FIN 500: Intro to Finance

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Recall

Last lecture we discussed futures and options basics.

- Cash/Spot, and Forwards;
- Futures and Futures Curves;
- Swaps;
- Option Markets;
- Option Basics and Payoffs;
- Put-Call Parity;
- Embedded and Exotic Options.

Today we will talk about option valuation and (some) credit.

Option Valuation

Chapter 21, *A Quantitative Primer on Investments with R*

Introduction

- Today we will discuss option valuation.
- In particular, we will discuss:
  - Binomial Tree Option Pricing;
  - The Black-Scholes-Merton Option Model;
  - Divergences from Black-Scholes;
  - Recovering the Risk-Neutral Density; and,
  - Real Options.

Option Valuation History

- While options seem esoteric, pricing them was a recurring question.
- Early traders knew of put-call parity.
- Bachelier (1900) priced options for arithmetic Brownian motion.
  - Option pricing used Brownian motion before Einstein did!
- Bronzin (1908) developed similar formulas as Bachelier.
- Others tried, but the issue was the underlier expected return.
- Black and Scholes realized there was a replication argument.
  - That meant discounting could be done at the risk-free rate.
- Advent of handheld calculators made pricing feasible as well.

Simulation

- Easiest approach (from programmer perspective): *simulation*.
  - Use *pseudorandom numbers* to create *random variates*.
  - Use these to simulate final payoff (*Monte Carlo approach*).
- This approach lets us estimate the option value.
- Sometimes, value is based on unusual/extreme events.
- We can then use *importance sampling* for better estimate.
  - Create many rare/extreme scenarios;
  - Those yield better clarity on behavior in those scenarios.
  - Downweight those results to rare/extreme event likelihood.

Simulation: Random Process Choice

Biggest question: what random process is most accurate?

1. *Arithmetic Brownian motion*: $dS_t = \mu dt + \sigma_A dW_t$,

$$S_t - S_0 \sim N(\mu t, \sigma_A^2 t). \tag{1}$$

2. *Geometric Brownian motion*: $dS_t = \mu S_t dt + \sigma_G S_t dW_t$,

$$\log(S_t/S_0) \sim N((\mu - \sigma_G^2/2)t, \sigma_G^2 t). \tag{2}$$

3. *Ornstein-Uhlenbeck process*: $dS_t = \gamma(S_t - \bar{S})dt + \sigma_O dW_t$,

$$S_t \sim N\left(\bar{S} + \frac{S_0 - \bar{S}}{e^{\gamma t}}, \sigma_O^2 \frac{1 - e^{-2\gamma t}}{2\gamma}\right). \tag{3}$$

4. *Brownian bridge* for known B_T: $dB_t = \frac{\bar{B}_t - B_t}{1 - t/T} dt + \sigma_B dW_t$,

$$B_t - B_0 \sim N\left(\bar{B}_t, \sigma_B^2 \frac{t(T-t)}{T^2}\right), \quad \text{for } \bar{B}_t = B_0 + \frac{(B_T - B_0)t}{T}. \tag{4}$$

Simulation: Easy Computation

- With a process model, simulation is very easy.
- For example, can easily simulate call value:

1. $\{S_{iT}\}_{i \in \{1 \ldots m\}} \leftarrow$ generate m random variates.
2. $\bar{V}_T = \frac{1}{m} \sum_{i=1}^{m} (S_{iT} - K)^+$
3. $\hat{V} = \bar{V}_T e^{-r_f(T-t)}$.

Binomial Trees

- Another intuitive way to price options: *binomial tree*.
  - At discrete times, underlier moves up or down.
  - Up-then-down may (or may not) be same as down-then-up.
  - If same, may refer to tree as a *mesh* or *lattice*.
- At tree end, find option worth for different outcomes.
- Also find underlier – bond portfolio yielding option payout.
- Portfolio value must equal option value (by LOOP).
- Work backwards in time doing this until at current time.

