The Weibull Distributions

1. Characterisation
   - If $Y \sim Exp(\theta)$, then $X = Y^{1/\gamma} \sim W(\theta, \gamma)$.
   - Probability Density Function (*pdf*):

$$\begin{aligned} f_X(x; \theta, \gamma) &= f_Y(x^\gamma; \theta) \left| \frac{dx^\gamma}{dx} \right| \\ &= \frac{\gamma x^{\gamma-1}}{\theta} \exp\left(- \frac{x^\gamma}{\theta} \right) \end{aligned}$$

   - Cumulative Distribution Function (*CDF*):

$$\begin{aligned} F_X(x; \theta, \gamma) &= \int_0^x \frac{\gamma u^{\gamma-1}}{\theta} \exp\left(- \frac{u^\gamma}{\theta} \right) du \\ &= \left[- e^{-u^\gamma/\theta} \right]_{u=0}^{x} \\ &= 1 - e^{-x^\gamma/\theta} \end{aligned}$$

The Weibull Distributions

1. Characterisation *(Continued)*

   - Moments:

$$E_{\theta,\gamma}(X) = \theta^{1/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right) \qquad E_{\theta,\gamma}(X^2) = \theta^{2/\gamma}\Gamma\left(\frac{2}{\gamma}+1\right)$$

RECALL: $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(k) = (k-1)!$ and $\Gamma(1/2) = \sqrt{\pi}$.

· If $\gamma = 1$, then

$$E_{\theta,1}(X) = \theta\Gamma(2) = \theta; \qquad E_{\theta,1}(X^2) = \theta^2\Gamma(3) = 2\theta^2$$

· If $\gamma = 0.5$, then

$$E_{\theta,0.5}(X) = \theta^2\Gamma(3) = 2\theta^2 \qquad E_{\theta,0.5}(X^2) = \theta^4\Gamma(5) = 24\theta^4$$

· If $\gamma = 2$, then

$$E_{\theta,2}(X) = \sqrt{\theta}\Gamma(3/2) = \frac{1}{2}\sqrt{\pi\theta} \qquad E_{\theta,2}(X^2) = \theta\Gamma(2) = \theta$$

1. Characterisation *(Continued)*

   - Quantiles (Percentiles):

$$
\begin{aligned}
x_p \text{ solves} : Pr\{X \leq x_p\} &= p \\
\Longrightarrow \quad & Pr\{X^\gamma \leq x_p^\gamma\} = p \\
\Longrightarrow \quad & Pr\{Y \leq x_p^\gamma\} = p \\
\Longrightarrow \quad & 1 - e^{-x_p^\gamma/\theta} = p \\
\Longrightarrow \quad & x_p = \left\{ \theta \ln \left(\frac{1}{1-p} \right) \right\}^{1/\gamma},
\end{aligned}
$$

2. Estimation of Parameters based on $x_1, \ldots, x_n$:

- (Standard) Method of Moments (*MOM*):

  Solve system:

$$\theta^{1/\gamma}\Gamma\left(\frac{1}{\gamma}+1\right) = \overline{x}, \qquad \theta^{2/\gamma}\Gamma\left(\frac{2}{\gamma}+1\right) = \overline{x^2},$$

  · Requires iterative (computer-based) solution.

  · The MOM estimates for our data are found to be:

$$\hat{\theta}_{MOM} = 36.24 \qquad \hat{\gamma}_{MOM} = 0.4930.$$

2. Estimation of Parameters based on $x_1, \ldots, x_n$ *(Continued)*:

- Method of Percentiles (*MOP*):

  Solve:

$$\left\{\theta \ln\left(\frac{1}{1-p_1}\right)\right\}^{1/\gamma} = \hat{x}_{p_1}; \qquad \left\{\theta \ln\left(\frac{1}{1-p_2}\right)\right\}^{1/\gamma} = \hat{x}_{p_2},$$

  for some choice of p_1 and p_2

  So the *MOP* estimates are:

$$\hat{\gamma}_{MOP} = \frac{\ln\{-\ln(1-p_2)\} - \ln\{-\ln(1-p_1)\}}{\ln(\hat{x}_{p_2}) - \ln(\hat{x}_{p_1})}$$
$$\hat{\theta}_{MOP} = \exp\left[\hat{\gamma}_{MOP} \ln(\hat{x}_{p_1}) - \ln\{-\ln(1-p_1)\}\right]$$

2. Estimation of Parameters based on $x_1, \ldots, x_n$ *(Continued)*:

- Method of Percentiles (*MOP*) *(Continued)*:

  For our data, *MOP* estimates based on upper and lower quartiles ($p_1 = 0.25$, $p_2 = 0.75$) are:

$$\hat{\gamma}_{MOP} = \frac{\ln\{-\ln(0.25)\} - \ln\{-\ln(0.75)\}}{\ln(2836.75) - \ln(401)} = 0.8038$$

$$\hat{\theta}_{MOP} = \exp\left[0.8038 \ln(401) - \ln\{-\ln(0.75)\}\right] = 429.94$$

  since upper and lower quartiles of data are:

$$\hat{x}_{0.25} = x_{[24]} + 0.25(x_{[25]} - x_{[24]}) = 401$$

$$\hat{x}_{0.75} = x_{[72]} + 0.75(x_{[73]} - x_{[72]}) = 2836.75$$

2. Estimation of Parameters based on $x_1, \ldots, x_n$ *(Continued)*:

   - Maximum Likelihood Estimate (*MLE*)

