \sum_{i=1}^{\infty} \frac{min\{1, d_i(x_i, z_i)\}}{2^i} \\
&\leq \sum_{i=1}^{\infty} (\frac{min\{1, d_i(x_i, y_i)\}}{2^i} + \frac{min\{1, d_i(y_i, z_i)\}}{2^i}) \\
&= \sum_{i=1}^{\infty} \frac{min\{1, d_i(x_i, y_i)\}}{2^i} + \sum_{i=1}^{\infty} \frac{min\{1, d_i(y_i, z_i)\}}{2^i} \\
&= \rho(x, y) + \rho(y, z)
\end{aligned}$$

Hence, the triangle inequality holds and ρ is a metric.

Min was needed otherwise we would not satisfy the condition where $d(x, y) = 0$ iff $x = y$. Otherwise, the distance between x and $y = x$ would be greater than 1/2 as the first term in the infinite sum would be 1/2.

2. **2023 Assignment 2**

(a) Consider the set $X = \{x_1 = (0,1), x_2 = (-2,0), x_3 = (0,-3), x_4 = (4,0)\}$. Find the set Y such that $y \in Y$ iff $y = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4$ and $\lambda_i \geq 0 \; \forall i, \sum \lambda_i = 1$.

The set Y is

$$conv X\{y | y = \sum_{i=1}^{m} \lambda_i x_i, x_i \in X \forall i, \lambda_i \geq 0 \forall i, \sum \lambda_i = 1\}$$

(b) Consider an element $y = (0.5, 0.5)$ of the set Y above. Represent y in the form above (as $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4$) aiming to have as many $\lambda_i = 0$ as possible. You do not need to prove that you cannot have fewer $\lambda_i = 0$.

Consider $\frac{1}{2}(0,1) + \frac{1}{4}(-2,0) + \frac{1}{4}(4,0) = (0.5, 0.5)$.

(c) Consider the set $H((1,1),0)) = \{x \in \mathbb{R}^2 | (1,1) \cdot x = 0\}$, where $\cdot$ is a dot product. What points lie to the North East of this set?

This is the set of points such that $(1,1) \cdot (x_1, x_2) = 0$, i.e., $x_1 + x_2 = 0$. This is the line $x_1 = -x_2$. Any point such that $x_2 > -x_1$ lies to the North East of this set.

(d) Consider the set $C = \{x \in \mathbb{R}^2 | (x_1)^2 + (x_2)^2 = 1\}$ (subscripts represent coordinates).
  i. Find $H((1,1),a)) = \{x \in \mathbb{R}^2 | (1,1) \cdot x = a\}$ such that C and H has a single common point.
  ii. Find $H((1,1),b)) = \{x \in \mathbb{R}^2 | (1,1) \cdot x = b\}$ such that C and H do not intersect and C lies to the South West of H while $(3,3)$ lies to the North East.

The set H is the line such that $x_1 + x_2 = a$. This is a line of gradient -1 that passes through the point $(\frac{a}{2}, \frac{a}{2})$.

The set C is the circumference of a circle of radius 1, centered at the origin $(0,0)$.

For (i), for both sets to have only one common point, it must be that the point $(\frac{a}{2}, \frac{a}{2})$ is exactly 1 unit away from the origin. Otherwise, both sets would either intersect zero times or more than once.

By Pythagoras' theorem, we have that $2(\frac{a}{2})^2 = 1$ and hence $a = \pm\sqrt{2}$. The set H is thus $H((1,1), \pm\sqrt{2})) = \{x \in \mathbb{R}^2 | (1,1) \cdot x = \pm\sqrt{2}\}$.

For (ii), for C and H not to intersect, and C to lie to the Southwest of H while $(3,3)$ lies to the Northeast, it must be that:
  i. $\frac{a}{2}$ is positive, since the origin lies to the southwest.
  ii. The point $(\frac{a}{2}, \frac{a}{2})$ is more than 1 unit away from the origin, i.e., $a > \sqrt{2}$.
  iii. $\frac{a}{2} < 3$, since $(3,3)$ lies to the Northeast.

And hence the set H is $H((1,1),a))$ where $\sqrt{2} < a < 6$.

(e) Consider $f(x) = x^2$. For any given $(x^0, (x^0)^2)$, describe the set $H = \{(x,y) \in \mathbb{R}^2 | (p, p^0) \cdot (x,y) = (p, p^0) \cdot (x^0, (x^0)^2)\}$ where (p, p^0) is selected so that H and $Gf = \{(x, x^2) \in \mathbb{R}^2\}$ have only one common point.

Gf is the parabola such that $f(x) = x^2$.

H is the hyperplane such that $px + p^0 y = px^0 + p^0(x^0)^2$.

For H and Gf to have only one common point, it must be that H is tangent to Gf at the point $(x^0, (x^0)^2$. This also means that the slopes of Gf and H are equal at the point $(x^0, (x^0)^2)$.
  i. The slope of Gf at this point is $f'(x) = 2x = 2x^0$

ii. To find the slope of H at this point, we rewrite the equation:

$$
\begin{aligned}
px + p^0 y &= px^0 + p^0(x_0)^2 \\
y &= -\frac{p}{p^0}x + \frac{px^0 + p^0(x^0)^2}{p^0}
\end{aligned}
$$

where the slope $-\frac{p}{p^0}$ must equal the slope of Gf, which is $2x^0$, and hence $p = -2x^0p^0$. Substituting this into the equation for H and simplifying, we have

$$y - 2x^0x = -(x^0)^2$$

And hence the hyperplane H is $H = \{(x, y) \in \mathbb{R}^2 | y - 2x^0x = -(x^0)^2\}$.

(f) Show that the set $U_1 = \{x \in \mathbb{R}_+ | x^2 \geq 1\}$ is convex.

Consider any $x', x'' \in U_1$ and $\lambda \in [0, 1]$. x' and x'' are positive real numbers ≥ 1.

Let $x_\lambda = \lambda(x') + (1 - \lambda)(x'')$.

$$
\begin{aligned}
x_\lambda &= \lambda(x') + (1 - \lambda)(x'') \\
&\geq \lambda(1) + (1 - \lambda)(1) = 1
\end{aligned}
$$

And hence $x_\lambda \geq 1$, $x_\lambda^2 \geq 1$ and the convex combination of any $x', x'' \in U_1$ is also in U_1 and the set is convex.

(g) Prove that the transformation is linear only if 0 is mapped to 0.

(skip, not in syllabus)

(h) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x)$ is a rotation of x by 30 degrees clockwise.

(skip, not in syllabus)

(i) For an integer $n > 0$, let P_n be the vector space of polynomials of degree at most n. Let $T(f) = xf(x)$.
  i. Prove that it is a linear transformation.
  ii. What is $ker\ T$?
  iii. What is the matrix representation of T?

(skip, not in syllabus)

(j) Show that the closure of a convex set is convex.

(refer to 2024 Assignment 2 Q10).

(k) Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$, be a concave function and $g : \mathbb{R} \to \mathbb{R}$ be an increasing and concave function defined on an interval I that contains $f(x)$. Then the function $g[f(x)]$ is concave.

If f concave, given any $x', x'' \in X$, we have

$$tf(x') + (1 - t)f(x'') \leq f(tx' + (1 - t)x''), \forall t \in [0, 1]$$

We want to show that

$$t(g(f(x')) + (1 - t)g(f(x'')) \leq g(f(tx' + (1 - t)x''))$$

We apply g to the concavity of f. The inequality is preserved since g is increasing:

$$g(tf(x') + (1 - t)f(x'')) \leq g(f(tx' + (1 - t)x''))$$

Further, we can add an inequality to the LHS since g is concave and defined on the interval I that contains $f(x)$:

$$t(g(f(x')) + (1-t)g(f(x'')) \leq g(tf(x') + (1-t)f(x'')) \leq g(f(tx' + (1-t)x''))$$

and hence $g(f(x))$ is concave.

