Pricing Options
Using
Monte Carlo Methods

This is a project done as a part of the course Simulation Methods. Option contracts and the Black-Scholes pricing model for the European option have been briefly described. The Least Square Monte Carlo algorithm for pricing American option is discussed with a numerical example.
R codes of both the algorithms have been provided.

By
Shariq Mohammed
Graduate Student
M.Sc. Applications of Mathematics
Chennai Mathematical Institute

Under the guidance of
Dr. Samar Singh
Visiting Faculty
Chennai Mathematical Institute

Options Contracts

There are two basic types of options: calls and puts.

- A *call option* gives the holder the right, but not the obligation, to buy the underlying asset by a certain date for a certain price.
- A *put option* gives the holder the right, but not the obligation, to sell the underlying asset by a certain date for a certain price.

The price in the contract is known as the exercise price or strike price (K); the date in the contract is known as the expiration date or maturity time(T) and the price of the underlying asset at time t is S_t. The *pay-off* of an option can be considered as $max(S_{t_e} - K, 0)$ for a call option and $max(K - S_{t_e}, 0)$ for a put option, where t_e is the time when the option is exercised. It should be emphasized that an option gives the holder the right to buy or sell, but not the obligation. The holder may opt not to exercise the right. This is what distinguishes options from forwards and futures, where the holder is obligated to buy or sell the underlying asset. It costs nothing to enter into a forward or futures contract, whereas there is a cost to acquiring an option and in this project discusses couple of methods of pricing the options.

According to when options can be exercised, they are classified into mainly three groups; European, Bermudan and American Options. Additionally, Exotic options differ from common options in terms of the underlying assets or the calculation of the pay-off. Asian, Barrier, Rainbow, Spread, Basket and maximum options are some commonly traded options that are categorized as exotic ones.

American options can be exercised at any time up to the expiration date, whereas *European options* can be exercised only on the expiration date itself. Most of the options that are traded on exchanges are American. However, European options are generally easier to analyse than American options, and some of the properties of an American option are frequently deduced from those of its European counterpart. Bermudan options are similar in style to American options regarding the possibility of early exercise. However, the difference is the fact that a Bermudan option has predetermined discrete exercise dates. So, Bermudan options can be placed between European and American options. We shall only consider options with non-dividend paying underlying assets.

European Option Pricing: Black-Scholes Model

Black and Scholes (1973) built the derivative pricing theory based on geometric Brownian motion (GBM). The price of an asset at time t, S_t, is said to follow a GBM if it satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

where W_t is a Weiner process or Brownian motion, μ the percentage drift and σ the percentage volatility are constant.

Let $(\Omega, \mathcal{F}, P)$ denote a probability space. All random variables considered are $\mathcal{F}$ measurable. A continuous process X is a said to be a *Brownian motion* or *Weiner process* if

(i) $P(X = 0) = 1$

(ii) For $0 \le s < t < \infty$, the distribution of $X_t - X_s$ is Normal(Gaussian) with mean 0 and variance $t - s$

(iii) For $m \ge 1$, $0 \le t_0 < t_1 < \ldots < t_m$, $Y_1, Y_2, \ldots, Y_m$ are independent random variables, where $Y_j = X_{t_j} - X_{t_{j-1}}$

By the help of Ito's lemma, the analytic solution of the above stochastic differential equation is

$$S_t = S_0 \exp\Big((\mu - \frac{\sigma^2}{2}) t + \sigma \sqrt{t} Z \Big)$$

for an arbitrary initial value S_0, where Z is a standard normal random variable. Hence, $\log\left(\frac{S_t}{S_0}\right)$ are normally distributed with mean $\left(\mu - \frac{\sigma^2}{2}\right)t$ and variance $\sigma^2 t$.

