Exercise 3

Chiara Thien Thao Nguyen Ba 10727985

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1 RFID

A RFID system based on Dynamic Frame ALOHA is composed of N=4 tags.

1. Find the overall collision resolution efficiency η in the different cases in which the initial frame size is set to r1=1,2,3,4,5,6
   - Assume that after the first frame, the frame size is correctly set to the current backlog size
   - Assume as given the duration of the arbitration period with N=2,3 tags when r=N
2. After computing the values of the efficiency with the different frame sizes, produce a plot with values of η over r1
3. For what values of r1 we have the maximum value for η? Comment.

1.1 Answer 1

The efficiency η is calculated as:

$$\eta = \frac{N}{L_N}$$

Initial frame size r1 = 1. To find the duration of the arbitration period, since the size of the first frame is not equal to the number of tags to resolve, the recursive formula does not apply. We can write the duration as:

$$L_4^* = r_1 + \sum_{i=0}^{3} L_{4-i} P(S = i)$$

Where:

- $P(S = 0) = 1$
- $P(S = 1) = P(S = 2) = P(S = 3) = 0$

Thus, we can write:

$$L_4^* = 1 + L_4$$

The second frame, by assumption, is set to the correct backlog size. For L_4 we can use the recursive formula:

$$L_4 = 4 + \sum_{i=0}^{3} L_{4-i} P(S = i)$$

Where:

- $P(S=0) = \frac{1}{4}^4 \cdot 4 + \frac{1}{4}^4 \cdot \binom{4}{2} \cdot \frac{4!}{2 \cdot 2} = \frac{10}{64}$
- $P(S=1) = \frac{1}{4}^4 \cdot 4 \cdot 4 \cdot 3 = \frac{3}{16}$
- $P(S=2) = \frac{1}{4}^4 \cdot \binom{4}{2} \cdot 4! = \frac{9}{16}$
- $P(S=3) = 0$
- $P(S=4) = \frac{1}{4}^4 \cdot 4! = \frac{6}{64}$

We obtain:

$$L_4 = 4 + \frac{10}{64} L_4 + \frac{3}{16} L_3 + \frac{9}{16} L_2$$

Using the recursive formulas for L_3 and L_2, we obtain:

$$L_3 = 3 + \sum_{i=0}^{2} L_{3-i} P(S = i)$$

Where:

- $P(S=0) = 3 \cdot \frac{1}{3}^3 = \frac{1}{9}$
- $P(S=1) = \frac{2}{3}$
- $P(S=2) = 0$
- $P(S=3) = 6 \cdot \frac{1}{3}^3 = \frac{2}{9}$

We obtain:

$$L_3 = 3 + \frac{1}{9} L_3 + \frac{2}{3} L_2$$

Iterating for L_2, we get:

$$L_2 = 2 + \sum_{i=0}^{1} L_{2-i} P(S = i)$$

Where:

- $P(S=0) = 2 \cdot \frac{1}{2}^2 = \frac{1}{2}$

- $P(S=1)=0$
- $P(S=2)=\frac{1}{2}$

We obtain:

$$L_2 = 2 + \frac{1}{2}L_2$$

which leads to $L_2 = 4$. We substitute the value in L_3 and we get:

$$L_3 = 3 + \frac{1}{9}L_3 + \frac{8}{3} = \frac{51}{8}$$

Then, we substitute the values of L_3 and L_2 in L_4 and get:

$$L_4 = 4 + \frac{10}{64}L_4 + \frac{3}{16} \cdot \frac{51}{8} + \frac{9}{16} \cdot 4 = \frac{953}{108} = 8.824$$

We substitute the value of L_4 to get the duration of the arbitration process:

$$L_4^* = 1 + L_4 = 1 + 8.824 = 9.824$$

We can calculate the efficiency as:

$$\eta = \frac{4}{9.824} \simeq 0.4072$$

Initial frame size r1 = 2. We use the same previous formula, but we need to recalculate the probabilities for this initial frame size.

$$L_4^* = r_1 + \sum_{i=0}^{3} L_{4-i} P(S=i)$$

Where:

- $P(S=0) = \frac{1}{2}^4 \cdot 2 + \frac{1}{2}^4 \cdot \binom{4}{2} = \frac{1}{2}$
- $P(S=1) = \frac{1}{2}^4 \cdot 4 \cdot 2 = \frac{1}{2}$
- $P(S=2) = P(S=3) = P(S=4) = 0$

Thus, by using the previous results and the new probabilities we obtain:

$$L_4^* = 2 + \frac{1}{2}L_4 + \frac{1}{2}L_3 = 2 + \frac{1}{2} \cdot \frac{953}{108} + \frac{1}{2} \cdot \frac{51}{8} = 9.599$$

We can calculate the efficiency as:

$$\eta = \frac{4}{9.599} \simeq 0.4167$$

Initial frame size r1 = 3. We use the same previous formula, but we need to recalculate the probabilities for this initial frame size.

$$L_4^* = r_1 + \sum_{i=0}^{3} L_{4-i} P(S = i)$$

Where:

- $P(S=0) = \frac{1}{3}^4 \cdot \binom{4}{2} \cdot 3 + \frac{1}{3}^4 \cdot 3 = \frac{7}{27}$
- $P(S=1) = \frac{1}{3}^4 \cdot \binom{4}{2} \cdot 4 = \frac{8}{27}$
- $P(S=2) = \frac{1}{3}^4 \cdot \binom{4}{2} \cdot 3! = \frac{4}{9}$
- $P(S=3) = P(S=4) = 0$

