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Option Pricing - Alternative Binomial Models

This tutorial discusses several different versions of the binomial model as it may be used for option pricing. A discussion of the mathematical fundamentals behind the binomial model can be found in the Binomal Model tutorial. As introduced in that tutorial there are primarily three parameters -- p, u and d -- that need to be calculated to use the binomial model.

The Binomal Model tutorial discusses the way that p, u and d are chosen in the formulation originally proposed by Cox, Ross, and Rubinstein. In the tutorials presented here several alternative methods for choosing p, u and d are presented. The methods discussed here are those proposed by,

- Jarrow-Rudd: This is commonly called the equal-probability model.
- Tian: This is commonly called the moment matching model.
- Jarrow-Rudd Risk Neutral: This is a modification of the original Jarrow-Rudd model that incorporates a risk-neutral probablity rather than an equal probability.
- Cox-Ross-Rubinstein With Drift: This is a modification of the original Cox-Ross-Runinstein model that incorporates a *drift* term that effects the symmetry of the resultant price lattice.
- Leisen-Reimer: This uses a completely different approach to all the other methods, relying on approximating the normal distrbution used in the Black-Scholes model.

Jarrow-Rudd

For reasons that will become self-evident, the binomial model proposed by Jarrow and Rudd is often refered to as the *equal-probability* model.

In the Binomal Model tutorial two equations are given that ensure that over a small period of time the expected mean and variance of the binomial model will match those expected in a risk neutral world. Since there are three unknowns in the binomial model (p, u and d) a third equation is required to calculate unique values for them.

The third equation proposed by Jarrow and Rudd is

$$p = 1/2$$

Equation 1: Third Equation for the Jarrow-Rudd Binomial Model and hence there is an equal probability of the asset price rising or falling.

This leads to the equations,

$$
\begin{aligned}
p &= 1/2 \\
u &= e^{\left(r-\sigma^2/2\right)\Delta t+\sigma\sqrt{\Delta t}} \\
d &= e^{\left(r-\sigma^2/2\right)\Delta t-\sigma\sqrt{\Delta t}}
\end{aligned}
$$

Equation 2: Parameters for the Jarrow-Rudd Binomial Model

The *p*, *u* and *d* calculated from Equation 2 may then be used in a similar fashion to those discussed in the [Binomal Model](http://www.goddardconsulting.ca/option-pricing-binomial-index.html) tutorial to generate a price tree and use it for pricing options. Note that a consequence of Equation 1 is that the Jarrow-Rudd model is no longer risk neutral. The alternative [Jarrow-Rudd Risk Neutral](#) model, discussed shortly, addresses this drawback.

Tian

In the [Binomal Model](http://www.goddardconsulting.ca/option-pricing-binomial-index.html) tutorial two equations are given that ensure that over a small period of time the expected mean and variance of the binomial model will match those expected in a risk neutral world. Note that the mean an variance are called the first and second moments of a distribution.

The model proposed by Tian exactly matches the first three moments of the binomial model to the first three moments of a lognormal distribution. Hence the three equations used by Tian are

$$
\begin{aligned}
pu + (1-p) &= e^{r\Delta t} \\
pu^2 + (1-p)d^2 &= \left(e^{r\Delta t}\right)^2 e^{\sigma^2\Delta t} \\
pu^3 + (1-p)d^3 &= \left(e^{r\Delta t}\right)^3 \left(e^{\sigma^2\Delta t}\right)^3
\end{aligned}
$$

Equation 3: Three Equation for the Tian Binomial Model

This leads to the parameters,

$$
\begin{aligned}
p &= \frac{e^{r\Delta t}-d}{u-d} \\
u &= 0.5e^{r\Delta t}\upsilon\left(\upsilon+1+\sqrt{\upsilon^2+2\upsilon-3}\right) \\
d &= 0.5e^{r\Delta t}\upsilon\left(\upsilon+1-\sqrt{\upsilon^2+2\upsilon-3}\right)
\end{aligned}
$$

$$\upsilon = e^{\sigma^2 \Delta t}$$

Equation 4: Parameters for the Tian Binomial Model

The *p*, *u* and *d* calculated from Equation 4 may then be used in a similar fashion to those discussed in the Binomal Model tutorial.

