# Through the Lens of Sequence Submodularity

## Supplementary Material

### 1 Recursive Form of $F(S)$ for S&T

Following previous work by Bernardini et al. (2016) and Piacentini, Bernardini, and Beck (2019), we report here how to derive the recursive form of $F(S)$.

Given a sequence of search patterns $S = (S_1, S_2, \dots, S_n)$, we call $\tilde{\mathcal{F}}_{S_k}$ the event of finding the target in a search area covered by the pattern $S_k$ (i.e. $\omega_k = 1$) and $\tilde{\mathcal{F}}_{S_k}$ its negation (i.e. $\omega_k = 0$). Let $\mathcal{F}_{S|_1^k}$ represent the event of finding the target within $k$ steps of the execution of $S$, and $\tilde{\mathcal{F}}_{S|_1^k}$ its negation. We have:

$$\mathcal{F}_{S|_1^k} = \mathcal{F}_{S_1} \cup \dots \cup \mathcal{F}_{S_k} \qquad \tilde{\mathcal{F}}_{S|_1^k} = \tilde{\mathcal{F}}_{S_1} \cap \dots \cap \tilde{\mathcal{F}}_{S_k}$$

$F(S|_1^k)$ is defined as the probability of finding the target by executing the sequence of patterns $S|_1^k$:

$$F(S|_1^k) \coloneqq P_S(\mathcal{F}_{S|_1^k}) = 1 - P_S(\tilde{\mathcal{F}}_{S|_1^k})$$

We define $P^*_{S|_1^k}(\gamma)$ as the probability that the target is following the path $\gamma$ conditioned to the failure of the patterns in $S|_1^k$:

$$P^*_{S|_1^k}(\gamma) \coloneqq P_S(\gamma | \tilde{\mathcal{F}}_{S|_1^k})$$

Given an a-priori probability distribution for each path $PD(\gamma)$, we have that $P^*_{\varnothing}(\gamma) = PD(\gamma)$. Using a total probability argument we can write:

$$\begin{aligned} P^*_{S|_1^{k-1}}(\gamma) = P^*_{S|_1^k}(\gamma) P_S(\tilde{\mathcal{F}}_{S_k} | \tilde{\mathcal{F}}_{S|_1^{k-1}}) + \\ P_S(\gamma | \tilde{\mathcal{F}}_{S|_1^{k-1}} \cap \mathcal{F}_{S_k}) P_S(\mathcal{F}_{S_k} | \tilde{\mathcal{F}}_{S|_1^{k-1}}) \end{aligned} \tag{1}$$

The different terms in this equation can be rewritten as follows. We call $P_{S|_1^k *}$ the probability that the target is found during the execution of a search pattern $S_k$, conditioned to the event that it has not been discovered earlier:

$$P_{S|_1^{k-1} *} \coloneqq P_S(\mathcal{F}_{S_k} | \tilde{\mathcal{F}}_{S|_1^{k-1}})$$

This probability is the product of two terms: 1. the probability that the target is following a path compatible with $S_k$ (i.e. $\gamma \in \Gamma_{S_k}$) computed according to the distribution $P^*_{S|_1^{k-1}}(\gamma)$, which encodes the fact that the target has not been discovered earlier; and 2. the probability that the observer finds the target when it is in view, i.e. the detection probability $\phi_{S_k}$. Thus, we have:

$$P_{S_1^{k-1} *} = \phi_{S_k} \sum_{\gamma \in \Gamma_{S_k}} P^*_{S|_1^{k-1}}(\gamma) \tag{2}$$

Consider now the term $P_S(\gamma | \tilde{\mathcal{F}}_{S|_1^{k-1}} \cap \mathcal{F}_{S_k})$. If $\gamma \notin \Gamma_{S_k}$, this term is equal to 0, otherwise it can be computed by simply conditioning the probability distribution $P^*_{S|_1^{k-1}}(\gamma)$ on the subset of the paths $\Gamma_{S_k}$ which are compatible with $S_k$. In formula:

$$P_S(\gamma | \tilde{\mathcal{F}}_{S|_1^{k-1}} \cap \mathcal{F}_{S_k}) = \frac{P^*_{S|_1^{k-1}}(\gamma)}{\sum_{\eta \in \Gamma_{S_k}} P^*_{S|_1^{k-1}}(\eta)} \mathbb{1}_{\Gamma_{S_k}} \tag{3}$$

where $\mathbb{1}$ is the indicator function: $\mathbb{1}_A(x) = 1$ if $x \in A$ and 0 otherwise.

