SPURIOUS MODES OF ELECTROMAGNETIC VECTOR POTENTIAL FINITE ELEMENTS

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ABSTRACT

The mode spectra of individual electromagnetic vector potential finite elements are studied in detail. Theoretical spectra are deduced from the general properties of vector fields and the specific shape functions used in particular elements. Results are compared with spectra calculated using standard isoparametric tetrahedron and hexahedron elements. It is shown that spurious shear modes are present in addition to the more familiar spurious divergence and translation modes.

INTRODUCTION

The problem of nonphysical, *spurious* modes is well known in vector potential finite element calculations of electromagnetic fields [1]. A bewildering number of such modes are often combined with physical modes in cavity resonance and waveguide calculations. Spurious modes can also cause difficulties in magnetostatic and eddy current problems [2,3]. Various methods have been used to control spurious modes, including penalty function methods [4], constraints on element shape functions [5], and elements with special *degrees of freedom* (DOFs) [3,6]. Ratnajeevan *et al.* [7] have indicated that the form of these modes is related to element shape functions, but the precise nature of spurious modes is still not clear. The present work explores the detailed nature of spurious modes by examining the mode spectra of individual vector potential finite elements.

Maxwell's equations of electrodynamics reduce to a single equation for the vector potential in the absence of charges, permanent magnets, conductivity, and the scalar potential [1],

$$\nabla \times \left([\nu]\nabla \times \vec{A}\right) = \vec{J}_o - [\epsilon]\ddot{\vec{A}}, \tag{1}$$

where $[\nu]$ is the reluctivity; $[\epsilon]$ is the permittivity; and $\vec{J}_o$ represents excitation currents. This partial differential equation is reduced to a matrix equation for unknown vector potential DOFs $\{\vec{A}\}$ using standard finite element techniques,

$$[M]\left\{\ddot{\vec{A}}\right\} + [K]\left\{\vec{A}\right\} = \left\{\vec{J}_o\right\}, \tag{2}$$

where $[M]$ represents dielectric properties; $[K]$ represents reluctivity properties and the curl-curl operator; and $\{\vec{J}_o\}$ represents loads due to currents. Cavity resonance is analyzed by solving

$$\left(-\omega_i^2[M] + [K]\right)\{\vec{A}_i\} = 0 \tag{3}$$

for the eigenvalues ω_i^2 and eigenvectors $\vec{A}_i$. An artificial *penalty energy* equal to $\alpha(\nabla \cdot \vec{A})^2$, where α is an arbitrary *penalty parameter*, is sometimes added to suppress spurious divergence modes [4]. Contributions from this energy, represented by $[K_p]$, are added to the curl-curl matrix $[K]$.

This paper deals with spurious modes in their simplest form, i.e., within individual elements. Element modes represent all possible functional dependencies, and, as such, constitute the basic building blocks of more complicated E&M field solutions. The analysis begins with a derivation of element mode spectra by considering the basic properties of vector fields and element shape functions. These analytical results are then compared with computed results from standard vector potential elements. The agreement found between analytic and numerical results gives insight into the detailed nature of spurious modes.

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VECTOR POTENTIAL SINGULARITIES

It is convenient to depict element modes by interpreting the vector potential $\vec{A}$ as a physical *displacement* vector. According to this analogy, the addition of a constant vector $\vec{A}_o$ to $\vec{A}$ is interpreted as *uniform translation* (though no physical motion actually takes place). The nine spatial derivatives of $\vec{A}$ (e.g., $\frac{\partial A_x}{\partial y} = A_{x,y}$) can be combined into three separate sets whose properties are invariant with respect to coordinate transformations. The first set contains only one operator, the divergence (or *dilitation*), given by:

$$d = A_{x,x} + A_{y,y} + A_{z,z}. \tag{4}$$

The second set contains the three components of the curl (or *rotation*),

$$c_x = A_{z,y} - A_{y,z} \tag{5}$$
$$c_y = A_{x,z} - A_{z,x} \tag{6}$$
$$c_z = A_{y,x} - A_{x,y}. \tag{7}$$

The last set contains the five remaining independent components of shear (or *deviatoric strain*),

$$s_x = A_{z,y} + A_{y,z} \tag{8}$$
$$s_y = A_{x,z} + A_{z,x} \tag{9}$$
$$s_z = A_{y,x} + A_{x,y} \tag{10}$$
$$s_{xz} = A_{x,x} - A_{z,z} \tag{11}$$
$$s_{yz} = A_{y,y} - A_{z,z}. \tag{12}$$

The last two shear components are not unique, and can be replaced by other equivalent expressions. The only requirements on the three components s_{xz}, s_{yz} and d are that they be independent and account for the three self-derivative terms, A_{x_i,x_i}. The terms "shear" and "translation" from elasticity are used here because electrodynamics has no equivalent terminology.

