- Approximating Compound Distributions:
  - Issues:
    - $G(s) = \sum_{n=0}^{\infty} F^{*(n)}(s) p_N(n)$ is VERY complicated
    - Just use *mgf*s? Some distributions for X_i have no *mgf*! (e.g., Pareto or Log-Normal)
    - $S = \sum_{i=1}^{N} X_i$ is a (random) sum of independent random variables, maybe CLT applies?

- Approximating Compound Distributions *(Continued)*:
  - Normal approximation to $G(s)$:
    - Form of approximation:

$$G(s) = Pr(S \leq s) \approx \Phi\left\{ \frac{s - E(S)}{\sqrt{Var(S)}} \right\}$$

    - Key advantages:
      - Only need $E(S) = \nu\mu_1$ and $Var(S) = \nu\mu_2 + \mu_1^2(\tau^2 - \nu)$

        (Don't need $p_N(n)$ or $f_X(x;\theta)$'s, just first two moments).

- Approximating Compound Distributions *(Continued)*:
  - Normal approximation to $G(s)$ *(Continued)*:
    - *Example*:

Let $N \sim Pois(\lambda)$, $X_i \overset{iid}{\sim} G(\alpha, \theta)$

So, $E(S) = E(N)E(X_i) = \lambda\alpha\theta$ and

$$\begin{aligned} Var(S) &= E(N)E(X_i^2) + \{E(X_i)\}^2\{Var(N) - E(N)\} \\ &= \lambda\alpha(\alpha+1)\theta^2 \end{aligned}$$

"Exact" answer (for $s \geq 0$):

$$\begin{aligned} G(s) &= \sum_{n=0}^{\infty} p_N(n) F^{*(n)}(s) \\ &= e^{-\lambda} + \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \int_0^s \frac{1}{\theta^{n\alpha}\Gamma(n\alpha)} x^{n\alpha-1} e^{-x/\theta} dx \\ &= e^{-\lambda} + \int_0^s e^{-x/\theta} e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n x^{n\alpha-1}}{n!\theta^{n\alpha}\Gamma(n\alpha)} dx. \end{aligned}$$

- Approximating Compound Distributions *(Continued)*:
  - Normal approximation to $G(s)$ *(Continued)*:
    - *Example (Continued)*:

Normal Approximation *CDF* Comparisons:

$(\lambda, \alpha, \theta) = (10, 1, 1)$			
s	$(s - \mu_S)/\sigma_S$	True	Nrm. Approx.
5	-1.118	0.1198	0.1318
10	0.000	0.5449	0.5000
15	1.118	0.8658	0.8682
20	2.236	0.9742	0.9873

$(\lambda, \alpha, \theta) = (20, 5, 0.1)$			
s	$(s - \mu_S)/\sigma_S$	True	Nrm. Approx.
5	-2.041	0.0125	0.0206
10	0.000	0.5190	0.5000
15	2.041	0.9725	0.9794
20	4.082	0.9998	1.0000

- Approximating Compound Distributions *(Continued)*:
  - · Normal approximation to $G(s)$ *(Continued)*:
    - · *Example (Continued)*:

Normal Approximation Quantile Comparisons:

Approximate 95th and 99th percentiles, $s_{0.95}$ and $s_{0.99}$:

$$\Phi\left(\frac{s_{0.95} - \lambda\alpha\theta}{\theta\sqrt{\lambda\alpha(\alpha+1)}}\right) = 0.95$$
$$\Phi\left(\frac{s_{0.99} - \lambda\alpha\theta}{\theta\sqrt{\lambda\alpha(\alpha+1)}}\right) = 0.99$$

Using facts: $\Phi^{-1}(0.95) = 1.645$, $\Phi^{-1}(0.99) = 2.33$, we have

$$s_{0.95} \approx \lambda\alpha\theta + 1.645\theta\sqrt{\lambda\alpha(\alpha+1)}$$
$$s_{0.99} \approx \lambda\alpha\theta + 2.33\theta\sqrt{\lambda\alpha(\alpha+1)}$$

			Nrm. Approx.		True	
λ	α	θ	$s_{0.95}$	$s_{0.99}$	$s_{0.95}$	$s_{0.99}$
20	5	0.1	14.029	15.707	14.221	16.194
100/3	1	0.3	14.029	15.707	14.270	16.340
10	1	1	17.357	20.420	18.122	22.494
20	1/3	3/2	17.357	20.420	18.221	22.855

