VECTOR FINITE ELEMENTS FOR ELECTROMAGNETIC FIELD COMPUTATION

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Abstract - A new structure is presented for the finite element analysis of vector fields. This structure employs the affine transformation to represent vectors and vector operations over triangular domains. Two-dimensional high order vector elements are derived that are consistent with Whitney forms. 1-form elements preserve the continuity of the tangential components of a vector field across element boundaries while 2-form elements preserve the continuity of the normal components. The 1-form elements are supplemented with additional variables to achieve p'th order completeness in the range space of the curl operator. The resulting elements are called tangential vector finite elements and provide consistent, reliable and accurate methods for solving electromagnetic field problems.

1. INTRODUCTION

The finite element method has proved to be successful in solving electromagnetic field problems involving scalar variables in both two and three dimensions [1]. However, the finite element analysis of field problems involving vector variables has been an enigma. The analysis of both two or three dimensional vector eddy current and microwave problems has generated numerical instabilities called spurious modes [2-6] while the application of gauge conditions for vector magnetostatic problems has been problematic [7-12]. It has been demonstrated that even simple variations in the finite element mesh can produce large errors in the vector potential solution of both eddy current [21] and magnetostatic field problems [13].

In this paper, we examine the representation of vectors over triangular finite elements. We show that the structure of vector finite elements is fundamentally different from that of scalar elements. A consistent relationship between scalar and vector elements is derived that matches the continuity conditions provided by variational calculus. This procedure eliminates the numerical instabilities that plague traditional solutions of vector electromagnetic field problems.

The structure presented in this paper is an extension of recent work in edge elements [14-16,32,33], in tangential vector elements [17-23], and in normal vector elements [24,25]. It is related to a mathematical construction known as Whitney forms [26,27,34]. In these forms, scalar functions that provide function but not derivative continuity are called 0-forms; vector functions that possess tangential but not normal continuity are called 1-forms; vector functions that possess normal but not derivative continuity are called 2-forms; and discontinuous scalar functions are called 3-forms.

The structure of electromagnetics parallels that of Whitney forms. The continuity requirement of a scalar variable such as the electric potential is that the potential is continuous but that its derivative may jump across inter-element boundaries. The electric field, which is the negative gradient of the potential, satisfies the continuity requirement that its tangential component is continuous but that its normal component may jump across inter-element boundaries. The electric flux, which is obtained by multiplying the electric field by the permittivity, satisfies the continuity requirement that its normal component is continuous but that its tangential component may jump across inter-element boundaries. Finally, the divergence of the electric flux is a charge distribution that is a discontinuous scalar.

In the numerical realm, if the potential is approximated by piecewise continuous polynomials - that is, separate polynomials defined over each finite elements possessing function but not derivative continuity across inter-element boundaries - then to be consistent and to avoid numerical instabilities, the electric field must be approximated by tangential vector finite elements in which the tangential component of the field across the element faces is continuous but the normal component may jump. Further, in this numerical structure, permittivity is an operator that converts tangential elements into normal elements in which the normal component of the field is continuous but the tangential component may jump across the element boundaries.

High-order vector elements in two-dimensions are presented that interpolate to the appropriate Whitney form. These elements mimic the inherent continuity of electromagnetic fields. We show that the resulting vector finite elements are consistent with the requirements of variational calculus and that they are complete to p'th order in the range space of the curl operator.

2. TRIANGLES

2.1 The Affine Transformation - As is well known, homogeneous or barycentric coordinates $\{\zeta_i, i = 1,2,3\}$ are defined over a triangle as follows [1,30]

$$X = \Delta F Z \tag{2.1}$$

where

$$X = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \qquad F = \frac{1}{\Delta}\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \qquad Z = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \tag{2.2}$$

Here (x_i, y_i), $i = 1,2,3$ are the three sets of triangle vertex coordinates and

$$\Delta = \sum_{i=1}^{3} (x_i y_{i1} - x_i y_{i2}) \tag{2.3}$$

with $(i, i1, i2) = \{(1,2,3), (2,3,1), (3,1,2)\}$. The transformation in (2.1) is due to Moebius and is known as the affine transformation.

The affine transformation is nonsingular and may be inverted to yield

$$Z = \frac{1}{\Delta} F^{-1} X \tag{2.4}$$

where the elements of the inverse matrix F^{-1} are

$$F^{-1} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{2.5}$$

Here

$$\begin{aligned} a_i &= x_{i1} y_{i2} - x_{i2} y_{i1} \\ b_i &= y_{i1} - y_{i2} \\ c_i &= x_{i2} - x_{i1} \end{aligned} \tag{2.6}$$

We note that

$$\Delta = \sum_{i=1}^{3} x_i b_i = \sum_{i=1}^{3} y_i c_i$$

Unit vectors tangent to the element edges are given by

$$\hat{t}_i = \frac{1}{l_i}(c_i \hat{x} - b_i \hat{y}) \tag{2.7}$$

where side i is opposite vertex i, $\hat{x}$ and $\hat{y}$ are the unit vectors in the x and y directions, respectively, and $l_i^2 = b_i^2 + c_i^2$ is the length of side i. See Figure 1. Unit normal vectors are given by

$$\hat{n}_i = \hat{t}_i \times \hat{z} = -\frac{1}{l_i}(b_i \hat{x} + c_i \hat{y}) \tag{2.8}$$

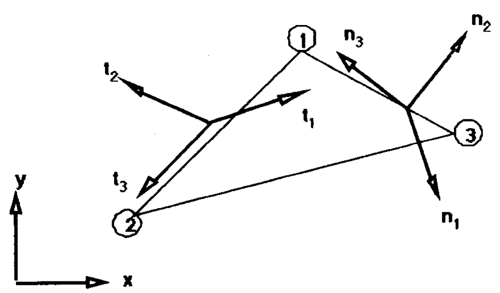

Figure 1. The unit tangent and unit normal vectors on a triangle where side i is opposite vertex i.

Writing (2.8) in matrix form gives

$$\hat{N} = -L^{-1} F^{-1} \hat{X} \tag{2.9}$$

where

$$\hat{N} = \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} \quad L = \begin{bmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{bmatrix} \quad \hat{X} = \begin{bmatrix} \hat{0} \\ \hat{x} \\ \hat{y} \end{bmatrix} \tag{2.10}$$

Even though the parameters a_i are multiplied by zero in equation (2.9), they are included in this equation to preserve all of the columns of the affine matrix F^{-1}. This allows (2.9) to be inverted

$$\hat{X} = -F L \hat{N} \tag{2.11}$$

In a similar way, the unit tangent vectors are written as

$$\hat{T} = L^{-1} G^{-1} \hat{X} \tag{2.12}$$

where G^{-1} is the matrix

$$G^{-1} = \begin{bmatrix} a_1 & c_1 & -b_1 \\ a_2 & c_2 & -b_2 \\ a_3 & c_3 & -b_3 \end{bmatrix} \tag{2.13}$$

