

# About an approximate formula for scattering amplitude by a disc

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## 1 An approximate solution for reduced wave problem by FSM

Consider the following reduced wave problem in the exterior domain of a disc with Dirichlet boundary condition. Let  $a$  be the radius of a disc, and  $k$  a wave number. Then the problem is represented as follows.

$$\left\{ \begin{array}{ll} -\Delta u - k^2 u = 0 & \text{in } \Omega_e, \\ u = f & \text{on } \Gamma_a, \\ \lim_{r \rightarrow \infty} \sqrt{r} \left\{ \frac{\partial u}{\partial r} - iku \right\} = 0, \end{array} \right.$$

where

$$\Omega_e = \{\mathbf{r} \in \mathbb{R}^2; |\mathbf{r}| > a\}, \quad \Gamma_a = \{\mathbf{a} \in \mathbb{R}^2; |\mathbf{a}| = a\},$$

and  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^2$ .

A positive number  $\rho$  is the radius of a disc containing all source points. Let  $N$  be a fixed positive integer. Then we define a basis function  $G_j(\mathbf{r})$  through

$$G_j(\mathbf{r}) = H_0^{(1)}(k|r e^{i\theta} - \rho e^{i\theta_j}|), \quad \theta_j = j \frac{2\pi}{N}, \quad 0 \leq j \leq N-1,$$

where  $H_0^{(1)}(\cdot)$  is the zeroth order Hankel function of the first kind, and the points  $(r, \theta)$  and  $(\rho, \theta_j)$  correspond to the complex numbers  $r e^{i\theta}$  and  $\rho e^{i\theta_j}$ , respectively.

An approximate solution of the problem above is given as follows[4].

$$u^{(N)}(\mathbf{r}) = \sum_{j=0}^{N-1} Q_j G_j(\mathbf{r}),$$

where  $Q_j$  is the intensity of sources, and  $\mathbf{r}$  corresponds to the polar coordinate  $(r, \theta)$ .

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The intensity of sources  $Q_j$  is computed as follows. Introduce the following normalized parameters:

$$\gamma = \frac{\rho}{a}, \quad \delta = \frac{r}{a}, \quad \kappa = ka.$$

Then the basis function is represented as follows.

$$G_j(\mathbf{r}) = H_0^{(1)}(\kappa|\delta - \gamma e^{-i(\theta - \theta_j)}|), \quad 0 \leq j \leq N-1.$$

Introduce the kernel function:

$$g(\theta) = H_0^{(1)}(\kappa|1 - \gamma e^{-i\theta}|).$$

The intensity of sources  $Q_j$  is given as follows.

$$Q_j = \frac{1}{N} \sum_{k=0}^{N-1} \frac{F_k^{(N)}}{G_k^{(N)}} e^{ij\theta_k} \quad \text{for } 0 \leq j \leq N-1,$$

where

$$F_k^{(N)} = \frac{1}{N} \sum_{j=0}^{N-1} f(\mathbf{a}_j) e^{-ik\theta_j}, \quad G_n^{(N)} = \frac{1}{N} \sum_{j=0}^{N-1} g(\theta_j) e^{-in\theta_j},$$

where  $\mathbf{a}_j$  corresponds to the polar coordinate  $(a, \theta_j)$ .

## 2 An approximate formula for scattering amplitude

An approximate scattering amplitude  $A^{(N)}(\theta)$  for the above problem is given as follows[5],[6].

$$A^{(N)}(\theta) = \lim_{r \rightarrow \infty} \left( \frac{e^{ir}}{\sqrt{r}} \right)^{-1} u^{(N)}(\mathbf{r}) = \sum_{j=0}^{N-1} \sqrt{\frac{2}{\pi k}} e^{-i\frac{\pi}{4}} Q_j e^{-i\kappa\gamma \cos(\theta - \theta_j)}$$

with  $\theta_j = \frac{2\pi j}{N}$ ,

where an asymptotic formula of Hankel functions[1] is used.

