Coordinate matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

Tetrahedron volume

$$V = \frac{1}{6}\mathrm{abs}(|A|)$$

Affine transformation matrix

$$F = \frac{1}{|A|}A$$

Notice $|A|$ here doesn't have $abs()$, it is the determinant of A. s.t.

$$X = |A|\, FZ$$

where

$$X = \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} \qquad , \qquad Z = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix}$$

Unit tangent vector

$$\hat{t}_{ij} = \frac{1}{l_{ij}} \begin{pmatrix} (x_j - x_i)\hat{x} \\ (y_j - y_i)\hat{y} \\ (z_j - z_i)\hat{z} \end{pmatrix}$$

where l_{ij} is the length of the edge.
Define

$$\begin{aligned} (i, i_1, i_2, i_3) =& (1\,2\,3\,4) \\ & (2\,3\,4\,1) \\ & (3\,4\,1\,2) \\ & (4\,1\,2\,3) \end{aligned}$$

Unit normal vector:
Step I:

$$\hat{n}_i = \frac{t_{i_1 i_2} \times t_{i_1 i_3}}{|t_{i_1 i_2} \times t_{i_1 i_3}|}$$

Step II:

$$\hat{n}_i = \mathrm{sign}\left(\hat{n}_i \cdot \hat{t}_{ii_1}\right)\hat{n}_i$$

Tetrahedron integration

$$\int_V \zeta_1^a \zeta_2^b \zeta_3^c \zeta_4^d \,\mathrm{d}V = \frac{a!\,b!\,c!\,d!}{(a+b+c+d+3)!} 6V$$

Twice the opposite triangle area

$$S_i = \left| l_{i_1 i_2} \hat{t}_{i_1 i_2} \times l_{i_1 i_3} \hat{t}_{i_1 i_3} \right|$$

Area matrix

$$S = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & S_3 & \\ & & & S_4 \end{pmatrix}$$

Unit vectors

$$\hat{\hat{X}} = \begin{pmatrix} \hat{0} \\ \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \hat{\hat{N}} = \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \\ \hat{n}_4 \end{pmatrix}$$

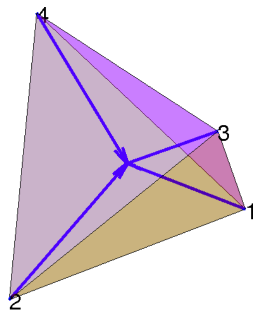


Figure 1: Outward pointing normal vectors

Normal vector outward pointing correction:
Type I tetrahedron

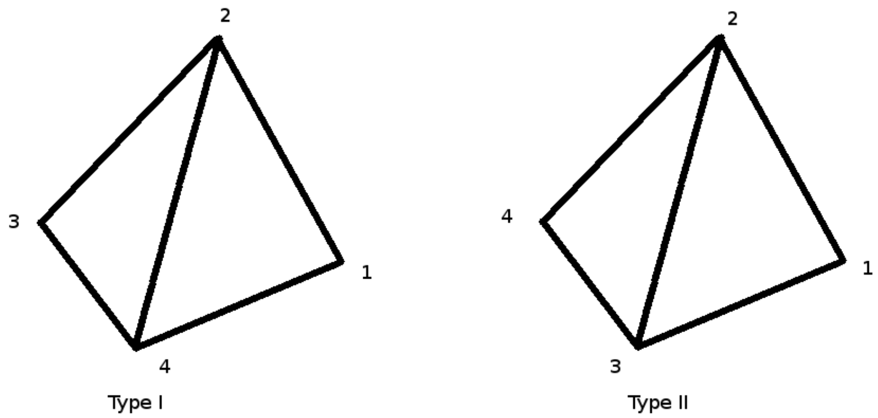
$$\gamma = 1$$

Type II tetrahedron

$$\gamma = -1$$

Criterion

$$\begin{array}{ccccc} det(A) > 0 & \leftrightarrow & (t_{12} \times t_{13}) \cdot t_{14} > 0 & \rightarrow & \text{type I} \\ det(A) < 0 & \leftrightarrow & (t_{12} \times t_{13}) \cdot t_{14} < 0 & \rightarrow & \text{type II} \end{array}$$