Inverting (2.12) gives

$$\hat{X} = G L \hat{T} \tag{2.14}$$

where

$$G = \frac{1}{\Delta} \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ -x_1 & -x_2 & -x_3 \end{bmatrix} \tag{2.15}$$

From (2.11) and 2.14), the zero "unit vector" in $\hat{X}$ equals

$$\hat{0} = \sum_i l_i \, \hat{n}_i = \sum_i l_i \, \hat{t}_i \tag{2.16}$$

Also, combining the above equations provides $\hat{N} = -L^{-1} F^{-1} G L \hat{T}$ and $\hat{T} = -L^{-1} G^{-1} F L \hat{N}$ where the elements of the product matrices $F^{-1}G$ and $G^{-1}F$ are

$$\begin{aligned} (F^{-1}G)_{ij} &= (a_i + b_i y_j - c_i x_j)/\Delta \\ (G^{-1}F)_{ij} &= (a_i + c_i x_j - b_i y_j)/\Delta \end{aligned} \tag{2.17}$$

2.2 Angles Between Unit Vectors - The angles between the unit tangent and unit normal vectors are obtained by using the dot product

$$\hat{N} \cdot \hat{N}^T = \hat{T} \cdot \hat{T}^T = C \tag{2.18}$$

where the matrix C is given by

$$C = \begin{bmatrix} 1 & -\cos\theta_3 & -\cos\theta_2 \\ -\cos\theta_3 & 1 & -\cos\theta_1 \\ -\cos\theta_2 & -\cos\theta_1 & 1 \end{bmatrix} \tag{2.19}$$

Here θ_i represents the included angle at vertex i. We also find that

$$\hat{N} \cdot \hat{T}^T = \hat{T} \cdot \hat{N}^T = \begin{bmatrix} 0 & -\sin\theta_3 & \sin\theta_2 \\ \sin\theta_3 & 0 & -\sin\theta_1 \\ -\sin\theta_2 & \sin\theta_1 & 0 \end{bmatrix} \tag{2.20}$$

Equation (2.20) is cast into a simpler form by noting that

$$\sin\theta_i = \frac{h_{i1}}{l_{i2}} = \frac{h_{i2}}{l_{i1}} \tag{2.21}$$

where h_i is the altitude to vertex i. Thus

$$\hat{N} \cdot \hat{T}^T = \hat{T} \cdot \hat{N}^T = H S L^{-1} = L^{-1} S H \tag{2.22}$$

where

$$H = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \tag{2.23}$$

3. ANALYSIS

3.1 Vectors - A vector v is expressed in terms of $\hat{X}$ as

$$v = v_x \hat{x} + v_y \hat{y} = \hat{X}^T V^{(X)} = V^{(X)T} \hat{X} \tag{3.1}$$

where

$$V^{(X)} = [\, q \quad v_x \quad v_y \,]^T \tag{3.2}$$

and q is to be determined. To convert (3.1) to normal and tangential forms, use (2.11) and (2.14). The result is

$$v = \hat{N}^T V^{(N)} = \hat{T}^T V^{(T)} \tag{3.3}$$

where

$$\begin{aligned} V^{(N)} &= -L F^T V^{(X)} \\ V^{(T)} &= L G^T V^{(X)} \end{aligned} \tag{3.4}$$

Solving for $V^{(X)}$ from (3.4) gives

$$\begin{aligned} V^{(X)} &= -F^{-T} L^{-1} V^{(N)} \\ V^{(X)} &= G^{-T} L^{-1} V^{(T)} \end{aligned} \tag{3.5}$$

The top row of (3.5) yields

$$q = -\sum_{i=1}^{3} \frac{a_i}{l_i} V_i^{(N)} = \sum_{i=1}^{3} \frac{a_i}{l_i} V_i^{(T)} \tag{3.6}$$

The components of $\mathbf{v}$ in the three directions tangent to the element edges are

$$\begin{aligned}\mathbf{v}_t &= \hat{T}\cdot\hat{N}^T V^{(N)} = H\,S\,L^{-1}V^{(N)} = L^{-1}\,S\,H\,V^{(N)} \\ &= \hat{T}\cdot\hat{T}^T V^{(T)} = C\,V^{(T)}\end{aligned} \tag{3.7}$$

where $\mathbf{v}_t$ is a vector containing the total components of $\mathbf{v}$ in the three tangential directions. Notice that the components of $\mathbf{v}_t$ are in general not the same as the components of $V^{(T)}$. Referring to Figure 1, we see that the representation of a vector $\mathbf{v}$ in terms of the unit vectors $\hat{N}$ and $\hat{T}$ is not unique. The unique components of the vector $\mathbf{v}$ in the three directions $\hat{t}_i$ are provided by the three components of $\mathbf{v}_t$.

Substituting (3.4) into (3.7) yields

$$\mathbf{v}_t = -\,H\,S\,F^T V^{(X)} = C\,L\,G^T V^{(X)} \tag{3.8}$$

It follows that

$$C = -\,H\,S\,F^T G^{-T} L^{-1} \tag{3.9}$$

In a similar way, the components of $\mathbf{v}$ in the directions normal to the element edges are

$$\mathbf{v}_n = \hat{N}\cdot\hat{T}^T V^{(T)} = H\,S\,L^{-1}V^{(T)} = H\,S\,G^T V^{(X)} \tag{3.10}$$

or

$$\mathbf{v}_n = \hat{N}\cdot\hat{N}^T V^{(N)} = C\,V^{(N)} = -\,C\,L\,F^T V^{(X)} \tag{3.11}$$

Let $D^{-1} = \Delta\,S\,F^T$ and $E^{-1} = -\,\Delta\,S\,G^T$. Direct evaluation yields

$$D^{-1} = \begin{bmatrix} 0 & c_1 & -b_1 \\ 0 & c_2 & -b_2 \\ 0 & c_3 & -b_3 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{bmatrix} \tag{3.12}$$

Thus we may write

$$\begin{aligned}\mathbf{v}_t &= -\,L^{-1}\,D^{-1}\,V^{(X)} \\ \mathbf{v}_n &= -\,L^{-1}\,E^{-1}\,V^{(X)}\end{aligned} \tag{3.13}$$

An example of the above is provided by decomposing the vector $\mathbf{v} = v_y\,\hat{y}$ in a triangle with vertices located at (0, 0), $(x_2, 0)$ and (x_3, y_3). In this case, F^T is lower triangular and

$$V^{(N)} = -\frac{L}{\Delta}\begin{bmatrix} 1 & 0 & 0 \\ 1 & x_2 & 0 \\ 1 & x_3 & y_3 \end{bmatrix}\begin{bmatrix} q \\ 0 \\ v_y \end{bmatrix} = -\begin{bmatrix} h_1^{-1}\,q \\ h_2^{-1}\,q \\ h_3^{-1}\,q + v_y \end{bmatrix} \tag{3.14}$$

