Aggregate Claims Modelling - Individual Risk Model

- Idea:
  - Consider each policy individually; n policies in portfolio
  - Let Y_i = amount of claim(s) on i^{th} policy (during some period)
  - Let $N_i = I_{(Y_i>0)}$; $q_i = Pr(N_i = 1) < 1$
  - Let $X_i = (Y_i|N_i = 1)$; $F_i(x) = Pr(X_i \leq x) = Pr(Y_i \leq x|N_i = 1)$;
    $\mu_i = E(X_i) = E(Y_i|N_i = 1)$; $\sigma_i^2 = Var(X_i) = Var(Y_i|N_i = 1)$
  - Aggregate claim amount is:

$$S = \sum_{i=1}^{n} Y_i$$

Aggregate Claims Modelling - Individual Risk Model

- Issues:
  - Appears that each policy can make at most one claim.
  - Multiple claims per policy handled if distribution of X_i compound.
  - Each policy has own q_i and $F_i(x)$; too many to estimate
  - Will approximate S using collective risk model approach.

- Basic characterisation of S:
  - $N_i \sim Binomial(1, q_i)$
  - We can write $Y_i = \sum_{i=1}^{N_i} X_i \sim CompBin\{1, q_i, F_i(x)\}$

    So,

$$\begin{aligned} E(Y_i) = q_i \mu_i; \quad Var(Y_i) &= q_i(\sigma_i^2 + \mu_i^2) + \mu_i^2 q_i (1 - q_i) \\ &= q_i\{\sigma_i^2 + (1 - q_i)\mu_i^2\} \end{aligned}$$

  - Unfortunately, $S = \sum_{i=1}^{n} Y_i$ is sum of independent $CompBin$ distributed quantities with *different* q_i's and $F_i(x)$'s. So, S not $CompBin$ distributed.
  - However,

$$E(S) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} q_i \mu_i$$

$$Var(S) = Var\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} Var(Y_i) = \sum_{i=1}^{n} \left[q_i\{\sigma_i^2 + (1 - q_i)\mu_i^2\}\right]$$

- Poisson Collective Risk Approximation to Individual Risk Model:
  - $N_i \sim Binomial(1, q_i) \dot{\sim} Pois(q_i)$
  - Let $\tilde{N}_i \sim Pois(q_i)$ and

$$\tilde{Y}_i = \sum_{j=1}^{\tilde{N}_i} X_{ij}$$

    where X_{i1}, X_{i2}, etc. are *iid* random variables with *CDF* $F_i(x)$, and are independent of $\tilde{N}_i$.
  - So, $\tilde{Y}_i \sim CompPois\{q_i, F_i(x)\}$ and:

$$\begin{aligned}
Pr(\tilde{Y}_i \leq y) &= \sum_{n=0}^{\infty} Pr(\tilde{Y}_i \leq y | \tilde{N}_i = n) Pr(\tilde{N}_i = n) \\
&\approx Pr(\tilde{Y}_i \leq y | \tilde{N}_i = 0) Pr(\tilde{N}_i = 0) \\
&\qquad + Pr(\tilde{Y}_i \leq y | \tilde{N}_i = 1) Pr(\tilde{N}_i = 1) \\
&= Pr(\tilde{N}_i = 0) + Pr(X_{i1} \leq y | \tilde{N}_i = 1) Pr(\tilde{N}_i = 1) \\
&= Pr(\tilde{N}_i = 0) + Pr(X_{i1} \leq y) Pr(\tilde{N}_i = 1) \\
&\approx Pr(N_i = 0) + Pr(X_i \leq y) Pr(N_i = 1) \\
&= Pr(Y_i \leq y | N_i = 0) Pr(N_i = 0) \\
&\qquad + Pr(X_i \leq y | N_i = 1) Pr(N_i = 1) \\
&= Pr(Y_i \leq y | N_i = 0) Pr(N_i = 0) \\
&\qquad + Pr(Y_i \leq y | N_i = 1) Pr(N_i = 1) \\
&= Pr(Y_i \leq y),
\end{aligned}$$

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - Define $\tilde{S} = \sum_{i=1}^{n} \tilde{Y}_i \sim CompPois\{Q, F(x)\}$ where

$$Q = \sum_{i=1}^{n} q_i \qquad \text{and} \qquad F(x) = Q^{-1} \sum_{i=1}^{n} q_i F_i(x).$$

    Since $Pr(\tilde{Y}_i \leq y) \approx Pr(Y_i \leq y)$:

$$Pr(\tilde{S} \leq s) = Pr\left(\sum_{i=1}^{n} \tilde{Y}_i \leq s\right) \approx Pr\left(\sum_{i=1}^{n} Y_i \leq s\right) = Pr(S \leq s)$$

