ON FINITE ELEMENT INTEGRATION IN NATURAL CO-ORDINATES

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SUMMARY

Integration formulas for polynomial expressions in terms of natural co-ordinates are extensively quoted without proof in standard references on finite element theory. A derivation of these relations is presented and their application is extended to non-integral exponents.

DISCUSSION

Natural co-ordinate systems are widely used in the formulation of finite element models. One of the advantages of such co-ordinate systems is the existence of the following integration formulae

$$\int_L \xi_1^a \xi_2^b \,\mathrm{d}l = \frac{a!\,b!}{(a+b+1)!}L \tag{1}$$

$$\int_A \xi_1^a \xi_2^b \xi_3^c \,\mathrm{d}A = \frac{a!\,b!\,c!}{(a+b+c+2)!}2A \tag{2}$$

$$\int_V \xi_1^a \xi_2^b \xi_3^c \xi_4^d \,\mathrm{d}V = \frac{a!\,b!\,c!\,d!}{(a+b+c+d+3)}6V \tag{3}$$

applicable to linear, triangular and tetrahedral domains, respectively. Equations (1)–(3) are widely quoted without proof in standard reference works for finite element theory.[1,2] Oden[3] attributes equation (2) to Stricklin,[4] who asserts this result on the basis of observation and verifies its validity for several values of a, b and c.

The integral in question may be evaluated by means of the parallelogram area element shown in Figure 1. The differential area element is given by

$$\mathrm{d}A = \frac{(h_1\,\mathrm{d}\xi_1)(h_2\,\mathrm{d}\xi_2)}{\sin\alpha_3} = 2A\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$

where $\xi_i = s_i/h_i$ are the area co-ordinates. Thus

$$\int_A \xi_1^a \xi_2^b \xi_3^c \,\mathrm{d}A = 2A\int_0^1\left[\int_0^{1-\xi_1} \xi_1^a \xi_2^b (1-\xi_1-\xi_2)^c \,\mathrm{d}\xi_2\right]\mathrm{d}\xi_1$$

The substitution of $t = \xi_2/(1-\xi_1)$ in the inner integral results in

$$\int_A \xi_1^a \xi_2^b \xi_3^c \,\mathrm{d}A = 2A\int_0^1 \xi_1^a (1-\xi_1)^{b+c+1}\,\mathrm{d}\xi_1 \int_0^1 t^b(1-t)^c\,\mathrm{d}t$$

Each of the integrals on the right-hand side is of the form of the beta function[5]

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,\mathrm{d}t = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)} \tag{4}$$

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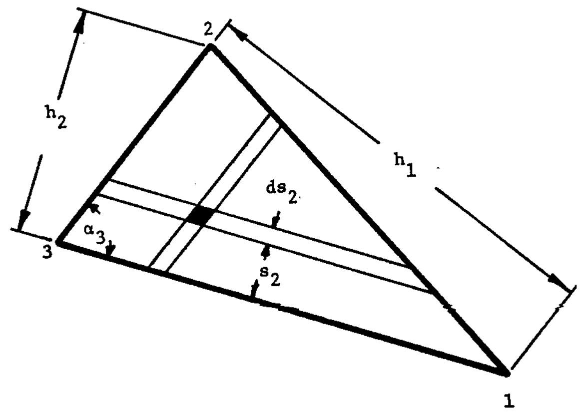

Figure 1. Integration domain

where Γ denotes the gamma function, which satisfies $\Gamma(n+1) = n!$ for integers $n \geqslant 0$. Thus

$$\int_A \xi_1^a \xi_2^b \xi_3^c \, \mathrm{d}A = 2A \frac{\Gamma(a+1)\,\Gamma(b+1)\,\Gamma(c+1)}{\Gamma(a+b+c+3)} \tag{5}$$

for complex numbers a, b, c with real parts greater than -1. Further restriction to non-negative integers a, b, c gives equation (2) as a special case of equation (5). Demonstrations for the linear and tetrahedral domains of equations (1) and (3) follow similar lines.

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EXACT MISES LIMIT LOADS FOR CYLINDRICAL SHELLS

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SUMMARY

Numerical solutions for limit loads of pressurized cylindrical cantilever shells are given for the exact Il'yushin yield surface.

It is suggested that the numerical approach can be used for more complex problems for which the exact solution based on equation (2) has not been obtained to date.

INTRODUCTION

Recently Ivanov[1] has proposed two approximations to the yield surface of Il'yushin.[2] The two

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