Improved Techniques for SAT-to-Ising encoding applied to Hybrid Quantum Annealing

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Background: Propositional Satisfiability (SAT)

Given a Boolean formula with n variables, is there a truth assignment μ satysfing the formula?

$$\phi \doteq (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee x_2)$$

In this case:

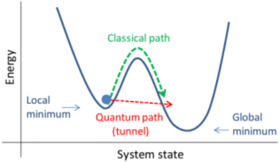
- If $\mu = \{x_1 = \top, x_2 = \bot\}$ then $\mu \not\models \phi$
- If $\mu = \{x_1 = \top, x_2 = \top\}$ then $\mu \models \phi$

In the worst case up to 2^n truth assignments have to be checked to verify its unsatisfiability
$\implies$ SAT is NP-complete!

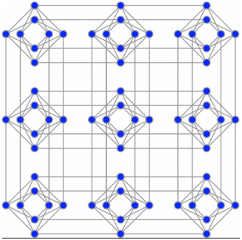
Background: Quantum Annealers (QA)

Quantum annealers are devices specialized in reaching the minimum energy of an Ising model and capable of exploiting quantum effects such as tunneling:

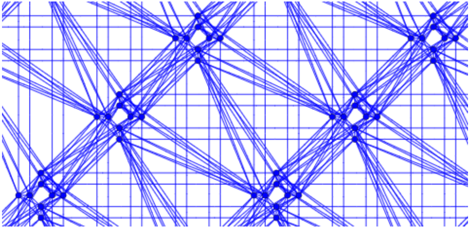
$$H(\underline{x}|\underline{h},\underline{J}) = \sum_{i \in V} h_i x_i + \sum_{(i,j) \in E} J_{ij} x_i x_j$$



- $x_i \in \{-1, 1\}$ (qubit) is the value of the vertex i.
- $h_i \in [-2, 2]$ is called bias of i, $J_{ij} \in [-1, 1]$ is called coupling between i and j, G(V,E) represents the connection graph.



Old Chimera architecture



New Pegasus architecture

SAT-to-Ising

GOAL: determine a reduction from SAT to an Ising model $P_F(\underline{x}|\underline{\theta})$, determining the parameters θ such that:

$$P_F(\underline{\mathbf{x}}|\underline{\theta}) = \theta_0 + \sum_{i \in V} \theta_i x_i + \sum_{(i,j) \in E} \theta_{ij} x_i x_j = \left\{ \begin{array}{ll} = 0, & \textit{if } SAT(F(\underline{\mathbf{x}})) \\ \geq g_{min}, & \textit{if } UNSAT(F(\underline{\mathbf{x}})) \end{array} \right\}$$

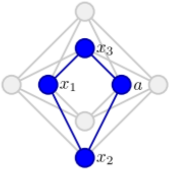
- The Ising model is also NP-complete!
- Multiple encodings are valid for a Boolean function
  $\implies$ we should use optimized encodings, less prone to quantum side-effects (e.g. co-tunneling)

SAT-to-Ising (cont.)

For simple Boolean functions $F(\underline{x})$, we can determine a corresponding penalty function such that:

$$\exists\underline{\theta}\forall\underline{\mathbf{x}}\left[\begin{array}{l}(F(\underline{\mathbf{x}}) \rightarrow \forall\underline{\mathbf{a}}.(P_F(\underline{\mathbf{x}},\underline{\mathbf{a}}|\underline{\theta}) \geq 0))\wedge \\ (F(\underline{\mathbf{x}}) \rightarrow \exists\underline{\mathbf{a}}.(P_F(\underline{\mathbf{x}},\underline{\mathbf{a}}|\underline{\theta}) = 0))\wedge \\ (\neg F(\underline{\mathbf{x}}) \rightarrow \forall\underline{\mathbf{a}}.(P_F(\underline{\mathbf{x}},\underline{\mathbf{a}}|\underline{\theta}) \geq g_{min}))\end{array}\right]$$

s.t. g_{min} is maximized

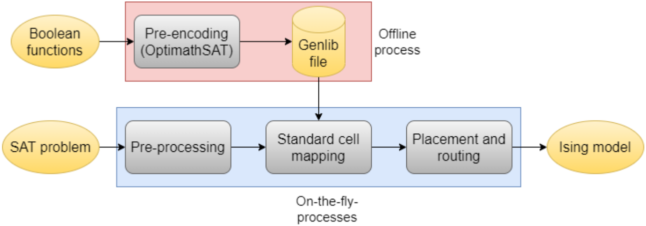


Example of SAT-to-Ising reduction

$F(\underline{x}) = x3 \leftrightarrow (x_1 \wedge x_2)$
$P_F(\underline{x}|\underline{\theta}) = \frac{5}{2} - \frac{1}{2}x_1 - \frac{1}{2}x_2 + x_3 + \frac{1}{2}x_1x_2 - x_1x_3 - x_2a - x_3a$

- $\underline{a}$ are additional nodes, called **ancillas**, necessary to prevent over-constrainedness.
- The general formula is $\exists\forall\exists$-quantified
  $\Longrightarrow$ Its complexity is worse than NP!
- We can apply (optimized) Shannon Expansion and solve it using Optimization Modulo Theories (OMT) only when few variables are involved.

Original workflow



Idea: divide-and-conquer $F(\underline{x})$ into $\bigwedge_k F_k(\underline{x})$, s.t. $P_F(\underline{x})$ into $\sum_k P_{F_k}(\underline{x})$

Offline process (via OMT):

- Pre-encoding: create a library of basic simple Boolean formulas $P_F(\underline{x})$, computed offline using the quantified formulation.

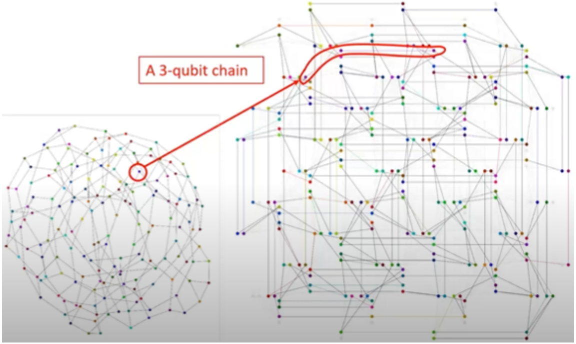
Online process:

- Preprocessing: partition $F(\underline{x})$ into a conjunction of simpler formulas $\bigwedge_k F_k(\underline{x})$.
- Standard cell mapping: match each $F_k(\underline{x})$ with a basic penalty functions P_{F_k} from the pre-computed library.
- Placement: each P_{F_k} is placed into a disjoint subgraph of the QA graph.
- Routing: equivalence chains of qubits $(x_i \leftrightarrow x_i')$, whose penalty is $1 - x_i x_i'$, are built to connect variables shared by P_{F_k}s.

Future work

Dwave recently announced a new Quantum Annealers, **Advantage Performance**.

- 5000+ qubits and about 40000 couplings.
- Up to 15*15*12 lattice, guaranteeing shorter chains.



Future work

- What kind of formal verification problems can we solve using annealers?
- Are there classes of problems where there is **quantum supremacy**?
- Are there better approaches to encode SAT problems into Ising models?