     Log-Likelihood Function:

$$l(\theta, \gamma) = n \ln \gamma - n \ln \theta + (\gamma - 1) \sum_{i=1}^{n} \ln x_i - \theta^{-1} \sum_{i=1}^{n} x_i^{\gamma}$$

     Score equations:

$$\frac{\partial l(\theta, \gamma)}{\partial \theta} = -n\theta^{-1} + \theta^{-2} \sum_{i=1}^{n} x_i^{\gamma} = 0$$
$$\frac{\partial l(\theta, \gamma)}{\partial \gamma} = n\gamma^{-1} + \sum_{i=1}^{n} \ln x_i - \theta^{-1} \sum_{i=1}^{n} x_i^{\gamma} \ln x_i = 0$$

     · *MLE*s require iterative (computer-based) solution method.

     · For our data:

$$\hat{\gamma}_{MLE} = 0.7131; \qquad \hat{\theta}_{MLE} = 245.44$$

3. Goodness-of-Fit Testing:

- Pearson Chi-Squared Test:

  · Use equal-count bins from previous exponential calculations.

Table 2.5: Observed and Expected Claim Amounts (in £'s) using the Weibull Distribution

Bin Range	O_i	$E_{i,MLE}$	$E_{i,MOM}$	$E_{i,MOP}$	Bin Range	O_i	$E_{i,MLE}$	$E_{i,MOM}$	$E_{i,MOP}$
0-260	12	18.6	33.4	17.7	2072-2618	5	5.9	3.9	6.5
260-545	18	10.7	10.7	11.9	2618-3285	6	5.6	3.7	6.0
545-860	10	8.7	7.5	10.0	3285-4145	6	5.4	3.6	5.5
860-1212	8	7.6	5.9	8.8	4145-5357	3	5.5	3.6	5.1
1212-1612	7	6.8	5.0	7.9	5357-7429	4	5.8	4.1	4.8
1612-2072	10	6.3	4.3	7.1	7429+	7	9.2	10.3	4.8

$X^2_{MLE} = 12.07$, $df = 12 - 1 - 2 = 9$, p-value $= 0.209$

$X^2_{MOM} = 32.98$, $df = 12 - 1 - 2 = 9$, p-value $= 0.000135$

$X^2_{MOP} = 8.69$, $df = 12 - 1 - 2 = 9$, p-value $= 0.466$

3. Goodness-of-Fit Testing *(Continued)*:

- Pearson Chi-Squared Test *(Continued)*:

  · Calculating $E_{i,MLE}$'s:

$$
\begin{aligned}
E_{(a,b)} &= nPr_{\theta,\gamma}(a < X \leq b) = n\{F_X(b;\theta,\gamma) - F_X(a;\theta,\gamma)\} \\
&= n\big\{e^{-a^{\gamma}/\theta} - e^{-b^{\gamma}/\theta}\big\}
\end{aligned}
$$

  · Using MLEs yields:

$$E_{(0,260.15),MLE} = 96\big\{1 - e^{-260.15^{0.7131}/245.44}\big\} = 18.57$$

3. Goodness-of-Fit Testing *(Continued)*:

- Pearson Chi-Squared Test *(Continued)*:

  · Equal-count bin construction, 12 bins each with $E_i = 8$:

  · For first bin, $(0, b)$, solve:

$$\begin{aligned} 8 &= 96 Pr_{\theta,\gamma}(0 < X \leq b) \\ &= 96\left\{1 - e^{-b^{\gamma}/\theta}\right\} \\ \Longrightarrow \quad b &= \left\{\theta \ln(12/11)\right\}^{1/\gamma} \end{aligned}$$

  Using $\hat{\gamma}_{MLE}$ and $\hat{\theta}_{MLE}$ yields $b = 73.16$

  · For next bin, $(73.16, b)$, solve:

$$96 = 8\left\{e^{-73.16^{0.7131}/245.44} - e^{-b^{0.7131}/245.44}\right\}.$$

  Yields $b = 206.51$. Continue.

  · $X^2_{MLE} = 16.5$, $df = 12 - 1 - 2 = 9$, p-value $= 0.057$

  · $X^2_{MOM} = 36$, $df = 12 - 1 - 2 = 9$, p-value $= 0.0000396$

  · $X^2_{MOP} = 12$, $df = 12 - 1 - 2 = 9$, p-value $= 0.213$

3. Goodness-of-Fit Testing *(Continued)*:

   - Pearson Chi-Squared Test *(Continued)*:

     ASIDE: Test $H_0 : \gamma = 1$ vs. $H_A : \gamma \neq 1$.

     · Tests whether Exponential is valid.

     · Could use *MLE*-based confidence interval.

     · Compare X^2's for Weibull model vs. Exponential model.

     · Difference in X^2's should have χ^2 distribution with

$$df = \text{ diff. in \# of parameters for } H_0 \text{ and } H_A$$

     · For our example, df=1 and

$$X^2_{MLE,H_0} - X^2_{MLE,H_A} = 23 - 12.07 = 10.97$$

     · p-value $= 0.000926$

     · Must compare X^2 calculated using same bin structure.