The Black-Scholes formula (Black and Scholes, 1973), which is driven by the Black-Scholes PDE, gives the exact value of a European call or put option, whereas American options do not have any closed form solution. The price of European call C^{eu} and European put P^{eu} on a non-dividend paying asset, currently trading at price S_0 can be calculated by the Black-Scholes formula as:

$$C^{eu} = S_0 \Phi(d_1) - K \mathrm{e}^{-rT} \Phi(d_2)$$

$$P^{eu} = K \mathrm{e}^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)$$

where $\Phi(.)$ denotes the cumulative distribution function of standard normal distribution and

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

Monte Carlo Simulation

Monte Carlo methods are a class of computational algorithms that are based on repeated computation and random sampling. Monte Carlo simulation is used in finance to value and analyse instruments, portfolios and investments by simulating the sources of uncertainty that affect their value.

Options can be priced by Monte Carlo simulation. First, the price of the underlying asset is simulated by random number generation for a number of paths. Then, the value of the option is found by calculating the average of discounted returns over all paths. Since the option is priced under risk-neutral measure, the discount rate is the risk-free interest rate.

In order to get a good estimate from simulation, the variance of the estimator should go to zero and thus the number of samples should go to infinity, which is computationally not feasible. Therefore, variance reduction techniques such as antithetic variates and control variates help us to obtain a better estimate in simulation.

It is emphasized that Bermudan options can be positioned between American and European options. In simulation, due to the inability to simulate in continuous time, an American option is priced under the assumption that it has Bermudan style, and thus only discrete exercise opportunities exist. So, if an American option is exercisable at any time t where $0 \leq t \leq T$, we restrict the option such that it can be exercised only at a fixed set of exercise opportunities $0 < t_1 < t_2 < \ldots < t_d$, where d is the number of exercisable time steps.

In addition, the American option holder compares the pay-off from early exercise and expected continuation value to make decision to keep the option alive or not, because the rational behaviour for the holder is to maximize the income by this comparison. The pay-off function is determined according to the option type. Hence, the key insight in simulation for pricing American options is to estimate the continuation value.

American Option Pricing: Least Square Monte Carlo Method

At maturity, the optimal exercise strategy for an American option is to exercise the option if it is in-the-money. Prior to maturity, the optimal strategy is to compare the immediate exercise value with the expected cash flows from continuing to hold the option, and then exercise if immediate exercise is more valuable. Thus, the key to optimally exercising an American option is identifying the conditional expected value of continuation. In the Least Square Monte Carlo(LSM) approach, Longstaff and Schwartz(2001) use the cross-sectional information in the simulated paths to identify the conditional expectation function. This is done by regressing the subsequent realised cash flows from continuation on a set of basis functions of the values of the relevant state variables. The fitted value of this regression is an efficient unbiased estimate of the conditional expectation function and allows us to accurately estimate the optimal stopping rule for the option.

A Numerical Example

As described in Longstaff and Schwartz(2001), perhaps the best way to convey the intuition of the LSM approach is through a simple numerical example. Consider an American put option on a share of non-dividend-paying stock. The put option is exercisable at a strike price of 1.10 at times 1, 2 and 3 where time three is the expiration date of the option. The risk-less rate is 6%. For simplicity, the algorithm is illustrated using only eight sample paths for the price of the stock. These sample paths are generated under the risk-neutral measure and are shown in the table 1.

Table 1: Stock Price Paths

Path	t = 0	t = 1	t = 2	t = 3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

Our objective is to solve for the stopping rule that maximizes the value of the option at each point along each path. Conditional on not exercising the option before the maturity date at time 3, the cash flows realized by the option holder from following the optimal strategy at time 3 are given in table 2. These cash flows are identical to the cash flows that would be received if the option were European rather than American.

If the put is in-the-money at time 2, the option holder must then decide whether to exercise the option immediately or continue the option's life until the final expiration date at time 3. From the stock-price matrix, there are only five paths for which the option is in-the-money at time 2. Let X denote the

Table 2: Cash flow matrix at time 3

Path	t = 1	t = 2	t = 3
1	-	-	0.00
2	-	-	0.00
3	-	-	0.07
4	-	-	0.18
5	-	-	0.00
6	-	-	0.20
7	-	-	0.09
8	-	-	0.00

stock prices at time 2 for these five paths and Y denote the corresponding discounted cash flows received at time 3 if the put is not exercised at time 2. We use only in-the-money paths since it allows us to better estimate the conditional expectation function in the region where exercise is relevant and significantly improves the efficiency of the algorithm. The vectors X and Y are given by the entries in table 3.

