About an approximate formula for scattering amplitude by a disc

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1 An approximate solution for reduced wave problem by FSM

Consider the following reduced wave problem in the exterior domain of a disc with Dirichlet boundary condition. Let a be the radius of a disc, and k a wave number. Then the problem is represented as follows.

$$\begin{cases} -\Delta u - k^2 u = 0 \quad \text{in } \Omega_e, \\ u = f \quad \text{on } \Gamma_a, \\ \displaystyle\lim_{r\to\infty} \sqrt{r}\left\{\frac{\partial u}{\partial r} - \mathrm{i}ku\right\} = 0, \end{cases}$$

where

$$\Omega_e = \{\boldsymbol{r} \in \mathbb{R}^2;\ |\boldsymbol{r}| > a\},\ \Gamma_a = \{\boldsymbol{a} \in \mathbb{R}^2;\ |\boldsymbol{a}| = a\},$$

and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$.

A positive number ρ is the radius of a disc containing all source points. Let N be a fixed positive integer. Then we define a basis function $G_j(\boldsymbol{r})$ through

$$G_j(\boldsymbol{r}) = H_0^{(1)}(k|r\,\mathrm{e}^{\mathrm{i}\theta} - \rho\,\mathrm{e}^{\mathrm{i}\theta_j}|), \quad \theta_j = j\frac{2\pi}{N}, \quad 0 \le j \le N-1,$$

where $H_0^{(1)}(\cdot)$ is the zeroth order Hankel function of the first kind, and the points (r, θ) and (ρ, θ_j) correspond to the complex numbers $r\,\mathrm{e}^{\mathrm{i}\theta}$ and $\rho\,\mathrm{e}^{\mathrm{i}\theta_j}$, respectively.

An approximate solution of the problem above is given as follows[4].

$$u^{(N)}(\boldsymbol{r}) = \sum_{j=0}^{N-1} Q_j G_j(\boldsymbol{r}),$$

where Q_j is the intensity of sources, and $\boldsymbol{r}$ corresponds to the polar coordinate (r, θ).

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The intensity of sources Q_j is computed as follows. Introduce the following normalized parameters:

$$\gamma = \frac{\rho}{a}, \quad \delta = \frac{r}{a}, \quad \kappa = ka.$$

Then the basis function is represented as follows.

$$G_j(\boldsymbol{r}) = H_0^{(1)}(\kappa|\delta - \gamma\,\mathrm{e}^{-\mathrm{i}(\theta-\theta_j)}|), \quad 0 \leq j \leq N-1.$$

Introduce the kernel function:

$$g(\theta) = H_0^{(1)}(\kappa|1 - \gamma\,\mathrm{e}^{-i\theta}|).$$

The intensity of sources Q_j is given as follows.

$$Q_j = \frac{1}{N}\sum_{k=0}^{N-1}\frac{F_k^{(N)}}{G_k^{(N)}}\,\mathrm{e}^{\mathrm{i}j\theta_k} \quad \text{for } 0 \leq j \ \leq N-1,$$

where

$$F_k^{(N)} = \frac{1}{N}\sum_{j=0}^{N-1} f(\boldsymbol{a}_j)\,\mathrm{e}^{-\mathrm{i}k\theta_j}, \quad G_n^{(N)} = \frac{1}{N}\sum_{j=0}^{N-1} g(\theta_j)\,\mathrm{e}^{-\mathrm{i}n\theta_j},$$

where $\boldsymbol{a}_j$ corresponds to the polar coordinate (a, θ_j).

2 An approximate formula for scattering amplitude

An approximate scattering amplitude $A^{(N)}(\theta)$ for the above problem is given as follows[5],[6].

$$A^{(N)}(\theta) = \lim_{r\to\infty}\left(\frac{\mathrm{e}^{\mathrm{i}r}}{\sqrt{r}}\right)^{-1} u^{(N)}(\boldsymbol{r}) = \sum_{j=0}^{N-1}\sqrt{\frac{2}{\pi k}}\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} Q_j\,\mathrm{e}^{-\mathrm{i}\kappa\gamma\cos(\theta-\theta_j)}$$

$$\text{with } \theta_j = \frac{2\pi j}{N},$$

where an asymptotic formula of Hankel functions[1] is used.