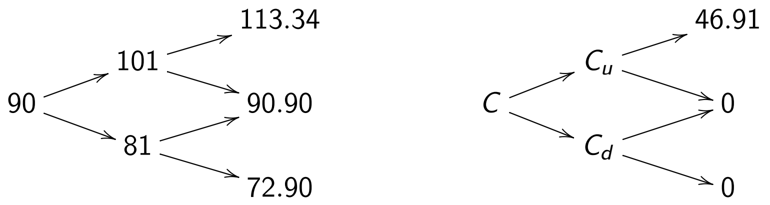
Binomial Tree: One-Step Example

- Assume the following stock, option payoffs in one year:

$$\begin{tikzcd}[row sep=small] & 101 \\ 90 \arrow[ur] \arrow[dr] & \\ & 81 \end{tikzcd} \qquad\qquad \begin{tikzcd}[row sep=small] & 21 \\ C \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd}$$

- If $r = 3\%$, what is C? Note: PV(\$81) $=$ \$78.64.
- Buy 1 share of stock for \$90, sell \$78.64 of bonds.
- In one year, sell stock and buy back bond:
  - If stock @ \$101, P&L $=$ \$101 $-$ \$81 $=$ \$20.
  - If stock @ \$81, P&L $=$ \$81 $-$ \$81 $=$ \$0.
- Cost of stock $-$ bond portfolio: \$90 $-$ \$78.64 $=$ \$11.36.
- To get worth of option, scale portfolio by $21/20$.
- Thus option is worth $\$11.36 \times \frac{21}{20} = \$11.93$.

Binomial Tree: Two-Step Example



- Extend to 2 years: If $r = 3\%$, what is C, C_u, and C_d?
- PV(\$72.90, \$81, \$90.90) = (\$70.78, \$78.64, \$88.25).
- At $t = 1$, up state: Buy stock for \$101, sell \$88.25 of bonds.
- In one year, sell stock and buy back bond:
  - If stock @ \$113.34, P&L = \$113.34 − \$90.90 = \$22.44.
  - If stock @ \$90.90, P&L = \$90.90 − \$90.90 = \$0.
- Cost of stock − bond portfolio: \$101 − \$88.25 = \$12.75.
- To get worth of C_u, scale portfolio by 46.91/22.44.
- Thus C_u is worth $\$12.75 \times \frac{46.91}{22.44} = \$26.65$.

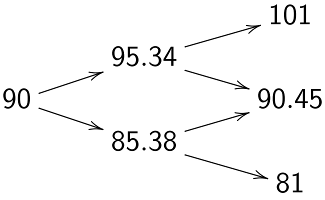
Binomial Tree: Two-Step Example Continued

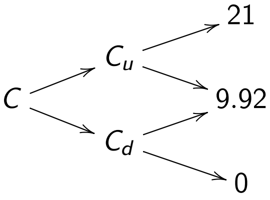
$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & 113.34 \\
 & 101 \arrow[ur] \arrow[dr] & \\
90 \arrow[ur] \arrow[dr] & & 90.90 \\
 & 81 \arrow[ur] \arrow[dr] & \\
 & & 72.90
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & 46.91 \\
 & 26.65 \arrow[ur] \arrow[dr] & \\
C \arrow[ur] \arrow[dr] & & 0 \\
 & C_d = 0 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

- Obviously, C_d worth \$0.
- If $r = 3\%$, what is C?
- Notice this is just a scaled version of One-Step Example.
  - Example I call paid off \$21 or \$0; worth \$11.93.
- Thus $C = \frac{26.65}{21} \times \$11.93 = \$15.14$.

Binomial Tree: Interpolated One-Step Example

- Suppose we interpolate the one-step example for more accuracy.
- Let up, down multipliers be u, d. $90u^2 = 101, 90d^2 = 81$.
- One-step example had strike of \$81, multiplier of 1.05.



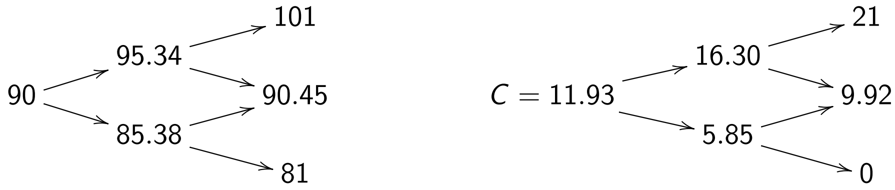


- Finding C_u, C_d is tricky. Solve simultaneous equations.

$$C_u = HSu + FV(B), \qquad C_d = HSd + FV(B); \tag{5}$$

$$\implies H = \frac{C_u - C_d}{Su - Sd}, \qquad B = \frac{uC_d - dC_u}{u - d} PV(1). \tag{6}$$

Binomial Tree: Interpolated One-Step Example Results



- With C_u, C_d we can work backward.
- Results of solving equations relate to replicating portfolio.
- H = *hedge ratio* = how much underlier to hold.
- B = bond holdings (financing).
- Option price = cost of replicating portfolio: $C = HS + B$.