On the other hand, the decomposition in (3.13) yields

$$\mathbf{v}_n = -\,L^{-1}\begin{bmatrix} 0 & -y_3 & x_3-x_2 \\ 0 & y_3 & -x_3 \\ 0 & 0 & x_2 \end{bmatrix}\begin{bmatrix} q \\ 0 \\ v_y \end{bmatrix} = -\begin{bmatrix} \cos\theta_2\, v_y \\ \cos\theta_1\, v_y \\ v_y \end{bmatrix} \tag{3.15}$$

From (3.7) we note that in general the vector $V^{(N)} = [\,0 \quad 0 \quad V_3\,]^T$ has a zero tangential component in the direction $\hat{t}_3$

$$\mathbf{v}_t = H\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ V_3/l_3 \end{bmatrix} = \begin{bmatrix} \sin\theta_2\, V_3 \\ -\sin\theta_1\, V_3 \\ 0 \end{bmatrix} \tag{3.16}$$

Also, a vector $V^{(N)} = [\,V_1 \quad V_2 \quad 0\,]^T$ satisfying the ratio $V_1 / V_2 = \sin\theta_1 / \sin\theta_2$ has a zero tangential component in the direction $\hat{t}_3$

$$\mathbf{v}_t = \begin{bmatrix} -\sin\theta_3\, V_2 \\ \sin\theta_3\, V_1 \\ \sin\theta_1 V_2 - \sin\theta_2 V_1 \end{bmatrix} = \begin{bmatrix} -\sin\theta_3\, V_2 \\ \sin\theta_3\, V_1 \\ 0 \end{bmatrix} \tag{3.17}$$

3.2 The Gradient - In two dimensions, the gradient operator is written in terms of the vector $\hat{X}$ as

$$\nabla\phi = \hat{X}^T\,\partial W\,\phi = \partial W^T\,\phi\,\hat{X} \tag{3.18}$$

where

$$\partial W = [\;w \quad \partial_x \quad \partial_y\;]^T \tag{3.19}$$

Here we adopt the notation $\partial x \equiv \partial/\partial x$, $\partial y \equiv \partial/\partial y$ et cetera and w is to be determined.

Applying the chain rule gives

$$\partial x = \frac{1}{\Delta}\sum_{i=1}^{3} b_i\,\partial\zeta_i \qquad \partial y = \frac{1}{\Delta}\sum_{i=1}^{3} c_i\,\partial\zeta_i \tag{3.20}$$

Thus we may write

$$\partial W = \frac{1}{\Delta}\;F^{-T}\;\partial Z \tag{3.21}$$

where

$$\partial Z = [\;\partial\zeta_1 \quad \partial\zeta_2 \quad \partial\zeta_3\;]^T \tag{3.22}$$

Equation (3.21) defines w. Substituting (3.21) into (3.18) gives

$$\nabla\phi = \hat{X}^T\,\Delta^{-1}\,F^{-T}\,\partial Z\,\phi \tag{3.23}$$

To express $\nabla\phi$ in terms of the unit normal vectors, use (2.11)

$$\nabla\phi = -\,\hat{N}^T\,H^{-1}\,\partial Z\,\phi \tag{3.24}$$

where we have employed $h_i\,l_i = \Delta$. Thus, the gradient is expressed "naturally" (i.e. as partials of homogeneous coordinates) in terms of the unit normals to the sides of the element.

The gradient is given in terms of the unit tangent vectors by substituting (2.14) into (3.23). The result is

$$\nabla\phi = \hat{T}^T\,H^{-1}\,G^T\,F^{-T}\,\partial Z\,\phi \tag{3.25}$$

The components of $\nabla\phi$ in the directions tangent to the element edges are obtained by taking the dot product of (3.24) with $\hat{T}$

$$\nabla\phi_t = -\,\hat{T}\cdot\hat{N}^T\,H^{-1}\,\partial Z\,\phi = -\,L^{-1}\,S\,\partial Z\,\phi \tag{3.26}$$

Similarly, the components of ϕ in the normal directions are

$$\nabla\phi_n = -\,C\,H^{-1}\,\partial Z\,\phi \tag{3.27}$$

3.3 The Divergence - In two dimensions, we may write the divergence of the vector $\mathbf{v}$ as

$$\nabla\cdot\mathbf{v} = \partial X^T\;V^{(X)} \tag{3.28}$$

where

$$\partial X = [\;0 \quad \partial x \quad \partial y\;]^T \tag{3.29}$$

Using (3.20) gives

$$\partial X^T = \partial Z^T\,E^{-1}\,/\,\Delta \tag{3.30}$$

Thus

$$\nabla\cdot\mathbf{v} = \partial Z^T\,\Delta^{-1}\,E^{-1}\;V^{(X)} \tag{3.31}$$

In terms of $V^{(T)}$, this yields from (3.5)

$$\nabla \cdot \mathbf{v} = \partial Z^T \Delta^{-1} L^{-1} E^{-1} G^{-T} V^{(T)} \tag{3.32}$$

Using (3.12), we find that

$$E^{-1} G^{-T} = - \Delta S \tag{3.33}$$

Thus (3.32) becomes

$$\nabla \cdot \mathbf{v} = - \partial Z^T S L^{-1} V^{(T)} \tag{3.34}$$

It is interesting to note that the divergence operator expressed in terms of unit tangent vectors is

$$\mathrm{div}^{(T)} = - \partial Z^T S L^{-1} \tag{3.35}$$

while the transpose of the tangential components of the gradient operator is from (3.26)

$$\mathrm{grad}^{(T)^T} = - \partial Z^T S L^{-1} \tag{3.36}$$

Thus, as expected, the adjoint of the gradient is equal to the divergence.

To write the divergence in terms of $V^{(N)}$, substitute (3.5) into (3.31). This gives

$$\nabla \cdot \mathbf{v} = - \partial Z^T K L^{-1} V^{(N)} \tag{3.37}$$

where $K = E^{-1} F^{-T} / \Delta$. The elements of K are

$$K_{ij} = (b_i b_j + c_i c_j) / \Delta \tag{3.38}$$

Using the cotangent identity [1], K may be written as

$$K = \begin{bmatrix} \cot\theta_2 + \cot\theta_3 & \cot\theta_3 & \cot\theta_2 \\ \cot\theta_3 & \cot\theta_1 + \cot\theta_3 & \cot\theta_1 \\ \cot\theta_2 & \cot\theta_1 & \cot\theta_1 + \cot\theta_2 \end{bmatrix} \tag{3.39}$$

3.4 The Curl - The curl operator is similar to the divergence operator in two dimensions

$$\nabla \times \mathbf{v} = \hat{z} \, \partial Y^T V^{(X)} \tag{3.40}$$

where

$$\partial Y = [\, 0 \quad -\partial y \quad \partial x \,]^T \tag{3.41}$$

We may write this as

$$\nabla \times \mathbf{v} = - \hat{z} \, \partial Z^T D^{-1} V^{(X)} \tag{3.42}$$

Using (3.5) and recognizing that $D^{-1} F^{-T} = \Delta S$ and that $D^{-1} G^{-T} = \Delta K$ allows us to write (3.42) in terms of unit normal and tangent vectors

$$\begin{aligned} \nabla \times \mathbf{v} &= \hat{z} \, \partial Z^T S H V^{(N)} \\ \nabla \times \mathbf{v} &= - \hat{z} \, \partial Z^T K H V^{(T)} \end{aligned} \tag{3.43}$$

3.5 The Laplacean - Using (3.34) and (3.25) we find that

$$\begin{aligned} \nabla \cdot \nabla \phi &= - \Delta^{-1} \partial Z^T S G^T F^{-T} \partial Z \phi \\ &= \Delta^{-1} \partial Z^T K \partial Z \phi \end{aligned} \tag{3.44}$$

Thus the relationship of the Laplacean operator to the triangle cotangents [1] is derived directly from the affine transformation.

