Macro 1 notes

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1. Lecture 1

   (a) Summary of Kuhn-Tucker conditions:

   For a minimisation (maximisation) problem, write all inequality constraints in the form:

   $$f_i(x) \leq 0 \qquad (f_i(x) \geq 0)$$

   Write the Lagrange function as the sum of the objective function and the weighted constraints. The partial derivatives of the Lagrange function:

   (i) with respect to the nonnegative variables are nonnegative (nonpositive) for a minimisation (maximisation) problem and the complementary slackness condition

   $$x\frac{\partial \phi}{\partial x} = 0$$

   is fulfilled;

   (ii) with respect to the free variables are equal to 0;

   (iii) with respect to the Lagrange multipliers corresponding to the inequality constraints are nonpositive (nonnegative) for a minimisation (maximisation) problem and the complementary slackness condition

   $$u\frac{\partial \phi}{\partial u} = 0$$

   is fulfilled;

   (iv) with respect to the Lagrange multipliers corresponding to the equality constraints are equal to 0.

   (b) Example: Minimise $x_1^2 - 4x_1 + x_2^2 - 6x_2$ subject to $x_1 + x_2 \leq 3$ and $-2x_1 + x_2 \leq 2$.

   $$\begin{aligned}
   \mathcal{L}(x_1, x_2, \lambda, \phi) &= x_1^2 - 4x_1 + x_2^2 - 6x_2 + \lambda(x_1 + x_2 - 3) + \phi(-2x_1 + x_2 - 2) \\
   \frac{\partial \mathcal{L}}{\partial x_1} &= 2x_1 - 4 + \lambda - 2\phi = 0 \\
   \frac{\partial \mathcal{L}}{\partial x_2} &= 2x_2 - 6 + \lambda + \phi = 0 \\
   \frac{\partial \mathcal{L}}{\partial \lambda} &= x_1 + x_2 - 3 \leq 0 \\
   \frac{\partial \mathcal{L}}{\partial \phi} &= -2x_1 + x_2 - 2 \leq 0 \\
   \lambda\frac{\partial \mathcal{L}}{\partial \lambda} &= \lambda(x_1 + x_2 - 3) = 0 \\
   \phi\frac{\partial \mathcal{L}}{\partial \phi} &= \phi(-2x_1 + x_2 - 2) = 0 \\
   \lambda &\geq 0 \\
   \phi &\geq 0
   \end{aligned}$$

   (c) Cake-eating problem (SP):

   Cake size is W_1. At each period $t = 1, 2, ..T$, consume c_t. Flow of utility is $u(c_t)$.

$u(.)$ is real valued, differentiable, strictly increasing, strictly concave, and satisfies the Inada condition $lim_{c\to 0}u'(c) = \inf$.

Lifetime utility is $\sum_{t=1}^{T}\beta^{t-1}u(c_t)$, $\beta \in (0,1)$.

Cake does not depreciate or grow, $W_t + 1 = W_t - c_t$ for $t = 1, 2, ...T$ (these are T different flow constraints).

Direct attack: Solve the Sequence Problem (SP):

$$\max_{\{c_t\}_{t=1}^{T},\{W_{t+1}\}_{t=1}^{T}} \sum_{t=1}^{T}\beta^{t-1}u(c_t)$$

subject to $c_t \geq 0, W_t \geq 0$, and given W_1.

We combine the flow constraints to the single resource constraint:

$$\sum_{t=1}^{T} c_t + W_{T+1} = W_1$$

And we rewrite the SP:

$$\begin{aligned}
\text{(SP)} \qquad & \max_{\{c_t\}_{t=1}^{T},\{W_{t+1}\}_{t=1}^{T}} \sum_{t=1}^{T}\beta^{t-1}u(c_t) \\
\text{s.t.} \qquad & \sum_{t=1}^{T} c_t + W_{T+1} = W_1 \\
& W_{T+1} \geq 0 \qquad \text{(no-Ponzi-game condition)} \\
& c_t \geq 0, \forall t, W_1 \text{ is given.}
\end{aligned}$$

The Lagrangian and Kuhn-Tucker conditions are:

$$\mathcal{L} = \sum_{t=1}^{T}\beta^{t-1}u(c_t) + \lambda(W_1 - \sum_{t=1}^{T} c_t - W_{T+1}) + \phi(W_{T+1}) \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^{t-1}u'(c_t) - \lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial W_{T+1}} = -\lambda + \phi = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = W_1 - \sum_{t=1}^{T} c_t - W_{T+1} = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = W_{T+1} \geq 0 \tag{5}$$

$$\phi\frac{\partial \mathcal{L}}{\partial \phi} = \phi W_{T+1} = 0 \tag{6}$$

Since (4) holds for all $t = 1, 2, ...T$:

$$\begin{aligned}
\lambda = u'(c_1) = \beta u'(c_2) = ... = \beta^{T-1}u'(c_T) \\
\implies u'(c_t) = \beta u'(c_{t+1}), t = 1, ...T-1
\end{aligned}$$

From (5), $\phi = \lambda = \beta^{t-1}u'(c_t) > 0$, then (6) implies $W_{T+1} = 0$.

In summary, the optimal consumption path $\{c_t^*\}_{t=1}^{T}$ is determined by:
$(T-1)$ Euler equations $u'(c_t) = \beta u'(c_{t+1}), t = 1, ...T-1$, and
the resource constraint $\sum_{t=1}^{T} c_t = W_1$, since $W_{T+1} = 0$

Define the value function:

$$V_T(W_1) = \sum_{t=1}^{T} \beta^{t-1} u(c_t^*)$$

Where $V_T(W_1)$ represents the maximum lifetime utility from a T-period problem given a cake of initial size W_1.

By the Envelope theorem,

$$V_T'(W_1) = \frac{\partial \mathcal{L}}{\partial W_1} = \lambda = \beta^{t-1} u'(c_t), t = 1, 2, .., T$$
$$V_T'(W_1) > 0 \text{ since } \lambda > 0$$

(d) Cake-eating problem (FE):

Using the value function from a T-period problem, the T+1-period problem can be formed as:

$$\begin{aligned} V_{T+1}(W_0) &= \max_{c_0} \{u(c_0) + \beta V_T(W_1)\} \\ \text{s.t.} \qquad W_1 &= W_0 - c_0 \end{aligned}$$

i.e., we convert the multiple-period problem into a 2-period problem by rewriting the objective function as the sum of utility from consumption in the current period and the continuation utility from consumption in all future periods.

We can rewrite the problem and take FOC:

$$\begin{aligned} & V_{T+1}(W_0) = \max_{c_0} \{u(c_0) + \beta V_T(W_0 - c_0)\} \\ \text{FOC:} \qquad & u'(c_0) = \beta V_T'(W_1) \end{aligned}$$

From the earlier envelope theorem, we have $V_T'(W_1) = \lambda = \beta^{t-1} u'(c_t), t = 1, 2, ..., T$ so we have the same Euler equations $u'(c_t) = \beta u'(c_{t+1}), t = 1, ...T - 1$.

In summary, we can solve an arbitrary T-period cake-eating problem by recursively solving a sequence of static optimisation problems, starting from the single-period problem for the very last period (backward deduction).

Each problem has the form:

$$V_{j+1}(W) = \max_{c \in [0,W]} \{u(c) + \beta V_j(W - c)\}$$

Where $j = 0, 1, 2..., T - 1$ and $V_0(.) = 0$.

(e) Exercise 1.1: Assume $u(c) = log(c), T = 3$, find $V_T(W)$ and the optimal consumption path.

At each period, we are maximising:

$$V_{T+1}(W) = \max_{c \in [0,W]} \{log(c) + \beta V_T(W - c)\}$$

Start with a 1-period problem. Since $W_{T+1} = 0$, the problem is:

$$\begin{aligned} V_1(W) &= \max_{c \in [0,W]} \{log(c) + \beta V_0(W - c)\} \\ c_1(W) &= W \\ V_1(W) &= log(W) \end{aligned}$$

For a 2-period problem, the problem is:

$$\begin{aligned}
&V_2(W) = \max_{c\in[0,W]} \{log(c) + \beta log(W-c)\} \\
&\text{FOC:} \quad \frac{1}{c} - \frac{\beta}{W-c} = 0 \\
&c_1(W) = \frac{W}{1+\beta} \\
&c_2(W) = W - \frac{W}{1+\beta} = \frac{\beta W}{1+\beta} \\
&V_2(W) = log(\frac{W}{1+\beta}) + \beta log(\beta\frac{W}{1+\beta})
\end{aligned}$$

For a 3-period problem, the problem is:

$$\begin{aligned}
&V_3(W) = \max_{c\in[0,W]} \{log(c) + \beta(log(\frac{W-c}{1+\beta}) + \beta^2 log(\beta\frac{W-c}{1+\beta}))\} \\
&\text{FOC:} \quad \frac{1}{c} - \frac{\beta}{W-c} - \frac{\beta^2}{W-c} = 0 \\
&c_1 = \frac{w}{1+\beta+\beta^2} \\
&c_2 = \frac{\beta w}{1+\beta+\beta^2} \\
&c_3 = \frac{\beta^2 w}{1+\beta+\beta^2}
\end{aligned}$$

2. Lecture 2

   (a) Direct attack for infinite horizon problem:

   We want to find the infinite sequence of consumption that maximises the agent's lifetime utility. We can solve the infinite horizon sequence problem given by:

$$\begin{aligned}
\text{(SP)} \qquad &\max_{\{c_t\}_{t=1}^{\infty},\{W_{t+1}\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&W_{t+1} = W_t - c_t \text{ for t = 0,1,2,...} \\
&W_0 \text{ is given.}
\end{aligned}$$

   With infinite horizon, the transversality condition (TVC) ensures that:

$$\lim_{T\to\infty} W_{T+1} = 0$$

   We can combine the transition equations as:

$$\sum_{t=0}^{\infty} c_t = W_0$$

   The problem simplifies to

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \text{ subject to } \sum_{t=0}^{\infty} c_t = W_0$$

   We can approach the problem via direct attack (Lagrangian) and work out the intertemporal consumption Euler equations:

$$u'(c_t) = \beta u'(c_{t+1}), t = 1, 2...$$

   (b) Dynamic programming for infinite horizon problem:

Denote $V(W)$ as the **value function** for the infinite horizon problem with starting cake size W. The recursive formulation of the problem is:

$$V(W) = \max_{c\in[0,W]} \{u(c) + \beta V(W-c)\} \text{ for all W}$$

State variable: W at the beginning of each period.
Control variable: c for the current period.
Transition equation: $W' = W - c$

We can also specify the DP problem as a **functional equation** to choose W':

$$\text{(FE)} \qquad V(W) = \max_{W'\in[0,W]} u(W-W') + \beta V(W') \text{ for all W}$$

If $V(.)$ is known, c or W' can be solved as functions of the state variable W:

$$W' = g(W)$$
$$c = W - g(W) \equiv \phi(W)$$

Functions like g or ϕ are **policy functions** or decision rules.

Assuming a solution $V(W)$ exists, the FOC is given by:

$$u'(c) = \beta V'(W') = u'(W-W') = u'(c) \text{ by Envelope Theorem, assuming V is differentiable}$$

Since the FOC holds for all W, it holds for the next period as well, yielding the same Euler equation:

$$u'(c) = \beta u'(c')$$

(c) Exercise 2.1: Let $u(c) = log(c)$, show that the FE can be solved analytically and the value function has the form:

$$V(W) = A + Blog(W)$$

where A and B are coefficients to be determined. Find A and B, and find the policy function for c.

We conjecture that the solution to V(W) takes the form $A + Blog(W)$ for all W.

$$V(W) = \max_{W'\in[0,W]} \{log(W-W') + \beta(A + Blog(W')\}$$

$$\text{FOC:} \qquad \frac{-1}{W-W'} + \frac{\beta B}{W'} = 0$$

$$W' = \frac{\beta BW}{1+\beta B}$$

Sub the optimal W' back into the FE and dropping the max operator:

$$V(W) = A + Blog(W) = log(W\frac{1+\beta B - \beta B}{1+\beta B}) + \beta(A + Blog\frac{\beta BW}{1+\beta B})$$

By comparing coefficients,

$$B = \frac{1}{1-\beta}$$

$$A = \frac{\frac{\beta}{1-\beta}log\beta - log\frac{1}{1-\beta}}{1-\beta}$$

$$V(W) = A + \frac{1}{1-\beta}log(W)$$

By substituting B into the FOC,

$$W' = \beta W$$
$$C = W(1-\beta)$$

(d) Converting SP to FE (see lecture notes for full descriptions):

(i) Deterministic model of optimal growth

$$
\begin{aligned}
\text{(SP)} \quad & \max_{\{c_t,k_{t+1}\}_{t=1}^{\infty},} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
& c_t + k_{t+1} = f(k_t), \text{ t = 0,1,2,...} \\
& c_t \geq 0, k_{t+1} \geq 0 \text{ for all } t, k_0 \text{ given.} \\
\text{(FE)} \quad & v(k) = \max_{k' \in [0,f(k)]} \{u(f(k)-k') + \beta v(k')\} \text{ for all k}
\end{aligned}
$$

(ii) Growth with technical progress

$$
\begin{aligned}
\text{(SP)} \quad & \max_{\{c_t,k_{t+1}\}_{t=1}^{\infty},} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\rho}}{1-\rho} \\
& C_t + K_{t+1} \leq F(K_t, L_t) + (1-\rho)K_t, \text{ t = 0,1,2,...} \\
& L_{t+1} = (1+\lambda)L_t, \text{ t = 0,1,2,...} \\
& C_t \geq 0, K_{t+1} \geq 0 \text{ for all } t, K_0, L_0 \text{ given.} \\
\text{(FE)} \quad & V(K,L) = \max_{K' \in [0,F(K,L)+(1-\rho)K]} \{\frac{C_t^{1-\rho}}{1-\rho} + \beta V(K',L')\} \\
& C = F(K,L) + (1-\rho)K - K' \\
& L' = (1+\lambda)L
\end{aligned}
$$

(iii) Optimal consumption-saving problem

$$
\begin{aligned}
\text{(SP)} \quad & \max_{\{c_t,x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & c_t + \frac{x_{t+1}}{R} = I + x_t, \text{ t = 0,1,2,...} \\
& c_t \geq 0, x_{t+1} \geq 0, \forall t, x_0 \text{ given.} \\
\text{(FE)} \quad & v(x) = \max_{x' \in [0,R(I+x)]} \{u(I + x - \frac{x'}{R}) + \beta v(x')\}
\end{aligned}
$$

(iv) Optimal consumption with habit persistence

$$
\begin{aligned}
\text{(SP)} \quad & \max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [log(c_t) + \gamma log(c_{t-1})] \\
\text{s.t.} \quad & c_t + k_{t+1} \leq Ak_t^{\alpha}, \text{ t = 0,1,2,...} \\
& c_t \geq 0, K_{t+1} \geq 0, \forall t, k_0, c_{-1} \text{ given.} \\
\text{(FE)} \quad & v(k, c_{-1}) = \max_{k' \in [0,Ak^{\alpha}]} \{log(c) + \gamma log(c_{-1}) + \beta v(k', c)\} \\
& c = Ak^{\alpha} - k'
\end{aligned}
$$

3. Lecture 3

(a) A metric space (S, ρ) is a set S, together with a metric $\rho : S \times S \to R$, such that for all $x, y, z \in S$:

i. $\rho(x, y) \geq 0$, with equality iff $x = y$ (nonnegativity);
ii. $\rho(x, y) = \rho(y, x)$ (symmetry);
iii. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle inequality)

A vector space is a set of elements that is closed under addition and scalar multiplication and that has a "zero" element denoted by θ.

A normed vector space $(S, ||\cdot||)$ is a vector space S, together with a norm $||\cdot|| : S \to R$, such that for all $x, y \in S$ and $\alpha \in R$:

i. $||x|| \geq 0$, with equality iff $x = \theta$;
ii. $||\alpha x|| = |\alpha| \cdot ||x||$; and

iii. $||x+y|| \leq ||x|| + ||y||$ (the triangle inequality).

It is standard to view any normed vector space $(S, ||\cdot||)$ as a metric space, with $\rho(x,y) \equiv ||x-y||$, all $x, y \in S$.

Commonly used metric spaces in macroeconomics:

i. The space $(C(X), ||\cdot||_{sup})$, where $||f||_{sup} = sup_{x\in X}|f(x)|$, for all $f \in C(X)$, the set of all continuous functions.

ii. The space $(\mathbb{R}^l, ||\cdot||_p)$, where $||\cdot||_p \equiv (\sum_{i=1}^{l} |x_i|^p)^{\frac{1}{p}}$, $p = 1, 2, \infty$

(b) Convergence: A sequence $\{x_n\}_{n=0}^{\infty}$ in a metric space (S,ρ) converges to $x \in S$, if for each $\epsilon > 0$, there exists N_ϵ such that

$$\rho(x_n, x) < \epsilon, \forall n \geq N_\epsilon$$

A sequence $\{x_n\}_{n=0}^{\infty}$ in a metric space (S,ρ) is a Cauchy sequence if for each $\epsilon > 0$, there exists N_ϵ such that

$$\rho(x_n, x_m) < \epsilon, \forall n, m \geq N_\epsilon$$

(c) Complete spaces: A metric space S, ρ is complete if every Cauchy sequence in S converges to an element in S. In a complete metric space, verifying a sequence is a Cauchy sequence is a way of proving convergence.

Theorem 3.1 in SLP: Let $X \subseteq \mathbb{R}^l$, and let $C(X)$ be the set of bounded continuous functions $f : X \to \mathbb{R}$ with the sup norm, $||f|| = sup_{x\in X}|f(x)|$. Then $C(X)$ is a complete normed vector space. (Note: this is useful for value functions since each value function is a bounded continuous function)

Let (S,ρ) be a metric space and $A \subset S$. A is closed if any convergent sequence of elements from A has its limit point in A. A is open if its complement is closed.

(d) Compactness: A subset A of a metric space S, ρ is compact if every sequence contained in A has a subsequence which converges to a point in A.

A subset of $(\mathbb{R}^l, ||\cdot||_p), p \in \{1, 2, \infty\}$ is compact iff it is closed and bounded.

A set $X \in \mathbb{R}^l$ is bounded if there exists a real number $M > 0$ such that $||x|| \leq M$ for all $x \in X$.

The state space in the dynamic programming problem is usually defined as a compact set.

(e) Contraction mapping and Blackwell's Sufficient Conditions for a Contraction: Let (S,ρ) be a metric space and $T : S \to S$ be a function mapping S into itself. T is a **contraction mapping** (with modulus β if for some $\beta \in (0,1)$:

$$\rho(Tx, Ty) \leq \beta\rho(x,y), \forall x, y, \in S$$

Example: Let $S = [a,b]$, with $\rho(x,y) = |x-y|$. Let $T : S \to S$ be a function with derivatives $0 < T'(x) < 1$ for all $x \in S$, then T is a contraction mapping.

Blackwell's sufficient conditions for a contraction: Let $X \subset R^l$, and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$ with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying:

i. (monotonicity): for $f, g \in B(X)$ satisfying $f(x) \leq g(x)$ for all $x \in X$,

$$(Tf)(x) \leq (Tg)(x), \forall x \in X$$

ii. (discounting) there exists some $\beta \in (0,1)$ such that

$$[T(f+a)](x) \leq (Tf)(x) + \beta(a), \text{ for all} f \in B(X), a \geq 0, x \in X.$$

Then T is a contraction with modulus β.

(f) Exercise 3.4: Show that if (S, ρ) is a complete metric space and $S' \subset S$, then S', ρ is a complete metric space iff S' is closed.

We first show that if S' is closed, then (S, ρ) is a complete metric space (i.e., every Cauchy sequence $\{x_n\}$ in (S', ρ) converges to some point in (S', ρ)).

Let $\{x_n\}$ be a Cauchy sequence in (S', ρ). Since (S, ρ) is complete, $\{x_n\}$ converges to a point x in S. Since S' is closed (i.e., every convergent sequence of elements in S' has its limit point in S') and $\{x_n\}$ is a convergent sequence, its limit x must be in $S' \subset S$.

Now we show the causality in the other direction, i.e., if (S, ρ) is a complete metric space, then S' is closed. To show this, we need to show that any convergent sequence $\{x_n\}$ in S' has its limit point $x \in S'$.

Since $\{x_n\}$ is a convergent sequence in S', it is also a Cauchy sequence in (S, ρ). Since (S', ρ) is a complete metric space and $\{x_n\}$ is a Cauchy sequence in it, $\{x_n\}$ must converge to a point in (S', ρ). This point must be x, as the limit is unique. That is, the limit $x \in S'$.

(g) Exercise 3.5: Recall the recursive formulation of the infinite-horizon cake-eating problem can be written as

$$\text{(FE)} \qquad V(W) = \max_{W' \in [0,W]} \{u(W - W') + \beta V(W')\}, \text{ for all W.}$$

Assume $W \in X \equiv [0, \bar{W}]$. Define an operator T as:

$$(TV)(W) = \max_{W' \in [0,W]} \{u(W - W') + \beta V(W')\}, \text{ for all } W \in X$$

Verify the following:

i. for any $V \in B(X), TV \in B(X)$, i.e., $T : B(X) \to B(X)$;
ii. T satisfies the "monotonicity condition";
iii. T satisfies the "discounting condition".

(refer to Assignment 1 Q3a for solution)

Therefore, T is a contraction (and hence there is a fixed point where TV = V, which is the solution to the functional equation).

4. Lecture 4

(a) **Contraction Mapping Theorem (CMT)**: Used to show that there exists a unique value function V in the (FE).

Let (S, ρ) be a metric space and $T : S \to S$. A point $x \in S$ is a **fixed point** of T if it satisfies $Tx = x$.

For example, for the infinite-horizon cake-eating problem, we can define the operator T:

$$(TV)(W) = \max_{W' \in [0,W]} \{u(W - W') + \beta V(W')\}, \text{ for all } W \in [0, \bar{W}]$$

Then, a fixed point of T is a solution to the FE. CMT proves existence and uniqueness of the value function in the (FE).

Theorem 3.2 in SLP: If **(i)** (S, ρ) **is a complete metric space** and **(ii)** $T : S \to S$ **is a contraction mapping with modulus** β, then

i. T has exactly one fixed point v in S, and
ii. for any $v_0 \in S, \rho(T^n v_0, v) \leq \beta^n \rho(v_0, v), n = 0, 1, 2...$

This provides a way to numerically solve for the value function through value function iteration.

Proof of CMT (not in these notes!)

Corollary of the CMT: Let S, ρ be a complete metric space, and let $T : S \to S$ be a contraction mapping with fixed point $v \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$, then $v \subseteq S'$. Further, if in addition $T(S') \subseteq S'' \subseteq S'$, then $v \in S''$.

The corollary is useful for establishing qualitative properties of a fixed point. In some situations we will apply the CMT twice: once on a larger space S to establish the existence and uniqueness of the solution, and again on a smaller space S' to characterise the qualitative properties of the fixed point (monotonicity, concavity, etc.)

(b) **Extreme Value Theorem (EVT)**: if f is a continuous function on a closed interval $[a, b]$, then f has both a maximum and a minimum on $[a, b]$. Further, if f is strictly concave (convex), then f has a unique maximum (minimum).

Consider a maximisation problem of the form:

$$v(x) = \max_{y \in \Gamma(x)} f(x, y), \text{ for all } x \in X \subseteq \mathbb{R}^l$$

Where $\Gamma(X) \to Y (Y \subseteq \mathbb{R}^m)$ is a non-empty **correspondence** (set function) and $f : X \times Y \to \mathbb{R}$ is a function.

If for each $x \in X, f(x, y)$ is continuous in y and the set $\Gamma(x)$ is compact, the EVT implies that $f(x, y)$ attains a maximum at some $y \in \Gamma(x)$ (for the cake-eating problem, this is like $W' \in [0, W]$.

That is, $v(x)$ is well-defined, and the set of optimal y's that attain the maximum, defined as

$$G(x) = y \in \Gamma(x) : f(x, y) = v(x)$$

is non-empty.

(c) **Theorem of the Maximum:** A generalised version of the Extreme Value Theorem, to show that there exists a unique policy function $g(x)$ (i.e., the choice variable as a function of the state variable) that solves the maximisation problem in the (FE).

Berge's Theorem of the Maximum: Let **(i)** $X \subseteq \mathbb{R}^l, Y \subseteq \mathbb{R}^m, f : X \times Y \to \mathbb{R}$ **be a continuous function**, and **(ii)** $\Gamma : X \to Y$ **be a nonempty, compact-valued, continuous correspondence**. Then $v : X \to \mathbb{R}$ is continuous and the correspondence $G : X \to Y$ is nonempty, compact-valued and upper hemi-continuous.

i. Γ is nonempty: the set $\Gamma(X)$ is nonempty for all $x \in X$.
ii. Γ is compact-valued: the set Γ is closed and bounded for all $x \in X$.
iii. Γ is continuous: as x varies, the set $\Gamma(x)$ becomes larger or smaller in a continuous way.
iv. A correspondence is continuous if it is both upper hemi-continuous and lower hemi-continuous.

Furthermore, if **(iii)** $f(x, y)$ **is strictly concave in** y, and **(iv)** Γ **is convex-valued**, then $G(x)$ is single-valued and continuous, denoted as g (i.e., $g(x)$ the solution to the maximisation problem is unique and a maximum):

$$g(x) = argmax_{y \in \Gamma(x)} f(x, y)$$

i. $f(x, y)$ (the objective function) is strictly concave in y:

$$f(x, \lambda y_1 + (1 - \lambda) y_2) \geq \lambda f(x, y_1) + (1 - \lambda) f(x, y_2)$$

for all $\lambda \in (0, 1), x \in X$, and $y_1, y_2 \in \Gamma(x)$ and the inequality is strict if $y_1 \neq y_2$.

ii. Γ is convex-valued: $\Gamma(x)$ is a convex set for all $x \in X$.

Summary of Theorem of the Maximum:

i. f (the objective function) is continuous in both x and y
ii. Γ (the constrained set) is nonempty, compact-valued, continuous
iii. f is strictly concave in y
iv. Γ is convex-valued

If (i) and (ii) are true, then there exists some policy function $g(x)$ (i.e., the choice variable as a function of the state variable) that is the solution to the maximisation problem, i.e.:

$$G(x) \equiv \{y \in \Gamma(x) : v(x) = f(x, y)\} \text{ is nonempty and v is continuous}$$

If (iii) and (iv) are true, then the solution to the maximisation problem is unique, i.e.:

$$G(x) \text{ is single-valued and continuous: } g(x) = argmax_{y \in \Gamma(x)} f(x, y)$$

5. Lecture 5

(a) General forms of the (SP) and (FE):

(SP): The infinite-horizon dynamic optimisation problem generally takes the form:

$$\begin{aligned} \text{(SP)} \qquad & \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \equiv v^*(x_0) \\ & \text{s.t. } x_{t+1} \in \Gamma(x_t), t = 0, 1, 2, ... \\ & x_0 \in X \text{ given.} \end{aligned}$$

Where:
$X \subseteq \mathbb{R}^l$ is the set of feasible values for $x_t, t = 0, 1, 2, ...$
$F : X \times X \to \mathbb{R}$ is the one-period return function.
$\Gamma : X \to X$ is a correspondence that describes the feasibility constraints.
Note: the general form is well-defined as long as Γ is non-empty.

For example, for the infinite-horizon cake-eating problem:

$$\begin{aligned} \text{(SP)} \qquad & \max_{\{c_t, W_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & \text{s.t. } W_{t+1} = W_t - c_t, t = 0, 1, 2, ... \\ & c_t \geq 0, W_{t+1} \geq 0, \text{ for all } t \\ & W_0 \text{ is given.} \end{aligned}$$

We can rewrite the cake-eating problem as:

$$\text{(SP)} \qquad \max_{\{W_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(W_t - W_{t+1})$$

Then,
$F(x_t, x_{t+1})$ maps to $u(W_t, W_{t+1})$
$x_{t+1} \in \Gamma(x_t)$ maps to $W_{t+1} \in [0, W_t] \equiv \Gamma(W_t)$
$x_0 \in X$ given maps to $W_0 \in [0, \bar{W}]$ given (this just means that W is finite).