Then an approximate far-field coefficient[2] $P^{(N)}(\theta)$ is given as follows.

$$P^{(N)}(\theta) = \sqrt{\frac{\pi k}{2}}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} A^{(N)}(\theta).$$

The scattering cross section $\sigma(\theta)$ is computed as follows[2].

$$\sigma(\theta) = \lim_{r\to\infty} 2\pi r\left|\frac{u(\boldsymbol{r})}{u_i(\boldsymbol{r})}\right|^2,$$

where $u_i(\boldsymbol{r})$ is an incident wave.

Suppose:

$$\lim_{r\to\infty}|u_i(\boldsymbol{r})| = 1.$$

The scattering wave $u(\boldsymbol{r})$ is expected to behave in the far-field as follows.

$$u(\boldsymbol{r}) \sim \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}}A(\theta) \quad \text{as } r \to \infty,$$

where $A(\theta)$ is the scattering amplitude. Then $\sigma(\theta)$ is represented as follows.

$$\sigma(\theta) = 2\pi|A(\theta)|^2.$$

Define an approximate scattering cross section $\sigma^{(N)}(\theta)$:

$$\sigma^{(N)}(\theta) = 2\pi|A^{(N)}(\theta)|^2.$$

3 Octave programs

GNU Octave is an array oriented software for numerical computing[3]. You can download the below Octave programs from `http://web.me.com/chibaf/math/octave/ffc/`

Let an incident wave $f = \mathrm{e}^{ikx}$. Then Dirichlet data dd on Γ_a is $dd = -\mathrm{e}^{\mathrm{i}\kappa\cos\theta}$, where $\kappa = ka$, k is a wave number, and a a radius of Γ. The following programs compute far-field coefficient and scattering cross section for dd.

3.1 ffcpl: Plotting profile of far-field coefficient

```
ffcpl(n, k, a, gamma)
n: number of collocation points (number of computation points)
k: wave number
a: radius of circle (obstacle)
gamma: tuning parameter, 0<gamma<1
(gamma=rho/a, rho is the radius of a circle containing source points)
```

3.2 scspl: Plotting profile of scattering cross section

```
scspl(n, k, a, gamma)
n: number of collocation points (number of computation points)
k: wave number
a: radius of circle (obstacle)
gamma: tuning parameter, 0<gamma<1
(gamma=rho/a, rho is the radius of a circle containing source points)
```

3.3 Arguments and tuning parameter

A positive k means that an incident wave comes from the left, and a negative k means that an incident wave comes from the right.

Tuning parameter γ is a positive number such that $0 < \gamma < 1$. Large γ and n are recommended for a large wave number k. You may need trial and error to select these parameters.

For example

- $\gamma = 0.5$ and $n = 256$ for $\kappa = k \times a$ with $|k \times a| = 10$.
- $\gamma = 0.9$ and $n = 8192$ for $\kappa = |k \times a| = 500$.

These Octave programs may be available for $\kappa = k \times a$ with $0 < |k \times a| \leq 600$.