(l) Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is a convex set, be a quasiconcave function and $g : \mathbb{R} \to \mathbb{R}$ an increasing and concave function defined on an interval I that contains $f(X)$. Then the function $g[f(x)]$ is quasiconcave.

f is quasiconcave, given any $x', x'' \in X$, we have

$$f(tx' + (1-t)x'') \geq \min\{f(x'), f(x'')\}$$

We want to show that

$$g(f(tx' + (1-t)x'')) \geq min\{gf(x'), gf(x'')\}$$

Since g is an increasing function, we can apply g to f and the inequality is preserved:

$$g(f(tx' + (1-t)x'') \geq g(min\{f(x'), f(x'')\})$$

Further, we can add an inequality to the RHS since g is quasiconcave and defined on the interval I that contains $f(x)$:

$$g(f(tx' + (1-t)x'') \geq g(min\{f(x'), f(x'')\}) \geq min\{gf(x'), gf(x'')\}$$

and hence $g(f(x))$ is quasiconcave.

Weekly problems

1. **Week 2: 04 Mar 2024**

   (a) **Problem 1**: Formally show that the sup metric in $\mathbb{R}^n$, defined as $d_{\infty(x,y)} = max_{i\in\{1,\dots,n\}}\{\mid x_i - y_i \mid\}$, satisfies triangle inequality.

   Let i^* be the value of i where $\mid x_i - z_i \mid$ reaches the maximum.

$$\sup_i\{\mid x_i - z_i \mid\} =\mid x_i^* - z_i^* \mid \leq \mid x_i^* - y_i^* \mid + \mid y_i^* - z_i^* \mid \leq \sup_i\{\mid x_i - y_i \mid\} + \sup_i\{\mid y_i - z_i \mid\}$$

   Hence, the sup metric satisfies triangle inequality.

   (b) **Problem 2** Is $\mid\mid f \mid\mid = \int_1^0 \mid f(x) \mid \, dx$ a metric on the space of all integrable functions $f : [0,1] \to \mathbb{R}$? Prove triangle inequality for $\mid\mid f \mid\mid$.

   No, it is not a metric on the space of all integrable functions of this form. We show using a counterexample:

$$f = \begin{cases} 0 & x \neq 1/2 \\ 1 & x = 1/2 \end{cases}$$

   In this example, $\mid\mid f \mid\mid$ is not a metric as the metric can $= 0$ even if $f \neq 0$. It is only a metric on the space of all continuous functions $f : [0,1] \to \mathbb{R}$.

   However, we can still prove triangle inequality for $\mid\mid f \mid\mid$.

$$\begin{aligned}\mid\mid f+g \mid\mid &= \int_1^0 \mid f(x) + g(x) \mid \, dx \\ &\leq \int_1^0 \mid f(x) \mid + \mid g(x) \mid \, dx = \int_1^0 \mid f(x) \mid \, dx + \int_1^0 \mid g(x) \mid \, dx = \mid\mid f \mid\mid + \mid\mid g \mid\mid\end{aligned}$$

   (c) **Problem 3** Let $x, y \in \mathbb{R}$. Prove that for any $\delta > 0$ there is $\epsilon > 0$ such that if $d'(x,y) = \mid x - y \mid < \epsilon$, then $d(x,y) = \mid \frac{x}{1+|x|} - \frac{y}{1+|y|} \mid < \delta$.

   Consider:

$$\frac{x}{1+\mid x \mid} - \frac{y}{1+\mid y \mid} = \frac{x(1+|y|) - y(1+|x|)}{(1+|x|)(1+|y|)} = \frac{(x-y) + x|y| - y|x|}{(1+|x|)(1+|y|)}$$

   If x and y have the same signs, then $x|y| - y|x| = 0$ and the inequality is trivial.

   If x and y have different signs, then, without loss of generality, we take $x > 0$ and $y < 0$. Then, since $|x-y| < \epsilon$, so $y = x - \epsilon < 0$.

   This means that $x|y| - y|x| = x(x-\epsilon) + (x-\epsilon)x = 2x(x-\epsilon) \leq 2\epsilon^2$ because $x \leq \epsilon$ and $x - \epsilon \leq \epsilon$.

   Let $\epsilon < 1$, then $x|y| - y|x| < 2\epsilon$.

$$\frac{(x-y) + x|y| - y|x|}{(1+|x|)(1+|y|)} < \frac{\epsilon + 2\epsilon}{(1+|x|)(1+|y|)} < 3\epsilon$$

   So we set $\epsilon = min\{\delta/3\}$, and we can set $3\epsilon \leq \delta$.

   (d) **Problem 4** Use Cauchy-Schwartz inequality to show that the Euclidean metric $d = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ satisfies a triangle inequality.

We want to show that: $\sqrt{\sum_{i=1}^n (x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2}$

$$\begin{aligned}
\sum_{i=1}^n (x_i - z_i)^2 &= \sum_{i=1}^n ((x_i - y_i) + (y_i - z_i))^2 \\
&= \sum_{i=1}^n (x_i - y_i)^2 + 2\sum_{i=1}^n (x_i - y_i)(y_i - z_i) + \sum_{i=1}^n (y_i - z_i)^2 \\
&\leq \sum_{i=1}^n (x_i - y_i)^2 + 2\sqrt{\sum_{i=1}^n (x_i - y_i)^2}\sqrt{\sum_{i=1}^n (y_i - z_i)^2 + \sum_{i=1}^n (y_i - z_i)^2} \\
&\text{(the middle terms are an inequality by CS inequality)} \\
&= \sum_{i=1}^n (x_i - y_i)^2 + \sum_{i=1}^n (y_i - z_i)^2
\end{aligned}$$

Taking square roots on both sides, we have

$$\sqrt{\sum_{i=1}^n (x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2}$$

2. **Week 3: 10 Mar 2024**

   (a) **Problem 1**: Show that two norms in $\mathbb{R}^n$, sup norm, $||x||_{sup} = max_{i \in \{1 \dots n\}} |x_i|$, and Euclidean norm, $||x||_E = \sqrt{\sum_{i \in \{1 \dots n\}} x_i^2}$, are Lipschitz equivalent. To this end, show that

   $$|| \cdot ||_{sup} \leq || \cdot ||_E \leq \sqrt{n} || \cdot ||_{sup}$$

   We first show that $|| \cdot ||_{sup} \leq || \cdot ||_E$. This can be shown by squaring both sides (inequalities are preserved since both norms are nonnegative):

   $$(|| \cdot ||_{sup})^2 = (max(x_i))^2 \leq \sum_{i \in \{1 \dots n\}} x_i^2 = (||x||_E)^2$$

   Now, we show that $|| \cdot ||_E \leq \sqrt{n} || \cdot ||_{sup}$. Again, we square both sides (inequalities are preserved since both norms are nonnegative):

   $$(||x||_E)^2 = \sum_{i \in \{1 \dots n\}} x_i^2 \leq n * (max(x_i))^2 = (\sqrt{n} || \cdot ||_{sup})^2$$

   Given the two inequalities above, both norms are Lipschitz-equivalent as there exist $(m, M) = (1, \sqrt{n})$ such that

   $$m || \cdot ||_{sup} \leq || \cdot ||_E \leq M || \cdot ||_{sup}$$

   (b) **Problem 2**: Prove that a union of any finite collection of bounded sets is bounded.

   A set S is bounded if $\exists \epsilon > 0 : S \in B_\epsilon(x) \subset S$.

   Consider the union of n different sets, i.e., $\cup_{i=1}^n S_i$, where each S_i is contained in a closed ball of centre x_i and radius ϵ_i.