(FE): The recursive formulation (i.e., the functional equation) generally takes the form:

$$\text{(FE)} \qquad \sup_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}, \text{ for all } x \in X.$$

Where:
x is the state variable at the beginning of the current period.
y is the state variable at the beginning of the next period (to be chosen in the current period).
X is the state space for x.
$\Gamma(x) \subset X$ is the set of feasible y values, given x.
$F(x, y)$ is the current period return.

$\beta v(y)$ is the discounted continuation value.

For example, for the infinite-horizon cake-eating problem:

$$\text{(FE)} \qquad \max_{W' \in [0,\bar{W}]} \{u(W - W') + \beta V(W')\}, \text{ for all } W \in [0, \bar{W}]$$

Then,
x maps to W
y maps to W'
X maps to $W \in [0, \bar{W}]$
$\Gamma(x) \subset X$ maps to $W' \in [0, W]$
$F(x, y)$ maps to $u(W - W')$
$\beta v(y)$ maps to $\beta v(W')$.

(b) **Correspondence between solutions to the (FE) and (SP)**:

A solution to the (SP) consists of:

i. an **optimal plan** (there may be multiple optimal plans) $\{x_t^*\}_{t=0}^{\infty}$ (with $x_0^* = x_0$) satisfying:

$$x_{t+1}^* \in \Gamma(x_t^*), t = 0, 1, 2, ...$$

ii. the **supremum attained** at $\{x_t^*\}$ for initial state x_0 (this is the maximum value of the objective function):

$$\sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) \equiv v^*(x_0)$$

A solution to the (FE) consists of:

i. a **value function** v satisfying the (FE):

$$v(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\} \text{ for all } x \in X$$

ii. a **policy correspondence** (i.e., set of optimal y given x) $G : X \to X$:

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}$$

The connection between solutions is the Principle of Optimality. If both solutions correspond, then we can focus on (FE), developing the main results for dynamic programming under certainty: existence, uniqueness, and properties of solutions to (FE).

Given $x_0 \in X$, let

$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t), t = 0, 1, ...\}$$

be the set of plans that are feasible from x_0.

i. **Assumption 1**: $\Gamma(x)$ is nonempty, for all $x \in X$.
This ensures $\Gamma(x_0)$ is nonempty, for all $x_0 \in X$.

ii. **Assumption 2**: For all $x_0 \in X$ and $\{x_t\} \in \Pi(x_0)$, $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ exists (although it may be $+\infty$ or $-\infty$).
A sufficient condition for this to hold: F is bounded and $0 < \beta < 1$.

Under Assumptions 1 and 2, the (SP) is well-defined and has a solution. We now want to show the equivalence between the solutions to the (SP) and the (FE).

i. Equivalence between v* (from (SP)) and v (from (FE)): v* is always uniquely defined under Assumptions 1 and 2, but there may be zero, one or many v's that satisfy the (FE).

Theorem 1 (Theorem 4.2 in SLP): Let X, Γ, F, β satisfy Assumptions 1 and 2. Then the supremum function v* satisfies the (FE).

Theorem 2 (Theorem 4.3 in SLP): Let X, Γ, F, β satisfy Assumptions 1 and 2. If v is a solution to (FE) and satisfies

$$lim_{t\to\infty}\beta_v^t(x_t) = 0, \text{ for all } \{x_t\} \in \Pi(x_0)$$

Then $v = v^*$ (intuitively, this is because $v(x_t)$ does not grow too fast).

ii. Equivalence between $\{x_t^*\}$ (from (SP)) and G (from (FE)), provided that $v = v^*$:

Given v = v*, we call G* the optimal policy correspondence:

$$G^*(x) = \{y \in \Gamma(x) : v^*(x) = F(x, y) + \beta v^*(y)\}$$

Theorem 3 (Theorem 4.4 in SLP): Let X, Γ, F, β satisfy Assumptions 1-2. Let $\{x_t^*\}_{t=0}^\infty \in \Pi(x_0)$ be a feasible plan that attains the supremum in (SP) for initial state x_0. Then $x_{t+1}^* \in G^*(x_t)$, i.e.,

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*), t = 0, 1, 2, ...$$

Where $G^*(x)$ is as defined above. That is, the optimal plan that solves (SP) also satisfies the optimal policy correspondence.

Theorem 4 (Theorem 4.5 in SLP): Let X, Γ, F, β satisfy Assumptions 1-2. Let $\{x_t^*\}_{t=0}^\infty \in \Pi(x_0)$ be a feasible plan from x_0 satisfying the condition above, i.e., let $\{x_t^*\}$ be a plan generated from x_0 by the optimal policy correspondence, and satisfy

$$lim \sup_{t\to\infty} \beta^t v^*(x_t^*) \leq 0$$

then $\{x_t^*\}$ attains the supremum in (SP) for initial state x_0.

If v is bounded or does not increase too fast as $t \to \infty$, the conditions in Theorem 2 and 4 can be satisfied.

In summary, under the conditions in Theorem 2 and 4,
A. $v(x_0)$ attains the supremum in (SP)
B. a plan $\{x_t^*\}_{t=0}^\infty$ generated by G from x_0 solves (SP).

(c) Existence and uniqueness of v satisfying (FE)

$$\text{(FE)} \qquad \max_{y\in\Gamma(x)} \{F(x, y) + \beta v(y)\}, \text{ for all } x \in X.$$

We assume that F is bounded. If X is chosen to be a compact set, then F is bounded, because as long as F is continuous, a continuous function on a compact set is bounded.

We assume that X, Γ, F, β satisfy the following two assumptions:

i. **Assumption 3**: X is a convex subset of $\mathbb{R}^l$, and the correspondence $\Gamma : X \to X$ is nonempty, compact-valued and continuous.

ii. **Assumption 4**: $F : A \to \mathbb{R}$ is bounded and continuous, and $0 < \beta < 1$.
$A \equiv \{(x, y) \in X \times X : y \in \Gamma(x)\}$. (i.e., A is the set of feasible combinations of (x, y). Both x and y are in X, but in addition y is in the constrained set $\Gamma(X)$).

Under Assumptions 3-4, Assumptions 1-2 hold, so the (SP) is well-defined, i.e., v^* is well-defined. Moreover, since F is bounded and continuous, v^* is also bounded and continuous (since v^* is just a sum of F discounted by β where $0 \leq \beta \leq 1$.

Since v^* is bounded and continuous, we seek solutions to (FE) in $(C(X), ||\cdot||_{sup})$, the set of continuous bounded functions with the sup norm, which is complete.

With both v^* and v being bounded, the conditions on v in Theorem 2 and Theorem 4 are satisfied.

Theorems 1-4 imply that under Assumptions 3-4, the solutions to the (FE) and (SP) coincide.

A value function v that satisfies the (FE) is a fixed point of an operator T defined as:

$$(Tv)(x) = \max_{y\in\Gamma(x)} F(x,y) + \beta v(y), \text{ for all } v \in C(X)$$

Then, solutions to the (FE) are fixed points of the operator T.

Theorem 5 (Theorem 4.6 in SLP): Let X, Γ, F, β satisfy Assumptions 3 and 4, and let $C(X)$ be the space of bounded continuous functions on X with the sup norm.

Then,

i. the operator T maps $C(X)$ into itself, $T : C(X) \to C(X)$;

ii. T has a unique fixed point $v \in C(X)$; and for all $v_0 \in C(X)$,

$$||T_n v_0 - v||_{sup} \leq \beta^n ||v_0 - v||_{sup}, n = 0, 1, 2, ...$$

iii. Moreover, given v, the optimal policy correspondence $G : X \to X$ defined by

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x,y) + \beta v(y)\}$$

is nonempty, compact-valued and upper hemi-continuous.

This means that $T : C(X) \to C(X)$ is a contraction mapping with modulus β, so it has a unique fixed point by the CMT, i.e., there is a unique v that satisfies the (FE).

Given this v, the corresponding policy correspondence is non-empty, i.e., there exists a solution to the maximisation problem of the (FE) for any $x \in X$.

Theorem 5 justifies the value function iteration algorithm to solve the (FE) numerically.

(d) Exercise 5.3

Consider a consumer whose objective function is simply discounted consumption. The consumer has initial wealth $x_0 \in X = \mathbb{R}$, and he can borrow or lend at the net interest rate $\beta^{-1} - 1$, where $\beta \in (0, 1)$. There are no constraints on borrowing, so his problem is simply:

$$\text{(SP)} \qquad \max_{\{c_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t c_t$$
$$\text{s.t. } 0 \leq c_t \leq x_t - \beta x_{t+1}, t = 0, 1, 2, ...$$
$$x_0 \text{ is given.}$$

x_t denotes the consumer's asset holding. The price of 1 unit of the asset is β.

i. Write down the functional equation corresponding to the sequence problem.

The (FE) is:

$$\text{(FE)} \qquad v(x) = \max_{y \leq \beta^{-1} x} \{x - \beta y + \beta v(y)\}$$

ii. What is the supremum (maximum) function for the sequence problem? Does it satisfy the functional equation? (relate your answer to Theorem 1)

Since there are no constraints on borrowing such that consumption is unbounded, the supremum function is

$$v^*(x) = +\infty, \forall x$$

As Theorem 1 implies, it satisfies the functional equation, since

$$+\infty = v^*(x) = \max_{y \le \beta^{-1}x} [x - \beta y + \beta v^*(y)] = \max_{y \le \beta^{-1}x} [x - \beta y + \infty] = +\infty$$

iii. Can you find another value function v that also satisfies the functional equation?

Yes, for example the function $v(x) = x$. Note that:

$$x = v(x) = \max_{y \le \beta^{-1}x} [x - \beta y + \beta v(y)] = \max_{y \le \beta^{-1}x} [x - \beta y + \beta y] = \max_{y \le \beta^{-1}x} x = x$$

iv. Does this v satisfy the condition in Theorem 2,

$$\lim_{t \to \infty} \beta^t v(x_t) = 0, \text{ for all } \{x_t\} \in \Pi(x_0)?$$

If not, give a counterexample. As a consequence, is $v(x_0)$ the maximum for the sequence problem? (relate to Theorem 2)

No. Notice that the sequence $x_t = \beta^{-t}x_0, t = 0, 1, ...$, is in $\Pi(x_0)$, however, we have

$$\lim_{t \to \infty} \beta^t v(x_t) = \lim_{t \to \infty} \beta^t x_t = \lim_{t \to \infty} \beta^t \beta^{-t} x_0 = x_0 \neq 0$$

Therefore, the condition in Theorem 2 does not hold. As a consequence, $v(x_0) = x_0$ is not the maximum for the sequence problem.

The problem with this example is that the customer can borrow without limit, leading to an explosive solution $v^* = +\infty$. To rule this out, we usually impose the "no-Ponzi-game" condition which states that any debt that has been accumulated has to be paid off eventually.

v. Now let us prohibit indebtedness by requiring $x_t \ge 0$, for all t. Rewrite the sequence problem and the functional equation.

The (SP) is:

$$\text{(SP)} \qquad \max_{\{c_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (x_t - \beta x_{t+1})$$
$$\text{s.t. } 0 \le x_{t+1} \le \frac{1}{\beta} x_t, t = 0, 1, 2, ...$$
$$x_0 \text{ is given.}$$

The (FE) is:

$$\text{(FE)} \qquad v(x) = \max_{y \in [0, \beta^{-1}x]} \{x - \beta y + \beta v(y)\}, \text{ for all } x$$

vi. Find the new supremum function. Does it satisfy the functional equation?

The objective function in (SP) is:

$$\sum_{t=0}^{\infty} \beta^t (x_t - \beta x_{t+1}) = x_0 - \beta x_1 + \beta x_1 - \beta^2 x^2 ...$$

If we cancel all the offsetting terms, the supremum function is

$$v^*(x_0) = x_0, \text{ for all } x \ge 0$$

$v*$ satisfies the functional equation, as Theorem 1 implies.

vii. Can you find the optimal plans that attain the supremum in (SP)? Do they satisfy $x_{t+1} = G^*(x_t)$? (relate to Theorem 3)

Given any $x_0 \geq 0$, the set of feasible plans $\Pi(x_0)$ consists of the sequences

$$(x_0, 0, 0...), (x_0, \beta_{-1}x_0, 0, ...), (x_0, \beta^{-1}x_0, \beta^{-2}x_0, 0, ...)...$$

and all convex combinations thereof. Every feasible plan $\{x_t\} \in \Pi(x_0)$ satisfies $x_{t+1} \in G^*(x_t)$, since

$$x_t = v^*(x_t) = x_t - \beta x_{t+1} + \beta v^*(x_{t+1}) = x_t - \beta x_{t+1} + \beta x_{t+1} = x_t$$

For a feasible plan $\{x_t\}$, if consumption occurs in finite time, i.e., the consumer consumes all his wealth within T, a finite number of periods, then $x_t = 0$ for all $t > T$. Then $\{x_t\}$ satisfies the condition in Theorem 4,

$$\limsup_{t\to\infty} \beta^t v^*(x_t) = \limsup_{t\to\infty} \beta^t x_t = 0$$

such that it is an optimal plan that attains the maximum in (SP). So, the optimal plans are those feasible plans with consumption occurring in finite time.

viii. Is every feasible plan generated from x_0 by G^* an optimal plan for (SP)? If not, find a counterexample. What condition does it violate? (relate your answer to Theorem 4).

No. consider the feasible plan $x_t = \beta^{-t}x_0, t = 0, 1, ...$ (i.e., in each period the consumer invests everything and consumes nothing). This yields discounted utility of zero, for all $x \geq 0$. Therefore, this is not an optimal plan. This violates the condition in Theorem 4 since

$$\limsup_{t\to\infty} \beta^t v^*(x_t) = \limsup_{t\to\infty} \beta^t x_t = \limsup_{t\to\infty} \beta^t \beta^{-t} x_0 = x_0$$

The problem can be fixed if the one-period return function satisfies the Inada condition, which rules out zero consumption in every period.

(e) Exercise 5.4: Proof of Theorem 5

Sketch of proof:

i. For each $v \in C(X)$, Tv is bounded.

Specifically, $|F(x, y) + \beta v(y)| \leq |F(x, y)| + \beta|v(y)|$, and since both F and v are bounded, Tv must be bounded.

Tv is also continuous following the Theorem of the Maximum.

Specifically, since $F(x, y) + \beta v(y)$ is continuous, and Γ is non-empty, compact and continuous, then Tv is continuous following the Theorem of the Maximum.

So, $T : C(X) \to C(X)$ (shown point (i) above).

ii. T satisfies Blackwell's sufficient conditions such that T is a contraction mapping.

Since $C(X)$ is a complete space and T is a contraction mapping, T has a unique fixed point (shown point (ii) above).

The properties of G follow from the Theorem of the Maximum (shown point (iii) above).

6. Lecture 6

(a) **Uniqueness of the policy function**: We want to ensure that the policy function is unique (i.e., single-valued G).

Theorem 6 (Theorem 4.8 in SLP): Let X, Γ, F, β satisfy Assumptions 3 and 4; let F be strictly concave and Γ be convex. Then, v is strictly concave and G is a continuous, single-valued function (denoted as g)

i. F is strictly concave (i.e., in all its arguments jointly) if

$$F(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') \geq \lambda F(x,y) + (1-\lambda)F(x',y')$$
$$\text{for all } x, x' \in X, y \in \Gamma(x), y' \in \Gamma(x'), \text{ and } \lambda \in (0,1), \text{ and the inequality is strict if } x \neq x'$$

Here, x, x' are any two points in X and y, y' are any two points in Y.

ii. Γ is convex if for any $\lambda[0,1]$ and $x, x' \in X, y \in \Gamma(x)$, and $y' \in \Gamma(x')$, we have

$$\lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)x')$$

If Γ is convex, Γ is convex-valued.

We also want to ensure that the policy function is strictly increasing (i.e., monotonic).

Theorem 7 (Theorem 4.7 in SLP): Let X, Γ, F, β satisfy Assumptions 3 and 4; let $F(x,y)$ be strictly increasing in each element of $x(x = (x_1, ..., x_l))$ and let Γ be monotone (i.e., if the state variable increases, the constraint set expands) in the sense that

$$x \leq x' \text{ implies } \Gamma(x) \subseteq \Gamma(x')$$

Then, v is strictly increasing.

Once the existence and uniqueness of a solution $v \in C(X)$ to the (FE) have been established, we can treat the maximisation problem in the (FE) as an ordinary optimisation problem.

However, before we can apply the standard optimisation method to characterise the value function g, we need v to be differentiable.

Theorem 8 (Theorem 4.11 in SLP): Let X, Γ, F, β satisfy Assumptions 3 and 4; let Γ be convex, F be strictly concave and continuously differentiable on the interior of A (where $A \equiv \{(x,y) \in X \times X : y \in \Gamma(x)\}$ (i.e., A is the set of feasible combinations of (x,y)). Then, v is continuously differentiable at all $x \in$ int X and $g(x) \in$ int $\Gamma(x)$ with derivatives given by

$$v_i(x) = F_i(x, g(x)), i = 1, 2, ..., l$$

Note: the result above follows from applying the Envelope Theorem to the (FE).

(b) **Convergence of the policy function**: by Theorem, 5, starting from any $v_0 \in C(X)$, the sequence $T^n v_0$ converges (uniform convergence, i.e., the max difference between the two functions converges to 0) to the unique v that satisfies the (FE). We also need the policy function to converge at the same time.

Theorem 9 (Theorem 4.9 in SLP): Let X, Γ, F, β satisfy Assumptions 3 and 4; let F be strictly concave and Γ be convex. Let $C'(X)$ be the set of bounded, continuous, concave functions on X, and let $v_0 \in C'(X)$. Let $\{(v_n, g_n)\}$ be defined by

$$v_{n+1} = Tv_n, n = 0, 1, 2, ..., \text{ and}$$
$$g_n(x) = argmax_{y \in \Gamma(x)}[F(x,y) + \beta v_n(y)], n = 0, 1, 2, ...$$

Then, $g_n \to g$ pointwise. If X is compact, then the convergence is uniform.

i. $g_n \to g$ pointwise: $g_n(x) \to g(x)$ for given $x \in X$
ii. $g_n \to g$ uniformly: $||g_n - g||_{sup} \to 0$
iii. Pointwise convergence does not imply uniform convergence.

This theorem suggests that the value function iteration procedure also gives us the policy function. In other words, v and g can be solved for numerically by value function iteration, where starting from any $v_0 \in C(X)$,

$$v_{n+1}(x) = \sup_{y \in \Gamma(x)} \{F(x,y) + \beta v_n(y)\}, \text{ all } x \in X$$
$$g_n(x) = argmax_{y \in \Gamma(x)}[F(x,y) + \beta v_n(y)], n = 0, 1, 2, ...$$

(c) **Euler equation and transversality condition**

Suppose the (FE) yields a unique value function $v(x)$ and a unique policy function $y = g(x)$. We want to study properties of the optimal $\{x_t^*\}$.

Euler equation: Derive the first-order condition (assuming interior solution for y for any given x and the Envelope condition from the (FE):

$$\begin{aligned} F_2(x, g(x)) + \beta v'(g(x)) = 0, &\qquad F_2 \text{ means the partial wrt } y = g(x) \\ v'(x) = F_1(x, g(x)) &\qquad F_1 \text{ means the partial wrt x} \end{aligned}$$

Combining the two equations yields the **Euler equations**

$$F_2(x, g(x)) + \beta F_1(g(x), g(g(x))) = 0$$

Or equivalently,

$$F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0, t = 0, 1, 2, ...$$

This equation can also be derived from the (SP) directly. Assuming interior solutions for the (SP), the Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

And the FOC with respect to x_{t+1} is:

$$\frac{\partial \mathcal{L}}{\partial x_{t+1}} \beta_t F_2 x_t, x_{t+1} + \beta^{t+1} F_1(x_{t+1}, x_{t+2}) = 0, t = 0, 1, 2, ...,$$

which is equivalent.

If $\{x_t^*\}$ is an optimal plan satisfying $x_{t+1}^* \in \text{int } \Gamma(x_t^*)$ for all t (i.e., interior solutions), then $\{x_t^*\}$ satisfies the Euler equations.

Note that the Euler equations are a system of second-order difference equations. To pin down the system, we need 2 boundary conditions.

One boundary condition is given by x_0, and the other is the **transversality condition (TVC)**, which is given by

$$\lim_{t \to \infty} \beta^t F_1(x_t, x_{t+1}) \cdot x_t = 0$$

Note that $F_1(x_t, x_{t+1}) = v'(x_t)$, so $F_1(x_t, x_{t+1})$ can be interpreted as the shadow price of the state variables.

Then, the inner product $F_1(x_t, x_{t+1}) \cdot x_t$ measures the total value of the state variables in period t.

Theorem 10 (Theorem 4.15 in SLP): Let $X \subseteq \mathbb{R}_+^l$, and let F and Γ satisfy Assumptions 3 and 4; let F be strictly concave and continuously differentiable and Γ be convex. Then, the sequence $\{x_{t+1}^*\}_{t=0}^{\infty}$ with $x_{t+1}^* \in int\ \Gamma(x_t^*), t = 0, 1, ...$ is optimal for the (SP) given x_0, if it satisfies the Euler equations and the transversality condition.

TVC is a necessary condition for optimality in an infinite-horizon dynamic model.

(d) Deterministic dynamics

To characterise the dynamic behaviour of the economic system, we need to examine the dynamic behaviour of the state variables x_t, which are governed by the policy function $x_{t+1} = g(x_t)$, or the Euler equations and TVC if g cannot be solved for explicitly.

We want to find out whether the system has a unique stationary point or **steady state**, and whether the system converges to that point from any initial point (**stability**).

If g can be solved explicitly, a fixed point of g is a **steady state** of the state variables x_t:

$$\bar{x} = g(\bar{x})$$

More generally, any $\bar{x}$'s such that $x_t \equiv \bar{x}$ for all t satisfy the Euler equation

$$F_2(\bar{x}, \bar{x}) + \beta F_1(\bar{x}, \bar{x}) = 0$$

and the TVC

$$\lim_{t\to\infty} \beta^t F_1(\bar{x}, \bar{x}) \cdot \bar{x} = 0$$

is a **steady state** (there can be multiple steady states).

The steady state $\bar{x}$ is **globally stable** if for all $x_0 \in X$ and $\{x_t\}_{t=0}^{\infty}$ satisfying the Euler equations and TVC for all t, we have

$$\lim_{t\to\infty} x_t = \bar{x}$$

Note: There is no general theory for global stability. In some special cases (e.g., a strictly concave g), global stability can be determined).

If the Euler equation defines a linear policy function, i.e.,

$$x_{t+1} = a_0 + Ax_t$$

then we have a well developed theory on the stability of a linear system.

Local stability: Let $\bar{x}$ be an interior steady state. $\bar{x}$ is **locally stable** if for all $x_0 \in X$ in a neighbourhood of $\bar{x}$ and $\{x_t\}_{t=0}^{\infty}$ satisfying $x_{t+1} = g(x_t)$ for all t, $\lim_{t\to\infty} x_t = \bar{x}$.

Global stability implies local stability, but not vice-versa.

Typically the policy function g cannot be solved explicitly, but we can discuss the local stability around the steady state:

i. solve for the steady state $\bar{x}$ implied by the Euler equations and TVC,
ii. linearise/ log-linearise the Euler equation around $\bar{x}$ (analogous to local approximation),
iii. discuss the stability of the linear dynamic system.

(e) Appendix: Proof of Theorem 6

We apply the corollary to the CMT from Lecture 4 to show that v is strictly concave.

Let $C'(X) \subset C(X)$ be the set of bounded, continuous and concave functions on X, and $C''(X) \subseteq C'(X)$ be the set of strictly concave functions. Since $C'(X)$ is a closed subset of the complete metric space $C(X)$, $C'(X)$ is a complete space such that $v \in C'(X)$. That is, v is concave.

To show that v is strictly concave, i.e., $v \in C''(X)$, it is sufficient to show that $T[C'(X)] \subseteq C''(X)$. That is, for any $f \in C'(X)$, we need to show $Tf \in C''(X)$, i.e., Tf is strictly concave.

Consider any $x, x' \in X, x \neq x', \lambda \in (0,1)$, we need to show

$$(Tf)(\lambda x + (1-\lambda)x') > \lambda(Tf)(x) + (1-\lambda)(Tf)(x')$$

As X is a convex set, $\lambda x + (1-\lambda)(Tf)(x')$.

Let $y \in \Gamma(x)$ attain the maximum $(Tf)(x)$, and $y' \in \Gamma(x')$ attain $(Tf)(x')$, i.e.,

$$\begin{aligned}(Tf)(x) &= F(x,y) + \beta f(y) \\ (Tf)(x') &= F(x,y') + \beta f(y')\end{aligned}$$

Since Γ is convex, $\lambda y + (1-\lambda)y' \in \Gamma(\lambda x + (1-\lambda)x')$. So,

$$\begin{aligned}(Tf)(\lambda x + (1-\lambda)x') &= \max_{y \in \Gamma(\lambda x + (1-\lambda)x')} F(\lambda x + (1-\lambda)x', y) + \beta f(y) \\ &\geq F(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') + \beta f(\lambda y + (1-\lambda)y') \\ &> [\lambda F(x,y) + (1-\lambda)F(x',y')] + \beta[\lambda f(y) + (1-\lambda)f(y')] \\ &= \lambda[F(x,y) + \beta f(y)] + (1-\lambda)[F(x',y') + \beta f(y')] \\ &= \lambda(Tf)(x) + (1-\lambda)(Tf)(x')\end{aligned}$$

And hence v is strictly concave.

Consider the maximisation problem on the RHS of the (FE): since $v(y)$ is strictly concave in y, $F(x,y)$ is strictly concave in x and y jointly, the objective function $F(x,y) + \beta v(y)$ is strictly concave in y. In addition, Γ is convex- valued.

It follows from the corollary to the Theorem of the Maximum that the maximum is attained at a unique y value. Hence, G is a continuous and single-valued function.

(f) Exercise 6.1 Write down the Euler equations and TVC for the deterministic growth model. Let

$$u(c) = log(c), F(k,n) = k^\alpha n^{1-\alpha}, \delta = 1$$

then $f(k) \equiv F(k,1) + (1-\delta)k = k^\alpha$. Conjecture that the policy function is of the form $k_{t+1} = ak_t^\alpha$, and find a.