4. VECTOR ELEMENTS

4.1 Compatibility - The finite element solution of electromagnetic field problems requires that the approximation functions satisfy the proper continuity conditions across element boundaries. Finite elements satisfying the proper continuity conditions are said to be *compatible*. For 1-form elements, compatibility requires that the tangential components of the vector field be continuous while for 2-form elements the normal components must be continuous.

Consider 1-form elements first. To set the tangential component of the field to be continuous, we need to evaluate the tangential components of the field along the element edges. From (3.7), we see that the simplest expression for the tangential component of the field is given in terms of the unit normal vectors

$$\mathbf{v}_t = L^{-1} S H V^{(N)} \tag{4.1}$$

A two dimensional vector function $\mathbf{v}(x,y)$ is expressed in terms of normal unit vectors as

$$\mathbf{v}(x,y) = \hat{N}^T V^{(N)}(\zeta_1, \zeta_2, \zeta_3) \tag{4.2}$$

We now approximate this function as follows

$$V^{(N)}(\zeta_1, \zeta_2, \zeta_3) = H^{-1} \, \Gamma(\zeta_1, \zeta_2, \zeta_3) \, \underset{\sim}{\tau} \tag{4.3}$$

Here the columns of the matrix Γ provide the desired vector basis functions and $\underset{\sim}{\tau}$ is a column vector containing coefficients that interpolate to the tangential components of the vector field, as indicated in Figure 2. The matrix H^{-1} is included in (4.3) to simplify (4.1). Combining (4.2) with (4.3) and (2.9) gives

$$\mathbf{v}(x,y) = \hat{N}^T H^{-1} \Gamma \underset{\sim}{\tau} = - \frac{1}{\Delta} \hat{X}^T F^{-T} \Gamma \underset{\sim}{\tau} \tag{4.4}$$

Written in block form, (4.3) is

$$V^{(N)} = \begin{bmatrix} h_1^{-1} & 0 & 0 \\ 0 & h_2^{-1} & 0 \\ 0 & 0 & h_3^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\gamma}_{11} & \tilde{\gamma}_{12} & \tilde{\gamma}_{13} \\ \tilde{\gamma}_{21} & \tilde{\gamma}_{22} & \tilde{\gamma}_{23} \\ \tilde{\gamma}_{31} & \tilde{\gamma}_{32} & \tilde{\gamma}_{33} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\tau}^1 \\ \underset{\sim}{\tau}^2 \\ \underset{\sim}{\tau}^3 \end{bmatrix} \tag{4.5}$$

where the contribution to $V^{(N)}$ in direction i along side j is given by

$$\begin{aligned} \tilde{\gamma}_{ij} &= \left[\gamma_0^{ij} \quad \gamma_1^{ij} \quad \cdots \quad \gamma_p^{ij} \right] \\ \underset{\sim}{\tau}^i &= \left[\tau_0^i \quad \tau_1^i \quad \cdots \quad \tau_p^i \right]^T \end{aligned} \tag{4.6}$$

The product matrix $S\Gamma$ arising from (4.1) and (4.3) equals

$$S\Gamma = \begin{bmatrix} \tilde{\gamma}_{31} - \tilde{\gamma}_{21} & \tilde{\gamma}_{32} - \tilde{\gamma}_{22} & \tilde{\gamma}_{33} - \tilde{\gamma}_{23} \\ \tilde{\gamma}_{11} - \tilde{\gamma}_{31} & \tilde{\gamma}_{12} - \tilde{\gamma}_{32} & \tilde{\gamma}_{13} - \tilde{\gamma}_{33} \\ \tilde{\gamma}_{21} - \tilde{\gamma}_{11} & \tilde{\gamma}_{22} - \tilde{\gamma}_{12} & \tilde{\gamma}_{23} - \tilde{\gamma}_{13} \end{bmatrix} \tag{4.7}$$

Consider the variables associated with side 1 first. We want the basis functions in the top row of the matrix $S\Gamma$ are to interpolate to the components indicated in Figure 2. Since $\mathbf{v}_t = L^{-1} S \Gamma \underset{\sim}{\tau}$ we need to impose the following conditions

$$\tilde{\gamma}_{31} - \tilde{\gamma}_{21} = l_1 \, \tilde{\beta}(\zeta_2, \zeta_3) \quad \text{along } \zeta_1 = 0 \tag{4.8a}$$

$$\tilde{\gamma}_{32} - \tilde{\gamma}_{22} = 0 \quad \text{along } \zeta_1 = 0 \tag{4.8b}$$

$$\tilde{\gamma}_{33} - \tilde{\gamma}_{23} = 0 \quad \text{along } \zeta_1 = 0 \tag{4.8c}$$

where $\tilde{\beta}(\zeta_2, \zeta_3)$ is the row vector containing the interpolation polynomials along side 1. Similar equations are obtained for the other two sides

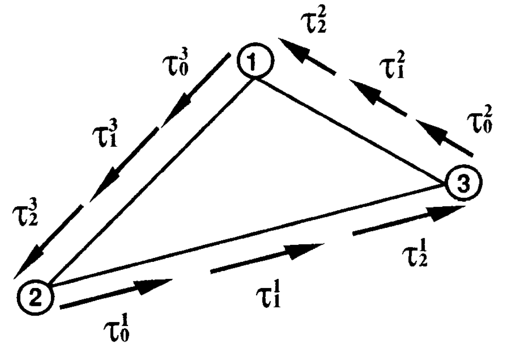

Figure 2. The tangential component interpolation coefficients τ_i^j where j is the triangle side and i is the node number.

$$\tilde{\gamma}_{11} - \tilde{\gamma}_{31} = 0 \quad \text{along } \zeta_2=0 \tag{4.8d}$$
$$\tilde{\gamma}_{12} - \tilde{\gamma}_{32} = l_2\ \tilde{\beta}(\zeta_1,\zeta_3) \quad \text{along } \zeta_2=0 \tag{4.8e}$$
$$\tilde{\gamma}_{13} - \tilde{\gamma}_{33} = 0 \quad \text{along } \zeta_2=0 \tag{4.8f}$$
$$\tilde{\gamma}_{21} - \tilde{\gamma}_{11} = 0 \quad \text{along } \zeta_3=0 \tag{4.8g}$$
$$\tilde{\gamma}_{22} - \tilde{\gamma}_{12} = 0 \quad \text{along } \zeta_3=0 \tag{4.8h}$$
$$\tilde{\gamma}_{23} - \tilde{\gamma}_{13} = l_3\ \tilde{\beta}(\zeta_1,\zeta_2) \quad \text{along } \zeta_3=0 \tag{4.8i}$$