Then an approximate far-field coefficient[2]  $P^{(N)}(\theta)$  is given as follows.

$$P^{(N)}(\theta) = \sqrt{\frac{\pi k}{2}} e^{i\frac{\pi}{4}} A^{(N)}(\theta).$$

The scattering cross section  $\sigma(\theta)$  is computed as follows[2].

$$\sigma(\theta) = \lim_{r \rightarrow \infty} 2\pi r \left| \frac{u(\mathbf{r})}{u_i(\mathbf{r})} \right|^2,$$

where  $u_i(\mathbf{r})$  is an incident wave.

Suppose:

$$\lim_{r \rightarrow \infty} |u_i(\mathbf{r})| = 1.$$

The scattering wave  $u(\mathbf{r})$  is expected to behave in the far-field as follows.

$$u(\mathbf{r}) \sim \frac{e^{ikr}}{\sqrt{r}} A(\theta) \quad \text{as } r \rightarrow \infty,$$

where  $A(\theta)$  is the scattering amplitude. Then  $\sigma(\theta)$  is represented as follows.

$$\sigma(\theta) = 2\pi |A(\theta)|^2.$$

Define an approximate scattering cross section  $\sigma^{(N)}(\theta)$ :

$$\sigma^{(N)}(\theta) = 2\pi |A^{(N)}(\theta)|^2.$$

### 3 Octave programs

GNU Octave is an array oriented software for numerical computing[3]. You can download the below Octave programs from <http://web.me.com/chibaf/math/octave/ffc/>

Let an incident wave  $f = e^{ikx}$ . Then Dirichlet data  $dd$  on  $\Gamma_a$  is  $dd = -e^{i\kappa \cos \theta}$ , where  $\kappa = ka$ ,  $k$  is a wave number, and  $a$  a radius of  $\Gamma$ . The following programs compute far-field coefficient and scattering cross section for  $dd$ .

#### 3.1 ffcpl: Plotting profile of far-field coefficient

```
ffcpl(n, k, a, gamma)
```

$n$ : number of collocation points (number of computation points)

$k$ : wave number

$a$ : radius of circle (obstacle)

$\gamma$ : tuning parameter,  $0 < \gamma < 1$

( $\gamma = \rho/a$ ,  $\rho$  is the radius of a circle containing source points)

#### 3.2 scspl: Plotting profile of scattering cross section

```
scspl(n, k, a, gamma)
```

$n$ : number of collocation points (number of computation points)

$k$ : wave number

$a$ : radius of circle (obstacle)

$\gamma$ : tuning parameter,  $0 < \gamma < 1$

( $\gamma = \rho/a$ ,  $\rho$  is the radius of a circle containing source points)

#### 3.3 Arguments and tuning parameter

A positive  $k$  means that an incident wave comes from the left, and a negative  $k$  means that an incident wave comes from the right.

Tuning parameter  $\gamma$  is a positive number such that  $0 < \gamma < 1$ . Large  $\gamma$  and  $n$  are recommended for a large wave number  $k$ . You may need trial and error to select these parameters.

For example

- $\gamma = 0.5$  and  $n = 256$  for  $\kappa = k \times a$  with  $|k \times a| = 10$ .
- $\gamma = 0.9$  and  $n = 8192$  for  $\kappa = |k \times a| = 500$ .

These Octave programs may be available for  $\kappa = k \times a$  with  $0 < |k \times a| \leq 600$ .

