Aggregate Claims Modelling - Collective Risk Model

- Compound Distributions and Reinsurance:
  - Definitions

    $S_Y = \sum_{i=1}^{N} Y_i$

    $S_Z = \sum_{i=1}^{N} Z_i$
  - Suppose $S = \sum_{i=1}^{N} X_i \sim CompDist\{\theta, F(x)\}$

    where $\theta = \lambda$ if $CompDist = CompPois$;

    $\theta = (m, q)$ if $CompDist = CompBinomial$;

    $\theta = (k, q)$ if $CompDist = CompNeqBin$.

    [NOTE: Recall θ-part of parameters deals with N only,

    while reinsurance generally deals with X_i's]
  - Then, $S_Y \sim CompDist\{\theta, F_Y(y)\}$ and $S_Z \sim CompDist\{\theta, F_Z(z)\}$

    where $F_Y(y) = Pr\{Y_i \leq y\}$; $F_Z(z) = Pr\{Z_i \leq z\}$

- Compound Distributions and Reinsurance *(Continued)*:

  · For proportional reinsurance with retention p:

$$F_Y(y) = Pr(Y_i \le y) = Pr(pX_i \le y) = Pr(X_i \le y/p)$$
$$= F(y/p)$$
$$F_Z(z) = Pr(Z_i \le z) = Pr\{(1-p)X_i \le z\} = Pr\{X_i \le z/(1-p)\}$$
$$= F\{z/(1-p)\}$$

  · For excess-of-loss reinsurance with retention M:

$$F_Y(y) = Pr(Y_i \le y) = Pr\{X_i I_{(X_i \le M)} + M I_{(X_i > M)} \le y\}$$
$$= \begin{cases} F(y) & \text{if } y < M \\ 1 & \text{if } y \ge M \end{cases}$$
$$F_Z(z) = Pr(Z_i \le z) = Pr\{(X_i - M) I_{(X_i > M)} \le z\}$$
$$= \begin{cases} F(z+M) & \text{if } z \ge 0 \\ 0 & \text{if } z < 0 \end{cases}$$

[NOTES: 1. $Pr(Z=0) = F(M) > 0$, so $S_Z = \sum_{i=1}^{N} Z_i$ is random sum where terms may be zero. More on this later.

2. Distributions of Y_i, Z_i not continuous.]

- Compound Distributions and Reinsurance *(Continued)*:
  - · For excess-of-loss reinsurance with retention M *(Continued)*:
    - · *Example*: Suppose $S \sim CompPois\{\lambda, Pareto(\alpha, \delta)\}$, $\alpha > 1$.

$$\begin{aligned} E(S_Y) = \lambda E(Y_i) &= \lambda \left\{ E(X_i) - \frac{\delta^\alpha}{(\alpha-1)(\delta+M)^{\alpha-1}} \right\} \\ &= \lambda \left\{ \frac{\delta}{\alpha-1} - \frac{\delta^\alpha}{(\alpha-1)(\delta+M)^{\alpha-1}} \right\} \\ &= \frac{\lambda\{\delta(\delta+M)^{\alpha-1} - \delta^\alpha\}}{(\alpha-1)(\delta+M)^{\alpha-1}} \end{aligned}$$

Also, we clearly have $E(S_Z) = E(S) - E(S_Y)$, so that

$$E(S_Z) = \frac{\lambda\delta}{\alpha-1} - E(S_Y) = \frac{\lambda\delta^\alpha}{(\alpha-1)(\delta+M)^{\alpha-1}}$$

Alternatively, we can calculate

$$\begin{aligned} E(Z_i) &= E(Z_i|Z_i > 0)Pr(Z_i > 0) \\ &\qquad + E(Z_i|Z_i = 0)Pr(Z_i = 0) \\ &= \left(\frac{\delta+M}{\alpha-1}\right) Pr(X_i > M) + 0 \\ &= \left(\frac{\delta+M}{\alpha-1}\right) \{1 - F(M)\} \\ &= \left(\frac{\delta+M}{\alpha-1}\right) \left\{ \frac{\delta^\alpha}{(\delta+M)^\alpha} \right\} \\ &= \frac{\delta^\alpha}{(\alpha-1)(\delta+M)^{\alpha-1}} \end{aligned}$$

[Recall: Z_i not continuous; $Z_i|Z_i > 0 \sim Pareto(\alpha, \delta+M)$]

So, $E(S_Z) = \lambda E(Z_i)$, same as before.

