Using Debye's contour, Debye shows the following asymptotic expansions (1) and (3) of order α for $H_\alpha^{(2)}(x)$ and $J_\alpha(x)$ from their integral representations[1],[2],[3], where $\alpha > x > 0$ and $H_\alpha^{(2)}(x)$ is the αth order Hankel function of the second kind. Watson gives an explanation on Debye's contour in his book[3].

$$\begin{aligned} H_\alpha^{(2)}(x) \quad &\sim \quad \frac{\mathrm{i}}{\pi}\,\mathrm{e}^{-\mathrm{i}x(\sin\tau_0 - \tau_0\cos\tau_0)} \left[\frac{\Gamma(\frac{1}{2})}{(\mathrm{i}\frac{x}{2}\sin\tau_0)^{\frac{1}{2}}} - \left(\frac{1}{8} + \frac{5}{24}\cot^2\tau_0\right) \frac{\Gamma(\frac{3}{2})}{(\mathrm{i}\frac{x}{2}\sin\tau_0)^{\frac{3}{2}}} \right. \\ &\left. + \left(\frac{3}{128} + \frac{7}{576}\cot^2\tau_0 + \frac{385}{3456}\cot^4\tau_0\right) \frac{\Gamma(\frac{5}{2})}{(\mathrm{i}\frac{x}{2}\sin\tau_0)^{\frac{5}{2}}} + \cdots \right], \end{aligned} \tag{1}$$

where τ_0 is a saddle point and defined through

$$\tau_0 = -\mathrm{i}\log\left(\frac{\alpha}{x} + \frac{\alpha}{x}\sqrt{1 - \left(\frac{x}{\alpha}\right)^2}\right). \tag{2}$$

Since

$$\cos\tau_0 = \frac{\alpha}{x}, \quad \sin\tau_0 = -\mathrm{i}\frac{\alpha}{x}\sqrt{1 - \left(\frac{x}{\alpha}\right)^2},$$

we have for a fixed $x > 0$

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}x(\sin\tau_0 - \tau_0\cos\tau_0)} \quad &= \quad \exp\left\{-\alpha\sqrt{1 - \left(\frac{x}{\alpha}\right)^2} + \alpha\log\left(\frac{\alpha}{x} + \frac{\alpha}{x}\sqrt{1 - \left(\frac{x}{\alpha}\right)^2}\right)\right\} \\ &= \quad \exp\left(-\alpha\sqrt{1 - \left(\frac{x}{\alpha}\right)^2}\right) \times \left(\frac{\alpha}{x} + \frac{\alpha}{x}\sqrt{1 - \left(\frac{x}{\alpha}\right)^2}\right)^\alpha \\ &\sim \quad \mathrm{e}^{-\alpha} \times \left(\frac{2\alpha}{x}\right)^\alpha = \left(\frac{2\alpha}{\mathrm{e}x}\right)^\alpha \quad \text{as } \alpha \to \infty. \end{aligned}$$

On the other hand, the following formulae hold for a fixed $x > 0$.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \text{and} \quad \left(\mathrm{i}\frac{x}{2}\sin\tau_0\right)^{1/2} \sim \sqrt{\frac{\alpha}{2}} \quad \text{as } \alpha \to \infty.$$

Then the first term of (1) is asymptotically equal to

$$\mathrm{i}\sqrt{\frac{2}{\pi\alpha}}\left(\frac{2\alpha}{\mathrm{e}x}\right)^\alpha.$$

Hence we have

$$H_\alpha^{(2)}(x) \sim \mathrm{i}\sqrt{\frac{2}{\pi\alpha}}\left(\frac{2\alpha}{\mathrm{e}x}\right)^\alpha \quad \text{as } \alpha \to \infty.$$

In the same manner as this discussion, from the following Debye's formula[1] we have the asymptotic expansion of $J_\alpha(x)$ for a fixed positive number x as $\alpha \to \infty$.

$$\begin{aligned} J_\alpha(x) \quad &\sim \quad \frac{1}{\pi}\,\mathrm{e}^{\mathrm{i}x(\sin\tau_0 - \tau_0\cos\tau_0)} \left[\frac{\Gamma(\frac{1}{2})}{(\mathrm{i}\frac{x}{2}\sin\tau_0)^{\frac{1}{2}}} + \left(\frac{1}{8} + \frac{5}{24}\cot^2\tau_0\right) \frac{\Gamma(\frac{3}{2})}{(\mathrm{i}\frac{x}{2}\sin\tau_0)^{\frac{3}{2}}} \right. \\ &\left. + \left(\frac{3}{128} + \frac{7}{576}\cot^2\tau_0 + \frac{385}{3456}\cot^4\tau_0\right) \frac{\Gamma(\frac{5}{2})}{(\mathrm{i}\frac{x}{2}\sin\tau_0)^{\frac{5}{2}}} + \cdots \right], \end{aligned} \tag{3}$$

where τ_0 is a saddle point and defined through

$$\tau_0 = -\mathrm{i}\log\left(\frac{\alpha}{x} - \frac{\alpha}{x}\sqrt{1-\left(\frac{x}{\alpha}\right)^2}\right). \tag{4}$$

Bibliography

[1] Debye, P. Nährungsformeln für die Zylinderfunktionen für groe Werte des Arguments und unbeschränkt veränderliche Werte des Index. *Math. Ann. I.XVII* (1909), 535–558.

[2] Olver, F. W. J. *Asymptotics and Special Functions*. AKP Classics. A K Peters, Ltd., Wellesley, Massachusetts, 1996.

[3] Watson, G. N. *A Treatise on the Theory of Bessel Functions*, second ed. Cambridge University Press, Cambridge, 1966.