Introduction to Module

Coventry University

Module overview

- Introduction
  - Introduction to digital communications
- Source coding
- Modulation
  - Introduction to modulation
  - Digital modulation techniques
  - Binary modulation
  - Multilevel modulation
- BER and capacity
- Introduction to channel coding
  - Block codes
  - Cyclic codes
  - Convolutional codes
  - Viterbi decoding

Module overview

- Channel
  - Wireless channel
  - Optical Communication Channel
  - Optical wireless channel
- Security and Emerging technologies
  - 5G architecture
  - Security in network

Assessment

- This is a 10 credit module and final grade will be determined by 100% examination.
  - Exam will be during the exam period in December
  - It will be a Time Constrained Assessment
    - Open book online assessment.
    - There will be 3 questions covering all topics and you will have to answer all of them.
- Overall module pass mark is 40%.
- There will be 2 more attempts if failed in the first attempt.
- Mark will be kept to 40 for 2nd and third attempt.

Ethics

- Please attend the sessions on time and read/listen all the uploaded materials before the lecture.
- You are encouraged to ask relevant questions during the lecture and also share your view with others
- Try to read latest journal/conference papers related to the topic
- Do not copy or cheat in the TCA . This will be dealt very seriously by the university
- At last, respect each other and consider that different people might have different background

Introduction to Communication

Learning outcomes

- At the end of this video you will:
  - Have an overall idea of communication system
  - Why digital communication.
  - Be able to describe the different blocks of a digital communication systems

Introduction to communication

- **Communication** is transferring data **reliably** from one point to another

Information Source → **Transmitter** → **Channel** → **Receiver** → **Destination sink**

- **Information source:**
  - The source of data
    - Data could be: Human voice, CD, video etc..

Introduction to communications(cont..)

- **Transmitter**
  - Converts the source information into a suitable form for transmission through a channel(media)
- **Channel**
  - The physical medium used to send the signal
  - Physical Mediums (Channels):
    - Wired : twisted pairs, coaxial cable, fiber optics
    - Wireless: Air, vacuum and water
- **Receiver**
  - Extracting the message/code in the received signal
- **Information Sink**
    - The final stage
    - The user

Noise

- Noise is **undesired** signal that corrupts the original signal and degrades it

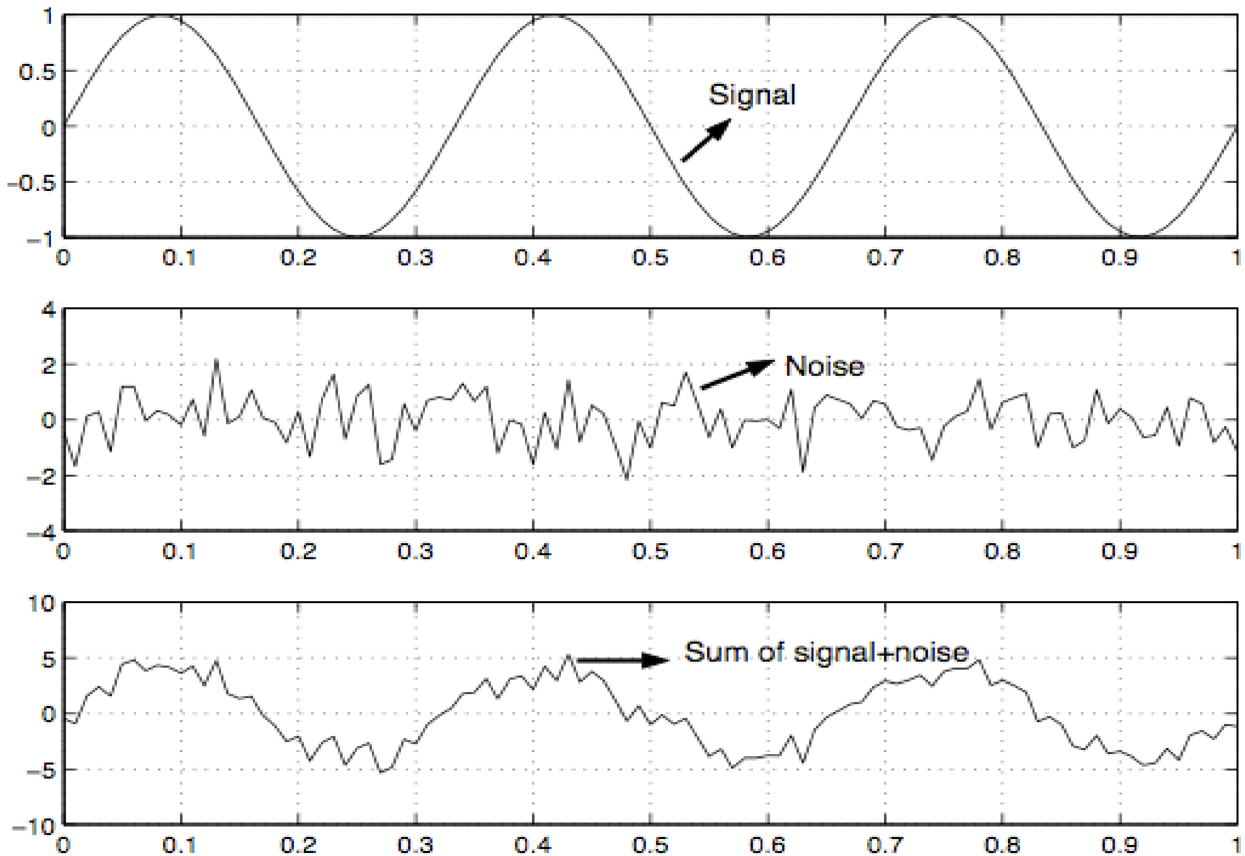
- Noise sources: can be different for different mediums

  - Thermal noise-
    - random fluctuation of a charge carrier in a resister

  - Atmospheric electromagnetic noise (Interference with another signals that are being transmitted at the same channel)

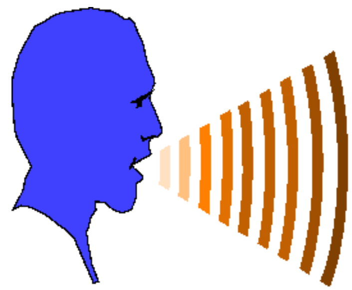
  - Sunlight – for visible light communication system

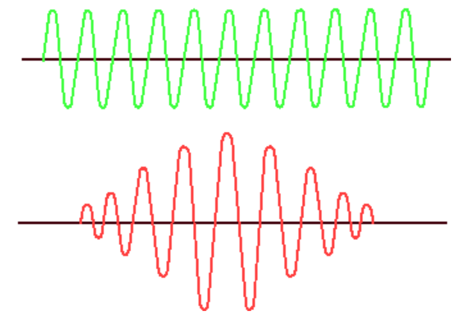
Noise (cont..)



Transmission systems

- Analogue system
  - Continuous modulation
  - Reliability is measured in terms of SNR





- Digital system
  - Signals made up of discrete symbols
  - Accuracy is specified in terms of bit error rate (Probability of making a bit error).

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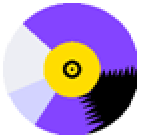
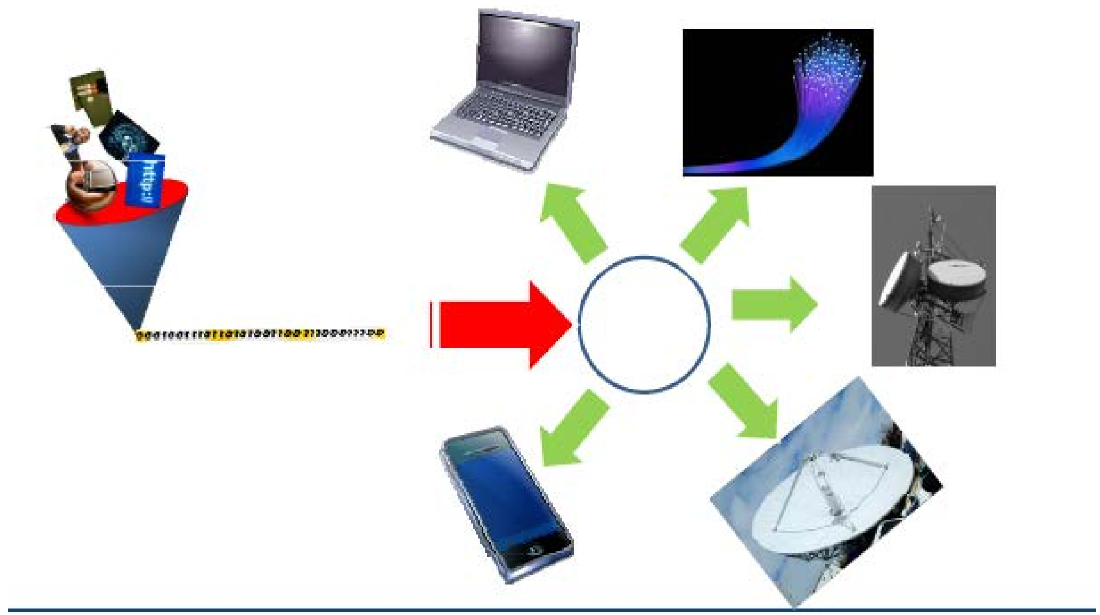


Why digital communication?

- Noise introduces distortion to analog signals.
- Better tradeoff between bandwidth efficiency and energy efficiency than analogue .
- Allows integration of voice, video, and data on a single system (multiplexing).
- Many signal processing techniques are available to improve system performance
  - Source coding, channel (error-correction) coding, equalization, encryption, etc.

Modern approach

Convert the source data into a binary sequence and then convert the binary sequence into a suitable form to transmit

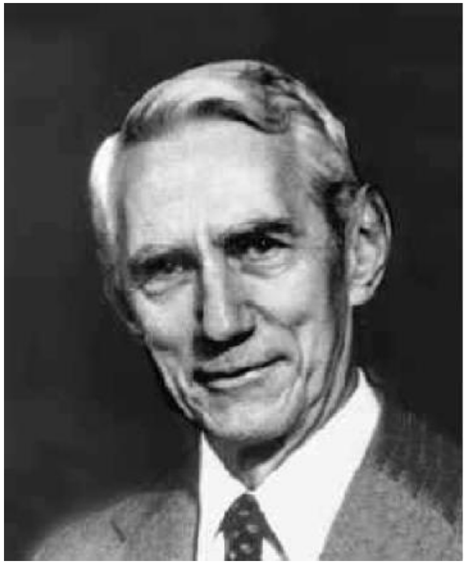


Theory

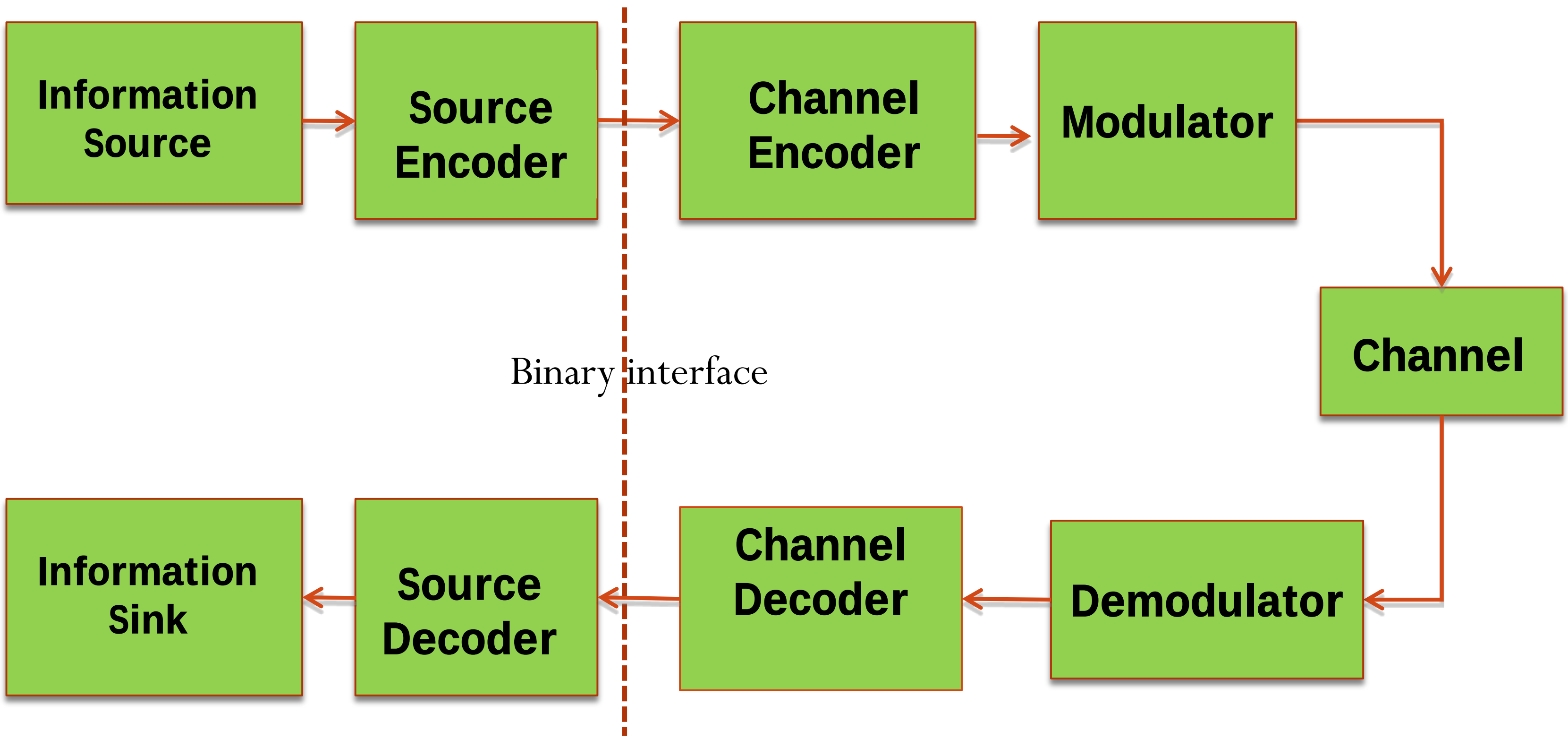
- **Information theory – A mathematical theory of communication**

  "The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point."

  -- C. Shannon, "A Mathematical Theory of Communication," *The Bell System Technical Journal*, vol. 27, 1948.



Simplified block diagram of a digital communication system

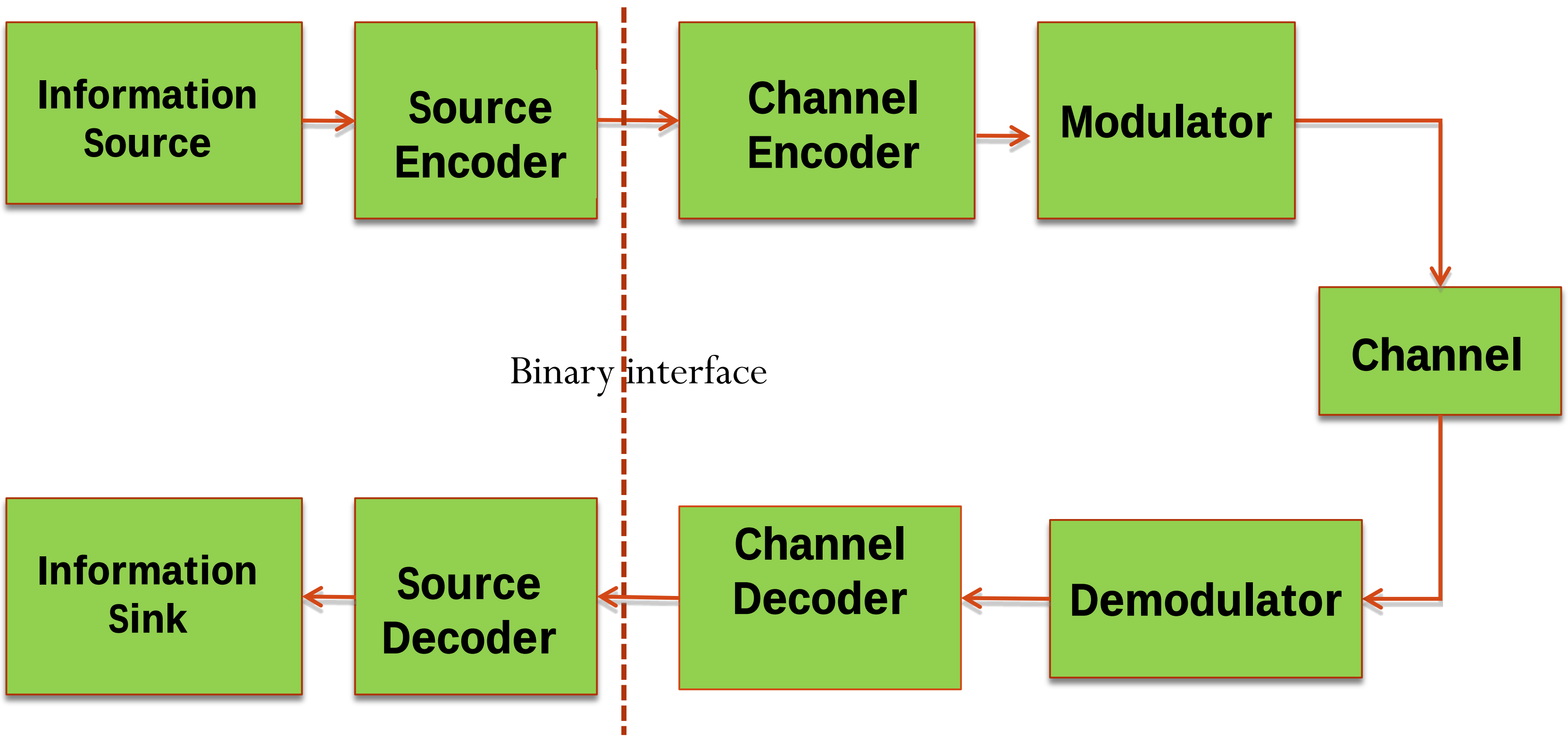


Source coding

Learning outcomes of today's lecture

- At the end of this lecture you will:
  - Know what is source coding
  - Understand different source coding techniques

Simplified block diagram of a digital communication system



What is source coding?

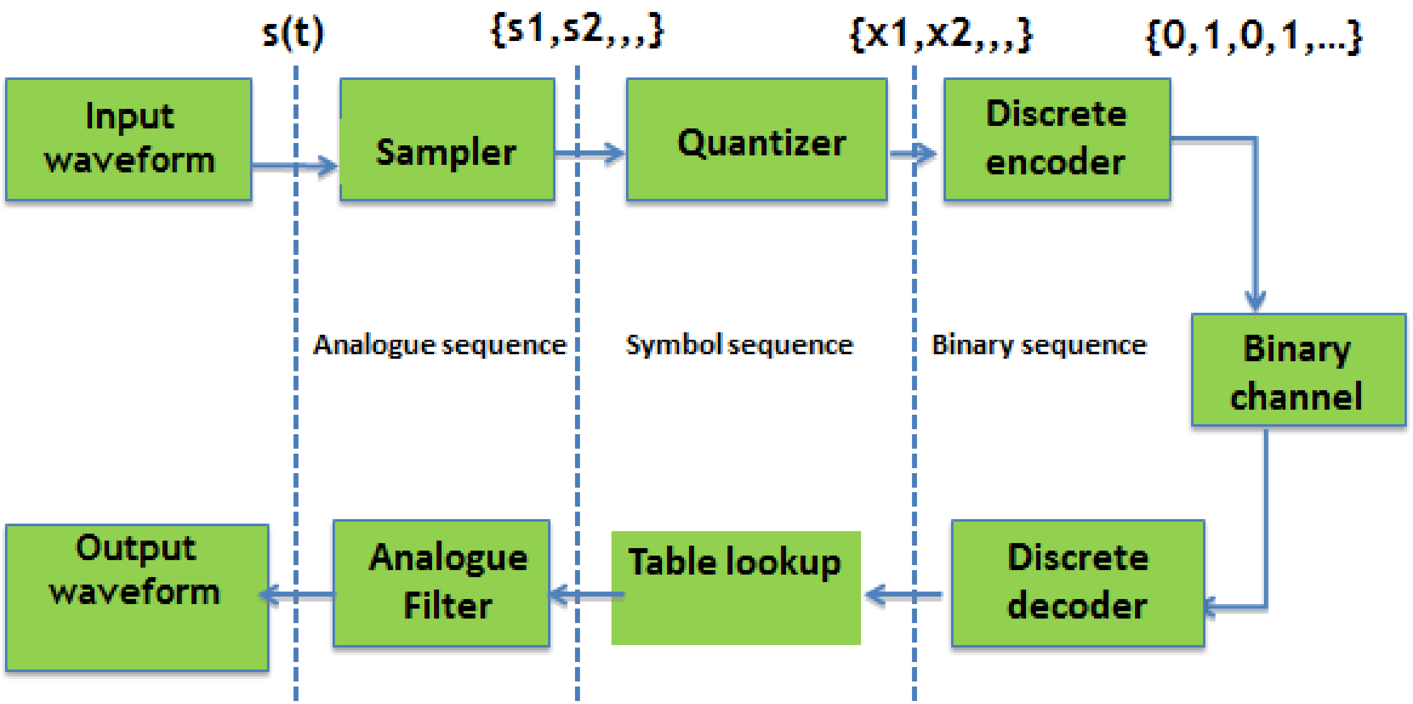
- A digital communication system is designed to transmit information in digital form
- Source coding: the conversion of the source output to a digital form, performed by a source encoder, which produces a sequence of binary digits

- **Design goal:** ***represent a source with the fewest bits such that best recovery of the source from the compressed data is possible***

Source coding for different sources

- **Source coding for a discrete source**
  - The information source can be uniquely retrieved from the encoded string of bits
  - Uniquely decodable
  - Lossless coding
- **Source coding for analog sources**
  - Quantization is necessary
  - It introduces distortion
  - Lossy compression
  - A tradeoff between the bit rate and the distortion