Conversion

$$\hat{\hat{N}} = -\gamma S^{-1} F^{-1} \hat{\hat{X}}$$
$$\hat{\hat{X}} = -\gamma F S \hat{\hat{N}}$$

Nabla

$$\nabla \zeta_i = -\frac{S_i}{6V} \hat{n}_i$$
$$\nabla \times \nabla \zeta_i = 0$$

where S_i is twice the triangle area.
Zeroth-order basis vector

$$\hat{N}_{ij} = l_{ij} \left(\zeta_i \nabla \zeta_j - \zeta_j \nabla \zeta_i \right)$$

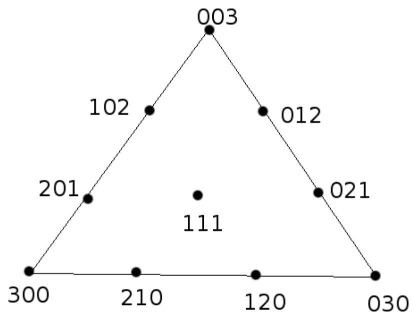


Figure 2: *Split code: n=3, split into three segments; m=3, three bits (2D). Put n numbers into m bits. Suitable for both global tetrahedron index and sub tetrahedron index.*

```
% all combinations that put n numbers into m bits
%   e.g. n = 3, m = 3, returns
%     003 300 030 102 012 201 111 021 210 120
function bits = codegen(n, m)
    if n == 0,
        bits = zeros(1,m);
        return;
    end

    if m == 1,
        bits = n;
        return;
    end

    bits = [];
    for ii = 0 : n,     % number put in first bit
        subcode = codegen(n-ii, m-1);
        num = size(subcode,1);
        combcode = [ii * ones(num,1), subcode];
        bits = [bits; combcode];
    end

    return;
end
```

Figure 3: *Recursive split code generation, Matlab*

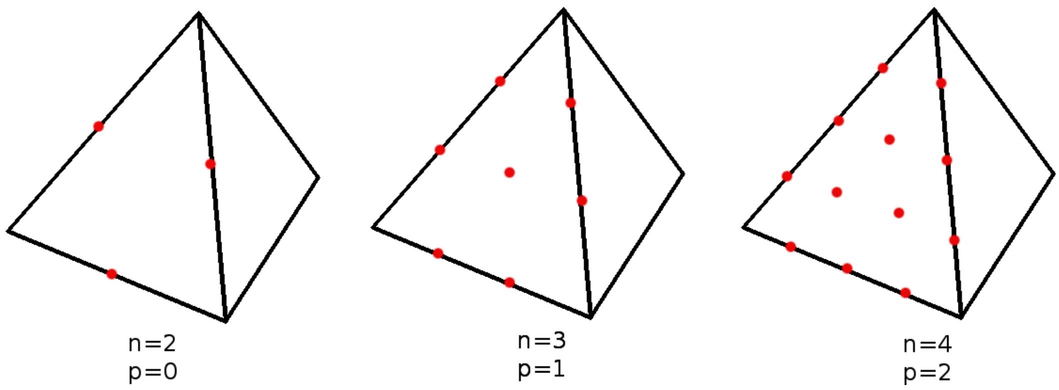


Figure 4: Interpolatory points in a global tetrahedron. Notice NOT *all points are used for generating Lagrangian polynomials for a given edge. Only points associated with a sub tetrahedron are used for interpolation.*