Table 3: Regression at time 2

Path	Y	X
1	0.00×0.94176	1.08
2	-	-
3	0.07×0.94176	1.07
4	0.18×0.94176	0.97
5	-	-
6	0.20×0.94176	0.77
7	0.09×0.94176	0.84
8	-	-

To estimate the expected cash flow from continuing the option's life conditional on the stock price at time 2, we regress Y on a constant, X, and X^2. This specification is one of the simplest possible. The resulting conditional expectation function is $E[Y|X] = -1.070 + 2.983X - 1.813X^2$. With this conditional expectation function, we now compare the value of immediate exercise at time 2 with the value from continuation from table 4.

The value of immediate exercise equals the intrinsic value $(1.10 - X)$ for the in-the-money paths, while the value from continuation is given by substituting X into the conditional expectation function. This comparison implies that it is optimal to exercise the option at time 2 for the fourth, sixth, and seventh paths. This leads to the table 5, which shows the cash flows received by the option holder conditional on not exercising prior to time 2.

Observe that when the option is exercised at time 2, the cash flow in the final column becomes zero. This is because once the option is exercised there are no further cash flows since the option can only be exercised once.

Table 4: Optimal early exercise decision at time 2

Path	Exercise	Continuation
1	0.02	0.0369
2	-	-
3	0.03	0.0461
4	0.13	0.1176
5	-	-
6	0.33	0.1520
7	0.26	0.1565
8	-	-

Table 5: Cash flow at time 2

Path	t = 1	t = 2	t = 3
1	-	0.00	0.00
2	-	0.00	0.00
3	-	0.00	0.07
4	-	0.13	0.00
5	-	0.00	0.00
6	-	0.33	0.00
7	-	0.26	0.00
8	-	0.00	0.00

Proceeding recursively, now examine whether the option should be exercised at time 1. From the stock price matrix, there are again five paths where the option is in-the-money at time 1. For these paths, we again define Y as the discounted value of subsequent option cash flows. Note that in defining Y, we use actual realized cash flows along each path and not the conditional expected value of Y estimated at time 2. Discounting back the conditional expected value rather than actual cash flows can lead to an upward bias in the value of the option.

Since the option can only be exercised once, future cash flows occur at either time 2 or time 3, but not both. Cash flows received at time 2 are discounted back one period to time 1, and any cash flows received at time 3 are discounted back two periods to time 1. Similarly X represents the stock prices at time 1 for the paths where the option is in-the-money. The vectors X and Y are given in table 6.

The conditional expectation function at time 1 is estimated by again regressing Y on a constant, X and X^2. The estimated conditional expectation function is $E[Y|X] = 2.038 - 3.335X + 1.356X^2$. Substituting the values of X into this regression gives the estimated conditional expectation function. These estimated continuation values and immediate exercise values at time 1 are given in the first and second columns of table 7 below. Comparing the two columns shows that exercise at time 1 is optimal for the fourth, sixth, seventh, and eighth paths.

Table 6: Regression at time 1

Path	Y	X
1	0.00×0.94176	1.09
2	-	-
3	-	-
4	0.13×0.94176	0.93
5	-	-
6	0.33×0.94176	0.76
7	0.26×0.94176	0.92
8	0.00×0.94176	0.88

Table 7: Optimal early exercise decision at time 1

Path	Exercise	Continuation
1	0.01	0.0139
2	-	-
3	-	-
4	0.17	0.1092
5	-	-
6	0.34	0.2866
7	0.18	0.1175
8	0.22	0.1533

Having identified the exercise strategy at times 1, 2, and 3, the stopping rule can now be represented by table 8, where the ones denote exercise dates at which the option is exercised.

Table 8: Stopping Rule

Path	t = 1	t = 2	t = 3
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

With this specification of the stopping rule, it is now straightforward to determine the cash flows realized by following this stopping rule. This is done by simply exercising the option at the exercise dates where there is a one in table 8. This leads to the following optimal cash flow table.