The (FE) for the deterministic growth model is

$$\begin{aligned}v(k) &= \max_{k' \in [0, f(k)]} \{u(f(k) - k') + \beta v(k')\} \text{ for all k} \\ f(k) &= F(k,1) + (1-\delta)k\end{aligned}$$

For the TVC, since the general form is:

$$\lim_{t \to \infty} \beta^t F_1(x_t, x_{t+1}) \cdot x_t = 0$$

So for the deterministic growth model, the form is:

$$\lim_{t \to \infty} \beta^t u'(f(k_t) - k_{t+1}) f'(k_t)(k_t) = 0$$

The Euler equation is:

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}), t = 0, 1, 2, ...$$

Since $u(c) = log(c)$ and $f(k) \equiv F(k,1) + (1-\delta)k = k^\alpha$, the Euler equation can be rewritten as:

$$\begin{aligned}\frac{1}{c_t} &= \beta \frac{1}{c_{t+1}} \alpha k_{t+1}^{\alpha-1} \\ \frac{1}{k_t^\alpha - k_{t+1}} &= \frac{\alpha\beta k_{t+1}^{\alpha-1}}{k_{t+1}^\alpha - k_{t+2}}\end{aligned}$$

Since we conjecture that the policy function is of the form $k_{t+1} = ak_t^\alpha$, then $k_{t+2} = ak_{t+1}^\alpha$. We plug this expression back into the Euler equation to get:

$$\frac{1}{k_t^\alpha - k_{t+1}} = \frac{\alpha\beta k_{t+1}^{\alpha-1}}{(1-\alpha)k_{t+1}^\alpha} = \frac{\alpha\beta}{(1-\alpha)k_{t+1}}$$

Equivalently,

$$\begin{aligned}
&(1-a)k_{t+1} = \alpha\beta(k_t^\alpha - k_{t+1}) \\
&\implies (1-a+\alpha\beta)k_{t+1} = \alpha\beta k_t^\alpha \\
&\implies k_{t+1} = \frac{\alpha\beta}{(1-a+\alpha\beta)}k_t^\alpha
\end{aligned}$$

By the conjecture, $k_{t+1} = ak_t^\alpha$, so:

$$\begin{aligned}
&\frac{\alpha\beta}{(1-a+\alpha\beta)} = a \\
&\implies (1-a)(\alpha\beta - a) = 0 \\
&a = 1 \text{ or } a = \alpha\beta
\end{aligned}$$

That is, $k_{t+1} = k_t^\alpha$ or $k_{t+1} = \alpha\beta k_t^\alpha$.

But $k_{t+1} = k_t^\alpha$ cannot be a solution, because under this solution,

$$c_t = k_t^\alpha - k_{t+1} = 0$$

such that the TVC

$$\lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t = 0$$

cannot hold. So, the policy function is given by

$$k_{t+1} = \alpha\beta k_t^\alpha$$

(g) Exercise 6.2: Find the steady state of the deterministic growth model discussed in Lecture 2.

(refer to Tutorial 3 for solution).

7. Lecture 7

(a) Stochastic dynamic programming

An infinite-horizon dynamic stochastic optimisation problem generally takes the form:

$$\begin{aligned}
\text{(SP)} \qquad &\sup_{\{x_{t+1}\}_{t=0}^\infty} \mathbb{E}_0\{\sum_{t=0}^\infty \beta^t F(x_t, x_{t+1}, z_t)\} \\
&\text{s.t. } x_{t+1} \in \Gamma(x_t, z_t), t = 0, 1, 2, ... \\
&x_0 \in X \text{ and } z_0 \in Z \text{ given.}
\end{aligned}$$

z_t is the period-t value of the exogenous stochastic variable, which is known at the time x_{t+1} is chosen.
Z is the set of possible values for $z_t, t = 0, 1, ...$
$S \equiv X \times Z$ is the set of possible states for (x_t, z_t), or the **state space**.
The expectation $\mathbb{E}$ is over all possible realisations of $\{z_t\}_{t=1}^\infty$.

Typically, z is assumed to follow a first-order Markov process, or Markov chain (i.e., z_t only depends on z_{t+1}.

A discrete-time stochastic process $\{z_t\}$ is a first-order Markov process if for all $t = 0, 1, ...,$:

$$Prob(z_{t+1}|z_t, z_{t-1}, ..., z_{t-k}) = Prob(z_{t+1}|z_t) \equiv Q(z_t, z_{t+1})$$

for all $k \geq 1$, where $Q(z, z')$ is the **transition function** of the Markov process.

If in addition, z_t takes only finite values $\{e_1, ..., e_n\}$, z_t follows a n-state Markov chain, with transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & ... & \pi_{1n} \\ \pi_{21} & \pi_{22} & ... & \pi_{2n} \\ ... & ... & ... & ... \\ \pi_{n1} & \pi_{n2} & ... & \pi_{nn} \end{bmatrix}$$

Where $\pi_{ij} = Prob(z_{t+1} = e_j | z_t = e_i)$, and $\sum_{j=1}^{n} \pi_{ij} = 1$ (i.e., each row sums to 1).

Now consider the functional equation corresponding to the sequence problem:

$$\text{(FE)} \qquad v(x, z) = \sup_{y \in \Gamma(x,z)} \{F(x, y, z) + \beta E_{z'|z} v(y, z')\}, \forall (x, z) \in S$$

i. The state variables are $x \in X$ and $z \in Z$. z' is not realised yet, but z is already realised.
ii. $\beta E_{z'|z} v(y, z')$ is the discounted expected continuation value, where

$$E_{z'|z} v(y, z') = \int_z v(y, z') Q(z, z') dz'$$

Note: $E_{z'|z} v(y, z')$ is just another way of writing $E[v(y, z')|z]$.

The integral expression assumes that z follows a continuous Markov process. If z_t takes only finite values and hence follows a n-state Markov chain, then instead of the integral we could use $\sum_{j=1}^{n} v(y_i, e_j)\pi_{ij}$ if $z = e_i$.

iii. The contraction mapping form is:

$$(Tv)(x, z) = \sup_{y \in \Gamma(x,z)} \{F(x, y, z) + \beta E_{z'|z} v(y, z')\}, \forall (x, z) \in S$$

$T : C(S) \to C(S)$, i.e., if $v \in C(S)$, we want $Tv \in C(S)$ as well.

We assume that $F(x, y, z)$ is bounded and continuous, so we need $E_{z'|z} v(y, z')$ to be bounded and continuous as well (i.e., we need boundedness and continuity to be preserved under expectation).

Like in the deterministic case, we care about the equivalence of solutions to the (SP) and (FE), and characterisation of solutions to the (FE) and their properties (see SLP Chapter 9; not covered in detail in this course).

Basically, in addition to the previous assumptions, if the exogenous shock z_t satisfies **either** of the following two conditions, then all the previous theorems still hold:

i. Z is a countable set (can be finite or infinite, but still countable). This allows the expectation to be written as a summation where $E_{z'|z} v(y, z') = \sum_{j=1}^{n} v(y_i, e_j)\pi(e_j|z)$.
ii. Z is a compact set and the transition function Q has the Feller property:

$$E(f(z')|z) = \int_z f(z') Q(z, z') dz' \in C(Z), \ \forall f \in C(Z)$$

i.e., $E_{z|z'} v(y, z') = \int_z f(z') Q(z, z') dz'$ is a bounded continuous function in the space of z. z' will be integrated out and the function will be of y and z'.

The conditions above ensure that the required properties of the value function are preserved under integration.

(b) **Stochastic Euler equation and transversality condition**

Suppose the optimal y is in the interior of $\Gamma(x,z)$, then the FOC for the maximisation problem in the (FE) is given by:

$$F_2(x,y,z) + \beta E_{z'|z} v_1(y,z') = 0, \text{ i.e.,}$$
$$F_2(x_t, x_{t+1}, z_t) + \beta E_{z_{t+1}|z_t} v_1(x_{t+1}, z_{t+1}) = 0$$

Note: F_2 denotes the derivative with respect to the second term. We are changing the order of expectation and differentiation, which is permitted because expectation is either a summation or an integral.

By the Envelope Theorem, the derivative of the value function is equal to the derivative of the objective function evaluated at the optimal point:

$$v_1(x,z) = F_1(x,y,z), \text{ i.e.:}$$
$$v_1(x_t, z_t) = F_1(x_t, x_{t+1}, z_t)$$
$$v_1(x_{t+1}, z_{t+1}) = F_1(x_{t+1}, z_{t+2}, z_{t+1})$$

Since these hold for all t, substituting back into the FOC, we get the **stochastic Euler equation**:

$$F_2(x_t, x_{t+1}, z_t) + \beta E_{z_{t+1}|z_t} F_1(x_{t+1}, x_{t+2}, z_{t+1}) = 0, t = 0, 1, ...$$

Similarly, we can write the **stochastic TVC** as:

$$\lim_{t\to\infty} \beta_t E_0[F_1(x_t, x_{t+1}, z_t) \cdot x_t] = 0$$

The optimal sequence must satisfy both the stochastic Euler equation and the stochastic TVC.

The behaviour of the sequence of state variables $\{s_t\} = \{(x_t, z_t)\}$ from a given $s_0 = (x_0, z_0)$ is described by the transition function Q (the stochastic process which governs the evolution of z_t)and the policy function $g : S \to X$ (which governs the evolution of x_t).

Theorem 9.13 in SLP establishes that $\{(x_t, z_t)\}$ is a Markov process, with a transition function P defined in terms of Q and g, both of which are one-period dependent.

In the presence of stochastic shocks, the steady state features a stationary or invariant or ergodic distribution over (x_t, z_t) (i.e., not a fixed point steady state but a stationary distribution over (x_t, z_t).

The key questions then are:

i. Under what conditions on P does there exist a unique stationary distribution over (x_t, z_t)?
ii. Under what conditions on P does the system converge to this stationary distribution?

These are covered in SLP Chapter 11-12 (not covered in this course).

(c) Application 1: Cake-eating problem with taste shocks

Suppose the agent's appetite is subject to uncertainty, i.e., utility over consumption is given by $\epsilon u(c)$, where ϵ is a random variable.

Assume that the agent knows the value of the taste shock when making current decisions but does not know the future values of this shock.

Assume that the taste shock takes on only two values: $\epsilon \in \{\epsilon_h, \epsilon_l\}$, with $\epsilon_h > \epsilon_l > 0$. Let the transition probability (i.e., conditional probability of the next period's ϵ) be:

$$\pi_{ij} \equiv Prob(\epsilon' = \epsilon_j | \epsilon = \epsilon_i), i, j \in \{h, l\}$$

This follows a 2-state Markov chain with a transition matrix given by

$$\Pi = \begin{bmatrix} \pi_{hh} & \pi_{lh} \\ \pi_{lh} & \pi_{ll} \end{bmatrix}$$

The state of the system is described by 2 variables: size of the cake, and the taste shock. The (FE) is written as:

$$V(W, \epsilon_i) = \max_{W' \in [0,W]} \{\epsilon_i u(W - W') + \beta E_{\epsilon'|\epsilon_i} V(W', \epsilon')\}$$

for all $W \in [0, \hat{W}]$ and $i \in h, l$, where:

$$E_{\epsilon'|\epsilon_i} V(W', \epsilon') = \sum_{j=h,l} \pi_{i,j} V(W', \epsilon_j)$$

Note: the second term is basically $\pi_{i,h} V(W', \epsilon_h) + \pi_{i,l} V(W', \epsilon_l)$

Since ϵ takes on finite values, the assumptions that u is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions ensure that the value function and the policy function from the (FE) are unique and have nice properties.

(d) Exercise 7.1: Write down the intertemporal consumption Euler equations for the cake-eating problem:

$$\epsilon_t u'(W_t - W_{t+1}) = \beta E_{\epsilon_{t+1}|\epsilon_t}[\epsilon_{t+1} u'(W_{t+1} - W_{t+2})],\ t = 0, 1, 2...$$

The Euler equations explicitly or implicitly determine the optimal policy function

$$W_{t+1} = g(W_t, e_t)$$

(e) Application 2: Stochastic growth model:

Time is discrete, and the time horizon is infinite, $t = 0, 1, ...$

Population: The economy is populated with a large number of identical, infinitely-lived households. Each household is endowed with 1 unit of time, to allocate between leisure l_t and work n_t. In addition, the households own an initial stock of capital K_0, which they rent to firms and may augment through investment.

Preferences: All households have identical preferences over consumption and leisure. For a given sequence of $\{(c_t, l_t)\}_{t=0}^{\infty}$, a household's lifetime utility is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \beta \in (0, 1) \tag{7}$$

Technology: Production has constant returns to capital and labour:

$$Y_t = z_t F(K_t, n_t)$$

F is strictly increasing in both inputs, exhibits constant returns to scale, and is strictly concave. The total factor productivity z_t is subject to exogenous shocks. With $z_0 > 0$ given, z_t evolves according to

$$log(z_{t+1}) = \rho log(z_t) + \epsilon_{t+1}$$

where $0 < \rho < 1$ and e_t are iid $\sim N(0, \sigma_\epsilon^2)$ on a truncated closed interval (e.g., $[-4\sigma_\epsilon, 4\sigma_\epsilon]$.
The law of motion of the aggregate capital stock is

$$K_{t+1} = (1 - \rho)K_t + I_t$$

where $\delta \in [0, 1]$ is the depreciation rate of capital, and I_t is the aggregate investment in capital.

Market structure: Labour, capital and goods markets are all perfectly competitive.

Timing of events in period t:

i. ϵ_t is realised, i.e., z_t is realised and observated to all parties.
ii. Firms hire labour N_t and rent capital K_t from households.

iii. Firms produce goods $Y_t = z_t F(K_t, N_t)$ and pay labour and capital.
iv. Households divide their income into consumption C_t and savings S_t (in equilibrium, $S_t = I_t$). In making this decision, households need to predict z_{t+1}.

Equilibrium concept: Since there are no distortions in the economy, the Pareto optimum coincides with the competitive equilibrium.

The social planner's problem is to maximise the expected utility of a representative household subject to the resource feasibility constraints:

$$C_t + K_{t+1} \leq z_t F(K_t, N_t) + (1-\delta)K_t, t = 0, 1, 2...$$

Dividing by the number of households N, transforms it into per-capita terms:

$$c_t + k_{t+1} \leq z_t F(k_t, n_t) + (1-\delta)k_t, t = 0, 1, 2...$$

The social planner's problem is given by the sequence problem

$$\begin{aligned}
&\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} E_0[\sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)] \\
&\text{s.t.} \qquad c_t + k_{t+1} \leq z_t F(k_t, n_t) + (1-\delta)k_t \\
&0 \leq n_t \leq 1 \\
&log(z_{t+1}) = \rho log(z_t) + \epsilon_{t+1} \\
&k_0, z_0 \text{ given, } c_t, k_{t+1} \geq 0
\end{aligned}$$

where an optimum requires that the first constraint holds with equality.

If we assume that u does not depend on n_t, i.e., households do not value leisure, then an optimum requires $n_t = 1$, for all t.

Define

$$f(k_t) = F(k_t, 1)$$

Then the sequence problem can be reduced to

$$\begin{aligned}
&\max_{\{k_{t+1}\}_{t=0}^{\infty}} E_0\{\sum_{t=0}^{\infty} \beta^t u[z_t f(k_t) + (1-\delta)k_t - k_{t+1}]\} \\
&\text{s.t.} \qquad 0 \leq k_{t+1} \leq z_t f(k_t) + (1-\delta)k_t, t = 0, 1, ... \\
&log(z_{t+1}) = \rho log(z_t) + \epsilon_{t+1} \\
&k_0, z_0 \text{ given.}
\end{aligned}$$

The corresponding (FE) is:

$$v(k, z) = \max_{k' \in [0, zf(k)+(1-\delta)k]} u[zf(k) - (1-\delta)k - k'] + \beta E_{z'|z} v(k', z')$$

for all (k, z), where z' evolves following the law of motion

$$log(z') = \rho log(z) + \epsilon', \epsilon' \sim iid\ N(0, \sigma_\epsilon)$$

We can set the state space for k as $X \equiv [0, \bar{k}]$ for some large enough $\bar{k}$.

As ϵ follows a truncated normal distribution, the state space for z can also be set as a compact set. Then, conditions that ensure existence and uniqueness of a value function and policy function and nice properties of the value function hold. Further, conditions that ensure convergence to a unique stationary distribution over (k, z) also hold.

(f) Exercise 7.2: Write down the stochastic Euler equation and TVC for the stochastic growth model, and the equation that determines the deterministic steady state.

Note: Tutorial 4 illustrates that a special version of the stochastic growth model has a closed form solution and a unique stationary distribution.

8. Lecture 8

(a) Solution methods

In general, there are two classes of numerical methods for solving dynamic macro models:

i. One class works on the functional equation
ii. Another class works on Euler equations

For linear quadratic dynamic programming, the (FE) can be quickly solved using linear algebra. Other tools for solving the (FE) include value function iteration and collocation.

Methods for solving Euler equations include projection and perturbation models.

Numerical techniques involved: function approximation, numerical integration, Markov chain approximation to an AR(1) process, etc.

Linear quadratic dynamic programming refers to a class of dynamic programming problems where the return function (F) is quadratic and the transition function (Γ, the constraint set) is linear.

i. e.g., linear-quadratic optimal control problems.
ii. for a LQ optimal control problem, the solution is characterised by an optimal linear regulator (i.e., the optimal policy function is linear).

We can use LQ dynamic programming to approximate one that is non linear-quadratic.

A filtering or estimation problem that shares similar mathematical structure with the optimal linear regulator is the Kalman filter.

(b) **Optimal linear regulator problem**

The optimal linear regulator problem is to maximise over the choice of $\{u_t\}_{t=0}^{\infty}$ the criterion

$$-\sum_{t=0}^{\infty} \beta^t \{x_t^{'} R x_t + u_t' Q u_t\}, 0 < \beta < 1$$

subject to

$$x_{t+1} = Ax_t + Bu_t, x_0 \text{ given}$$

where:

i. x_t is a $n \times 1$ vector of state variables
ii. u_t is a $k \times 1$ vector of control variables
iii. R is a $n \times n$ positive semidefinite symmetric matrix
iv. Q is a $k \times k$ positive definite symmetric matrix
v. A is a $n \times n$ matrix
vi. B is a $n \times k$ matrix

Note: R and Q are invertible and the quadratic forms $x_t' R x_t \geq 0$ and $u_t' Q u_t \geq 0$.

The corresponding functional equation is:

$$V(x) = \max_u [-x'Rx - u'Qu + \beta V(Ax + Bu)]$$

We can solve the functional equation by "guess and verify". We guess that the value function is quadratic, i.e., $V(x) = -x'Px$ where P is a positive semidefinite symmetric matrix (i.e., $V(Ax + Bu) = -x'Px$.

Then, the functional equation becomes

$$-x'Px = \max_u [-x'Rx - u'Qu - \beta(Ax + Bu)'P(Ax + Bu)]$$

Since $\frac{\partial u'Qu}{\partial Q} = 2Qu$ and $\frac{\partial \beta(Ax+Bu)'P(Ax+Bu)}{\partial Q} = -2\beta B'P(Ax+Bu)$, the FOC for the maximisation problem with respect to u is:

$$-2Qu - 2\beta' B'P(Ax + Bu) = 0$$
$$(Q + \beta B'PB)u = -\beta B'PAx$$

Which implies the following feedback rule (i.e., policy function) for u:

$$u = \beta(Q + \beta B'PB)^{-1}B'PAx$$

or $u = -Fx$ (where F is the optimal control), where

$$F = \beta(Q + \beta B'PB)^{-1}B'PA$$

Substituting u into the functional equation and rearranging gives

$$P = R + \beta A'PA - \beta^2 A'PB(Q + \beta B'PB)^{-1}B'PA$$

This equation is the algebraic matrix **Riccati equation**. Under certain stability conditions, the Riccati equation has a unique positive semidefinite solution, which is approached by iterations on the equation:

$$P_{j+1} = R + \beta A'P_jA - \beta^2 A'P_jB(Q + \beta B'P_jB)^{-1}B'P_jA$$

when starting from a zero matrix P_0. The policy function associated with P_j is

$$F_{j+1} = \beta(Q + \beta B'P_jB)^{-1}B'P_jA$$

Stability: Upon substituting the optimal control $u_t = -Fx_t$ into the transition function $x_{t+1} = Ax_t + Bu_t$, we obtain

$$x_{t+1} = (A - BF)x_t$$

which governs the evolution under the optimal control (also, at steady state, $\bar{x} = (A - BF)\bar{x}$).

i. The system is globally stable for all $x_0 \in \mathbb{R}^n$ if the eigenvalues of $(A - BF)$ are all strictly less than unity in absolute value.
ii. If some eigenvalues are less than 1 and the rest equal or greater than 1, the system is saddle-path stable.

A modified version of the optimal linear regulator problem:

$$\max_{\{u_t\}_{t=0}^{\infty}} \{-\sum_{t=0}^{\infty} \beta^t\{x_t'Rx_t + u_t'Qu_t + 2u_t'Hx_t\}\}$$
$$\text{subject to} \qquad x_{t+1} = Ax_t + Bu_t$$
$$x_0 \text{ given.}$$

Note: The difference is the addition of a cross product H; in the first optimal regulator problem it's additively separable in x and u. By redefining the control variable, we can convert this problem into the previous form (see LS page 1020-1021).

The optimal policy has the form

$$u_t = -(Q + \beta B'PB)^{-1}\beta B'PA + H)x_t$$

where P solves the Riccati equation

$$P = R + \beta A'PA - (\beta A'PB + H')(Q + \beta B'PB)^{-1}(\beta B'PA + H)$$

(c) **Stochastic optimal linear regulator problem**

The stochastic discounted optimal linear regular problem is:

$$\max_{\{u_t\}_{t=0}^{\infty}} \{-E_0 \sum_{t=0}^{\infty} \beta^t \{x_t' R x_t + u_t' Q u_t\}\}$$

subject to x_0 given and the law of motion

$$x_{t+1} = Ax_t + Bu_t + C\epsilon_{t+1}$$

where ϵ_t is a $n \times 1$ vector of random variables $IID \sim N(0, I)$

The value function for the problem is

$$V(x) = -x'Px - \beta(1-\beta)^{-1} trace(PCC')$$

where P is the unique positive semidefinite solution of the Riccati equation.

Furthermore, the optimal policy function is still given by $u_t = -Fx_t$, where

$$F = \beta(Q + \beta B'PB)^{-1} B'PA$$

A notable feature of this solution is that the feedback rule is identical to that for the corresponding deterministic problem (i.e., the optimal decision rule is independent of the problem's noise statistics). This is called the **certainty equivalence** principle, an important property of the LQ optimal control problem.

An important use of linear-quadratic approximations is to approximate the solution of more complicated dynamic problems, e.g.:

i. finding the nonstochastic steady state (using the Euler equations)
ii. converting the constraints into linear form (by redefining variables or linearisation around the steady state)
iii. converting the objective function into the quadratic form

see LS, page 129-131 for details.

(d) Exercise 8.1

A household seeks to maximise

$$-\sum_{t=0}^{\infty} \beta^t \{(c_t - b)^2 + \gamma i_t^2\}$$

subject to

$$\begin{aligned} c_t + i_t &= ra_t + y_t \\ a_{t+1} &= a_t + i_t \\ y_{t+1} &= \rho_1 y_t + \rho_2 y_{t-1} \end{aligned}$$

Here, c_t, i_t, a_t, y_t are the household's consumption, investment, asset holdings and exogenous labour income at t, while $b > 0, \gamma > 0, r > 0, \beta \in (0,1)$ and ρ_1, ρ_2 are parameters and y_0, y_{-1} are initial conditions.

Assume that ρ_1, ρ_2 are such that $z^2 - \rho_1 z - \rho_2 = 0$ implies $|z| \leq 1$ (to ensure stability).

i. Map this problem into a modified optimal linear regulator problem.

   The linear quadratic problem is of the form:

$$-\sum_{t=0}^{\infty} \beta^t \{x_t' R x_t + u_t' Q u_t + 2u_t' H x_t\}$$
$$\text{subject to} \qquad x_{t+1} = Ax_t + Bu_t$$

For this specific problem,

$$x_t = (1, a_t, y_t, y_{t-1})'$$
$$u_t = i_t$$

We write the constraints into linear form:

$$\begin{bmatrix} 1 \\ a_{t+1} \\ y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho 1 & \rho 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ a_t \\ y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} i_t$$

which is of the form $x_{t+1} = Ax_t + Bu_t$.

We now write the return function $-[(c_t - b)^2 + \gamma i_t^2]$ into a quadratic. Based on the first constraint, $c_t = ra_t + y_t - i_t$, and hence

$$c_t - b = (-b, r, 1, 0)x_t - u_t \equiv Ex_t - u_t$$

where $E = (-b, r, 1, 0)$

The return function can be rewritten as

$$\begin{aligned} -[(c_t - b)^2 + \gamma i_t^2] &= -[(Ex_t - u_t)^2 + \gamma u_t^2] \\ &= -[(Ex_t - u_t)'(Ex_t - u_t) + \gamma u_t^2] \\ &= -[x_t' E' E x_t + (1 + \gamma)u_t^2 - 2u_t E x_t] \\ &\equiv -[x_t' R x_t + u_t' Q u_t + 2u_t' H x_t] \end{aligned}$$

where $R = E'E, Q = 1 + \gamma, H = -E$

Then, the value function is given by

$$v(x_t) = -x_t' P x_t$$

where P solves the Riccati equation

$$P = R + \beta A'PA - (\beta A'PB + H')(Q + \beta B'PB)^{-1}(\beta B'PA + H)$$

and the policy function is given by

$$u_t = -(Q + \beta B'PB)^{-1}\beta B'PA + H)x_t$$

ii. For parameter values $[\beta, 1 + r, b, \gamma, \rho_1, \rho_2] = [0.95, 0, 95^{-1}, 30, 1, 1.2, -0.3]$, compute the household's optimal policy function.

(refer to Tutorial 4, to be done in Matlab)

(e) **Kalman filter**

The Kalman filter is a recursive algorithm for estimating the expectation

$$E[x_t | y_t, ..., y_0]$$

of a hidden state vector x_t, conditional on observing a history $(y_t, ..., y_0)$ of a vector of noisy signals on the hidden state (i.e., we are trying to infer x_t when we can only observe y_t.

The Kalman filter can be used to formulate or simplify a variety of signal-extraction and prediction problems.

The setting for the Kalman filter is the following **linear state-space system**: Given x_0, let the transition equations for x_t and y_t be:

$$x_{t+1} = Ax + C\omega_{t+1}$$
$$y_t = Gx_t + v_t$$

where:

i. x_t is a $n \times 1$ state vector,

ii. ω_t is an iid Gaussian vector with $E\omega_t\omega_t' = I$,

iii. v_t is an iid Gaussian vector orthogonal to ω_s for all t, s with $Ev_tv_t' = R$

Assume that the initial condition x_0 is unobserved but is known to have a Gaussian distribution with mean $\hat{x}_0$ and covariance matrix Σ_0 (i.e., this is our Bayesian prior).

Our intent is to forecast x_{t+1} at time t (i.e., our forecast is $\hat{x}_{t+1} = E[x_{t+1}|y^t]$ given the history of observations $y^t \equiv [y_t, ..., y_0]$ while minimising the mean squared forecasting error $(x_{t+1} - \hat{x}_{t+1})^2$.

The recursive algorithm for computing $\hat{x}_{t+1}$ is the Kalman filter:

$$\begin{aligned}
\hat{x}_{t+1} &= (A - K_tG)\hat{x}_t + K_ty_t \\
K_t &= A\Sigma_tG'(G\Sigma_tG' + R)^{-1} \\
\Sigma_{t+1} &= A\Sigma_tA' + CC' - A\Sigma_tG'(G\Sigma_tG' + R)^{-1}G\Sigma_tA'
\end{aligned}$$

where $\Sigma_t = E(x_t - \hat{x_t})(x_t - \hat{x_t})'$ is the mean squared forecast error, K_t is the **Kalman gain**, x_0 and Σ_0 are given (the latter is from the Bayesian prior).

Note that the equation for Σ_{t+1} has similar structure to the Riccati equation.

(f) Exercise 8.2

Let the model be given by

$$\begin{aligned}
x_{t+1} = x_t + \omega_{t+1}, w_{t+1} \sim N(0, Q) \\
y_t = x_t + v_t, v_t \sim N(0, R)
\end{aligned}$$

where w_t and v_t are independent. Apply the Kalman filter to compute $\hat{x}_{t+1}$. What does the expression converge to as $t \to \infty$?