Along side i, the desired interpolation polynomials are given by

$$\beta_k(\zeta_{i1}, \zeta_{i2}) = P_{p-k}(\zeta_{i1})\, P_k(\zeta_{i2}) \tag{4.9}$$

where p is the polynomial order and $P_m(\zeta)$ are the Silvester polynomials [1]

$$\begin{aligned} P_o(\zeta) &= 1 \\ P_m(\zeta) &= \prod_{j=1}^{m} \left(\frac{p\zeta - j + 1}{j} \right), \quad m \neq 1 \end{aligned} \tag{4.10}$$

The solution of equations (4.8) is not unique. For the first column of (4.7), consider the solution

$$\tilde{\gamma}_{11} = \tilde{0} \qquad \tilde{\gamma}_{21} = \tilde{0} \qquad \tilde{\gamma}_{31} = l_1\tilde{\beta}(\zeta_2,\zeta_3) \tag{4.11}$$

Equations (4.11) satisfy equations (4.8a) and (4.8g) everywhere but violate equation (4.8d) at the one point $\zeta_3 = 1$. On the other hand, the solution

$$\tilde{\gamma}_{11} = 0 \qquad \tilde{\gamma}_{21} = - l_1\tilde{\beta}(\zeta_2,\zeta_3) \qquad \tilde{\gamma}_{31} = 0 \tag{4.12}$$

satisfies (4.8a) and (4.8d) everywhere but violates (4.8g) at the one point $\zeta_2 = 1$. As a result, provided that we use r + s = 1, we may write

$$\Gamma_1 = l_1\, B_1 \tag{4.13}$$

where Γ_1 is the side 1 part of Γ and

$$B_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & -r\beta_1(\zeta_2,\zeta_3) & \cdots & -\beta_p(\zeta_2,\zeta_3) \\ \beta_0(\zeta_2,\zeta_3) & s\,\beta_1(\zeta_2,\zeta_3) & \cdots & 0 \end{bmatrix} \tag{4.14}$$

Similar arguments with the variables on sides 2 and 3 lead to

$$\Gamma = [\ l_1\, B_1 \quad l_2\, B_2 \quad l_3\, B_3\ ] \tag{4.15}$$

where

$$B_2 = \begin{bmatrix} \beta_0(\zeta_3,\zeta_1) & s\beta_1(\zeta_3,\zeta_2) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & -r\beta_1(\zeta_3,\zeta_1) & \cdots & -\beta_1(\zeta_3,\zeta_1) \end{bmatrix} \tag{4.16}$$

$$B_3 = \begin{bmatrix} 0 & -r\beta_1(\zeta_1,\zeta_2) & \cdots & -\beta_p(\zeta_1,\zeta_2) \\ \beta_0(\zeta_1,\zeta_2) & s\beta_1(\zeta_1,\zeta_2) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.17}$$

2-form elements possessing continuity of the normal components are derived in a similarly with the letters referring to normal and tangential directions interchanged. The final result is

$$v(x,y) = \hat{T}^T H^{-1}\, \Gamma\, \tilde{\eta} = -\frac{1}{\Delta}\, \hat{N}^T F^{-T}\, \Gamma\, \tilde{\eta} \tag{4.18}$$

where $\tilde{\eta}$ is a column vector containing the coefficients that interpolate to the normal components of the vector field.

4.2 Low Order Elements - For first order, (4.15) becomes

$$\Gamma = \begin{bmatrix} 0 & 0 & l_2\zeta_3 & 0 & 0 & -l_3\zeta_2 \\ 0 & -l_1\zeta_3 & 0 & 0 & l_3\zeta_1 & 0 \\ l_1\zeta_2 & 0 & 0 & -l_2\zeta_1 & 0 & 0 \end{bmatrix} \tag{4.19}$$

The six functions in (4.19) satisfy the compatibility requirements of 1 and 2-form finite elements. However, these functions do not provide finite elements that are complete in the range space of the curl operator. As we shall see below, we need to add two extra functions to this set to make curl **v** complete to first order.

Zeroth order elements are derived from first order by setting $\tau_0^1 = \tau_1^1$, $\tau_0^2 = \tau_1^2$ and $\tau_0^3 = \tau_1^3$ in the p = 1 approximation for **v**(x,y). The result is

$$\Gamma = \begin{bmatrix} 0 & l_2\zeta_3 & -l_3\zeta_2 \\ -l_1\zeta_3 & 0 & l_3\zeta_1 \\ l_1\zeta_2 & -l_2\zeta_1 & 0 \end{bmatrix} \tag{4.20}$$

These elements are equivalent to the edge elements of [28,14] and to the vector elements of [24,25]. Interpolation polynomials of the type (4.20) were first given in [16].

4.3 Unisolvence - As already noted, the above analysis is useful but incomplete. Nedelec [28,29] makes the point that vector elements must satisfy both compatibility and unisolvence conditions. Compatibility requires that the proper interface conditions be satisfied along element boundaries; this is achieved in a 1-form sense by using the above tangential elements and in a 2-form sense by using the above normal elements. Unisolvence means that the finite element basis functions must be linearly independent to provide a unique, particular solution of the operator equation.

The range space of the curl operator is

$$R\ (\text{curl}) = \{\, \mathbf{r} : \mathbf{r} = \nabla \times \mathbf{v} \ \ \forall\, \mathbf{v} \in C^T \} \tag{4.21}$$

where C^T is the space of tangentially continuous 1-form vectors. The nullspace of the curl operator is

$$N\ (\text{curl}) = \{\, \mathbf{v} : \mathbf{v} = -\nabla\phi \ \ \forall\ \phi \in C^o \} \tag{4.22}$$

where C^o is the space of 0-form continuous scalars. Obviously, the solution **v** of the curl equation $\mathbf{r} = \nabla \times \mathbf{v}$ is not unique since $\mathbf{v} = \mathbf{v}' - \nabla\phi$ will satisfy the curl equation for any ϕ.

There is a second issue however. Even if one takes care of model $\nabla\phi$ correctly, it is possible to satisfy $\nabla \times \mathbf{v} = 0$ with $\mathbf{v} \neq 0$ and $\mathbf{v} \neq -\nabla\phi$ if the functions approximating $\mathbf{v}$ are not linearly independent. Thus, to get correct solutions, we must ensure the linear independence of the approximating functions. Unisolvence requires that the only non-trivial solution of $\nabla \times \mathbf{v} = 0$ is $\mathbf{v} = -\nabla\phi$ for some ϕ. Thus, if $\nabla \times \mathbf{v} = 0$ and $\mathbf{v} \neq -\nabla\phi$ then $\mathbf{v} = 0$ may be the only solution.