Binomial Tree: Generalizing

- We can generalize what we just did to any binomial tree.
  - Recursively handle sub-trees; work backwards.
- Discrete dividends, path-dependency complicate matters.
  - Typically, we assume a dividend yield δ.
  - Path-dependencies? Price sub-trees or just simulate.
- What are sensible u, d (for geometric Brownian motion)?
  - Cox-Ross-Rubinstein: $u = e^{\sigma\sqrt{\Delta t}}, d = 1/u.$
  - Jarrow-Rudd: $u = e^{(r_f - \delta - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}}, d = e^{(r_f - \delta - \sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}}.$
  - Could even match skewness with Tian (1993) approach.
- All are asymptotically equivalent; but, some converge faster.

Drift and Diffusion

- Assume underlier follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{7}$$

- So $\log(S_T) - \log(S_t) \sim N(\mu(T-t), \sigma^2(T-t))$? No.
  - Geometric Brownian motion means correct drift by $-\sigma^2/2$.
- Then $\log(S_T) \sim N(\log(S_t) + (\mu - \sigma^2/2)(T-t), \sigma^2(T-t))$.
- Subtract $\log(K)$ from each side:

$$\log(\frac{S_T}{K}) \sim N\left(\log(\frac{S_t}{K}) + (\underbrace{\mu}_{=r_f} - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right). \tag{8}$$

- This give distribution of return beyond strike K.

The Risk-Neutral World

- If drift $\mu = r_f$, preceding is *risk-neutral density* ($\mathbb{Q}$-measure).
- *Physical probabilities*: actual probabilities of events, $\mathbb{P}$-measure.
- However, $\mathbb{P}$-measure requires pricing kernel $k > r_f$.
- Example of difference betweeen $\mathbb{P}$ and $\mathbb{Q}$ worlds:
  - Consider 1Y bond w/payoff \$1000 w.p. 0.9 and \$400 w.p. 0.1.
  - If 1Y risk-free rate is 5%, PV(expected cashflows) = \$895.24.
  - Implies YTM = 11.7%; but, market would demand risk premium.
  - If market demands 15%, price = \$940/1.15 = \$817.39.
  - $\mathbb{Q}$-measure is then given by:

$$P_{\mathbb{Q}}(\text{no default}) = \left\{ q : \frac{1000q + 400(1-q)}{1.05} = 817.39 \right\}. \tag{9}$$

$\implies q = 0.76$ and P(default) $= 0.24 > 0.1$.

  - Risk-neutral probabilities assign higher likelihood to bad outcomes.

Black-Scholes-Merton (1973) Model

- Then in the risk-neutral $\mathbb{Q}$ world:

$$P_Q(S_T > K) = \Phi\left(\frac{\ell n(\frac{S_t}{K}) + (r_f - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) = \Phi(d_2); \tag{10}$$

$$E_Q(S_T | S_T > K) = S_t\Phi\left(\frac{\ell n(\frac{S_t}{K}) + (r_f + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \tag{11}$$

$$= S_t\Phi(d_1).$$

- Thus the Black and Scholes (1973) and Merton (1973) model:

$$C_{t,T} = \underbrace{S_t\Phi(d_1)}_{E_Q(S_T|S_T>K)} - \underbrace{Ke^{-r_f(T-t)}\Phi(d_2)}_{PV(K)\cdot P_Q(S_T>K)}. \tag{12}$$

- Put price follows: $P_{t,T} = Ke^{-r_f(T-t)}\Phi(-d_2) - S_t\Phi(-d_1)$.
- Sort of a complicated DCF valuation.

Option Models for Other Underliers

- Can modify B-S-M for stocks with dividends, commodities.
- Merton (1973) model: dividend yield (like convenience yield).
  - Stock with dividends: dividend yield δ.
  - Commodities: $\delta = y - c =$ convenience yield $-$ storage cost.

$$C_{t,T} = S_t e^{-\delta(T-t)}\Phi(d_1) - Ke^{-r_f(T-t)}\Phi(d_2) \tag{13}$$
$$d_1 = \frac{\ell n(S_t/K) + (r_f - \delta + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

- Black (1976) model: Options on futures maturing at $T_1 > T$.

$$C_{t,T} = e^{-r_f(T-t)}[F_{t,T_1}\Phi(d_1) - K\Phi(d_2)] \tag{14}$$
$$d_1 = \frac{\ell n(F_{t,T_1}/K) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Sensitivities and Risk Factors

- We might trade options because we think they are mispriced.
- Or, we might be a market maker carrying options inventory.
- In these cases, hedging risk factors is critical.
- Analyze option price sensitivity to factors $\Rightarrow$ exposures.
- Use derivatives of option price V wrt factor (*Greeks*).
  - $\Delta = \frac{\partial V}{\partial S}$; often $\Phi(d_1)$ (underlier exposure);
  - $\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}$ (indicates frequency of adjusting Δ hedge);
  - vega $= \frac{\partial V}{\partial \sigma}$ (exposure to 1% σ change; and, no: *not* a Greek letter);
  - $\rho = \frac{\partial V}{\partial r_f}$ (related to DV01);
  - $\psi = \frac{\partial V}{\partial \delta}$ (dividend yield sensitivity);
  - $\theta = \frac{\partial V}{\partial t}$ (time decay; unhedgeable).

