Consider optimizing on finite time horizon 0 to T ($T \to \infty$ has similar argument). Action is $\{a\} = \{a_1, a_2, \cdots, a_T\}$, where unit time represents a tiny physical time δt. Unkonwn state indicated by b_t. At time 0, Monte Carlo sample prior b_0 and get $s_1, \cdots, s_N$, and obtain optimal control

$$\{a^0\} = \text{argmax}_a \sum_{i=0}^{T} J(a, s_i)$$

Adopt this control and evolve the true (uncertain) system (e.g. oil reservoir) for a unit time, obtain observation o. Assume state s distributed according to b_0, we try to find out a 'transport map' $f(s)$ (Tarek 2012, Bayesian Inference with Optimal Maps) that distributes according to

$$\propto b_0(s)\mathbb{P}[o|s, a] \approx \frac{1}{N} \sum_{i} \mathbb{P}[o|s_i, a]$$

The map 'transports' samples s_i to $f(s_i)$. At the next time step, we seek to obtain optimal control from an updated belief

$$\{a^1\} = \text{argmax}_a \sum_{i=1}^{T} J(a, s^1)$$

where

$$s^1 = f(s_i) \approx s_i + c_i \delta t$$

Here δt is the physical time increment. This is based on the assumption that information 'arrives gradually' (sorry for the poor wording). We use adjoint to find $a^1 - a^0$'s sensitivity to c_i's.
Write the reward for time 0 to 1 as r_0, and reward from t to T as J_t, then $J_0 = r_0 + J_1$
we have

$$\left.\frac{\partial r_0}{\partial a} + \frac{\partial J_1}{\partial a}\right|_{a^0} = 0 \tag{1}$$

$$\left.\frac{\partial J_1}{\partial a}\delta a + \frac{\partial J_1}{\partial s} c\delta t\right|_{a^1} = 0 \tag{2}$$

After approximation

$$\left.\frac{\partial J_1}{\partial a}\right|_{a^0} \approx \left.\frac{\partial J_1}{\partial a}\right|_{a^1}$$

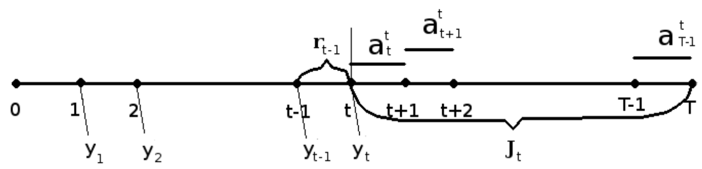
we get

$$\delta a = \left(\frac{\partial r_0}{\partial a}\right)^{-1} \frac{\partial J}{\partial s} c\delta t \tag{3}$$

This gives a sequential update formulation of optimal control.

0.1 Remarks

1.The key step is to construct the optimal transport map f. Tarek parameters f by parametric chaos on the whole domain of f. However, by noticing $f \approx I + \delta$ for every time increment, I feel there may be some way to simplify the construction. Not sure how to do it yet. (requires knowledge of manifold)?
2.The resulting policy is not optimal (the optimal can only be obtained by backward induction on belief space). Is here any remedy?



1 History Matching with Ensemble Kalman Filter

Take incompressible reservoir as an example. States $x^i_{0|0} = [s^i_0, k^i_0]$ are the $t = 0$ 'augmented' samples consisted of saturation and permeabilities. y_t is the observations from the true reservoir. The subscript $0|0$ means samples at time 0 (first 0) based on observations **at and before** time 0 (second 0, which means no observations yet). Superscript index $i = 1, \cdots, N$ indicates the samples.

1.1 Forecast Step

Obtain $x^i_{t|t-1}$ from $x^i_{t-1|t-1}$ given control u_{t-1}. The s^i_t part is updated through reservoir simulator

$$\begin{aligned} s^i_{t|t-1} &= f(x^i_{t-1|t-1}, u_{t-1}) \\ k^i_{t|t-1} &= k^i_{t-1|t-1} \end{aligned} \tag{4}$$

Forecast observation

$$y^i_{t|t-1} = g(x^i_{t|t-1}, u_t) \tag{5}$$

1.2 Update Step

$$x^i_{t|t} = x^i_{t|t-1} + Cov[x_{t|t-1}, y_{t|t-1}]Cov^{-1}[y_{t|t-1}, y_{t|t-1}] \left(y_t - y^i_{t|t-1} \right) \tag{6}$$

2 Sequential Optimal Control

Denote $\tilde{J} = \sum_i J^i$ as the ensemble objective function. $\tilde{J}_{t|t}$ indicates the objective function after time t given the optimal control obtained with the observations $y_1, \cdots, y_t$.

Denote

$$a^t = \{a^t_t, a^t_{t+1}, \cdots, a^t_{T-1}\}$$

as the optimal control starting from time t with observations up to y_t.

$$\bar{a}^t = \{a^{t-1}_t, a^{t-1}_{t+1}, \cdots, a^{t-1}_{T-1}\}$$

as the optimal control starting from time t with observations up to y_{t-1}.

$$\begin{aligned} \tilde{J}_{t-1|t-1} &= \sum_i J_{t-1} \left(\{a^{t-1}_{t-1}, \bar{a}^t\}; x^i_{t-1|t-1} \right) \\ &= \sum_i \left(r_{t-1}(a^{t-1}_{t-1}; x^i_{t-1|t-1}) + J_t \left(\bar{a}^t; x^i_{t|t-1} \right) \right) \end{aligned} \tag{7}$$

$$x^i_{t|t-1} = F(x^i_{t-1|t-1}, a^{t-1}_{t-1}) \tag{8}$$

Eqn(8) can be obtained by EnKF forecast step. Also,

$$\tilde{J}_{t|t} = \sum_i J_t \left(a^t; x^i_{t|t} \right) \tag{9}$$

We have

$$\begin{aligned} \frac{d\tilde{J}_{t-1|t-1}}{da_{t-1}^{t-1}} &= 0 \\ \frac{\partial \tilde{J}_{t-1|t-1}}{\partial \bar{a}^{t}} &= 0 \end{aligned} \tag{10}$$

and

$$\frac{\partial \tilde{J}_{t}}{\partial a^{t}} = 0 \tag{11}$$