    So, Individual Risk Model distribution of S, with parameters $\{q_i\}_{i=1,\ldots,n}$ and $\{F_i(x)\}_{i=1,\ldots,n}$, can be approximated by Collective Risk Model distribution of $\tilde{S}$, which is Compound Poisson with rate Q and "portfolio-wide" (mixture) claim distribution $F(x)$

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - Assessment of approximation accuracy:

$$E(\tilde{S}) = E\left(\sum_{i=1}^{n} \tilde{Y}_i\right) = \sum_{i=1}^{n} E(\tilde{Y}_i) = \sum_{i=1}^{n} q_i \mu_i = E(S)$$

and

$$\begin{aligned} Var(\tilde{S}) = Var\left(\sum_{i=1}^{n} \tilde{Y}_i\right) &= \sum_{i=1}^{n} Var(\tilde{Y}_i) = \sum_{i=1}^{n} q_i(\sigma_i^2 + \mu_i^2) \\ &= \sum_{i=1}^{n} q_i\{\sigma_i^2 + (1-q_i)\mu_i^2 + q_i\mu_i^2\} \\ &= Var(S) + \sum_{i=1}^{n} q_i^2 \mu_i^2 \end{aligned}$$

Approximation will generally be good when $\sum_{i=1}^{n} q_i^2 \mu_i^2$ is small *relative* to the size of $Var(S)$.

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - *Example*:
    - Portfolio of 6000 term-life policies
    - Death benefit $\sim\ G(\alpha_i, \theta_i)$

Age	No. of Employees	Prob. of Death	α	θ
25-40	2500	0.0007	48000	0.25
41-50	2000	0.0025	65250	0.23
51+	1500	0.0085	81000	0.22

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - *Example (Continued)*:
    - Mean and Variance under Individual Risk Model:

$$\begin{aligned}
E(S) &= \sum_{i=1}^{n} q_i \mu_i = \sum_{i=1}^{n} q_i \alpha_i \theta_i \\
&= \sum_{i=1}^{2500} 0.0007(48000)(0.25) + \sum_{i=2501}^{4500} 0.0025(65250)(0.23) \\
&\quad + \sum_{i=4501}^{6000} 0.0085(81000)(0.22) \\
&= 2500(8.4) + 2000(37.52) + 1500(151.47) \\
&= 323242.5
\end{aligned}$$

and:

$$\begin{aligned}
Var(S) &= \sum_{i=1}^{n} \left[q_i \{ \sigma_i^2 + (1 - q_i)\mu_i^2 \} \right] \\
&= \sum_{i=1}^{n} \left[q_i \{ \alpha_i \theta_i^2 + (1 - q_i)\alpha_i^2 \theta_i^2 \} \right] \\
&= \sum_{i=1}^{2500} 0.0007(48000)(0.25^2)\{1 + (0.9993)(48000)\} \\
&\quad + \sum_{i=2501}^{4500} 0.0025(65250)(0.23^2)\{1 + (0.9975)(65250)\} \\
&\quad + \sum_{i=4501}^{6000} 0.0085(81000)(0.22^2)\{1 + (0.9915)(81000)\} \\
&= 2500(100731.54) + 2000(561663.61) + 1500(2676285.56) \\
&= 5389584420.42
\end{aligned}$$

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - · *Example (Continued)*:
    - · Approximate $Pr(S > 500000)$
      - · Normal approximation:

$$
\begin{aligned}
Pr(S > 500000) &\approx 1 - \Phi\left(\frac{500000 - 323242.5}{73413.79}\right) \\
&= 1 - \Phi(2.41) \\
&= 0.00798
\end{aligned}
$$

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - *Example (Continued)*:
    - Approximate $Pr(S > 500000)$ *(Continued)*
      - Poisson Collective Risk approximation:

        $\tilde{S} \sim CompPois\{Q, F(x)\}$, with:

$$\begin{aligned} Q &= \sum_{i=1}^{n} q_i \\ &= 2500(0.0007) + 2000(0.0025) + 1500(0.0085) \\ &= 19.5 \end{aligned}$$

and $F(x)$ a mixture of Gamma distributions:

$$\begin{aligned} F(x) &= Q^{-1} \sum_{i=1}^{n} q_i F_i(x) \\ &= (19.5)^{-1}\{2500(0.0007)G(x; 48000, 0.25) \\ &\quad + 2000(0.0025)G(x; 65250, 0.23) \\ &\quad\quad + 1500(0.0085)G(x; 81000, 0.22)\} \end{aligned}$$