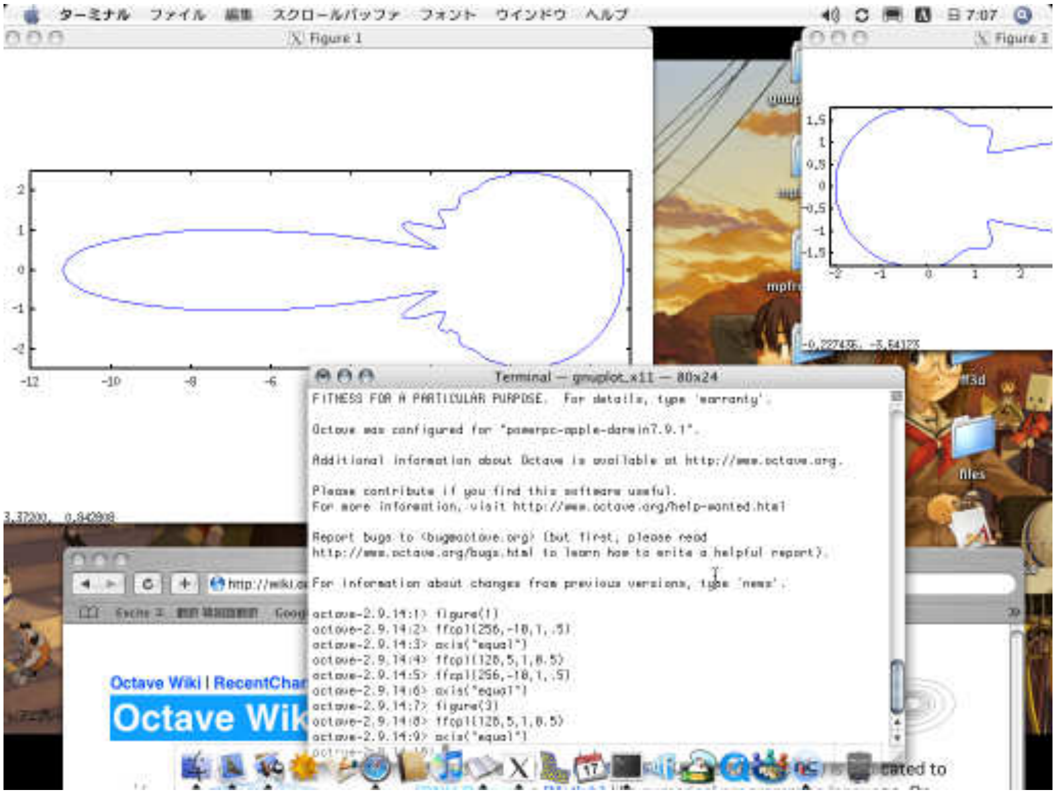


Figure 1: Example of outputs for the programs

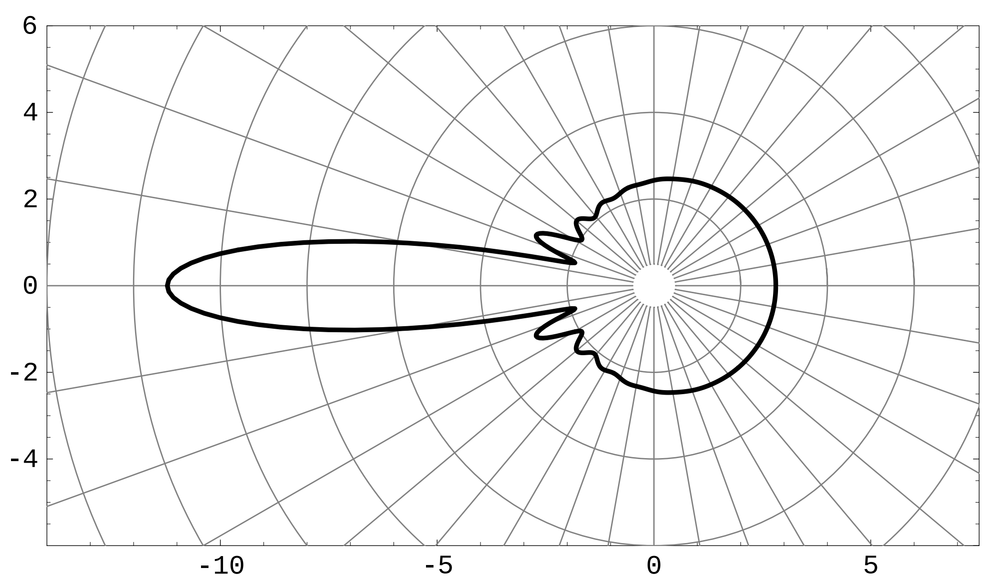
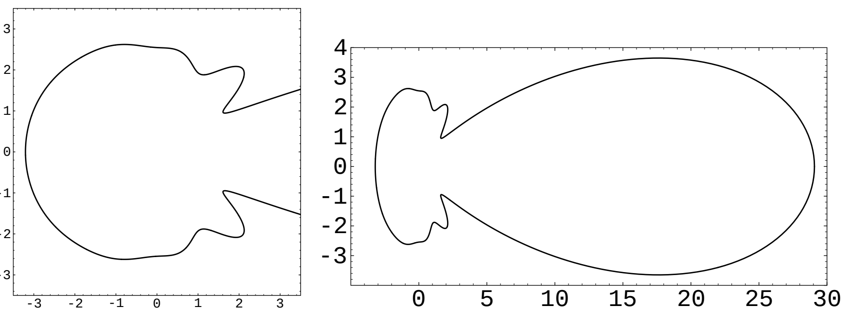
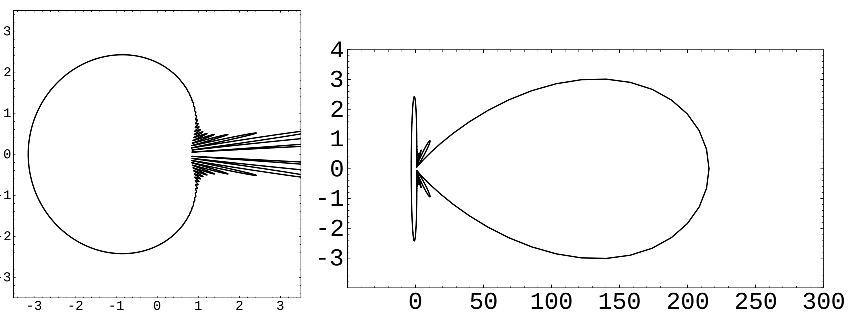


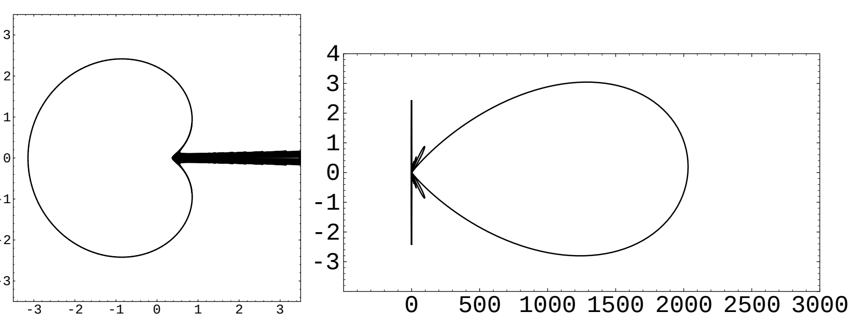
Figure 2: Profile of $|P(\theta)|$ with $\kappa = ka = 10$



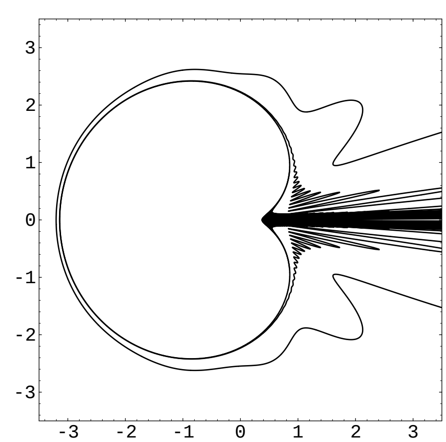
A: $\kappa = 5$



B: $\kappa = 50$



C: $\kappa = 500$



D: Composite of A, B and C

Figure 3: Scattering cross sections

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