Having identified the cash flows generated by the American put at each date along each path, the option can now be valued by discounting each cash flow in the option cash flow matrix back to time zero, and averaging over all paths.

Table 9: Optimal cash flow

Path	t = 1	t = 2	t = 3
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.07
4	0.17	0.00	0.00
5	0.00	0.00	0.00
6	0.34	0.00	0.00
7	0.18	0.00	0.00
8	0.22	0.00	0.00

Applying this procedure results in a value of 0.1144 for the American put. This is roughly twice the value of 0.0564 for the European put obtained by discounting back the cash flows at time 3 from table 1.

This example illustrates how least squares can use the cross-sectional information in the simulated paths to estimate the conditional expectation function. In turn, the conditional expectation function is used to identify the exercise decision that maximizes the value of the option at each date along each path. As shown by this example, the LSM approach is easily implemented since nothing more than simple regression is involved.

The valuation framework of LSM algorithm

The valuation framework underlying the LSM algorithm is based on the general derivative pricing paradigm of Black and Scholes (1973), Merton (1973), Harrison and Kreps (1979). Harrison and Pliska (1981), Cox. Ingersoll, and Ross (1985), Heath, Jarrow, and Morton (1992) and others.

Assume an underlying complete probability space $(\Omega, \mathcal{F}, P)$ and finite time horizon $[0, T]$, where the state space Ω is the set of all possible realizations of the stochastic economy between time 0 and T and has typical element ω representing a sample path, $\mathcal{F}$ is the sigma field of distinguishable events at time T ie., $\mathcal{F}_T = \mathcal{F}$ and P is a probability measure defined on the elements of $\mathcal{F}$. We define $F = \{\mathcal{F}_t \mid t \in [0, T]\}$ to be the augmented filtration generated by the relevant price processes for the securities in the economy. Consistent with the no-arbitrage paradigm, we assume the existence of an equivalent martingale measure Q for this economy.

We restrict our attention to derivatives with pay-offs that are elements of the space of square-integrable or finite-variance functions $L^2(\Omega, \mathcal{F}, Q)$. Standard results such as Bensoussan (1984) and Karatzas (1988) imply that the value of an American option can be represented by the Snell envelope; the value of an American option equals the maximized value of the discounted cash flows from the option, where the maximum is taken over all stopping times with respect to the filtration $\mathcal{F}$. Let $C(\omega, s; t, T)$ denote the path of cash flows generated by the option, conditional on the option not being exercised at or prior to time t and on the option holder following the optimal stopping strategy $\forall\ s,\ t < s \leq T$. This function is analogous to the intermediate cash flow tables of the previous section.

LSM algorithm provides a path wise approximation to the optimal stopping rule that maximizes the value of the American option. Assume that the option can only be exercised at the N discrete times $0 < t_1 \leq t_2 \leq \ldots \leq t_N = T$, and consider the optimal stopping policy at each exercise date. In practice, of course, many American options are continuously exercisable; the LSM algorithm can be used to approximate the value of these options by taking N to be sufficiently large.

At the final expiration date of the option, the investor exercises the option if it is in-the-money or allows it to expire if it is out-of-the-money. At exercise time t_n, prior to the final expiration date, however, the option holder must choose whether to exercise immediately or to continue the life of the option and revisit the exercise decision at the next exercise time. The value of the option is maximized path wise, and hence unconditionally, if the investor exercises as soon as the immediate exercise value is greater than or equal to the value of continuation.

At time t_n, the cash flow from immediate exercise is known and its value equals this cash flow but, the cash flows from continuation are unknown at time t_n. No-arbitrage valuation theory implies that the value of continuation, or equivalently, the value of the option assuming that it cannot be exercised until after t_n, is the expectation of the remaining discounted cash flows $C(\omega, s; t_n, T)$ with respect to the risk-neutral pricing measure Q. At time t_n, the value of continuation $F(\omega; t_n)$ can be expressed as

$$F(\omega; t_n) = E_Q\left[\sum_{i=n+1}^{N} exp\left(-\int_{t_n}^{t_i} r(\omega, s)ds\right)C(\omega, t_i; t_n, T)\ \middle|\ \mathcal{F}_{t_n}\right] \qquad (\dagger)$$

where $r(\omega, t)$ is the (possibly stochastic) risk-less discount rate, and the expectation is taken conditional on the information set $\mathcal{F}_{t_n}$ at time t_n. Now the problem of optimal exercise reduces to comparing the immediate exercise value with this conditional expectation, and then exercising as soon as the immediate exercise value is positive and greater than or equal to the conditional expectation.