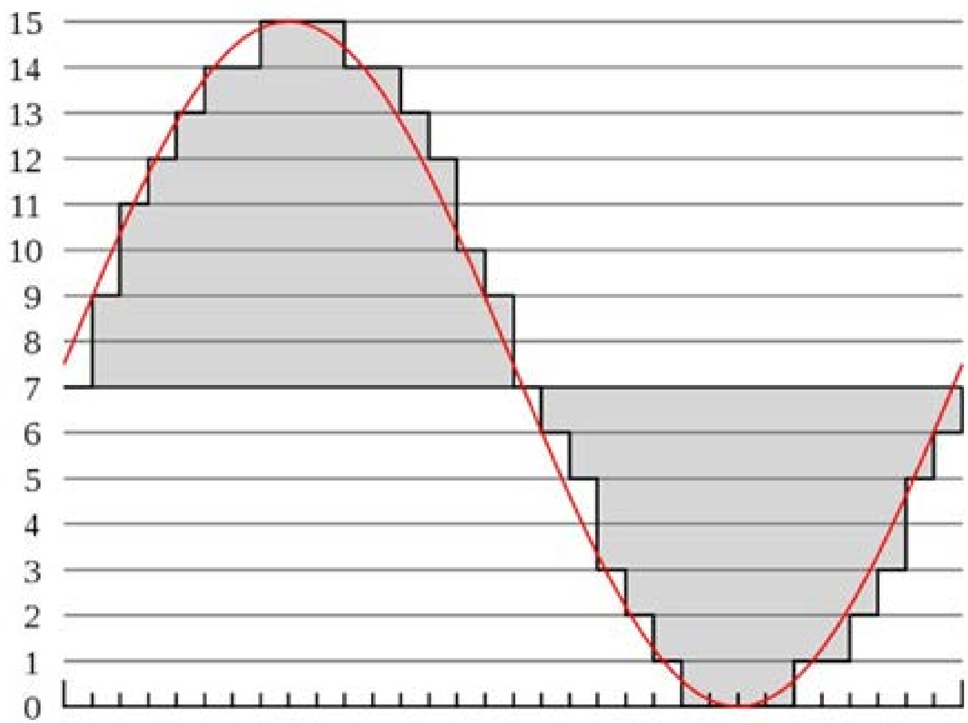
A general diagram for source coding



An example

- **Pulse-code modulation (PCM)**
  - Digital representation of an analog signal
  - The standard form for digital audio and various Blu-ray, DVD and CD formats
- **Operation**
  - Sample the magnitude of the analog signal regularly at uniform intervals
  - Each sample is then quantized to the nearest value within a range of digital steps
- **The PCM process is commonly implemented on a single integrated circuit generally referred to as an analog-to-digital converter (ADC).**

PCM



- **In telephony, a standard audio signal for a single phone call is encoded as 8,000 analog samples per second, of 8 bits each, giving a 64 kbit/s digital signal**

Discrete source coding

Fixed length code for discrete source

- Map sequence of symbol into binary sequence with unique decodability
- Simple approach
  - Map each source symbol into an L–tuple of binary digits
- For example if X consist of the 6 symbols {a,b,c,d,e,f} then following fixed-length code of block length L=3 can be used

C(a)=000

C(b)=001

C(c)=010

C(d)=011

C(e)=100

C(f)=101

Discrete source coding

Variable length code for discrete source

- Probable symbols should have shorter codewords than improbable to reduce bpss
- A variable length source code C encodes each symbols x in source alphabet X to a binary codeword C(x) of length l(x)
- For example for X={a, b, c}

  C(a)=0

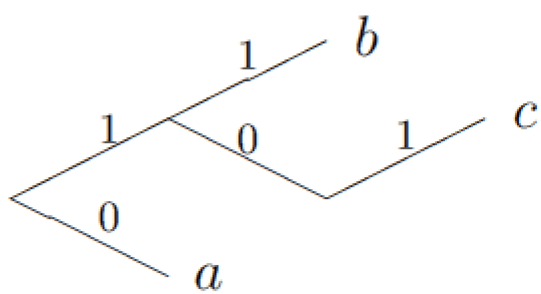
  C(b)=10

  C(c)=11

Discrete source coding

Prefix free code for discrete source

- Prefix-Free – No codeword can be a prefix of any other codeword



$$a \rightarrow 0$$
$$b \rightarrow 11$$
$$c \rightarrow 101$$

Huffman Code

- Characteristics of Huffman Codes:
  - Prefix-free, variable length code that can achieve the shortest average code length for an alphabet
  - Most frequent symbols have short codes
- Procedure
  - List all symbols and probabilities in descending order
  - Merge branches with two lowest probabilities, combine their probabilities
  - Repeat until one branch is left

Huffman Code Example

Huffman tree generated from the exact frequencies of the text **"this is an example of a huffman tree"**

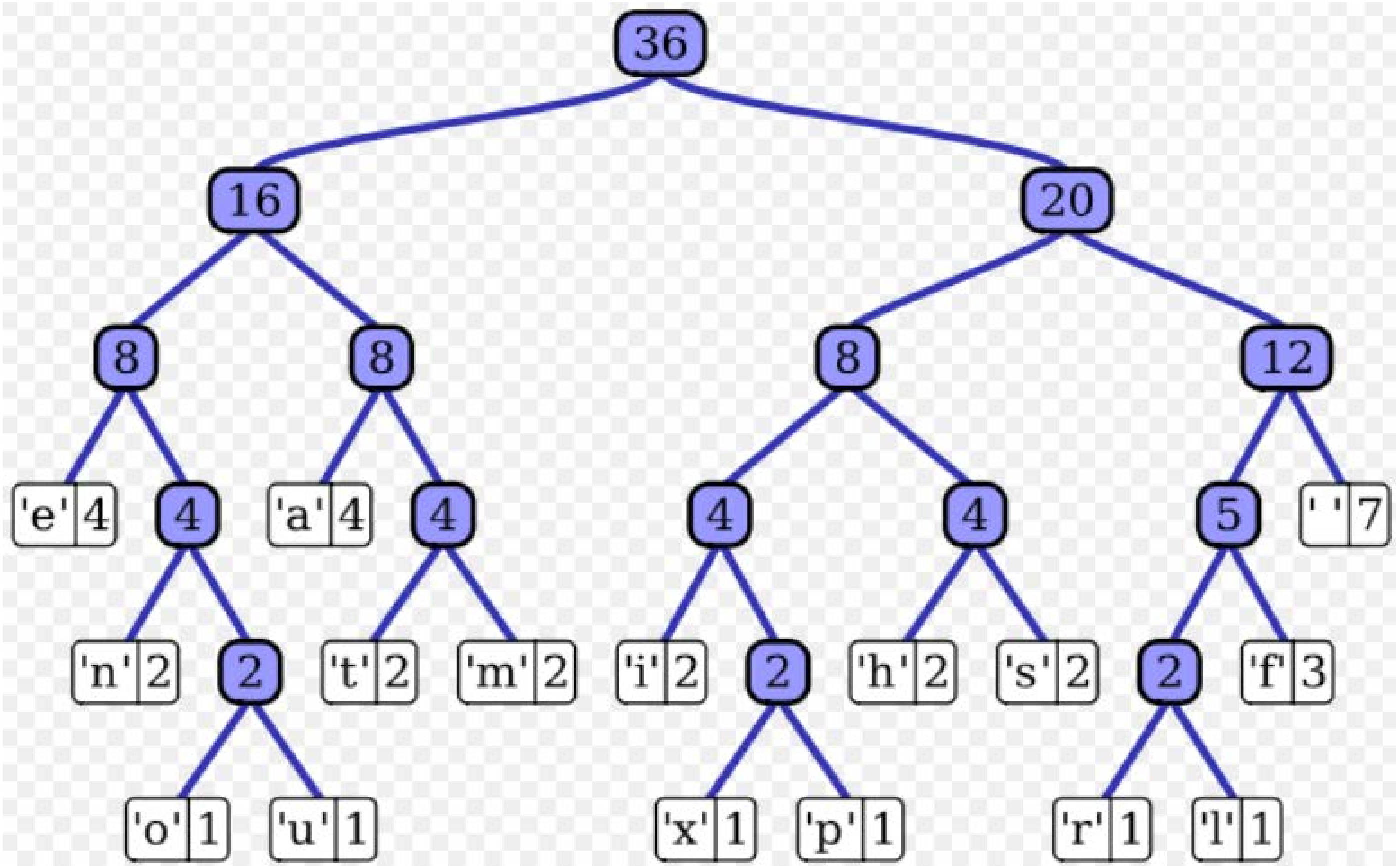


Figure from Wikipedia

Lempel-Ziv universal data compression

- **In Huffman coding** we need to know the source statistics
- **Lempel-Ziv algorithm**
  - Belongs to the class of *universal source coding algorithms*
  - Does not need to know the source statistics
  - Applications: UNIX compress, GIF, TIFF, PDF, etc
- Look through a code dictionary with already coded segments
  - If matches segment,
    - send <dictionary address, next character> and store segment + new character in dictionary
  - If no match,
    - store in dictionary, send <0,symbol>

LZ coding example

- Encode [a b a a b a b b b b b b a b b b b b a]

Code Dictionary

Address	Contents	Encoded Packets
1	a	< 0 , a >
2	b	< 0 , b >
3	aa	< 1 , a >
4	ba	< 2 , a >
5	bb	< 2 , b >
6	bbb	< 5 , b >
7	bba	< 5 , a >
8	bbbb	< 6 , b >
		< 4 , - >

Note: 9 code words, 3 bit address, 1 bit for new character,

Lossy data compression

- For continuous-amplitude analog sequences
  - Lossless compression is impossible
  - Lossy compression through scalar or vector quantization
- Quantization
  - Uses a finite number of levels to represent each continuous-amplitude symbol
  - Introduces distortion, a loss of signal fidelity
  - Then we may use Huffman coding to improve the efficiency
- Tradeoff between bit rate and distortion!
  - Minimise bit rate for a given distortion
  - Minimise distortion for a given rate

Summary

- Digital communications' system blocks
  - Information source
  - Source encoder
- Source encoder
  - Discrete encoder
    - Huffman coding
    - LZ coding

Self study

- **The Kraft inequality**
- **Unique decodability**
- **Entropy**

7057CEM : Digital communiation Systems

Introduction and source coding

1. The signals

$$g_1(t) = 10\cos(100\pi t)$$

and

$$g_2(t) = 10\cos(50\pi t)$$

are both sampled at times $t_n = n/f_s$, where $n = 0, \pm1, \pm2, \ldots..$, and $f_s = 75$ samples per second. Show that the obtained $g_1(t_n)$ and $g_2(t_n)$ are identical. What is this phenomenon called?

2. Encode the following phrase using Huffman coding, then calculate the saving percentage of the compression.
The phrase is : MISSISSIPPI RIVER

3. The characters a to h have the set of frequencies based on the first 8 Fibonacci numbers as follows:
a : 1, b : 1, c : 2, d : 3, e : 5, f : 8, g : 13, h : 21
A Huffman code is used to represent the characters. What is the sequence of characters corresponding to the following code?
110111100111010

4. Encode the following using Lampel Ziv algorithm.
AABABBBABAABABBBABBABB

7057CEM : Digital communiation Systems

Introduction and source coding

1. The signals

$$g_1(t) = 10\cos(100\pi t)$$

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Solution:

For $n \in$ integers and $f_s = 75$, $t_n = n/f_s = n/75$, then

$$g_1(t_n) = 10\cos((100/75)\pi n) = 10\cos(1.333\pi n)$$
$$g_2(t_n) = 10\cos((50/75)\pi n) = 10\cos(0.666\pi n)$$

Remember that $\cos(\theta) = \cos(\theta + 2\pi k)$, where $k \in$ integers.
$\cos(-\theta) = \cos(\theta)$

Example: when $k = -1$, then $\cos(\theta) = \cos(\theta - 2\pi) = \cos(2\pi - \theta)$

Since the cos(.) function is periodic over 2π , then

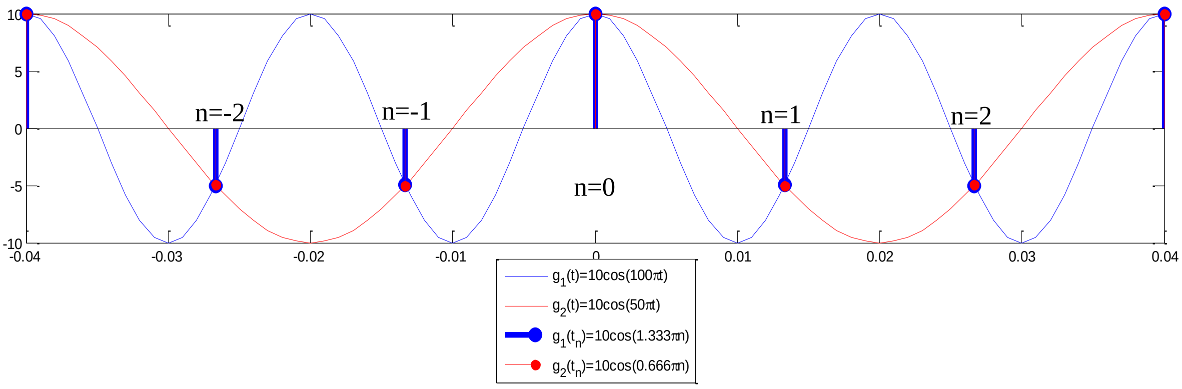
$$\begin{aligned} g_1(t_n) &= 10\cos(1.333\pi n) \\ &= 10\cos((1.333\pi - 2\pi)n) \\ &= 10\cos(-0.666\pi n) = 10\cos(0.666\pi n) = g_2(t_n) \end{aligned}$$

Hence, $g_1(t_n)$ and $g_2(t_n)$ are identical for the given sampling frequency

For further checks let's take $n = 0, \pm 1, \pm 2, \pm 3$ as an example, substituting n in $g_1(t_n)$ and $g_2(t_n)$ results in

n	-3	-2	-1	0	1	2	3
$g_1(t_n)$	10	-5	-5	10	-5	-5	10
$g_2(t_n)$	10	-5	-5	10	-5	-5	10

It is apparent that $g_1(t_n)$ and $g_2(t_n)$ are identical for the above n substituted values, see



This phenomenon is called **ALIASING** where each of a set of signals' **frequencies** that when sampled at a given **uniform rate**, would give the same set of **sampled values**, and thus might be incorrectly substituted for one another when reconstructing the original signal.

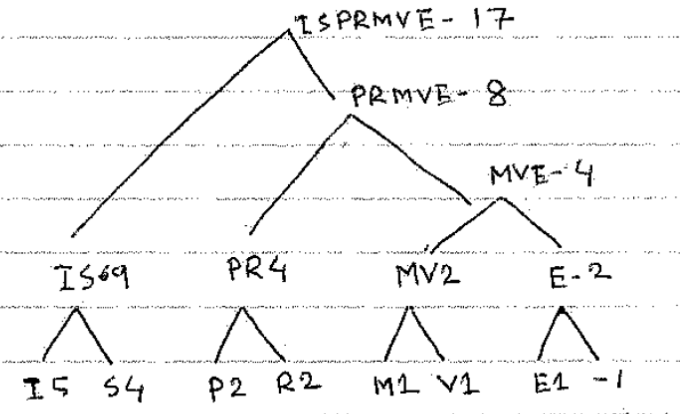
2. Encode the following phrase using Huffman coding, then calculate the saving percentage of the compression.
The phrase is : MISSISSIPPI RIVER

Solution:

MISSISSIPPI RIVER = 17 characters

For ASCII ; 8 × 17 = 136 bits

M = 1
I = 5
S = 4
P = 2
R = 2
V = 1
E = 1
– = 1



Encoding → 00 01 100 101 1100 1101 1110 1111

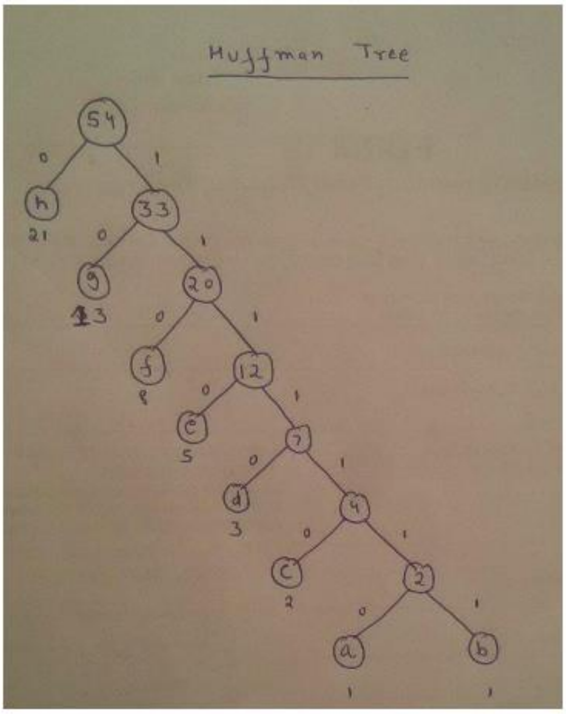
MISSISSIPPI RIVER

1100 00 01 01 00 0101 100 100 00 1111 10100 1101 1110 101

Saving percentage = $\frac{46}{136} = 33\% \Rightarrow 67\%$

3. The characters a to h have the set of frequencies based on the first 8 Fibonacci numbers as follows:
a : 1, b : 1, c : 2, d : 3, e : 5, f : 8, g : 13, h : 21
A Huffman code is used to represent the characters. What is the sequence of characters corresponding to the following code?
110111100111010

Solution: Using frequencies given in the question, huffman tree can be generated as:



Following are the codes:

Character	Code
a	1111110
b	1111111
c	111110
d	11110
e	1110
f	110
g	10
h	0

Using prefix matching, given string can be decomposed as

```
110 11110 0 1110 10
f   d     h  e   g
```

4. Encode the following using Lampel Ziv algorithm.
AABABBBABAABABBBABBABB

Solution:

A---0
B---1

Dictionary content	A	AB	ABB	B	ABA	ABAB	BB	ABBA	BB	
Code address	1	2	3	4	5	6	7	8	9	
Encoded Packet	0A	1B	2B	0B	2A	5B	4B	3A	7	
Codeword	,0	1,1	10,1	00,1	010,0	101,1	100,1	011,0	0111	

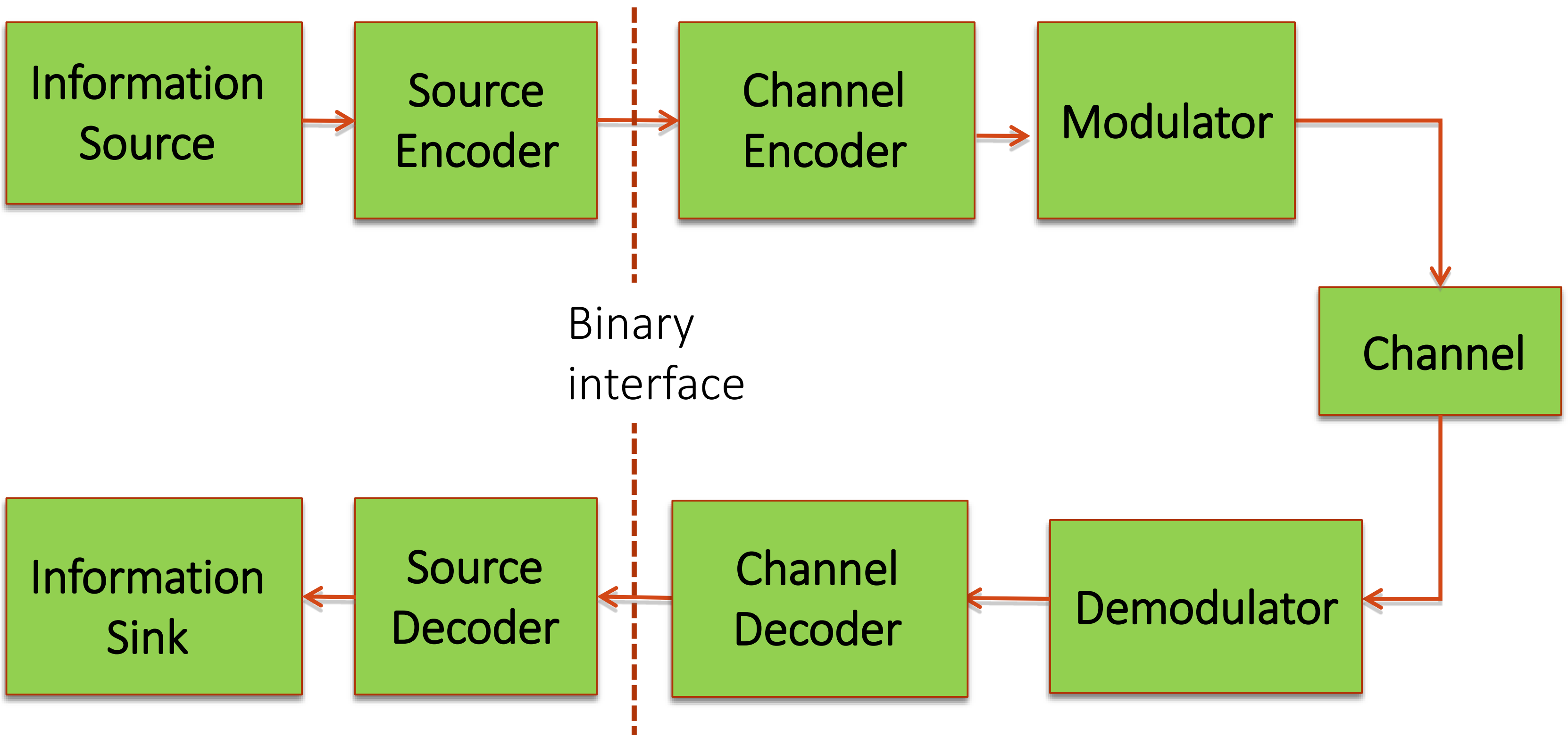
Digital Communication Systems 7057CEM

Binary Modulation

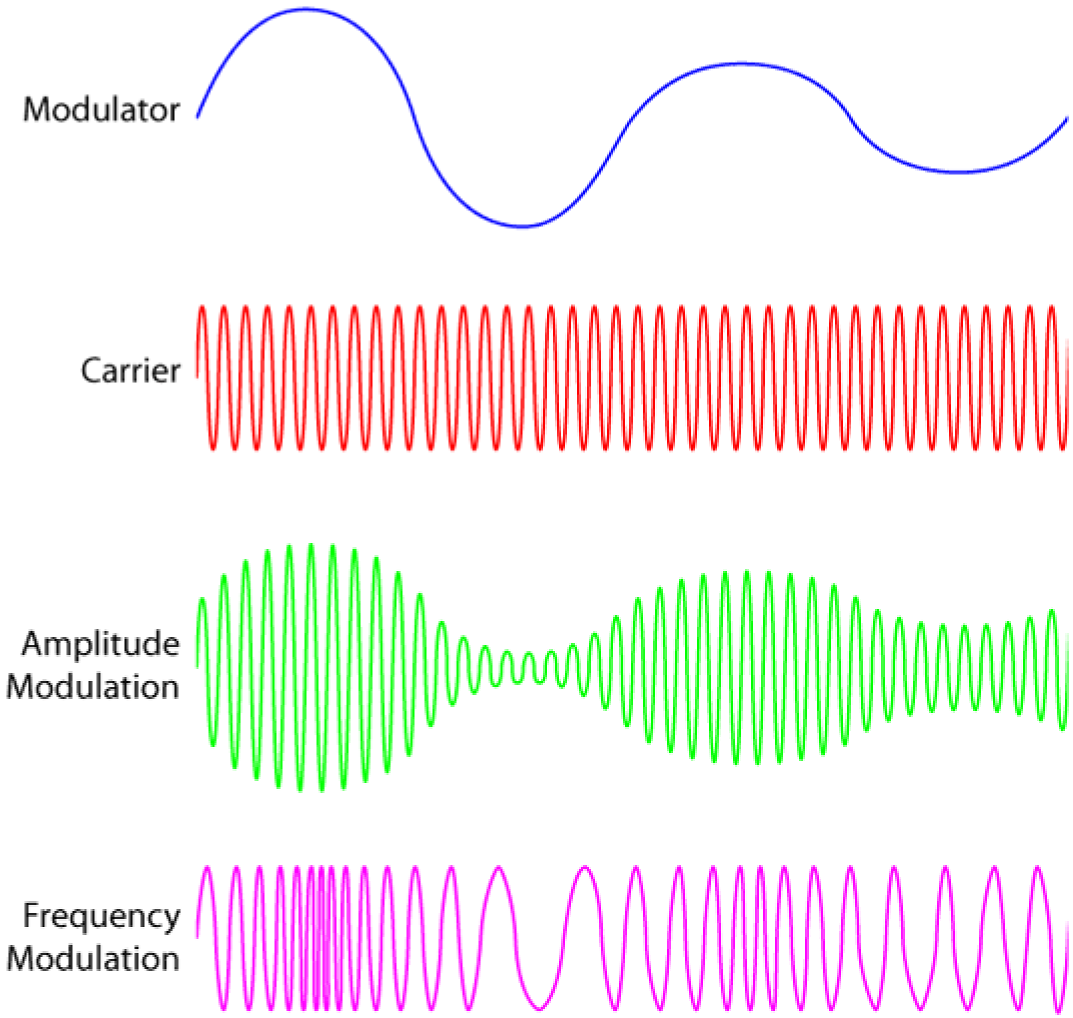
Learning outcomes

- At the end of this session, you will:
  - Gain knowledge of modulation and demodulation techniques
  - Be able to differentiate between coherent and non-coherent demodulation

Recap from last lecture



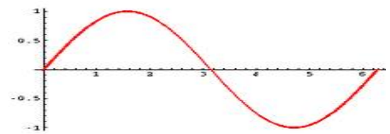
What is Modulation



The message signal, which contains the **information** is used to control the parameters of a **carrier** signal, so as to impress the information onto the carrier.