4.4 Bezier Forms - The unisolvence of vector finite elements is most easily understood by representing the finite element approximation functions in terms of Bezier polynomials. Bezier polynomials have a number of advantages over ordinary interpolation polynomials: they are simpler, easier to evaluate, and have a clearer geometric interpretation. Their disadvantage is that only the Bezier coefficients corresponding to the element vertices equal field values.

In general, p'th order Bezier polynomials in two-dimensions are given by [30]

$$\beta_{ijk}(\zeta_1,\zeta_2,\zeta_3) = \frac{p!}{i!\,j!\,k!}\,\zeta_1^i\,\zeta_2^j\,\zeta_3^k \tag{4.23}$$

where $i + j + k = n$. We may approximate a scalar $\phi(x,y)$ as

$$\phi(x,y) = \sum_{\substack{i,j,k \\ i+j+k=p}} \phi_{ijk}\,\beta_{ijk}(\zeta_1,\zeta_2,\zeta_3) \tag{4.24}$$

As noted, the parameters ϕ_{ijk} in (4.24) are not equal to the values of the function $\phi(x,y)$ except at the vertices. Instead, the ϕ_{ijk} form a faceted surface called the "control net" for $\phi(x,y)$. This control net may be loosely thought of as a first-order finite element approximation of the p'th order polynomial where $\phi(x,y)$ fits inside the convex hull of the control net.

As is well known [1], the number of terms m_s in a complete polynomial of order p is

$$m_s = (p+1)(p+2)/2 \tag{4.25}$$

We now ask the following questions:

1. How many independent variables are there in $\mathbf{v} = \mathbf{v}' - \nabla\phi$ if ϕ is a p'th order polynomial?
2. Of these variables, how many are required to make $\mathbf{r} = \text{curl } \mathbf{v}$ complete to order (p-1) ?

Consider p=1 first. In this case, we may write

$$\phi = \sum_{i=1}^{3} \phi_i\,\zeta_i \tag{4.26}$$

Equation (3.24) implies that $\nabla\phi$ is a constant. We find that

$$\mathbf{v} = \nabla\phi = -\hat{N}^T L \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \tag{4.27}$$

Thus there are three parameters in a single element in $\nabla\phi$ and U parameters in a connected mesh, where U is the number of vertices.

Thinking geometrically, we know that ϕ has a different, constant slope along each edge and can therefore be represented by zeroth order tangential elements. The number of variables in this representation is equal to the number of edges in the mesh. The number of edges E in a triangular mesh is given by E = 3(U - 1) - B where U is the total number of vertices and B is the number of vertices on the boundary.

The curl of a zeroth order edge element is equal to a constant in each element. Since the number of independent variables in a single zeroth order edge element is three, we are led to the conclusion that three independent variables are required to represent $\mathbf{v} = \mathbf{v}' - \nabla\phi$ such that $\mathbf{r} = \text{curl } \mathbf{v}$ is complete to zeroth order. Note that ϕ is linear in this expression and that $\nabla \times \nabla\phi = 0$ as required.

In a connected mesh of T triangles, the number of parameters in $\nabla\phi$ is equal to the number of vertices U while the number of parameters in $\mathbf{v}'$ is equal to the number of edges E. The number of independent curls is T = 2(U - 1) - B = E - U + 1.

4.5 Higher-Order - The number of variables in $\nabla\phi$ for higher order polynomials is derived by noticing that the control net of a Bezier polynomial $\phi(x,y)$ consists of a triangular set of first order finite elements. Since the control net determines the associated Bezier polynomial completely, we can determine the number of variables in $\nabla\phi$ by counting the variables in the gradient of the control net. This number is equal to the number of edges in the Bezier control net and is given by $m_b = 3p(p+1)/2$.

To answer question two, note that in high order elements the number of independent curls is equal to the number of facets in the control net and is given by $m_f = p^2$. Yet $\mathbf{r} = \nabla \times \mathbf{v}$ should be a complete polynomial of order (p-1) in which there are $m_r = p(p+1)/2$ variables. It follows that $m_x = m_f - m_r = p(p-1)/2$ of the variables are redundant. Thus, the number of independent variables in $\mathbf{v}$ complete to order (p-1) is

$$m_c = m_b - m_x = p\,(p+2) \tag{4.28}$$

Equation (4.28) was derived previously by Nedelec using a different approach [28]. We note that the first four terms in the series for m_c are 3, 8, 15, and 24.

4.6 Tangential Elements - Equation (4.28) shows that the polynomials derived in Section 4.1 are incomplete. The number of variables in these edge elements is $m_e = 3p$. We must add $m_c - m_e = p(p-1)$ additional variables to achieve completeness. The first four terms in this series are 0, 2, 6, 12. Note that only the second of the first four terms in this sequence is not divisible by three.

No additional terms beyond the three in (4.20) are needed to generate zeroth order complete tangential elements. However, two additional terms must be added to the first order elements in (4.19) to make it complete. These additional basis functions may not contribute to the tangential components of the field along the edges of the element. We therefore use (3.16) to introduce functions that have a zero tangential component along the appropriate side.

Consider the matrix formed by augmenting the matrix Γ with two additional columns such that

$$\Gamma = \begin{bmatrix} l_1 B_1 & l_2 B_2 & l_3 B_3 & \Lambda \end{bmatrix} \tag{4.29}$$

where Λ is the matrix

$$\Lambda = \begin{bmatrix} 4\,\zeta_2\,\zeta_3 & 0 \\ 0 & 4\,\zeta_1\,\zeta_3 \\ 0 & 0 \end{bmatrix} \tag{4.30}$$

The two extra functions in (4.29) do not contribute to the tangential components of $\mathbf{v}(x,y)$: the function in column 1 of Λ is zero along edges 2 and 3 of the triangle and has a zero tangential component along edge 1 while the function in column 1 of Λ is zero along edges 2 and 3 of the triangle and has a zero tangential component along edge 2. Thus the element formed by (4.29) retains the linear tangential components on the three sides. However, its normal components are quadratic on two of the three sides with the addition of Λ. Since column 1 of Λ equals 1 at $\zeta_2 = \zeta_3 = 1/2$ while

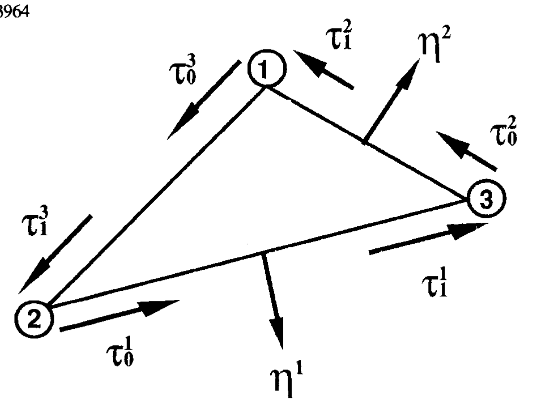

Figure 3. A first order complete tangential vector finite element. Note that only two of the three sides have a parameter in the normal direction and that these normal parameters are not interpolatory.