Substituting Equations (2) and (3) into (1), we obtain the following recursive structure for the computation of $P^*_{S|_1^k}(\gamma)$:

$$P^*_{S|_1^k}(\gamma) = \frac{P^*_{S|_1^{k-1}}(\gamma) \cdot (1 - \phi_{S_k} \cdot \mathbb{1}_{\Gamma_{S_k}}(\gamma))}{1 - P_{S|_1^{k-1} *}} \tag{4}$$

A recursive structure for the computation of $F(S|_1^k)$ can now be obtained as follows:

$$F(S|_1^k) = P_S(\mathcal{F}_{S|_1^{k-1}}) + P_S(\mathcal{F}_{S_k} | \tilde{\mathcal{F}}_{S|_1^{k-1}}) \cdot P_S(\tilde{\mathcal{F}}_{S|_1^{k-1}}) \tag{5}$$

or, in a more compact notation,

$$\begin{aligned} &F(S|_1^k) = F(S_1^{k-1}) + P_{S|_1^{k-1} *} \cdot (1 - F(S|_1^{k-1})) \\ &F(\varnothing) = 0 \end{aligned} \tag{6}$$

The exact recursive formula for the computation of $F(S)$ is obtained combining Equations (2), (3) and (6).

### 2 Dataset Generation

Problems instances are generated mimicking search-and-tracking missions. In every problem, a target can follow up to 10 possible directions, stemming from the last position known to the searcher (LKP). We identify up to 4 different destination points along each direction. The searcher can

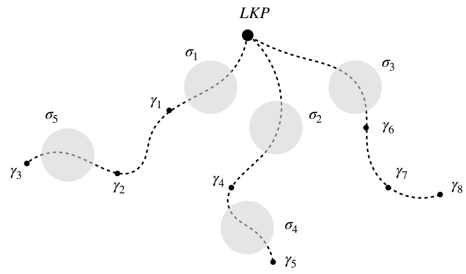


Figure 1: Example of problem instance generated.

perform search patterns localized in 20 different areas at different time stamps.

Time stamps are generated sequentially by taking a random sequence of all the search patterns $\bar{S} = (\sigma_1, \ldots, \sigma_n)$ and imposing that a search pattern $t(\sigma_i) = t(\sigma_{i-1}) + r$, where $r$ is a random number, uniformly generated between 0 and a maximum value. Each search pattern is associated with one direction and the destination points coming after it along the same route. The detection probability is a linear function of the indexes of the sequence of search patterns in $\bar{S}$ with different angular coefficients: $\phi_{\sigma_i} = m \cdot i + q$, where $m$ is a value between $-1$ and $1$, and $q$ is such that $\sum_i \phi_{\sigma_i}$ is constant across the scenarios. A scenario with $m = -1$ corresponds to patterns with a lower time stamp having higher detection probabilities, while, a scenario with $m = 1$, represent patterns with higher time stamp having higher detection probabilities. When $m = 0$, all the patterns have the same detection probability.

Figure 1 shows an example of an instance with three different directions, eight destinations and five patterns. Pattern $\sigma_1$ is associated with $\gamma_1$, $\gamma_2$ and $\gamma_3$, while the subsequent pattern $\sigma_5$ is associated with $\gamma_5$.

Code and problem instances are included in the Supplementary Material.

## 3 Additional Results

Figure 2 shows the comparison between the standard and the generalized greedy algorithms when limiting the number of possible occurrences of the same search pattern to different values. We report the median, the 1%, the 99% percentile of the ratio of the objective values and the average solution qualities. The numeric values are reported in Table 1.

## References

Bernardini, S.; Fox, M.; Long, D.; and Piacentini, C. 2016. Leveraging probabilistic reasoning in deterministic planning for large-scale autonomous search-and-tracking. In *Proceedings of the 26th International Conference on Automated Planning and Scheduling (ICAPS)*.

Piacentini, C.; Bernardini, S.; and Beck, C. 2019. Autonomous Target Search with Multiple Coordinated UAVs. *Journal of Artificial Intelligence Research* 65:519–568.