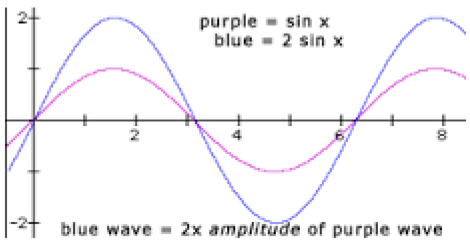
Carrier

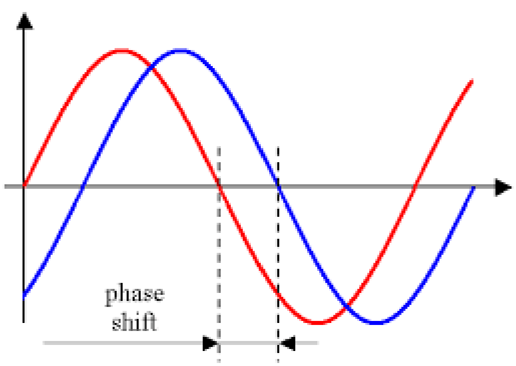
$$\boxed{c(t) = A\sin(2\pi f t + \varphi)}$$

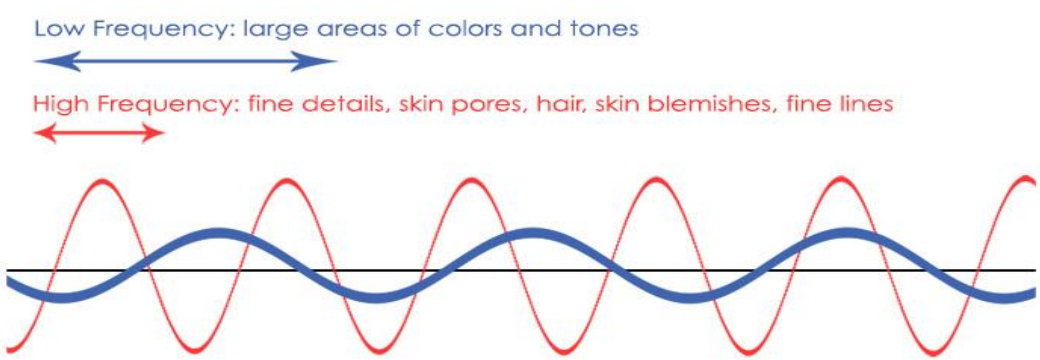


What are the parameters of the sinusoidal wave

1) Amplitude
2) Frequency
3) Phase



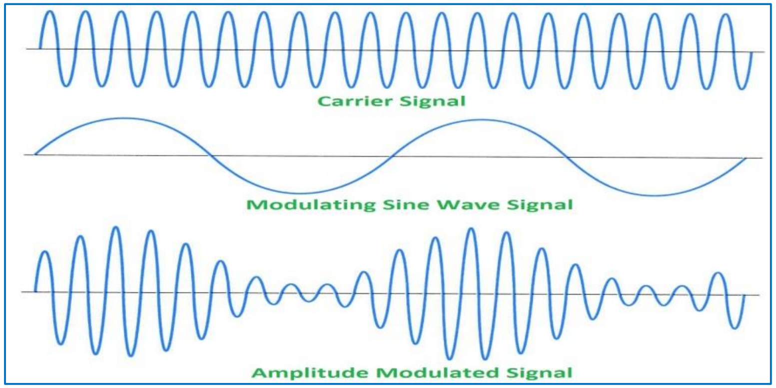




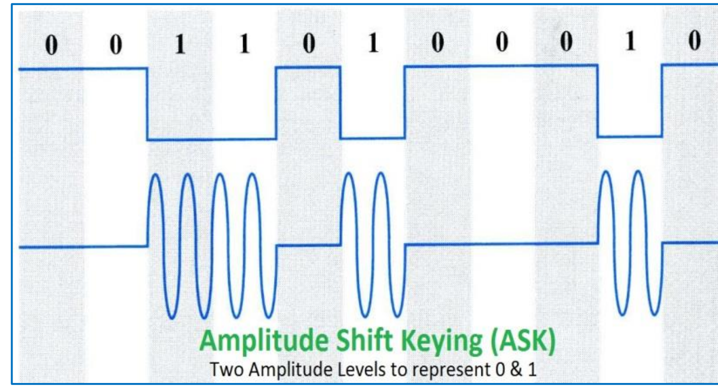
Types of information Signals

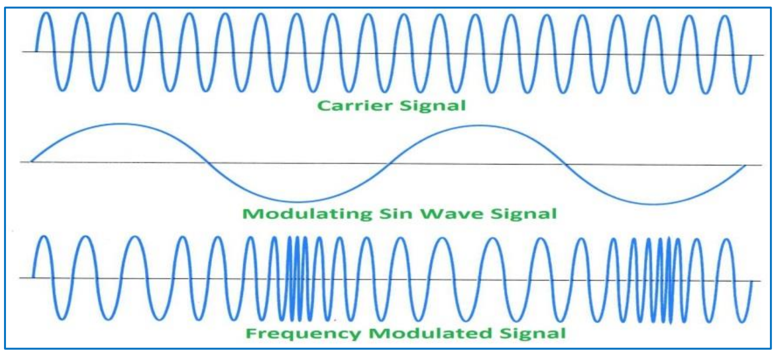
Analogue

Digital

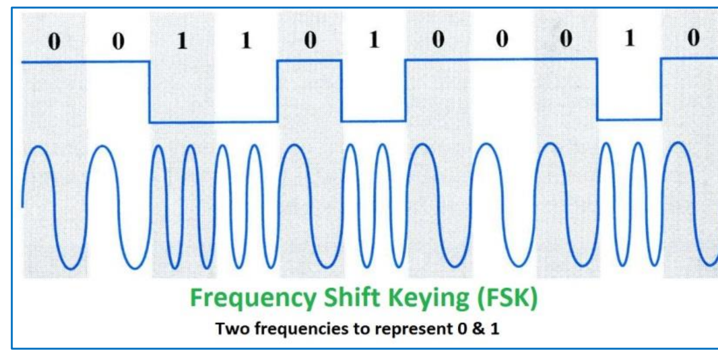


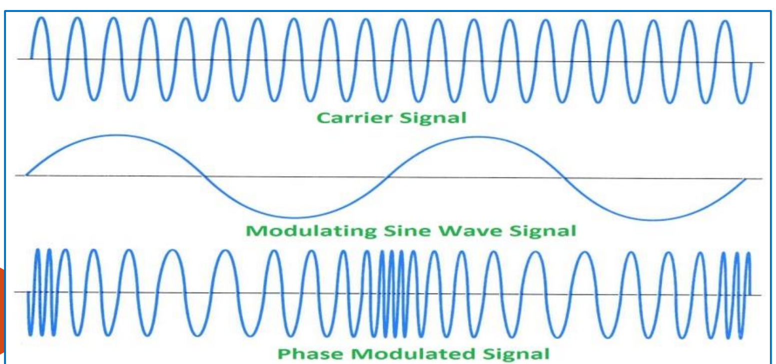
AM
Amplitude
ASK



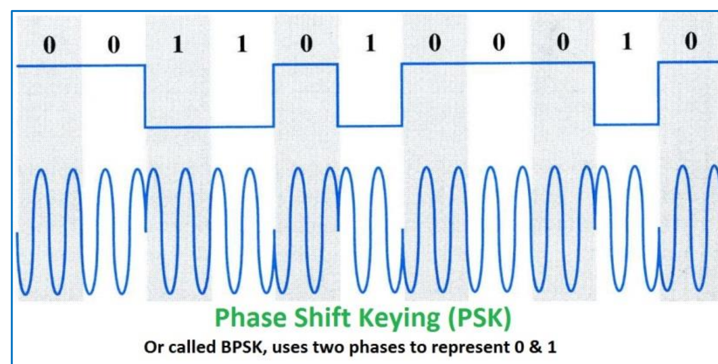


FM
Frequency
FSK



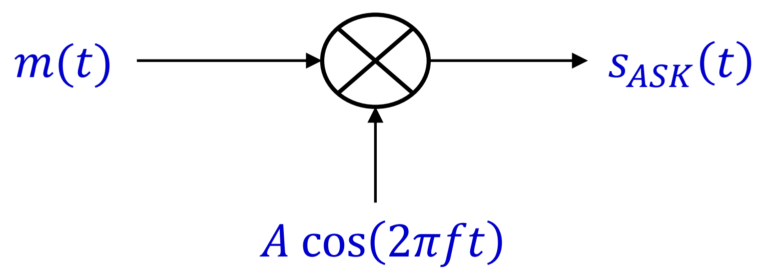


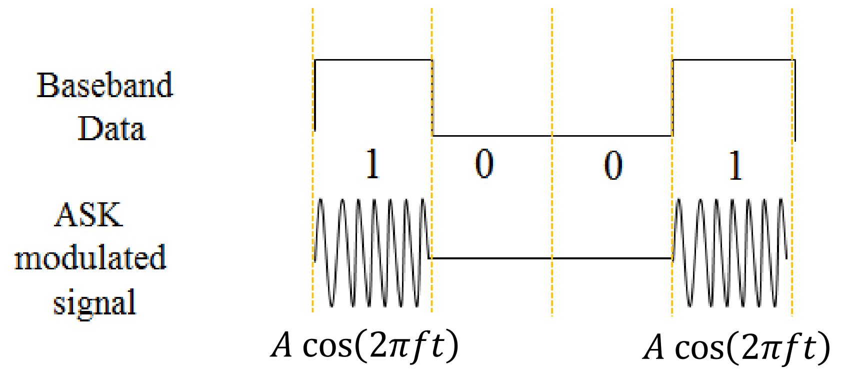
PM
Phase
PSK



Amplitude Shift Keying (ASK)

- The **amplitude** of a carrier signal changes according to the amplitude of message signal.
- The simplest and most common form to operate as a switch.
- ASK is often called On-Off Keying (OOK).
- Used in optical fiber communication.





$$s_{ASK}(t) = m(t)A\cos(2\pi f t) = \begin{cases} A\cos(2\pi f t) & if\ m(t) = 1 \\ 0 & if\ m(t) = 0 \end{cases}$$

ASK – Average energy per bit

- Average energy per bit $(E_b)\ = \frac{E+0}{2} \rightarrow E = 2E_b$
- Signal power $(S) = \frac{A^2}{2} = \frac{E}{T_b}$ where E = energy, assume R = 1Ω

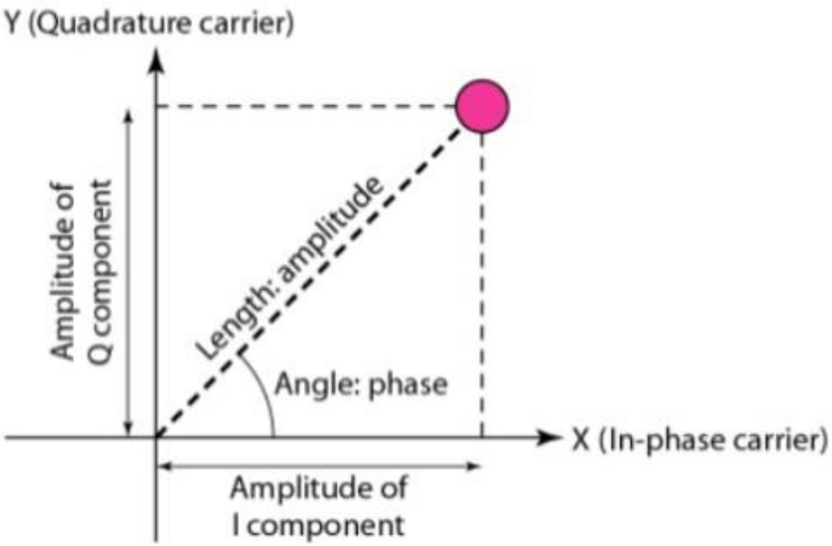
$$A = \sqrt{\frac{4E_b}{T_b}} \longrightarrow \quad s_{ASK}(t) = \begin{cases} \sqrt{\frac{4E_b}{T_b}}\cos(2\pi f t) & if\ m(t) = 1 \\ 0 & if\ m(t) = 0 \end{cases}$$

Exercise :

- An ASK signal has an amplitude of 20 Volts and carrier frequency of 900 MHz , the data signal has a bit rate of 48.6 Kbit/s.
  - Assuming 1Ω load , what is the energy per bit?
  - Assuming 50 Ω load, what is the energy per bit?

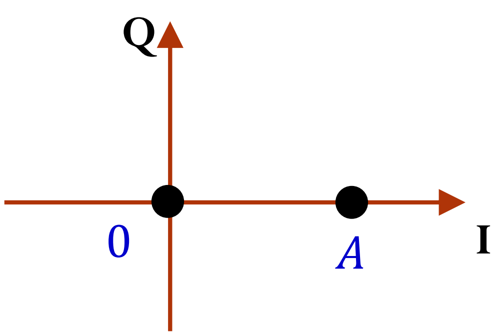
ASK – Constellation diagram

- A constellation diagram is a representation of signal modulated by a digital modulation scheme.



- Concept of a constellation diagram
  - ➢ This helps us to define the amplitude and phase of the signal

- ASK Constellation diagram

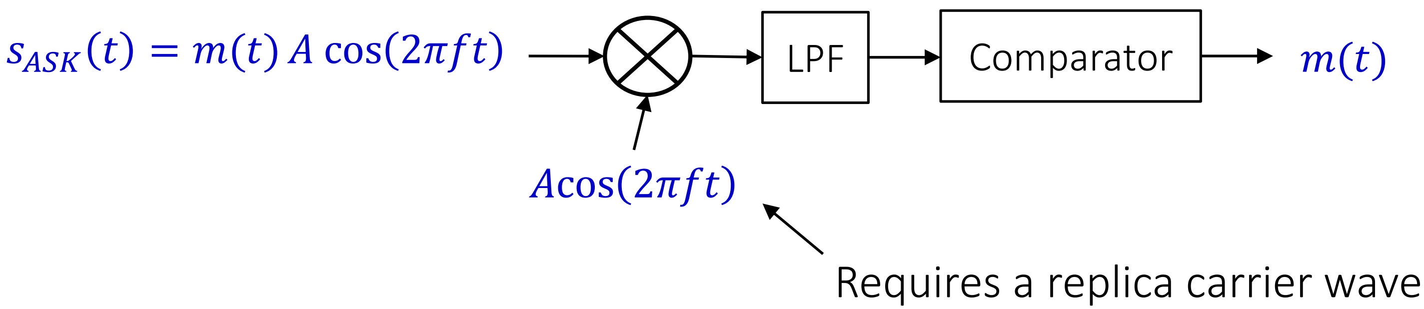


➢ X-axis: **In-phase** (I) carrier components i.e. $\cos(2\pi f t)$

➢ Y-axis: **Quadrature** (Q) carrier components (i.e. $\sin(2\pi f t)$)

ASK Demodulator

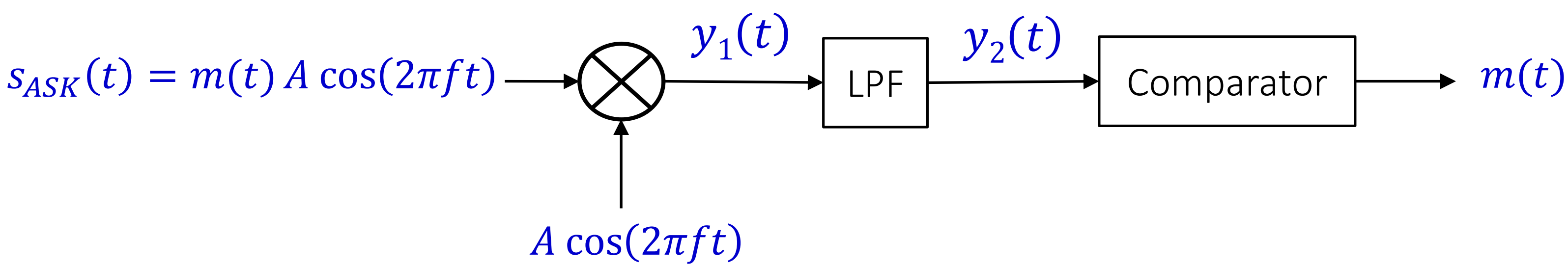
- Coherent (Synchronous)



- Non-Coherent (non-synchronous)

ASK Demodulator

- Coherent (Synchronous)



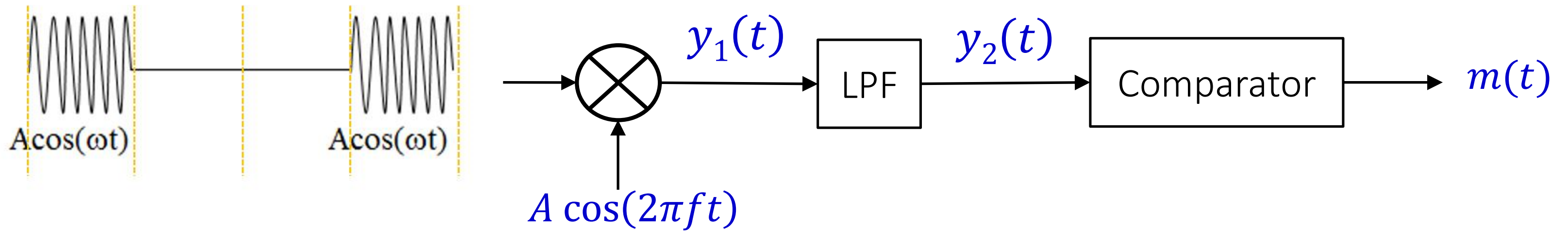
$$
\begin{aligned}
y_1(t) &= m(t)\, A^2 \cos^2(2\pi f t) \\
&= m(t)\, A^2\, \frac{1+\cos(4\pi f t)}{2} \\
&= m(t)\,\frac{A^2}{2} + m(t)\,\frac{A^2}{2}\cos(4\pi f t) \\
&= \begin{cases} \frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f t) & if\ m(t) = 1 \\ 0 & if\ m(t) = 0 \end{cases}
\end{aligned}
$$

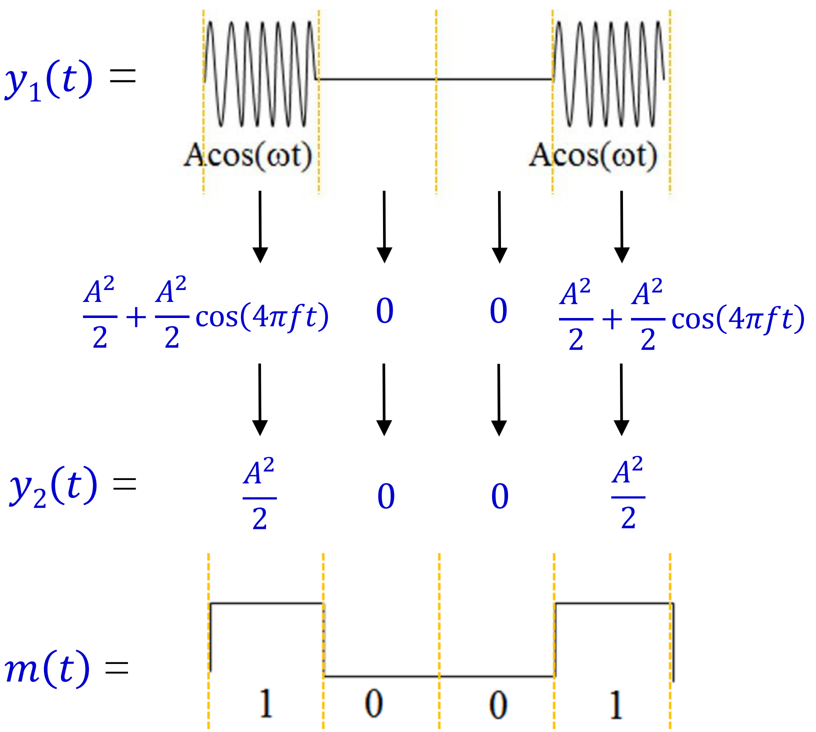
Using $\cos^2(x) = \frac{1+\cos(2x)}{2}$

$$
y_2(t) = \begin{cases} \frac{A^2}{2} & if\ m(t) = 1 \\ 0 & if\ m(t) = 0 \end{cases}
$$

