

About an approximate formula for scattering amplitude by a disc

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Abstract

A fundamental solution method gives an analytic representation of the approximate solution for the wave problem in the exterior region of a disc. The asymptotic behavior of this representation yields an approximate formula for the scattering amplitude.

1 An approximate solution for reduced wave problem by FSM

Consider the following reduced wave problem in the exterior domain of a disc with Dirichlet boundary condition. Let a be the radius of a disc, and k a wave number. Then the problem is represented as follows.

$$\left\{ \begin{array}{ll} -\Delta u - k^2 u = 0 & \text{in } \Omega_e, \\ u = f & \text{on } \Gamma_a, \\ \lim_{r \rightarrow \infty} \sqrt{r} \left\{ \frac{\partial u}{\partial r} - iku \right\} = 0, \end{array} \right.$$

where

$$\Omega_e = \{\mathbf{r} \in \mathbb{R}^2; |\mathbf{r}| > a\}, \quad \Gamma_a = \{\mathbf{a} \in \mathbb{R}^2; |\mathbf{a}| = a\},$$

and $|\cdot|$ is the Euclidean norm in \mathbb{R}^2 .

A positive number ρ is the radius of a disc containing all source points. Let N be a fixed positive integer. Then we define a basis function $G_j(\mathbf{r})$ through

$$G_j(\mathbf{r}) = H_0^{(1)}(k|r e^{i\theta} - \rho e^{i\theta_j}|), \quad \theta_j = j \frac{2\pi}{N}, \quad 0 \leq j \leq N-1,$$

where $H_0^{(1)}(\cdot)$ is the zeroth order Hankel function of the first kind, and the points (r, θ) and (ρ, θ_j) correspond to the complex numbers $r e^{i\theta}$ and $\rho e^{i\theta_j}$, respectively.

An approximate solution of the problem above is given as follows[6].

$$u^{(N)}(\mathbf{r}) = \sum_{j=0}^{N-1} Q_j G_j(\mathbf{r}),$$

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where Q_j is the intensity of sources, and \mathbf{r} corresponds to the polar coordinate (r, θ) .

The intensity of sources Q_j is computed as follows. Introduce the following normalized parameters:

$$\gamma = \frac{\rho}{a}, \quad \delta = \frac{r}{a}, \quad \kappa = ka.$$

Then the basis function is represented as follows.

$$G_j(\mathbf{r}) = H_0^{(1)}(\kappa|\delta - \gamma e^{-i(\theta - \theta_j)}|), \quad 0 \leq j \leq N-1.$$

Introduce the kernel function:

$$g(\theta) = H_0^{(1)}(\kappa|1 - \gamma e^{-i\theta}|).$$

The intensity of sources Q_j is given as follows.

$$Q_j = \frac{1}{N} \sum_{k=0}^{N-1} \frac{F_k^{(N)}}{G_k^{(N)}} e^{ij\theta_k} \quad \text{for } 0 \leq j \leq N-1,$$

where

$$F_k^{(N)} = \frac{1}{N} \sum_{j=0}^{N-1} f(\mathbf{a}_j) e^{-ik\theta_j}, \quad G_n^{(N)} = \frac{1}{N} \sum_{j=0}^{N-1} g(\theta_j) e^{-in\theta_j},$$

where \mathbf{a}_j corresponds to the polar coordinate (a, θ_j) .

2 An approximate formula for scattering amplitude

An approximate scattering amplitude $A^{(N)}(\theta)$ for the above problem is given as follows[7],[8].

$$A^{(N)}(\theta) = \lim_{r \rightarrow \infty} \left(\frac{e^{ir}}{\sqrt{r}} \right)^{-1} u^{(N)}(\mathbf{r}) = \sum_{j=0}^{N-1} \sqrt{\frac{2}{\pi k}} e^{-i\frac{\pi}{4}} Q_j e^{-i\kappa\gamma \cos(\theta - \theta_j)}$$

with $\theta_j = \frac{2\pi j}{N}$,

where an asymptotic formula of Hankel functions[1] is used.