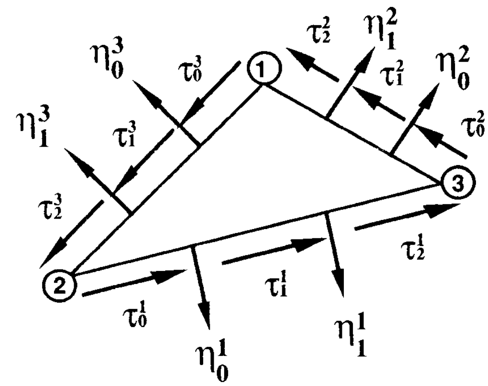

Figure 4. A second order complete tangential vector finite element. The tangential components of this element are quadratic and interpolate to 3 points on each side while the normal components are cubic and are not interpolated.

column 2 of Λ equals 1 at $\zeta_1 = \zeta_3 = 1/2$, these functions interpolate the quadratic part of the normal at the mid sides. This is depicted pictorially in Figure 3. Note, however, that the total function defined by Γ does **NOT** interpolate to the normal components of v(x,y) at the mid sides since the first six functions in Γ contribute non-zero normal components there.

Elements of the type given in (4.29) were first derived by Din Sun and the author [17]. A different derivation is provided in a companion paper [31]. An obvious disadvantage of first order tangential elements is that 2 variables cannot be located in a triangle in a symmetric way. This disadvantage does not occur with second order tangential elements. Using arguments similar to the above yields the element in Figure 4. The basis functions for this element are given in Table 1.

5. VARIATIONAL PRINCIPLES

5.1 The Vector Wave Equation - At this point, we need to determine how the elements derived above are used in finite element analysis. To make the analysis concrete, we examine the variational solution of the vector wave equation. The following analysis expands on that given in [3] and first appeared in this form in [17].

The vector wave equation is

$$\nabla\times\frac{1}{\mu}\nabla\times\mathbf{E} = \frac{k^2}{\mu}\mathbf{E} \qquad (5.1)$$

where $k^2 = \omega^2 \mu \varepsilon$. We shall show that the functional

$$F(\mathbf{E}) = \int \frac{1}{\mu}\left(\nabla\times\mathbf{E}\right)^2 d\Omega - \int \frac{k^2}{\mu}\mathbf{E}^2 d\Omega \qquad (5.2)$$

may be used to solve equation (5.1) provided that the tangential component of **E** is made continuous. To minimize F(**E**), let $\mathbf{E} = \mathbf{E}_{ex} + \beta\xi$ where $\mathbf{E}_{ex}$ is the exact solution of the wave equation (5.1), β is a number, and ξ is an arbitrary vector function. The first variation of F(**E**) is

$$\delta F(\mathbf{E}) = \frac{\partial F(\mathbf{E}_{ex} + \beta\xi)}{\partial\beta}\Big|_{\beta=0}$$
$$\int \frac{1}{\mu}\left(\nabla\times\xi\right)\cdot\left(\nabla\times\mathbf{E}_{ex}\right)d\Omega - \int \frac{k^2}{\mu}\xi\cdot\mathbf{E}_{ex}d\Omega. \qquad (5.3)$$

By integrating the vector identity

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = (\nabla\times\mathbf{a})\cdot\mathbf{b} - \mathbf{a}\cdot(\nabla\times\mathbf{b}) \qquad (5.4)$$

we find that

$$\oint (\mathbf{a}\times\mathbf{b}))\cdot d\mathbf{S} = \int \left[\left(\nabla\times\mathbf{a}\right)\cdot\mathbf{b} - \mathbf{a}\cdot\left(\nabla\times\mathbf{b}\right)\right]d\Omega \qquad (5.5)$$

Thus (5.3) is converted into

$$\delta F(\mathbf{A}) = \int \xi\cdot\left[\nabla\times\frac{1}{\mu}\nabla\times\mathbf{E}_{ex} - \frac{k^2}{\mu}\mathbf{E}_{ex}\right]d\Omega + \oint\left(\xi\times\frac{1}{\mu}\nabla\times\mathbf{E}_{ex}\right)\cdot d\mathbf{S}. \qquad (5.6)$$

Since $\mathbf{E}_{ex}$ satisfies (5.1) exactly, the volume integral in (5.6) drops out, leaving

$$\delta F(\mathbf{E}) = \oint\left(\xi\times\frac{1}{\mu}\nabla\times\mathbf{E}_{ex}\right)\cdot d\mathbf{S} = -jw\oint\left(\xi\times\mathbf{H}\right)\cdot d\mathbf{S} \qquad (5.7)$$

Finally, setting the first variation of F(**E**) equal to zero yields

$$\oint\left(\xi\times\mathbf{H}\right)\cdot d\mathbf{S} = 0 \qquad (5.8)$$

(5.8) provides the natural boundary conditions for the functional.

Table 1. The basis functions for the element in Figure 4.

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & l_2\zeta_3(2\zeta_3-1) & 4sl_2\zeta_1\zeta_3 & 0 & 0 & -4rl_3\zeta_1\zeta_2 & -l_3\zeta_2(2\zeta_2-1) & 9\zeta_2\zeta_3(2\zeta_3-1)/2 & 9\zeta_2\zeta_3(2\zeta_2-1)/2 & 0 & 0 & 0 & 0 \\ 0 & -4rl_1\zeta_2\zeta_3 & -l_1\zeta_3(2\zeta_3-1) & 0 & 0 & 0 & l_3\zeta_1(2\zeta_1-1) & 4sl_1\zeta_1\zeta_2 & 0 & 0 & 0 & 9\zeta_1\zeta_3(2\zeta_1-1)/2 & 9\zeta_1\zeta_3(2\zeta_3-1)/2 & 0 & 0 \\ l_1\zeta_2(2\zeta_2-1) & 4sl_1\zeta_2\zeta_3 & 0 & 0 & -4rl_2\zeta_1\zeta_3 & -l_2\zeta_1(2\zeta_1-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9\zeta_2\zeta_3(2\zeta_2-1)/2 & 9\zeta_2\zeta_3(2\zeta_2-1)/2 \end{bmatrix}$$

5.2 Interface Conditions - At interfaces, the electric field **E** satisfies the condition that its tangential component is continuous

$$\hat{n} \times \mathbf{E}_1 = \hat{n} \times \mathbf{E}_2 \tag{5.9}$$

while the flux $\mathbf{D} = \varepsilon \mathbf{E}$ satisfies the condition that its normal component of is continuous

$$\hat{n} \cdot \varepsilon_1 \mathbf{E}_1 = \hat{n} \cdot \varepsilon_2 \mathbf{E}_2 \tag{5.10}$$