The LSM algorithm

The LSM approach uses least squares to approximate the conditional expectation function at $t_{N-1}, t_{N-2}, \ldots, t_1$. We work backwards since $C(\omega, s; t, T)$ is defined recursively and $C(\omega, s; t_n, T)$ can differ from $C(\omega, s; t_{n+1}, T)$ since it may be optimal to stop at time t_{n+1}, thereby changing all subsequent cash flows along a realized path ω. At time t_{N-1}, assume that the unknown functional form of $F(\omega; t_{N-1})$ in equation ($\dagger$) can be represented as a linear combination of a countable set of $\mathcal{F}_{t_{N-1}}$-measurable basis functions. For more details, refer to Longstaff and Schwartz (2001).

To implement the LSM approach, we approximate $F(\omega; t_{N-1})$ using the first $M(< \infty)$ basis functions, and denote it as $F_M(\omega; t_{N-1})$. Once this subset of basis functions has been specified, $F_M(\omega; t_{N-1})$ is estimated by regressing the discounted values of $C(\omega, s; t_{N-1}, T)$ onto the basis functions for the paths where the option is in-the-money at time t_{N-1}. Denote the fitted value of this regression by $\widehat{F_M}(\omega; t_{N-1})$. For further properties of this estimator refer to White (1984) and Amemiya (1985).

For all the in-the-money paths ω compare $\widehat{F_M}(\omega; t_{N-1})$ with the immediate exercise value and determine if exercise at time t_{N-1} is optimal. Once the exercise decision is identified, $C(\omega, s; t_{N-2}, T)$ can then be approximated. The recursion proceeds by rolling back to time t_{N-2}, and repeating the procedure until the exercise decisions at each exercise time along each path have been determined. The American option is then valued by starting at time zero, moving forward along each path until the first stopping time occurs, discounting the resulting cash flow from exercise back to time zero, and then taking the average over all the paths ω.

Choice of basis functions

Extensive numerical tests indicate that the results from the LSM algorithm are remarkably robust to the choice of basis functions. Types of basis functions generally used are Laguerre, Hermite, Legendre, Chebyshev, Gegenbauer, and Jacobi polynomials. Numerical tests indicate that Fourier or trigonometric series and even simple powers of the state variables also give accurate results.

While the results are robust to the choice of basis functions, it is important to be aware of the numerical implications of the choice. The choice of basis functions also has implications for the statistical significance of individual basis functions in the regression. In particular, some choices of basis functions are highly correlated with each other, resulting in estimation difficulties for individual regression coefficients akin to the multicollinearity problem in econometrics. This difficulty does not affect the LSM algorithm since the focus is on the fitted value of the regression rather than on individual coefficients; the fitted value of the regression is unaffected by the degree of correlation among the explanatory variables. However, if the choice of basis functions leads to a cross-moment matrix that is nearly singular, then numerical inaccuracies in some least squares algorithms may affect the functional form of estimated conditional expectation function.

