Excess-of-Loss Reinsurance - Specific Examples

1. Exponential Claims, $X_i \sim Exp(\theta)$:
   - Expected Claim Size

$$\begin{aligned} E_\theta(Y) &= E_\theta(X) - \int_0^\infty y f_X(y+M;\theta)dy \\ &= \theta - \int_0^\infty \frac{y}{\theta} e^{-(y+M)/\theta} dy \\ &= \theta - e^{-M/\theta} \int_0^\infty \frac{y}{\theta} e^{-y/\theta} dy \\ &= \theta(1 - e^{-M/\theta}) \end{aligned}$$

   - Moment Generating Function

$$m_Y(t) = (1-\theta t)^{-1}\left\{1 - t\theta e^{M(t-\theta^{-1})}\right\}, \qquad t < \theta^{-1}$$

1. Exponential Claims, $X_i \sim Exp(\theta)$ *(Continued)*:

    - Likelihood Estimation:

$$
\begin{aligned}
L_1(\theta) &= \prod_{i=1}^{n} \theta^{-1} \exp(-y_i\theta^{-1}) \prod_{j=1}^{m} \left[1 - \{1 - \exp(-M\theta^{-1})\}\right] \\
&= \theta^{-n} \exp\left(-\theta^{-1} \sum_{i=1}^{n} y_i\right) \exp(-mM\theta^{-1}) \\
\Longrightarrow l_1(\theta) &= \ln\{L_1(\theta)\} \\
&= -n \ln\theta - \theta^{-1} \sum_{i=1}^{n} y_i - mM\theta^{-1} \\
&= -n \ln\theta - \theta^{-1} \sum_{i=1}^{n+m} y_i,
\end{aligned}
$$

since $y_{n+1} = \ldots = y_{n+m} = M$ (if y_i's sorted).

So, MLE solves:

$$
\begin{aligned}
\frac{dl_1(\theta)}{d\theta} &= -n\theta^{-1} + \theta^{-2} \sum_{i=1}^{n+m} y_i = 0 \\
\Longrightarrow \quad \hat{\theta} &= \frac{1}{n} \sum_{i=1}^{n+m} y_i = \frac{n+m}{n}\overline{y} = \left(1 + \frac{m}{n}\right)\overline{y}
\end{aligned}
$$

1. Exponential Claims, $X_i \sim Exp(\theta)$ *(Continued)*:

   - Likelihood Estimation *(Continued)*:

     Variance of MLE is:

$$\begin{aligned}\left[-E_\theta\left\{\frac{d^2 l_1(\theta)}{d\theta^2}\right\}\right]^{-1} &= \left[-E_\theta\left\{n\theta^{-2} - 2\theta^{-3}\sum_{i=1}^{n+m} Y_i\right\}\right]^{-1} \\ &= \left[\theta^{-2}E_\theta(n) - 2\theta^{-3}(n+m)E_\theta(Y_i)\right]^{-1} \\ &= \left[-(n+m)\theta^{-2}(1-e^{-M\theta^{-1}})\right. \\ &\qquad \left. + 2(n+m)\theta^{-3}\{\theta(1-e^{-M\theta^{-1}})\}\right]^{-1} \\ &= \theta^2\{(n+m)(1-e^{-M\theta^{-1}})\}^{-1},\end{aligned}$$

     since $n \sim \text{Binomial}[n+m, Pr_\theta(X_i \leq M) = 1 - e^{-M/\theta}]$

   - Numerical Example:

     · Use our data, assume $M = 15000$.

     · So, $m = 4$ and the MLE of θ is:

$$\frac{1}{92}\left\{\sum_{i=1}^{92} y_i + 4(15000)\right\} = 2491.38$$

     · With approximate standard error:

$$\frac{2491.38}{\sqrt{96(1-e^{-15000/2491.38})}} = 254.58$$

     · Compare to $\hat{\theta}_{MLE} = 2898.83$, $\hat{\sigma}(\hat{\theta}_{MLE}) = 305.15$ for full data.