ASK Demodulator

- Coherent (Synchronous)



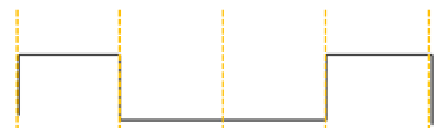


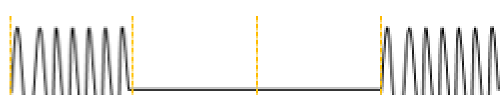
ASK Demodulator

- Non-Coherent (non-synchronous)

$$s_{ASK}(t) = m(t)\,A\cos(2\pi f t) \rightarrow \boxed{\text{Envelope Detector}} \rightarrow \boxed{\text{Comparator}} \rightarrow m(t)$$

$$s_{ASK}(t) = m(t)\,A\cos(2\pi f t) \rightarrow \boxed{\text{Half-wave Rectifier}} \rightarrow \boxed{\text{LPF}} \rightarrow \boxed{\text{Comparator}} \rightarrow m(t)$$

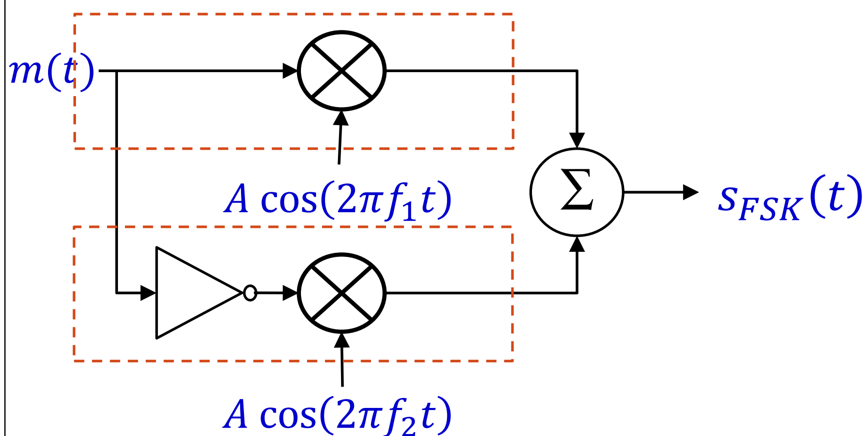


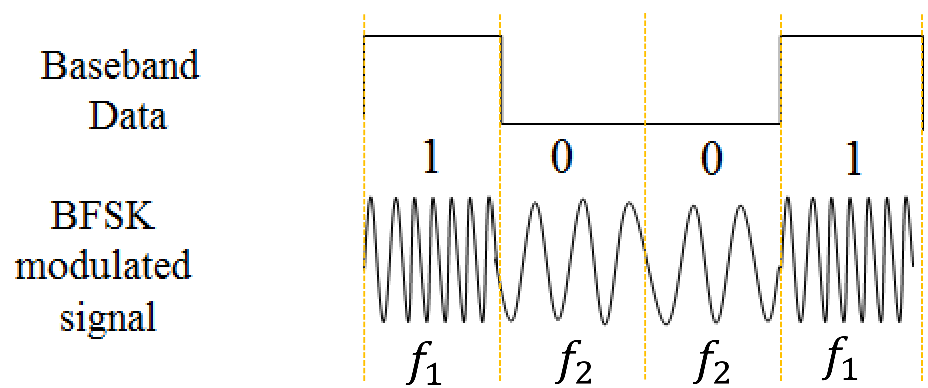


1 0 0 1

Frequency Shift Keying (FSK)

- A form of frequency modulation.
- The **frequency** of a carrier signal changes according to the amplitude of message signal.
- According to the digital signal 0 or 1, FSK changes the frequency of carrier wave.
- Amplitude is not changeable.





$$s_{FSK}(t) = m(t)A\cos(2\pi f_1 t) + \overline{m(t)}\,A\cos(2\pi f_2 t)$$

$$= \begin{cases} A\cos(2\pi f_1 t) & if\ m(t) = 1 \\ A\cos(2\pi f_2 t) & if\ m(t) = 0 \end{cases}$$

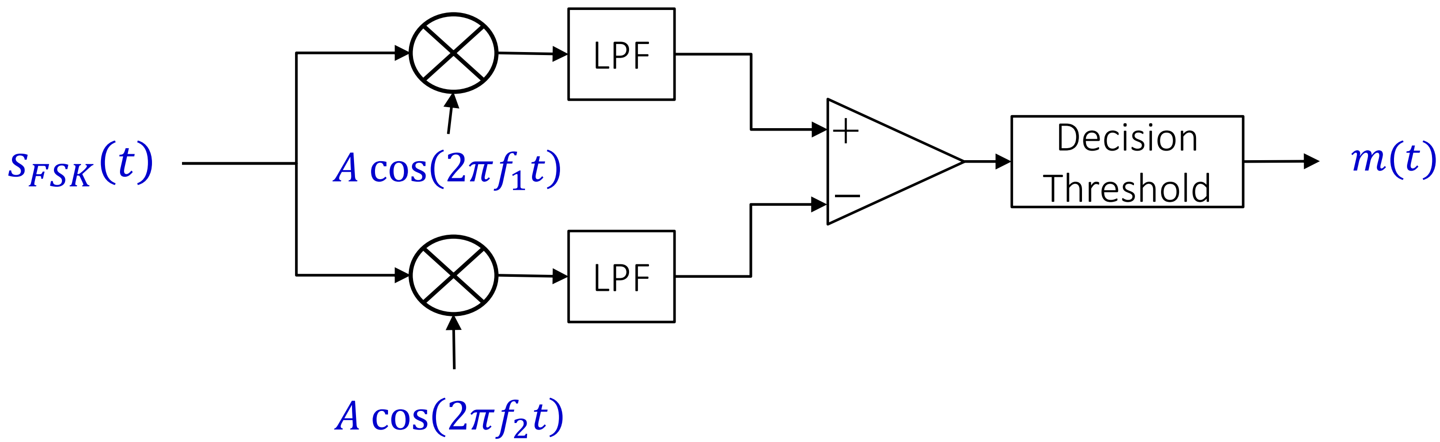
FSK – Average energy per bit

- Average energy per bit $(E_b)\ = \frac{E+E}{2} \rightarrow E\ = E_b$
- Signal power $(S) = \frac{A^2}{2} = \frac{E}{T_b}$ where E = energy, assume R = 1Ω

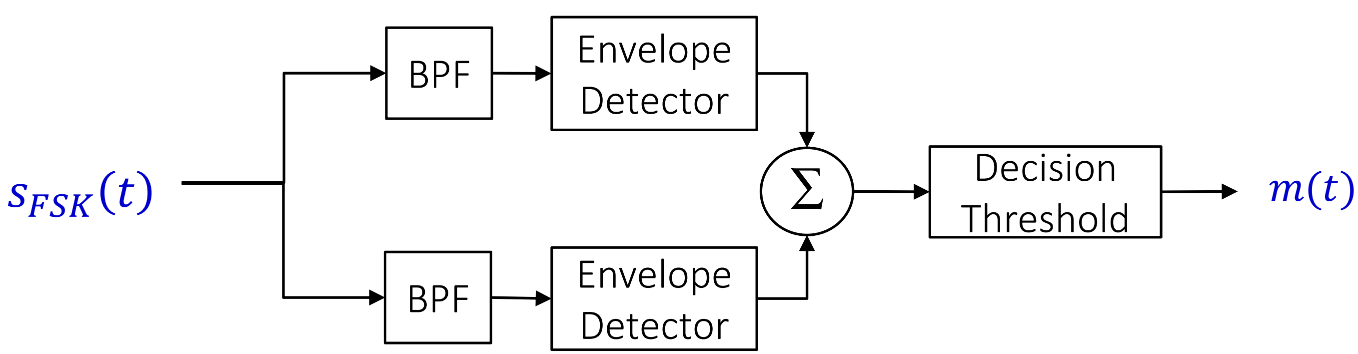
$$A = \sqrt{\frac{2E_b}{T_b}} \longrightarrow s_{FSK}(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_1 t) & if\ m(t) = 1 \\ \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_2 t) & if\ m(t) = 0 \end{cases}$$

FSK Demodulator

- Coherent (Synchronous)

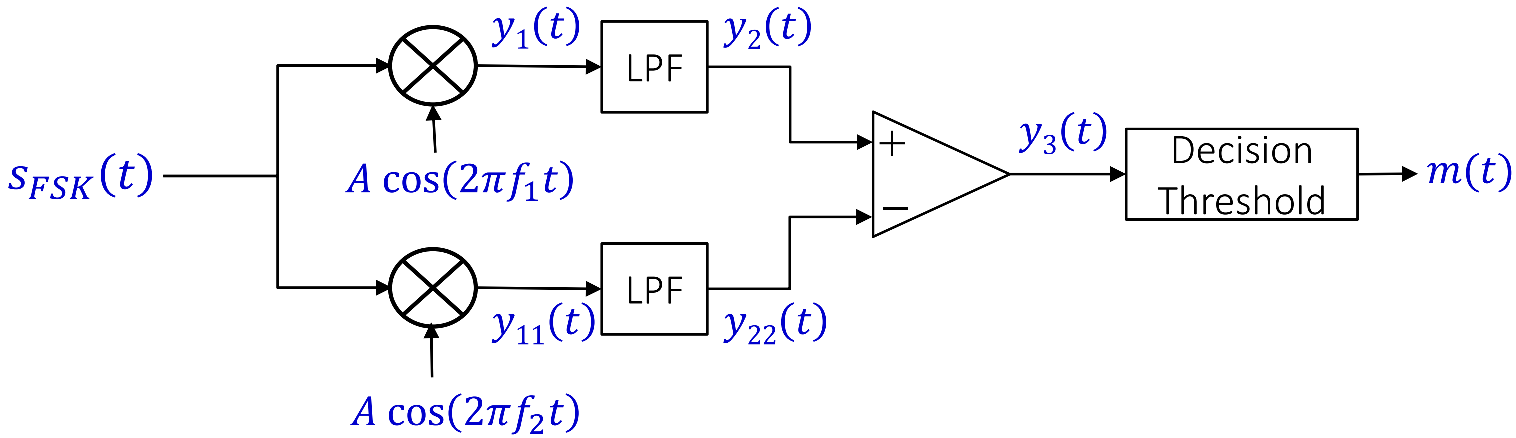


- Non-Coherent



FSK Demodulator

- Coherent (Synchronous)



(i) $s_{FSK}(t) = A\cos(2\pi f_1 t)$

$$y_1(t) = A^2 \cos^2(2\pi f_1 t)$$

$$= A^2 \frac{1+\cos(4\pi f_1 t)}{2} = \frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f_1 t)$$

$$y_2(t) = \frac{A^2}{2}$$

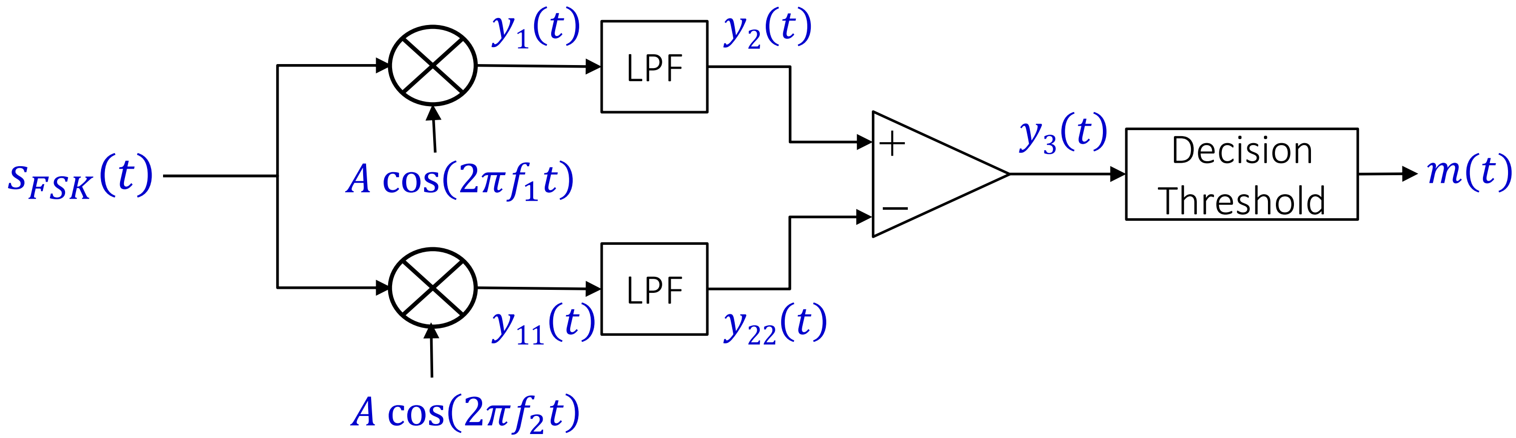
$\longrightarrow$ $y_3(t) = \frac{A^2}{2}$ $\longrightarrow$ $m(t) = $ Logic '1'

$$y_{11}(t) = A^2\cos(2\pi f_1 t)\ \cos(2\pi f_2 t)$$

$$y_{22}(t) = 0$$

FSK Demodulator

- Coherent (Synchronous)



(ii) $s_{FSK}(t) = A\cos(2\pi f_0 t)$

$$y_1(t) = A^2\cos(2\pi f_2 t)\ \cos(2\pi f_1 t)$$

$$y_2(t) = 0$$

$$y_{11}(t) = A^2\cos^2(2\pi f_2 t)$$

$$= A^2\,\frac{1+\cos(4\pi f_2 t)}{2} = \frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f_2 t)$$

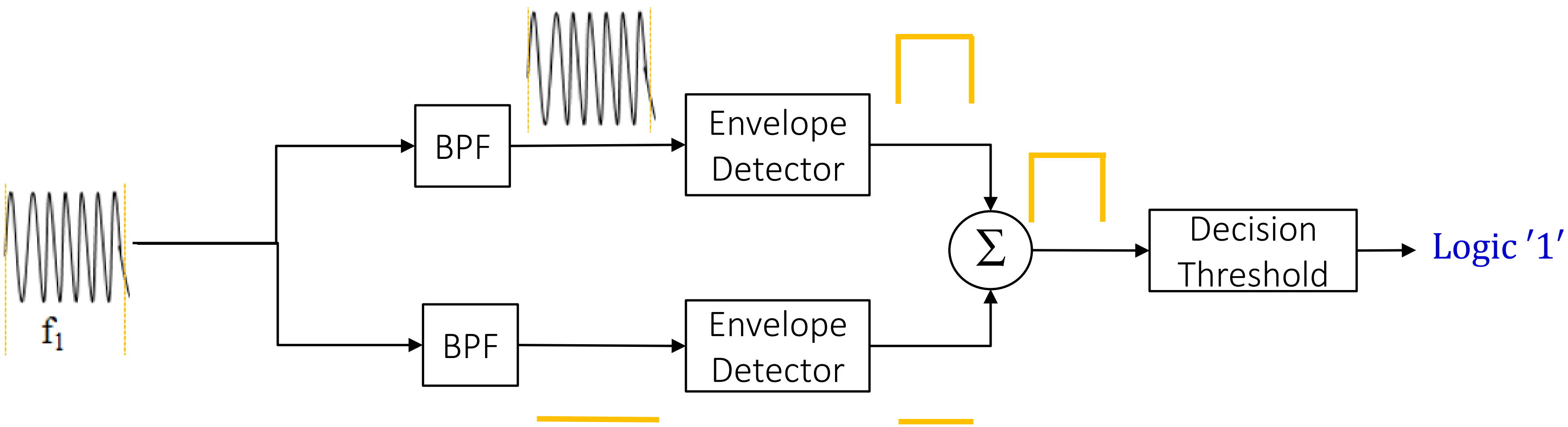
$$\longrightarrow y_3(t) = -\frac{A^2}{2} \longrightarrow m(t) = \text{Logic '0'}$$

$$y_{22}(t) = \frac{A^2}{2}$$

FSK Demodulator

- Non-Coherent

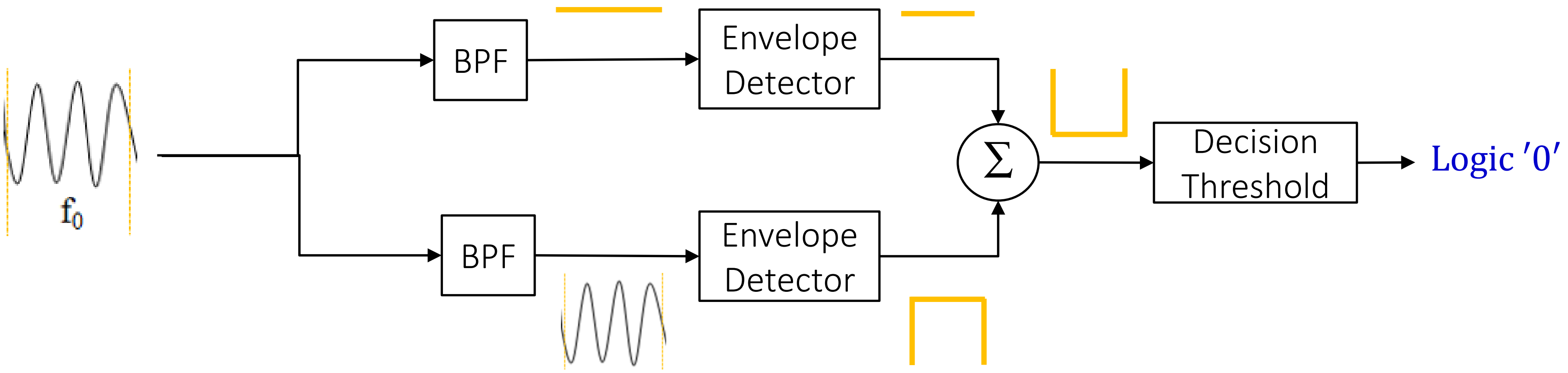
(i) $s_{FSK}(t) = A\cos(2\pi f_1 t)$



FSK Demodulator

- Non-Coherent

(ii) $s_{FSK}(t) = A\cos(2\pi f_2 t)$

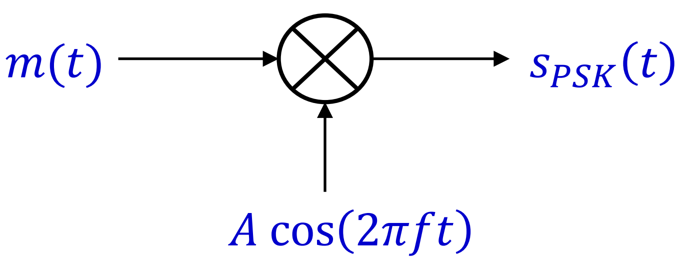


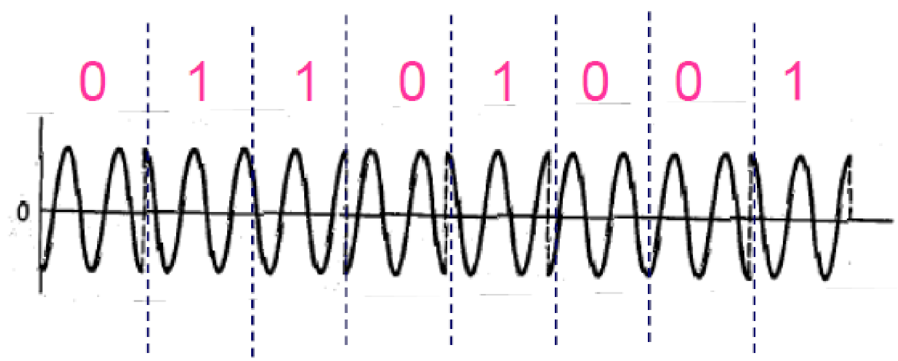
Exercise:

- A certain data source has bit rate of 3.84 Mbit/s. Identify the frequencies of the upper and lower nulls in the frequency spectrum and the corresponding null-to-null bandwidths for:
  - a) ASK with a carrier frequency of 1800 MHz
  - b) FSK with frequencies of 1800 MHz and 1808 MHz for 0s and 1s respectively

Phase Shift Keying (PSK)

- The **phase** of a carrier signal changes according to the amplitude of message signal.
- Binary PSK: Two phases represent two binary digits





$$s_{PSK}(t) = \begin{cases} A\cos(2\pi f t) & if\ m(t) = 1 \\ A\cos(2\pi f t + \pi) & if\ m(t) = 0 \end{cases}$$

$$= \begin{cases} A\cos(2\pi f t) & if\ m(t) = 1 \\ -A\cos(2\pi f t) & if\ m(t) = 0 \end{cases}$$

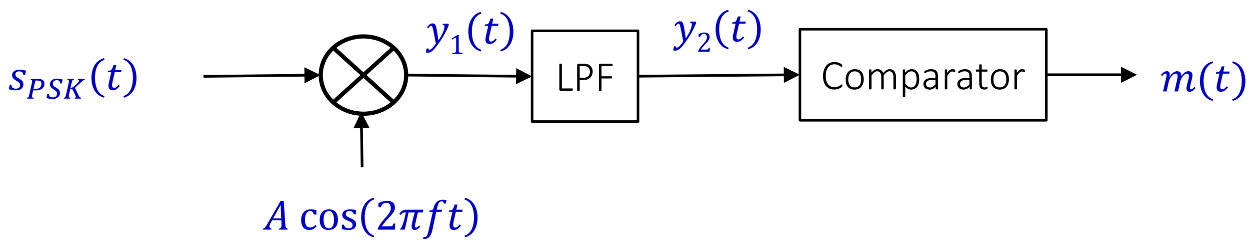
PSK – Average energy per bit

- Average energy per bit $(E_b)\ = \frac{E+E}{2} \rightarrow E\ = E_b$
- Signal power $(S) = \frac{A^2}{2} = \frac{E}{T_b}$ where E = energy, assume R = 1Ω

$$A = \sqrt{\frac{2E_b}{T_b}} \longrightarrow s_{PSK}(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f t) & if\ m(t) = 1 \\ \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f t + \pi) & if\ m(t) = 0 \end{cases}$$

