Aggregate Claims Modelling

1. The idea:
   - Model total amount, S, made on entire portfolio for some fixed period
     - · Need model for claim sizes, X_i
     - · Need model for claim numbers, N
     - · Assumptions:
       - · Claim sizes and rate constant over time period
       - · Claim sizes and number independent
     - · Collective Risk Model:
       - · S is a random sum

$$S = \sum_{i=1}^{N} X_i$$

       - · $X_i \overset{iid}{\sim} f_X(x)$, "portfolio-wide" distribution
       - · $N \sim p_N(n)$
         - · Typically choose Poisson(λ), Binomial(m,q) or Negative Binomial(k,q)
       - · If $N = 0$, define $S = 0$

2. Some notation:

   - Claim sizes:

     · *pdf* $f_X(x)$, *CDF* $F_X(x)$

     · *mgf* $m_X(t)$, raw moments $\mu_k = E(X_i^k) = \frac{d^k}{dt^k} m_X(t)\Big|_{t=0}$

   - Claim number:

     · *pmf* $p_N(n)$, *mgf* $m_N(t)$

     · central moments $\nu = E(N)$, $\tau^2 = Var(N)$

   - Aggregate claims:

     · *CDF* $G(s) = Pr(S \leq s)$

     · *mgf* $m_S(t)$

- The *CDF* of S, $G(s)$
  - If $X_i \sim F_X(x)$ and $N \sim p_N(n)$

$$\begin{aligned} G(s) &= \sum_{n=0}^{\infty} Pr\{S \leq s | N = n\} p_N(n) \\ &= \sum_{n=0}^{\infty} Pr\Big\{ \sum_{i=1}^{N} X_i \leq s \Big| N = n \Big\} p_N(n) \\ &= \sum_{n=0}^{\infty} Pr\Big\{ \sum_{i=1}^{n} X_i \leq s \Big| N = n \Big\} p_N(n) \\ &= \sum_{n=0}^{\infty} Pr\Big\{ \sum_{i=1}^{n} X_i \leq s \Big\} p_N(n) \\ &= \sum_{n=0}^{\infty} F_X^{*(n)}(s) p_N(n), \end{aligned}$$

where $F_X^{*(n)}(x)$ is *n-fold convolution* of $F_X(x)$ defined recursively:

$$F_X^{*(0)}(x) = I_{(x \geq 0)}$$

and $$F_X^{*(n)}(x) = \int_0^{\infty} F_X^{*(n-1)}(x-u) f_X(u) du, \quad n = 1, 2, 3, \ldots$$

- The *CDF* of S, $G(s)$ *(Continued)*
  - An example:
    - If $X_i \sim Exp(\theta)$ and $N \sim Geometric(p)$, then:

$$p_N(n) = p(1-p)^{n-1}, \qquad n \geq 1$$

and:

$$\begin{aligned}
F_X^{*(1)}(x) &= \int_0^\infty I_{(x-u\geq 0)} \frac{1}{\theta} e^{-u/\theta} du \\
&= \int_0^x \frac{1}{\theta} e^{-u/\theta} du \\
&= 1 - e^{-x/\theta}, \qquad x \geq 0 \\
F_X^{*(2)}(x) &= \int_0^\infty F_X^{*(1)}(x-u) f_X(u) du \\
&= \int_0^x \left\{1 - e^{-(x-u)/\theta}\right\} \frac{1}{\theta} e^{-u/\theta} du \\
&= \int_0^x \frac{1}{\theta}\left(e^{-u/\theta} - e^{-x/\theta}\right) du \\
&= \left[- e^{-u/\theta} - \frac{1}{\theta} u e^{-x/\theta} \right]_{u=0}^x \\
&= 1 - \left(1 + \frac{1}{\theta} x\right) e^{-x/\theta} \\
&\vdots \\
F_X^{*(n)}(x) &= 1 - e^{-x/\theta} \sum_{r=1}^n \frac{x^{n-r}}{(n-r)!\theta^{n-r}}
\end{aligned}$$

[NOTE: Differentiation shows:

$$f_X^{*(n)}(x) = \frac{d}{dx} F_X^{*(n)}(x) = \frac{1}{(n-1)!\theta^n} x^{n-1} e^{-x/\theta}$$

as it should, so we can also write

$$F_X^{*(n)}(x) = \int_0^x \frac{1}{(n-1)!\theta^n} u^{n-1} e^{-u/\theta} du$$

as in the course notes.]