- Compound Distributions and Reinsurance *(Continued)*:
  - · For excess-of-loss reinsurance with retention M *(Continued)*:
    - · New perspective on S_Z, focus on Z_j's > 0 only:

      Assume for simplicity, Z_i's ordered so zeroes are last.

      Define $N_Z = \sum_{i=1}^{N} I_{(X_i > M)}$ = number of non-zero Z_i's.

      Then,

$$S_Z = \sum_{i=1}^{N} Z_i = \sum_{j=1}^{N_Z} Z_j$$

- Compound Distributions and Reinsurance *(Continued)*:
  - For excess-of-loss reinsurance with retention M *(Continued)*:
    - New perspective on S_Z, focus on Z_j's > 0 only *(Continued)*:

      Distribution of N_Z?

      Note: $m_I(t) = E\left\{e^{tI(X_i > M)}\right\} = F(M) + e^t\{1 - F(M)\}$.

      So, $m_{N_Z}(t) = m_N[\ln\{m_I(t)\}]$

      If $N \sim Pois(\lambda)$, so $m_N(t) = \exp\{\lambda(e^t - 1)\}$, then:

$$
\begin{aligned}
m_{N_Z}(t) &= \exp[\lambda\{m_I(t) - 1\}] \\
&= \exp\big(\lambda[F(M) + e^t\{1 - F(M)\} - 1]\big) \\
&= \exp\big[\lambda\{1 - F(M)\}(e^t - 1)\big] \\
&= \exp\big[\lambda_1(e^t - 1)\big].
\end{aligned}
$$

Thus, $N_Z \sim Pois[\lambda\{1 - F(M)\}]$

Similarly,

- $N \sim Binomial(m, q)$

  $\implies \quad N_Z \sim Binomial[m, q\{1 - F(M)\}]$

- $N \sim NegBin(k, q)$

  $\implies \quad N_Z \sim NegBin[k, q/\{1 - (1 - q)F(M)\}]$

So, $S_Z = \sum_{i=1}^{N} Z_i \sim CompDist\{\theta, F_Z(z)\}$ means

$$
S_Z = \sum_{j=1}^{N_Z} Z_j \sim CompDist\{\theta_1, F_{Z|Z>0}(z)\}
$$

For appropriate new θ_1.

This is like a “reparameterisation”

- Compound Distributions and Reinsurance *(Continued)*:
  - For excess-of-loss reinsurance with retention M *(Continued)*:
    - New perspective on S_Z, focus on Z_j's > 0 only *(Continued)*:
      - Example *(Continued)*:

        Recall $S \sim CompPois\{\lambda, Pareto(\alpha, \delta)\}$, $\alpha > 1$.

        We saw, $E(S_Z) = E(N)E(Z_i) = \frac{\lambda\delta^\alpha}{(\alpha-1)(\delta+M)^{\alpha-1}}$

        Now we see, $E(S_Z) = E(N_Z)E(Z_i|Z_i > 0)$ and

        $$E(N_Z) = \lambda\{1 - F(M)\} = \lambda\left\{\frac{\delta^\alpha}{(\delta+M)^\alpha}\right\}$$

        And

        $$\begin{aligned} & X_i \sim Pareto(\alpha, \delta) \\ \implies \quad & Z_i|Z_i > 0 \sim Pareto(\alpha, \delta + M) \end{aligned}$$

        So,

        $$\begin{aligned} E(S_Z) &= E(N_Z)E(Z_i|Z_i > 0) \\ &= \lambda\left\{\frac{\delta^\alpha}{(\delta+M)^\alpha}\right\}\frac{\delta+M}{\alpha-1} \\ &= \frac{\lambda\delta^\alpha}{(\alpha-1)(\delta+M)^{\alpha-1}} \end{aligned}$$

        the same as before.

        Moreover, we now see S_Z has a

        $$CompPois\left\{\frac{\lambda\delta^\alpha}{(\delta+M)^\alpha}, Pareto(\alpha, \delta+M)\right\}$$

        distribution