   Define $r := max\{\epsilon_i + d(x_1, x_i) : i \in \{1, ..., n\}\}$. This is the maximum value of "the radius of any of the closed balls + the distance between the centre of the same closed ball and the centre of ball 1".

   Now choose a point a in any of the sets, i.e., in $B_{\epsilon_i}(x_i)$ for some i.

   By triangle inequality, the distance from the centre of ball 1 to point a is smaller than r, i.e., $d(x_1, a) \leq d(x_1, x_i) + d(x_i, a) \leq r$.

   Hence, the union is bounded by the closed ball of centre x_1 with radius r.

(c) **Problem 3**: Consider a sequence $\{1, 0, 1, 0, 1, 0\}$. Does it converge? Does it have a subsequence that converges? How many of them? What are the limits of these convergent subsequences?

The sequence is finite and converges to 0. For any $\epsilon > 0$, there exists an $N(\epsilon) \in \mathbb{N}$ such that $x_n \in B_\epsilon(x),\ \forall n > N(\epsilon)$.

We define $N(\epsilon) = 5$ and $x = 0$, and for all $n > 5, x_n \in B_\epsilon(0)$.

Subsequences $\{1, 0, 1, 0\}, \{1, 0\}$ etc. converge to 0; subsequences $\{1, 0, 1\}, \{1, 0, 1, 0, 1\}$ etc. converge to 1.

Note: If instead we are discussing infinite sequences, then there are subsequences that converge to 0 and 1; the full sequence does not converge.

(d) **Problem 4**: Construct an infinite sequence of real numbers, $X_i \in \mathbb{R}$, such that (i) $x_i < x_{i+1}$, (ii) $x_i < 5$ for all $i \in \mathbb{N}$ and (iii) it does not converge to 5.

Consider a sequence that converges to 4, e.g., $\{x_i\} = 4 - 1/i$.

The terms are: $\{3, 3 + 1/2, 3 + 2/3, 3 + 3/4\}$ etc.

3. **Week 4: 17 Mar 2024**

(a) **Problem 1:** Prove that a sequence $\{x_n\}$ is bounded (according to Definition 18) if and only if there exists $M \in \mathbb{R}_+$ such that $d(x_n, 0) < M$ for all $n \in \mathbb{N}$.

Definition 18: A sequence $\{x_n\}$ is bounded if there exists $M \in \mathbb{R}$ such that $d(x_n, x_m) < M$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

We first show that if Definition 18 is true, then there exists $M \in \mathbb{R}_+$ such that $d(x_n, 0) < M$ for all $n \in \mathbb{N}$.

By triangle inequality, $d(x_n, 0) \leq d(x_n, x_m) + d(x_m, 0)$.

Let $\epsilon = 1$ (i.e., the distance between terms x_n and x_m) and $m = N + 1$.

Combined with triangle inequality, this gives $d(x_n, 0) \leq d(x_n, x_m) + d(x_m, 0) < 1 + d(x_{N+1}, 0)$. This holds $\forall n > N$, and means that all terms after x_N are bounded.

For all terms up to $x_N, d(x_n, 0) \leq max\{d(x_0, 0), d(x_1, 0), ...d(x_N, 0)\}$. This means that all terms up to x_N are bounded.

Hence, $max\{d(x_n, 0) \leq max\{d(x_0, 0), d(x_1, 0), ...d(x_N, 0), 1 + d(x_{N+1}, 0)\}$ bounds all the terms of the sequence and the sequence is bounded.

Now, we show that if there exists $M \in \mathbb{R}_+$ such that $d(x_n, 0) < M$ for all $n \in \mathbb{N}$, then Definition 18 is true.

Again by triangle inequality, $d(x_n, x_m) \leq d(x_n, 0) + d(x_m, 0)$.

Since both terms on the RHS are bounded, the term on the LHS is bounded, and hence Definition 18 is true.

(b) **Problem 2:** Consider $\bar{\mathbb{R}} = \mathbb{R} \cup \infty \cup \{-\infty\}$ with $||x|| = \frac{|x|}{1+|x|}$ if $x \in \mathbb{R}$ and $||x|| = 1$ of $x = \infty$ or $x = -\infty$.

i. Define the correspondent distance $d(x, y)$.

ii. Prove that the sequence $\{\frac{1}{n}\}_{n=0}^{\infty}$ is bounded.

i. The distance is:

$$d(x,y) = \begin{cases} |\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}| & x, y, \in \mathbb{R} \\ |\frac{|x|}{1+|x|} - 1| & x \in \mathbb{R}, y = \pm\infty \\ |1 - \frac{|y|}{1+|y|}| & x = \pm\infty, y \in \mathbb{R} \\ 2 & x, y = \pm\infty, x \neq y \end{cases}$$

ii. To prove that the sequence $\{\frac{1}{n}\}_{n=0}^{\infty}$ is bounded, we can show that there exists $M \in \mathbb{R}_+$ such that $d(x_n, 0) < M$ for all $n \in \mathbb{N}$.

This is straightforward because all distances in this space are less than 3.

(c) **Problem 3:** Consider a sequence of functions $\{f_n\}_{n=1}^{\infty}, f_n = x^n$, in the space $C[0,1]$ of continuous functions defined on $[0,1]$.

In questions (a), (b), below, determine if the sequence converges in the given metric. If it does, provide a proof. If it does not, explain why.

(a) $d(f,g) = \sup_{x\in[0,1]}\{|f(x) - g(x)|\}$

This is the maximum distance between the function $\{x^n\}_{n=1}^{\infty}$ and its limit (also a function).

If we take the limit to be 0 (since $x \in [0,1]$), then the distance is $\sup|1^n - 0| = 1$ and hence the sequence does not converge.

(b) $d(f,g) = \int_0^1 |f(x) - g(x)|dx$

This is the "area under the curve" between the function and its limit (also a function).

If we take the limit to be 0, then the distance is $\int_0^1 |x^n - 0|dx = \frac{1}{n+1}$. From $[0,1)$, the sequence converges since for any ϵ we can find a N such that $\forall n > N, d(f_n, 0) < \epsilon$. However, the function must be equal to 0 arbitrarily close to 0, but equal to 1 at $x = 1$, hence the function that it converges to is not continuous and the sequence of functions does not converge to a continuous function in this metric.

4. **Week 5: 22 Mar 2024**

Problem 1

We have shown (a clean proof to be posted on Saturday) that $\{f_n(x)\}_{n=1}^{\infty}$ with $f_n(x) = x^n$ converges to $f(x) = 0$ in L^1 norm $d(f,g) = ||f(x) - g(x)|| = \int_0^1 |f(x) - g(x)|dx$.

We also noticed that the same argument would carry through if $\tilde{f}(0)$ is defined as

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

(We noticed that $\tilde{f}$ is not a continuous function, so we need to consider something more general than the space of all continuous functions on $[0,1]$, $C[0,1]$, but you can take whatever space is convenient.)

So, $\{f_n(x)\}$ converges both to $f(x)$ and $\tilde{f}(x)$. We, however, have Theorem 17 which says that there should be at most one limit.

Something's gotta give. Either our proof of convergence is wrong, or the proof of Theorem 17, or something else. What is wrong?

Problem 1: Under the L^1 norm, $d(f, \tilde{f}) = 0$, so this is not a distance, hence the sequence can converge to multiple limit points.

Problems 2-3

Consider the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ in $C[0,1]$ defined as

$$f_n(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2 \\ 1 - n\left(x - \frac{1}{2}\right), & \text{if } 1/2 \le x \le 1/2 + 1/n \\ 0, & \text{if } 1/2 + 1/n \le x \le 1 \end{cases}$$

Note that $f_n(x) \in C[0,1]$.

Problem 2

Show that for any ϵ there exists N such that for any $m, n > N$: $||f_{n+1} - f_m|| < \epsilon$.

