1 Problem Statement

Spaces

$$X^D = \{v \in H^1(\Omega) \mid v|_{\partial\Omega^D} = u^D\}$$
$$X = \{v \in H^1(\Omega) \mid v|_{\partial\Omega^D} = 0\}$$

Notation

$$A_e^R = \partial\Omega^R \cap A_e$$
$$A_e^N = \partial\Omega^N \cap A_e$$

Solution space

$$u \in X^D$$

Test space

$$v \in X$$

Equation

$$\nabla \cdot (a\nabla u) + k^2 u = s$$

with 3 kinds of boundary conditions (Dirichlet, Neumann, Robin)

$$u\Big|_{\partial\Omega^D} = u^D$$
$$\hat{n} \cdot \nabla u\Big|_{\partial\Omega^N} = f$$
$$\left(\gamma u + \frac{\partial u}{\partial \hat{n}}\right)\Big|_{\partial\Omega^R} = g$$

Weak form

$$-\int_\Omega a\nabla u \cdot \nabla v + \int_\Omega k^2 uv - \int_{\partial\Omega^R} \gamma auv = \int_\Omega sv - \int_{\partial\Omega^R} agv - \int_{\partial\Omega^N} afv$$

Proceed as $u^D = 0$, i.e. homogeneous Dirichlet, and use explicit elimination in the last step. Assume f, g, u^D, s, a, γ elemental / piecewise constant.

Elemental contribution (A_e)

- Bilinear matrix

$$-\int_{A_e} a\Big(J^{-T}\nabla_L L_i\Big)^T \cdot \Big(J^{-T}\nabla_L L_j\Big) + \int_{A_e} k^2 L_i L_j - \int_{A_e^R} \gamma a L_i L_j$$

- Load vector

$$\int_{A_e} sL_i - \int_{A_e^R} agL_i - \int_{A_e^N} afL_i$$

In which,

$$(J^T)^{-1} = \frac{1}{2A_e}\begin{pmatrix} y_2 - y_3 & -y_1 + y_3 \\ -x_2 + x_3 & x_1 - x_3 \end{pmatrix} \tag{1}$$

$$\nabla_L \vec{L} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \tag{2}$$

$$\int_{A_e} L_1^a\, L_2^b\, L_3^c = a!\, b!\, c!\, \frac{2A_e}{(a+b+c+2)\,!} \tag{3}$$

For a boundary edge of length L, let i, j denote the nodal basis index on the edge.

$$\int_e L_i L_j = \frac{L}{6}(1 + \delta_{ij})$$
$$\int_e L_i = \frac{L}{2}$$

Explicit elimination. Denote $\{I^D\}$ as the nodal indices on $\partial\Omega^D$

- Remove $\{I^D\}$ rows of the equation
- Load vector
$$F = F - M\{:, I^D\} \cdot u^D$$
- Remove $\{I^D\}$ columns of the bilinear matrix

Finally, solve
$$Mu = F$$
For the general eigenvalue problem,
$$A_1 u = \lambda A_2 u$$
we have
$$Au \;\to\; A(\neg I^D, \neg I^D)u(\neg I^D) + A(\neg I^D, I^D)u(I^D) \equiv \bar{A}\bar{u} + f$$
Thus the general eigenvalue problem reduces to
$$\left(\bar{A}_1 - \bar{A}_2\right) \bar{u} = f_2 - f_1$$
The solution can be unique for non-homogeneous Dirichlet boundary conditions.