The standard form is

$$\begin{aligned}
x_{t+1} &= Ax + C\omega_{t+1}, \qquad \omega_{t+1} \sim \text{Gaussian } (0, I) \\
y_t &= Gx_t + v_t, \qquad V_t \sim \text{Gaussian } (0, R)
\end{aligned}$$

In this specific question, $A = 1, C = 1, G = 1, CC' = Q$

The standard form of the Kalman filter is:

$$\begin{aligned}
\hat{x}_{t+1} &= (A - K_tG)\hat{x}_t + K_ty_t \\
K_t &= A\Sigma_tG'(G\Sigma_tG' + R)^{-1} \\
\Sigma_{t+1} &= A\Sigma_tA' + CC' - A\Sigma_tG'(G\Sigma_tG' + R)^{-1}G\Sigma_tA'
\end{aligned}$$

So for this specific question:

$$\begin{aligned}
\hat{x}_{t+1} &= (1 - K_tG)\hat{x}_t + K_ty_t \\
K_t &= \Sigma_t(\Sigma_t + R)^{-1} \\
\Sigma_{t+1} &= \Sigma_t + Q - \Sigma_t(\Sigma_t + R)^{-1}\Sigma_t \\
\Sigma_{t+1} &= Q + \frac{\Sigma_tR}{\Sigma_t + R}, \text{ with } \Sigma_0 \text{ given.}
\end{aligned}$$

As $t \to \infty$, the $\{\Sigma_t\}$ process converges to a steady state, Σ, where Σ is a fixed point of $\Sigma_{t+1} = Q + \frac{\Sigma_t R}{\Sigma_t + R}$:

$$\begin{aligned}\Sigma &= Q + \frac{\Sigma_t R}{\Sigma_t + R} \\ \implies \Sigma^2 &- \Sigma Q - QR = 0 \\ \implies \Sigma &= \frac{Q + \sqrt{Q^2 + 4QR}}{2} \text{ (the negative root solution} < 0\text{, and is discarded)}\end{aligned}$$

Then, $K_t \to \frac{\Sigma}{\Sigma + R} \in (0, 1)$, and $\hat{x}_{t+1} = (1 - K_t G)\hat{x}_t + K y_t$.

9. Lecture 9

   (a) **Discrete-state dynamic programming**

   We can discretise the continuous state space X to convert the problem into a discrete-state dynamic programming problem, then solve it by value function iteration.

   We use the deterministic optimal growth model as an example, where:

$$\text{FE} \qquad v(k) = \max_{k' \in [0, f(k)]} \{u[f(k) - k'] + \beta v(k')\}$$

   for all $k > 0$, where $f(k) = F(k, 1) + (1 - \delta)k$.

   The Euler equation is

$$u'(c) = \beta u'(c') f'(k')$$

   To solve the (FE) numerically, we first need to specify a state space for k to work with. This state space should be a compact set containing feasible and meaningful values of k:

$$X \equiv [\underline{k}, \bar{k}]$$

   We can take $\underline{k}$ as k_0 (or a small positive number), while $\bar{k}$ should be higher than k_{ss}, where k_{ss} is the steady state value of k satisfying

$$1 = \beta f'(k_{ss})$$

   Then, we approximate the continuous state space using a discrete one (i.e., discretisation of the state space.
   Suppose we divide $[\underline{k}, \bar{k}]$ into n points, $\underline{k} = k_1, k_2, ...k_n = \bar{k}$. Then, the state space for k becomes

$$X_D \equiv \{k_1, k_2, ..., k_n\}$$

   In discretising the state space, we may consider using finer grids for the lower region of the state space (to improve precision).

   The (FE) for the discrete-state dynamic programming is:

$$\begin{aligned}v(k_i) = \max_{k_j \in X_D} &\{u[f(k_i) - k_j] + \beta v(k_j)\} \\ \text{subject to } &k_j \leq f(k_i)\end{aligned}$$

   for all $i = 1, 2, ..., n$, i.e., for all $k_i \in X_D$.
   Note: $k_j \leq f(k_i)$ is thus analogous to $k' \in [0, f(k)]$, but we don't need the lower bound at 0 because we have already set the lower bound at k_0 or a small positive number.
   Note: The RHS problem then becomes finding the maximum of a finite set of numbers.

   (b) Discrete-state value function iteration

By the CMT, the unique value function v satisfying the (FE) can be computed by value function iteration, i.e.:

$$v^{s+1}(k_i) = \max_{k_j \in X_D} \{u[f(k_i) - k_j] + \beta v^s(k_j)\}$$
$$\text{subject to } k_j \leq f(k_i)$$

starting from a given v^0 until $||v^{s+1} - v^s||_{sup} < \epsilon(1 + ||v^s||_{sup})$.
The CMT ensures uniform convergence to v.
ϵ can be set to a very small number, and $(1 + ||v^s||)$ is added for normalisation.
Note: In writing the computer code for VFI, use matrix operations instead of loops as much as possible.

(c) Exercise 9.1

Solve the following deterministic growth model using discrete-state value function iteration:

$$v(k) = \max_{k' \in [0, k^\alpha]} \{log(k^\alpha - k') + \beta v(k')\}$$

Let $\beta = 0.9932, \alpha = 0.36$. Plot the value function $v(k)$ and policy function $k'(k)$ against the true functions.

Basic idea for coding: Discrete-state VFI involves the following steps:

i. Define the state space $X \equiv [\underline{k}, \bar{k}]$. Note that $\bar{k}$ is defined based on k_{ss}, where $k_{ss} = \frac{\alpha\beta}{1/(1-\alpha)}$ since at the steady state, $1 = \beta f'(k_{ss})$.
ii. Discretise the state space and represent it by a state vector $(k_1, k_2, ..., k_n)'$
iii. Key step: Prepare matrices for value function iteration.
   A. In the RHS maximisation problem, $log(k^\alpha - k')$ does not involve v, so it can be defined before value function iteration.
   B. Define a matrix for capital stock

$$kM = \begin{bmatrix} k_1 & k_2 & ... & k_n \\ k_1 & k_2 & ... & k_n \\ ... & ... & ... & ... \\ k_1 & k_2 & ... & k_n \end{bmatrix}$$

   and define a matrix for all possible consumption levels

$$cM = \begin{bmatrix} k_1^\alpha - k_1 & k_2^\alpha - k_1 & ... & k_n^\alpha - k_1 \\ k_1^\alpha - k_2 & k_2^\alpha - k_2 & ... & k_n^\alpha - k_2 \\ ... & ... & ... & ... \\ k_1^\alpha - k_n & k_2^\alpha - k_n & ... & k_n^\alpha - k_n \end{bmatrix}$$

   where $cM(j, i)$ refers to consumption when $k = k_i$ and $k' = k_j$.
   To get $k' \leq k^\alpha$ to ensure $c > 0$, we define a matrix for $log(k^\alpha - k')$ as

$$uM = log(cM) \text{ if } cM > 0, = -\infty \text{ if } cM \leq 0$$

iv. Choose the initial guess for v, represented by a vector $(v_1, ..., v_n)'$ corresponding to the state vector, and a value for ϵ.
v. In each loop, construct the objective function as

$$vM = uM + \beta \begin{bmatrix} v_1 & v_1 & ... & v_1 \\ v_2 & v_2 & ... & v_2 \\ ... & ... & ... & ... \\ v_n & v_n & ... & v_n \end{bmatrix}$$

   Taking the maximum of vM gives the optimal $k'(k_i)$ and a new v_i for each k_i. Calculate the maximum difference between this new v and v.
vi. Stop the loop when the difference is smaller than the desired ϵ.

Matlab code for Exercise 9.1:

```
clear
clc

%% assign parameters

beta=0.9932;
alpha=0.36;

% compute kss
kss=(alpha*beta)^(1/(1-alpha));

% select state space [klb kub]
klb=0.001; % lower bound
kub=1.5*kss; % upper bound

%% define vectors and matrices

% discretise the state space
N=401;
inc=(kub-klb)/(N-1); % increments
kvec=[klb:inc:kub]'; % state vector

% initial guess for value function
V=kvec;
VNEW=zeros(N,1); % update of value function

% define matrices to record current utility
kM=ones(N,1)*kvec';
cM=kM.^alpha-kM'; % cM(i,j) gives consumption when k=k_j, k'=k_i
UM=zeros(N,N);  % records current utilities
UM(cM<=0)=-inf; % for nonfeasible c, utility is negative infinity
UM(cM>0)=log(cM(cM>0)); % for feasible c, u(c)=log(c)

%% beginning value function iteration
Vtol=100;
epsilon=1.e-6;

while Vtol>epsilon
    VpM=V*ones(1,N);
    VM=UM+beta*VpM;
    [VNEW I]=max(VM); % I records the index of the optimal k'
    VNEW=VNEW'; % transform into a column vector
    I=I';

    Vtol=max(abs(VNEW-V))/(1+max(abs(V))) % sup norm

    V=VNEW;
end
kp=kvec(I); % optimal k'

% define the true value function
a=alpha/(1-alpha*beta);
b=1/(1-beta)*(log(1-alpha*beta)+beta*a*log(alpha*beta));

figure
plot(kvec,V,kvec,a*log(kvec)+b,':')
title('value function')
xlabel('k')
ylabel('v')
```

```
legend('value function iteration', 'true value function','Location','southeast')

figure
plot(kvec,kp,kvec,alpha*beta*kvec.^alpha,':')
title('policy function')
xlabel('k')
ylabel('k prime')
legend('value function iteration', 'true policy function','Location', 'southeast')
```

10. Lecture 10

   (a) General algorithm for continuous-state value function iteration

   Consider the general form of the functional equation

   $$\text{(FE)} \qquad v(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\} \text{ for all } x \in X$$

   Instead of discretisation (as was done in Lecture 9), we can approximate continuous functions with continuous functions.

   Let $v(x) \approx \hat{v}(x;a)$, where $\hat{v}$ is some continuous function with parameters a.

   i. What functional form of $\hat{v}$ to choose depends on which functional approximation method we want to use.
   ii. Once that is determined, we want to find coefficients a such that $\hat{v}(x;a)$ approximately solves the (FE).

   The algorithm for continuous-state value function iteration generally involves the following steps:

   i. Initialise: Choose functional form for $\hat{v}(x;a)$ and choose the approximation grid $X_A = \{x_1, ..., x_n\}$. Make initial guess $\hat{v}(x;a^0)$ and choose stopping criteria $\epsilon > 0$.
   ii. Maximisation step: Given $\hat{v}(x;a^s)$, solve the RHS maximisation problem in the (FE) for each $x_i \in X_A$, and let v_i^{s+1} denote the maximum attained:
   $$v_i^{s+1} = \max_{y \in \Gamma(x_i)} \{F(x_i, y) + \beta \hat{v}(y;a^s)\}$$
   iii. Fitting step: Using the chosen function approximation method, compute new coefficients a^{s+1} such that $\hat{v}(x;a^{s+1})$ fits the (x_i, v_i^{s+1}) points.
   iv. Calculate approximation error: If $||\hat{v}(x;a^{s+1}) - \hat{v}(x;a^s)||_{sup} < \epsilon(1 + ||\hat{v}(x;a^s)||_{sup})$, stop; otherwise go back to maximisation step.

   (b) Piecewise linear interpolation

   In the fitting step, an exact fit with (x_i, v_i) would involve interpolation, i.e., connecting successive (x_i, v_i) points.

   i. The simplest interpolation is to draw straight lines between successive points, i.e., **piecewise linear interpolation**.
   ii. That is, on each interval $[x_i, x_{i+1}]$, the value function v is approximated by a linear function, but the linear functions for different intervals are different.
   iii. We can also connect points by higher order polynomials; e.g., by 3rd-order polynomials (cubic splines).

   Suppose we start with a guess v_i^s for each x_i, then in the maximisation step

   $$v_i^{s+1} = \max_{y = \Gamma(x_i)} \{F(x_i, y) + \beta \hat{v}^s(y)\}$$

   we can compute $\hat{v}^s(y)$ according to the piecewise linear relationship defined by all (x_i, v_i^s) points. For instance, if $y \in [x_j, x_{j+1}]$, then

   $$\hat{v}^s(y) = v_j^s + \frac{v_{j+1}^s - v_j^s}{x_{j+1} - x_j}(y - x_j)$$

In Matlab, this is done using the command $interp1.m$.
Then, solve the maximisation problem to obtain a new maximum v_i^{s+1} for each x_i.
The new (x_i, v_i^{s+1}) points define a new piecewise linear relationship to be used in the next loop.
COnvergence is achieved when v_i^{s+1} and v_i^s are close enough for all i.

Piecewise linear interpolation is easy to implement and is shape-preserving (i.e., approximation preserves the monotonicity and concavity of the value function).
However, the kinks between linear functions for different intervals make the objective of the maximisation problem not differentiable, forcing us to use slower optimisation algorithms (such as $fminbnd.m$).

(c) Cubic spline interpolation

Recall that a drawback of piecewise linear interpolation is that $\hat{v}$ is not differentiable. More efficient polynomial interpolations use splines.
A spline is any smooth function that is piecewise polynomial but also smooth where the polynomial pieces connect (i.e., no kinks; differentiable). The **cubic spline** is the most popular.

Suppose we have the points $(x_i, v_i), i = 1, ..., n$. On each interval $[x_i, x_{i+1}]$, the cubic spline is a cubic function

$$a_i + b_i x + c_i x^2 + d_i x^3$$

So a cubic spline is represented as a list of the coefficients (a_i, b_i, c_i, d_i) along with a list of nodes x_i.
The coefficients differ for each "piece" of the function.

In Matlab, the command *spline.m* can calculate the coefficients and hence pin down the cubic spline for us, given $\{(x_i, v_i)\}_{i=1}^n$.
Compared with piecewise linear interpolation, the only difference in the algorithm is that when solving the maximisation problem

$$v_1^{s+1} = max_{y \in \Gamma(x_i)} \{F(x_i, y) + \beta \hat{v}(y)\}$$

$v\hat{(y)}$ is computed using the cubic spline that approximates the (x_i, v_i^s) points. Again, *spline.m* can do the job for us.

(d) Chebyschev Regression/ interpolation

In piecewise linear and cubic spline interpolation, the continuous value function is approximated by piecewise polynomials.
By Weierstrass' Theorem, any continuous function can be approximated by a polynomial function.

As a polynomial has the form:

$$p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

it is natural to use the monomials $x^n, n = 0, 1, ...$ as the basis functions to construct approximations to continuous functions.
However, this choice is not the best choice since these monomials are nearly linearly dependent.

We consider **orthogonal polynomials** as a base for the space of continuous functions, which would yield more efficient and precise approximation results.
The family of polynomials $\{\phi_n(x)\}$ is mutually orthogonal with respect to a weighting function $\omega(x)$ iff the inner product

$$< \phi_n, \phi_m > \equiv \int_a^b \phi_n(x) \phi_m(x) \omega(x) dx = 0, \text{ for } n \neq m$$

where $\omega(x)$ is a function on $[a, b]$ that is positive almost everywhere and has a finite integral on $[a, b]$.

The most popular orthogonal polynomials are **Chebyshev polynomials** $\{T_n(x)\}$, for $x \in [-1, 1]$, with weighting function $\omega(x) = (1 - x^2)^{-1/2}$
They can be constructed recursively:

$$\begin{aligned}
&T_0(x) = 1, \\
&T_1(x) = x, \\
&T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1, ...
\end{aligned}$$

or equivalently

$$T_n(x) = cos(ncos^{-1}(x)), n = 0, 1, 2, ...$$

We can verify that $< T_n, T_m >= 0$ for $n \neq m$.

Taking Chebyshev polynomials as the base, $v(x)$, $x \in [\underline{x}, \bar{x}]$ can be approximated by

$$\hat{v}(x, a) = \sum_{l=0}^{m} a_l T_l(\frac{2(x - \underline{x})}{\bar{x} - \underline{x}} - 1), \text{ for some } m$$

where $\frac{2(x-\underline{x})}{\bar{x}-\underline{x}} - 1$ transforms x to $[-1, 1]$ and $a \equiv (a_0, a_1, ..., a_m)$ are coefficients to be determined.
We can incorporate **Chebyshev regression** into the value function iteration:

i. Compute the $n \geq m + 1$ Chebyshev interpolation nodes on $[-1, 1]$:

$$z_i = -cos(\frac{2i - 1}{2n}\pi), i = 1, ..., n$$

These nodes are the n roots of $T_n(x) = 0$.

ii. Adjust the nodes to the state space for x, $[\underline{x}, \bar{x}]$:

$$x_i = \underline{x} + (z_i + 1)(\frac{\bar{x} - \underline{x}}{2}), i = 1, ..., n$$

iii. Maximisation step: With $\hat{v}(x; a^s)$ given by the equation for the Chebyshev polynomials above, solve the RHS maximisation problem in the (FE) for all nodes x_i to obtain the maximum v_i:

$$v_i^{s+1} \equiv \max_{y \in \Gamma(x)} \{F(x_i, y) + \beta\hat{v}(y; a^s)\}, i = 1, ..., n$$

iv. Fitting step: Compute the updated Chebyshev coefficients a^{s+1} to fit the $(x_i, v_i^{s+1}$ points:

$$a_l^{s+1} = \frac{\sum_{i=1}^{n} T_l(z_i)v_i^{s+1}}{\sum_{i=1}^{n} T_l(z_i)^2}, l = 0, 1, ..., m$$

Note that a^{s+1} are regression coefficients from the following regression:

$$\begin{bmatrix} v_1^{s+1} \\ v_2^{s+1} \\ ... \\ v_n^{s+1} \end{bmatrix} = \begin{bmatrix} T_0(z_1) & T_1(z_1) & ... & T_m(z_1) \\ T_0(z_2) & T_1(z_2) & ... & T_m(z_2) \\ ... & ... & ... & ... \\ T_0(z_n) & T_1(z_n) & ... & T_m(z_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ ... \\ a_m \end{bmatrix} + \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ ... \\ \epsilon_n \end{bmatrix}$$

Then the updated value function is given by

$$\hat{v}(x; a^{s+1}) = \sum_{l=0}^{m} a_l^{s+1} T_l[\frac{2(x - \underline{x})}{\bar{x} - \underline{x}} - 1]$$

Convergence is achieved if a^{s+1} and a^s are close enough or $\{v_i^{s+1}\}$ and $\{v_i^s\}$ are close enough. Otherwise, return to the maximisation step.
If $n = m + 1$, then $\hat{v}(x; a^s)$ passes through (x_i, v_i) points in every fitting step, so the approximation is an interpolation (**Chebyshev interpolation**).

(e) Exercise 10.1:

Solve the deterministic growth model in Exercise 9.1 using piecewise linear interpolation and cubic spline.

Matlab code for Exercise 10.1:

```
clear
clc

%% assign parameter values and define state space

global beta alpha;
beta=.9932;
alpha=0.36;

% compute kss
kss=(1/(beta*alpha))^(1/(alpha-1));

% define the state space
klb=0.001; % lower bound
kub=1.5*kss; % upper bound

% define nodes for k
global kvec;
N=51;
inc=(kub-klb)/(N-1); % increments
kvec=[klb:inc:kub]'; % state vector

%% initial guess for value function
global V;
V=kvec; % initial guess for the value function

VNEW=zeros(N,1); % update of value function
kp=zeros(N,1); % k'(k)

%% value function iteration
Vtol=100;
tic
while Vtol>1.e-6;
    % for each k, solve for k'
    for i=1:N
        % find the optimal k', for given k_i
        %[kp(i) v]=fminbnd(@linearinterp_vfun,0,kvec(i)^alpha,[],kvec(i)); % piecewise linear
        [kp(i) v]=fminbnd(@splineinterp_vfun,0,kvec(i)^alpha,[],kvec(i)); % cubic spline
        VNEW(i)=-v;
    end;
    Vtol=max(abs(VNEW-V))/(1+max(abs(V))) % use sup norm
    V=VNEW;
end;
toc

%% plotting

% define true value function
a=alpha/(1-alpha*beta);
b=1/(1-beta)*(log(1-alpha*beta)+beta*a*log(alpha*beta));

figure
plot(kvec,V,kvec,a*log(kvec)+b,':')
title('value function')
xlabel('k')
ylabel('v')
legend('computed', 'true','Location','southeast')

figure
plot(kvec,kp,kvec,alpha*beta*kvec.^alpha,':')
title('policy function')
```

```
xlabel('k')
ylabel('k prime')
legend('computed', 'true','Location', 'southeast')

%% save results
save exercise10_1;
```

11. Lecture 11

(a) Projection method

The methods we have introduced so far all seek to solve the functional equations.

In Lecture 11 and 12, we will introduce solution methods that seek to find the policy functions from equilibrium conditions without solving the value function:

i. projection methods (global approximation)
ii. perturbation methods (local approximation)

Projection methods take basis functions to construct an approximated policy function that approximately solves the Euler equations. This method can also be used to solve the (FE).

Using the simple deterministic growth model as an example, the (SP) is:

$$\begin{aligned}&\text{(SP):} \qquad \max_{\{k_{t+1}\}_{t=0}^{\infty}} \{\sum_{t=0}^{\infty} \beta^t u[f(k_t) - k_{t+1}]\} \\ &\text{subject to } 0 \leq k_{t+1} \leq f(k_t), t = 0, 1, ... \\ &k_0 \text{ given}\end{aligned}$$

And the Euler equation is

$$u'[f(k_t) - k_{t+1}] = \beta u'[f(k_{t+1} - k_{t+2}]f'(k_{t+1})$$

The unknown true policy function $k_{t+1} = g(k_t)$ satisfies the Euler equation:

$$u'[f(k_t) - g(k_t)] = \beta u'[f(g(k_t)) - g[g(k_t)]]f'(g(k_t))$$

As the equation above holds for all t, we can drop the subscript t and rewrite as

$$u'[f(k) - g(k)] - \beta u'[f(g(k)) - g(g(k))]f'(g(k)) = 0 \text{ for all } k$$

or equivalently,

$$H(g(k)) \equiv u'[f(k) - g(k)] - \beta u'[f(g(k)) - g(g(k))]f'g(k)) = 0$$

where the only unknown function in H is g.

Our goal is to find an approximation $\hat{g}(k)$ of $g(k)$ for which the Euler equation is approximately satisfied. There are two issues to resolve:

i. We need to choose a functional form for $\hat{g}(k)$.
ii. We need to define a metric to evaluate the fit of the approximation.

Choosing the functional form for $\hat{g}(k)$:

i. Let $\{\phi_l(k)\}$ be a set of basis functions defined on $k \in X \equiv [\underline{k}, \bar{k}]$. We can approximate $g(k)$ by

$$\hat{g}(k; a) = \sum_{l=0}^{m} a_l \phi_l(k)$$

ii. As discussed in Lecture 10, orthogonal polynomials constitute better basis functions for approximating continuous functions.

Defining a metric to evaluate the fit of an approximation

i. For a given approximation $\hat{g}(k;a)$, we can compute $H(\hat{g}(k;a)$ for all k. Then we need to define a metric to measure how close $H(\hat{g})$ is to zero (a **"residual function"**).

ii. In general, this metric is defined as the inner product of $H(\hat{g})$ and some weighting function $w(k)$ on X:

$$< H(\hat{g}(\cdot;a)), w >= \int_x H(\hat{g}(k;a))w(k)dk$$

So this projection method is also known as a **weighted residual method**.

iii. Once a metric is defined to measure the distance from the residual function to zero, we seek to find parameters a such that this metric is minimised.

The choice of w defines different projection methods. The broad classes of projection methods include:

i. The **least square method**: A simple choice for w is the residual function itself. In our example, the problem amounts to finding a that solves

$$\min_a < H(\hat{g}(\cdot;a)), H(\hat{g}(\cdot;a)) >$$

ii. The **collocation method**: This method, by selecting the weighting function, sets the residuals to zero on the nodes such that the approximation is exact on these nodes, and nothing is imposed outside these nodes. In our example,

$$w(k) = \begin{cases} 1, & k = k_i \\ 0, & k \neq k_i \end{cases}$$

So the problem amounts to finding a such that

$$H(\hat{g}(k_i;a)) = 0, i = 1, 2, ...n$$

(b) Chebyshev collocation

We apply the collocation method to solve the Euler equation of the growth model where we approximate the policy function with Chebyshev polynomials.

The algorithm of the Chebyshev collocation:

i. Choose the degree m of Chebyshev polynomial approximation to $g(k)$ on $[\underline{k}, \bar{k}]$, i.e., the basis functions for constructing $\hat{g}$ are $\{T_l\}_l^m = 0$.

ii. Compute the $m+1$ collocation points on $[-1, 1]$:

$$z_i = -cos(\frac{2_i - 1}{2(m+1)}\pi), i = 1, ..., m+1$$

iii. Adjust the nodes to the $[\underline{k}, \bar{k}]$ interval:

$$k_i = \underline{k} + (z_i + 1)(\frac{\bar{k} - \underline{k}}{2}), i = 1, ..., m+1$$

iv. For a given set of parameters $a = (a_0, a_1, ..., a_m)$, construct the Euler residual function:

$$H(\hat{g}(k_i;a)) \equiv u'[f(k_i) - \hat{g}(k_i;a)] - \beta u'[f(\hat{g}(k_i;a)) - \hat{g}(\hat{g}(k_i,a);a)]f'(\hat{g}(k_i;a))$$

where

$$\hat{g}(k_i;a) = \sum_{l=0}^m a_l T_l[2\frac{k_i - \underline{k}}{\bar{k} - \underline{k}} - 1]$$
$$\hat{g}(\hat{g}(k_i;a);a) = \sum_{l=0}^m a_l T_l[2\frac{\hat{g}(k_i;a) - \bar{k}}{\bar{k} - \underline{k}} - 1]$$

for all $i = 1, ..., m+1$.

v. The optimal a is the solution to

$$H(\hat{g}(k_i;a)) = 0, i = 1, ..., m+1$$

Note that this is a system of $m+1$ nonlinear equations in $m+1$ unknowns: $a_0, ..., a_m$. The Matlab function $fsolve.m$ can be used to solve for a.

The Chebyshev collocation method is very fast when it works, but Chebyshev polynomials tend to display oscillating behaviour at higher orders (so can start with smaller m instead).
In implementation, solving the system of nonlinear equations can be tricky and convergence to the true solution often hinges on a good initial guess for a.

Can we apply the projection method to solve the (FE)?

i. In solving the (FE), we can avoid value function iteration by using the collocation method.
ii. The basic idea is to find the coefficients a such that the approximation to the value function is exact at the chosen nodes:

$$\hat{v}(k_i;a) = \max_{k' \in [0, f(k_i)]} \{u(f(k_i) - k') + \beta \hat{v}(k';a)\} \tag{Eq 6}$$

Suppose $\hat{v}(x;a)$ is constructed as

$$\hat{v}(k;a) = \sum_{l=0}^{m} a_l \phi_l(k)$$

where $\{\phi_l\}_{l=0}^m$ denotes the $m+1$ known basis functions chosen to construct the approximation.
iii. Then, according to Eq 6, $a \equiv (a_0, a_1, ..., a_m)$ satisfies

$$\sum_{l=0}^{m} a_l \phi_l(k_i) = \max_{k' \in [0, f(k_i)]} \{u(f(k_i) - k' + \beta \sum_{l=0}^{m} a_l \phi_l(k')\}, i = 1, ..., m+1 \tag{Eq 7}$$

Note that Eq 7 represents $m+1$ linear equations in $m+1$ unknowns, $a_0, a_1, ..., a_m$.
iv. So, the collocation method transforms the problem into solving a system of nonlinear equations. We can use the command $fsolve.m$ in Matlab to do this.

(c) Endogenous grid method

The **endogenous grid method** (EGM) is a projection method which can be much faster than standard projection methods.

The basic idea:

i. Construct a grid for k'
ii. Suppose the policy function for c is given by $c = h(k)$, then

$$c' = h(k')$$

which can be constructed given the grid for k'.
iii. By the Euler equation

$$u'(c) = \beta u'(c') f'(k')$$

we can easily solve for c corresponding to each k'.
iv. By the resource constraint

$$c + k' = k^\alpha$$

we can easily solve for the current state k corresponding to each k'. Then we have an updated $c = h(k)$.
v. Convergence is achieved if the implied c' or k converges.