Problem 3

Suppose that if $\{f_n\} \to_{n\to\infty} f$, then $f(x) = 1$ on $[0, 1/2]$ and $f(x) = 0$ on $(1/2, 1]$.

Problem 2: We want to show that at large enough N, f_{n+1} and f_m are so close together that for any $m, n > N : ||f_n - f_m|| < \epsilon$, where $||f_n||$ is the L^1 norm.

Recall that the L^1 norm is $||f||| := \int_0^1 ||f(t)||dt$. We can think of the area as the difference between two triangles, i.e., $(\frac{1}{2})(\frac{1}{2})(\frac{1}{m} - \frac{1}{n})$. Since we need this distance to be $< \epsilon$, we pick a N where $\frac{1}{N} > (\frac{1}{m} - \frac{1}{n})$. Then:

$$(\frac{1}{2})(\frac{1}{2})(\frac{1}{m} - \frac{1}{n}) < (\frac{1}{2})(\frac{1}{2})(\frac{1}{N}) < \epsilon$$

So we pick $N > \frac{1}{4\epsilon}$.

Problem 3: For the interval $[0, \frac{1}{2}]$, $f_n(x) = 1$, so this limit is trivial. The same is true for the interval $[1/2 + 1/n, 1]$ where $f_n(x) = 0$, so we only need to look at the interval $[1/2 \le x \le 1/2 + 1/n]$.

As $n \to \infty$, $1/2 + 1/n \to 1/2$, so the limit of the interval is $x \in [1/2, 1/2]$, or just the value $x = 1/2$. We already know that at $x = 1/2, f_n(x) = 1$. Right after the point $x = 1/2$, i.e., at $x = 1/2 + 1/n, f_n(x) = 0$. This holds for all values $x \in (1/2, 1]$ since we are taking the limit as $n \to \infty$.

So, the function converges to $f(x) = 1$ on $[0, 1/2]$ and $f(x) = 0$ on (1/2,1].

5. **Week 9: 29 Apr 2024**

Problem 1

The space $\mathbb{R}$ endowed with norm $||x|| = |x|$ is complete.
Show that a subset $(0, 1)$ of that space is not complete.

Problem 2

Consider a sequence $\{(1 + 1/n)^n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ endowed with the norm $||x|| = |x|$. This sequence converges in e in that space and, hence, is Cauchy.
Consider the space $\mathbb{Q}$ with the same norm. Prove that this sequence is Cauchy in that space.

Problem 3

Is the space $\mathbb{Q}$ with the norm defined in Problem 1 complete?

Problem 4

Define correspondence $\Psi(x) = [x, x + 1]$. Find $\{x \in \mathbb{R} : \Psi(x) \subseteq [3, 5]\}$ and $\{x \in \mathbb{R} : \Psi(x) \cup [3, 5] \neq \emptyset\}$

(a) **Problem 1:**

Consider the sequence $\{a_n\} = \frac{1}{n}$. This sequence is Cauchy but does not converge in $(0, 1)$, hence the space is not complete.

(b) **Problem 2:**

For the sequence to be Cauchy in $\mathbb{Q}, \forall\epsilon > 0, \exists N(\epsilon)$ such that $\forall m, n > N(\epsilon), |x_m - x_n| \leq \epsilon$.

Since both terms are in $\mathbb{Q}$, and their difference is in $\mathbb{Q}$, the sequence is Cauchy in this space as long as we can show that we can pick a $N(\epsilon)$ such that $|(1 + \frac{1}{n})^n - (1 + \frac{1}{m})^m| \leq \epsilon$.

We pick N such that $|(1 + \frac{1}{N})^N - (1 + \frac{1}{n})^n| \leq |(1 + \frac{1}{N})^N - e| \leq \epsilon, \forall n > N$.

Note: we should also include that the sequence is monotonic and not oscillating, otherwise the inequalities may not hold.

The main insight here is that whether the sequence is Cauchy or not depends on the metric and less on the space we are in.

(c) **Problem 3:**

No, $\mathbb{Q}$ is not complete with the absolute value norm. Consider the sequence $\{3, 3.1, 3.14, ...\}$ which converges to $\pi \notin \mathbb{Q}$.

Alternatively, we can say that e is not in $\mathbb{Q}$, so the sequence that converges to e converges to a limit that is not in the space, and hence there exists an ϵ-ball around the limit that does not contain elements of the sequence.

(d) **Problem 4:**

To find the set $\{x \in \mathbb{R} : \Psi(x) \subseteq [3, 5]\}$, we need to find all real numbers x such that the interval $[x, x + 1]$ is entirely contained within the interval $[3, 5]$. This means that both endpoints of $[x, x + 1]$ must be within $[3, 5]$. We have:

$x \geq 3$ (to ensure that x is within $[3, 5]$)
$x + 1 \leq 5$ (to ensure that $x + 1$ is within $[3, 5]$)
Combining the inequalities, we have $x \in [3, 4]$.

To find the set $\{x \in \mathbb{R} : \Psi(x) \cup [3, 5] \neq \emptyset\}$, we need to find all real numbers x such that the union of $[x, x + 1]$ and $[3, 5]$ is not an empty set. Since the union of two non-empty intervals is always non-empty, the answer is $\mathbb{R}$ (it would have been different if it was intersection).

6. **Week 10: 6 May 2024**

Problem 3

Consider the following coordination game $\Gamma(z)$

	l	r
L	(1,1)	(0,0)
R	(0,0)	(z, z)

(a) Find best response correspondence (BR) of this game for any $z \in \mathbb{R}$ (don't forget about mixed strategies).
(b) Fix z. What could be a metric on the space of strategies? How to re-write BR so that it is useable in the next problem?

For Thursday

Problem 4

Consider the game in Problem 3, and its BR.
(a) Fix z. Is BR upper/lower hemi-continuous?
(b) Is $BR(z)$ upper/lower hemi-continuous?

(a) **Problem 3:**

(a): Let the probability of L and R be α and $1 - \alpha$, and the probability of l and r be β and $1 - \beta$.

$$BR_1(\beta) = \begin{cases} z < 0 & \alpha = 1 & \forall \beta \\ z = 0 & \alpha = 1 & \forall \beta \\ z = 0 & \alpha \in [0, 1] & \beta = 0 \\ z > 0 & \alpha = 0 & \beta = 0 \\ & \alpha = 1 & \beta = 1 \\ & \alpha \in [0, 1] & \beta = \frac{z}{1+z} \end{cases}$$

XX

(b) **Problem 4:**

XX

2024 assignments

1. **2024 Assignment 1**

   (a) Let X consist of only two elements, x and y. Describe all metrics on X.

   A metric on X must satisfy the following properties:

   i. Non-negative function $d : X \times X \to \mathbb{R}_+$
   ii. $d(x, y) = 0$ iff $x = y$
   iii. $d(x, y) = d(y, x)$
   iv. $d(x, z) \leq d(x, y) + d(y, z)$

   To satisfy condition (b), it means we need some form of discrete metric, i.e.,:

$$d(x, y) = \begin{cases} a \neq 0 & x \neq y \\ 0 & x = y \end{cases}$$

   We also need a to be a non-negative real number to satisfy condition (a).

   To satisfy condition (c), we need $d(x, y) = d(y, x) = a$, which is satisfied with the discrete metric above.

   To satisfy condition (d), we need $d(x, y) \leq d(x, z) + d(y, z)$, which is satisfied with the discrete metric above (since the set X has only two elements, at least two of three of the elements in the triangle inequality have to be identical, and the discrete metric satisfies the triangle inequality).

   In summary, all metrics on X are some form of discrete metric, where the distance $d(x, y)$ is a non-negative real number a when $x \neq y$, and the distance is 0 when $x = y$.

   (b) Consider all infinite sequences of 0 and 1 with sup metric on that space. What is a simpler description of this metric?