Then an approximate far-field coefficient[2] $P^{(N)}(\theta)$ is given as follows.

$$P^{(N)}(\theta) = \sqrt{\frac{\pi k}{2}} e^{i\frac{\pi}{4}} A^{(N)}(\theta).$$

The scattering cross section $\sigma(\theta)$ is computed as follows[2].

$$\sigma(\theta) = \lim_{r \rightarrow \infty} 2\pi r \left| \frac{u(\mathbf{r})}{u_i(\mathbf{r})} \right|^2,$$

where $u_i(\mathbf{r})$ is an incident wave.

Suppose:

$$\lim_{r \rightarrow \infty} |u_i(\mathbf{r})| = 1.$$

The scattering wave $u(\mathbf{r})$ is expected to behave in the far-field as follows.

$$u(\mathbf{r}) \sim \frac{e^{ikr}}{\sqrt{r}} A(\theta) \quad \text{as } r \rightarrow \infty,$$

where $A(\theta)$ is the scattering amplitude. Then $\sigma(\theta)$ is represented as follows.

$$\sigma(\theta) = 2\pi|A(\theta)|^2.$$

Define an approximate scattering cross section $\sigma^{(N)}(\theta)$:

$$\sigma^{(N)}(\theta) = 2\pi|A^{(N)}(\theta)|^2.$$

3 Octave programs

GNU Octave is an array oriented software for numerical computing[4]. If you are Windows or Mac OS X user, you can obtain an executable program from Octave-forge[5]. You can download the below Octave programs from my web site[3].

Let an incident wave $f = e^{ikx}$. Then Dirichlet data dd on Γ_a is $dd = -e^{i\kappa \cos \theta}$, where $\kappa = ka$, k is a wave number, and a a radius of Γ . The following programs compute far-field coefficient and scattering cross section for dd .

3.1 ffcpl: Plotting profile of far-field coefficient

```
ffcpl(n, k, a, gamma)
n: number of collocation points (number of computation points)
k: wave number
a: radius of circle (obstacle)
gamma: tuning parameter, 0<gamma<1
(gamma=rho/a, rho is the radius of a circle containing source points)
```

3.2 scspl: Plotting profile of scattering cross section

```
scspl(n, k, a, gamma)
n: number of collocation points (number of computation points)
k: wave number
a: radius of circle (obstacle)
gamma: tuning parameter, 0<gamma<1
(gamma=rho/a, rho is the radius of a circle containing source points)
```

3.3 Arguments and tuning parameter

A positive k means that an incident wave comes from the left, and a negative k means that an incident wave comes from the right.

Tuning parameter γ is a positive number such that $0 < \gamma < 1$. Large γ and large n are recommended for a large wave number k . You may need trial and error to select these parameters.

For example

- $\gamma = 0.5$ and $n = 256$ for $\kappa = k \times a$ with $\kappa = 10$.
- $\gamma = 0.9$ and $n = 8192$ for $\kappa = 500$.

These Octave programs may be available for $\kappa = k \times a$ with $0 < \kappa \leq 600$.