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - *Example (Continued)*:
    - Approximate $Pr(S > 500000)$ *(Continued)*
      - Poisson Collective Risk approximation *(Continued)*:

        Let $Z \sim F(x)$, so that: $E(\tilde{S}) = QE(Z)$; $Var(\tilde{S}) = QE(Z^2)$; $Skew(\tilde{S}) = QE(Z^3)$;

        Now,

$$
\begin{aligned}
F = {} & \frac{1.75}{19.5} G(48000, 0.25) + \frac{5}{19.5} G(65250, 0.23) \\
& + \frac{12.75}{19.5} G(81000, 0.22)
\end{aligned}
$$

        So,

$$
\begin{aligned}
E(Z) &= \frac{1.75}{19.5}(48000)(0.25) + \frac{5}{19.5}(65250)(0.23) \\
&\quad + \frac{12.75}{19.5}(81000)(0.22) \\
&= 16576.54 \\
E(Z^2) &= \frac{1.75}{19.5}(48000)(48001)(0.25^2) \\
&\quad + \frac{5}{19.5}(65250)(65251)(0.23^2) \\
&\quad + \frac{12.75}{19.5}(81000)(81001)(0.22^2) \\
&= 278307224.36 \\
E(Z^3) &= \frac{1.75}{19.5}(48000)(48001)(48002)(0.25^3) \\
&\quad + \frac{5}{19.5}(65250)(65251)(65252)(0.23^3) \\
&\quad + \frac{12.75}{19.5}(81000)(81001)(81002)(0.22^3) \\
&= 4721920844246.51
\end{aligned}
$$

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - *Example (Continued)*:
    - Approximate $Pr(S > 500000)$ *(Continued)*
      - Poisson Collective Risk approximation *(Continued)*:

        So,

$$\begin{aligned} E(\tilde{S}) &= QE(Z) = 323242.5 \\ Var(\tilde{S}) &= QE(Z^2) = 5426990874.98 \\ \rho_{\tilde{S}} &= Skew(\tilde{S})/Var(\tilde{S})^{3/2} \\ &= \{QE(Z^3)\}/\{QE(Z^2)\}^{3/2} = 0.230311 \end{aligned}$$

For translated Gamma approximation, solve:

$$\begin{aligned} 0.230311 &= \frac{2}{\sqrt{\alpha_g}} \\ 5426990874.98 &= \alpha_g \theta_g^2 \\ 323242.5 &= \alpha_g \theta_g + k \end{aligned}$$

Yields $\alpha_g = 75.41$, $\theta_g = 8483.29$ and $k = -316484.76$

So,

$$\begin{aligned} &Pr(S > 500000) \approx Pr(\tilde{S} > 500000) \\ &\quad \approx Pr\{G(75.41, 8483.29) > 500000 + 316484.76\} \\ &\quad = 0.01235 \end{aligned}$$

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - *Example (Continued)*:
    - Approximate $Pr(S > 500000)$ *(Continued)*
      - Poisson Collective Risk approximation *(Continued)*:

        Alternatively,

$$Pr\{G(75.41, 8483.29) > 500000 + 316484.76\}$$
$$\approx Pr\{\chi^2_{(151)} > 2(500000 + 316484.76)/8483.29\}$$
$$= 0.01268$$

Or, since $\chi^2_{(df)} \approx N(df, 2df)$ when df large:

$$Pr\{\chi^2_{(151)} > 2(500000 + 316484.76)/8483.29\}$$
$$\approx \Phi\left(\frac{192.49 - 151}{\sqrt{302}}\right)$$
$$= \Phi(2.3875)$$
$$= 0.00848$$

- Poisson Collective Risk Approximation to Individual Risk Model *(Continued)*:
  - *Example (Continued)*:
    - Approximate $Pr(S > 500000)$ *(Continued)*
      - Poisson Collective Risk approximation *(Continued)*:

        Accuracy of Poisson Collective Risk approximation:

$$\begin{aligned}\sum_{i=1}^{n} q_i^2 \mu_i^2 = \sum_{i=1}^{n} q_i^2 \alpha_i^2 \theta_i^2 \\ &= 2500(0.0007^2)(48000^2)(0.25^2) \\ &\quad + 2000(0.0025^2)(65250^2)(0.23^2) \\ &\quad\quad + 1500(0.0085^2)(81000^2)(0.22^2) \\ &= 37406454.55,\end{aligned}$$

        This is 0.7% of $Var(S)$; approximation is good here.