   The sup metric simplifies to the discrete metric:

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

   That is, when the two sequences $\{x\}$ and $\{y\}$ are exactly identical, the distance $d(x, y)$ is 0. Otherwise, the distance is 1.

   (c) Let (X, d_1) and (Y, d_2) be metric spaces, and $(Z, d_\pi) = (X \times Y, d_\pi)$ be a product space with the metric d_π defined as

$$d_\pi(z, z') = d_\pi((x, y), (x', y')) = \sqrt{(d_1(x, x'))^2 + (d_2(y, y'))^2}$$

   Show that product metric d_π is a metric.

   The product metric must satisfy the following properties:

   i. Non-negative function $d : X \times X \to \mathbb{R}_+$
   ii. $d(x, y) = 0$ iff $x = y$
   iii. $d(x, y) = d(y, x)$
   iv. $d(x, z) \leq d(x, y) + d(y, z)$

   (a) is satisfied for the product metric is non-negative as it is taking the positive root.

   For (b), we can verify that if $(x, y) = (x', y')$, and d_1 and d_2 are both metrics, then the product metric is:

$$\sqrt{(d_1(x, x))^2 + (d_2(y, y))^2} = \sqrt{(0)^2 + (0)^2} = 0$$

Hence, (b) is satisfied for the product metric.

For (c), we can verify that $d_\pi((x,y),(x',y')) = d_\pi((x',y'),(x,y))$, as long as d_1 and d_2 are both metrics. Specifically:

$$\sqrt{(d_1(x,x'))^2 + (d_2(y,y'))^2} = \sqrt{(d_1(x',x))^2 + (d_2(y',y))^2}$$

For (d), the inequality to be proven is:

$$\sqrt{(d_1(x_1,x_3))^2 + (d_2(y_1,y_3))^2} \leq \sqrt{(d_1(x_1,x_2))^2 + (d_2(y_1,y_2))^2} + \sqrt{(d_1(x_1,x_2))^2 + (d_2(y_1,y_2))^2} \tag{A}$$

Since all the terms ≥ 0, we square both sides to obtain:

$$\begin{aligned}(d_1(x_1,x_3))^2 + (d_2(y_1,y_3))^2 \leq{}& (d_1(x_1,x_2))^2 + (d_2(y_1,y_2))^2 + (d_1(x_2,x_3))^2 + (d_2(y_2,y_3))^2 \\ &+ 2\sqrt{(d_1(x_1,x_2))^2 + (d_2(y_1,y_2))^2}\sqrt{(d_1(x_1,x_2))^2 + (d_2(y_1,y_2))^2}\end{aligned} \tag{B}$$

By the triangle inequality for d_1, we know that:

$$(d_1(x_1,x_3))^2 \leq (d_1(x_1,x_2))^2 + (d_1(x_2,x_3))^2 \tag{C}$$

Similarly, by the triangle inequality for d_2, we know that:

$$(d_2(y_1,y_3))^2 \leq (d_2(y_1,y_2))^2 + (d_2(y_2,y_3))^2 \tag{D}$$

Given that (C) and (D) hold, and the remaining term in (B), which is 2 multiplied by two positive roots, is ≥ 0, the inequality in (B) holds and thus the inequality in (A) holds.

This means that triangle inequality is satisfied for the product metric and thus we have shown that the product metric is a metric.

Triangle inequality is a real challenge here. We need to show that

$$\sqrt{d_1(x,x'')^2 + d_2(y,y'')^2} \leq \sqrt{d_1(x,x')^2 + d_2(y,y')^2} + \sqrt{d_1(x',x'')^2 + d_2(y',y'')^2} \tag{1}$$

We can square both sides (they are positive) to get

$$d_1(x,x'')^2 + d_2(y,y'')^2 \leq \tag{2}$$

$$d_1(x,x')^2 + d_2(y,y')^2 + d_1(x',x'')^2 + d_2(y',y'')^2 + 2\sqrt{d_1(x,x')^2 + d_2(y,y')^2}\sqrt{d_1(x',x'')^2 + d_2(y',y'')^2} \tag{3}$$

From triangle inequalities of each individual metric, we know that $d_1(x,x'') \leq d_1(x,x') + d(x',x'')$, thus $d_1(x,x'')^2 \leq d_1(x,x')^2 + d(x',x'')^2 + 2d_1(x,x')d_1(x',x'')$. Unfortunately, it does not imply $d_1(x,x'')^2 \leq d_1(x,x')^2 + d(x',x'')^2$, which is what we need. Using the inequality above for d_1 and similar for d_2, cancelling out terms, and dividing by two, we have

$$d_1(x,x')d_1(x',x'') + d_2(y,y')d_2(y',y'') \leq \sqrt{d_1(x,x')^2 + d_2(y,y')^2}\sqrt{d_1(x',x'')^2 + d_2(y',y'')^2} \tag{4}$$

Let me square both sides (they are still positive) to arrive to the standard statement of C-S:

$$(d_1(x,x')d_1(x',x'') + d_2(y,y')d_2(y',y''))^2 \leq (d_1(x,x')^2 + d_2(y,y')^2)(d_1(x',x'')^2 + d_2(y',y'')^2) \tag{5}$$

This completes the proof.

(d) Let (X, d_1) and (Y, d_2) be metric spaces, and (Z, d_π) be a product space with the metric d_π defined in the previous problem.
Show that the sequence $\{z_n\} = \{(x_n, y_n\}$ converges to $z = (x, y)$ in (Z, d_π) iff $\{x_n\}$ converges to x in (X, d_1) and $\{y_n\}$ converges to y in (Y, d_2).

First, we show that if $\{x_n\}$ converges to x in (X, d_1) and $\{y_n\}$ converges to y in (Y, d_2), then $\{z_n\} = \{(x_n, y_n\}$ converges to $z = (x, y)$ in (Z, d_π).

For reference, the metric $d_\pi(z, z')$ is defined as:

$$d_\pi(z, z') = d_\pi((x, y), (x', y')) = \sqrt{(d_1(x, x'))^2 + (d_2(y, y'))^2}$$

Given that $\{x_n\}$ converges to x, this means that $\forall \epsilon_x > 0$, there must exist $n_1 \in \mathbb{N}$ such that

$$d(x_n, x) < \epsilon_x, \forall n \in \mathbb{N}, n > n_1$$

Similarly, given that $\{y_n\}$ converges to y, this means that $\forall \epsilon_y > 0$, there must exist $n_2 \in \mathbb{N}$ such that

$$d(y_n, y) < \epsilon_y, \forall n \in \mathbb{N}, n > n_2$$

Hence, for all $n > \max\{n_1, n_2\}$, and defining z as the limit of $\{z_n\}$, we have:

$$d_\pi(z_n, z) \leq \sqrt{(\epsilon_x)^2 + (\epsilon_y)^2}, \forall n \in \mathbb{N}, n > \max\{n_1, n_2\}$$

Since this holds for arbitrarily small ϵ_x and ϵ_y, we can set $\epsilon_z = \sqrt{(\epsilon_x)^2 + (\epsilon_y)^2}$ to get:

$$d_\pi(z_n, z) \leq \sqrt{(\epsilon_x)^2 + (\epsilon_y)^2} = \epsilon_z$$

In summary, $\forall \epsilon_z, \exists n_1, n_2 \in \mathbb{N}$ such that $d_\pi(z_n, z) \leq (\epsilon_z), \forall n \in \mathbb{N}, n > \max\{n_1, n_2\}$. Hence, $\{z_n\}$ converges to z.

Now, we show that if $\{z_n\} = \{(x_n, y_n\}$ converges to $z = (x, y)$ in (Z, d_π), then $\{x_n\}$ converges to x in (X, d_1) and $\{y_n\}$ converges to y in (Y, d_2).