The spatial dependence of any vector field may be described as a linear combination of fundamental modes. In general, the vector potential components may be expanded as polynomials:

$$A_\xi = \sum_{j,k,l=0}^{\infty} a_{\xi,j,k,l}\ x^j y^k z^l, \qquad \xi = x, y, z. \tag{13}$$

Including only zero and first order terms, there are exactly twelve independent coefficients corresponding to twelve funda-

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mental modes. Zero order displacements are decomposed into three translation modes, which represent uniform displacement along the three coordinate axes. As has been shown, the nine remaining first order displacements correspond to three uniform curl modes, five uniform shear modes, and one uniform divergence mode. The modes through first order are *pure* in the sense that they contain only one type of motion (e.g., curl modes have no translation, shear, or divergence). Higher order displacements are decomposed into similar (but nonuniform) curl, shear and divergence modes[2].

In elasticity [9], the divergence and shear components carry the elastic energy. This leaves the translation and curl modes as singular *rigid body modes.* In electrodynamics, the situation is reversed. Only curl modes carry magnetic energy, while translation, divergence, and shear represent singular "rigid body modes" of the vector potential. An artificial penalty energy proportional to $\nabla \cdot \vec{A}$ is often added to make the divergence modes

mode				*curl*			*shear*					*div*
ID	A_x	A_y	A_z	x	y	z	x	y	z	xz	yz	
zero order												
T1	1	.	.	.	.	.	.	.	.	.	.	.
T2	.	1	.	.	.	.	.	.	.	.	.	.
T3	.	.	1	.	.	.	.	.	.	.	.	.
first order												
C1-1	.	-z	y	2	.	.	.	.	.	.	.	.
C1-2	z	.	-x	.	2	.	.	.	.	.	.	.
C1-3	-y	x	.	.	.	2	.	.	.	.	.	.
S1-1	y	x	.	.	.	.	.	.	2	.	.	.
S1-2	.	z	y	.	.	.	2	.	.	.	.	.
S1-3	z	.	x	.	.	.	.	2	.	.	.	.
S1-4	x	.	-z	.	.	.	.	.	.	2	.	.
S1-5	.	y	-z	.	.	.	.	.	.	.	2	.
D1-1	x	y	z	.	.	.	.	.	.	.	.	3

Table 1: The Spatial Dependence of the Twelve Normal Modes Expected in a Four-Node TETRA Element

nonsingular. However, singular translation and shear modes remain.

First consider the simplest three-dimensional element, a standard isoparametric tetrahedron or TETRA element [8]. Shape functions are exactly complete through first order, i.e., the shape functions associated with each vector component have terms proportional to $(1, x, y, z)$ with no higher order terms. The element has three DOFs (A_x, A_y and A_z) at each of four grid points, and thus has twelve independent modes.

The modes of a single TETRA element correspond exactly to the three zero order and nine first order modes described above. These twelve modes are summarized in Table 1. The spatial dependence of each mode is shown with its curl, shear and divergence. The three modes designated as T1-T3 represent zero order uniform translation. Three curl modes (C1-1 through C1-3), five shear modes (S1-1 through S1-5) and one divergence mode (D1-1) are also shown. In electrodynamics, the curl modes represent uniform magnetic fields in each of the three coordinate directions, while the translation, shear and divergence modes do not represent physical quantities.

