Aggregate Claims Modelling - Collective Risk Model

- Specific choices for the distribution of N
  1. Compound Poisson Distributions, $N \sim Poisson(\lambda)$:

$$p_N(n;\lambda) = Pr_\lambda(N = n) = \frac{\lambda^n e^{-\lambda}}{n!}, \qquad n = 0, 1, 2, 3, \ldots$$

· So, $\nu = E(N) = \lambda$, $\tau^2 = Var(N) = \lambda$,

and *mgf* of N is $m_N(t) = \exp\{\lambda(e^t - 1)\}$

· Thus

$$\begin{aligned} E(S) &= \lambda\mu_1 \\ Var(S) &= \lambda\mu_2 + \mu_1^2(\lambda - \lambda) = \lambda\mu_2 \\ Skew(S) &= E\big[\{S - E(S)\}^3\big] = \lambda\mu_3 \\ m_S(t) &= \exp\left(\lambda\big[e^{\ln\{m_X(t)\}} - 1\big]\right) = \exp\big[\lambda\{m_X(t) - 1\}\big] \end{aligned}$$

· Notation $S \sim CompPois\{\lambda, F_X(x)\}$ or $S \sim CompPois\{\lambda, m_X(t)\}$

[NOTE: Second "parameter" of distribution is a *FUNCTION*]

- Specific choices for the distribution of N *(Continued)*
  1. Compound Poisson Distributions, $N \sim Poisson(\lambda)$ *(Continued)*:
     - Addition Property of Compound Poissons:

       If $S_1, \ldots, S_K$ are independent, $S_i \sim CompPois\{\lambda_i, F_i(x)\}$,

       then

$$S = \sum_{i=1}^{K} S_i \sim CompPois\{\Lambda, F(x)\}$$

where $\Lambda = \sum_{i=1}^{K} \lambda_i$, $F(x) = \Lambda^{-1} \sum_{i=1}^{K} \lambda_i F_i(x)$

Proof: See course notes

Idea: Combining portfolios:

- Rate of claims adds.
- New "portfolio-wide" distribution = weighted average

Example:

$S_1 \sim CompPois(\lambda, 1 - e^{-x/\theta_1})$;

$S_2 \sim CompPois(\lambda, 1 - e^{-x/\theta_2})$

If S_1, S_2 independent, then $S_1 + S_2 \sim CompPois\{2\lambda, F(x)\}$

where

$$\begin{aligned} F(x) &= \frac{1}{2\lambda}\left\{\lambda(1 - e^{-x/\theta_1}) + \lambda(1 - e^{-x/\theta_2})\right\} \\ &= 1 - \tfrac{1}{2}e^{-x/\theta_1} - \tfrac{1}{2}e^{-x/\theta_2} \end{aligned}$$

- Specific choices for the distribution of N *(Continued)*

  2. Compound Binomial Distributions, $N \sim Binomial(m, q)$:

$$p_N(n; m, q) = \frac{m!}{n!(m-n)!} q^n (1-q)^{m-n}, \qquad n = 0, 1, 2, \ldots, m$$

· So, $\nu = E(N) = mq$, $\tau^2 = Var(N) = mq(1-q)$,
and *mgf* of N is $m_N(t) = (qe^t + 1 - q)^m$

· Thus

$$E(S) = mq\mu_1$$

$$Var(S) = mq\mu_2 + \mu_1^2\{mq(1-q) - mq\} = mq(\mu_2 - q\mu_1^2)$$

$$Skew(S) = E\big[\{S - E(S)\}^3\big] = mq\mu_3 - 3mq^2\mu_2\mu_1 + 2mq^3\mu_1^3$$

$$m_S(t) = \{qm_X(t) + 1 - q\}^m$$

· Notation $S \sim CompBin\{m, q, F_X(x)\}$ or
$S \sim CompBin\{m, q, m_X(t)\}$

- Specific choices for the distribution of N *(Continued)*
  2. Compound Binomial Distributions, $N \sim Binomial(m, q)$ *(Continued)*:

     · Addition Property of Compound Binomials:

     If $S_1, \ldots, S_K$ are independent, $S_i \sim CompBin\{m_i, q, F(x)\}$, then

     $$S = \sum_{i=1}^{K} S_i \sim CompBin\{M, q, F(x)\}$$

     where $M = \sum_{i=1}^{K} m_i$.

     Proof: See course notes

     Note: If q's and/or $F(x)$'s different for S_i's, then
     S does not have Compound Binomial distribution

- Specific choices for the distribution of N *(Continued)*
  2. Compound Negative Binomial Distributions, $N \sim NegBin(k, q)$:

$$p_N(n; k, q) = \frac{(k+n-1)!}{n!(k-1)!} q^k (1-q)^n, \qquad n = 0, 1, 2, 3, \ldots$$

OR, if k not an integer:

$$p_N(n; k, q) = \frac{\Gamma(k+n)}{n!\Gamma(k)} q^k (1-q)^n, \qquad n = 0, 1, 2, 3, \ldots$$

· So, $\nu = E(N) = k(1-q)/q$, $\tau^2 = Var(N) = k(1-q)/q^2$,

and *mgf* of N is $m_N(t) = q^k\{1-(1-q)e^t)^{-k}$

· Thus

$$\begin{aligned} E(S) &= k\mu_1(1-q)/q \\ Var(S) &= k\mu_2(1-q)/q + \mu_1^2[k(1-q)/q^2 - \{k(1-q)/q\}] \\ &= k(1-q)\{q\mu_2 + (1-q)\mu_1^2\}/q^2 \\ Skew(S) &= k(1-q)\{q^2\mu_3 + 3q(1-q)\mu_2\mu_1 + 2(1-2q+q^2)\mu_1^3\}/q^3 \\ m_S(t) &= q^k\{1-(1-q)m_X(t)\}^{-k} \end{aligned}$$

· Notation $S \sim CompNegBin\{k, q, F_X(x)\}$ or

$S \sim CompNegBin\{k, q, m_X(t)\}$

- Specific choices for the distribution of N *(Continued)*
  2. Compound Negative Binomial Distributions, $N \sim NegBin(k, q)$ *(Continued)*:

     · Addition Property of Compound Negative Binomials:

     If $S_1, \ldots, S_K$ are independent, $S_i \sim CompNegBin\{k_i, q, F(x)\}$, then

     $$S = \sum_{i=1}^{K} S_i \sim CompNegBin\{k, q, F(x)\}$$

     where $k = \sum_{i=1}^{K} k_i$.

     Proof: See course notes

     Note: If q's and/or $F(x)$'s different for S_i's, then
     S does not have Compound Negative Binomial distribution