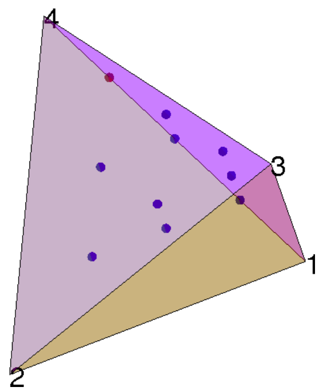


Figure 5: Interpolatory points in global tetrahedron along edge 14, n=4 (p=2), m=4 (3D)

which satisfies

$$\hat{t}_{ij} \cdot \hat{N}_{ij} = 1$$

Alert $\hat{N}_{ij}$ does not normalize with itself !
Define order of accuracy p, and global split segments (refer to split code) n, then

$$n = p + 2$$

Relation between global tetrahedron split segments and sub tetrahedron split segments

$$n' = n - 2$$

and

$$n' = p$$

Natural coordinates

$$Z = \frac{1}{n}(\text{split\_code})$$

Differential operators in x-y-z coordinates

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

$$\nabla \cdot = \begin{pmatrix} \partial_x & \partial_y & \partial_z \end{pmatrix}$$

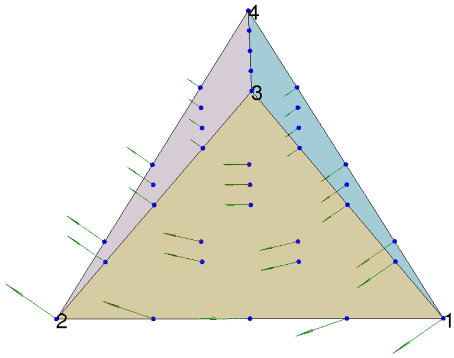


Figure 6: $\hat{N}_{12}$

$$\nabla \times = \begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}$$

$$\nabla \cdot \hat{N}_{ij} = 0$$

$$\nabla \times \hat{N}_{ij} = 2l_{ij} \nabla \zeta_i \times \nabla \zeta_j$$

For a global tetrahedron, define the split code of an interpolation point to be

$$(i,\, j,\, k,\, l)$$

where

$$i + j + k + l = n$$

For the sub tetrahedron, define split code to be

$$(i',\, j',\, k',\, l')$$

where

$$i' + j' + k' + l' = n'$$

The interpolation polynomial for point (i', j', k', l') is

$$\Phi^{n'}_{i'j'k'l'} \equiv P^{n'}_{i'}(\zeta'_1) P^{n'}_{j'}(\zeta'_2) P^{n'}_{k'}(\zeta'_3) P^{n'}_{l'}(\zeta'_4)$$

in which

$$P^{n'}_{i'}(\zeta') = \frac{1}{i'!} \prod_{s=0}^{i'-1} (n'\zeta' - s)$$

$$P^{n'}_{0}(\zeta) = 1$$

Φ is normalized in the sense that

$$\Phi^{n'}_{i'j'k'l'}\left(\frac{i'}{n'}, \frac{j'}{n'}, \frac{k'}{n'}, \frac{l'}{n'}\right) = 1$$

$$\Phi^{n'}_{i'j'k'l'}(\text{otherwise}) = 0$$

Although $n' = 0$ is not used, we still define its polynomial

$$P^0_0 = 1$$

$$\nabla P^0_0 = 0$$

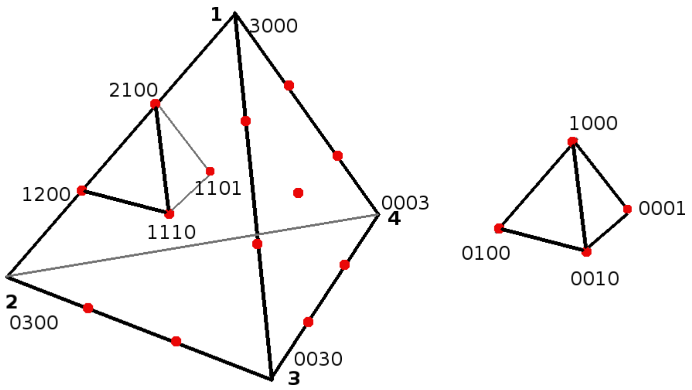


Figure 7: $n' = 1$, $p = 1$, $n = 3$. *Sub tetrahedron along edge 12. The left (global) tetrahedron uses global split code, the right (sub) tetrahedron uses sub split code.*