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- Approximating Compound Distributions *(Continued)*:
  - Normal approximation to $G(s)$ *(Continued)*:
    - Accuracy of Normal Approximation depends on skewness.
      - Standardised coefficient of skewness $\rho_S = \frac{Skew(S)}{\{Var(S)\}^{3/2}}$.
      - For $CompPois\{\lambda, G(\alpha, \theta)\}$,

$$Var(S) = \lambda\mu_2 = \lambda\alpha(\alpha+1)\theta^2$$
$$Skew(S) = \lambda\mu_3 = \lambda\alpha(\alpha+1)(\alpha+2)\theta^3$$

So,

$$\rho_S = \frac{Skew(S)}{\{Var(S)\}^{3/2}} = \frac{\alpha+2}{\sqrt{\lambda\alpha(\alpha+1)}}$$

For $(\lambda, \alpha, \theta) = (10, 1, 1)$, $\rho_S = \frac{3}{\sqrt{20}} = 0.671$

For $(\lambda, \alpha, \theta) = (20, 5, 0.1)$, $\rho_S = \frac{7}{\sqrt{600}} = 0.286$

Rule of Thumb: If $\rho_S < 0.5$, Normal Approximation good

- Approximating Compound Distributions *(Continued)*:
  - Translated Gamma approximation to $G(s)$:
    - If $Y \sim G(\alpha_g, \theta_g)$, then $X = Y + k \sim transG(k, \alpha_g, \theta_g)$.
    - If $X = Y + k \sim transG(k, \alpha_g, \theta_g)$, then

$$E(X) = k + \alpha_g \theta_g; \quad Var(X) = \alpha_g \theta_g^2; \quad \rho_X = \frac{2}{\sqrt{\alpha_g}}$$

    - Approximate $G(s)$ by $F_X(s; k, \alpha_g, \theta_g)$ such that

$$E(S) = k + \alpha_g \theta_g; \quad Var(S) = \alpha_g \theta_g^2; \quad \rho_S = \frac{2}{\sqrt{\alpha_g}}$$

    - *Example (Continued)*:

For $S \sim CompPois\{10, G(1,1)\}$, we saw

$$E(S) = 10, \quad Var(S) = 20 \quad \rho_S = 0.671$$

Translated Gamma approximation uses (k, α_g, θ_g) such that

$$10 = k + \alpha_g \theta_g, \quad 20 = \alpha_g \theta_g^2, \quad 0.671 = \frac{2}{\sqrt{\alpha_g}},$$

which yields $\alpha_g = 8.889$, $\theta_g = 1.5$ and $k = -3.333$

Thus, our approximation is:

$$\begin{aligned} G(s) &\approx Pr(X \leq s) = Pr(-3.333 + Y \leq s) \\ &= Gamma(s + 3.333; 8.889, 1.5) \\ &= Pr\{\chi^2_{(17.778)} \leq 2(s + 3.333)/1.5\} \end{aligned}$$

- Approximating Compound Distributions *(Continued)*:
  - · Translated Gamma approximation to $G(s)$ *(Continued)*:
    - · *Example (Continued)*:

	$(\lambda, \alpha, \theta) = (10, 1, 1)$		
s	True	Nrm. Approx.	Gamma Approx.
5	0.1198	0.1318	0.1183
10	0.5449	0.5000	0.5446
15	0.8658	0.8682	0.8668
20	0.9742	0.9873	0.9742
	$(\lambda, \alpha, \theta) = (20, 5, 0.1)$		
5	0.0125	0.0206	0.0128
10	0.5190	0.5000	0.5190
15	0.9725	0.9794	0.9724
20	0.9998	1.0000	0.9998

- Approximating Compound Distributions *(Continued)*:
  - Translated Gamma approximation to $G(s)$ *(Continued)*:
    - *Example (Continued)*:

      Approximate percentiles:

$$s_p = \gamma_{p;(\alpha_g,\theta_g)} + k = \frac{1}{2}\theta_g \chi^2_{2\alpha_g}(p) + k$$

where $Pr\{Gamma(\alpha_g, \theta_g) \leq \gamma_{p;(\alpha_g,\theta_g)}\} = p$

			Gamma. Approx.		True	
λ	α	θ	$s_{0.95}$	$s_{0.99}$	$s_{0.95}$	$s_{0.99}$
20	5	0.1	14.218	16.207	14.221	16.194
100/3	1	0.3	14.268	16.350	14.270	16.340
10	1	1	18.105	22.539	18.122	22.494
20	1/3	3/2	18.209	22.877	18.221	22.855

    - Try Java Applet.