Black-Scholes-Merton: Complications

- These results work reasonably well for European options.
- And, we can use these results for many index options.
- However, prices are rarely log-normal; kurtosis usually > 3.
  - Important for commodities; B-S value should be lower bound.
- American options: must consider value of "early exercise."
  - Typically, must price American options on a lattice (tree).
- Replication may fail if there are price jumps.
- Also need to worry about *pin risk*:
  - Delta hedging changes price distribution near expiry.
  - Price may be attracted to/repelled from strike prices.
- Replication can also fail during large market moves/data delays.

Portfolio Insurance

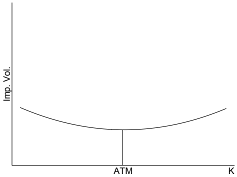
- We can dynamically replicate options with stock, bond.
- If we have a custom portfolio, probably no options listed on it.
- Can trade portfolio and bonds to synthesize an option.
- Create protective put for portfolio, aka *portfolio insurance*.
- Sheer brilliance! Or is it?
- Problem: dynamic replication has certain assumptions.
  - Know bond, stock prices, volatility; ability to transact quickly.
- In a market crash, these assumptions all break.
  - Prices jump; prices/fills delayed; some trading halted; volatility spikes.
- Post-1987 crash, portfolio insurance became less popular.

Implied Volatility

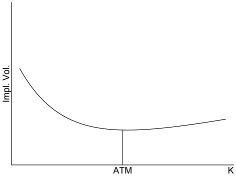
- What if we turn pricing upside-down?
  - Find volatilities which would imply market prices.
- Why would we do this?
  - Volatility is the one parameter whose value we never see.
  - But we see prices, so we examine *implied volatilities*.
- If we plot implied volatilities versus K, often see a curve.
  - *Volatility curve* partly due to using normal distribution.
  - True distribution has fatter tails; B-S-M is lower bound.
  - Also caused by leverage effects and risk aversion.

Volatility Curves

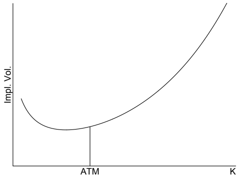
- Volatility curves adjust B-S-M for invalid assumptions.
- Curves may exhibit a *smile* or *smirk*.



Smile for stocks
pre-1987 crash



Smirk for stocks
post-1987 crash



Smirk for
commodities

- Note leverage, reverse leverage, risk aversion effects.
- Often fit volatility curve, then use it for pricing and inference.
- Commodities curve less stable; varies with supply elasticity.

Beyond the Volatility Curve

- Can create a dynamic volatility model which implies curve.
  - Heston (1993) *stochastic volatility* model:

$$dS_t = rS_t dt + \sqrt{\Sigma_t} S_t dW_t^S \tag{15}$$

$$d\Sigma_t = \underbrace{a(b - \Sigma_t)dt}_{\text{OU mean reversion}} + \gamma\sqrt{\Sigma_t} dW_t^\Sigma \tag{16}$$

$$dW_t^\Sigma \cdot dW_t^S = \rho \qquad (\rho > 0 \Rightarrow \text{leverage effect}). \tag{17}$$

- Can also plot implied volatility vs. K, T to get *volatility surface*.
  - Surface profile typically flattens as $T \uparrow$.
  - If we think surface is roughly stationary, can try to fit it.
  - May want fit to have smooth first, second derivatives.