1. Exponential Claims, $X_i \sim Exp(\theta)$ *(Continued)*:

   - The Reinsurer's Perspective:

     · Distribution of amounts for which reinsurer is liable, $Z|Z > 0$:

$$\begin{aligned} f_{Z|Z>0}(z;\theta) &= f_X(z+M;\theta)\{1 - F_X(M;\theta)\}^{-1} \\ &= \theta^{-1}\exp\{-(z+M)\theta^{-1}\}\left[1 - \{1 - \exp(-M\theta^{-1})\}\right]^{-1} \\ &= \theta^{-1}\exp\{-(z+M)\theta^{-1}\}\exp(M\theta^{-1}) \\ &= \theta^{-1}\exp(-z\theta^{-1}) \end{aligned}$$

     · Distribution remains $Exp(\theta)$, "memoryless" property.

2. Pareto Claims, $X_i \sim Pareto(\alpha, \delta)$:

   - Expected Claim Size

$$
\begin{aligned}
E_{\alpha,\delta}(Y) &= E_{\alpha,\delta}(X) - \int_0^\infty y f_X(y+M;\alpha,\delta)dy \\
&= \frac{\delta}{\alpha-1} - \int_0^\infty y \frac{\alpha\delta^\alpha}{\{\delta+(y+M)\}^{\alpha+1}}dy \\
&= \frac{\delta}{\alpha-1} - \frac{\delta^\alpha}{(\delta+M)^\alpha}\int_0^\infty y \frac{\alpha(\delta+M)^\alpha}{\{(\delta+M)+y\}^{\alpha+1}}dy \\
&= \frac{\delta}{\alpha-1} - \left\{\frac{\delta^\alpha}{(\delta+M)^\alpha}\right\}\left(\frac{\delta+M}{\alpha-1}\right) \\
&= \frac{\delta\{(\delta+M)^{\alpha-1} - \delta^{\alpha-1}\}}{(\alpha-1)(\delta+M)^{\alpha-1}}
\end{aligned}
$$

   - Numerical Example

     · With $M = 15000$, we see that

$$E_{\alpha,\delta}(X) - E_{\alpha,\delta}(Y) = \frac{\delta^\alpha}{(\alpha-1)(\delta+15000)^{\alpha-1}}$$

     · For our data, $\hat{\alpha}_{MLE} = 1.909$ and $\hat{\delta}_{MLE} = 2704.47$

     · Predict a drop in expected claim size using these MLEs as:

$$\frac{\hat{\delta}_{MLE}^{\hat{\alpha}_{MLE}}}{(\hat{\alpha}_{MLE}-1)(\hat{\delta}_{MLE}+15000)^{\hat{\alpha}_{MLE}-1}} = 539.23.$$

     · Actual drop is $2989.83 - 2387.57 = 602.26$

2. Pareto Claims, $X_i \sim Pareto(\alpha, \delta)$ *(Continued)*:

- Likelihood Estimation:

$$l_1(\alpha,\delta) = n\ln\alpha + (n+m)\alpha\ln\delta - (\alpha+1)\sum_{i=1}^{n}\ln(\delta+y_i) - m\alpha\ln(\delta+M)$$

· Score equations:

$$\frac{\partial l_1(\alpha,\delta)}{\partial\alpha} = \frac{n}{\alpha} + (n+m)\ln\delta - \sum_{i=1}^{n+m}\ln(\delta+y_i)$$

$$\frac{\partial l_1(\alpha,\delta)}{\partial\delta} = \frac{(n+m)\alpha}{\delta} - (\alpha+1)\sum_{i=1}^{n+m}\frac{1}{\delta+y_i} + \frac{m}{\delta+M}$$

· Require iterative (computer-based) solution

· For our data, with $M = 15000$: $\hat{\alpha}_{MLE} = 1.853$, $\hat{\delta}_{MLE} = 2602.47$

[Compare to no censoring case: $\hat{\alpha}_{MLE} = 1.909$, $\hat{\delta}_{MLE} = 2704.47$]

2. Pareto Claims, $X_i \sim Pareto(\alpha, \delta)$ *(Continued)*:

- Likelihood Estimation *(Continued)*:

  · Fisher Information:

$$-E_{\alpha,\delta}\left\{\frac{\partial^2 l_1(\alpha,\delta)}{\partial\alpha^2}\right\} = E_{\alpha,\delta}\left(\frac{n}{\alpha^2}\right) = \frac{n+m}{\alpha^2}\left\{1 - \frac{\delta^\alpha}{(\delta+M)^\alpha}\right\}$$

$$-E_{\alpha,\delta}\left\{\frac{\partial^2 l_1(\alpha,\delta)}{\partial\alpha\partial\delta}\right\} = -E_{\alpha,\delta}\left(\frac{n+m}{\delta} - \sum_{i=1}^{n+m}\frac{1}{\delta+Y_i}\right)$$

$$= -\frac{n+m}{(\alpha+1)\delta}\left\{1 - \frac{\delta^{\alpha+1}}{(\delta+M)^{\alpha+1}}\right\}$$

$$-E_{\alpha,\delta}\left\{\frac{\partial^2 l_1(\alpha,\delta)}{\partial\delta^2}\right\} = E_{\alpha,\delta}\left\{\frac{(n+m)\alpha}{\delta^2} - \sum_{i=1}^{n+m}\frac{(\alpha+1)}{(\delta+Y_i)^2} + \frac{m}{(\delta+M)^2}\right\}$$

$$= \frac{(n+m)\alpha}{(\alpha+2)\delta^2}\left\{1 - \frac{\delta^{\alpha+2}}{(\delta+M)^{\alpha+2}}\right\}$$

Since $m \sim Binomial[n+m, \delta^\alpha(\delta+M)^{-\alpha}]$ and:

$$E_{\alpha,\delta}\{(\delta+Y_i)^{-k}\} = \int_0^M (\delta+y)^{-k}\alpha\delta^\alpha(\delta+y)^{-(\alpha+1)}dy + (\delta+M)^{-k}Pr_{\alpha,\delta}(Y_i = M)$$

$$= \alpha\delta^\alpha\int_0^M(\delta+y)^{-\{(\alpha+k)+1\}}dy + (\delta+M)^{-k}\left\{\frac{\delta^\alpha}{(\delta+M)^\alpha}\right\}$$

$$= \frac{\alpha}{(\alpha+k)\delta^k}\left\{1 + \frac{k\delta^{\alpha+k}}{\alpha(\delta+M)^{\alpha+k}}\right\}$$

  · Using MLEs and $M = 15000$ for our data:

$$\widehat{Var}_{\alpha,\delta}(\hat{\alpha}_{MLE}, \hat{\delta}_{MLE}) = \begin{pmatrix} 0.3546 & 670.16 \\ 670.16 & 1413396.83 \end{pmatrix}$$

  · So, 95% confidence intervals are:

$$\alpha: \quad 1.853 \pm 1.96\sqrt{0.3546} = (0.6859,\ 3.0201)$$

$$\delta: \quad 2602.48 \pm 1.96\sqrt{1413396.83} = (272.30,\ 4932.64)$$

2. Pareto Claims, $X_i \sim Pareto(\alpha, \delta)$ *(Continued)*:

   - The Reinsurer's Perspective:

     · Distribution of amounts for which reinsurer is liable, $Z|Z>0$:

$$
\begin{aligned}
f_{Z|Z>0}(z;\alpha,\delta) &= \frac{f_X(z+M;\alpha,\delta)}{1-F_X(M;\alpha,\delta)} \\
&= \frac{\alpha\delta^\alpha}{\{\delta+(z+M)\}^{\alpha+1}}\left\{\frac{\delta^\alpha}{(\delta+M)^\alpha}\right\}^{-1} \\
&= \frac{\alpha(\delta+M)^\alpha}{\{(\delta+M)+z\}^{\alpha+1}}
\end{aligned}
$$

     since

$$
Pr_{\alpha,\delta}(X_i > x) = 1 - F_X(x;\alpha,\delta) = \frac{\delta^\alpha}{(\delta+x)^\alpha}.
$$

     · Thus, $Z|Z>0$ is Pareto$(\alpha, \delta + M)$