$n' = 1$ polynomials

$$\begin{aligned} P_0^1(\zeta) &= 1 \\ P_1^1(\zeta) &= \zeta \end{aligned}$$

$$\begin{aligned} \nabla P_0^1 &= 0 \\ \nabla P_1^1 &= 1\nabla\zeta \end{aligned}$$

$n' = 2$ polynomials

$$\begin{aligned} P_0^2 &= 1 \\ P_1^2 &= 2\zeta \\ P_2^2 &= 2\zeta^2 - \zeta \end{aligned}$$

$$\begin{aligned} \nabla P_0^2 &= 0 \\ \nabla P_1^2 &= 2\nabla\zeta \\ \nabla P_2^2 &= (4\zeta - 1)\nabla\zeta \end{aligned}$$

$n' = 3$ polynomials

$$\begin{aligned} P_0^3(\zeta) &= 1 \\ P_1^3(\zeta) &= 3\zeta \\ P_2^3(\zeta) &= \frac{9}{2}\zeta^2 - \frac{3}{2}\zeta \\ P_3^3(\zeta) &= \frac{9}{2}\zeta^3 - \frac{9}{2}\zeta^2 + \zeta \end{aligned}$$

$$\begin{aligned} \nabla P_0^3(\zeta) &= 0 &&= Q_0^3\nabla\zeta \\ \nabla P_1^3(\zeta) &= 3\nabla\zeta &&= Q_1^3\nabla\zeta \\ \nabla P_2^3(\zeta) &= \left(9\zeta - \frac{3}{2}\right)\nabla\zeta &&= Q_2^3\nabla\zeta \\ \nabla P_3^3(\zeta) &= \left(\frac{27}{2}\zeta^2 - 9\zeta + 1\right)\nabla\zeta &&= Q_3^3\nabla\zeta \end{aligned}$$

$n' = 4$ polynomials

$$
\begin{aligned}
P_0^4(\zeta) &= 1 \\
P_1^4(\zeta) &= 4\zeta \\
P_2^4(\zeta) &= 8\zeta^2 - 2\zeta \\
P_3^4(\zeta) &= \frac{32}{3}\zeta^3 - 8\zeta^2 + \frac{4}{3}\zeta \\
P_4^4(\zeta) &= \frac{32}{3}\zeta^4 - 16\zeta^3 + \frac{22}{3}\zeta^2 - \zeta
\end{aligned}
$$

$$
\begin{aligned}
\nabla P_0^4(\zeta) &= 0 \quad = Q_0^4 \nabla\zeta \\
\nabla P_1^4(\zeta) &= 4\nabla\zeta \quad = Q_1^4 \nabla\zeta \\
\nabla P_2^4(\zeta) &= (16\zeta - 2)\,\nabla\zeta \quad = Q_2^4 \nabla\zeta \\
\nabla P_3^4(\zeta) &= \left(32\zeta^2 - 16\zeta + \frac{4}{3}\right)\nabla\zeta \quad = Q_3^4 \nabla\zeta \\
\nabla P_4^4(\zeta) &= \left(\frac{128}{3}\zeta^3 - 48\zeta^2 + \frac{44}{3}\zeta - 1\right)\nabla\zeta \quad = Q_4^4 \nabla\zeta
\end{aligned}
$$

$\nabla\Phi_{ijkl}^n$

$$
\begin{aligned}
\nabla\Phi_{ijkl}^n = {} & Q_i^n P_j^n P_k^n P_l^n \nabla\zeta_i \\
& + P_i^n Q_j^n P_k^n P_l^n \nabla\zeta_j \\
& + P_i^n P_j^n Q_k^n P_l^n \nabla\zeta_k \\
& + P_i^n P_j^n P_k^n Q_l^n \nabla\zeta_l
\end{aligned}
$$