The mode structure of a standard, isoparametric hexagon or HEXA element [8] is complicated by the presence of higher order shape functions. The HEXA element has three DOFs at each of eight grid points, for a total of 24 independent modes. The shape function[3] for each vector component contains constant, first order, bilinear and trilinear terms $(1, x, y, z, xy, yz, xz, xyz)$. The first twelve modes correspond to the twelve TETRA modes shown in Table 1. The remaining twelve higher order modes are

[2]Translation modes occur only in zero order. Curl and shear are mixed in higher order modes, but separate divergence and curl modes can always be constructed.

mode				*curl*			*shear*					*div*
ID	A_x	A_y	A_z	x	y	z	x	y	z	xz	yz	
second order												
C2-1	.	-xz	xy	2x	-y	-z	.	2y	-z	.	.	.
C2-2	xz	-yz	.	y	x	.	-y	x	.	z	-z	.
C2-3	-xy	.	yz	z	.	x	z	.	-x	-2y	-y	.
C2-4	xz	.	.	.	x	.	.	x	.	z	.	z
C2-5	yz	.	-xy	-x	2y	-z	-x	.	z	.	.	.
C2-6	.	xy	-xz	.	z	y	.	-z	y	x	2x	.
C2-7	.	.	yz	z	.	.	z	.	.	-y	-y	y
C2-8	.	xy	.	.	.	y	.	.	y	.	x	x
S2-1	yz	xz	xy	.	.	.	2x	2y	2z	.	.	.
third order												
C3-1	xyz	.	.	.	xy	-xz	xy	xz	.	yz	.	yz
C3-2	.	xyz	.	-xy	.	yz	-xy	.	yz	.	xz	xz
C3-3	.	.	xyz	xz	-yz	.	xz	yz	.	-xy	-xy	xy

Table 2: The Spatial Dependence of the Twelve Additional Normal Modes Expected in an Eight-Node HEXA Element

shown in Table 2. As can be seen, there are eleven additional curl modes (all with shear) and one additional pure shear mode; but no additional pure divergence modes. Some curl modes also have divergence because they are *incomplete.* If the missing quadratic and cubic terms were present in the shape functions, then curl and divergence would be contained in separate modes.

EXPERIMENTAL RESULTS

Experimental mode spectra were calculated for standard isoparametric TETRA and HEXA elements using a research version of MSC/NASTRAN, a large commercial structural analysis program [10]. This program has several standard, reliable eigenvalue solving routines. The results reported here were obtained using the Givens tridiagonalization method; but similar results were obtained using other methods, including the Inverse Power and Lanczos methods. In each case the finite element model consisted of only a single element with no constraints or loads applied. The HEXA element was given a simple unit cube geometry, while the TETRA element had vertices at (0,0,0), (1,0,0), (0,1,0) and (0,0,1) in Cartesian coordinates. The divergence penalty energy was added in order to evaluate curl and divergence separately. For simplicity, $\epsilon = \nu = 1$ in all cases.

The divergence and curl of each eigenvector was evaluated in order to better correlate results with theoretical mode spectra. The "curl energy" C was calculated from the eigenvectors $\{\vec{A}_i\}$ and curl-curl matrix $[K]$ using

$$C = \{\vec{A}_i\}^T [K] \{\vec{A}_i\}. \tag{14}$$

The curl energy is a positive semi-definite measure of the curl in the eigenvector (it is also the total magnetic field energy stored within the problem volume). The "divergence energy" D was calculated using the penalty matrix $[K_p]$ as follows:

$$D = \alpha^{-1} \{\vec{A}_i\}^T [K_p] \{\vec{A}_i\}. \tag{15}$$

In Eq. (15), the work associated with $[K_p]$ is divided by the penalty parameter α. As a result D represents the "geometric" divergence, a quantity independent of α. The ratio D/C is a relative measure of mode divergence.

The modes computed for the TETRA element are shown in Table 3. Modes are listed in order of increasing frequency along with the divergence and curl energies D and C. The first eight modes all have (machine) zero frequency. These modes, with neither divergence nor curl, are linear combinations of the translation modes (T1 through T3) and the shear modes (S1-1 through S1-5) shown in Table 1. Modes nine through eleven

[3]Strictly speaking, Cartesian (x, y, z) shape functions apply only to rectangular brick elements. Parametric mapping coordinates (ξ, η, ζ) are used for more general element shapes [8].