For the magnetic field **H**, the corresponding interface conditions are

$$\hat{n} \times \mathbf{H}_1 = \hat{n} \times \mathbf{H}_2 \tag{5.11}$$

$$\hat{n} \cdot \mu_1 \mathbf{H} = \hat{n} \cdot \mu_2 \mathbf{H}_2 \tag{5.12}$$

(5.12) is expressed in terms of **E** by using Faraday's law. This yields

$$\hat{n} \cdot \nabla \times \mathbf{E}_1 = \hat{n} \cdot \nabla \times \mathbf{E}_2 \tag{5.13}$$

Adopting an (n,τ,λ) coordinate system, where n is normal to the interface and τ and λ are parallel to it, (5.13) becomes

$$\left(\frac{\partial E_\lambda}{\partial \tau} - \frac{\partial E_\tau}{\partial \lambda}\right)_1 = \left(\frac{\partial E_\lambda}{\partial \tau} - \frac{\partial E_\tau}{\partial \lambda}\right)_2 \tag{5.14}$$

Notice that (5.14) involves only the tangential components and the tangential derivatives of **E**. We therefore conclude that continuity of the normal component of magnetic flux $\mathbf{B} = \mu \mathbf{H}$ is ensured by setting the tangential component of the electric field **E** to be continuous. A corollary of this is that continuity of the normal component of electric flux $\mathbf{D} = \varepsilon \mathbf{E}$ is ensured by setting the tangential component of the magnetic field **H** to be continuous. We express the tangential component of the magnetic field in terms of the electric field by using Faraday's law. The result is

$$\hat{n} \times \frac{1}{\mu_1}(\nabla \times \mathbf{E}_1) = \hat{n} \times \frac{1}{\mu_2}(\nabla \times \mathbf{E}_2) \tag{5.15}$$

Writing this out in components gives

$$\frac{1}{\mu_1}\left(\frac{\partial E_\tau}{\partial n} - \frac{\partial E_n}{\partial \tau}\right)_1 = \frac{1}{\mu_2}\left(\frac{\partial E_\tau}{\partial n} - \frac{\partial E_n}{\partial \tau}\right)_2 \tag{5.16}$$

$$\frac{1}{\mu_1}\left(\frac{\partial E_n}{\partial \lambda} - \frac{\partial E_\lambda}{\partial n}\right)_1 = \frac{1}{\mu_1}\left(\frac{\partial E_n}{\partial \lambda} - \frac{\partial E_\lambda}{\partial n}\right)_2 \tag{5.17}$$

Provided that (5.16) and (5.17) are satisfied, (5.10) is satisfied.

5.3 Natural Interface Conditions - Evaluating (5.8) by components yields

$$\oint (\xi_\tau H_\lambda - \xi_y H_\tau)\, dS = 0 \tag{5.18}$$

In terms of the electric field **E** this is

$$\oint \left[\xi_\tau \frac{1}{\mu}\left(\frac{\partial E_\tau}{\partial n} - \frac{\partial E_n}{\partial \tau}\right) - \xi_\lambda \frac{1}{\mu}\left(\frac{\partial E_n}{\partial \lambda} - \frac{\partial E_\lambda}{\partial n}\right)\right] dS = 0 \tag{5.19}$$

On exterior boundaries, if we set the tangential components of **E** equal to a given value, then ξ_τ and ξ_λ are zero and (5.19) is satisfied. On the other hand, along boundaries where we leave ξ_τ and ξ_λ to be arbitrary, we will automatically obtain a solution for **E** that satisfies

$$\hat{n} \times \mathbf{H} = 0 \tag{5.20}$$

On interior boundaries, if we impose continuity of the tangential component of **E** then

$$\begin{aligned} \xi_\tau|_1 &= \xi_\tau|_2 \\ \xi_\lambda|_1 &= \xi_\lambda|_2 \end{aligned} \tag{5.21}$$

and (5.19) yields

$$\begin{aligned} &\int \xi_\tau \left[\frac{1}{\mu_1}\left(\frac{\partial E_\tau}{\partial n} - \frac{\partial E_n}{\partial \tau}\right)_1 - \frac{1}{\mu_2}\left(\frac{\partial E_\tau}{\partial n} - \frac{\partial E_n}{\partial \tau}\right)_2\right] dS \\ &- \int \xi_\lambda \left[\frac{1}{\mu_1}\left(\frac{\partial E_n}{\partial \lambda} - \frac{\partial E_\lambda}{\partial n}\right)_1 - \frac{1}{\mu_2}\left(\frac{\partial E_n}{\partial \lambda} - \frac{\partial E_\lambda}{\partial n}\right)_2\right] dS = 0 \end{aligned} \tag{5.22}$$

Since ξ_τ and ξ_λ are arbitrary on interior boundaries, (5.22) yields exactly the required interface conditions (5.16) and (5.17).

6. CONCLUSIONS

In view of the above analysis, specifying the continuity of the normal component of electric or magnetic field is *not* required in the finite element solution of the vector wave equation. As discussed in [16] and [23], this is also true in the magnetostatic and eddy current cases. In fact, imposing continuity on the normal component of the field is wrong with finite elements of polynomial order less than 5 since it is then impossible to have C^1 or derivative continuous finite elements with an arbitrary mesh. Tangential vector finite elements provide correct solutions because the continuity of the normal component of the electric field is *not* specified, so that the discontinuity in the normal component is correctly produced by the natural boundary conditions in the variational principle.

The zeroth and first order elements presented in this paper are used in references [14-25] to solve electromagnetic field problems. In all cases, the solutions obtained are reported to be accurate and reliable.

The analysis presented in Section 4 shows that spurious solutions in finite element methods arise from two sources: (1) approximations that do not satisfy the compatibility conditions between elements, and (2) approximations that do not satisfy the unisolvence condition within an element. This allows us to define two types of spurious modes:

Type 1 Spurious Modes - Incorrect solutions generated by poor approximations of the nullspace of the curl operator.

Type 2 Spurious Modes - Incorrect solutions formed by linearly dependent approximation functions.

References [20-21] show that tangential vector finite elements eliminate the problem of Type 1 spurious modes. These modes are formed with conventional Cartesian finite elements because polynomials of order less than five do not in general satisfy the proper compatibility conditions between elements. A characteristic of Type 1 spurious modes is that the number of such modes equals the dimension of the nullspace of the differential operator.

Type 2 spurious modes arise when one or more of the approximation functions in a finite element is a linear combination of the others. An example of an element generating this type of spurious mode is provided by adding the third normal on the vacant side in the element in Figure 3. While such an element is symmetric, the analysis in Section 4 shows that the resulting set of nine functions is linearly dependent. Computations confirm that adding this ninth function introduces Type 2 spurious modes. A characteristic of Type 2 spurious modes is that their number equals the product of the number of elements times the number of dependent approximation functions per element.

In summary, we find that the affine transformation provides a powerful tool for defining vector finite elements. This representation allows us to simplify the derivation of finite elements that interpolate to either the tangential or the normal components of vector fields. The methods presented in this paper can be extended to three dimensions; work is in progress on a paper describing this extension.

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