The EGM gets is name as the grid for the current state keeps changing in each iteration.

(d) Exercise 11.1: Consider the deterministic growth model as given in Exercise 9.1, and compute the policy function by solving the Euler equation with Chebyshev collocation.

12. Lecture 12

(a) Overview of perturbation method

Pertubation methods are used to find an approximate solution to a problem, starting from the exact solution to a related simpler problem.
Typically, the problem at hand can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

Two ways to apply linearisation:

i. Linearise the constraints and quadratically approximate the criterion function, i.e., to transform the problem into a linear-quadratic problem to solve.
ii. Derive the optimality conditions from the original problem and then **log-linearise** the optimality conditions. This is the most common approach to solve DSGE models.

However, both methods of linearisation may fail at occaions, calling for higher-order approximations.

i. In the first case, since approximation is performed before deriving the optimality conditions, the resulting optimality conditions may not be a correct linear representation of the original nonlinear optimality conditions.
ii. In the second case, a linear approximation of the nonlinear optimality conditions may omit important information such that the resulting policy function is far from being accurate.

Higher-order perturbation methods have been developed to approximate the polciy functions.

Some of the algorithms are encoded in software like Dynare, which is a program for solving, simulating and estimating dynamic models.

Compared with global approximation methods like value function iteration and projection methods:

i. The advantage of the pertubation method is that it is not subject to the curse of dimensionality and works well with large-scale models.
ii. The disadvantage is that the method is a local approximation method (based on small perturbations from the steady state) and hence may not fully capture the behaviour of the model further from the steady state.

(b) Log-linearisation

We use the simple deterministic growth model to illustrate the log-linearisation approach.

The sequence problem is given by:

$$\begin{aligned}&\max_{\{k_{t+1}\}_{t=0}^{\infty}} [\sum_{t=0}^{\infty} \beta^t u(c_t)]\\&\text{subject to}\\&c_t + k_t + 1 = f(k_t) + (1-\delta)k_t,\ \forall t\\&k_0 \text{ given, } k_{t+1} \geq 0\end{aligned}$$

With general u and f, the model does not possess an analytical solution for the policy function. We can pursue the following log-linearisation procedure to approximate the solution:

i. Find equilibrium conditions for the problem.
ii. Find the (deterministic) steady state.
iii. Log-linearise the conditions around steady state.

iv. Solve for policy functions using the **method of undetermined coefficients.**

Let $u(c) = log(c), f(k_t) = k_t^\alpha$, and we solve this model following the procedure above, and discuss its stability.

Equilibrium conditions: The equilibrium is characterised by the intertemporal consumption Euler equation and the resource constraint

$$\frac{1}{c_t} = \beta \frac{\alpha k_{t+1}^{\alpha-1} + (1-\delta)}{c_{t+1}} \tag{Eq 1}$$

$$c_t + k_{t+1} = k_t^\alpha + (1-\delta)k_t \tag{Eq 2}$$

Steady state: Denote the steady state values as $\bar{c}, \bar{k}$ and $\bar{y} \equiv \bar{k}^\alpha$, and let $c_t = c_{t+1} = \bar{c}$ and $k_t = k_{t+1} = \bar{k}$ in Eq1 and Eq2

$$\begin{aligned}(1) &\Rightarrow 1 = \beta[\alpha \bar{k}^{\alpha-1} + (1-\delta)] \\ &\Rightarrow \bar{k} = [\frac{\alpha}{\frac{1}{\beta} - 1 + \delta}]^{\frac{1}{1-\alpha}} \\ (2) &\Rightarrow \bar{c} = \bar{k}^\alpha - \delta \bar{k}\end{aligned}$$

Log-linearisation around the steady state

i. Two basic formulas: Define $\hat{x} = log(x) - log(\bar{x})$, then

$$x = \bar{x}e^{\hat{x}}$$

And by Taylor series expansion,

$$e^{\hat{x}} \approx 1 + \hat{x}$$

ii. First, linearise the Euler equation (Eq 1):

$$\begin{aligned} & \frac{1}{\bar{c}e^{\hat{c}_t}} = \beta \frac{\alpha(\bar{k}e^{\hat{k}_{t+1}})^{\alpha-1} + (1-\delta)}{\bar{c}e^{\hat{c}_{t+1}}} \\ \Rightarrow \quad & e^{\hat{c}_{t+1} - \hat{c}_t} = \beta[\alpha \bar{k}^{\alpha-1} e^{(\alpha-1)\hat{k}_{t+1}} + 1 - \delta] \\ \Rightarrow \quad & 1 + \hat{c}_{t+1} - \hat{c}_t \approx \beta[\alpha \bar{k}^{\alpha-1}(1 + (\alpha-1)\hat{k}_{t+1}) + 1 - \delta]\end{aligned}$$

Applying the steady state relationships

$$1 = \beta[\alpha \bar{k}_{\alpha-1} + (1-\delta)]$$

we have

$$\begin{aligned} \hat{c}_{t+1} - \hat{c}_t &\approx \beta \alpha \hat{k}^{\alpha-1}(\alpha-1)\hat{k}_{t+1}, \text{ i.e.,} \\ \hat{c_{t+1}} &\approx \hat{c}_t - \beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\hat{k}_{t+1} \end{aligned} \tag{Eq 3}$$

iii. Second, log-linearise the resource constraint (2):

$$\begin{aligned} & \bar{c}e^{\hat{c}_t} + \bar{k}e^{\hat{k}_{t+1}} = (\bar{k}e^{\hat{k}_t})^\alpha + (1-\delta)\bar{k}e^{\hat{k}_t} \\ \Rightarrow \quad & \bar{c}e^{\hat{c}_t} + \bar{k}e^{\hat{k}_{t+1}} = \bar{k}^\alpha e^{\alpha\hat{k}_t} + (1-\delta)\bar{k}e^{\hat{k}_t} \\ \Rightarrow \quad & \bar{c}(1+\hat{c}_t) + \bar{k}(1+\hat{k}_{t+1} \approx \bar{k}^\alpha(1+\alpha\hat{k}_t) + (1-\delta)\bar{k}(1+\hat{k}_t) \\ \Rightarrow \quad & \bar{c} + \bar{c}\hat{c}_t + \bar{k} + \bar{k}\hat{k}_{t+1} \approx \bar{k}^\alpha + (1-\delta)\bar{k} + (\alpha\bar{k}^{\alpha-1} + (1-\delta))\bar{k}\hat{k}_t \end{aligned}$$

Applying the steady state relationships

$$\begin{aligned} \bar{c} + \bar{k} &= \bar{k}^\alpha + (1-\delta)\bar{k}, \\ \alpha\bar{k}^{\alpha-1} + (1-\delta) &= \frac{1}{\beta} \end{aligned}$$

we have

$$\begin{aligned} \bar{c}\hat{c}_t + \bar{k}\hat{k}_{t+1} &\approx \frac{1}{\beta}\bar{k}\hat{k}_t, \text{ i.e.,} \\ \hat{k}_{t+1} &\approx \frac{1}{\beta}\hat{k}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t \end{aligned} \tag{Eq 4}$$

iv. Substituting Eq 4 into Eq 3,

$$\hat{c}_{t+1} \approx \hat{c}_t - \beta\alpha(1-\alpha)\bar{k}^{\alpha-1}[\frac{1}{\beta}\hat{k}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t], \text{ i.e.,}$$

$$\hat{c}_{t+1} \approx -\alpha(1-\alpha)\bar{k}^{\alpha-1}\hat{k}_t + [1+\beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}}]\hat{c}_t \qquad \text{(Eq 5)}$$

Stability

i. Combining Eq 4 and Eq 5, we get the approximated linear system:

$$\begin{bmatrix}\hat{c}_{t+1}\\ \hat{k}_{t+1}\end{bmatrix} = \begin{bmatrix}1+\beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}} & -\alpha(1-\alpha)\bar{k}^{\alpha-1}\\ -\frac{\bar{c}}{\bar{k}} & \frac{1}{\beta}\end{bmatrix}\begin{bmatrix}\hat{c}_t\\ \hat{k}_t\end{bmatrix}$$

or in short,

$$x_{t+1} = Ax_t \qquad \text{(Eq 6)}$$

ii. The two eigenvalues of A, λ_1 and λ_2, are given by the roots of the characteristic polynomial

$$p(\lambda) \equiv \det(\lambda I - A) = \lambda^2 - [1+\frac{1}{\beta}+\beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}}]\lambda + \frac{1}{\beta} = 0$$

$$\lambda_1+\lambda_2 = 1+\frac{1}{\beta}+\beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}} > 2,$$

$$\lambda_1\lambda_2 = \frac{1}{\beta} > 1$$

Note: $\lambda_1+\lambda_2 = tr(A), \lambda_1\lambda_2 = det(A)$

iii. Hence, both λ_1 and λ_2 are positive and at least one of them is greater than 1

iv. Further, note that $p(1)$ (i.e., the characteristic polynomial at $\lambda = 1$ is:

$$p(1) = 1 - [1+\frac{1}{\beta}+\beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}}] + \frac{1}{\beta}$$

$$= -\beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}} < 0$$

Hence one of λ_1 and λ_2 is greater than 1 and the other is less than 1 (since $p(1)$ is negative).

v. Suppose (without loss of generality) that $0 < \lambda_1 < 1$ and $\lambda_2 > 1$, and the corresponding eigenvectors are given by $v_1 = \begin{bmatrix}v_{11}\\ v_{21}\end{bmatrix}$ and $v_2 = \begin{bmatrix}v_{12}\\ v_{22}\end{bmatrix}$, i.e.,

$$Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2$$

Define $V \equiv \begin{bmatrix}v_1 & v_2\end{bmatrix} = \begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}$ and $\Lambda \equiv \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$ then

$$A = V\Lambda V^{-1} \Rightarrow V^{-1}A = \Lambda V^{-1}$$

vi. Multiplying Eq 6 by V^{-1}, we have

$$V^{-1}x_{t+1} = V^{-1}Ax_t = \Lambda V^{-1}x_t$$

Define $\omega_t = V^{-1}x_t$, then the approximated linear system can be rewritten as

$$\omega_{t+1} = \Lambda\omega_t$$

implying that $\omega_t = \Lambda^t\omega_0$, and equivalently,

$$\begin{bmatrix}w_{1t}\\ w_{2t}\end{bmatrix} = \begin{bmatrix}\lambda_1^t & 0\\ 0 & \lambda_2^t\end{bmatrix}\begin{bmatrix}\omega_1 0\\ \omega_2 0\end{bmatrix} = \begin{bmatrix}\lambda_1^t\omega_{10}\\ \lambda_2^t\omega_{20}\end{bmatrix}$$

Since $\lambda_2 > 1$, to ensure the stability of the system, we must have $\omega_{20} = 0$ (otherwise $\lambda_{20}^t \omega_{20}$ is explosive when $t \to \infty$). Hence,

$$\omega_{2t} = 0, \text{ for all } t = 0, 1, 2, ... \tag{Eq 7}$$

Recall that

$$\begin{bmatrix} \omega_{1t} \\ \omega_{2t} \end{bmatrix} = V^{-1} \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} = \frac{1}{v_{11}v_{22} - v_{12}v_{21}} \begin{bmatrix} v_{22} & -v_{12} \\ -v_{21} & v_{11} \end{bmatrix} \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix}$$

so Eq 7 implies that

$$\hat{c}_t = \frac{v_{11}}{v_{21}} \hat{k}_t, \text{ for all } t = 0, 1, 2, ... \tag{Eq 8}$$

In particular, the initial consumption is given by

$$\hat{c}_0 = \frac{v_{11}}{v_{21}} \hat{k}_0$$

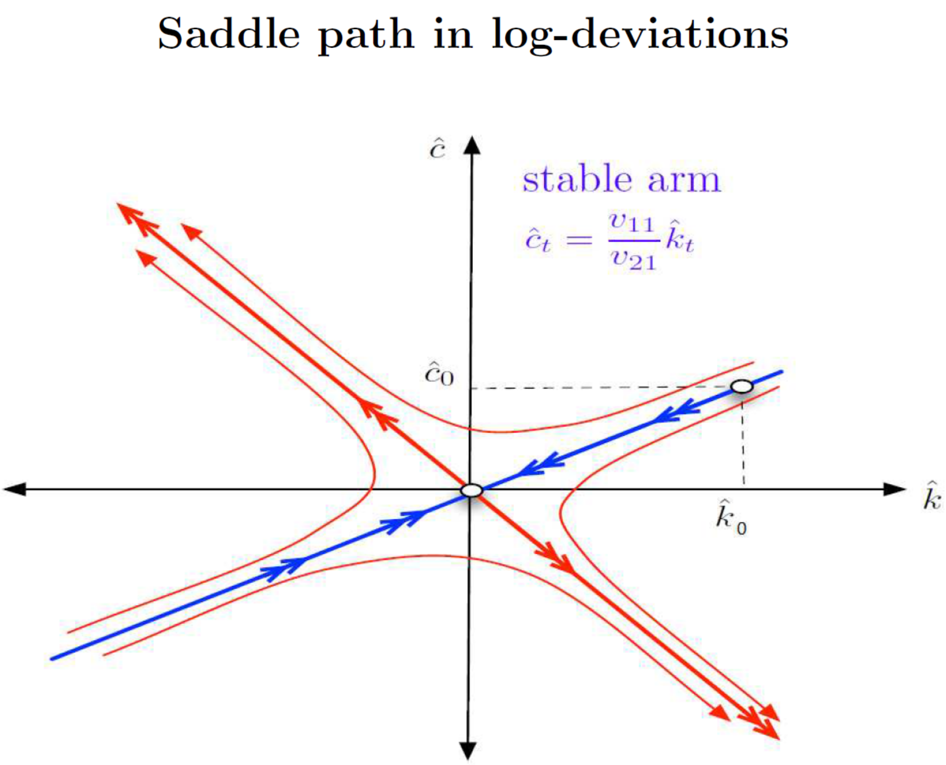
where initial capital k_0 and hence $\hat{k}_0$ is pre-determined.

vii. That is, the approximated linear system is saddle-path stable, where the **saddle-path** or stable arm is given by Eq 8 with $\hat{k}_0$ pre-determined.

viii. The initial consumption must jump on this stable arm to ensure convergence to the steady state.

ix. The dynamics of the linearised system is illustrated in the figure below. Both capital and consumption converge monotonically to their steady-state values.

x. Since the linear system is an approximation to the original nonlinear system at the steady state, the original system is **locally saddle-path stable**.



Find the policy functions that solve the linearised system in Eq 7.

i. One way is to utilise Eq 8 in Eq 6:

$$\hat{k}_{t+1} = -\frac{\bar{c}}{\bar{k}}\hat{c}_t + \frac{1}{\beta}\hat{k}_t = [\frac{1}{\beta} - \frac{\bar{c}}{\bar{k}}\frac{v_{11}}{v_{21}}]\hat{k}_t$$

As we may not discuss stability explicitly to derive Eq 8, we introduce a more common way using Eq 6 alone.

ii. We conjecture that the solution to the linear system is given by

$$\hat{k}_{t+1} = a_k \hat{k}_t, \hat{c}_t = a_c \hat{k}_t$$

where a_k and a_c are coefficients to be determined. Then,

$$\hat{c}_{t+1} = a_c \hat{k}_{t+1} = a_c a_k \hat{k}_t$$

iii. Using the expressions in Eq 6 above, we have

$$\begin{aligned} a_c a_k \hat{k}_t &= \{[1 + \beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}}]a_c - \alpha(1-\alpha)\bar{k}^{\alpha-1}\}\hat{k}_t \\ a_k \hat{k}_t &= [\frac{1}{\beta} - \frac{\bar{c}}{\bar{k}} a_c]\hat{k}_t \end{aligned}$$

Equating coefficients gives

$$\begin{aligned} a_c a_k &= [1 + \beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}}]a_c - \alpha(1-\alpha)\bar{k}^{\alpha-1} \\ a_k &= \frac{1}{\beta} - \frac{\bar{c}}{\bar{k}} a_c \end{aligned}$$

Simplifying yields

$$a_k^2 - [1 + \beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\frac{\bar{c}}{\bar{k}}]a_k + \frac{1}{\beta} = 0 \quad \text{(Eq 9)}$$

$$a_c = \frac{\bar{k}}{\bar{c}}(\frac{1}{\beta} - a_k) \quad \text{(Eq 10)}$$

iv. Note that a_k satisfies a quadratic function which is the characteristic polynomial $p(\lambda)$ i.e., a_k is the eigenvalue of A.

v. From the previous discussion, there are two solutions for a_k - one less than 1, and one greater than 1 (which was discarded).

vi. Then, Eq 9 and Eq 10 determine a unique $0 < a_k < 1$ and a_c, and the policy functions are given by

$$\hat{k}_{t+1} = a_k \hat{k}_t, \hat{c}_t = a_c \hat{k}_t$$

Starting from $\hat{k}_0$, we can use the policy functions to simulate the time paths of $\hat{c}_t$ and $\hat{k}_{t+1}, t = 0, 1, 2, ...$ They converge monotonically to their steady state values, zeroes.

(c) Exercise 12.1

Let $\beta = 0.9932, \alpha = 0.36, \delta = 0.015, k_0 = 0.1$. In Matlab, implement the following:

i. Compute the steady state values $\bar{k}$ and $\bar{c}$, and find $\hat{k}_0$.

ii. Compute the eigenvalues of A, λ_1 and λ_2, and the corresponding eigenvectors v_1 and v_2. Plot the stable arm as defined in Eq 8.

iii. Compute a_k and a_c by solving Eq 9 and Eq 10. Verify that the two values of a_k are eigenvalues of A and the policy function for $\hat{c}_t$ is identical to Eq 8. Plot the policy functions for $\hat{k}_{t+1}$ and $\hat{c}_t$.

iv. Simulate and plot the time paths of $\hat{c}_t$ and $\hat{k}_{t+1}, t = 0, 1, 2, ...$ and verify that $\hat{c}_t \to 0, \hat{k}_{t+1} \to 0$ as $t \to \infty$.

v. Transform the linear policy functions into policy functions for k_{t+1} and c_t as functions of k_t and plot them.

(refer to Tutorial 6)

13. Lecture 13

(a) Introduction to numerical tools for solving stochastic problems

For the stochastic growth model:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} E_0\{\sum_{t=0}^{\infty}\beta^t u[z_t f(k_t + (1-\delta)k_t - k_{t+1}]\}$$
$$\text{s.t.} \qquad 0 \le k_{t+1} \le z_t f(k_t) + (1-\delta)k_t, t = 0, 1, ...$$
$$log(z_{t+1}) = \rho log(z_t) + \epsilon_{t+1}, \epsilon_t \sim IID\ N(0, \sigma_\epsilon^2)$$
$$k_0, z_0 \text{ given}$$

The corresponding (FE) is

$$v(k,z) = \max_{k' \in 0, zf(k)+(1-\delta)k]} u[zf(k) + (1-\delta)k - k'] + \beta E_{z'|z} v(k', z')$$

for all (k, z), where z' evolves according to the law of motion

$$log(z') = \rho log(z) + \epsilon', \epsilon' \sim IID\ N(0, \sigma_\epsilon^2)$$

The Euler equation is

$$u'(c_t) = \beta E_t u'(c_{t+1})[z_{t+1} f'(k_{t+1}) + (1-\delta)]$$

To solve this stochastic problem we can use value function iteration (discrete or continuous), and projection or perturbation methods to solve the Euler equation.
However, to apply these methods ,we first need to compute expectations like $E_{z'|z} v(k', z')$.

i. One way is to approximate the autoregressive stochastic process for z using a finite-state Markov chain (i.e, make z discrete).
The advantage of this is that it keeps the state space manageable and simplifies the computation of expectations.
For the stochastic growth model, the continuous state space for Z can be approximated using a finite set of points $\{z^1, z^2, ..., z^K\}$.
The expectation can then be computed as

$$E_{z'|z^i} v(k', z') = \sum_{j=1}^{K} \pi_{ij} v(k', z^j)$$

ii. Another way is to compute the expectations numerically using some numerical integration technique.
This keeps the state space for k and z continuous.
The computation of expectation can be more accurate, but is also more time-consuming (curse of dimensionality).

(b) Approximation of autoregressions by Markov chains

Suppose z_t follows an autoregressive AR(1) process

$$z_{t+1} = (1-\rho)\mu + \rho z_t + \epsilon_{t+1}$$

where $|\rho| < 1$ (for z_t to be stationary), and $\epsilon_t \sim IID\ N(0, \sigma_\epsilon^2)$
The unconditional mean of z_t is μ and the unconditional variance of z_t is $\sigma_z^2 = \frac{\sigma_\epsilon^2}{1-\rho^2}$, so z_t follows $N(\mu, \sigma_z^2)$.

The task is to approximate z_t by a K-state Markov chain $\{Z, \Pi\}$, where $Z = \{z^1, z^2, ..., z^K\}$ and $\Pi = [\pi_{ij}]$.
i.e., we want to estimate $\pi_{ij} = Prob[z' = z^j | z = z^i]$.

i. Tauchen (1986) method:
   A. The nodes Z are equally spaced on $[\mu - m\sigma_z, \mu + m\sigma_z]$ (e.g., if $m = 3$, then the interval is 6 standard deviations).

B. m is user-specified, with

$$z^1 = \mu - m\sigma_z; z^K = \mu + m\sigma_z; z^i = z^1 + (i-1)s, i = 2, ..., K-1$$

where $s = \frac{z^K - z^1}{K-1}$.

C. The transition probabilities are defined as, for $i = 1, ..., n$,

$$\begin{aligned}
\pi_{i1} &= Prob(z' \in (-\infty, z^1 + \frac{s}{2}]|z = z^i) \\
\pi_{ij} &= Prob(z' \in [z^j - \frac{s}{2}, z^j + \frac{s}{2}]|z = z^i), j = 2, ..., K-1 \\
\pi_{iK} &= Prob(z' \in [z^K - \frac{s}{2}, \infty)|z = z^i)
\end{aligned}$$

ii. Tauchen and Hussey (1991) method:

A. Let $\{z^i\}$ and $\{w^i\}$ denote the Gaussian quadrature nodes and weights for $N(\mu, \hat{\sigma}^2)$ (this is the distribution for z_t), where $\hat{\sigma}$ is user-specified.

B. Several choices of $\hat{\sigma}$ are used in the literature: $\hat{\sigma} = \sigma_\epsilon, \hat{\sigma} = \sigma_z$, or $\hat{\sigma} = \omega\sigma_\epsilon + (1-\omega)\sigma_z$ with $\omega = \frac{1}{2} + \frac{\rho}{4}$.

C. To calculate the transition properties, let

$$\tilde{\pi}_{ij} = \frac{p(z^j|z^i)}{\hat{p}(z^j|z = \mu)} w^j$$

where $p(z^j|z^i)$ is the pdf of $N((1-\rho)\mu + \rho z^i, \sigma_\epsilon^2)$ at z^j, and $\hat{p}(z^j|z = \mu)$ is the pdf of $N(\mu, \hat{\sigma}^2)$ at z^j. Then

$$\pi_{ij} = \frac{\tilde{\pi}_{ij}}{\sum_{k=1}^K \tilde{\pi}_{ik}}$$

where $\sum_{j=1}^K \pi_{ij} = 1$.

iii. Adda and Cooper (2003) method:

A. Discretise the state space of z_t (i.e., the whole real space $\mathbb{R}$) into K intervals so that z_t has an equal probability $\frac{1}{K}$ of falling into them (i.e., not equal distance).
Denote the cutoff points as $z_c^i, i = 1, ..., K+1$, where $z_c^i = -\infty$ and $z_c^{K+1} = \infty$, then

$$\Phi(\frac{z_c^{i+1} - \mu}{\sigma_z}) - \Phi(\frac{z_c^i - \mu}{\sigma_z}) = \frac{1}{K}, i = 1, ..., K,$$

where Φ is the cdf of the standard normal distribution.

B. The node z^i is defined as the conditional mean of z_t on interval $[z_c^i, z_c^{i+1}]$, for $i = 1, ..., n$:

$$z^i = E[z_t | z_t \in [z_c^i, z_c^{i+1}]]$$

C. The transition probabilities are computed as

$$\pi_{ij} = Prob(z' \in [z_c^j, z_c^{j+1}]|z \in [z_c^i, z_c^{i+1}])$$

Floden (2008) compares the three methods and finds that:

i. AR(1) processes with relatively low persistence are well approximated with all methods even with a small number of nodes, but Tauchen and Hussey's quadrature based methods consistently deliver better approximations.
Note: We usually need an odd number of nodes (e.g., 3/5/7).

ii. The approximations are less precise when persistence is high and Tauchen's method appears to be relatively robust.

Properties of a Markov chain:

i. Given a Markov chain with state space $Z \equiv \{z^1, z^2, ..., z^K\}$ (i.e., K different states) and transition matrix Π, where Π satisfies

$$\sum_{j=1}^K \pi_{ij} = 1, \text{ for all } i = 1, 2, ..., K$$

ii. Suppose the distribution over states at t is given by

$$\mu_t \equiv (\mu_t^1, \mu_t^2, ..., \mu_t^K)'$$

where $\mu_t^i = Prob(z_t = z^i)$ and $\sum_{i=1}^K \mu_t^i = 1$. Then, μ_{t+1} is

$$\mu_{t+1} = \Pi' \mu_t$$

Implying that

$$\mu_t = (\Pi')^t \mu_0, \text{ where } \mu_0 \text{ is the initial distribution.}$$

The stationary distribution of z_t is

$$\mu^* = \Pi' \mu^* \Leftrightarrow (I - \Pi')\mu^* = 0$$

(i.e., at steady state, $\mu_t = \mu^*$).
We can observe from this form that μ^* is an eigenvector of Π' associated with a unit-eigenvalue of Π'.
Since Π is a transition matrix, Π' has at least one unit eigenvalue.
A sufficient condition for a unique stationary distribution μ^* is that $0 < \pi_{ij} < 1$ for all i, j.

Once the z_t process is approximated by a Markov chain, the model can be solved by value function iteration (see Tutorial 7) or a projection method.
Note that we approximate the $log(z_t)$ process by a Markov chain.
Once the model is solved, we need to simulate the model to see how well it matches certain features of the data.
This involves simulating the exogenous state variable z_t using its law of motion and simulating the endogenous state variable k_t using the policy function $k_{t+1} = \hat{g}(k_t, z_t)$ (i.e., $k_t \to k_{t+1}$ is governed by the policy function, while $z_t \to z_{t+1}$ is governed by the Markov chain).
Suppose z_t is approximated by a Markov chain with states $\{z^1, ..., z^K\}$ and transition matrix Π. We can use the following method to simulate a sequence of z_t, starting from some initial value:

i. From a random number generator with uniform distribution $U[0,1]$, draw $\{u_t\}_{t=1}^T$.
ii. Let $z_0 = z^m$, where z^m is the state closest to the initial value among $\{z^1, ..., z^K\}$. If the initial value is not given, simply let z_0 be the median value among $\{z^1, ..., z^K\}$.
iii. For $t = 1, ..., T$, suppose $z_{t-1} = z^i$, then calculate z_t as follows:
   A. If $u_t \leq \pi_{i1}$, then $z_t = z^1$
   B. If $\sum_{k=1}^{j-1} \pi_{ik} < u_t \leq \sum_{k=1}^{j} \pi_{ik}$, then $z_t = z^j$ for $j = 2, ..., K$.

(c) Numerical integration

Numerical integration is also called **quadrature**. The problem it concerns is to compute $\int_D f(x)\omega(x)dx$, where f is an integrable function and ω is some nonnegative weighting function.
It is often not possible to calculate the integral exactly. We can approximate the integral value by choosing an appropriate set of quadrature nodes $x_i \in D$ and weights $\omega_i \in R_+$ so that

$$\int_D f(x)\omega(x)dx \approx \sum_{i=1}^n \omega_i f(x_i)$$

Quadrature methods differ in how they choose the nodes and weights (e.g., Newton-Cotes, Gaussian, Monte Carlo, etc).