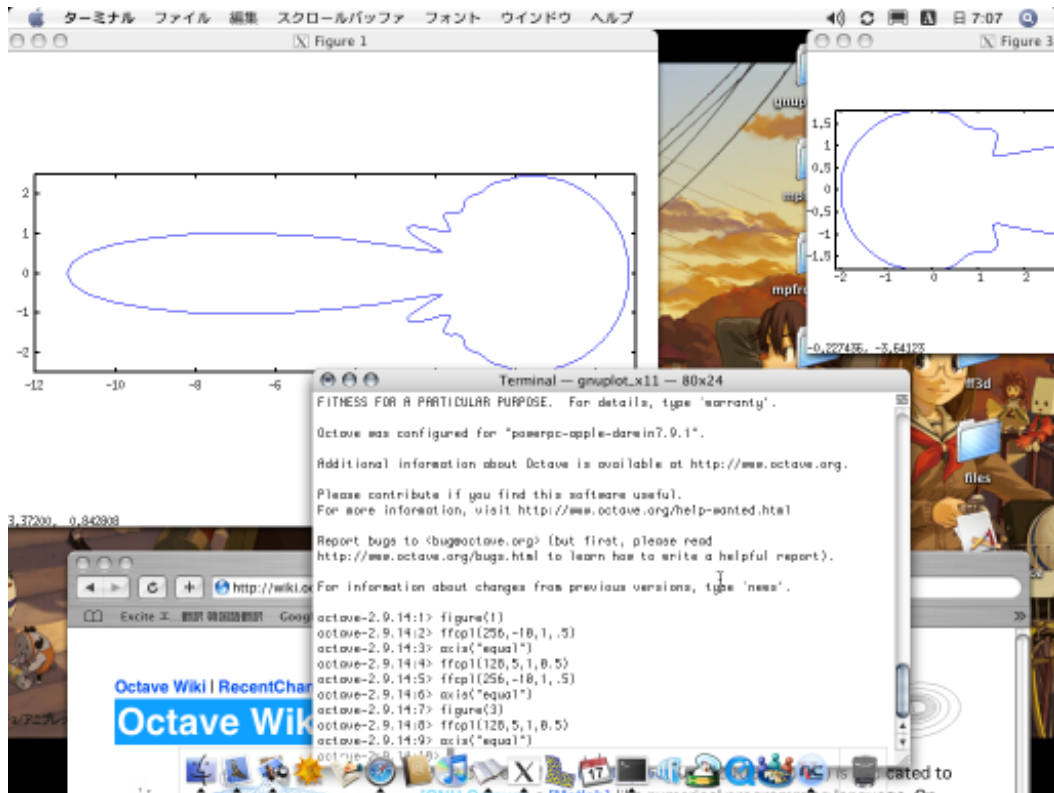


Figure 1: Example of outputs for the programs

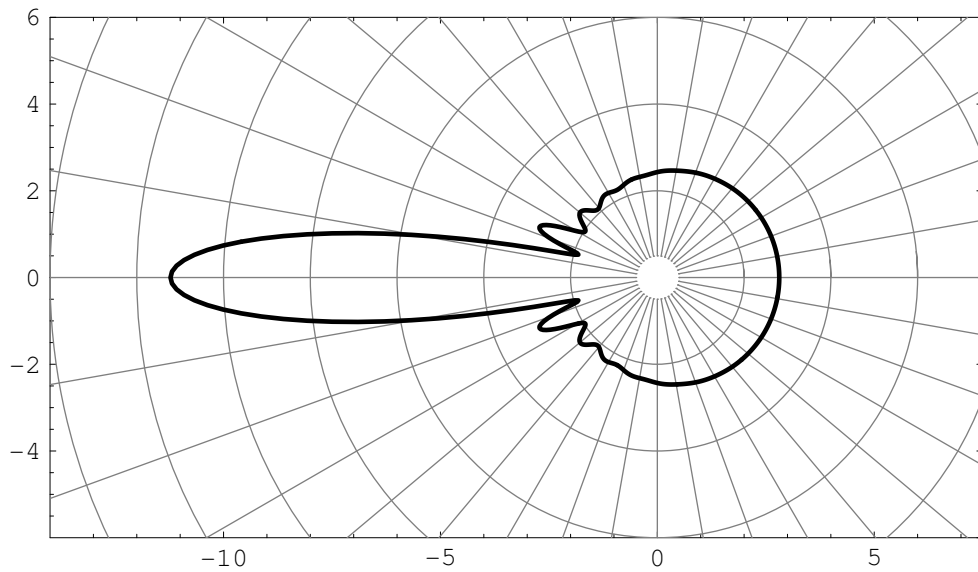
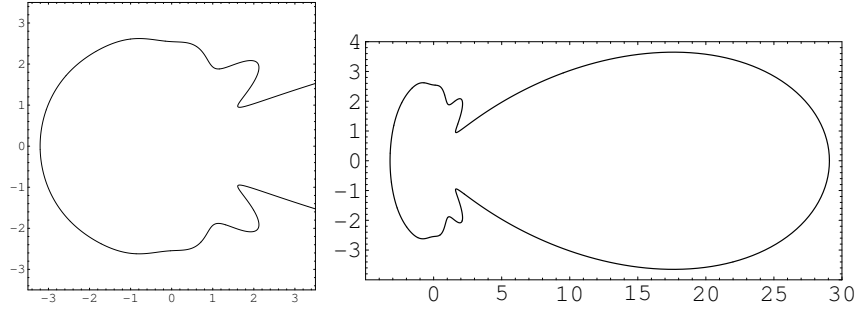
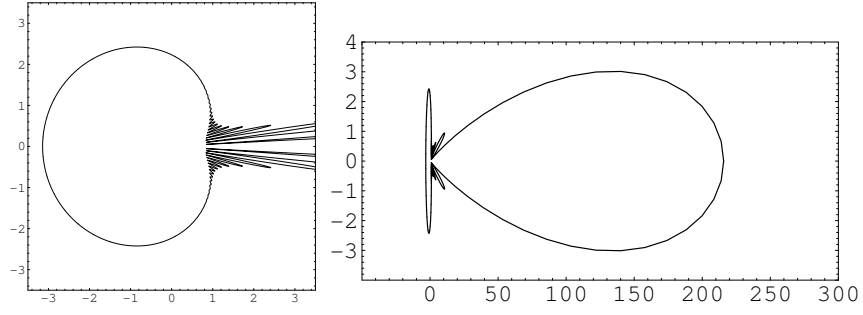


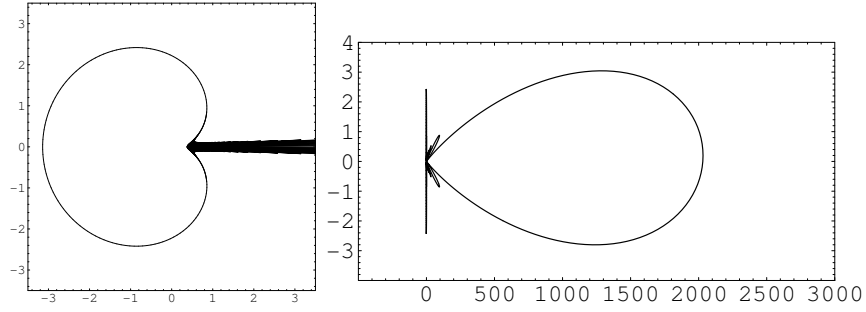
Figure 2: Profile of $|P(\theta)|$ with $\kappa = ka = 10$



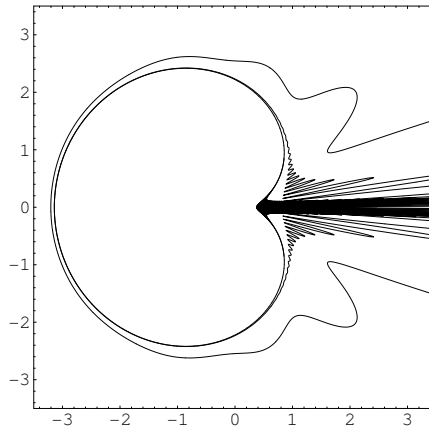
A: $\kappa = 5$



B: $\kappa = 50$



C: $\kappa = 500$



D: Composite of A, B and C

Figure 3: Scattering cross sections

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