Excess-of-Loss Reinsurance - Insurer's Perspective

1. Distribution of "Effective" Claim Size, Y:

   - Expected Claim Size

     $\cdot$ If $X \sim f_X(x;\theta)$,

$$
\begin{aligned}
E_\theta(Y) &= E_\theta\{XI_{(X\le M)}\} + E\{MI_{(X>M)}\} \\
&= \int_0^\infty xI_{(x\le M)} f_X(x;\theta)dx + MPr_\theta(X > M) \\
&= \int_0^M xf_X(x;\theta)dx + M\int_M^\infty f_X(x;\theta)dx \\
&= \int_0^\infty xf_X(x;\theta)dx - \int_M^\infty xf_X(x;\theta)dx + M\int_M^\infty f_X(x;\theta)dx \\
&= E_\theta(X) - \int_M^\infty (x-M)f_X(x;\theta)dx \\
&= E_\theta(X) - \int_0^\infty yf_X(y+M;\theta)dy,
\end{aligned}
$$

1. Distribution of "Effective" Claim Size, Y *(Continued)*:

   - Moment Generating Function

     · If X has *mgf* $m_X(t)$,

$$\begin{aligned} m_Y(t) &= \int_0^M e^{tx} f_X(x;\theta)dx + e^{tM} Pr_\theta(X > M) \\ &= m_X(t) - e^{tM} \int_0^\infty (e^{ty} - 1) f_X(y + M;\theta)dy \end{aligned}$$

2. Modelling Claim Sizes:
   - Key Issues
     - · Choose family for X-distribution, $f_X(x;\theta)$.
     - · Are X_i values recorded? Usually not.
     - · The Y_i values are *Right-Censored*.
     - · Y_i's not continuous; $Pr(Y_i = M) = Pr(X_i > M) > 0$.
     - · Modelling Y distribution directly very difficult (but possible)

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2. Modelling Claim Sizes *(Continued)*:

   · Estimate X distribution parameters using Y data?

      · $\overline{x}$, $\hat{x}_p$ generally not calculable from y's

      · MOM, MOP not feasible

      · MLE concepts still feasible

2. Modelling Claim Sizes *(Continued)*:
   - MLEs of X distribution parameters using Y data

     · Likelihood concept:

$$L(\theta) = \prod_i Pr_\theta(i^{\text{th}} \text{ data point observed})$$

"Product of chances of observing each data point"

· For n uncensored (i.e., $Y_i = X_i < M$);

and m censored (i.e., $Y_i = M$) observations:

· Contribution to likelihood of n uncensored observations

$$L(\theta; x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i; \theta)$$

where $X_i \sim f_X(x; \theta)$ is underlying *pdf* for claim model

· Contribution to likelihood of m censored observations

$$\{Pr_\theta(Y_i = M)\}^m = \{Pr_\theta(X > M)\}^m = \{1 - F_X(M; \theta)\}^m$$

· Likelihood of θ based on Y_i's:

$$\begin{aligned} L_1(\theta; y_1, \ldots, y_{n+m}) &= \prod_{i=1}^{n} f_X(x_i; \theta) \prod_{j=1}^{m} \{1 - F_X(M; \theta)\} \\ &= L(\theta; x_1, \ldots, x_n)\{1 - F_X(M; \theta)\}^m \end{aligned}$$

2. Modelling Claim Sizes *(Continued)*:

   · Log-likelihood:

$$\begin{aligned} l_1(\theta) &= \ln\{L_1(\theta; y_1, \ldots, y_{n+m})\} \\ &= \ln\{L(\theta; x_1, \ldots, x_n)\} + m \ln\{1 - F_X(M; \theta)\} \end{aligned}$$

   · MLE Theorem still holds! Can create confidence intervals as:

$$\hat{\theta}_1 \pm 1.96\sqrt{I_1^{-1}(\hat{\theta}_1)}$$

   where $\hat{\theta}_1$ solves $\frac{\partial}{\partial\theta} l_1(\theta) = 0$ and

$$I_1(\theta) = -E_\theta\left\{\frac{\partial^2}{\partial\theta\partial\theta^T} l_1(\theta)\right\}$$

   [NOTE: n and m are *random*, though $n + m$ is considered fixed]

Excess-of-Loss Reinsurance - Reinsurer's Perspective

1. All claim information known
   - No change to original claim modelling
   - Uncommon

2. Information regarding claims below retention level unknown
   - Total number of claims unknown
   - Leads to *truncated* datasets; only observe Z if $Z > 0$.
   - Estimate distribution parameters of $f_X(x;\theta)$ based observed Z_i's:

     · Use conditional distribution, which has *CDF*:

$$\begin{aligned} Pr_\theta(Z \leq z | Z > 0) &= Pr_\theta(X \leq z + M | X > M) \\ &= \frac{Pr_\theta(M < X \leq z + M)}{Pr_\theta(X > M)} \\ &= \frac{F_X(z + M;\theta) - F_X(M;\theta)}{1 - F_X(M;\theta)} \end{aligned}$$

     · And *pdf*:

$$f_{Z|Z>0}(z;\theta) = \frac{f_X(z + M;\theta)}{1 - F_X(M;\theta)}$$