Gaussian quadrature builds on the orthogonal polynomial approach to function approximation.
Given a nonnegative weighting function $w(x)$, Gaussian quadrature creates approximations of the form

$$\int_a^b f(x)w(x)dx \approx \sum_{i=1}^n \omega_i f(x_i)$$

where the n quadrature nodes $x_i \in [a, b]$ and positive weights ω_i are chosen efficiently (i.e., if $f(.)$ is a low-order polynomial function, the approximation will be exact).

For the stochastic growth model, the expected continuation value in the (FE) is given by

$$E_{z'|z}v(k',z') = \int_{-\infty}^{\infty} v(k', e^{\rho log(z)+\epsilon'})p(\epsilon')d\epsilon'$$

where $p(.)$ is the pdf for $N(0,\sigma_\epsilon^2)$.

i. With a normal density as the weighting function, the Gaussian quadrature nodes ϵ_i and weights ω_i can be obtained from the Matlab function *qnwnorm.m*.

ii. Then,

$$\int_{-\infty}^{\infty} v(k', e^{\rho log(z)+\epsilon'})p(\epsilon')d\epsilon' \approx \sum_{i=1}^{n} v(k', e^{\rho log(z)+\epsilon_i})\omega_i$$

iii. We can rewrite the (FE) as

$$\begin{aligned} v(k,z) &= \max_{k'}\{u[zf(k)+(1-\delta)k-k'] + \beta\sum_{i=1}^{n}\omega_i vk', e^{\rho log(z)+\epsilon_i})\} \\ &\text{s.t. } k' \in [0, zf(k)+(1-\delta)k] \end{aligned}$$

iv. We can solve the (FE) using any solution method for continuous state space.

v. If we solve the (FE) using value function iteration with piecewise linear or cubic spline interpolation, we need to approximate the value function using two-dimensional interpolation (since there are now two state variables).

vi. In Matlab, we can use *interpn.m* for two-dimensional interpolation.

(d) Exercise 13.1

Let $\mu = 0, \rho = 0.6, \sigma_\epsilon^2 = 0.013$.
Use the three methods to approximate the AR(1) process for z_t.

(e) Exercise 13.2

Consider a two-state Markov chain with transition matrix

$$\begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

Verify that μ^* is given by

$$\mu^* = \begin{bmatrix} \frac{q}{p+q} \\ \frac{p}{p+q} \end{bmatrix}$$

(f) Exercise 13.3

Assume that z_t follows a Markov chain with states $\{1.1, 1, 0.9\}$ and transition matrix

$$\Pi = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Suppose $z_0 = 1.1$ and we draw a random u_t sequence given by $\{0.9, 0.95, 0.6\}$. What is the simulated z_t sequence?

The simulated sequence is $\{1, 0.9, 0.9\}$:

$$\begin{aligned} z_0 &= 1.1 \text{ (this is given)} \\ z_1 &= z^2 = 1 \text{ (first row, second col)} \\ z_2 &= z^3 = 0.9 \text{ (second row, third col)} \\ z^3 &= z^3 = 0.9 \text{ (third row, third col)} \end{aligned}$$

14. Lecture 15

(a) **Complete Markets Models**

Assumptions:
Pure exchange economy, stochastic endowments, complete markets.

Two market structures/ trading arrangements:

i. Arrow-Debreu structure with complete markets in dated contingent claims, all traded at time 0
ii. Sequential-trading structure with complete markets in one-period Arrow securities

Both structures entail different assets and timings of trades, but lead to identical consumption allocations.
The competitive equilibrium is also Pareto efficient/ socially optimal.
These results hinge on the complete markets assumption.

(b) Setup

Time is discrete and the time horizon $t = 0, 1, 2, ...$ is infinite.

Events: In each period $t \geq 0$, there is a realisation of a stochastic event $s_t \in S$.

i. Let the history of events up and until time t be denoted

$$s^t = (s_0, s_1, ..., s_t) = (s^{t-1}, s_t)$$

This is the list of events/ history up to time t, which represents the state of the economy.

ii. The unconditional probability of observing a particular sequence of events s^t is denoted $\pi_t(s^t)$ (this is the probability of the specific "history" being realised).

iii. The conditional probabilities are denoted $\pi_t(s^t|s^\tau)$, which is the probability of observing s^t conditional upon the realisation of s^τ (where τ is any period $< t$; if $\tau = 0$, this is the unconditional probability.

iv. Assume that trading occurs after observing s_0 and set $\pi_0(s_0) = 1$ (so s_0 is observed with certainty).

Agents: There are I agents named $i = 1, ..., I$.

i. In every period t, agent i receives a stochastic endowment of one good, $y_t^i(s^t)$ (this is agent i's endowment in period t).
Let $y^i \equiv \{y_t^i(s^t)\}_{t=0}^\infty$ (i.e., endowment depends on the history of realised events, and can be heterogenous across agents).

ii. Agent i's consumption in period t is denoted $c_t^i(s^t)$. Let $c^i \equiv \{c_t^i(s^t)\}_{t=0}^\infty$.
   A. Feasible consumption allocations satisfy the resource constraint

$$\sum_{i=1}^{I} c_t^i(s^t) \leq \sum_{i=1}^{I} y_t^i(s^t) \equiv Y_t(s^t) \;\forall t, s^t \tag{Eq 1}$$

where $Y_t(s^t)$ is the aggregate endowment in period t.
(i.e., the market clears every period; in each period t and history s^t, total consumption is equal to the total endogenous endowment).

Preferences: Agent i chooses a history-dependent consumption plan c^i to maximise expected lifetime utility (i.e., expected over all histories)

$$U(c^i) \equiv \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t)$$

We assume $u'(c) > 0, u''(c) < 0$ and $u'(0) = +\infty$
Thus, agents have:

i. heterogenous endowments y_i
ii. identical utility functions: they agree on $\pi_t(s^t)$ and have the same $u(.), \beta$.

Information: The history s^t is publicly observable.

There are two alternative trading arrangements:

i. **Arrow-Debreu's Time 0 trading**
   A. Single enormous market at time $t = 0$ to trade a complete set of contingent claims for all possible realisations of s^t (known as **Arrow-Debreu securities**).
   We call one unit of AD securities $AD(t, s^t)$. These are purchased at time 0 and give one unit of consumption at time t if history s^t is realised. Such securities exist for every possible s^t.

   B. At subsequent periods, $t = 1, 2, ...$, agreed-upon trades are carried out but no further trading occurs.

ii. **Radner's sequential trading**
   A. Trade happens at every period. At each date $t = 0, 1, 2, ...$ and history s^t, there is a makret to trade a complete set of contingent claims for all possible realisations of s_{t+1}.

   B. There is a sequential trading of one-period-ahead, state-contingent claims (known as **Arrow securities** at each date t, given the realised history s^t.

(c) Pareto problem:

The social planner chooses c^i for all agents to maximise social welfare, subject to resource constraints:

$$\max_{\{c^i\}_{i=1}^I} W = \sum_i \lambda_i U(c^i)$$
$$\text{s.t.} \qquad \sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \ \forall t, s^t$$

where $\lambda_i \geq 0$ for $i = 1, ..., I$ are nonnegative Pareto weights (i.e., the social welfare is the weighted sum of all agents' utility).

i. A solution to this problem is Pareto efficient.
ii. By varying $\lambda \equiv (\lambda_1, ..., \lambda_I)$, we can trace out the set of Pareto efficient allocations.

Solving the Pareto problem

i. Write down the Lagrangian with stochastic multiplier (i.e., the Lagrangian multiplier) $\theta_t(s^t) \geq 0$ for each resource constraint:

$$\mathcal{L} = \sum_i \lambda_i \{\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t))\pi_t(s^t)\} + \sum_{t=0}^{\infty} \sum_{s^t} \{\theta_t(s^t) \sum_i [y_t^i(s^t) - c_t^i(s^t)]\}$$

   This can be written more compactly as

$$\mathcal{L} = \sum_i \sum_{t=0}^{\infty} \sum_{s^t} \{\lambda_i \beta^t u(c_t^i(s^t))\pi_t(s^t) + \theta_t(s^t)[y_t^i(s^t) - c_t^i(s^t)]\}$$

ii. The FOCs are

$$\lambda_i \beta^t u'(c_t^i(s^t))\pi_t(s^t) = \theta^t(s^t), \ \forall t, s^t$$

iii. The ratios of the FOCs for agent i and agent 1 are:

$$\frac{\lambda_i}{\lambda_1} \frac{u'(c_t^i(s^t))}{u'(c_t^i(s^t))} = 1$$

   We can invert this to write $c_t^i(s^t)$ in terms of $c_t^1(s^t)$:

$$c_t^i(s^t) = u'^{-1}(\frac{\lambda_1}{\lambda_i} u'(c_t^1(s^t)))$$

iv. Plugging this expression into the resource constraint gives:

$$\sum_i u'^{-1}(\frac{\lambda_1}{\lambda_i}u'(c_t^i(s^t))) = Y_t(s^t),\ \forall t, s^t$$

This is a single equation in $c_t^1(s^t)$, so we can solve for $c_t^1(s^t)$ and then use that to recover $c_t^i(s^t)$ for all other i.

Properties of the solution:

i. Given the realised aggregate endowment in period t, denoted as Y_t, $c_t^1(s^t)$ are the same for all s^t leading up to Y_t.
ii. Given the Pareto weights λ, $c_t^1(s^t)$, and hence any $c_t^i(s^t)$, is a function of the realised aggregate endowment Y_t:

$$c_t^i = f(\lambda_i, Y_t; \lambda),\ \forall i$$

where f is a time-invariant function that is defined by the given u and λ.
iii. This solution is **history-free** and **distribution-free**: it only depends on aggregate endowments Y_t; it does not depend on the specific history s^t nor on the realisation of individual endowments y_t^i.

Only the ratios of the Pareto weights matter (and not the absolute value of the weights), so we can normalise the weights, e.g., letting $\lambda_1 = 1$ or imposing $\sum_{i=1}^I \lambda_i = 1$.

iv. Next, we will characterise the competitive equilibrium (CE) under each of the two trading arrangements.
   A. The two trading arrangements lead to the same CE allocation.
   B. The CE allocation corresponds to the solution to a special Pareto problem.

(d) **CE with time 0 trading**

Let $q_t^0(s^t)$ denote the unit price at date $t = 0$ of an Arrow-Debreu security. Each unit of the security is a claim to one unit of consumption for delivery at t, contingent on history s^t.

Taking prices $q_t^0(s^t)$ as given, agent i chooses a consumption plan $c^i \equiv \{c_t^i(s^t)\}_{t=0}^{\infty}$ **at date 0** to maximise

$$U(c^i) \equiv \sum_{t=0}^{\infty}\sum_{s^t} \beta^t u(c_t^i(s^t))\pi_t(s^t)$$

subject to the single budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t) \qquad \text{(Eq 5)}$$

Equilibrium concept:

i. A price system is a sequence of prices of Arrow-Debreu securities $q \equiv \{q_t^0(s^t)\}_{t=0}^{\infty}$
ii. An allocation is a collection of consumption plans $c \equiv \{c^i\}_{i=1}^I$
iii. A competitive equilibrium is a price system q and a feasible allocation c such that, taking q as given, the allocation c solves each agent's problem.

The FOC for each agent's problem

i. The Lagrangian with a single multiplier $\mu_i \geq 0$ on budget constraint (taken from (Eq 5) from above):

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t u(c_t^i(s^t))\pi_t(s^t) + \mu_i \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)[y_t^i(s^t) - c_t^i(s^t)]$$

ii. This can be written more compactly as

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \{\beta^t u(c_t^i(s^t))\pi_t(s^t) + \mu_i q_t^0(s^t)[y_t^i(s^t) - c_t^i(s^t)]\}$$

iii. The FOCs for $c_t^i(s^t)$ are

$$\beta^t u'(c_t^i(s^t))\pi_t(s^t) = \mu_i q_t^0(s^t),\ \forall t, s^t \tag{Eq 6}$$

This implies that

$$q_t^0(s^t) = \beta^t \frac{u'(c_t^i(s^t))}{\mu_i}\pi_t(s^t) \tag{Eq 7}$$

Equilibrium allocation:

i. Similar to the Pareto problem, we take the ratio of (Eq 6) for agent i and agent 1:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^i(s^t))} = \frac{\mu_i}{\mu_1}$$

ii. We invert the expression to write $c_t^i(s^t)$ in terms of $c_t^1(s^t)$:

$$c_t^i(s^t) = u'^{-1}(\frac{\mu_i}{\mu_1}u'(c_t^1(s^t)))$$

For this to be an equilibrium allocation, it must be feasible, i.e., satisfy (Eq 1):

$$\sum_i u'^{-1}(\frac{\mu_i}{\mu_1}u'(c_t^1(s^t))) = Y_t(s^t)$$

This is a single nonlinear equation in $c_t^1(s^t)$.

iii. Once we have found $c_t^1(s^t)$ we can then recover $c_t^i(s^t)$ for all other i, This gives

$$c_t^i = g(\mu_i, Y_t; \mu) \tag{Eq 8}$$

where $\mu \equiv (\mu_1, ..., \mu_I)$ is the vector of Lagrange multipliers.

Similar to the Pareto problem, the equilibrium consumption is once again time-invariant, history matters only through realisation of aggregate endowment Y_t, etc.

iv. To find the μ_i's, for each agent i, evaluate the budget constraint (Eq 5) at $c_t^i = g(\mu_i, Y_t; \mu)$:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) g(\mu_i, Y_t; \mu) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t),\ \forall i \tag{Eq 9}$$

This is a system of I nonlinear equations in I unknowns μ_i, so the competitive equilibrium (i.e., $\mu_i, c_t^i, q_t^0(s^t)$) is determined by (Eq 7), (Eq 8), (Eq 9).

The CE can be computed using the Negeshi algorithm:

i. Fix μ_1. Guess a value for the remaining μ_i. Use these guesses to compute a tentative $c_t^i = g(\mu_i, Y_t; \mu)$

ii. Recover the price system from (Eq 7):

$$q_t^0(s^t) = \beta^t \frac{u'(g(\mu_i, Y_t; \mu))}{\mu_i}\pi_t(s^t)$$

(can use any i, e.g., $i = 1$)

iii. Given these $q_t^0(s^t)$, solve the system of budget constrants (Eq 9) to get new mu_i.

iv. Iterate on steps (i) to (iii) until the mu_i converge.

Efficiency of the CE with time 0 trading

i. Note that the CE allocation coincides with a particular planning solution, the one for which the planner has Pareto weights $\lambda_i = \frac{1}{\mu_i}$.

ii. If these specific weights are chosen, the planner's multipliers (i.e., shadow prices) $\theta_t(s^t)$ coincide with the equilibrium prices $q_t^0(s^t)$.

iii. Therefore the CE is Pareto efficient and consistent with the First Fundamental Welfare Theorem.

iv. We can also show that the Second Fundamental Welfare Theorem holds:

   A. In a CE, the multipliers μ_i are endogenously determined by the distribution of endowments $y_i = \{y_t^i(s^t)\}$ both directly and indirectly via the equilibrium prices q (see (Eq 9)).

   B. Different configurations of y_i imply different configurations of μ_i and hence different allocations c_t^i.

   C. If we have some desired outcome c_i in mind, the social planner can try to find the needed configuration of y_i through redistribution of wealth, which will then determine the μ_i's that would deliver c_i as an equilibrium outcome.

   D. That is, we can support any Pareto efficient allocation as a CE by an appropriate redistribution of wealth.

15. Lecture 16

(a) **CE with sequential trading**

Note: superscript ^{t} denotes history; subscript $_t$ denotes time period.

At each date t, after observing s^t, there is trade in a complete set of one-period-ahead contingent claims (**Arrow securities**).

i. $a_{t+1}^i(s^t, s')$: the units of an Arrow security that agent i purchases at (t, s^t). Each unit will deliver one unit of consumption at time $t+1$ if the state $s_{t+1} = s'$ is realised.

   A. $a_{t+1}^i(s^t, s') > 0$ denotes an asset

   B. $a_{t+1}^i(s^t, s') < 0$ denotes a liability/ debt

ii. $q_t(s^t, s')$ is the price of such a claim, also the price of one unit of time $t+1$ consumption contingent on the realisation of s' at $t+1$, given history s^t. This is also known as the **pricing kernel**.

Taking prices as given, agent i chooses the sequence of consumption $\{c_t^i(s^t)\}$ and the portfolio ("investment") $\{a_{t+1}^i(s^t, s')\}$ to maximise

$$U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t)$$

subject to the sequence of one-period budget constraints

$$c_t^i(s^t) + \sum_{s'} q_t(s^t, s') a_{t+1}^i(s^t, s') \leq y_t^i(s^t) + a_t^i(s^{t-1}, s_t), \ \forall t, s^t \qquad \text{(Eq 10)}$$

The Lagrangian with Lagrangian multipliers $\mu_t^i(s^t) \geq 0$ on (Eq 10) is:

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) \\ &+ \sum_{t=0}^{\infty} \sum_{s^t} \mu_t^i(s^t) [y_t^i(s^t) + a_t^i(s^t) - c_t^i(s^t) - \sum_{s'} q^t(s^t, s') a_{t+1}^i(s^t, s')]\end{aligned}$$

The FOC with respect to $c_t^i(s^t)$ is

$$\beta^t u'(c_t^i(s^t)) \pi_t(s^t) = \mu_t^i(s^t)$$

The FOC with respect to $a_{t+1}^i(s^t, s')$ is

$$q_t(s^t, s') \mu_t^i(s^t) = \mu_{t+1}^i(s^t, s')$$

We combine the FOCs to get

$$q_t(s^t, s_{t+1}) = \beta \frac{u'(c^i_{t+1}(s^{t+1}))}{u'(c^i_t(s^t))} \pi_{t+1}(s^{t+1}|s^t) \tag{Eq 11}$$

where $s^{t+1} = (s^t, s_{t+1})$
Note: $\pi_{t+1}(s^{t+1}|s^t) = \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)}$ since these are conditional probabilities.

The allocations $\{c^i_t(s^t)\}$ impied by this sequential trading arrangement coincide with the Arrow-Debreu allocations.

i. Recall that in the Arrow-Debreu problem

$$q^0_t(s^t) = \beta^t \frac{u'(c^i_t(s^t))}{\mu_i} \pi_t(s^t)$$

where μ_i denotes the multiplier on agent i's lifetime budget constraint.

ii. The no-arbitrage condition means that the Radner prices can be written as

$$q_t(s^t, s_{t+1}) = \frac{q^0_{t+1}(s^t, s_{t+1}}{q^0_t(s^t)}$$

Detailed explanation of the no-arbitrage condition:
If an individual wants 1 unit of consumption at time $t+1$, there are different choices of securities that he can buy:

A. Purchase 1 unit of AD securities in the time 0 trading model. The cost is $q^0_{t+1}(s^{t+1})$.
B. Purchase 1 unit of Arrow securities at time t. The cost is $q_t(s^t, s_{t+1})$.
C. At time 0, purchase $q_t(s^t, s_{t+1})$ units of AD securities, then in time t, use these AD securities to purchase 1 unit of (s_{t+1}) Arrow securities. The cost is $q_t(s^t, s_{t+1})q^0_t(s^t)$.

By the no-arbitrage condition, all of these costs must be equal.

iii. Given these prices, if $c_i \equiv \{c^i_t(s^t)\}$ is an Arrow-Debreu allocation, i.e., $c^i_t(s^t)$ satisfies (Eq 7) for all t, s^t, such that

$$q^0_{t+1}(s^t, s_{t+1}) = \beta^{t+1} \frac{u'(c^i_{t+1}(s^{t+1}))}{\mu_i} \pi_{t+1}(s^{t+1})$$

Then, we have

$$\frac{q^0_{t+1}(s^t, s_{t+1})}{q^0_t(s^t)} = \beta \frac{u'(c^i_{t+1}(s^{t+1}))\pi_{t+1}(s^{t+1})}{u'(c^i_t(s^t))\pi_t(s^t)}$$

This simplifies to (Eq 11), so the allocation $\{c^i\}$ satisfies the Radner FOCs.

iv. It can also be shown (not covered in these materials; see L&S Chapter 8) that the allocation $\{c_i\}$ is budget feasible in the Radner economy, and that there is no other allocation in the Radner economy that is preferred.
v. Therefore, the allocation $\{c_i\}$ is the equilibrium allocation in the Radner economy.

(b) Recursive equilibrium with Markov endowments

We established that the equilibrium allocations are identical in the Arrow-Debreu economy with **complete markets** in dated contingent claims all traded at time 0, and also in a sequential-trading economy with **complete** one-period Arrow securities.

This finding holds for arbitrary individual endowment processes $y^i_t(s^t)$ that are functions of the history of events (s^t), which in turn are governed by some arbitrary probability measure $\pi_t(s^t)$. But without further structure on $\pi_t(s^t)$ and $y^i_t(s^t)$, the model is too general to yield testable applications.

Now, we assume that the stochastic endowments are governed by a Markov chain.

i. Events $s \in S$
ii. Transition probabilities

$$\pi(s'|s) = Prob[s_{t+1} = s'|s_t = s]$$

so that the unconditional probabilities are

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})...\pi(s_1|s_0)\pi(s_0)$$

where $\pi(s_0) = 1$.
iii. Endowments are time-invariant functions of the current state:

$$y_t^i(s^t) = y^i(s^t)$$

such that y only varies by i and s^t but not t.

CE consumption allocation:
i. The consumption allocation $c_t^i(s^t)$ in a CE is a time-invariant function of the aggregate endowment $Y_t(s^t)$.
ii. Now, $Y_t(S^t)$ is a time-invariant function of current state s_t alone, so $c_t^i(s^t)$ is also a time-invariant function of current state s_t:

$$c_t^i(s^t) = c^i(s_t)$$

iii. The CE allocation is history free and distribution free. Agents are fully insured of their idiosyncratic risks.

(Eq 11) implies that the pricing kernels are time-invariant functions of current and next period states.

$$q(s_t, s_{t+1}) = \beta \frac{c^i(s^{t+1})}{c^i(s_t)} \pi(s^{t+1}|s^t)$$

With Markov endowments, the Radner problem has a recursive formulation:
i. Agent i has assets a and faces current state s.
ii. The agent chooses consumption c and a portfolio of Arrow securities $a'(s')$.
iii. Let $v^i(a, s)$ denote the value function of agent i in state (a, s), then the functional equation is given by

$$v^i(a, s) = \max_{c, a'(s')} [u(c) + \beta \sum_{s'} v^i(a'(s'), s')\pi(s'|s)]$$

where the maximisation problem for agent i is subject to the budget constraint

$$c + \sum_{s'} q(s, s')a'(s') = y^i(s) + a \tag{Eq 13}$$

To derive the Euler equation:
i. The FOC for $a'(s')$ is given by

$$u'(c)q(s, s') = \beta v_1^i(a'(s'), s')\pi(s'|s)$$

ii. By the Envelope condition,

$$v_1^i(a, s) = u'(c) \implies v_1^i(a'(s'), s') = u'(c')$$

where c' is agent i's consumption in next period if s' is realised and hence the agent's new asset holding becomes $a'(s')$.
iii. Substituting the expression for $v_1^i(a'(s'), s')$ into the FOC, we get:

$$u'(c)q(s, s') = \beta u'(c')\pi(s'|s) \implies q(s, s') = \beta \frac{u'(c')}{u'(c)} \pi(s'|s)$$

That is,

$$q(s_t, s_{t+1}) = \beta \frac{u'(c^i(s_{t+1}))}{u'(c^i(s_t))} \pi(s_{t+1}|s_t)$$

which suggests that the MRS are equalised across agents.

Let $c^i = g^i(s)$ (which does not depend on a) and $a'^i(s') = h^i(a, s, s')$ denote the poicy functions that achieve the RHS maximum of the functional equation.
A recursive competitive equilibrium is a set of prices $q(s, s')$, a set of value functions $v^i(a, s)$ and a set of policy functions $g^i(s)$ and $h^i(a, s, s')$ such that

i. taking prices as given, the value functions and policy functions solve each agent's problem
ii. the goods market and asset markets clear

Asset market clearing requires

$$\sum_i h^i(a, s, s') = 0, \ \forall a, s, s'$$

Asset market clearing implies goods market clearing. Note that the policy functions satisfy agent i's budget constraint (Eq 13):

$$g^i(s) + \sum_{s'} q(s, s')h^i(a, s, s') = y^i(s) + a$$

Summing over i and using the assets markets clearing conditions gives

$$\sum_i g^i(s) = \sum_i y^i(s), \ \forall s$$

which is the goods market clearing condition.

Note that agents' asset holdings are generally history dependent.

Given initial state s_0 and initial asset distribution a_0^i, can use the policy functions $g^i(s)$ and $h^i(a, s, s')$ to iterate forward to simulate c_t^i and $a_{t+1}^i(s')$ for all possible realisations of $s_1, s_2, ..., s_t, s_{t+1}$.

16. Lecture 17

(a) Introduction to asset pricing

Recall that the prices of Arrow-Debreu and Arrow securities are:

$$q_t^0(s^t) = \beta^t \frac{u'(c_t^i(s^t))}{\mu_i} \pi_t(s^t)$$

and

$$q_t(s^t, s_{t+1}) = \beta \frac{u'(c_{t+1}^i(s^{t+1}))}{u'(c_t^i(s^t))} \pi_{t+1}(s^{t+1}|s^t)$$

respectively.

These can be used to construct the price of any asset. For instance, suppose there is an asset which entitles a stream of consumption goods $\{d_t(s^t)\}_{t=0}^{\infty}$ (where d_t is the quantity), then the price of this asset at date 0 must be

$$p_0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t)$$

If this equation does not hold, the **no arbitrage condition** will be violated, i.e., there will be arbitrage opportunities. Such an asset is viewed as redundant, in the sense that it consists of a bundle of history-contingent dated claims, each of which has already been priced in the market.

We consider a representative agent model to study the price determination of a single asset. Then, we can derive testable implications on asset prices and returns, which we can bring to the data.

(b) **Lucas' asset-pricing model**

Setup of the Lucas tree model (Lucas, 1978):

i. Representative agent model
ii. Endowment economy ("fruit harvest" each period from assets)
iii. Time $t = 0, 1, 2, ...$
iv. A single type of durable asset (a fruit tree)
v. The asset delivers a flow of nondurable consumption goods (dividends); each tree produces a random harvest of fruit in every period.

At the beginning of period t, each tree yields fruit or dividends in the amount y_t, which follows a Markov process with a compact state space and transition function

$$F(y'|y) = Prob(y_{t+1} \leq y'|y_t = y)$$

The fruit is not storable.
vi. The economy is populated with a large number of identical infinitely-lived consumers.

Each consumer starts life at time 0 with a_0 trees.

Each consumer has expected lifetime utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

Because fruit is not storable, to smooth consumption, consumers need to transfer wealth across periods by trading assets (trees).

vii. Denote the price of the asset at date t as p_t. p_t is the "ex-dividend" price, i.e., the asset bought at time t will receive its first dividend $t + 1$.
viii. Because consumers are identical, and trees are identical, we can normalise them by a representative consumer and one tree, respectively.

Note: Since all consumers are identical, they have no incentive to trade, and hence they don't trade.

ix. We normalise $a_0 = 1$, i.e., the durable asset has a unit supply.

(c) Recursive competitive equilibrium

Because the states are Markov, the problem has a recursive formulation.