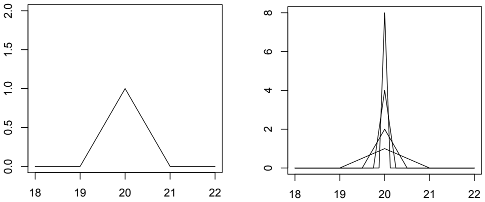
Extracting Risk-Neutral Densities

- Second derivative of option prices wrt K yields underlier density.
- Create a butterfly w/strikes at $K-h, K, K+h$.
- Triangle area: $\frac{1}{2} \cdot 2h \cdot h = h^2$. Normalize: hold $1/h^2$ butterflies.
- If we write out the butterfly portfolio value, $V_{B,h}$:

$$\begin{aligned} V_{B,h} &= \frac{C_{K-h,t,T} - 2C_{K,t,T} + C_{K+h,t,T}}{h^2} && (18) \\ &= \frac{P_{K-h,t,T} - 2P_{K,t,T} + P_{K+h,t,T}}{h^2}, && (19) \end{aligned}$$

- Centered difference formula: $V_{B,h} \approx \frac{\partial^2}{\partial K^2}$ of $C_{K,t,T}$ or $P_{K,t,T}$.
- Take limit to get second derivative:

$$\frac{\partial^2 C_{K,t,T}}{\partial K^2} = \frac{\partial^2 P_{K,t,T}}{\partial K^2} = \lim_{h \downarrow 0} V_{B,h}. \quad (20)$$



- Take limit as $h \downarrow 0$; get Dirac delta function δ.
- Finding the butterfly price = finding its expected value:

$$V_{B,h} = e^{-r_f(T-t)} \int_0^{\infty} f(S_T)\delta(S_T - K)dS_T = e^{-r_f(T-t)} f(K), \quad (21)$$

where f = risk-neutral density of S_T.

- Thus finding $\frac{\partial^2 C_{K,t,T}}{\partial K^2} = \frac{\partial^2 P_{K,t,T}}{\partial K^2}$ for all K gives us $fe^{-r_f(T-t)}$.

Extracting Risk-Neutral Densities: Improvements

- This can be tough if we lack option prices for all strikes.
- To fix that, we use vol curve to find all option prices.
- Then use those prices to extract risk-neutral density.
- Great — if you are a $\mathbb{Q}$-quant (arbitrage pricers).
- What about $\mathbb{P}$-quants (alpha believers)?
- Can we get physical density $\mathbb{P}$ from risk-neutral $\mathbb{Q}$?
- Not easily. Often must invert with an uncertain quantity.

Real Options

- *Real options* are option-like decisions we can make.
- May be embedded in contracts; typically, give possibility of:
  - Starting, stopping, canceling, or abandoning;
  - Increasing, decreasing, lengthening, or shortening;
  - Switching assets.
- Examples of real options:
  - Reactivate/lease a gold mine?
  - Store inventory for busy season?
  - Start and run "peaker" plant to create electricity?
  - Increase factory capacity to create more high-margin product?
  - Purchase more fuel in later months at agreed price?

Real Options: Valuation

- Valuing real options is usually done in one of many ways:
  - Comparative analysis;
  - Discounted cash flow (DCF) analysis;
  - Monte Carlo simulation analysis;
  - Using option models for similar options; or,
  - Stochastic optimization.
- We'll examine two of these approaches.

DCF Analysis of Real Options

- DCF analysis is common method for valuing decision/project:
  - Estimate/predict future cashflows; and,
  - Compute net present value (NPV) of those cashflows.
- The DCF approach neglects two key questions:
  - What rate to use for DCF analysis? (CAPM-based? WACC?)
  - What horizon do we analyze to?
- This does not even begin to consider term-structure of rates.
- Analysis/investment horizon is equally thorny:
  - What if horizon is random or affected by option exercise?
- DCF analysis also neglects current market prices, volatility.

Analyzing Real Options Using Option Models

- Can also use Black-Scholes-Merton and related models.
  - This is called a *real options approach*[2]
- A real options approach requires regularity conditions:
  1. Option must have starting, ending dates $t_0 < T \leq \infty$.
  2. Risk factors $S_1, S_2, \ldots$ must be clearly identified.
  3. $S_1, S_2, \ldots$ must be continuously traded (for hedging).
  4. Must have stochastic process models for $S_1, S_2, \ldots$.
  5. Must know exact option form (*e.g.* exercise type, payoff).
  6. Option cannot violate market completeness, measurability.
- Assumptions 3 and 4 are most critical.
- Assumption 5 rarely holds; try various models and guess/average.

[2] Real options and a real options approach are different. Confusing, eh?

Example: Value of Gold Mine Lease

- Consider leasing a gold mine for ten years.
  - Mine can produce 10,000 oz./year; extraction costs $1000/oz.
  - Cash gold @ $1300/oz; futures $\geq$1Y @ $1200/oz; vol. = 30%.
  - WACC = 8%; risk-free $r_f = 3\%$; storage $c = y$ conv. yld.
- DCF: Discount E(cashflows) ($300/oz, then $200/oz.) at WACC.
- Real options approach: Annuity of annual/quarterly options.[3]
- Simulate real option values:
  - Geometric Brownian motion (GBM) or Ornstein-Uhlenbeck (OU)?