Jarrow-Rudd Risk Neutral

The Jarrow-Rudd Risk Neutral model is a minor modification to the standard Jarrow-Rudd model. The same *u* and *d* are used, however instead of *p=1/2* the standard risk neutral value for *p* is chosen.

This gives the following parameters,

$$p = \frac{e^{r\Delta t} - d}{u - d}$$
$$u = e^{\left(r - \sigma^2/2\right)\Delta t + \sigma\sqrt{\Delta t}}$$
$$d = e^{\left(r - \sigma^2/2\right)\Delta t - \sigma\sqrt{\Delta t}}$$

Equation 5: Parameters for the Jarrow-Rudd Risk Neutral Binomial Model

The *p*, *u* and *d* calculated from Equation 4 may then be used in a similar fashion to those discussed in the Binomal Model tutorial. This model is a special case of the Cox-Ross-Rubinstein With Drift model.

Cox-Ross-Rubinstein With Drift

The derivation of the original binomial model equations as discussed in the Binomal Model tutorial holds even when an arbitrary drift is applied to the *u* and *d* terms. Hence, for an arbirary η, the following parameters give a valid binomial model

$$p = \frac{e^{r\Delta t} - d}{u - d}$$
$$u = e^{\eta\Delta t + \sigma\sqrt{\Delta t}}$$
$$d = e^{\eta\Delta t - \sigma\sqrt{\Delta t}}$$

Equation 6: Parameters for the Cox-Ross-Rubinstein With Drift Binomial Model

When the *drift* term *η* is set to the zero this model collapses to the original Cox-Ross-Rubinstein binomial model which generates a lattice of prices that is centred around the current asset price S_0. This is shown in Figure 3 of the Binomal Model tutorial.

However, with the original model, if the option is a long way out of the money then only a few of the resulting lattice points may have a non-zero payoff associated with them at expiry. The *drift* term can be used to drift (or skew) the lattice upwards or downwards to obtain a lattice where more of the nodes at expiry are in the money.

The Jarrow-Rudd Risk Neutral model is a specific case of the Cox-Ross-Rubinstein With Drift model. A common drift is $\eta = (ln(X)-ln(S_0))/T$, (where $ln(\cdot)$ is the natural logarithm and T is the time to expiry in years) which makes the resulting lattice symmetric about the strike at expiry. A drawback of that particular drift is that the underlying price tree is a function of the strike and hence must be recalculated for options with different strikes, even if all other factors remain constant.

Leisen-Reimer

Leisen and Reimer developed a model with the purpose of improving the rate of converegence of their binomial tree. (All of the models discussed above converge to the Black-Scholes solution in the limit as the size of the time step Δt is reduced to zero. However the convergence is not smooth.)

The Leisen-Reimer tree is generated using the parameters,

$$\bar{p} = h^{-1}(d_1)$$
$$p = h^{-1}(d_2)$$
$$u = e^{r\Delta t}\frac{\bar{p}}{p}$$
$$d = \frac{e^{r\Delta t} - pu}{1-p}$$

Equation 7: Parameters for the Leisen-Reimer Binomial Model

where $h^{-1}(\cdot)$ is a discrete approximation to the cummulative distribution function for a normal distribution. There are several ways this can be calculated. One suggested by Leisen and Reimer is to use,

$$h^{-1}(z) = 0.5 + \mathrm{sgn}(z)\left[0.25 - 0.25\exp\left\{-\left(\frac{z}{n+1/3+0.1/(n+1)}\right)(n+1/6)\right\}\right]^{\frac{1}{2}}$$

Equation 8: Cummulative Distribution Approximation for the Leisen-Reimer Binomial Model

where n is the number of time points in the model (including times 0 and T) which must be odd, and d_1 and d_2 are their usual definitions from the Black-Scholes formulation.

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