PSK Demodulator

- Coherent (Synchronous)



- ➢ Phase synchronisation: ensure local oscillator output at the receiver is synchronised to the carrier in modulator
- ➢ Timing synchronisation: ensure proper bit timing of the decision making operation

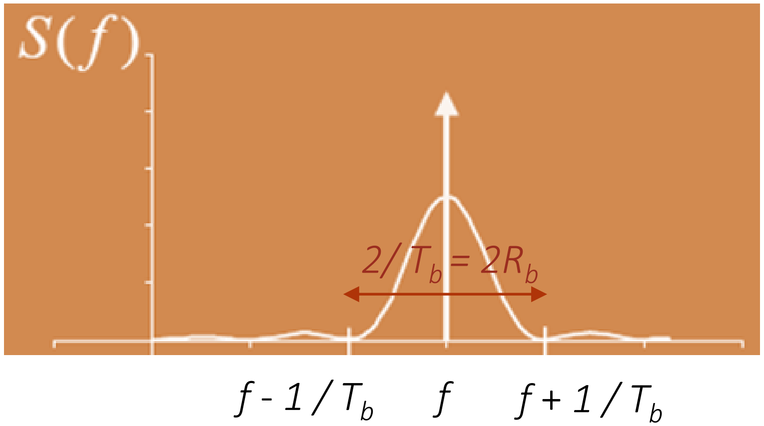
$$
\begin{aligned}
y_1(t) &= \pm A^2 \cos^2(2\pi f t) \\
&= \pm A^2 \, \frac{1+\cos(4\pi f t)}{2} \\
&= \pm \frac{A^2}{2} \pm m(t) \, \frac{A^2}{2} \cos(4\pi f t)
\end{aligned}
$$

Using $\cos^2(x) = \frac{1+\cos(2x)}{2}$

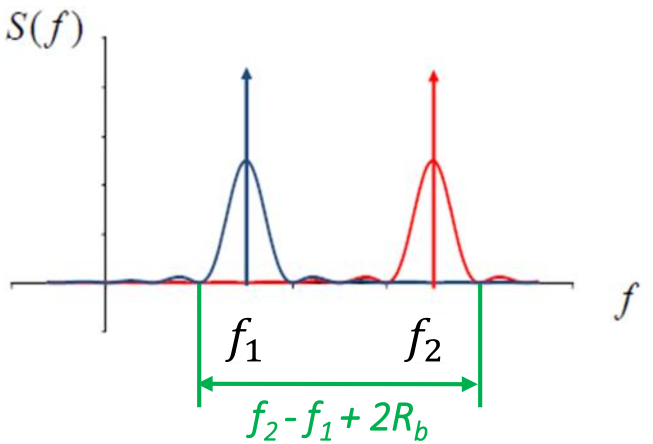
$y_2(t) = \pm \frac{A^2}{2} \longrightarrow m(t)$ = Logic ‘1’ or Logic ‘0’

Bandwidth

ASK & PSK



FSK



- Null-to-null Bandwidth
  - **ASK/PSK**: $B = 2/T_b$ or $2\ R_b$, where T_b = bit duration & R_b = bit rate
  - **FSK**: $B = f_2 - f_1 + 2\ R_b$

Bandwidth

- Bandwidth can be expressed as

$$B = (1 + \alpha)\, R_b$$

where α is the roll-off factor for bandwidth control $(0 \leq \alpha \leq 1)$
R_b is the bit rate

- ASK/PSK: $B = (1 + \alpha)R_b$
- FSK: $B = f_2 - f_1 + (1 + \alpha)R_b$

Exercise:

- A certain PSK modulated signal has a bandwidth of 30 kHz and is filtered with roll-off factor $\alpha = 0.35$. What is the maximum bit rate of the data signal?

Summary and Further Reading

- Digital Modulation Schemes

  (a) ASK

  (b) FSK

  (c) PSK
- Bandwidth and Average energy per bits
- Further Reading
  - Digital communications – John G. Proakis, Masoud Salehi 2008
  - Digital communications: Fundamental and application – Sklar, Bernard 2001

Coventry University
7057CEM Digital Communication Systems

Tutorial: Binary modulation

1. The data bits 10111 are to be transmitted. Draw waveforms of the modulated signal for:

   a) ASK
   b) FSK
   c) PSK

2. A certain data source has bit rate of 3.84 Mbit/s. Identify the frequencies of the upper and lower nulls in the frequency spectrum and the corresponding null-to-null bandwidths for:

   a) ASK with a carrier frequency of 1800 MHz
   b) FSK with frequencies of 1800 MHz and 1808 MHz for 0s and 1s respectively.
   c) PSK with a carrier frequency of 2 GHz.

3. A PSK signal has amplitude 20 volts and a carrier frequency of 900 MHz; the data signal has a bit rate of 48.6 Kbit/s

   a) Write down an expression for this signal
   b) Assuming a 1Ω load, what is the energy per bit?
   c) What is the energy per bit for a 50 Ω load?
   d) What is the null-to-null bandwidth of this signal?
   e) What is the Nyquist bandwidth of this signal?
   f) What is the bandwidth of this signal if a spectral shaping filter with roll-off factor α = 0.25 is used.

4. A certain FSK transmitter using a carrier of 500 kHz sending 10 kbps and a frequency deviation of 100 kHz. How much bandwidth do you need for your transmission?

5. A certain PSK modulated signal has a bandwidth of 30 kHz and is filtered with roll-off factor α = 0.35. What is the maximum bit rate of the data signal?

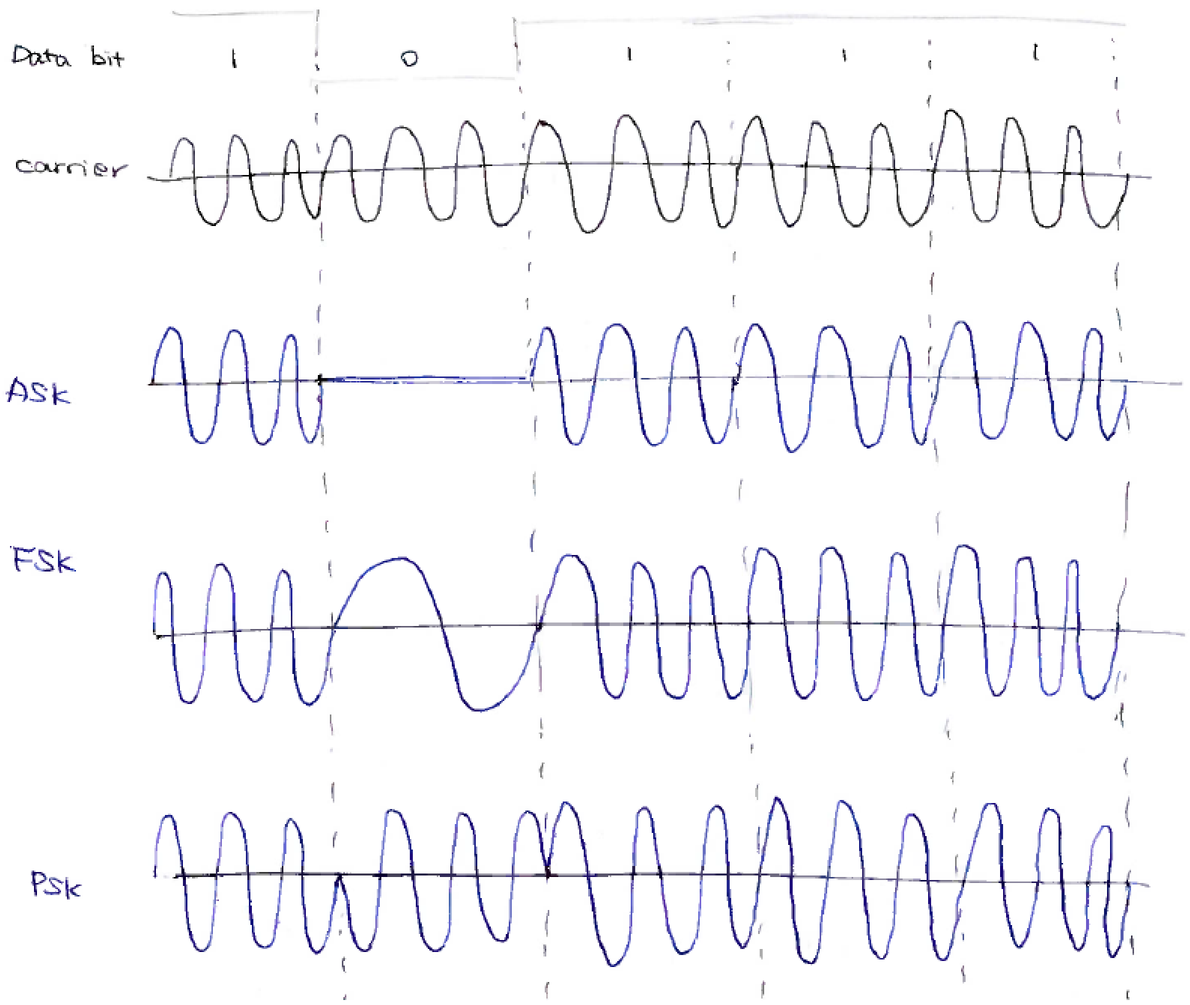
Coventry University

7057CEM Digital Communication Systems

Tutorial: Binary modulation

1. The data bits 10111 are to be transmitted. Draw waveforms of the modulated signal for:

a) ASK
b) FSK
c) PSK

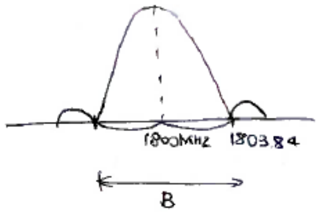


2. A certain data source has bit rate of 3.84 Mbit/s. Identify the frequencies of the upper and lower nulls in the frequency spectrum and the corresponding null-to-null bandwidths for:

a) ASK with a carrier frequency of 1800 MHz
b) FSK with frequencies of 1800 MHz and 1808 MHz for 0s and 1s respectively.
c) PSK with a carrier frequency of 2 GHz.

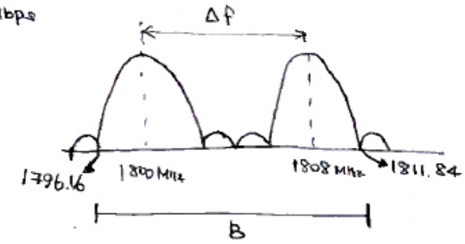
2. (a) ASK $B = 2R$ where R = data rate

$B = 2 \times 3.84 = 7.68$ MHz



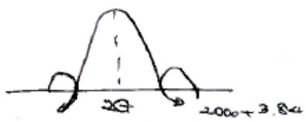
(2) FSK $f_{c_1} = 1800$ MHz, $f_{c_2} = 1808$ MHz, $R = 3.84$ Mbps

$B = \Delta f + 2R = 8 + 7.68 = 15.68$ MHz



(3) PSK, $f_c = \overset{2000\,MHz}{2\,GHz}$, $R = 3.84$ Mbps

$B = 2R = 7.68$ MHz



3. A PSK signal has amplitude 20 volts and a carrier frequency of 900 MHz; the data signal has a bit rate of 48.6 Kbit/s

a) Write down an expression for this signal

3. $V = 20V$, $f_c = 900$ MHz, $R_b = 48.6$ kbps

$S(t) = A\cos(2\pi f_c t)$

(a) $S_{psk}(t) = \begin{cases} 20\cos(1800\pi t) & \text{'1'} \\ 20\cos(1800\pi t + \pi) & \text{'0'} \end{cases}$

b) Assuming a 1Ω load, what is the energy per bit?

(b) $E_b = \dfrac{A^2 T_b}{2} = \dfrac{A^2}{2R_b} = \dfrac{20 \times 20}{2 \times 48.6 \times 10^3} = \dfrac{1}{24.3 \times 10} = 4.1\,mJ$

c) What is the energy per bit for a 50 Ω load?

(c) For 50Ω load,

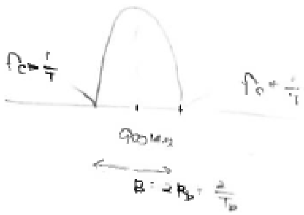
Signal power $= \dfrac{20^2}{2 \times 50} = 4W$ $\qquad S = \dfrac{A_c^2}{2 \cdot R}$

Energy $= 4 \cdot \dfrac{1}{48.6 \times 10^3} = 82\,\mu J$ $\qquad E_b = S \cdot T_b = 4 \cdot \dfrac{1}{R_b}$

d) What is the null-to-null bandwidth of this signal?

e) What is the bandwidth of this signal if a spectral shaping filter with roll-off factor α = 0.25 is used.

(d) $B = 2R = 2 \times 48.6 = 97.2$ KHz



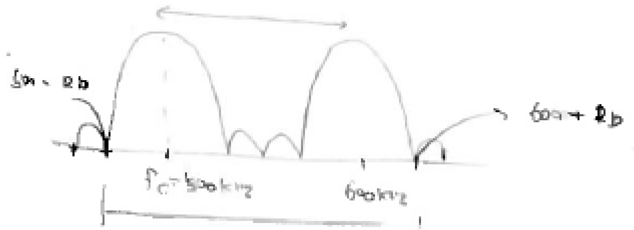
(e) $B = (1+\alpha)R$

$= (1+0.25) \cdot 48.6 \times 10^3 = 60.75$ kHz

4. A certain FSK transmitter using a carrier of 500 kHz sending 10 kbps and a frequency deviation of 100 kHz. How much bandwidth do you need for your transmission?

$f_c = 500$ kHz, $R_b = 10$ kbps, $\Delta f = 100$ kHz

$T_b = \frac{1}{10\text{ kbps}}$



$B = \Delta f + 2R = 120$ kHz

5. A certain PSK modulated signal has a bandwidth of 30 kHz and is filtered with roll-off factor α = 0.35. What is the maximum bit rate of the data signal?

$B = (1+\alpha)R \iff R = \frac{B}{1+\alpha} = \frac{30 \times 10^3}{1+0.35} = 22.2$ Kbps

A Method for the Construction of Minimum-Redundancy Codes*

DAVID A. HUFFMAN†, ASSOCIATE, IRE

***Summary*—An optimum method of coding an ensemble of messages consisting of a finite number of members is developed. A minimum-redundancy code is one constructed in such a way that the average number of coding digits per message is minimized.**

INTRODUCTION

ONE IMPORTANT METHOD of transmitting messages is to transmit in their place sequences of symbols. If there are more messages which might be sent than there are kinds of symbols available, then some of the messages must use more than one symbol. If it is assumed that each symbol requires the same time for transmission, then the time for transmission (length) of a message is directly proportional to the number of symbols associated with it. In this paper, the symbol or sequence of symbols associated with a given message will be called the "message code." The entire number of messages which might be transmitted will be called the "message ensemble." The mutual agreement between the transmitter and the receiver about the meaning of the code for each message of the ensemble will be called the "ensemble code."

Probably the most familiar ensemble code was stated in the phrase "one if by land and two if by sea." In this case, the message ensemble consisted of the two individual messages "by land" and "by sea", and the message codes were "one" and "two."

In order to formalize the requirements of an ensemble code, the coding symbols will be represented by numbers. Thus, if there are D different types of symbols to be used in coding, they will be represented by the digits $0, 1, 2, \cdots, (D-1)$. For example, a ternary code will be constructed using the three digits 0, 1, and 2 as coding symbols.

The number of messages in the ensemble will be called N. Let $P(i)$ be the probability of the ith message. Then

$$\sum_{i=1}^{N} P(i) = 1. \tag{1}$$

The length of a message, $L(i)$, is the number of coding digits assigned to it. Therefore, the average message length is

$$L_{av} = \sum_{i=1}^{N} P(i)L(i). \tag{2}$$

The term "redundancy" has been defined by Shannon[1] as a property of codes. A "minimum-redundancy code" will be defined here as an ensemble code which, for a message ensemble consisting of a finite number of members, N, and for a given number of coding digits, D, yields the lowest possible average message length. In order to avoid the use of the lengthy term "minimum-redundancy," this term will be replaced here by "optimum." It will be understood then that, in this paper, "optimum code" means "minimum-redundancy code."

The following basic restrictions will be imposed on an ensemble code:

(a) No two messages will consist of identical arrangements of coding digits.

(b) The message codes will be constructed in such a way that no additional indication is necessary to specify where a message code begins and ends once the starting point of a sequence of messages is known.

Restriction (b) necessitates that no message be coded in such a way that its code appears, digit for digit, as the first part of any message code of greater length. Thus, 01, 102, 111, and 202 are valid message codes for an ensemble of four members. For instance, a sequence of these messages 1111022020101111102 can be broken up into the individual messages 111-102-202-01-01-111-102. All the receiver need know is the ensemble code. However, if the ensemble has individual message codes including 11, 111, 102, and 02, then when a message sequence starts with the digits 11, it is not immediately certain whether the message 11 has been received or whether it is only the first two digits of the message 111. Moreover, even if the sequence turns out to be 11102, it is still not certain whether 111-02 or 11-102 was transmitted. In this example, change of one of the two message codes 111 or 11 is indicated.

C. E. Shannon[1] and R. M. Fano[2] have developed ensemble coding procedures for the purpose of proving that the average number of binary digits required per message approaches from above the average amount of information per message. Their coding procedures are not optimum, but approach the optimum behavior when N approaches infinity. Some work has been done by Kraft[3] toward deriving a coding method which gives an average code length as close as possible to the ideal when the ensemble contains a finite number of members. However, up to the present time, no definite procedure has been suggested for the construction of such a code

* Decimal classification: R531.1. Original manuscript received by the Institute, December 6, 1951.

† Massachusetts Institute of Technology, Cambridge, Mass.

[1] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. Jour.*, vol. 27, pp. 398–403; July, 1948.

[2] R. M. Fano, "The Transmission of Information," Technical Report No. 65, Research Laboratory of Electronics, M.I.T., Cambridge, Mass.; 1949.

[3] L. G. Kraft, "A Device for Quantizing, Grouping, and Coding Amplitude-modulated Pulses," Electrical Engineering Thesis, M.I.T., Cambridge, Mass.; 1949.

to the knowledge of the author. It is the purpose of this paper to derive such a procedure.

Derived Coding Requirements

For an optimum code, the length of a given message code can never be less than the length of a more probable message code. If this requirement were not met, then a reduction in average message length could be obtained by interchanging the codes for the two messages in question in such a way that the shorter code becomes associated with the more probable message. Also, if there are several messages with the same probability, then it is possible that the codes for these messages may differ in length. However, the codes for these messages may be interchanged in any way without affecting the average code length for the message ensemble. Therefore, it may be assumed that the messages in the ensemble have been ordered in a fashion such that

$$P(1) \geqq P(2) \geqq \cdots \geqq P(N-1) \geqq P(N) \qquad (3)$$

and that, in addition, for an optimum code, the condition

$$L(1) \leqq L(2) \leqq \cdots \leqq L(N-1) \leqq L(N) \qquad (4)$$

holds. This requirement is assumed to be satisfied throughout the following discussion.

It might be imagined that an ensemble code could assign q more digits to the Nth message than to the $(N-1)$st message. However, the first $L(N-1)$ digits of the Nth message must not be used as the code for any other message. Thus the additional q digits would serve no useful purpose and would unnecessarily increase L_{av}. Therefore, for an optimum code it is necessary that $L(N)$ be equal to $L(N-1)$.

The kth prefix of a message code will be defined as the first k digits of that message code. Basic restriction (b) could then be restated as: No message shall be coded in such a way that its code is a prefix of any other message, or that any of its prefixes are used elsewhere as a message code.

Imagine an optimum code in which no two of the messages coded with length $L(N)$ have identical prefixes of order $L(N)-1$. Since an optimum code has been assumed, then none of these messages of length $L(N)$ can have codes or prefixes of any order which correspond to other codes. It would then be possible to drop the last digit of all of this group of messages and thereby reduce the value of L_{av}. Therefore, in an optimum code, it is necessary that at least two (and no more than D) of the codes with length $L(N)$ have identical prefixes of order $L(N)-1$.

One additional requirement can be made for an optimum code. Assume that there exists a combination of the D different types of coding digits which is less than $L(N)$ digits in length and which is not used as a message code or which is not a prefix of a message code. Then this combination of digits could be used to replace the code for the Nth message with a consequent reduction of L_{av}. Therefore, all possible sequences of $L(N)-1$ digits must be used either as message codes, or must have one of their prefixes used as message code.

The derived restrictions for an optimum code are summarized in condensed form below and considered in addition to restrictions (a) and (b) given in the first part of this paper:

(c) $$L(1) \leqq L(2) \leqq \cdots \leqq L(N-1) = L(N). \qquad (5)$$

(d) At least two and not more than D of the messages with code length $L(N)$ have codes which are alike except for their final digits.

(e) Each possible sequence of $L(N)-1$ digits must be used either as a message code or must have one of its prefixes used as a message code.

Optimum Binary Code

For ease of development of the optimum coding procedure, let us now restrict ourselves to the problem of binary coding. Later this procedure will be extended to the general case of D digits.

Restriction (c) makes it necessary that the two least probable messages have codes of equal length. Restriction (d) places the requirement that, for D equal to two, there be only two of the messages with coded length $L(N)$ which are identical except for their last digits. The final digits of these two codes will be one of the two binary digits, 0 and 1. It will be necessary to assign these two message codes to the Nth and the $(N-1)$st messages since at this point it is not known whether or not other codes of length $L(N)$ exist. Once this has been done, these two messages are equivalent to a single composite message. Its code (as yet undetermined) will be the common prefixes of order $L(N)-1$ of these two messages. Its probability will be the sum of the probabilities of the two messages from which it was created. The ensemble containing this composite message in the place of its two component messages will be called the first auxiliary message ensemble.