Table 1: Comparison of standard $G_s$ and generalized $G_g$ greedy algorithms. We report median, 1%-percentile, 99%-percentile of $\frac{G_s}{G_g}$, average and standard deviation of the solution qualities for $G_s$ and $G_g$, respectively.

| id | med $\frac{G_s}{G_g}$ | 1%-p$\frac{G_s}{G_g}$ | 99%-p$\frac{G_s}{G_g}$ | mean $G_s$ | std $G_s$ | mean $G_g$ | std $G_g$ |
|---|---|---|---|---|---|---|---|
| max repetition = 1 | | | | | | | |
| -1.0 | 0.861 | 0.004 | 1.000 | 104 | 77 | 160 | 30 |
| -0.8 | 0.877 | 0.004 | 1.000 | 105 | 72 | 155 | 30 |
| -0.6 | 0.846 | 0.004 | 1.000 | 98 | 71 | 151 | 31 |
| -0.4 | 0.801 | 0.004 | 1.000 | 88 | 69 | 144 | 29 |
| -0.2 | 0.809 | 0.004 | 1.000 | 85 | 66 | 138 | 27 |
| 0 | 0.785 | 0.005 | 1.000 | 76 | 63 | 130 | 27 |
| 0.2 | 0.722 | 0.005 | 1.000 | 63 | 59 | 120 | 27 |
| 0.4 | 0.011 | 0.005 | 0.991 | 46 | 50 | 112 | 25 |
| 0.6 | 0.009 | 0.005 | 0.901 | 27 | 39 | 102 | 24 |
| 0.8 | 0.009 | 0.005 | 0.828 | 14 | 28 | 90 | 24 |
| 1.0 | 0.009 | 0.005 | 0.812 | 7 | 19 | 78 | 23 |
| total | 0.617 | 0.004 | 1.000 | 65 | 68 | 125 | 38 |
| max repetition = 2 | | | | | | | |
| -1.0 | 0.938 | 0.513 | 1.000 | 148 | 37 | 165 | 29 |
| -0.8 | 0.903 | 0.476 | 1.000 | 143 | 37 | 164 | 30 |
| -0.6 | 0.873 | 0.527 | 1.000 | 139 | 39 | 163 | 31 |
| -0.4 | 0.848 | 0.443 | 1.000 | 133 | 38 | 158 | 30 |
| -0.2 | 0.835 | 0.495 | 1.000 | 129 | 37 | 155 | 29 |
| 0 | 0.832 | 0.449 | 1.000 | 122 | 37 | 148 | 29 |
| 0.2 | 0.808 | 0.008 | 1.000 | 110 | 37 | 138 | 29 |
| 0.4 | 0.752 | 0.007 | 1.000 | 95 | 33 | 129 | 26 |
| 0.6 | 0.676 | 0.006 | 0.949 | 76 | 29 | 116 | 25 |
| 0.8 | 0.618 | 0.006 | 0.903 | 55 | 31 | 101 | 24 |
| 1.0 | 0.587 | 0.006 | 0.874 | 42 | 31 | 87 | 24 |
| total | 0.794 | 0.007 | 1.000 | 108 | 50 | 138 | 38 |
| max repetition = 3 | | | | | | | |
| -1.0 | 0.94 | 0.584 | 1.000 | 150 | 35 | 166 | 29 |
| -0.8 | 0.902 | 0.57 | 1.000 | 145 | 35 | 166 | 30 |
| -0.6 | 0.871 | 0.564 | 1.000 | 141 | 37 | 165 | 32 |
| -0.4 | 0.84 | 0.536 | 1.000 | 135 | 37 | 161 | 31 |
| -0.2 | 0.821 | 0.555 | 1.000 | 131 | 36 | 159 | 30 |
| 0 | 0.816 | 0.491 | 1.000 | 125 | 37 | 153 | 30 |
| 0.2 | 0.799 | 0.456 | 1.000 | 115 | 37 | 143 | 30 |
| 0.4 | 0.747 | 0.452 | 1.000 | 102 | 32 | 135 | 27 |
| 0.6 | 0.68 | 0.387 | 0.968 | 83 | 27 | 121 | 26 |
| 0.8 | 0.622 | 0.009 | 0.936 | 65 | 25 | 105 | 24 |
| 1 | 0.604 | 0.008 | 0.917 | 54 | 25 | 90 | 24 |
| total | 0.793 | 0.337 | 1.000 | 113 | 46 | 142 | 38 |