Method	DCF	GBM real opt.	OU real opt.	Sim GBM	Sim OU
Annually	$14.3 mn	$37.7 mn	$23.9 mn	$41.7 mn	$20.8 mn
Quarterly	$15.1 mn	$33.3 mn	$20.3 mn	$60.5 mn	$21.9 mn

- Production contraints: use tree or stochastic optimization.
- If gold $\leq$ $1000/oz., DCF valuation deeply flawed.

[3] I interpolated between cash and futures for 3M–9M.

Credit

Chapter 22, *A Quantitative Primer on Investments with R*

Introduction

- Today we will discuss credit.
- In particular, we will discuss:
  - Basics, Credit Measures, and Issues;
  - Credit Derivatives;
  - The Merton Model of the Firm;
  - Structural Credit Models;
  - Accounting-based Credit Models;
  - Default Intensity Credit Models; and,
  - Credit Instruments for Modeling and Trade.

Credit Basics

- Basic idea of credit: entity's ability to borrow money/value.
- *Credit risk*: risk of *default* (missing promised cashflow/repayment).
- Credit is both absolute and relative.
  - Entities with better/worse credit pay less/more to borrow.
  - When credit is *loose/tight*, everyone pays less/more.
- Credit is *procyclical*: looser in expansion, tighter in recession.
- Duality of lender perceptions leads to two equilibria:
  - If troubles look temporary: lend more, market self-heals
  - If troubles look long-lasting: lend less, crisis ensues.
- If lenders refuse to lend, we may end up in a *credit crunch*.

Measures of Credit and Yields

- There are many ways we can measure/compare creditworthiness.
- Can use yields, ratings, default probabilities, credit scores.
- Yields are commonly-used: higher yield means more risk.

$$\text{Promised YTM} = \text{Expected yield} + \text{Risk penalty.} \tag{22}$$

- We sometimes watch *credit spreads* = yield default premia:
  - Default premium = Promised YTM − Same-tenor govt YTM.
  - TED spread separates peaceful (*TED* <48bp), crises (*TED* >48bp).
  - TED = 3MUSDLIBOR − 3M US T-bills YTM
- Lending rates often quoted at idiosyncratic spread, *e.g.* "LIBOR+110"

Measures of Credit: Ratings

- *Rating agencies* try to estimate credit risk of bonds, issuers.
- Four agencies recognized by ECB for assessing borrower collateral.
  - By size: Standard & Poor's (S&P), Moody's, Fitch, DBRS.
- Range of (long-term debt) credit ratings:

	Investment Grade		Below Investment Grade	
Agency	Highest	High	Speculative	Very Poor
S&P	AAA,AA+→AA-	A+→BBB-	BB+→B-	CCC...D
Moody's	Aaa,Aa1→Aa3	A1→Baa3	Ba1→B3	Caa1...C
Fitch	AAA,AA+→AA-	A+→BBB-	BB+→B-	CCC,DDD,DD,D
DBRS	AAA,AAH→AAL	AH→BBBL	BBH→BL	CCCH→CL

- Short-term debt ratings are simpler, coarser.

Measures of Credit: Default Probabilities

- Default probabilities (PDs): not easily mapped from ratings.
  - Ratings are relative: ratings' PDs vary over business cycle.
- This is $\mathbb{P}$-measure; discount with (risky) kernel k.
- Rating-implied PDs also vary for sovereign vs corporate bonds.
  - Sovereign 5-year PDs: 0% (Aaa) to 1.3% (Baa) to 38% (Caa–C).
  - Corporate 5-year PDs: 0.1% (Aaa) to 3.7% (Baa) to 36% (Caa–C).
- Complication: *seasoning* = default rate falls after a few years.

Credit Scoring

- Bank lending may rely on other information, recourse:
  - Bank may have borrower as a banking/payments customer.
  - Bank may require more restrictive covenants (*e.g.* collateral).
  - Repayment is more likely to be amortized.
- Banks often use *credit scores* for lending decisions.
- Credit scoring models: typically quantitative, more objective.
- Credit scoring tends to increase lending to small businesses.
  - Some research suggests credit scoring reduces adverse selection.

Credit Issues

- Credit analysis may be complicated by issues/portfolio effects.
- *Absolute priority*: repayment order in liquidation (by *seniority*).
  - Most bonds say new bond issues cannot be higher priority
  - Priority: derivatives→bank loans→bonds by seniority→equity.
  - Covenants, info asymmetries make funding sources differ.
- Borrowers who share risk factors may have *correlated defaults*.
  - These are difficult to simulate; copulas are not enough.
- Defaults, recovery rates may correlate: more defaults=lower recovery.
- Mandate effects: safe asset holdings affect capital adequacy.