This newly created ensemble contains one less message than the original. Its members should be rearranged if necessary so that the messages are again ordered according to their probabilities. It may be considered exactly as the original ensemble was. The codes for each of the two least probable messages in this new ensemble are required to be identical except in their final digits; 0 and 1 are assigned as these digits, one for each of the two messages. Each new auxiliary ensemble contains one less message than the preceding ensemble. Each auxiliary ensemble represents the original ensemble with full use made of the accumulated necessary coding requirements.

The procedure is applied again and again until the number of members in the most recently formed auxiliary message ensemble is reduced to two. One of each of the binary digits is assigned to each of these two composite messages. These messages are then combined to form a single composite message with probability unity, and the coding is complete.

TABLE I

OPTIMUM BINARY CODING PROCEDURE

Message Probabilities												
Original Message Ensemble	*Auxiliary Message Ensembles*											
	1	2	3	4	5	6	7	8	9	10	11	12
												→1.00
											→0.60	
										→0.40	0.40	
									→0.36	0.36		
								→0.24	0.24	0.24		
0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20			
							→0.20	0.20	0.20			
0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18				
						→0.18	0.18	0.18				
					→0.14	0.14	0.14					
0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10					
0.10	0.10	0.10	0.10	0.10	0.10	0.10						
0.10	0.10	0.10	0.10	0.10	0.10	0.10						
				→0.10	0.10							
		→0.08	0.08	0.08	0.08							
			→0.08	0.08								
0.06	0.06	0.06	0.06	0.06								
0.06	0.06	0.06	0.06									
0.04	0.04	0.04	0.04									
*0.04	0.04	0.04										
0.04	0.04	0.04										
0.04	0.04											
	→0.04											
0.03												
0.01												

Now let us examine Table I. The left-hand column contains the ordered message probabilities of the ensemble to be coded. N is equal to 13. Since each combination of two messages (indicated by a bracket) is accompanied by the assigning of a new digit to each, then the total number of digits which should be assigned to each original message is the same as the number of combinations indicated for that message. For example, the message marked *, or a composite of which it is a part, is combined with others five times, and therefore should be assigned a code length of five digits.

When there is no alternative in choosing the two least probable messages, then it is clear that the requirements, established as necessary, are also sufficient for deriving an optimum code. There may arise situations in which a choice may be made between two or more groupings of least likely messages. Such a case arises, for example, in the fourth auxiliary ensemble of Table I. Either of the messages of probability 0.08 could have been combined with that of probability 0.06. However, it is possible to rearrange codes in any manner among equally likely messages without affecting the average code length, and so a choice of either of the alternatives could have been made. Therefore, the procedure given is always sufficient to establish an optimum binary code.

The lengths of all the encoded messages derived from Table I are given in Table II.

Having now determined proper lengths of code for each message, the problem of specifying the actual digits remains. Many alternatives exist. Since the combining of messages into their composites is similar to the successive confluences of trickles, rivulets, brooks, and creeks into a final large river, the procedure thus far described might be considered analogous to the placing of signs by a water-borne insect at each of these junctions as he journeys downstream. It should be remembered that the code which we desire is that one which the insect must remember in order to work his way back upstream. Since the placing of the signs need not follow the same rule, such as "zero-right-returning," at each junction, it can be seen that there are at least 2^{12} different ways of assigning code digits for our example.

TABLE II

RESULTS OF OPTIMUM BINARY CODING PROCEDURE

i	$P(i)$	$L(i)$	$P(i)L(i)$	*Code*
1	0.20	2	0.40	10
2	0.18	3	0.54	000
3	0.10	3	0.30	011
4	0.10	3	0.30	110
5	0.10	3	0.30	111
6	0.06	4	0.24	0101
7	0.06	5	0.30	00100
8	0.04	5	0.20	00101
9	0.04	5	0.20	01000
10	0.04	5	0.20	01001
11	0.04	5	0.20	00110
12	0.03	6	0.18	001110
13	0.01	6	0.06	001111
			$L_{av}=3.42$	

The code in Table II was obtained by using the digit 0 for the upper message and the digit 1 for the lower message of any bracket. It is important to note in Table I that coding restriction (e) is automatically met as long as two messages (and not one) are placed in each bracket.

Generalization of the Method

Optimum coding of an ensemble of messages using three or more types of digits is similar to the binary coding procedure. A table of auxiliary message ensembles similar to Table I will be used. Brackets indicating messages combined to form composite messages will be used in the same way as was done in Table I. However, in order to satisfy restriction (e), it will be required that all these brackets, with the possible exception of one combining the least probable messages of the original ensemble, always combine a number of messages equal to D.

It will be noted that the terminating auxiliary ensemble always has one unity probability message. Each preceding ensemble is increased in number by $D-1$ until the first auxiliary ensemble is reached. Therefore, if N_1 is the number of messages in the first auxiliary ensemble, then $(N_1-1)/(D-1)$ must be an integer. However $N_1=N-n_0+1$, where n_0 is the number of the least probable messages combined in a bracket in the original ensemble. Therefore, n_0 (which, of course, is at least two and no more than D) must be of such a value that $(N-n_0)/(D-1)$ is an integer.

In Table III an example is considered using an ensemble of eight messages which is to be coded with four digits; n_0 is found to be 2. The code listed in the table is obtained by assigning the four digits 0, 1, 2, and 3, in order, to each of the brackets.

TABLE III

Optimum Coding Procedure for $D=4$

Message Probabilities: *Original Message Ensemble*	*Auxiliary Ensembles*			$L(i)$	*Code*
			→1.00		
		→0.40			
0.22	0.22	0.22		1	1
0.20	0.20	0.20		1	2
0.18	0.18	0.18		1	3
0.15	0.15			2	00
0.10	0.10			2	01
0.08	0.08			2	02
	→0.07				
0.05				3	030
0.02				3	031

Acknowledgments

The author is indebted to Dr. W. K. Linvill and Dr. R. M. Fano, both of the Massachusetts Institute of Technology, for their helpful criticism of this paper.

Coding with Linear Systems*

JOHN P. COSTAS†, ASSOCIATE, IRE

***Summary*—Message transmission over a noisy channel is considered. Two linear networks are to be designed: one being used to treat the message before transmission and the second to filter the treated message plus channel noise at the receiving end. The mean-square error between the actual transmission circuit output and the delayed message is minimized for a given allowable average signal power by proper network design. Numerical examples are given and discussed.**

I. Introduction

THE PROBLEM to be considered here is that of message transmission over a noisy channel. As shown in Fig. 1, a message function, $f_m(t)$, is to be sent down a channel into which a noise function, $f_n(t)$, is introduced additively. The resultant system output is represented by $f_0(t)$. In most communication systems, the opportunity exists to code the message before its introduction into the transmission channel. Recently, Wiener, Shannon, and others have considered coding processes of a rather complex nature wherein the message function is sampled, quantized, and the resulting sample values converted into a pulse code for transmission. Although this technique may be quite useful in many instances, its application will be restricted by the complexity of the terminal equipment required. In this discussion, the coding and decoding systems will be limited to linear networks. In Fig. 1, network $H(\omega)$ will be used to code the message before transmission and network $G(\omega)$ will perform the necessary decoding. Network $H(\omega)$ must be designed so that the message is predistorted or coded in such a way as to enable the decoding or filtering network $G(\omega)$ to give a better system output than would have been possible had the message itself been sent without modification.

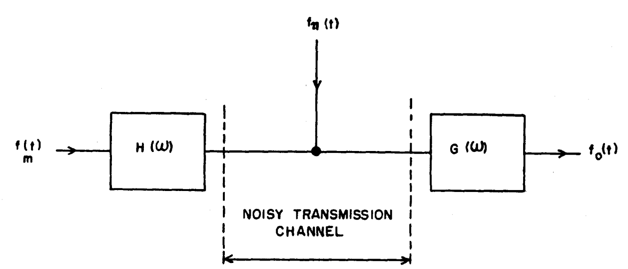


Fig. 1—Transmission system.

Before going further, a criterion of performance must be chosen for the transmission system. That is, some measurable quantity must be decided upon to enable us to determine whether one particular network pair $H(\omega)-G(\omega)$ is more satisfactory than some other pair. No single performance criterion can be expected to apply in all situations and no such claim is made for the mean-square error measure of performance which is to be used

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Digital Communication Systems 7057CEM

M-ary Modulation

Learning outcomes

- At the end of this lecture you will:
  - Gain knowledge of M-ary ASK, FSK and PSK schemes
  - Gain knowledge of M-ary QAM

M-ary modulation

- In ‘binary’ ASK, PSK and FSK, each modulated carrier transmits one bit of information (i.e., two message, each carrying one bit – 0 or 1).
- *M*-ary Modulation: To increase the bit transmission rate, each carrier signal is modulated with more than one bit of information, i.e. *M* messages, each carrying $\log_2(M)$ bits
  - e.g. 4-ary modulation: four message, each carrying two bits – 00, 01, 10, 11
- Requires more average power.

M-ary ASK

- The amplitude of the transmitted carrier signal is levelled into *M* **different levels**.

- If the maximum allowed voltage amplitude is A, then all *M* possible values are in the range[0, A] is given by

$$v_i = \frac{A\,i}{M-1} \qquad ;\boldsymbol{where\ i = 0{,}1{,}..M-1}$$

- The difference between two symbols is given by

$$\delta = \frac{A}{M-1}$$

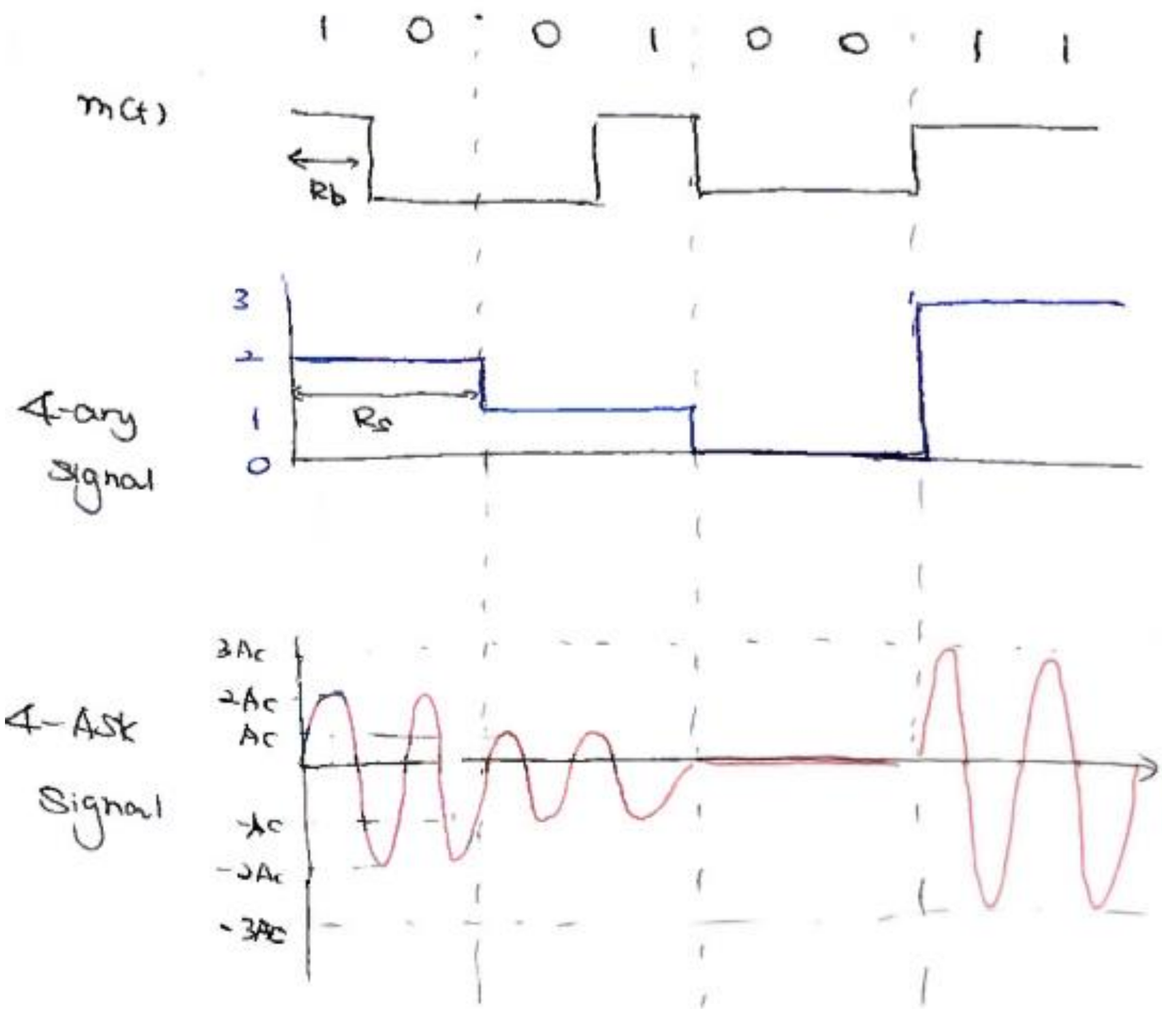
M-ary ASK: Example

- **4-ary ASK** with the message m=10010011 and the maximum amplitude of 3 Volts.

  - $\log_2(4) = 2$: each message carries 2 bits
  - All possible symbols are in the range [0 3]
  - $\delta = \frac{A}{M-1} = \frac{3}{4-1} = 1\,V$
  - $v_i = \frac{A\,i}{M-1}$

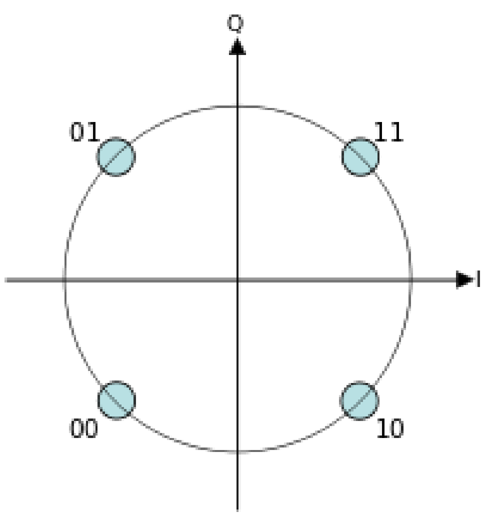
$$v_0 = 0\,V\ (S_0{:}\,00)$$
$$v_1 = 1\,V\ (S_1{:}\,01)$$
$$v_2 = 2\,V\ (S_2{:}\,10)$$
$$v_3 = 3\,V\ (S_3{:}\,11)$$

M-ary ASK: Example

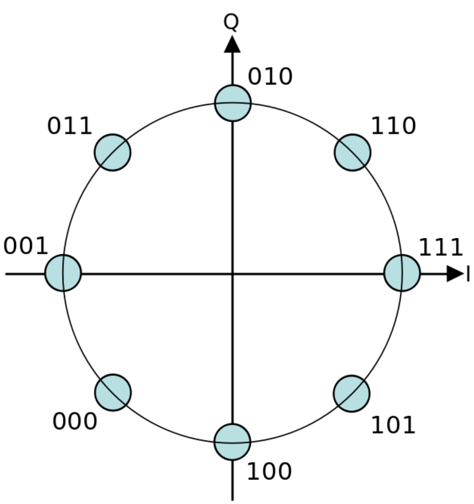


M-ary PSK

- In "Phase Shift Keying", the magnitude of X_k is constant, and the information is in the phase of the symbol.
- 2π phase is divided into M-phases with equal spacing
- M-PSK allows to send two or more bits at a time
  - 4-PSK (Quadrature PSK (QPSK): k =2, a phase shift of $\pi/2$
  - 8-PSK: k = 3, a phase shift of $\pi/4$



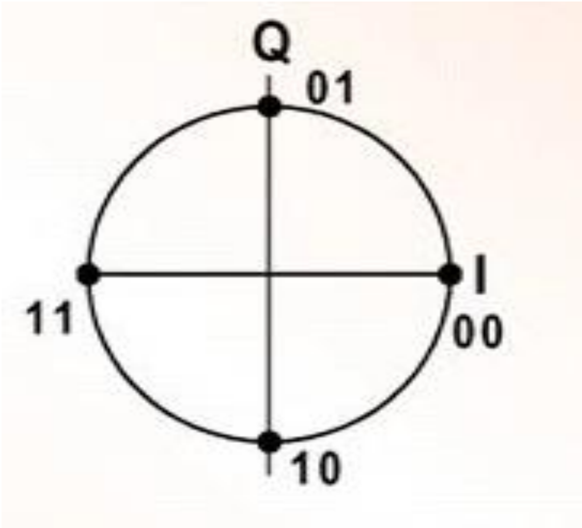
4-ary PSK (QPSK)



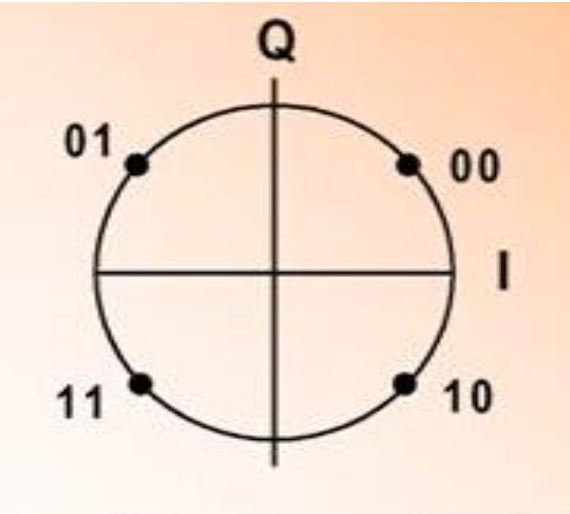
8-ary PSK

Quadrature PSK (QPSK)

- 4-ary PSK modulation technique, i.e., 2 bits per symbol
- A minimum phase change of $\pi/2$
- Can be interpreted as two independent BPSK system (same performance but twice the bandwidth efficiency of BPSK)



Phase of carrier: $\frac{\pi}{2}, \pi, \frac{3\pi}{2},\ 2\pi$



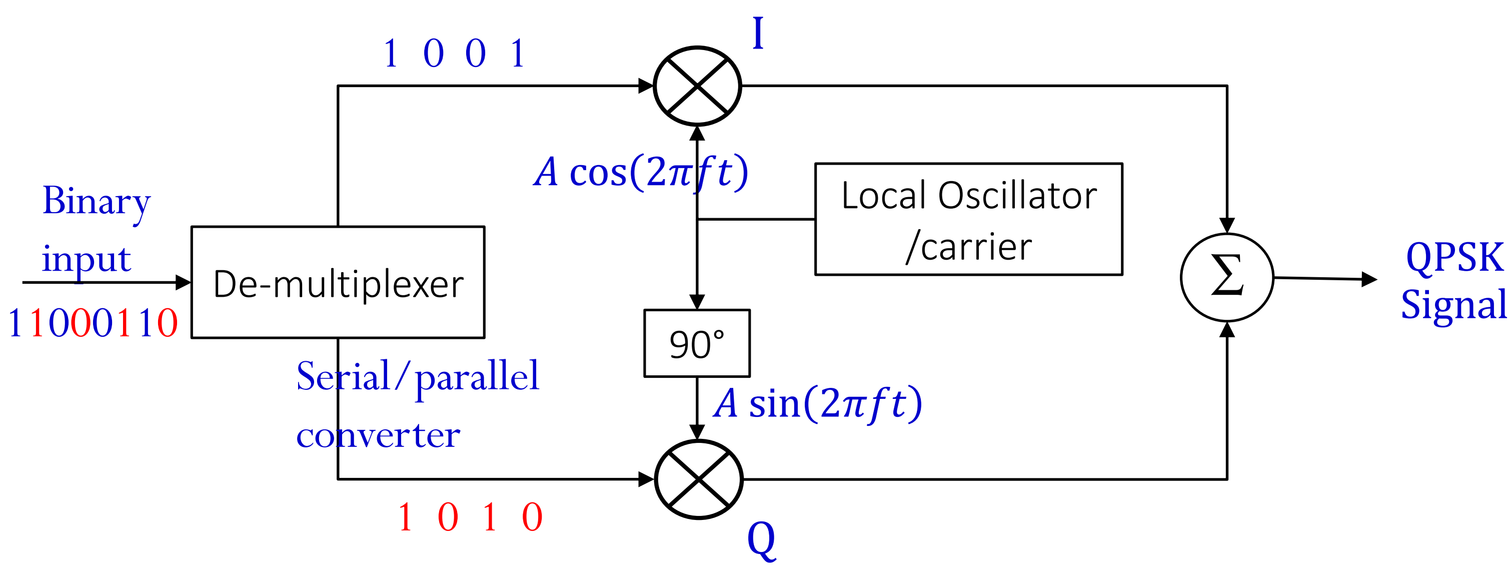
Phase of carrier: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Activity:

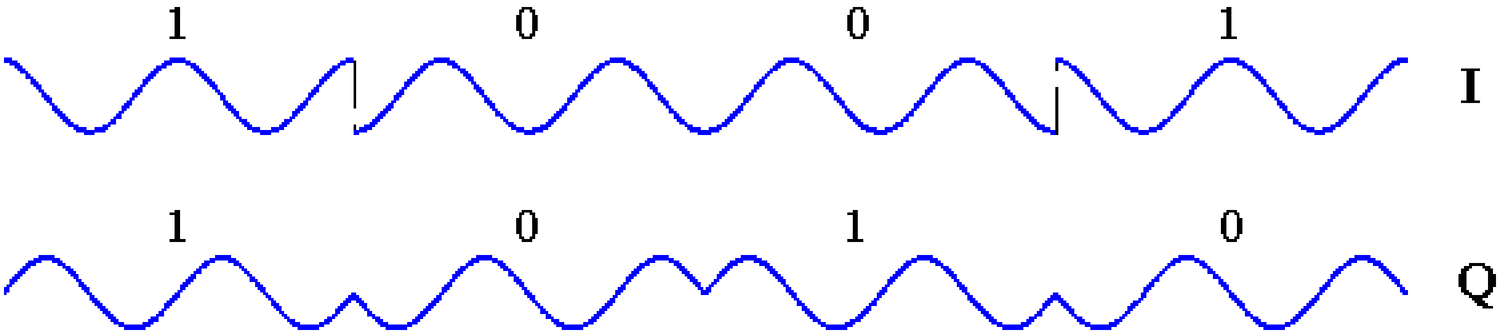
- Research why symbols are not sequential in constellation diagram?