R code for pricing European call & put options using Black Scholes formula.

```
# S0 = initial asset price
# K = strike price
# r = risk-free interest rate
# sigma = volatility
# t = maturity time

EU_call_bs = function(S0 = 100, K = 100, r = 0.1, sigma = 0.25,
                                                              t = 1)
{
  d1 = (log(S0/K)+(r+((sigma)^2)/2)*t)/(sigma*sqrt(t))
  d2 = d1-(sigma*sqrt(t))

  return((S0*pnorm(d1))-(K*exp(-r*t)*pnorm(d2)))
}

EU_put_bs = function(S0 = 100, K = 100, r = 0.1, sigma = 0.25,
                                                              t = 1)
{
  d1 = (log(S0/K)+(r+((sigma)^2)/2)*t)/(sigma*sqrt(t))
  d2 = d1-(sigma*sqrt(t))

  return((K*exp(-r*t)*pnorm(-d2))-(S0*pnorm(-d1)))
}
```

Output

```
> EU_call_BS()
[1] 14.97579

> EU_put_BS()
[1] 5.459533
```

R Code for the implementation of the LSM algorithm using the first three Laguerre polynomials

```
lsm = function(n = 1000, d = 252, S0 = 1, K = 1.1, sigma = 0.2
                                                , r = 0.06, T = 1)
{
  # S0 = initial asset price
  # K = strike price
  # r = risk-free interest rate
  # sigma = volatility
  # T = maturity time
  # n = Number of paths simulated
  # d  = Number of time steps in the simulation

    s0 = S0/K
    dt = T/d
    z = rnorm(n)
    s.t = s0*exp((r - 0.5*sigma^2)*T + sigma*z*(T^0.5))
    s.t[(n+1):(2*n)] = s0*exp((r - 0.5*sigma^2)*T - sigma*z*(T^0.5))
    CC = pmax(1-s.t, 0)

    payoffeu = exp(-r*T)*(CC[1:n]+CC[(n+1):(2*n)])/2*K
    euprice = mean(payoffeu)


    for(k in (d-1):1)
    {
        z = rnorm(n)
        mean = (log(s0)+k*log(s.t[1:n]))/(K+1)
        vol = (k*dt/(K+1))^0.5*z
        s.t.1 = exp(mean+sigma*vol)
        mean = (log(s0)+k*log(s.t[(n+1):(2*n)]))/(k+1)
        s.t.1[(n+1):(2*n)] = exp(mean-sigma*vol)
        CE = pmax(1-s.t.1, 0)
        idx = (1:(2*n))[CE > 0]
        discountedCC = CC[idx]*exp(-r*dt)
        basis1 = exp(-s.t.1[idx]/2)
        basis2 = basis1*(1-s.t.1[idx])
        basis3 = basis1*(1-2*s.t.1[idx]+(s.t.1[idx]^2)/2)
        p = lm(discountedCC ~ basis1 + basis2 + basis3)$coefficients
        estimatedCC = p[1] + p[2]*basis1 + p[3]*basis2 + p[4]*basis3
        EF = rep(0, 2*n)
        EF[idx] = (CE[idx] > estimatedCC)
        CC = (EF == 0)* CC * exp(-r*dt) + (EF == 1)*CE
        s.t = s.t.1
    }
```

```
    payoff = exp(-r*dt)*(CC[1:n]+CC[(n+1):(2*n)])/2
    usprice = mean(payoff*K)
    error = 1.96*sd(payoff*K)/sqrt(n)
    earlyex = usprice - euprice

    data.frame(usprice, error, euprice)
}
```

Output

```
> lsm()
    usprice      error  euprice
1 0.5395756 0.01716987 0.100417
```

References

- Glasserman, P, 2003, *Monte Carlo Methods in Financial Engineering*, Cambridge University Press, Cambridge.
- Hull, J C, 2003, *Options, Futures and Other Derivatives*, 7^{th} edition, Pearson Prentice Hall, New Jersey.
- Longstaff, F A and Schwartz, E S, 2001, "Valuing American Options by Simulation: A Simple Least-Squares Approach", *Review of Financial Studies*, Vol. 14, No. 1, pp. 113-147.
- Hull, J C, 2002, *Fundamentals of Futures and Options Markets*, 4^{th} edition, Prentice Hall, New Jersey.
- Beketov, M A, 2013, *American options pricing with Least Squares Monte Carlo method*, CRAN.
- Coskan, C, 2006, *Pricing American Options by Simulation*, Boğaziçi University, Istanbul.
- Iacus, S M, 2011, *Option Pricing and Estimation of Financial Models with R*, 1^{st} edition, Wiley, New Jersey.