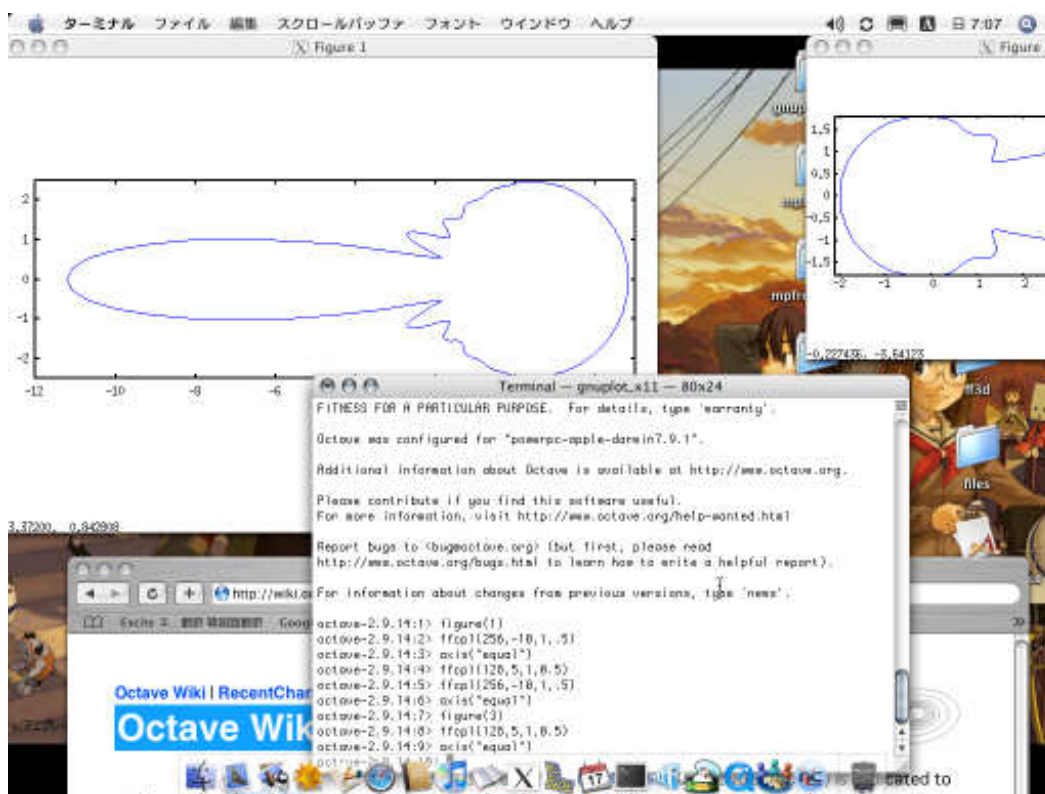


Figure 1: Example of outputs for the programs

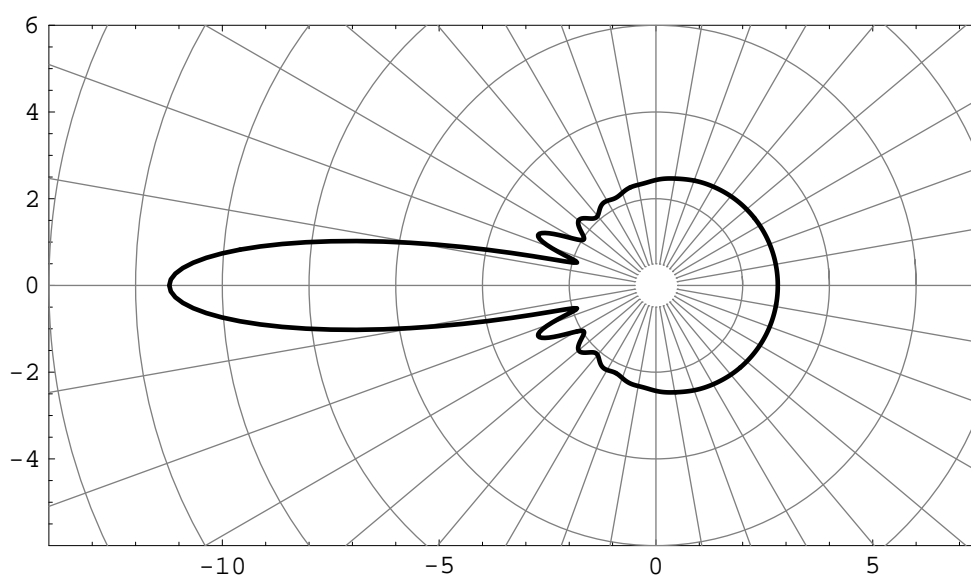
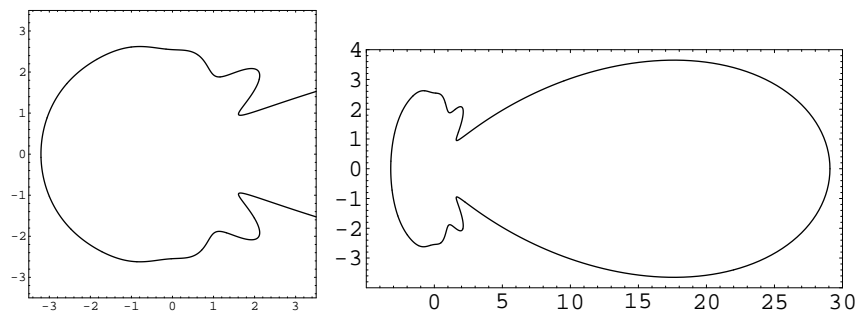
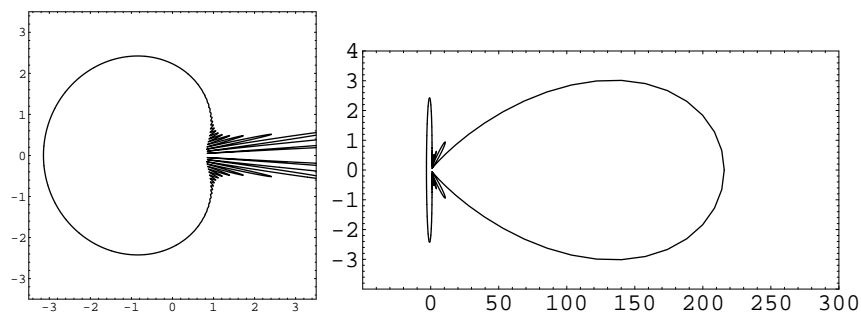


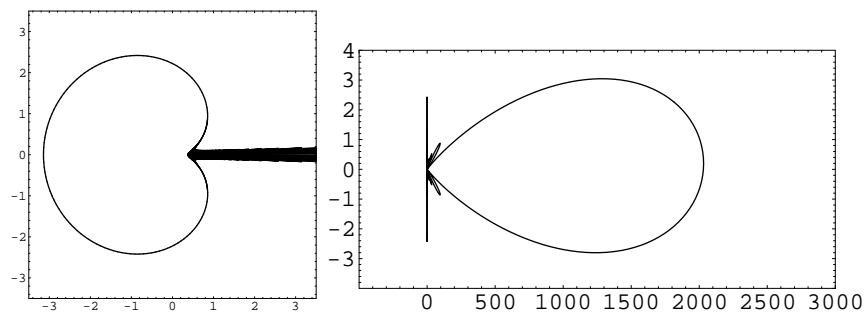
Figure 2: Profile of  $|P(\theta)|$  with  $\kappa = ka = 10$



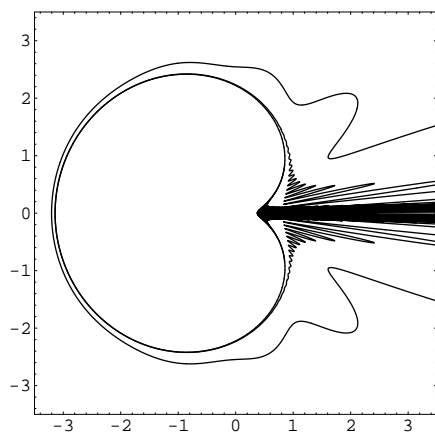
A:  $\kappa = 5$



B:  $\kappa = 50$



C:  $\kappa = 500$



D: Composite of A, B and C

Figure 3: Scattering cross sections

## References

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