QPSK Modulator

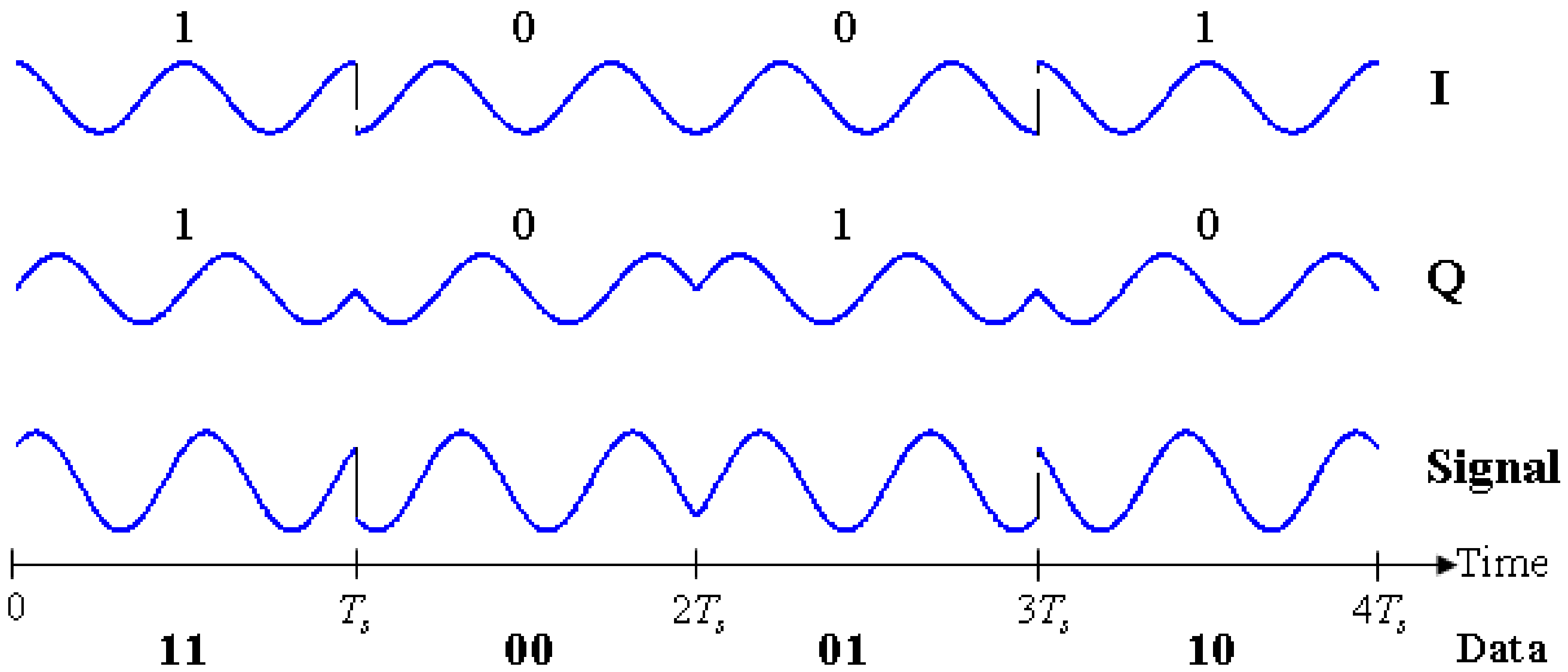
- A de-multiplexer (serial to parallel converter) is used to separate bits from the input.
- Signal on the in-phase arm is multiplied by cosine and the signal on the quadrature arm is multiplied by sine component.
- Adding both the signal gives the QPSK modulated signal



QPSK Modulator (cont..)



QPSK Modulator (cont..)

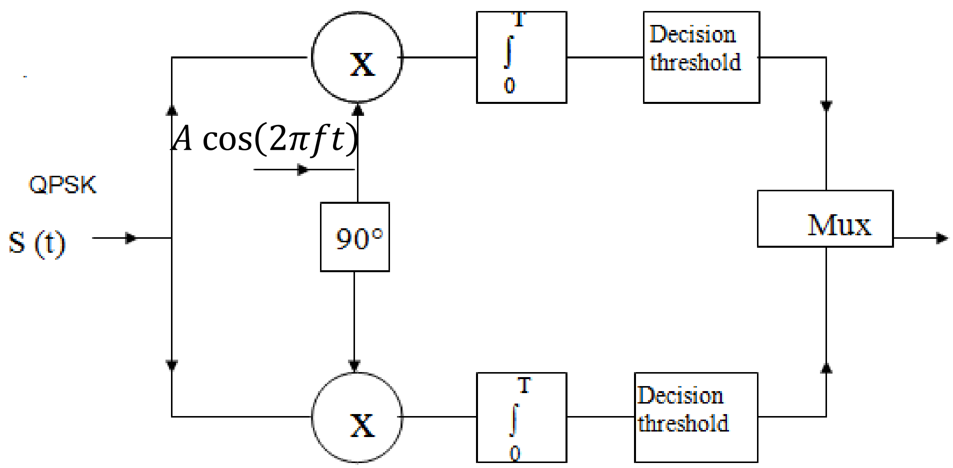


QPSK - Demodulator

- The received signal is multiplied by a reference carrier $\cos(2\pi f t)$ and $\sin(2\pi f t)$ on separate arms (in-phase and quadrature arms).
- The multiplied output on each arm is integrated over one bit period using an integrator.
- A threshold detector makes a decision on each integrated bit based on the signal level.

Note: Each branch is similar to BPSK (or ASK) demodulator.

- Bits on the in-phase arm (even bits) and on the quadrature arm (odd bits) are remapped to form detected information stream.



M-ary FSK

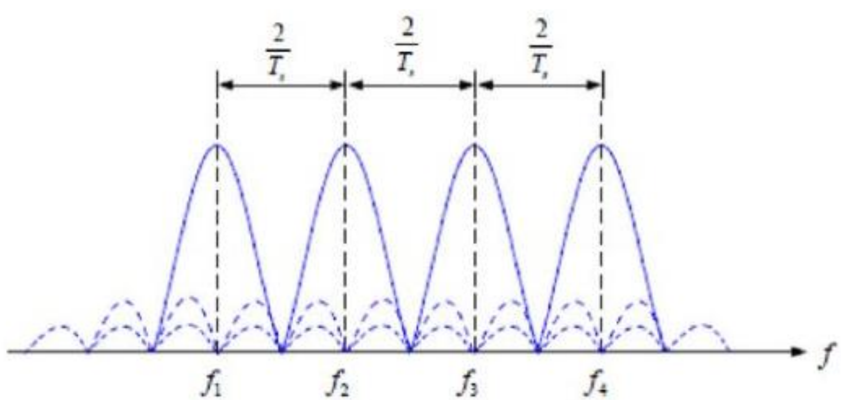
- Another method of sending two or more bits per symbol is to use FSK with **more than two frequencies**.

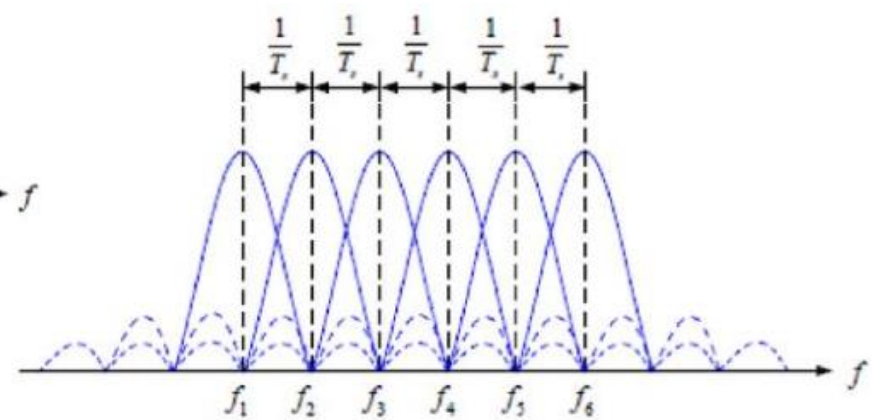
$$s_{MFSK}(t) = A\cos(2\pi f_i t), \quad 0 \le t \le T_s, \; i = 1, 2, \cdots, M$$

- This increases the bandwidth of the modulated signal. It has been used because of its **better performance** in noise.

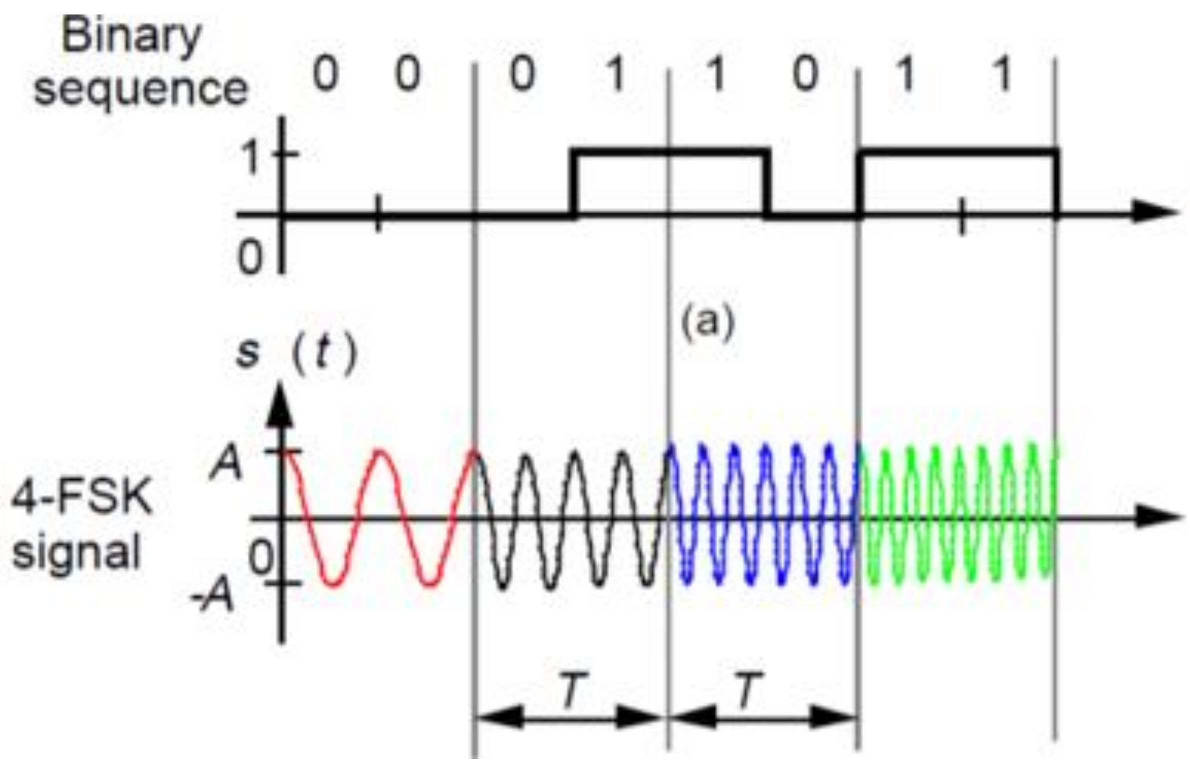
M-ary FSK (cont...)

- The optimum spacing between frequencies is the symbol rate for coherent detection (known as orthogonal).
  - ➢ Frequency spacing = 2×symbol rate
  - ➢ Orthogonal frequency spacing = 2×symbol rate/2 = symbol rate



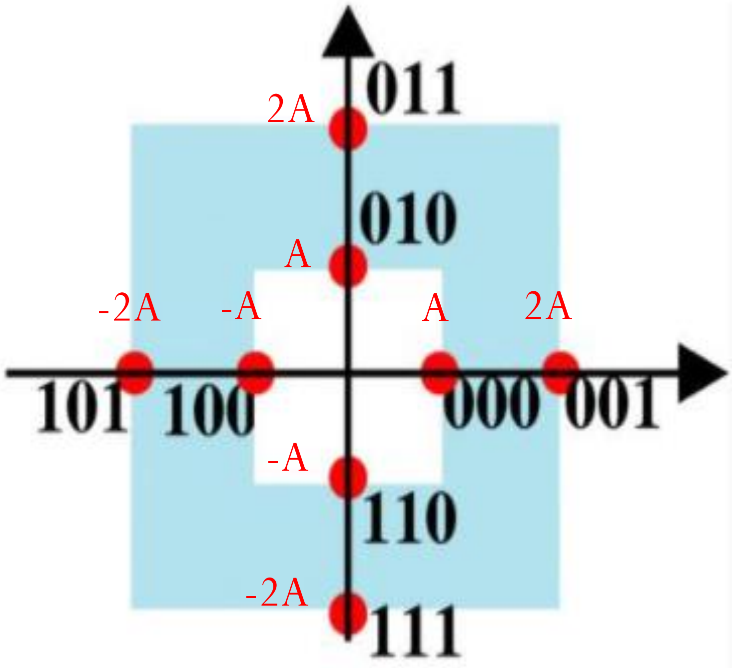


Example: 4-FSK signal



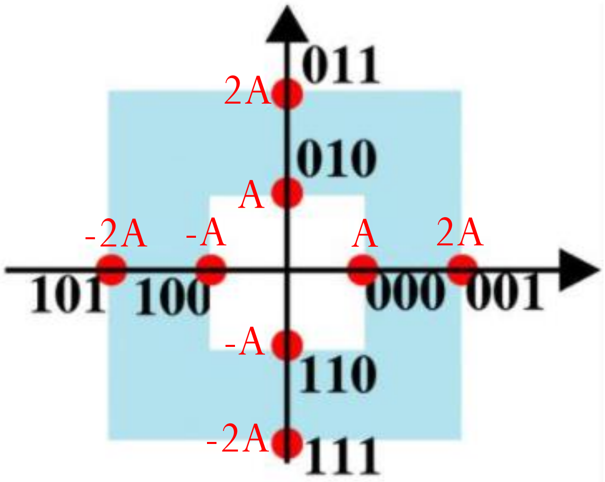
Quadrature amplitude modulation(QAM)

- Hybrid modulation technique: Combination of ASK and PSK, i.e., Both **amplitude** and **phase of a carrier signal** are varied
- This offers better BER compared to ASK and PSK schemes.
- Example: 8-QAM : 1 symbol= n = 3 bits

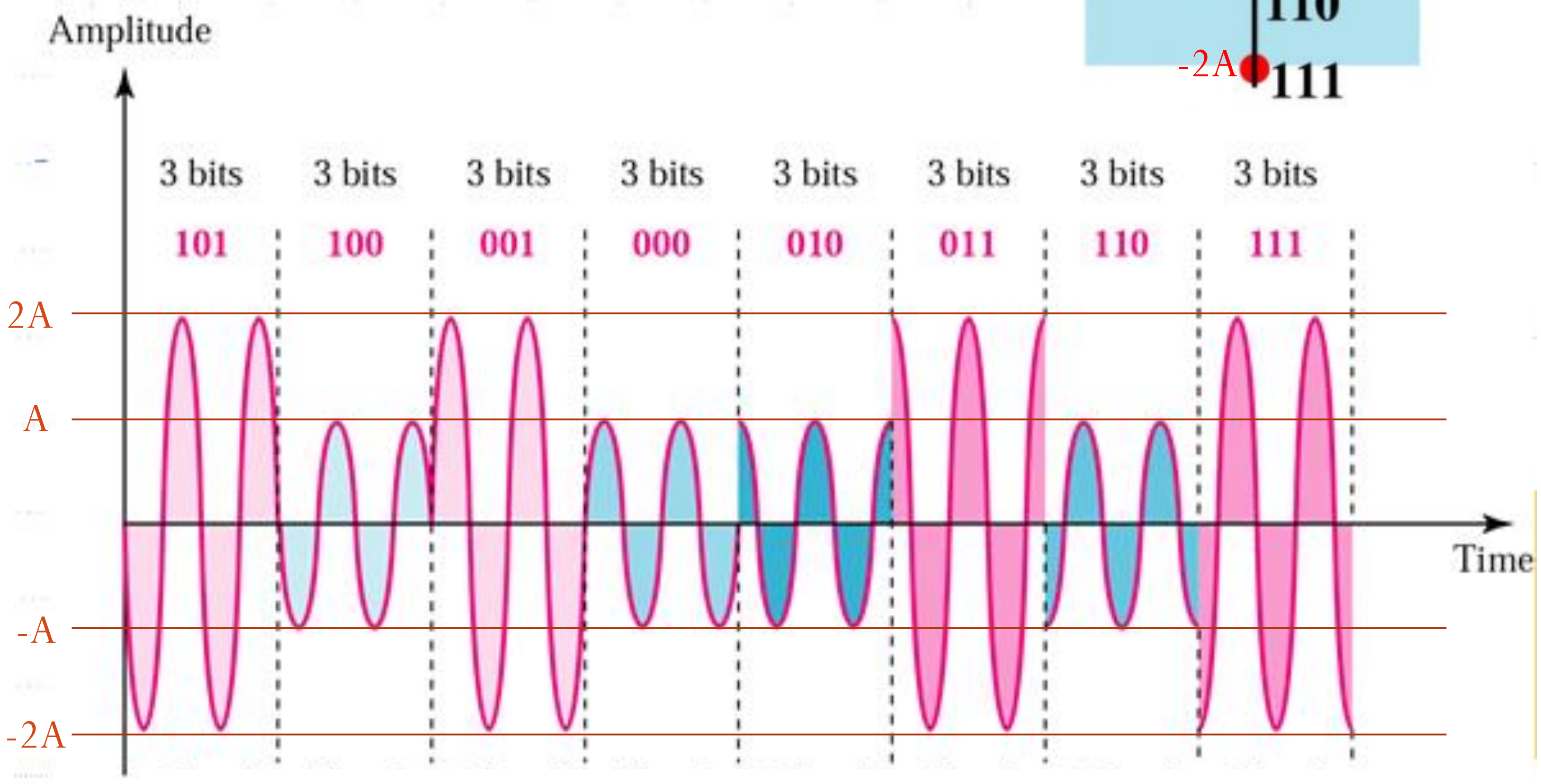


8-QAM Constillation diagram

QAM(cont.)



2A
011
010
A
-2A
-A
A
2A
101
100
000
001
-A
110
-2A
111



Amplitude
3 bits
3 bits
3 bits
3 bits
3 bits
3 bits
3 bits
3 bits
101
100
001
000
010
011
110
111
2A
A
-A
-2A
Time

Activity 2:

- Find out the applications of QAM modulation technique in communication systems?

Summary and Further Reading

- M-ary modulation

  (a) M-ary ASK

  (b) M-ary FSK

  (c) M-ary PSK

  (d) QAM
- Further Reading
  - Digital communications – John G. Proakis, Masoud Salehi 2008
  - Digital communications: Fundamental and application – Sklar, Bernard 2001

Digital Communication Systems 7057CEM

Symmetric Communication channel and Bit Error Rate (BER)

Coventry University

Learning outcomes

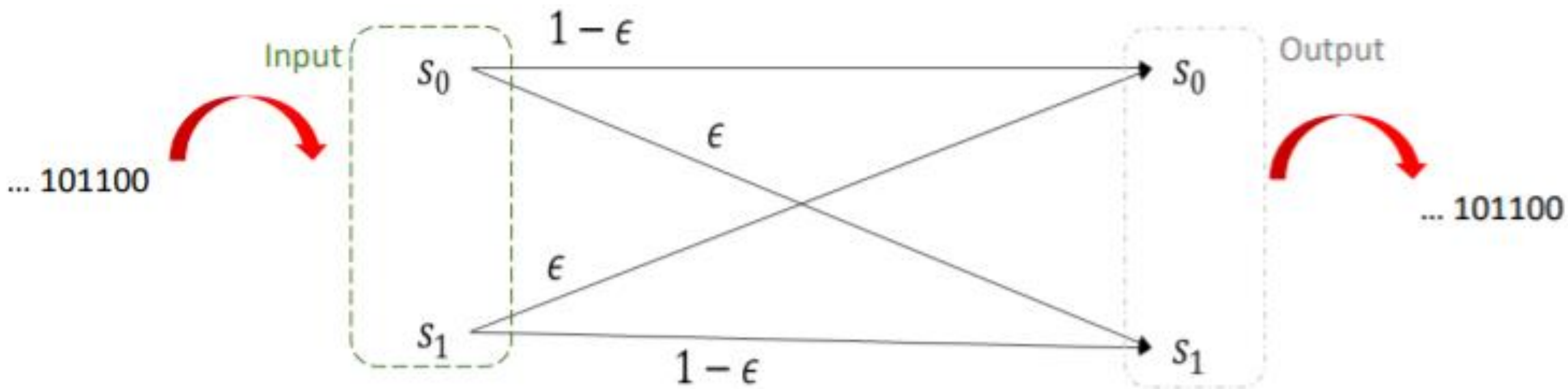
- **At the end of this lecture you will be able to:**
  - Analyse Binary Symmetric Channel (BSC)
  - Apply probability theorem in BSC channel
  - Develop BER formula for different modulation techniques
  - Compare BER for different modulation techniques

Recap from last lecture

- Digital Modulation Schemes

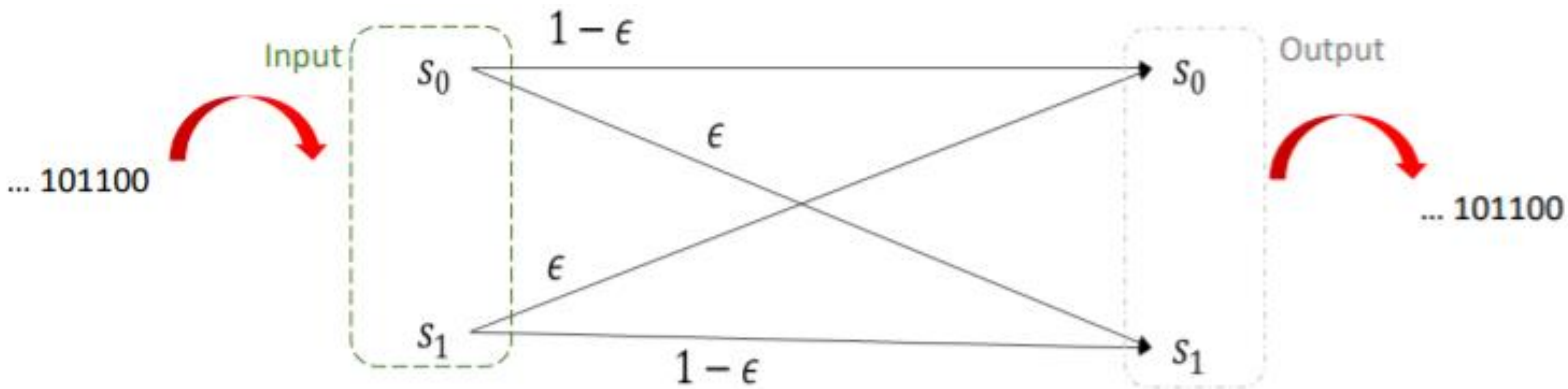
  (a) ASK , (b) FSK and (c) PSK
- Bandwidth and Average energy per bits
- M-ary modulation
- QAM