Credit Derivatives

- *Credit derivatives* protect buyer from default.
- *Asset swap*: swap bond for most of face+buyback option.
  - This gives default protection and repo-like financing.
- *Credit default swap*: in default, exchange bond for face.
- Credit default swaps also exist on indices:
  - CDX (N. Am., Em. Mkt) and iTraxx (Europe, Asia).

The Value of Firm Equity

- We saw that Bachelier, Bronzin modeled stock prices as:

$$dS_t = \mu dt + \sigma dW_t. \tag{23}$$

- Also noted a flaw: stocks could go below 0. Is it a flaw?
- Limited liability: Equity = (Assets - Liabilities)$^+$.
  - Merton idea: Equity is a call option on the value of the firm.
- Model firm w/assets A_t, liabilities L_t, asset vol σ_A:

$$dA_t = \mu A_t dt + \sigma_A A_t dW_t, \tag{24}$$

$$S_t = (A_t - L_t)^+. \tag{25}$$

The Value of Firm Equity: Implications

- Equity as a call on firm assets explains some stylized facts:
  - Some asymmetry (negative skewness) of equity returns.
  - *Leverage effect*: volatility rises when price falls.
- Should we estimate volatilities for assets instead of equities?
- This led to a whole range of capital-structure-based models.

Structural Credit Models: Merton, Moody's KMV

- Structural credit models use model of firm capital structure.
- Merton model: one zero-coupon bond, maturing at τ.
- Callable bonds? Consider all permutations of early calls.
- Moody's KMV models: equate equity and option value, volatility:

$$S_t = A_t e^{-\delta(\tau-t)}\Phi(d_1) - L_t e^{-r_f(\tau-t)}\Phi(d_2), \tag{26}$$

$$\sigma_S S_t = \sigma_A A_t \Phi(d_1), \tag{27}$$

where d_1, d_2 are as in Black-Scholes-Merton for asset vol σ_A.
- Moody's KMV infers $\hat{L}_t$; may be unknown/shadow liabilities.

Structural Models: Samuelson-McKean

- Problem: When should equity call option expire? (Never, probably.)
- Samuelson and McKean considered a perpetual American option.

$$A_t^* = \frac{L_t \eta}{\eta - 1} \quad \text{(early exercise boundary)} \tag{28}$$

$$\eta = \frac{r_f - \delta + \frac{\sigma_A^2}{2} + \sqrt{(\delta - r_f - \frac{\sigma_A^2}{2})^2 + 2\delta\sigma_A^2}}{\sigma_A^2}, \tag{29}$$

$$S_t = C_{L_t,t,\infty} = \begin{cases} (A_t^* - L_t)(\frac{A_t}{A_t^*})^\eta & A_t \leq A_t^*, \\ A_t - L_t & A_t > A_t^*. \end{cases} \tag{30}$$

- Equity might not seem to be an American option, however:
  - Shareholders can choose to buyout lenders at any time.

Structural Models: Takeaway

- We could (and do) build much more complicated firm models.
- Key outcome of structural models: *distance to default*.

$$d_{\text{default}} = \frac{\log(A_t/L_t)}{\sigma_A} \quad \text{or,} \tag{31}$$

$$d^{\mathbb{Q}}_{\text{default}} = d_{2,\text{Merton}} = \frac{\log(A_t/L_t) + (r_f - \delta - \frac{\sigma_A^2}{2})(T-t)}{\sigma_A\sqrt{T-t}}. \tag{32}$$

- Distance to default is used in other credit/factor models.
- Could also use equity options and vol curve to infer σ_A, L_t.

Accounting-Based Models: Ratios

- Accounting-based models look at "structural-ish" measures.
- Often use accounting ratios: how many times can X pay for Y?
- *Coverage ratios*: measure earnings vs fixed costs.
    - *Times-interest-earned ratio* = EBIT/interest.
    - *Fixed-charge coverage ratio* = $\frac{\text{EBIT}}{\text{interest}+\text{leases}+\text{sinking funds}}$.
- *Liquidity ratios*: measure bills vs liquid assets.
    - *Current ratio* = current assets/current liabilities.
    - *Quick ratio* = $\frac{\text{current assets}-\text{inventory}}{\text{current liabilities}}$.
- *Cashflow-debt ratio* measures need for short-term funding.
- *Leverage ratios*: indicate excessive debt (debt- or assets-to-equity).
- *Profitability ratios*: measure returns vs capital.
    - *Return on assets* = ROA = EBIT/total assets.
    - *Return on equity* = ROE = net income/equity.