$\epsilon = \nu = 1; \alpha = 1 \times 10^5$

mode no.	frequency Hz	D	C
1	5.03E-09	3.18E-35	2.76E-25
2	6.81E-09	0	7.41E-19
3	1.76E-07	8.28E-33	1.76E-18
4	9.47E-09	1.54E-33	-2.23E-18
5	1.06E-06	3.08E-33	-3.81E-19
6	2.20E-06	1.54E-33	3.14E-17
7	3.03E-06	1.54E-33	2.29E-18
8	3.14E-06	0	-3.07E-18
9	1.01E+00	4.71E-39	2.00E+00
10	1.59E+00	6.15E-31	2.28E+00
11	1.59E+00	5.89E-31	2.19E+00
12	5.51E+02	6.00E+00	2.69E-30

Table 3: Normal Modes Calculated for a Single Four-Node TETRA Element

correspond to the curl modes C1-1 through C1-3. Finally, the twelfth mode corresponds to the divergence mode D1-1. In the absence of the divergence penalty energy, this mode is also singular. Here, it has an artificially high frequency due to the large penalty parameter, $\alpha = 1 \times 10^5$.

The 24 modes calculated for the HEXA element are shown in Table 4. There are nine zero frequency modes representing linear combinations of the singular modes T1-T3, S1-1 through S1-5, and S2-1. Modes 10 through 17 represent the eight divergence-free curl modes: C1-1, C1-2, C1-3, C2-1, C2-2, C2-3, C2-5, and C2-6. Modes 18 through 23 represent the six coupled curl/divergence modes: C2-4, C2-7, C2-8, C3-1, C3-2 and C3-3. Finally, mode 24 represents the only pure divergence mode, D1-1.

DISCUSSION

These results show good agreement between calculated mode spectra and those predicted on the basis of vector field properties and shape functions. Close agreement was found with respect to divergence and curl characteristics, and with respect to the number of modes. (Mode shapes were also confirmed directly using a constraint technique; results are not shown here.) It is clear that individual elements exhibit a number of singular shear and translation modes in addition to the usual spurious divergence modes. Control of these modes in larger models is a key factor in obtaining unique, nonsingular solutions to vector potential E&M problems.

Shear and translation modes are similar to rigid body modes frequently seen in structural mechanics. If a body is unconstrained, then it will exhibit six zero frequency modes: three translation and three rotation. The zero frequency modes found in E&M elements also represent unconstrained "motions" of the vector potential. However, since the energies involved are different from those in structural mechanics, the number and nature of these modes is different. Like mechanical rigid body modes, singular E&M translation and shear modes can be removed using constraint techniques.

CONCLUSIONS

Divergence modes are not the only singular modes of the electromagnetic vector potential. A detailed analysis of mode structure in individual finite elements shows that most singular modes are related to translation and shear. The presence of these modes is deduced from the general properties of vector fields and the specific shape functions used. Much effort has been devoted to suppressing the obvious divergence singularities of Maxwell's equations. These results show that translation and shear singularities must also be considered.

$\epsilon = \nu = 1; \alpha = 1 \times 10^5$

mode no.	frequency Hz	D	C
1	4.02E-07	7.68E-18	-3.98E-16
2	7.12E-07	1.22E-16	-2.87E-17
3	7.67E-07	9.20E-17	-1.12E-16
4	7.89E-07	8.05E-17	-5.30E-16
5	9.95E-07	4.43E-17	-5.63E-16
6	1.05E-06	-3.26E-17	2.29E-17
7	1.18E-06	7.91E-17	4.70E-13
8	1.27E-06	9.71E-17	3.34E-12
9	1.34E-06	1.75E-13	4.98E-07
10	5.51E-01	1.36E-16	1.61E+00
11	5.51E-01	1.01E-16	1.44E+00
12	5.51E-01	-1.38E-16	1.73E+00
13	7.80E-01	-1.78E-16	9.86E+00
14	7.80E-01	-2.62E-16	8.67E+00
15	7.80E-01	-2.14E-16	1.01E+01
16	9.95E-01	4.19E-19	6.19E+00
17	9.95E-01	2.47E-17	7.68E+00
18	1.74E+02	6.31E-01	1.26E+00
19	1.74E+02	5.52E-01	1.10E+00
20	1.74E+02	5.08E-01	1.02E+00
21	2.47E+02	3.57E+00	1.78E+00
22	2.47E+02	3.44E+00	1.72E+00
23	2.47E+02	3.87E+00	1.94E+00
24	3.02E+02	3.60E+01	3.52E-16

Table 4: Normal Modes Calculated for a Single Eight-Node HEXA Element

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