We prove by contrapositive, i.e., we show that if either $\{x_n\}$ does not converge to x or $\{y_n\}$ does not converge to y, $\{z_n\}$ does not converge to (x, y).

Recall that the metric $d_\pi(z, z')$ is defined as:

$$d_\pi(z, z') = d_\pi((x, y), (x', y')) = \sqrt{(d_1(x, x'))^2 + (d_2(y, y'))^2}$$

Assume without loss of generality that $\{x_n\}$ is the sequence that does not converge to any x. In this case, $\forall x \in X, \exists \epsilon_x > 0, \forall N \in \mathbb{N}, \exists n \geq N$ such that $|x_n - x| \geq \epsilon_x$.

In this case, assuming that $\{y_n\}$ converges to y, the infimum distance between $\{z_n\}$ and any potential limit point z is

$$\sqrt{(\epsilon_x)^2} = \epsilon_x$$

We set $\epsilon_x = 2\epsilon_z$ and can show that $\{z_n\}$ does not converge to any z:

$$\forall z \in Z, \exists \epsilon_z > 0, \forall N \in \mathbb{N}, \exists n \geq N \text{ such that } |z_n - z| \geq \epsilon_x = 2\epsilon_z$$

Question 4

4 — 0.9 / 1 pt

✔ **– 0.1 pts** See annotation

5 This is not exactly right. You set ϵ_z based on ϵ_x and ϵ_y, so you cannot claim "$\forall \epsilon_z$. But this is easily fixable; you can either introduce ϵ_z from the begining and select ϵ_x and ϵ_y accordingly (if you are writing a paper), or have what you have, and then say that for any ϵ_z I select ϵ_x and ϵ_y in such and such way. It is also counter-productive to select different ϵ_x and ϵ_y.

Note: This also holds if both $\{x_n\}$ and $\{y_n\}$ do not converge, in which case the infimum distance between $\{z_n\}$ and any potential limit point z is $\epsilon_x + \epsilon_y$, which is even larger.

(e) Prove a generalised triangle inequality:

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{n-1}, x_n)$$

We prove by induction, starting from the triangle inequality where $n = 3$.

$$d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$$

The $n = 3$ case is true by definition of the triangle inequality for distance.

Now, we show that if the triangle inequality is true for the case where $n = k$, it is true for the case where $n = k + 1$.

The $n = k$ case is:

$$d(x_1, x_k) \leq d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{k-1}, x_k)$$

We assume that this is true to consider the $n = k + 1$ case.

The $n = k + 1$ case is:

$$d(x_1, x_{k+1}) \leq d(x_1, x_2) + ... + d(x_{k-1}, x_k) + d(x_k, x_{k+1})$$

We consider two scenarios: $d(x_1, x_k) < d(x_1, x_{k+1})$ and $d(x_1, x_k) \geq d(x_1, x_{k+1})$.

In the first scenario, we use the standard definition of the triangle inequality to consider distances between x_1 and x_{k+1}:

$$\begin{aligned} d(x_1, x_{k+1}) &\leq d(x_1, x_k) + d(x_k, x_{k+1}) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{k-1}, x_k) + d(x_k, x_{k+1}) \end{aligned}$$

And hence for this scenario, if the triangle inequality is true for the case where $n = k$, it is true for the case where $n = k + 1$.

In the second scenario,

$$\begin{aligned} d(x_1, x_{k+1}) &\leq d(x_1, x_k) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{k-1}, x_k) \\ &\leq d(x_1, x_2) + ... + d(x_{k-1}, x_k) + d(x_k, x_{k+1}) \end{aligned}$$

And hence for this scenario, if the triangle inequality is true for the case where $n = k$, it is true for the case where $n = k + 1$.

Overall, since the $n = 3$ case is true, and if the triangle inequality is true for the case where $n = k$, it is true for the case where $n = k + 1$, the generalised triangle inequality is true for cases where $n \geq 3$.

Note: the triangle inequality is true (with equality) for the case where $n = 1$, in which case all the distances are 0, and also true (with equality) in the case where $n = 2$, in which case at least two of the points x_1, x_2, x_3 are identical.

(f) Let $\{x_n\}$ be a sequence of positive real numbers. Show that $\{x_n\} \to \infty$ iff $\{\frac{1}{x_n}\} \to 0$.

First, we show that $\{x_n\} \to \infty$ implies $\{\frac{1}{x_n}\} \to 0$.

Since $\{x_n\} \to \infty$, for $\epsilon > 0$, there exists a $N(\epsilon) \in \mathbb{N}$ such that $x_n > \frac{1}{\epsilon}$ for all $n > N(\epsilon)$. Taking reciprocals, we have $0 < \frac{1}{x_n} < \frac{1}{\epsilon}$ for all $n > N(\epsilon)$ and hence $\{\frac{1}{x_n}\} \to 0$ since we can choose any arbitrarily small number for ϵ.

Second, we show that $\{\frac{1}{x_n}\} \to 0$ implies $\{x_n\} \to \infty$.

Since $\{\frac{1}{x_n}\} \to 0$, this means that for any small $\epsilon > 0$, there exists a $N(\epsilon) \in \mathbb{N}$ such that $\frac{1}{x_n} < \epsilon$ for all $n > N(\epsilon)$. Taking reciprocals, we have $x_n > \frac{1}{\epsilon}$ for all $n > N(\epsilon)$ and hence $\{x_n\} \to \infty$ since we can choose any arbitrarily large number for $\frac{1}{\epsilon}$.

(g) Consider the space of all m-dimensional vectors $\mathbb{R}^m$ with the Euclidean metric; de la Fuente calls it $\mathbb{E}^m$. Show that every bounded sequence in $\mathbb{E}^m$ contains at least one convergent subsequence.

Since m is finite, we break the proof up into m steps. For each step i, we will look only at the i^{th} coordinate of each term of the sequence to find a sequence convergent in the i^{th} coordinate.

i. **Step 1:** We take the first coordinate of each term in the sequence to form the sequence $\{x_n\}$, which is a sequence in $\mathbb{R}^1$. We define j as a peak of the sequence such that all subsequent terms are of smaller value, i.e., $k > j \implies \{x_k\} < \{x_j\}$.

Case 1: Suppose $\{x_n\}$ has an infinite number of peaks, at term number $p_1, p_2, p_3, \ldots$, i.e.:

$$p_1 < p_2 < p_3 < \ldots \qquad \text{where } x_{p_1} > x_{p_2} > x_{p_3} \ldots$$

In this case, if we consider the subsequence $\{x_{p_n}\}$, it must be strictly decreasing since for each peak, all subsequent terms are of smaller value, i.e.,

$$p_j > p_k \implies x_{p_j} < x_{p_k}$$

Since the subsequence $\{x_{p_n}\}$ is bounded (given that it is a subsequence of $\{x_n\}$ which is bounded), and it is strictly decreasing, it must converge to its infimum.

Case 2: Suppose $\{x_n\}$ has an finite number of peaks and let $N \in \mathbb{N}$ be the term number of the last peak. Then, the term number $q_1 = N + 1$ is not a peak and so there exists a subsequence with term numbers $q_1, q_2, q_3, \ldots$, where:

$$q_1 < q_2 < q_3 < \ldots \qquad \text{where } x_{q_1} < x_{q_2} < x_{q_3} \ldots$$

In this case, if we consider the subsequence $\{x_{q_n}\}$, it must be strictly increasing since after each term, we can find a term later in the sequence that is of larger value, since there are no further peaks after q_1.

Since the subsequence $\{x_{q_n}\}$ is bounded (given that it is a subsequence of $\{x_n\}$ which is bounded), and it is strictly increasing, it must converge to its supremum.

