Computational Homework 3

Due: Monday, April 15 (end-of-day 11.59pm)

You should use C++ to write the code. Submit code and summary of results via Compass2g.
Remark: There is also a mathematical hw (#4) due on the same date.

1. **Monte Carlo Pricing of European and Asian Calls.** Consider the following option data for a stock:

$$\mu = 10\%,\ \sigma = 15\%,\ S_0 = \$15,\ K = \$16,\ T = 1 \text{ year},\ r = 0\%$$

Assume that S_t satisfies a Geometric Brownian motion, i.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

(a) Write a code to simulate $\{S_t\}_t \in [0, T]$ using an *exact* simulation algorithm.

(b) Apply Monte Carlo to price both European call and put options for $n = 10,000$, $\Delta t = 1/24$ (i.e. half month). What is the Monte Carlo variance for each option? Which one is more volatile? Explain why intuitively.

(c) Apply plain Monte Carlo to compute the price of an Asian call option with payoff

$$\max\{\bar{S}_T - K, 0\},\ \text{with } \bar{S}_T = \frac{1}{24}\sum_{n=1}^{24} S_n.$$

Is the variance of the Asian call option smaller or bigger than that of the European? Explain why intuitively.

2. **Exchange Option.** We have two assets U and V with dynamics given by

$$dU_t = rU_t dt + \sigma_U U_t dW_t^U$$
$$dV_t = rV_t dt + \sigma_V V_t dW_t^V,$$

where $Corr(W_t^V, W_t^U) = \rho$. Consider the *exchange option* with payoff

$$\max\{V_T - U_T, 0\},$$

that is an option that allows you to exchange one asset for the other at maturity.

(a) Implement your code from Homework 4/Problem 3 to simulate two correlated Brownian motions.

(b) Price the exchange option using Monte Carlo simulations and plot the option price as a function of n, comparing with the option's corresponding Black-Scholes price given by

$$V_0\mathcal{N}(d_1) + U_0\mathcal{N}(-d_2)$$

with $d_1 = \left(ln(V_0/U_0) + \hat{\sigma}^2 T/2\right)/\hat{\sigma}\sqrt{T}$, $d_2 = d_1 - \hat{\sigma}\sqrt{T}$, and $\hat{\sigma} = \sqrt{\sigma_V^2 + \sigma_U^2 - 2\rho\sigma_V\sigma_U}$.

For your simulations use $V_0 = 50, U_0 = 60, \sigma_V = 0.3, \sigma_U = 0.4, \rho = 0.7, r = 0.05, T = 5/12$.

3. **Cox-Ingersoll-Ross Model (CIR).** One of the most popular short rate models in the literature is the CIR model with dynamics given by

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$$

   (a) Simulate the CIR model via a direct application of the Euler scheme. What do you observe? Comment.

   (b) In your simulation method in (a), replace r_t with $\sqrt{\max\{r_t, 0\}}$ in both the drift and diffusion coefficients of the SDE. Plot the simulated sample paths.
   Remark: This is the so-called *full-truncation scheme* and it has first-order weak convergence $\mathcal{O}(h)$, where h is the discretization step.

For the simulations use $\sigma = 2$, $a = 0.1$, b=0.4 , and $r_0 = 0.3$.