Thus, by using the previous results and the new probabilities we obtain:

$$L_4^* = 4 + \frac{7}{27} L_4 + \frac{8}{27} L_3 + \frac{4}{9} L_2 = 3 + \frac{7}{27} \cdot \frac{953}{108} + \frac{8}{27} \cdot \frac{51}{8} + \frac{4}{9} \cdot 4 = 8.954$$

We can calculate the efficiency as:

$$\eta = \frac{4}{8.954} \simeq 0.4467$$

Initial frame size r1 = 4. For this case, since we have that the number of tags is 4 and we have previously computed the probabilities and L_4. We have that:

$$L_4 = 8.824$$

We can calculate the efficiency as:

$$\eta = \frac{4}{8.824} \simeq 0.4533$$

Initial frame size r1 = 5. We use the same previous formula, but we need to recalculate the probabilities for this initial frame size.

$$L_4^* = r_1 + \sum_{i=0}^{3} L_{4-i} P(S = i)$$

Where:

- $P(S=0) = \frac{1}{5}^4 \cdot 5 + \frac{1}{5}^4 \cdot 3 \cdot \binom{5}{2} \cdot 2! = \frac{13}{125}$
- $P(S=1) = \frac{1}{5}^4 \cdot 4 \cdot \binom{5}{2} \cdot 2! = \frac{16}{125}$
- $P(S=2) = \frac{1}{5}^4 \cdot \binom{4}{2} \cdot \binom{5}{3} \cdot 3! = \frac{72}{125}$

- $P(S=3)=0$
- $P(S=4)=\frac{1}{5}^4 \cdot 5 \cdot 4! = \frac{24}{125}$

Thus, we obtain:

$$L_4^* = 5 + \frac{13}{125}L_4 + \frac{16}{125}L_3 + \frac{72}{125}L_2 = 5 + \frac{13}{125}\cdot\frac{953}{108} + \frac{16}{125}\cdot\frac{51}{8} + \frac{72}{125}\cdot 4 = 9.038$$

We can calculate the efficiency as:

$$\eta = \frac{4}{9.038} \simeq 0.4426$$

Initial frame size r1 = 6. We use the same previous formula, but we need to recalculate the probabilities for this initial frame size.

$$L_4^* = r_1 + \sum_{i=0}^{3} L_{4-i} P(S=i)$$

Where:

- $P(S=0)=\frac{1}{6}^4 \cdot 6 + \frac{1}{6}^4 \cdot 3 \cdot \binom{6}{2} \cdot 2! \ = \frac{2}{27}$
- $P(S=1)=\frac{1}{6}^4 \cdot 4 \cdot \binom{6}{2} \cdot 2! = \frac{5}{54}$
- $P(S=2)=\frac{1}{6}^4 \cdot \binom{4}{2} \cdot \binom{6}{3} \cdot 3! = \frac{5}{9}$
- $P(S=3)=0$
- $P(S=4)=\frac{1}{6}^4 \cdot \binom{6}{4} \cdot 4! = \frac{5}{18}$

Thus, we obtain:

$$L_4^* = 6 + \frac{2}{27}L_4 + \frac{5}{54}L_3 + \frac{5}{9}L_2 = 6 + \frac{2}{27}\cdot\frac{953}{108} + \frac{5}{54}\cdot\frac{51}{8} + \frac{5}{9}\cdot 4 = 9.466$$

We can calculate the efficiency as:

$$\eta = \frac{4}{9.466} \simeq 0.4226$$

1.2 Answer 2

The following picture illustrates the plot of the efficiency η based on the different values of the initial frame r_1.

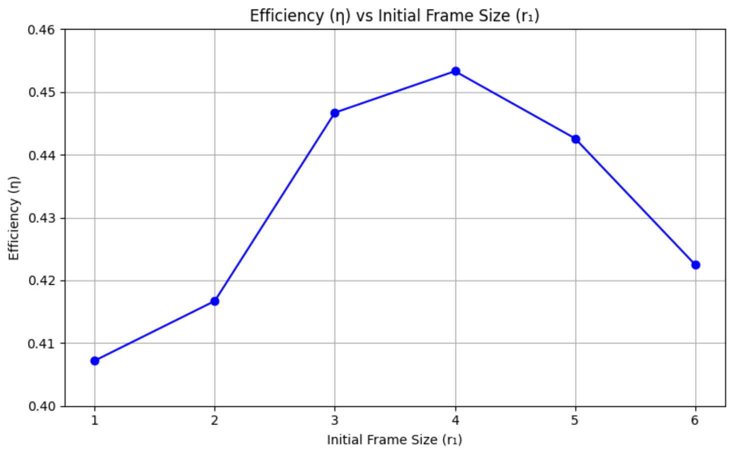

Figure 1: Plot of η

1.3 Answer 3

As we can see from the calculations and the obtained results, the value of r_1 for which η is maximum is when $r_1 = N$, where N is the number of tags. So, in this case we have $N = 4$ and the maximum value of r_1 for which η is maximum is 4.

This is because setting $r_1 = N$ provides an optimal balance between collision and idle probabilities. If r_1 is smaller than N, the probability of collision increases due to too many tags competing for limited slots. On the other hand, if r_1 is larger than N, the frame becomes underutilized, resulting in many empty slots. Therefore, the maximum number of successful transmissions occurs when $r_1 = N$, which aligns with the theoretical result that the expected efficiency is maximized under this condition.