Representative consumer's dynamic programming problem:
i. Holding one asset (the Lucas tree) entitles the owner to a dividend y' tomorrow, although y' varies across states (this contrasts to the Arrow securities, where the payment is made only if s' is realised in the next period).
ii. The asset can be viewed as a portfolio of **future dividend-contingent Arrow securities**, and its price is a function of current state y only.
iii. The representative consumer takes as given a pricing function $p(y)$ for the asset (i.e., consumers are price takers).
iv. The state variables are (a, y) where a is the consumer's current asset holdings (i.e., the consumer has a trees). The (FE) is:

$$\begin{aligned} v(a, y) = \max_{a'} [u(c) + \beta \int v(a', y') dF(y'|y)] \\ \text{s.t.} \qquad c + p(y)a' = (p(y) + y)a \end{aligned}$$

(i.e., the budget constraint is that consumption + tomorrow's assets = price of assets held since the end of the last period + dividends from the assets held since the end of the last period).
Note: $\int v(a', y') dF(y'|y)$ is the same as $E_{y'|y} v(a', y')$

v. Alternatively, we can define the consumer's beginning of period wealth as

$$\omega \equiv (p(y) + y)a$$

and use ω ("cash on hand") instead of a as a state variable, and rewrite the (FE) as:

$$v(\omega, y) = \max_{c \geq 0}[u(c) + \beta \int v(w', y')dF(y'|y)]$$
$$\text{s.t.} \qquad w' = R(y', y)(w - c)$$

where $R(y', y)$ is the gross rate of return on the asset:

$$R(y'y) \equiv \frac{p(y') + y'}{p(y)}$$

A **recursive competitive equilibrium** is a value function $v(a, y)$, policy function $a' = g(a, y)$ and pricing function $p(y)$ such that

i. Taking $p(y)$ as given, $v(a, y)$ and $g(a, y)$ solve the consumer's dynamic programming problem
ii. the asset market clears, i.e.,

$$g(a, y) = 1 \; \forall a, y$$

If the asset market clears, the budget constraint implies

$$c + p(y) = p(y) + y$$

so that we have the goods market clearing condition ($c = y$) (only need to consider market clearing conditions for $n - 1$ markets, then the n^{th} market also clears).

In equilibrium, the consumption and asset holdings of the representative consumer are:

$$c_t = y_t, a_{t+1} = 1, \;\; \forall t$$

(d) Equilibrium asset prices

The (FE) for the Lucas tree model is

$$v(a, y) = \max_{a'}[u(c) + \beta \int v(a', y')dF(y'|y)]$$
$$\text{s.t.} \qquad c + p(y)a' = (p(y) + y)a$$

To derive the consumption Euler equation, we first take the FOC for the RHS maximisation problem:

$$u'(c)p(y) = \beta \int v_1(a', y')dF(y'|y)$$

By the Envelope condition:

$$v_1(a, y) = u'(c)(p(y) + y) \implies v_1(a', y') = u'(c')(p(y') + y')$$

Substituting $v_1(a', y')$ into the FOC gives the Euler equation:

$$u'(c) = \beta \int u'(c')\frac{p(y') + y'}{p(y)}dF(y'|y)$$

where c and c' are evaluated at the optimum.

Since $R(y', y) = \frac{p(y')+y'}{p(y)}$, the Euler equation is given by

$$u'(c_t) = \beta E_t\{u'(c_{t+1}R_{t+1}\}, \qquad R_{t+1} = \frac{p_{t+1} + y_{t+1}}{p_t}$$

In equilibrium, $c = y$, so equilibrium prices $p(y)$ solve

$$u'(y) = \beta \int u'(y')\frac{p(y') + y'}{p(y)}dF(y'|y)$$

or

$$p(y) = \beta \int \frac{u'(y')}{u'(y)}(p(y') + y')dF(y'|y) \tag{Eq 2}$$

The equilibrium pricing function $p(y)$ is a fixed point of (Eq 2).
(Eq 2) is linear in $p(y)$ so this basically reduces to solving a linear algebra problem.

Iterated Euler equations:

i. In time-series notation, we have the Euler equations

$$p_t = E_t\{\beta \frac{u'(y_{t+1})}{u'(y_t)}(p_{t+1} + y_{t+1})\}$$
$$p_{t+1} = E_{t+1}\{\beta \frac{u'(y_{t+2})}{u'(y_{t+1}}(p_{t+2} + y_{t+2})\}$$

Substituting p_{t+1} in the second equation into the first equation,

$$\begin{aligned} p_t &= E_t\{\beta \frac{u'(y_{t+1})}{u'(y_t)}(E_{t+1}\{\beta \frac{u'(y_{t+2})}{u'(y_{t+1}}(p_{t+2} + y_{t+2})\} + y_{t+1})\} \\ &= E_t\{\beta \frac{u'(y_{t+1})}{u'(y_t)} y_{t+1} + \beta^2 \frac{u'(y_{t+2})}{u'(y_t)} y_{t+2} + \beta^2 \frac{u'(y_{t+2})}{u'(y_t)} p_{t+2}\} \end{aligned}$$

Note: these are all conditional expectations but when we iterate $\mathbb{E}$, they all become conditional on y_t.

ii. More generally, iterating forward T times, we have:

$$p_t = E_t\{\sum_{j=1}^{T} \beta^j \frac{u'(y_{t+j})}{u'(y_t)} y_{t+j} = E_t\{\beta^T \frac{u'(y_{t+T})}{u'(y_t)} p_{t+T}\}$$

So in the limit we have

$$p_t = E_t\{\sum_{j=1}^{\infty} \beta^j \frac{u'(y_{t+j})}{u'(y_t)} y_{t+j}\} + E_t\{\lim_{T\to\infty} \beta^T \frac{u'(y_{t+T})}{u'(y_t)} p_{t+T}\}$$

We can interpret p_t as the sum of a fundamental component (the first term) and a speculative component (the second term).
In equilibrium, the speculative component is zero, as the TVC implies that (see Appendix):

$$\lim_{T\to\infty} E_t\{\beta^T u'(y_{t+T})p_{t+T}\} = 0$$

Thus, in this model, equilibrium asset prices are given by the fundamental component:

$$p_t = E_t\{\sum_{j=1}^{\infty} \beta^j \frac{u'(y_{t+j})}{u'(y_t)} y_{t+j}\}$$

where dividends are discounted from $t + j$ back to t using the stochastic discount factors

$$M_{t,t+j} = \beta^j \frac{u'(y_{t+j})}{u'(y_t)}$$

The equilibrium price of an asset is given by the expected discount value of the dividend stream delivered by the asset; this implication can be tested in the data.

Example: log utility

A. Suppose $u(c) = log(c)$, so that $u'(c) = \frac{1}{c}$. Then,

$$\begin{aligned} p_t &= E_t\{\sum_{j=1}^{\infty} \beta^j \frac{1/y_{t+j}}{1/y_t} y_{t+j}\} \\ &= E_t\{\sum_{j=1}^{\infty} \beta^j y_t\} = \frac{\beta}{1-\beta} y_t \end{aligned}$$

B. So, for log utility, the equilibrium pricing function is

$$p(y) = \frac{\beta}{1-\beta} y$$

When y is high (low), consumers seek to smooth consumption by buying (selling) assets, and asset prices rise (fall) to ensure $a' = 1$.

(e) Appendix:
Recall that the standard form of the TVC for a stochastic problem is

$$\lim_{t\to\infty} \beta^t E_0\{F_1(x_t, x_{t+1}, z_t) \cdot x_t\} = 0$$

where $F(x, x_{t+1}, z_t)$ is the one period return function. In the Lucas asset model, $x_t \sim a_t, x_{t+1} \sim a_{t+1}, z_t \sim y_t$, and thus

$$F(x_t, x_{t+1}, z_t) = u(c_t) = u((p_t + y_t)a_t - p_t a_{t+1})$$

Hence,

$$F_1(x_t, x_{t+1}, z_t) = u'(c_t)(p_t + y_t)$$

In equilibrium, $c_t = y_t, a_t = 1, \ \forall t$. So the TVC is given by

$$\lim_{t\to\infty} \beta^t E_0\{u'(y_t)(p_t + y_t)\} = 0$$

Note that

$$\lim_{t\to\infty} \beta^t E_0\{u'(y_t)(p_t + y_t)\} = \lim_{t\to\infty} \beta^t E_0\{u'(y_t)p_t\} + \lim_{t\to\infty} \beta^t E_0\{u'(y_t)y_t\}$$

As the dividend y_t is finite (as the state space for y_t is compact, we have

$$\lim_{t\to\infty} \beta^t E_0\{u'(y_t)y_t\} = 0$$

Hence the TVC reduces to

$$\lim_{t\to\infty} \beta^t E_0\{u'(y_t)p_t\} = 0$$

This equation holds not only from the perspective at date 0, but at all dates t, so more generally, we have

$$\lim_{T\to\infty} \beta^T E_t\{u'(y_{t+T})p_{t+T}\} = 0$$

Note that the TVC implies that the asset's price p_t cannot grow too fast (cannot grow faster than $\frac{1}{\beta^t}$). Therefore in asset pricing models, the TVC rules out the possibility of a permanent financial bubble.

(f) Exercise 17.1:
Show that with CRRA utility and IID dividend growth:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \qquad g_{t+1} \equiv \frac{y_{t+1}}{y_t} \text{ is IID over time}$$

the equilibrium pricing function is

$$p(y) = \frac{\beta\delta}{1-\beta\delta} y, \qquad \delta \equiv E[g^{1-\sigma}]$$

The price/dividend ratio is again a constant which depends on the distribution of g_{t+1} and the risk aversion of the consumer.

We conjucture that $p(y) = \kappa y$, and plug this into

$$p(y) = \int \beta \frac{u'(y')}{u'(y)} (p(y') + y') dF(y'|y)$$

And then we get

$$\begin{aligned}\kappa y &= \int [\beta \frac{y'^{-\sigma}}{y^{-\sigma}}](\kappa y' + y')dF(y'|y) \\ &= \beta(1+\kappa)\int \frac{y'^{1-\sigma}}{y^{-\sigma}}dF(y'|y)\end{aligned}$$

And hence

$$\begin{aligned}\kappa &= \beta(1+\kappa)\int (\frac{y'}{y})^{1-\sigma}dF(y'|y) \\ &= \beta(1+\kappa)\int g'^{1-\sigma}dF(y'|y) \\ &= \beta(1+\kappa)\mathbb{E}[g^{1-\sigma}] \qquad \text{(since g is iid)}\end{aligned}$$

We define $\delta = \mathbb{E}[g^{1-\sigma}]$ to get the required form:

$$\begin{aligned}\kappa y = p(y) &= \frac{\beta\delta}{1-\beta\delta}y \\ \delta &\equiv \mathbb{E}[g^{1-\sigma}]\end{aligned}$$

17. Lecture 18

(a) Introduction to Incomplete Markets Models

In complete markets models, the optimal consumption allocation is not history dependent.

In incomplete markets models, we do not have a complete set of state-contingent claims, borrowing constraints, etc., so these reduced opportunities make allocations history-dependent since risks are not completely insured for.

Exogenously incomplete markets models (also known as Bewley models) feature a large number of ex-ante identical but ex-post heterogenous households who face idiosyncratic risks which they can only self-insure by trading a limited number of assets.

Focus on three Bewley models:

i. Huggett (1993): endowment economy, no aggregate uncertainty
ii. Aiyagari (1994): production economy, no aggregate uncertainty
iii. Krussel-Smith (1998): production economy, with aggregate uncertainty (i.e., aggregate productivity shocks).

Another type of incomplete markets models involve endogenous incompleteness, which results from information frictions or limited enforcement of contracts.

(b) Setup of **Huggett (1993)**

Discrete time, infinite time horizon, $t = 0, 1, 2, ...$

The economy is populated with a continuum of heterogenous agents, indexed by $i \in [0, 1]$

i. Idiosyncratic endowment risk: In every period, each household receives a stochastic endowment y_{it} which follows a Markov chain:

$$\pi(y'|y) = Prob[y_{it+1} = y'|y_{it} = y]$$

Note: endowments differ across households but everyone's distribution follows the same Markov chain.

ii. Endowment risks are independent across households.
iii. Aggregate endowment is a constant, $\int_i y_{it} = Y$ (no aggregate uncertainty).

Households can only trade a single riskless bond (this is an extreme form of market incompleteness):

i. q_t is the price at which the bond can be bought or sold. Individuals pay q_t to purchase the bond in time t, to get one unit of goods in time $t+1$.
The interest rate is therefore $\frac{1}{q^t} - 1$.
ii. Let a_{it+1} denote agent i's end-of-period t asset holdings. If this is negative, the agent is borrowing; if this is positive, the agent is saving.
iii. Households try to acquire a buffer stock of savings in an attempt to self-insure against their idiosyncratic risk (i.e., they have a precautionary savings motive).

Borrowing constraints:

i. If current y_{it} is low, households may want to borrow to keep c_{it} smooth.
ii. Such borrowing is limited by a constraint

$$a_{it+1} \geq \underline{a}, \qquad \underline{a} < 0$$

$\underline{a}$ is like the maximum debt that each i can take. If we set $\underline{a} = 0$, then households cannot take a negative (borrowing) position.

Households differ in their asset holdings and endowments, (a_{it}, y_{it}).
There is a cross-sectional distribution of households over asset holdings and endowments, $\mu_t(a, y)$.
$\mu_t(a, y)$ gives the mass of households that are of type (a, y) in period t. This distribution changes over time, starting from an initial distribution.
Note: We can think of $\mu_t(a, y)$ as a vector with the share of households of each type in period t. For example, if a takes only the values a_1 and a_2, and y takes only the values y_1 and y_2:

$$\mu_t(a, y) = \begin{bmatrix} \%a_1, y_1 \\ \%a_1, y_2 \\ \%a_2, y_1 \\ \%a_2, y_2 \end{bmatrix}$$

However, with a constant aggregate endowment, μ_t will converge to a **stationary distribution**. That is, the economy will reach a **stationary equilibrium**.
Note: It is still possible that in the stationary equilibrium, households move across types. The main point is that the proportions of each type does not change.

(c) Stationary equilibrium

In the steady-state equilibrium, the distribution of households over types $\mu(.)$ is constant, and as a result bond price q_t is constant (since demand and supply for the bond is determined by households' types, and the distribution of households' types is constant.

However, individual-level variables y_{it}, c_{it}, a_{it+1} are not constant, and can change even in the steady state.

An initial distribution $\mu_0(a, y) \neq \mu(a, y)$ would induce transitional dynamics, but our focus is on the "long run" where such transitional dynamics of the distribution have played out.

Dynamic programming problem of an individual household:

i. Taking the bond price q as given, the Bellman equation for an agent of type (a, y) is

$$v(a, y; q) = \max_{a' \geq \underline{a}} [u(c) + \beta \sum_{y'} v(a', y'; q) \pi(y'|y)]$$

subject to the budget constraint

$$c + qa' \leq a + y$$

Note: the semicolon in $v(a, y; q)$ denotes that a and y are individual state variables, but q is an aggregate state variable.

ii. Let $a' = g(a, y; q)$ denote the policy function that solves the RHS maximisation problem. Note: q is constant and households do not need to know $\mu(.)$ to solve their problem.

iii. Let $\lambda \geq 0$ denote the Lagrangian multiplier on the borrowing constraint $a' \geq \underline{a}$, then the Lagrangian for the maximisation problem is:

$$\mathcal{L} = u(a + y - qa') + \beta \sum_{y'} v(a', y'; q)\pi(y'|y) + \lambda(a' - \underline{a})$$

where the second term is the borrowing constraint.

iv. The FOC with respect to a' is

$$-qu'(c) + \beta \sum_{y'} v_1(a', y'; q)\pi(y'|y) + \lambda = 0$$

The envelope condition is

$$v_1(a, y; q) = u'(c)$$

And hence

$$qu'(c) - \beta \sum_{y'} u'(c')\pi(y'|y) = \lambda \geq 0$$

where λ, c, c' are evaluated at the optimum.

v. We now have two scenarios: either (i) the borrowing constraint is slack ($a' \geq \underline{a}$ and $\lambda = 0$) and we have the usual consumption Euler equation

$$\begin{aligned} qu'(c) &= \beta \sum_{y'} u'(c')\pi(y'|y) \\ &= \beta E_{y'|y} u'(c') \end{aligned}$$

or (ii) the borrowing constraint binds ($a' = \underline{a}$ and $\lambda > 0$) so that we have the Euler inequality

$$qu'(c) > \beta \sum_{y'} u'(c')\pi(y'|y)$$

In case (ii), since $c + qa' = a + y$, then $c = a + y - q\underline{a}$.

A stationary equilibrium is a value function $v(a, y)$, policy function $g(a, y)$, distribution $\mu(a, y)$ and price q such that:

i. taking q as given, $v(a, y)$ and $g(a, y)$ solve the dynamic programming problem for an agent of type (a, y)
ii. the asset market clears (i.e., net supply/demand is zero):

$$\sum_a \sum_y g(a, y)\mu(a, y) = 0$$

iii. the distribution $\mu(a, y)$ is stationary:

$$\mu(a', y') = \sum_a \sum_y Prob[a', y'|a, y]\mu(a, y)$$

where the conditional distribution $Prob[a', y'|a, y]$ is defined by $a' = g(a, y)$ (i.e., evolution of a is governed by the policy function) and $\pi(y'|y)$ (i.e., evolution of y is governed by the Markov chain).

(d) Solution algorithm

The stationary equilibrium can be computed by:

i. Start with an initial guess q^0 and specify a tolerance $\epsilon > 0$

ii. Solve individual's problem for $v(a, y; q^s)$ and $g(a, y; q^s)$
iii. Solve for the stationary distribution $\mu(a, y; q^s)$ implied by $g(a, y; q^s)$ and the exogenous $\pi(y'|y)$
iv. Compute the error on the market-clearing condition:

$$||\sum_a \sum_y g(a, y; q^s)\mu(a, y; q^s)||$$

Stop if the error is less than ϵ; otherwise update the price q^{s+1} and repeat steps (ii) - (iv).

The solution algorithm involves an inner problem to solve the individual agent's dynamic programming problem and find the stationary distribution for a given q (steps (ii) and (iii)), and an outer problem to find the market-clearing price q given individual optimality (steps (i) and (iv)).

The individual's problem can be solved by discrete-state value function iteration:

i. Discretise the state space for a, $[\underline{a}, \bar{a}]$ for some upper bound $\bar{a}$.
ii. If y is a continuous Markov process, approximate it by a finite-state Markov chain (see Lecture 13).

To compute the stationary distribution $\mu(a, y; q)$:

i. Note that the state vector $x_t \equiv (a_t, y_t)$ follows an endogenous Markov chain, because y_t follows an exogenous Markov chain and a is determined by the policy function.
ii. If we know the transition matrix for x_t, P, then we can find the stationary distribution of x_t, μ^*:

$$\mu^* = P'\mu^*$$

Note: (refer to Lecture 13) - basically, μ^* will be the eigenvector associated with the eigenvalue of 1.

To compute the transition probabilities for (a_t, y_t):

i. Write the transition probabilities for the state vector

$$Prob[a_{t+1}, y_{t+1}|a_t, y_t]$$

ii. The distribution of y_{t+1} is independent of a_{t+1}, since y is iid, so this is

$$Prob[a_{t+1}|a_t, y_t] \times Prob[y_{t+1}, y_t]$$

This probability is actually just an indicator function:

$$\begin{cases} 1 & a_{t+1} = g(a_t, y_t) \\ 0 & \text{otherwise} \end{cases}$$

iii. a_{t+1} is given by the policy function $a_{t+1} = g(a_t, y_t)$, so

$$Prob[a_{t+1}|a_t, y_t] = \mathbf{1}[a_{t+1} = g(a_t, y_t)]$$

where **1** denotes the indicator function.
iv. Hence,

$$Prob[a_{t+1}, y_{t+1}|a_t, y_t] = \mathbf{1}[a_{t+1} = g(a_t, y_t)] \times Prob[y_{t+1}|y_t]$$

v. So once we have computed the policy function $g(a, y)$ we can then compute these transition probabilities, which form P.
vi. In this sense, the endogenous Markov process for the state vector $x_t \equiv (a_t, y_t)$ is a coupling of the exogenous process for y_t with the endogenous policy function.

To update the price q:

i. We are trying to find q such that the asset market clears.
ii. If for some $q^s, s = 0, 1, 2, ...$ we have

$$\sum_a \sum_y g(a, y; q^s)\mu(a, y; q^s) > 0$$

then there is excess demand and so we should increase the price, updating to some $q^{s+1} > q^s$.

iii. Likewise, if for some $q^s, s = 0, 1, 2, ...$ we have

$$\sum_a \sum_y g(a, y; q^s)\mu(a, y; q^s) < 0$$

then there is excess supply and so we should decrease the price, updating to some $q^{s+1} < q^s$.

iv. A simple bisection method (i.e., splitting the function into positive and negative sections) can be used to find the market-clearing q, where the excess demand function is

$$F(q) = \sum_a \sum_y g(a, y; q^s)\mu(a, y; q^s)$$

(e) Results for the Huggett Model:

(see Tutorial 10 for details on parameterisation and code. Also note that figures in the tables are annualised r with 6 periods in a year, so instead of $r = \frac{1}{q} - 1$, the rates of return here are $(1 + r)^6$)

We want to compare results for the Huggett model to the benchmark of the complete markets model, which has:

i. Complete risk-sharing: $c_{it} = Y, \ \forall i, t$
ii. Implied bond price $q = \beta$ (see the Euler equation)
iii. Assets are (see the budget constraint):

$$a_{it+1} = (1 + r)(a_{it} + y_{it} - Y)$$

where $r \equiv \frac{1}{q-1}$ denotes the risk-free rate.

In a scenario with low risk aversion:

i. As borrowing constraint becomes tighter (higher $\underline{a}$), there is a high demand for saving, pushing up equilibrium q and pushing down r.
ii. As borrowing constraint becomes slack, there is more borrowing, pushing down equilibrium q and pushing up r.
iii. This approaches the complete market case $q \approx \beta$ if $\underline{a}$ is low enough.

In a scenario with high risk aversion:

i. Higher risk aversion raises q and hence reduces r, for all $\underline{a}$.
ii. Higher risk aversion plus tight borrowing constraints leads to massive demand for saving and hence very low interest rates.

18. Lecture 19

(a) Introduction to **Aiyagari (1994)'s Incomplete Markets Model**

Aiyagari (1994) presents a qualitative and quantitative analysis of the standard deterministic growth model, modified to include:

i. Heterogenous agents facing uninsurable idiosyncratic risks, and no aggregate uncertainty.
ii. A production economy version of Huggett (1993).

Aiyagari's model quantifies the impact on the aggregate saving rate, the importance of asset trading to individuals, and therelative inequality of wealth and income distributions.

Production (similar to the deterministic growth model):

i. There is a representative firm with production function

$$Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha} \qquad 0 < \alpha < 1$$

ii. Capital depreciates at rate δ, $0 < \delta < 1$.

iii. The firm chooses capital K_t and labour demand N_t taking as given the rental rate r_t and wage rate w_t. Then, profit maximation implies:

$$\text{(FOC wrt } K_t\text{)} \qquad r_t + \delta = F_{K,t} = \alpha(\frac{K_t}{N_t})^{\alpha-1}$$
$$\text{(FOC wrt } N_t\text{)} \qquad \omega_t = F_{K,t} = (1-\alpha)(\frac{K_t}{N_t})^{\alpha}$$

Note that in steady state, profits = 0, so:

$$Y + (1-\delta)K - (1-r)K - wN = 0$$
$$Y - \delta k = rK + wN$$

Households (similar to the Huggett model)

i. There is a continuum $i \in [0,1]$ of heterogenous households, each of which maximises

$$\mathbb{E}\{\sum_{t=0}^{\infty} \beta^t u(c_{it})\}, \qquad 0 < \beta < 1$$

subject to the budget constraints

$$c_{it} + a_{it+1} \leq (1+r_t)a_{it} + \omega_t n_{it}$$

where r_t is the real risk-free return on their savings.

ii. Households also face the borrowing constraint

$$a_{it+1} \geq -\phi$$

where ϕ is the upper bound of borrowing.

(b) Stationary Equilibrium

We focus on the stationary (steady-state equilibrium), where, similar to the Huggett model,

i. Aggregate variables are constant, i.e.,

$$\omega, r, N, K, \mu(a,n), \pi(n)$$

ii. But individual-level variables are not constant, i.e.,

$$n_{it}, c_{it}, a_{it}$$

iii. In this equilibrium, aggregate labour is

$$N = \sum_n n\pi(n)$$

which is essentially exogenous.

In contrast to the Huggett model, there is a physical store of value (capital), i.e., aggregate assets are not in zero net supply.
Aggregate assets or aggregate capital stock is given by

$$r + \delta = \alpha(\frac{K}{N})^{\alpha-1}$$
$$\implies K = (\frac{\alpha}{r+\delta}^{\frac{1}{1-\alpha}})N \equiv K(r)$$

The equilibrium wage rate is

$$\omega = (1-\alpha)(\frac{K}{N})^{\alpha} = (1-\alpha)(\frac{\alpha}{r+\delta}^{\frac{\alpha}{1-\alpha}} \equiv w(r)$$

i.e., the real wage rate is a function of the real interest rate.

The individual's dynamic programming problem is:

i. Given r, the Bellman equation for an individual of type (a, n) is:

$$v(a,n;r) = \max_{a' \geq -\phi} [u(c) + \beta \sum_{n'} v(a',n';r)\pi(n'|n)]$$
$$\text{s.t.} \qquad c + a' \leq (1+r)a + \omega(r)n$$

ii. Note that given r, individuals do not need to know either K or $\mu(.)$ to solve their problem.
iii. Let $a' = g(a, n; r)$ denote the policy function that solves the RHS maximisation problem.

A **stationary equilibrium** is a value function $v(a, n)$, policy function $g(a, n)$, distribution $\mu(a, n)$, price r and capital stock K such that

i. taking r as given, $v(a, n)$ and $g(a, n)$ solve the dynamic programming problem for an agent of type (a, n)
ii. taking r as given, K solves the firm's profit maximisation problem, i.e., K solves (Eq 1) below:

$$K = (\frac{\alpha}{r+\delta})^{\frac{1}{1-\alpha}} N \tag{Eq 1}$$

iii. the asset market clears (i.e., total saving = capital stock)

$$K = \sum_a \sum_n g(a,n)\mu(a,n)$$

iv. the distribution $\mu(a, n)$ is stationary

$$\mu(a',n') = \sum_a \sum_n Prob[a',n'|a,n]\mu(a,n)$$

where the conditional distribution $Prob[a', n'|a, n]$ is given by $a' = g(a, n)$ and $\pi(n'|n)$

Assets are in positive net supply, i.e., in aggregate there is more saving than there is borrowing. When the asset market clears, the goods market also clears:

$$\begin{aligned}
&c + a' = (1+r)a + \omega n \\
&\implies C + K = (1+r)K + wN \qquad \text{(this is the aggregate)} \\
&\implies C = rK + wN \\
&\implies C = Y - \delta K \qquad \text{(from above, since profits = 0)} \\
&\implies C + \delta K = K^{\alpha} N^{1-\alpha}, \qquad C \equiv \sum_a \sum_n c(a,n)\mu(a,n)
\end{aligned}$$

where $c(a, n)$ denotes the consumption policy implied by $g(a, n)$ and $I = \delta K$ is steady state investment.
Note: the last line also implies that at steady state, $C + I = Y$.

(c) Solution Algorithm

Similar to the Huggett model, the stationary equilibrium can be computed by:

i. Choose $\epsilon > 0$, and an initial guess r^0 (which then implies $K(r^0)$ and $\omega(r^0)$.
ii. Solve individual's dynamic problem for $v(a, n; r^s)$ and $g(a, n; r^s)$, given r^s.
iii. Solve for the stationary distirbution $\mu(a, n; r^s)$ implied by $g(a, n; r^s)$ and the exogenous $\pi(n'|n)$.
iv. Compute the error on the market-clearing condition

$$|| \sum_a \sum_n g(a,n;r^s)\mu(a,n;r^s) - K(r^s)||$$

If this error is less than ϵ, stop. Otherwise, update to r^{s+1} and repeat steps (ii) to (iv).