Accounting-Based Models: Ratio Models

- Beaver first put ratios in linear models; used market-wide coefficients.
- *Altman's Z-score*: $Z > 3$ is safe, $Z < 1.8$ is distressed.

$$\begin{aligned} Z = & \overbrace{1.2\frac{\text{working capital}}{\text{total assets}} + 1.4\frac{\text{retained earnings}}{\text{total assets}}}^{\text{liquidity ratios}} \\ & + \underbrace{3.3 ROA + \frac{\text{sales}}{\text{total assets}}}_{\text{profitability ratios}} + \underbrace{0.6\frac{\text{equity value}}{\text{total liabilities}}}_{\text{(inverse) leverage}} . \end{aligned} \tag{33}$$

- *Ohlsen's O-score*:

$$\begin{aligned} O = & -1.3 - \text{size } (0.4 \text{ assets/GNP}) - \text{profitability index} \\ & - \text{liquidity index} + \text{leverage index} \end{aligned} \tag{34}$$

$$P(\text{Bankruptcy}) = \frac{1}{1 + e^{-O}}. \tag{35}$$

- *Piotrowski's F-score*: uses 9 indicator variables.
  - 4 profitability, 3 financial performance, 2 operating efficiency.

Default Intensity Models

- *Default intensity models* estimate default rate/P(default) (aka PD).
- Idea: model *time to default* which may be *censored* (unobserved).
  - Left censoring: people too risky do not get loans.
  - Right censoring: loan may be repaid before default occurs.
- *Survival analysis* used for handling censored data.
- Often model time to default as Exp(λ) or Gamma(m, λ).
  - Gamma better if multiple risk sources/default after multiple failures.
- Then, we build models for conditional λ, m.
  - Compound exponential/gamma model or gamma GLM.
- Recent work uses Hawkes processes to get correlated defaults.

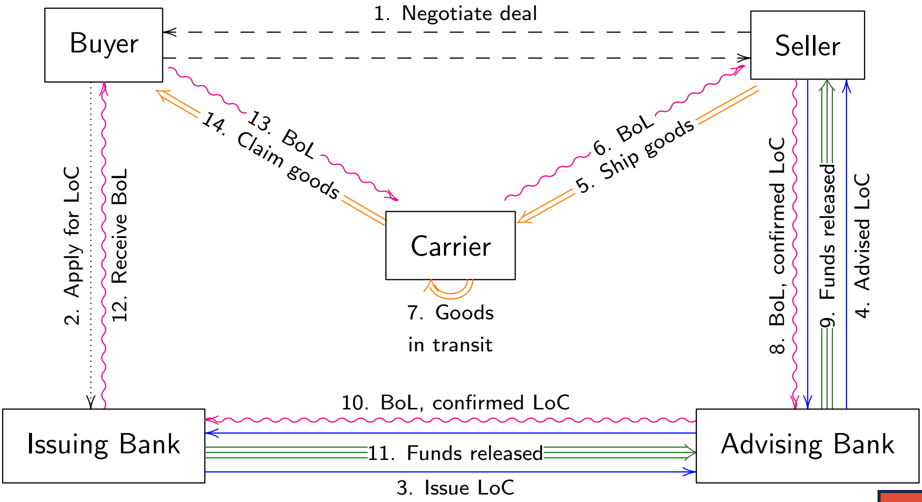
Market-Based Credit Models

- Can also look at what market prices of credit derivatives imply.
- Market prices: especially nice since they are forward-looking.
- Credit default swap (CDS) prices are especially informative.
  - CDSs yield nearly direct measures of risk-neutral PDs.
- Some research suggests stock returns may also be informative.
- And yet... we can model CDS prices.
- So perhaps some combination of methods would work best?
  - Research suggests: Yes; combine different approaches.

Letters of Credit

- Finally: useful to see how credit works in arduous situations.
- In international trade, parties often use *letters of credit* (LoCs, LCs).
  - LoC is a promise by issuer's bank to pay seller's banks for goods.
  - Complex set of handshakes, sign-offs to reduce risk.
  - *Bill of lading* (BoL) is claim for goods at end of transport.
- LoCs, BoLs used even w/very low-credit counterparties.
- A diagram is informative to show interactions.

Letters of Credit: Diagram



N.B. 8–9–10–11 may proceed as 8–10–11–9.

The Road Ahead

We have covered option valuation and credit.
On to investment management firms next time!

- All Together Now: Investment Firms.