- The *CDF* of S, $G(s)$ *(Continued)*
  - An example *(Continued)*:
    - So,

$$
\begin{aligned}
G(s) &= \sum_{n=1}^{\infty} p(1-p)^{n-1}\left\{1 - e^{-s/\theta}\sum_{r=1}^{n}\frac{s^{n-r}}{(n-r)!\theta^{n-r}}\right\} \\
&= \sum_{n=1}^{\infty} p(1-p)^{n-1} - \sum_{n=1}^{\infty} p(1-p)^{n-1} e^{-s/\theta}\sum_{r=1}^{n}\frac{s^{n-r}}{(n-r)!\theta^{n-r}} \\
&= 1 - pe^{-s/\theta}\sum_{r=1}^{\infty}\sum_{n=r}^{\infty}(1-p)^{n-1}\frac{s^{n-r}}{(n-r)!\theta^{n-r}} \\
&= 1 - pe^{-s/\theta}\sum_{r=1}^{\infty}\sum_{m=0}^{\infty}(1-p)^{m+r-1}\frac{s^{m}}{m!\theta^{m}} \\
&= 1 - pe^{-s/\theta}\sum_{r=1}^{\infty}(1-p)^{r-1}\sum_{m=0}^{\infty}\frac{\{(1-p)s/\theta\}^{m}}{m!} \\
&= 1 - pe^{-s/\theta}\left\{\frac{1}{1-(1-p)}\right\}e^{(1-p)s/\theta} \\
&= 1 - \exp(-ps/\theta)
\end{aligned}
$$

    - Generally, won't be able to get exact formula

      (distributions for N and X_i were carefully chosen here)
    - We will learn to approximate $G(s)$

- The expectation and variance of S:
  - Expected value of S

$$\begin{aligned}E(S) &= E\{E(S|N)\} = E\left\{E\left(\sum_{i=1}^{N} X_i \middle| N\right)\right\}\\ &= E\left\{\sum_{i=1}^{N} E(X_i|N)\right\} = E\left\{\sum_{i=1}^{N} E(X_i)\right\}\\ &= E(N\mu_1) = \mu_1\nu\end{aligned}$$

So, expected total = (expected number)×(expected size of each)

  - Variance of S

$$\begin{aligned}Var(S) &= E\{Var(S|N)\} + Var\{E(S|N)\}\\ &= E\left\{Var\left(\sum_{i=1}^{N} X_i \middle| N\right)\right\} + Var(N\mu_1)\\ &= E\left\{\sum_{i=1}^{N} Var(X_i|N)\right\} + \mu_1^2\tau^2\\ &= E\left\{\sum_{i=1}^{N} Var(X_i)\right\} + \mu_1^2\tau^2\\ &= E\{N(\mu_2 - \mu_1^2)\} + \mu_1^2\tau^2\\ &= \nu(\mu_2 - \mu_1^2) + \mu_1^2\tau^2 = \nu\mu_2 + \mu_1^2(\tau^2 - \nu)\end{aligned}$$

So, variance of total > (expected number)×(variance of each)

This is referred to as the *overdispersion* property of random sums

- The *mgf* of S, $m_S(t)$:

$$\begin{aligned}
m_S(t) = E(e^{tS}) = E\{E(e^{tS}|N)\} &= E\left[E\left\{\exp\left(t\sum_{i=1}^{N} X_i\right)\middle|N\right\}\right] \\
&= E\left[E\left\{\prod_{i=1}^{N}\exp(tX_i)\middle|N\right\}\right] \\
&= E\left[\prod_{i=1}^{N} E\{\exp(tX_i)|N\}\right] \\
&= E\left[\prod_{i=1}^{N} E\{\exp(tX_i)\}\right] \\
&= E\left\{\prod_{i=1}^{N} m_X(t)\right\} \\
&= E\big[\{m_X(t)\}^N\big] \\
&= E\big(\exp[N\ln\{m_X(t)\}]\big) \\
&= m_N[\ln\{m_X(t)\}]
\end{aligned}$$