To update r:

i. We are trying to find r such that the asset market clears.

ii. If for any r^s (for $s = 0, 1, 2, ...$) we have

$$\sum_a \sum_n g(a, n; r^s)\mu(a, n; r^s) > K(r^s)$$

then there is excess savings and we should decrease the return, updating to some $r^{s+1} < r^s$.

iii. Likewise if for any r^s we have

$$\sum_a \sum_n g(a, n; r^s)\mu(a, n; r^s) < K(r^s)$$

then there is excess borrowing and we should increase the return, updating to some $r^{s+1} > r^s$.

(d) Parameterisation and main results

Aiyagari's parameterisation

i. One period is a year
ii. Time discount factor $\beta = 0.96$
iii. Cobb-Douglas production function with $\alpha = 0.36$
iv. CRRA utility $u(c)$ with risk aversion $\mu\{1, 3, 5\}$ (note that here, μ denotes risk aversion and not the distribution of household's types).
v. Seven-state Markov chain chosen to replicate AR(1), with innovation standard deviation $\sigma \in \{0.2, 0.4\}$ and persistence $\rho \in \{0, 0.3, 0.6, 0.9\}$
vi. Benchmark has borrowing constraint $\phi = 0$

Aiyagari's main results show that compared to the complete markets benchmark, aggregate savings rate is higher in the incomplete markets economy, due to uninsurable idiosyncratic risks.
Real interest rates are lower compared to in the case of complete markets.
Aggregate savings rate increases (and returns to capital fall) when persistence increases, innovation standard deviation increases, and risk aversion increases.

(e) Application: Optimal Capital Income Tax

Aiyagari (1995) considers an optimal tax problem faced by the government - the Ramsey problem, where households behaved as described in the Aiyagari model.

The optimal capital income tax in a stationary equilibrium is shown to be positive.

i. When there are idiosyncratic risks and markets are incomplete, households have a precautionary motive for accumulating assets in order to buffer earnings shocks and smooth consumption.
ii. The possibility of being borrowing-constrained in future periods serves to enhance this precautionary savings motive.
iii. These features lead to excess capital accumulation such that the equilibrium capital stock is greater than the optimal level of capital.
iv. A positive tax rate on capital income is needed to reduce capital accumulation to its socially optimal level.

To show this result, we look at the consumption Euler equation for the individual's problem (assuming the borrowing constraint is slack):

$$u'(c_t) = \beta(1 + r)\mathbb{E}_t u'(c_{t+1})$$

i. If $r = \frac{1}{\beta} - 1$, i.e., the rate of return on capital equals the rate of time preference, then

$$u'(c_t) = \mathbb{E}_t u'(c_{t+1})$$

ii. With the usual CRRA utility function, we have:

$$\begin{aligned}
u'(c_t) &= c_t^{-\mu} \\
u''(c_t) &= \mu c_t^{-\mu-1}) < 0 \\
u'''(c_t) &= (-\mu)(-\mu - 1)c_t^{-\mu-2} > 0
\end{aligned}$$

That is, $u(.)$ is strictly convex such that

$$u'(\mathbb{E}_t(c_{t+1})) < \mathbb{E}_t u'(c_{t+1}) = u'(c_t)$$

As u is strictly concave, this then implies that

$$\mathbb{E}_t(c_{t+1}) > c_t \tag{Eq 6}$$

iii. (Eq 6) implies that a household's consumption $c_t \to \infty$ as $t \to \infty$ almost surely. This is due to an over-accumulation of capital by households.

iv. To prevent this outcome, the rate of return on savings received by households should be lower than the rate of time preference, i.e., $r < \frac{1}{\beta} - 1$.

v. Aiyagari (1995) shows that the optimal capital income tax is positive such that the after-tax rate of return on capital is lower than the rate of time preference:

$$\bar{r} \equiv (1-\tau_k)r < \frac{1}{\beta} - 1, \qquad \tau_k > 0$$

vi. That is, a positive tax rate on capital is justified.

19. Lecture 20

(a) Introduction to **Krusell and Smith (1998)'s Incomplete Markets Model**

i. Krusell-Smith (1998) presents a heterogeneous agents version of the stochastic growth model (We previously studied the representative agents model stochastic growth model).

ii. Objective: Address how movements in the distribution of wealth and income affect the macroeconomy

iii. Key differences from Aiyagari:

A. There is aggregate risk posed by aggregate productivity shocks

B. The wealth distribution is time-varying, which is a state variable for the individual household's problem

C. In computation, the wealth distribution is approximated with a finite number of moments of the distribution

D. Aiyagari's model had a stationary equilibrium; the model here only has a recursive competitive equilibrium.

(b) Setup

i. Firms:

A. There is a representative firm with production function

$$Y_t = z_t F(K_t, N_t) = z_t K_t^{\alpha} N_t^{1-\alpha} \qquad 0 < \alpha < 1$$

B. Aggregate productivity $z_t \in \{z_g, z_b\}$ (i.e., good and bad states of the economy) follows a 2-state Markov chain with

$$\pi_z(z'|z) = Prob[z_{t+1} = z'|z_t = z]$$

C. Factor prices are derived by taking FOCs of the profit function:

$$\pi = z_t K_t^{\alpha} N_t^{1-\alpha} - r_t K_t - w_t N_t$$
$$r_t = \alpha z_t (\frac{K_t}{N_t})^{\alpha-1}$$
$$w_t = (1-\alpha) z_t (\frac{K_t}{N_t})^{\alpha}$$

ii. Households:

A. There is a continuum $i \in [0,1]$ of heterogenous households, each of which maximises

$$\mathbb{E}\{\sum_{t=0}^{\infty} \beta^t u(c_{it}), \qquad 0 < \beta < 1\}$$

subject to the budget constraints

$$c_{it} + k_{it+1} \leq w_t n_{it} + r_t k_{it} + (1-\delta) k_{it}$$

where r_t denotes the rental rate of capital.
Note: The depreciation term is moved to the household's problem rather than the firm's problem. We can think of this as the consumer's asset holdings being the portion of capital that did not depreciate in that period.
The rate of return on capital is $r_t - \delta$.

B. Households also face the constraint

$$k_{it} \geq 0$$

iii. Idiosyncratic risk

A. (Similar to Aiyagari) Households face idiosyncratic labour endowment risk

$$n_{it} \in \{n_l, n_h\}$$

where $n_l(n_h)$ can be interpreted as labour endowment when unemployed (employed).

B. Idiosyncratic risk is correlated with the aggregate state. Joint transition probabilities are given by

$$\pi(n', z'|n, z) = Prob[n_{t+1} = n', z_{t+1} = z'|n_t = n, z_t = z]$$

Note: This is $\pi_z(z'|z) \times \pi(n'|n, z, z')$. π and π_z are different distributions.

C. Let $\psi(n|z)$ denote the fraction of households with n if aggregate state is z.
Recall that in the Aiyagari model, the transition probabilities are $\pi(n'|n)$ which approaches $\psi(n)$ in the steady state. However, since z is now exogenous, the limiting distribution is $\psi(n|z)$.

(c) Recursive Competitive Equilibrium

i. State variables are:

A. Individual state variables: k_{it}, n_{it}.

B. Aggregate state variables: $z_t, \mu_t(k_{it}, n_{it})$

C. Factor prices r_t and w_t are functions of z_t, μ_t

D. Unlike the Aiyagari model, the distribution μ evolves over time:

$$\mu_{t+1} = H_t(\mu_t, z_t, z_{t+1})$$

μ_{t+1} depends on k_{it+1} and n_{it+1}.
k_{it+1} depends on k_{it}, z_t, μ_t
n_{it+1} depends on n_t, z_t, z_{t+1}.

ii. The household's dynamic problem is:

A. The Bellman equation for an agent of type k, n given z, μ is:

$$v(k, n; z, \mu) = \max_{k' \geq 0} [u(c) + \beta \sum_{n'} \sum_{z'} v(k', n'; z', \mu') \pi(n', z'|n, z)]$$

subject to

$$c + k' \leq w(z, \mu) n + r(z, \mu) k + (1-\delta) k$$

and the law of motion for the distribution μ

$$\mu' = H(\mu, z, z')$$

B. Note that z, μ matter for the individual's problem through the factor pricing functions $r(z, \mu)$ and $w(z, \mu)$

C. Let $k' = g(k, n; z, \mu)$ denote the policy function implied by the RHS maximisation problem.

iii. A **recursive competitive equilibrium** is a value function $v(k, n; z, \mu)$, policy function $g(k, n; z, \mu)$, pricing functions $r(z, \mu), w(z, \mu)$ and law of motion for the distribution $H(\mu, z, z')$ (*note that all of these functions are time-invariant*), such that

A. taking $r(z, \mu), w(z, \mu)$ as given, $v(k, n; z, \mu)$ and $g(k, n; z, \mu)$ solve the dynamic programming problem for an agent of type (k, n)

B. $r(z, \mu), w(z, \mu)$ solve the firm's profit maximisation problem

$$r(z, \mu) = \alpha z (\frac{K}{N})^{\alpha - 1}, \qquad w(z, \mu) = (1 - \alpha) z (\frac{K}{N})^{\alpha}$$

C. Capital and labour markets clear:

$$K = \sum_k \sum_n k \mu(k, n) \qquad N = \sum_n n \psi(n|z)$$

D. The law of motion $H(\mu, z, z')$ is generated by the policy function $k' = g(k, n; z, \mu)$ and the exogenous Markov chain $\mu(n', z'|n, z)$

iv. To solve for the equilibrium, we need to keep track of the distribution μ.

A. This can be seen from the household's dynamic programming problem.

B. This can also be seen from the Euler equation

$$u'(c) = \beta \sum_n^{\prime} \sum_z^{\prime} u'(c')(1 + r(z', \mu') - \delta) \pi(n', z'|n, z)$$

where

$$r(z', \mu') = \alpha z' (\frac{K'}{N'})^{\alpha - 1}$$

and K' is determined by μ'.

(d) Solution algorithm

i. The distribution μ is a high-dimensional state variable since there is a probability assigned for each $\mu(k_i, n_i)$.

ii. The computation strategy proposed by Krussel-Smith is to approximate the wealth distribution with a finite number of moments (e.g., mean, variance, skewness, kurtosis, etc.)

A. The wealth distribution is the distribution μ after n has been marginalised out, i.e., $\sum_n \mu(k, n) = \mu_k(k)$.

B. Krusell-Smith use the first moment only (i.e., the mean) to approximate the wealth distribution.

C. The mean of the wealth distribution is aggregate capital stock, K.

D. They consider a linear law of motion for K, which approximates the law of motion H:

$$logK' = a_z + b_z logK$$

Note that this law of motion does not depend on z' because we have marginalised out n' from k. The a and b parameters depend on z.

iii. With this approximation, the Bellman equation for a household of type (k, n) becomes

$$v(k, n; z, K) = \max_{k' \geq 0} [u(c) + \beta \sum_{n'} \sum_{z'} v(k', n'; z', K') \pi(n', z'|n, z)]$$

subject to the budget constraint

$$c + k' \leq w(z, K)n + r(z, K)k + (1 - \delta)k$$

and the law of motion for K

$$logK' = a_z + b_z logK$$

A. The distribution μ has been replaced by a single variable K.
B. A household does not know the law of motion for K. However, the household has a perceived law of motion and makes decisions based on that.
C. In equilibrium, the perceived law of motion of any household is the same as the actual law of motion generated by aggregating all households' decisions.

iv. This suggests the following iterative algorithm to find the equilibrium:
A. Start with initial guess for the coefficients in the law of motion, a_z^0, b_z^0.
B. Solve the individual's problem for the value function $v^0(k, n; z, K)$ and policy function $k' = g^0(k, n; z, K)$
C. Use exogenous $\pi_z(z'|z), \pi(n', z'|n.z)$, law of motion for K, and $g^0(k, n; z, K)$ to simulate a panel of capital stocks k_t and let

$$K_t = \frac{1}{I}\sum_{i=1}^{I} k_{it}, \qquad t = 1, ..., T$$

D. For each $z \in \{z_g, z_b\}$, run the regression

$$logK_{t+1} = a_z^1 + b_z^1 logK_t + \epsilon_{t+1}$$

on the simulated aggregate capital stock data.
(e.g., for the two-state case, we will run a regression for z_g and for z_b, and the parameters will be different for each case).
E. Check if $||(a_z^0, b_z^0) - (a_z^1, b_z^1)||$ is small enough (we would need to check across all states and ensure that ϵ has been reached across all states, i.e., the maximum value across states is below ϵ).
If not, update to a new guess for (a_z, b_z) and repeat steps (B) to (E).

v. Refinement of the algorithm
A. Calculate $||(a_z^0, b_z^0) - (a_z^1, b_z^1)||$ and calculate R^2 in the regression

$$logK_{t+1} = a_z^1 + b_z^1 logK_t + \epsilon_{t+1}$$

If $(a_z^0, b_z^0) \approx (a_z^1, b_z^1)$ and R^2 of regression is high, stop. Otherwise, update to a new guess and try again.
B. If coefficients $(a_z^0, b_z^0) \approx (a_z^1, b_z^1)$ but R^2 remains low, add more moments to perceived law of motion.

(e) Calibration and main results

i. (see slides for details)
ii. Authors are able to get close estimations for the law of motion for aggregate capital in the approximate equilibrium using the first moment alone.
This implies that this macroeconomic model with heterogeneity features **approximate aggregation** (since we approximated the whole distribution by aggregating capital).
iii. Recall that in the representative agent stochastic growth model, all endogenous variables are functions of aggregate capital stock and aggregate productivity.
iv. This still approximately holds in this heterogenous agents model; in equilibrium, all aggregate endogenous variables and relative prices can be almost perfectly described as a function of K (the mean of the wealth distribution) and z.
v. The Krusell-Smith model has this property because individual policy functions $k' = g(k, n; z, K)$ are nearly linear in k. There are nonlinearities, but only for very poor households who don't hold much capital.
vi. Krusell and Smith also extend the benchmark model to consider time-varying discount factors (i.e., different values for β). This version of the model leads to much more wealth inequality.

20. Lecture 21

(a) Introduction to **Hopenhayn (1992)'s model**

i. This is a workhorse model for studying firm and industry dynamics (as opposed to household decisions, which were the focus of the previous models).
ii. Competitive firms face idiosyncratic productivity shocks, but there is no aggregate uncertainty.
iii. Firms enter, grow, decline, and exit, generating an endogenous distribution of firm size, which is stationary in the long run.
iv. The paper establishes the existence and uniqueness of a stationary industry equilibrium with entry and exit and characterises comparative statics in such an equilibrium.

(b) The model and stationary equilibrium

i. Key elements of the model
   A. A continuum of competitive firms, no strategic interactions.
   B. Firms produce with a decreasing returns to scale production function.
   C. Individual firm productivities follow a first-order Markov process, but with no aggregate risk.
   D. Fixed cost for firms to enter, and fixed cost for firms to operate in each period.
ii. We focus on a stationary equilibrium which features constant output and factor prices and a time-invariant firm size distribution.

iii. Time $t = 0, 1, 2...$
iv. Competitive markets: Firms take output price p and input price (wage rate) w as given.
We can choose either p or w as the numeraire; we choose $w =$ so p will then be relative to w.

v. Firms face idiosyncratic productivity shocks in each period.
   A. Productivity draws follow a first-order Markov process with transition function $F(z'|z)$
   Note: The CDF is F, i.e., $F(z'|z) = Prob(z_{t+1} \leq z'|z_t = z)$. The PDF is $f(z'|z)$.

   B. $F(.|z)$ is assumed to be strictly decreasing in z:

   $$\text{if } z_1 > z_2, F(z'|z_1) \leq F(z'|z_2) \qquad \text{for all } z'$$

   i.e., $F(\cdot|z_1)$ first order stochastically dominates $F(\cdot|z_2)$.
   C. This assumption implies if $z_1 > z_2$, the expected value of z' conditional on z_1 is greater than conditional on z_2.

vi. Firms need to pay a fixed sunk cost $k_e > 0$ to enter the industry. Upon entry, a new firm draws initial productivity z_0 from distribution $G(z)$ (pdf $g(z)$).
vii. In every period each firm chooses labour input n to produce a single output y, given productivity z:

$$y = zf(n)$$

So the profit of the firm in the period is given by

$$\pi(z; p) = \max_n [pzf(n) - n - k]$$

where $f(.)$ is strictly increasing and strictly concave and parameter $k > 0$ is the per-period fixed cost of operating.
Notes: There is no capital in the production function. k is the per-period fixed cost; k_e is the fixed cost for entering the industry.

viii. Let $n(z; p)$ and $y(z; p)$ denote the optimal employment and associated output respectively; $n(z; p)$, $y(z; p)$, $\pi(z, p)$ are all strictly increasing in productivity z.

ix. Timing of events within a period:
   A. Firms enter the period with productivity z.
   B. Incumbents pay operating cost k, choose labour n, and produce output y.

C. Incumbents choose to stay or exit (after production); entrants decide to enter or not, and pay k_e to enter

D. Stayers draw z' from $F(z'|z)$, entrants draw z' from G
(they then enter next period with z')

x. Industry demand and supply

A. Industry demand is given by the exogenous demand function $D(p)$, which is strictly decreasing.

B. Two key aggregate variables: (i) the output price p, and (ii) the cross-sectional distribution of productivity types $\mu(z;p)$.

C. Industry supply is endogenous:

$$Y(p) = \int y(z;p)\mu(z;p)dz$$

D. The market clears when

$$Y(p) = D(p)$$

xi. Incumbent's problem

A. Let $v(z;p)$ denote the value function of an incumbent firm with current productivity z, given price p

B. The Bellman equation for an incumbent firm is

$$v(z;p) = \pi(z;p) + \beta \max[0, \int v(z';p)f(z'|z)dz'] \qquad \text{(Eq 1)}$$

where the first term is the firm's profit in the current term, and the second term is the maximum between 0 (if the firm exits) and the firm's profit in the next period (if it stays and produces).

C. Intuitively, if the firm's current productivity is low, its future productivity is expected to be low, and the firm is more likely to exit.

D. There exits an **exit threshold** $z^*(p)$ such that the firm exits if $z < z^*(p)$, where $z^*(p)$ solves

$$\int v(z';p)f(z'|z^*)dz' = 0$$

i.e., the firm exits if expected future profit is zero.

xii. Entrant's problem

A. Potential entrants are ex ante identical

B. Entrants pay $k_e > 0$ to enter, and make an initial draw z' from G after entry.

C. Entrants enter next period with z' and behave as an incumbent firm.

D. Let $m \geq 0$ denote the mass of entrants. **Free entry** implies that

$$\beta \int v(z';p)g(z')dz' - k_e \leq 0$$

with strict equality whenever $m > 0$
Note: m does not include the incumbents.
Note: In the stationary equilibrium, $m = m'$.

xiii. A stationary equilibrium is a value function $v(z;p)$, output policy $y(z;p)$, cutoff productivity $z^*(p)$, distribution $\mu(z;p)$, mass of entrants m and price p such that:

A. Taking p as given, $v(z;p)$, $y(z;p)$ and $z^*(p)$ solve the problem of an incumbent firm of type z.

B. The free entry condition

$$\beta \int v(z;p)g(z)dz - k_e \leq 0$$

is satisfied, with strict equality whenever $m > 0$

C. The goods market clears

$$\int y(z;p)\mu(z;p)dz = D(p)$$

D. The distribution $\mu(z;p)$ is stationary

$$\mu(z';p) = \int \phi(z'|z)\mu(z;p)dz + mg(z')$$

where $\phi(z'|z)$ is given by

$$\phi(z'|z) = f(z'|z) \times \mathbf{1}[z \geq z^*(p)]$$

Note: the ϕ term refers to the distribution of the incumbents; the mg term refers to the distribution of the entrants.

(c) Computing the stationary equilibrium

i. Hopenhayn (1992) establishes the existence and uniqueness of a stationary equilibrium with positive entry.

ii. General algorithm to compute the equilibrium:

A. Guess output price p^0, solve the incumbent's dynamic programming problem (Eq 1) to find the value function $v(z;p^0)$ and the optimal exit rule $z^*(p^0)$.

B. Check whether $p = p^0$ satisfies the free-entry condition with positive entry:

$$\beta \int v(z;p)g(z)dz - k_e = 0$$

This step can be done using a bisection method.
If the condition does not hold for some prespecified tolerance level, then guess a new price p^1 and go back to step A. Continue until a price p^* is found.

C. Guess a measure of entrants, m^0. Given $m = m^0$, calculate the stationary distribution $\mu^0(z;p^*)$ that solves

$$\mu(z';p^*) = \int \phi(z'|z)\mu(z;p^*)dz + mg(z') \qquad \text{(Eq 2)}$$

Note that the RHS depends on the price found in step B via the exit threshold $z^*(p^*)$.

D. Given this $\mu^0(z;p^*)$, calculate the total industry supply and check the market clearing condition

$$Y^0(p^*) = \int y(z;p^*)\mu^0(z;p)dz = D(p^*) \qquad \text{(Eq 3)}$$

For example, if $Y^0(p^*)$ is too low, then go back to step C and guess $m^1 > m^0$. Continue until until a m^* is found that solves the market-clearing condition.

iii. Now suppose productivity follows a N-state Markov chain with states $z_1 < z_2 < ... < z_n$ and transition probabilities f_{ij}, and the algorithm can be implemented as follows.

iv. Solve an incumbent's dynamic programming problem

A. Given price p, we have that v, π, y are vectors of N elements.
Denote $v_i(p) \equiv v(z_i;p); \pi_i(p) \equiv \pi(z_i;p); y_i(p) \equiv y(z_i;p), i = 1, ..., N$.

B. The discretised Bellman equation for an incumbent firm is then

$$v_i(p) = \pi_i(p) + \beta \max[0, \sum_{j=1}^{N} v_j(p) f_{ij}] \qquad \text{(Eq 4)}$$

where $f_{ij} = Prob(z' - z_j | z = z_i)$. The vector $\{v_i(p)\}$ can be solved by value function iteration.
(We can show that (Eq 4) defines a contraction mapping).

v. Find the equilibrium p^* that satisfies the free-entry condition

A. Given $\{v_i(p)\}$ that solves the incumbent's problem, the free-entry condition with positive entry is

$$v^e(p) \equiv \beta \sum_{i=1}^{N} v_i(p) g_i = k_e$$

where $g_i \equiv g(z_i)$ is the productivity distribution for entrants.

B. Easy to see that $v^e(0) < 0$ and $v^e(p)$ is monotone increasing in p, so p^* can be solved by bisection.

vi. Define the exit decision

A. Once p^* and the incumbent firm's value function $v_i(p^*)$ are solved, we can define the exit threshold $z^*(p^*)$:

$$z^*(p^*) = z_{i^*}, i^* \equiv argmin_i [\sum_{j=1}^{N} v_j(p^*) f_{ij} \geq 0]$$

B. All firms with productivity $z_i < z^*(p^*)$ (i.e., $i < i^*$) exit, all firms with $z_i \geq z^*(p^*)$ (i.e., $i \geq i^*$) continue.

C. Collect exit decisions into a vector $x(p^*)$ with elements

$$\begin{aligned} x_i(p^*) &= 1 \qquad \text{if } z_i < z^*(p^*) \qquad \text{(firms that exit)} \\ x_i(p^*) &= 0 \qquad \text{if } z_i \geq z^*(p^*) \qquad \text{(firms that stay)} \end{aligned}$$

vii. Compute the stationary distribution

A. Recall that the stationary distribution satisfies

$$m(z'; p) = \int \phi(z'|z) \mu(z; p) dz + mg(z')$$

where

$$Fdellandrea@student.unimelb.edu.au \phi(z'|z) = f(z'|z) \times \infty\{z \geq z^*(p)\}$$

B. Let $\mu_i \equiv \mu(z_i; p)$ denote the mass of firms with productivity z_i

C. The distribution evolution equation (Eq 2) is in fact N linear equations in $\mu_1, ..., \mu_N$, with N unknowns (the μ's):

$$\boldsymbol{\mu} = \boldsymbol{\Phi}(p^*)\boldsymbol{\mu} + m\boldsymbol{g}$$

where $\boldsymbol{\Phi}(p^*)$ has elements

$$\phi_{ji}(p^*) = (1 - x_i(p^*)) f_{ij}, \qquad i, j = 1, ..., N$$

Intuitively, ϕ is the mass of each incumbent type that stays in the market, multiplied by the mass of each type, while mg is the mass of new entrants.

D. The stationary distribution $\boldsymbol{\mu}$ (also a vector of N elements) is solved as

$$\boldsymbol{\mu} = m(\boldsymbol{I} - \boldsymbol{\Phi}(p^*))^{-1}\boldsymbol{g} \equiv \boldsymbol{\mu}(m, p^*)$$

E. Note that $\boldsymbol{\mu}$ is linearly homogeneous in m:

$$\begin{aligned} \boldsymbol{\mu}(1, p^*) &= (I - \boldsymbol{\Phi}(p^*))^{-1} g \\ \boldsymbol{\mu}(m, p^*) &= m(I - \boldsymbol{\Phi}(p^*))^{-1} g \\ \boldsymbol{\mu}(m, p^*) &= m\boldsymbol{\mu}(1, p^*) \end{aligned}$$

viii. Find the measure of entrants m^* using market clearing condition

A. The mass of entrants m^* equates industry supply to demand:

$$Y(m,p^*) = \sum_{i=1}^{N} y_i(p^*)\mu_i(m,p^*) = D(p^*)$$

B. Because $\boldsymbol{\mu}$ is lineary homogenous in m,

$$Y(m,p^*) = \sum_{i=1}^{N} y_i(p^*)[m\mu_i(1,p^*)] = mY(1,p^*)$$

C. So m^* is

$$m^* = \frac{D(p^*)}{Y(1,p^*)} = \frac{D(p^*)}{\sum_{i=1}^{N} y_i(p^*)\mu_i(1,p^*)}$$

ix. Aside: No entry case

A. If $m^* = 0$, there is no entry.

B. Then, in a stationary, equilibrium, there must also be no exit.

C. The stationary distribution of firms $\boldsymbol{\mu}$ is simply the stationary distribution of the Markov chain for productivities, denoted as $\bar{f}$ (this is just the Markov chain for the incumbent firms).

D. Then equilibrium price p^* is determined by the market clearing condition

$$Y(p) = \sum_{i=1}^{N} y_i(p)\bar{f}_i = D(p)$$

i.e., industry input = industry demand.
Note: Recall that the stationary distribution is the eigenvector corresponding to the unit eigenvalue of the transition matrix.
Note: We no longer use the free-entry condition since there is no more entry.

(d) Comparative statics

i. Implications of an increase in entry cost k_e:

A. Increases expected discounted profits, since from the free-entry condition

$$\beta \int v(z;p^*)g(z)dz = k_e$$

so if k^e increases, p^* must increase.

B. Decreases exit threshold, since from the exit equation (the second term of the Bellman equation),

$$\int v(z;p^*)f(z'|z^*)dz' = 0$$

so if k_e increases, exit threshold decreases, there is less selection, incumbents make more profits, and more continue.
This also increases the average age of firms.

C. Decreases entrants m^*

D. Decreases entry/exit rate

E. Increase price p^*

ii. An increase in entry cost k_e has ambiguous implications for firm-size distribution (distribution over employment)

A. **Price effect**: higher k_e increases price p^*, hence incumbents increase output $y(z;p)$ and employment $n(z;p^*)$

B. **Selection effect**: higher k_e reduces productivity threshold z^* for exit, hence more incumbent firms stay, and the average incumbent has a relatively lower productivity and employment.