Conversion between global tetrahedron and sub tetrahedron.
For example, basis of a sub tetrahedron along *edge 12*

$$
P_{i'}^p(\zeta_1') P_{j'}^p(\zeta_2') P_{k'}^p(\zeta_3') P_{l'}^p(\zeta_4') \hat{N}_{12}
$$

Notation: from now on we denote a basis by b_I, where I is the global index for the basis.
Range

$$
i',\, j',\, k',\, l' = 0, \cdots, p
$$

Index conversion

$$
\begin{aligned}
i' &= i - 1 \\
j' &= j - 1 \\
k' &= k \\
l' &= l
\end{aligned}
$$

Coordinate conversion (change of variables)

$$
\begin{aligned}
\zeta_1' &= \frac{p+2}{p}\zeta_1 - \frac{1}{p} \\
\zeta_2' &= \frac{p+2}{p}\zeta_2 - \frac{1}{p} \\
\zeta_3' &= \frac{p+2}{p}\zeta_3 \\
\zeta_4' &= \frac{p+2}{p}\zeta_4
\end{aligned}
$$

There are

$$
(p+1)(p+3)(p+4)/2
$$

basis vectors associated with (but not belong to) an element. They includes

$$6 \cdot (p+1)$$

basis vectors associated with edges;

$$4 \cdot p(p+1)$$

basis vectors associated with faces; and

$$\frac{1}{2} \cdot (p+1)(p+3)(p+4)$$

basis vectors associated with interior points.
It is verified

- An edge associated basis vector is shared by all elements that have that edge.
- A face associated basis vector is shared by two elements that have that face.
- An interior basis vector is owned by a single element.

The first two items ensures the tangential continuity. However, the same basis vector can represent different normal components in different elements that sharing it.

1 Finite Element Formulation

Boundary condition on S_1 (electrical conducting, Dirichlet condition)

$$\hat{n} \times \vec{E} = 0$$

Boundary condition on S_2 (third kind, natural condition)

$$\frac{1}{\mu_r}\hat{n} \times (\nabla \times \vec{E}) + \gamma \hat{n} \times (\hat{n} \times \vec{E}) = \vec{U}$$

Differential Equation

$$\nabla \times \left(\frac{1}{\mu_r} \nabla \times \vec{E} \right) - k_0^2 \epsilon_r \vec{E} = -jk_0 Z_0 \vec{J}$$

Functional

$$\begin{aligned} F(\vec{E}) = \frac{1}{2} \int_V & \left[\underbrace{\frac{1}{\mu_r} \left(\nabla \times \vec{E} \right) \cdot \left(\nabla \times \vec{E} \right)}_{1} - \underbrace{k_0^2 \epsilon_r \vec{E} \cdot \vec{E}}_{2} \right] \mathrm{d}V \\ & + \underbrace{\int_{S_2} \left[\frac{\gamma}{2} \left(\hat{n} \times \vec{E} \right) \cdot \left(\hat{n} \times \vec{E} \right) + \vec{E} \cdot \vec{U} \right] \mathrm{d}S}_{4} \\ & + jk_0 Z_0 \underbrace{\int_V \vec{E} \cdot \vec{J} \mathrm{d}V}_{3} \end{aligned}$$

Notices:

- Code should be carried out for complex number operations.
- μ_r, ϵ_r, γ, $\vec{J}$ are element-wise constant.

Enforce Dirichlet boundary condition: for the basis associating S_1 surfaces' edges, simply set it to zero. (not verified yet)
Elemental matrix from functional part 1:

$$\frac{1}{\mu_r^e}\int_{V^e}\left(\nabla\times\vec{b}_I\right)\cdot\left(\nabla\times\vec{b}_J\right)\,\mathrm{d}V$$

Elemental matrix from functional part 2:

$$k_0^2\epsilon_r^e\int_{V^e}\vec{b}_I\cdot\vec{b}_J\,\mathrm{d}V$$

Elemental vector from functional part 3:

$$\vec{J}^e\cdot\int_{V^e}\vec{b}_I\,\mathrm{d}V$$

Not consider part 4 yet.