Overall, since either Case 1 or Case 2 must be true, $\{x_n\}$ contains at least one convergent subsequence.

ii. **Step 2:** We now disregard all the terms from the original sequence in $\mathbb{R}^m$, apart from the terms whose 1st coordinate was part of the convergent sequence in **Step 1**. We then repeat **Step 1** but with the 2nd coordinate of each term in the sequence, to derive a sequence that is convergent in both the 1st and 2nd coordinates.

iii. **Step 3:** We now disregard all the terms from the original sequence in $\mathbb{R}^m$, apart from the terms whose 1st and 2nd coordinate was part of the convergent sequence in **Step 2**. We then repeat **Step 1** but with the 3rd coordinate of each term in the sequence, to derive a sequence that is convergent in the 1st, 2nd and 3rd coordinates.

iv. We repeat the process m times until we have a sequence that is convergent in m coordinates. We have found a convergent subsequence in $\mathbb{R}^m$.

(h) Construct a sequence of real numbers $\{x_n\}, x_n \in R$ such that

i. it does not have any cluster points; and

Consider the sequence $\{x_n\} = n$, which is $(1, 2, 3, \ldots)$. It has no cluster points.

ii. is unbounded and has one cluster point.

Consider the sequence

$$x_n = \begin{cases} n & \text{if n is odd} \\ 0 & \text{otherwise} \end{cases}$$

The sequence is unbounded and has a cluster point around 0.

(i) Let $\{f_n\}_{n=1}^{\infty}$ with $f_n = \frac{1}{n}\frac{nx}{1+nx}$ for $0 \leq x \leq 1$. Show that $f_n \to 0$ in sup-norm L^∞ on $C[0,1]$.

To show that $f_n \to 0$, we need to show that $||f_n - 0||_\infty \to 0$ as $n \to \infty$, where $||f_n||_\infty$ is the sup norm for f_n.

$||f_n - 0||_\infty$ on $C[0,1]$ is $\sup_{x\in[0,1]} \frac{x}{1+nx}$ which is maximised at $x = 1$, i.e., $\sup_{x\in[0,1]} \frac{x}{1+nx} = \frac{1}{1+n}$

As $n \to \infty$, $\sup_{x\in[0,1]} \frac{x}{1+nx} = \frac{1}{1+n} \to 0$, so $f_n \to 0$ in sup-norm L^∞ on $C[0,1]$.

(j) Let $B[0,1]$ be the set of all bounded functions on $[0,1]$. Consider the sup-norm L^∞. Show that $C[0,1]$, the set of all continuous functions, is a closed subset of $B[0,1]$.

First, we show that $C[0,1]$, the set of all continuous functions, is a subset of $B[0,1]$. Specifically, we will show that all continuous functions on $[0,1]$ are bounded on $[0,1]$ with the sup norm. We will prove this by contradiction.

Suppose there exists a function $f(x) \in C$ which is unbounded in the the closed interval $[0,1]$. This means that we cannot find a real number M such that $||f(x)||_\infty < M$, where $||f(x)||_\infty$ is the sup norm for $f(x)$.

This contradicts the Extreme Value Theorem, where a continuous function on the closed interval $[0,1]$ must attain a maximum and minimum, i.e., either $||f(x)||_\infty \leq ||f(x_{max})||_\infty$ or $||f(x)||_\infty \leq ||f(x_{min})||_\infty, \forall x \in [0,1]$.

Hence, since assuming that there exists an unbounded $f(x) \in C$ leads to a contradiction, it must be that $C[0,1]$, the set of all continuous functions, is a subset of $B[0,1]$.

Now, we show that $C[0,1]$, is closed. To do this, we need to show that it contains all its limit points, i.e., the limit of any sequence of continuous functions in $[0,1]$ is also a continuous function in $[0,1]$.

Let $\{f_n\}$ be a sequence of continuous functions in $[0,1]$ that converges to a limit f in $[0,1]$. This means that $\forall \epsilon_x > 0, \exists N$ such that $\forall n > N, x \in [0,1], ||f_n(x) - f(x)||_\infty < \epsilon_x$.

Furthermore, since $f_n(x)$ is continuous on $[0,1]$, it must be that $\forall \epsilon_y > 0, \exists \delta_n > 0$ such that $\forall x \in [0,1], |x-y| < \delta_n, ||f_n(x) - f_n(y)||_\infty < \epsilon_y$.

Since the above holds for any arbitrary ϵ_x and ϵ_y, we set $\epsilon_x = \epsilon_y = \frac{\epsilon}{4}$. Then, by the triangle inequality, $\forall n > N, \forall x, y \in [0,1]$ with $|x - y| < \min\{\delta_1, \delta_2, ...\delta_N\}$, we have:

$$||f(x) - f(y)||_\infty \leq ||f(x) - f_n(x)||_\infty + ||f_n(x) - f_n(y)||_\infty + ||f(y) - f_n(y)||_\infty < \frac{3\epsilon}{4} < \epsilon$$

Hence the limit of any sequence of continuous functions in $[0,1]$ is also continuous, i.e., $f \in C[0,1]$ and $C[0,1]$ is closed.

2. **2024 Assignment 2**

(a) Let $f : X \to Y$ and $g : Y \to Z$ be two continuous functions. Show that $h = g \circ f : X \to Z$ is a continuous function on X.

We use Theorem 46, i.e., $f : X \to Y$ is continuous on X iff $f^-1(C)$ is open for any open subset $C \in Y$, and equivalently $f^-1(C)$ is closed for any closed subset $C \in Y$.

Without loss of generality, we show that for every open set V in Z, $h^{-1}(V)$ is open in X.

Since g is continuous, $g^{-1}(V)$ is open in Y for every open set V in Z. Similarly, since f is continuous, $f^{-1}(g^{-1}(V))$ is open in X for every open set V in Z.

Since $h^{-1}(V) = f^{-1}(g^{-1}(V))$, and we have shown that this is open in X for every open set V in Z, it follows by the definition of continuity that h is a continuous function on X.

(b) In problems 2-4 we will study the following object:

$$S(\omega) = \{x \in [l(\omega), h(\omega)] | f(x, \omega) \geq f(x', \omega) \forall x' \in [l(\omega), h(\omega)]\}$$

where $f : [l(\omega), h(\omega)] \to \mathbb{R}, l : \Omega \to\to \mathbb{R}, h : \Omega \to\to \mathbb{R}$ are continuous functions.
Show that, for a given ω, $S(\omega)$ is non-empty.

We need to show that, for a given ω, there exists at least one $x \in [l(\omega), h(\omega)]$ such that $f(x, \omega) \geq f(x', \omega) \forall x' \in [l(\omega), h(\omega)]$.

Since f is continuous and $[l(\omega), h(\omega)]$ is closed and bounded, by the Extreme Value Theorem, there exists x^* such that $f(x', \omega) \leq f(x^*, \omega)\ \forall x' \in [l(\omega), h(\omega)]$. This implies that for a given ω there is at least one element in $S(\omega)$, which is x^*.

(c) Show that, for a given ω, $S(\omega)$ is a closed set.

We use Theorem 36, i.e., $S(\omega)$ is a closed set iff it contains all its limit points.

Let x_n be a sequence in $S(\omega)$ converging to a limit x.

Since x_n converges to x, we have $\lim_{n \to \infty} f(x_n, \omega) = f(x, \omega)$ since f is continuous.

Since $x_n \in S(\omega),\ \forall n = (1, 2, ...)$, we have $f(x_n, \omega) \geq f(x', \omega)$ for all $x' \in [l(\omega), h(\omega)]$.

Taking the limit as n approaches infinity, we have $f(x, \omega) \geq f(x', \omega)$ for all $x' \in [l(\omega), h(\omega)]$.

Hence $x \in S(\omega)$, which implies that $S(\omega)$ contains all its limit points and is closed.

(d) Consider a sequence $\{\omega_n\} \in \Omega$ such that $\{\omega_n\} \to \omega_0$ and a sequence $\{x_n\} \in S(\omega_n) \subseteq [l(\omega_n, h(\omega_n)]$ such that $\{x_n\} \to x_0$. Show that $x_0 \in S(\omega_0)$.