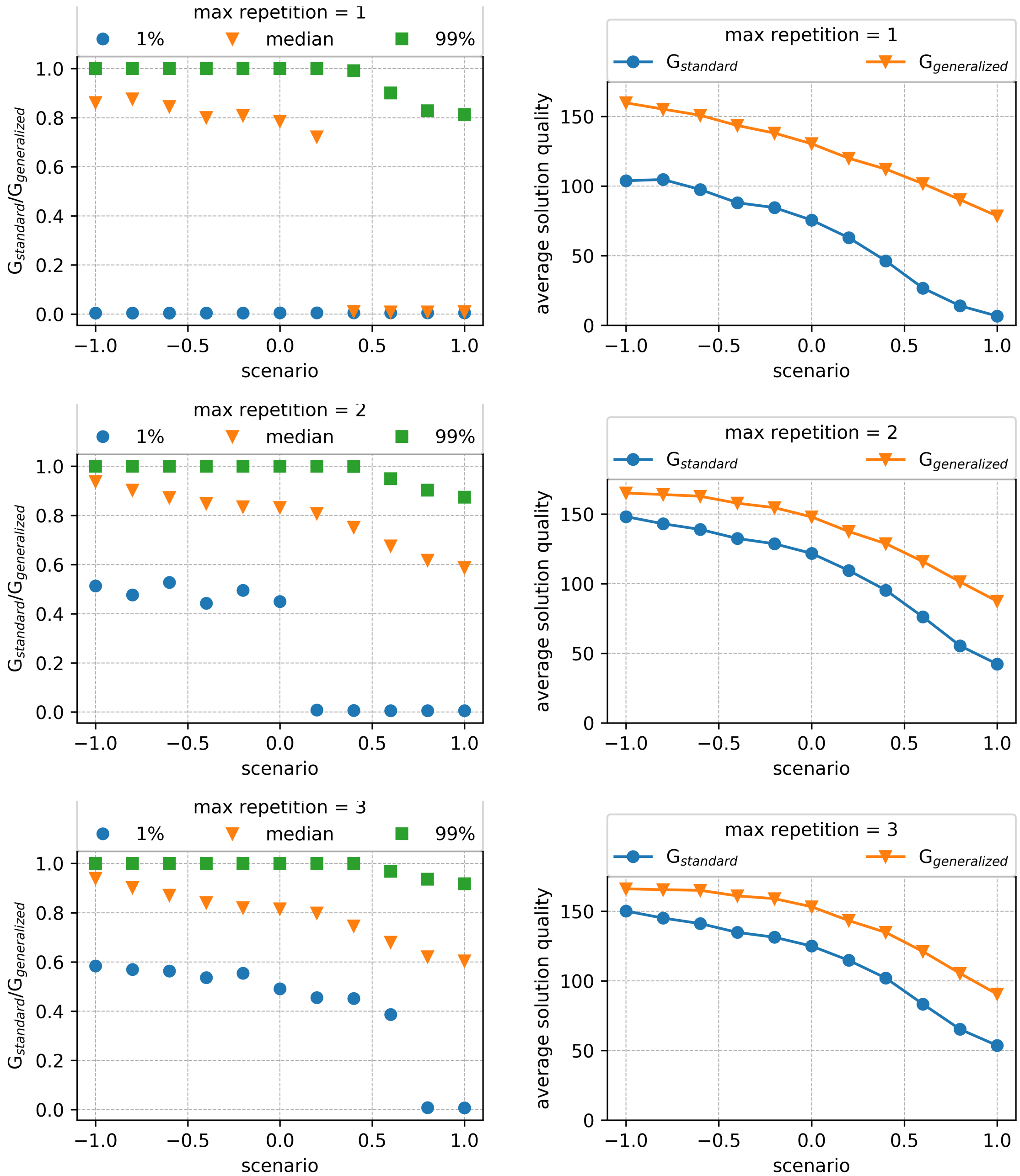


Figure 2: On the left: median, 1%-percentile, 99%-percentile of the ratio of the objective values obtained by the standard and the generalized greedy algorithms. On the right: average solution qualities.