Since $\{x_n\} \in S(\omega_n)\ \forall n$, it means that $f(x_n, \omega_n) \geq f(x', \omega_n)\ \forall x' \in [l(\omega_n, h(\omega_n)]$, for all n.

Since f is continuous and $\{\omega_n\} \to \omega_0$ and $\{x_n\} \to x_0$, it means that at the limit, $f(x_0, \omega_0) \geq f(x', \omega_0)\ \forall x' \in [l(\omega_n, h(\omega_n)]$, and hence $x_0 \in S(\omega_0)$.

(e) A budget correspondence is defined as $B(p, Y) = \{x \in \mathbb{R}^2_+ : x_1 + px_2 \leq Y\}$.
Prove that $B(p, Y)$ is upper hemicontinuous.

Consider an arbitrary point (p^0, Y^0) in (p, Y).

Let U be an open ball $Ball_\delta(p^0, Y^0)$ in $\mathbb{R}^2_+$. This means that for any $(p', Y') \in U$, we have $d((p', Y'), (p^0, Y^0)) < \delta$.

Now, consider an arbitrary open set V containing $B(p^0, Y^0)$. This is an open ϵ-ball around $B(p^0, Y^0) = \{x \in \mathbb{R}^2_+ : x_1 + p^0 x_2 \leq Y^0\}$, i.e., an ϵ-band around the line $x_1 + p^0 x_2 = Y^0$, but restricting $x \in \mathbb{R}^2_+$.

Since $x_1 + px_2 = Y$ is continuous, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $(p', Y') \in U$ with $d((p', Y'), (p^0, Y^0)) < \delta$, we have $B(p', Y') \in V$.

Therefore for any arbitrary $(p^0, Y^0) \in (p, Y)$, for every open set V containing $B(p^0, Y^0)$, there exists an open set U containing (p^0, Y^0) such that $B(p', Y') \subset V$ for every $(p', Y') \in U$. Hence, $B(p, Y)$ is upper hemicontinuous.

(f) Prove that $B(p, Y)$ is lower hemicontinuous.

Consider an arbitrary point (p^0, Y^0) in (p, Y).

Let U be an open ball $Ball_\delta(p^0, Y^0)$ in $\mathbb{R}^2_+$. This means that for any $(p', Y') \in U$, we have $d((p', Y'), (p^0, Y^0)) < \delta$.

Now, consider an arbitrary open set V that intersects $B(p^0, Y^0)$. This is an open ϵ-ball that intersects $B(p^0, Y^0) = \{x \in \mathbb{R}^2_+ : x_1 + p^0 x_2 \leq Y^0\}$, i.e., that intersects the line $x_1 + px_2 = Y$, but restricting $x \in \mathbb{R}^2_+$.

Since $x_1 + px_2 = Y$ is continuous, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $(p', Y') \in U$ with $d((p', Y'), (p^0, Y^0)) < \delta$, the set $B(p', Y')$ intersects V.

Therefore for any arbitrary $(p^0, Y^0) \in (p, Y)$, for every open set V that intersects $B(p^0, Y^0)$, there exists an open set U containing (p^0, Y^0) such that $B(p', Y')$ has a non-zero intersection with V for every $(p', Y') \in U$. Hence, $B(p, Y)$ is lower hemicontinuous.

(g) Consider $\Psi(x) = [x, x+1]$. Show that this correspondence is lower hemicontinuous.

Consider an arbitrary point $x^0 \in X$, and an arbitrary open set V such that $\Psi(x_0) \cap V \neq \emptyset$. In other words, $[x^0, x^0 + 1] \cap V \neq \emptyset$.

Now, let U be an open set containing x^0. Since U is open and contains x^0, there exists $\epsilon > 0$ such that $(x^0 - \epsilon, x^0 + \epsilon) \subset U$.

Consider any $x \in (x^0 - \epsilon, x^0 + \epsilon)$. Then, $x \in U$, and since $\Psi(x) = [x, x+1]$, we have $[x, x+1] \cap V \neq \emptyset$. This implies that $\Psi(U) \cap V \neq \emptyset$.

Therefore, for every open set V such that $\Psi(x^0) \cap V \neq \emptyset$, there exists an open set U containing x^0 such that $\Psi(x^0) \cap V \neq \emptyset$ for every $x \in U$. Hence, the correspondence Ψ is lower hemicontinuous.

(h) Let $f : X \to X$ be a contraction. Show that f is uniformly continuous.

If f is a contraction, then for any $x, y \in X$, we have $d(f(x), f(y)) \leq \beta d(x, y),\ \beta \in (0, 1)$.

We want to show that for any $\epsilon > 0, \exists \delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$ for all $x \in X$.

Pick an arbitrary $\epsilon > 0$, and choose $\delta = \frac{\epsilon}{\beta}$.

Now, consider any $x, y \in X$ such that $d(x, y) < \delta$. We want to show that $d(f(x), f(y)) < \epsilon$.

Since $d(x, y) < \delta$, we have $\beta d(x, y) < \beta\delta = \epsilon$ and hence $f(B_\delta(x)) \subset B_\epsilon(f(x))$ for all $x \in X$, and thus f is uniformly continuous.

(i) Show that a set X is convex iff every convex combination of X lies in X.

We first show that X is convex $\implies$ every convex combination of X lies in X.
Since X is convex, given any two points x' and x'' in X, the point

$$x^\lambda = (1 - \lambda)x' + \lambda x''$$

is also in X for every $\lambda \in [0, 1]$.
Now we consider an arbitrary convex combination of X. This can be described as

$$\sum_{i=1}^{n} \lambda_i x_i$$

where $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.
Since X is convex, each $x_i \in X$, and thus $\sum_{i=1}^{n} \lambda_i x_i \in X$. Therefore, if X is convex, every convex combination of points in X lies in X.

We now show that every convex combination of X lies in $X \implies X$ is convex. We prove by contradiction.
Suppose X is not convex. Then, there exist points $x', x'' \in X$ such that

$$x^{\lambda} = (1 - \lambda)x' + \lambda x'' \notin X$$

However, this is a contradiction because x^{λ} is a convex combination of x' and x'' and we assumed that every convex combination of X lies in X. Therefore, if every convex combination of X lies in X, then X is convex.

(j) Show that the closure of a convex set is convex.

Let A be a convex set. We want to show that $cl(A)$, the closure of A, is convex.

Let x^0, x^1 be arbitrary points in $cl(A)$. By the definition of closure, $\forall \epsilon > 0$, $\exists a^0, a^1 \in A$ such that $d(x^0 - a^0) < \epsilon$ and $d(x^1 - a^1) < \epsilon$.

Since A is convex, for any $\lambda \in [0, 1]$, the point $\lambda a^0 + (1 - \lambda)a^1$ is also in A.

Consider the point $\lambda x^0 + (1 - \lambda)x^1, \lambda \in [0, 1]$. Since $d(x^0 - a^0) < \epsilon$ and $d(x^1 - a^1) < \epsilon$, we have:

$$\begin{aligned} d(\lambda x^0 + (1 - \lambda)x^1 - (\lambda a^0 + (1 - \lambda)a^1)) &\leq d(\lambda(x^0 - a^0) + (1 - \lambda)(x^1 - a^1)) \\ &\leq \lambda(\epsilon) + (1 - \lambda)(\epsilon) \\ &= \epsilon \end{aligned}$$

Since this holds for any arbitrarily small $\epsilon > 0$, every convex combination of points in $cl(A)$ is arbitrarily close to a point in A for all $\lambda \in [0, 1]$. Therefore, $\lambda x^0 + (1 - \lambda)x^1 \in cl(A)$ and hence $cl(A)$ is convex.