Symmetric communication channel



- **Binary symmetric channel (BSC)**
  - Two different data symbols
  - The symbols are transferred via a BSC channel
  - Detection of the symbols could be incorrect with a small probability ϵ $(0 < \epsilon < \frac{1}{2})$
- Many problems in communication theory can be reduced to a BSC

Probability of the output symbols

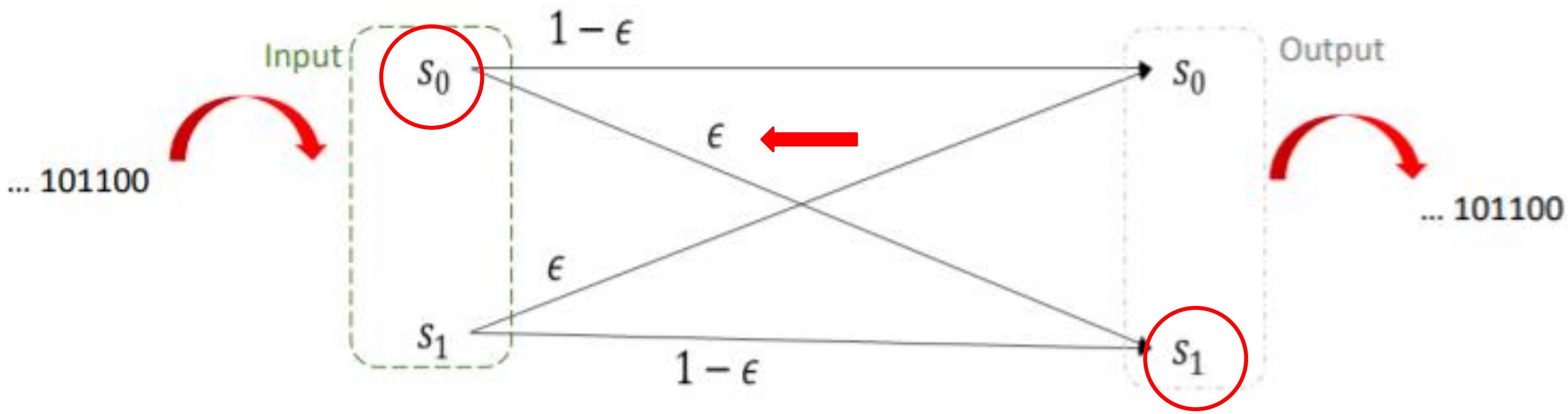


- Let x, y be the transmitted and received symbols
- For a given x, y,
  - $p(y|x)$ = 'likelihood function' – the probability that y is received given that x is transmitted via a channel

$$p(y = s_1 \mid x = s_0) =$$

$$p(y = s_0 \mid x = s_0) =$$

Probability of the output symbols

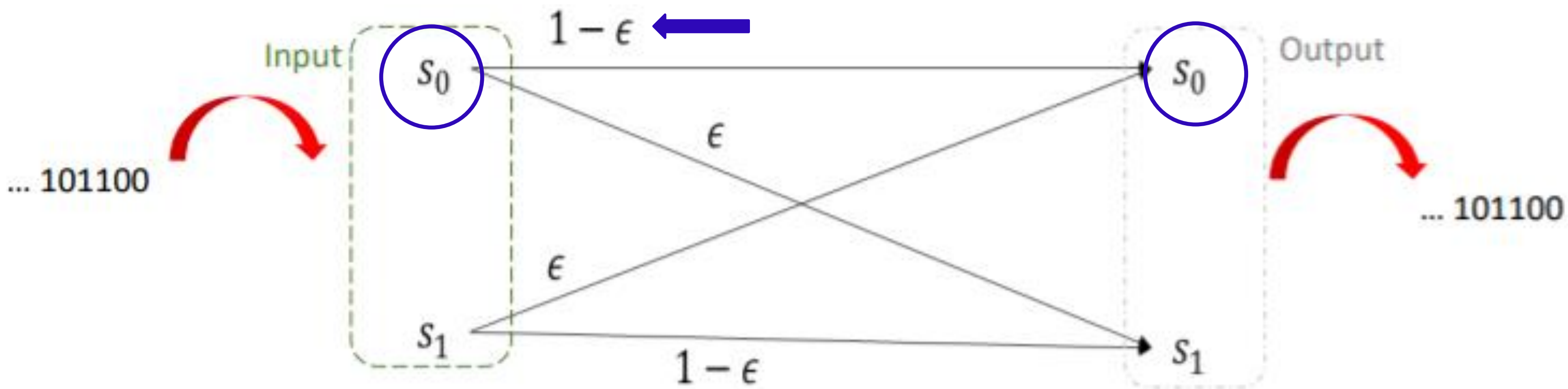


- Let x, y be the transmitted and received symbols
- For a given x, y,
  - $p(y|x)$ = 'likelihood function' – the probability that y is received given that x is transmitted via a channel

$$p(y = s_1 \mid x = s_0) = \epsilon$$

$$p(y = s_0 \mid x = s_0) =$$

Probability of the output symbols



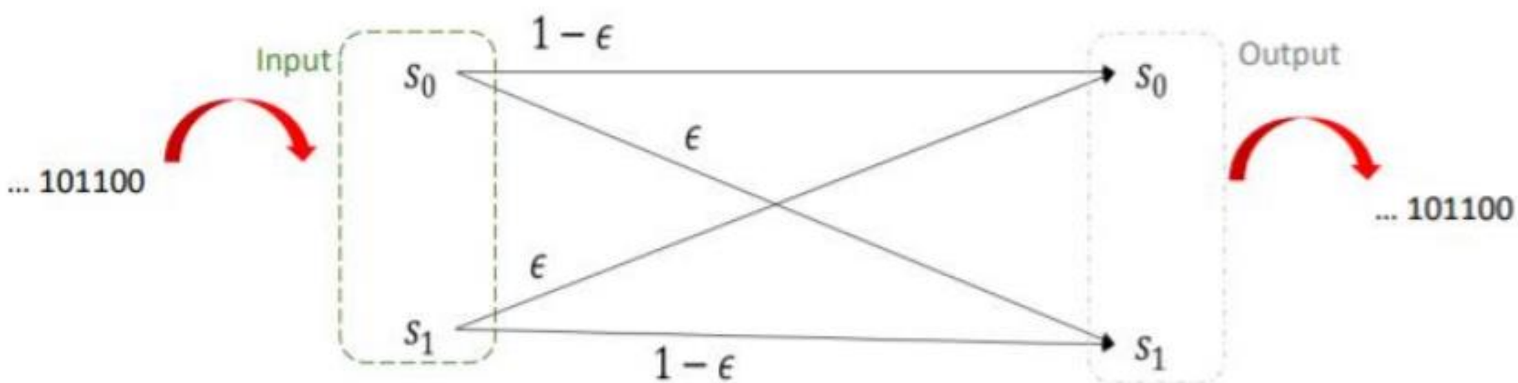
- Let x, y be the transmitted and received symbols
- For a given x, y,
  - $p(y|x)$ = 'likelihood function' – the probability that y is received given that x is transmitted via a channel

$$p(y = s_1 \mid x = s_0) =$$

$$p(y = s_0 \mid x = s_0) = 1 - \epsilon$$

Exercise 1:

Consider a transmission over a binary symmetric communications channel whose input is X= (s_0, s_1) and output Y= (s_0, s_1). Let p be the probability of transmitting s_0. (This implies the probability of transmitting s_1 is 1- p.)



Assume that the probabilities of transmitting s_0 and s_1 are 3/4 and 1/4, respectively. Find the probability that the output symbol equals to s_0 for a given ϵ = 1/4.

Important concept in communication

Bit error rate(BER):

- Performance of Digital Data Systems is dependent on the

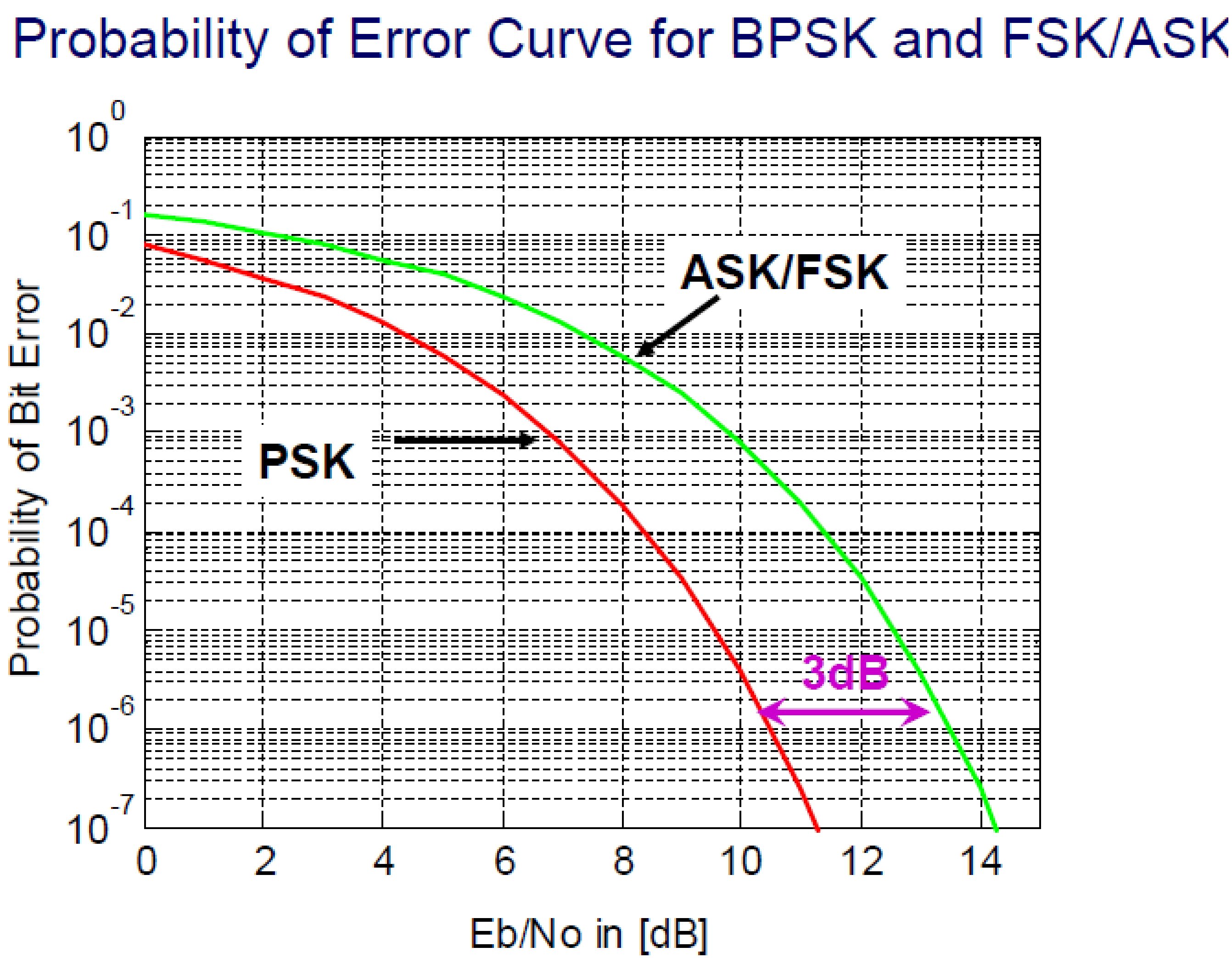
$$BER=\frac{No.of\ Errors}{Total\ no.of\ bits}$$

- For example, 10 bits in error out of 10000 received gives a BER of 10^{-3}

Measure of the Signal to Noise Ratio(SNR)

- SNR is the ratio between signal level and noise level
- Typically measured at the receiving end of the communications system
- Defined as SNR = 10 Log_{10} (Signal power/ Noise power) dB
- Here, **we use $\boldsymbol{\frac{E_b}{N_0}}$, why???**
  - ➢ The performance of different modulation schemes can be measured w.r.t $\frac{E_b}{N_0}$
  - ➢ A fair comparison can be provided, i.e., BER vs. $\frac{E_b}{N_0}$
- For a noise power of *N* watts in a bandwidth *B* Hz, there is a noise power spectral density, $N_0 = \frac{N}{B}$.

BER representation



Shannon Channel capacity

- One of the most important theorem
- Defined as the maximum information rate at which reliable communication can be achieved over a communications channel.
- For an AWGN channel, Shannon-Hartley theorem states that

$$C = B \log_2 (1 + SNR)$$

where C is channel capacity in bit/s and B is bandwidth in Hz

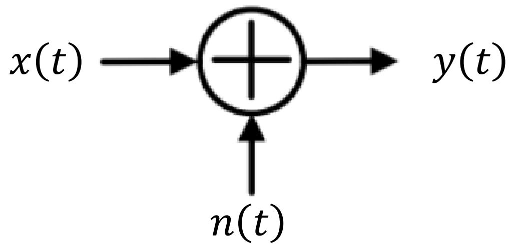
- Linearly proportional to available bandwidth
- Logarithmically proportional to available SNR

Exercise 2:

- What is the capacity of an AWGN channel that experience a signal to noise power ratio of 30 dB and has a bandwidth of 1kHz?

Additive White Gaussian Noise (AWGN)

- Additive
- White
- Gaussian



Noisy

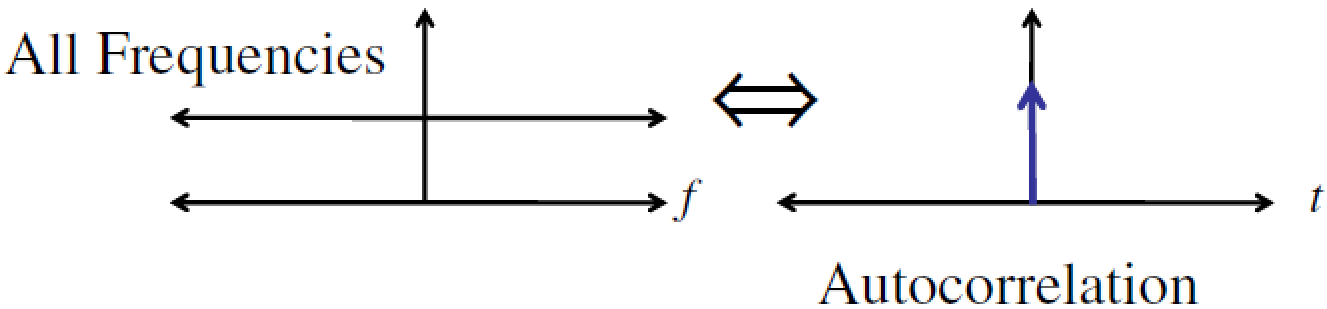
Noise Free

$$y(t) = x(t) \boxed{+\, n(t)}$$

Additive Noise

Additive White Gaussian Noise (AWGN)

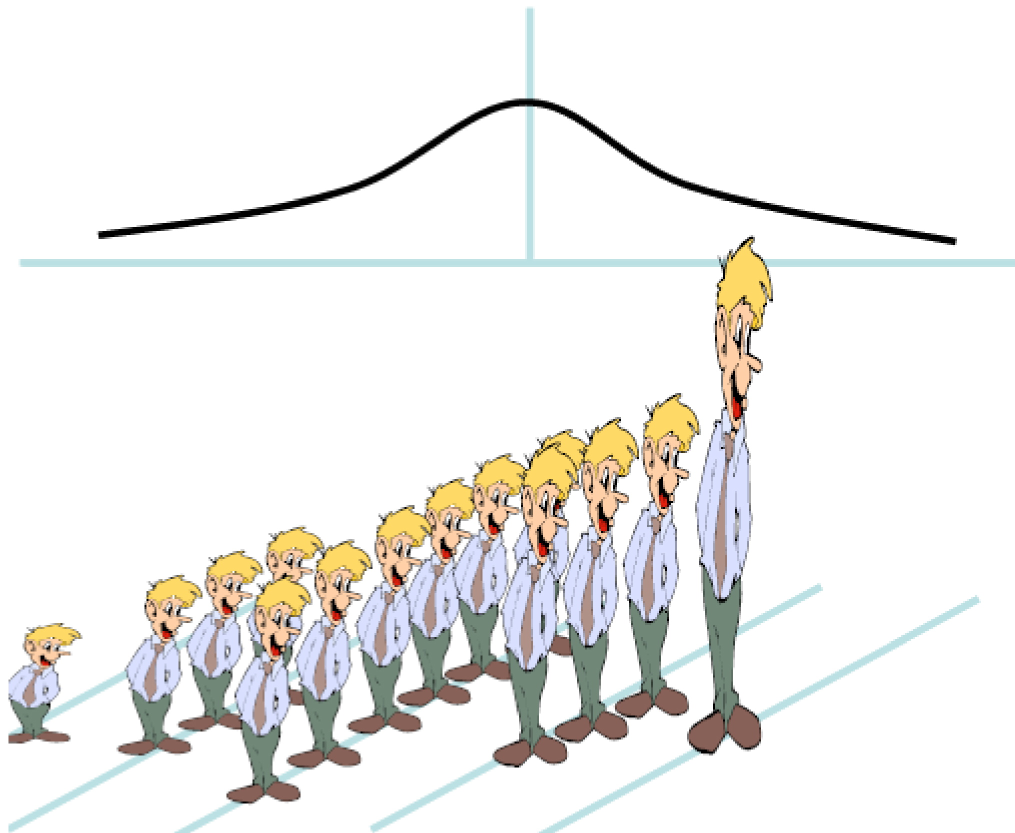
- Additive
- **White**
- Gaussian



Additive White Gaussian Noise (AWGN)

- Additive
- White
- Gaussian

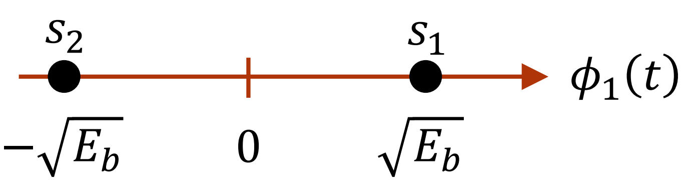
$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \text{ with } \mu = 0 \text{ and } \sigma 2 = \frac{N_0}{2}$$



BER- BPSK

$$s_{PSK}(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f t) & if\ m(t) = 1 \\ -\sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f t) & if\ m(t) = 0 \end{cases}$$

- Signal space representation



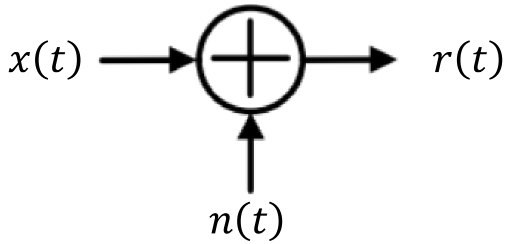
$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f t)$$

$$s_1(t) = \sqrt{E_b}\phi_1(t),\ s_2(t) = -\sqrt{E_b}\phi_1(t)$$

$$d_{12} = 2\sqrt{E_b}$$

BER- BPSK

- AWGN Channel

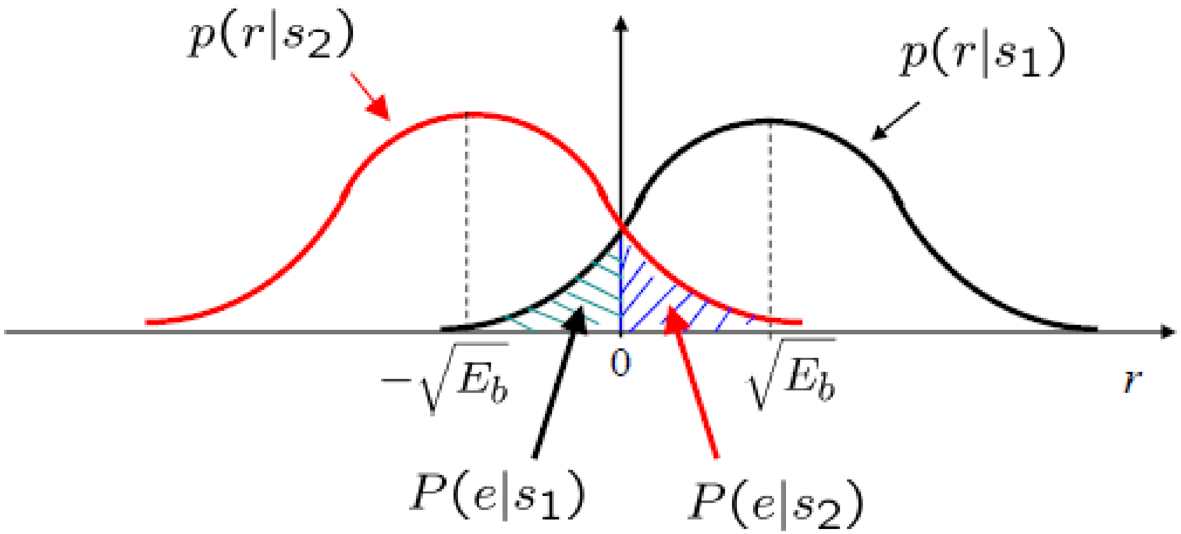


$\mathrm{r(t) = x(t) +\ n(t)}$ where n(t) is an additive white Gaussian noise with zero mean and variance $\frac{N_0}{2}$

$$p(n) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{n^2}{N_0}\right)$$

- (i) x = S_2 (bit 0) : $r = -\sqrt{E_b} + n$
- (ii) x = S_1 (bit 1) : $r = \sqrt{E_b} + n$

BER- BPSK



- $P(\mathrm{e}|S_2)$: Bit error occurs if $r > 0 \rightarrow \mathrm{n} > \sqrt{E_b}$

$$P(e|S_2) = \mathbb{p}\left[\mathrm{n} > \sqrt{E_b}\ \right]$$

$$= \int_{\sqrt{E_b}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{n^2}{N_0}\right) dn$$

BER- BPSK

- The BER for BPSK in the presence of additive white Gaussian noise is defined as

$$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2}\, dt$

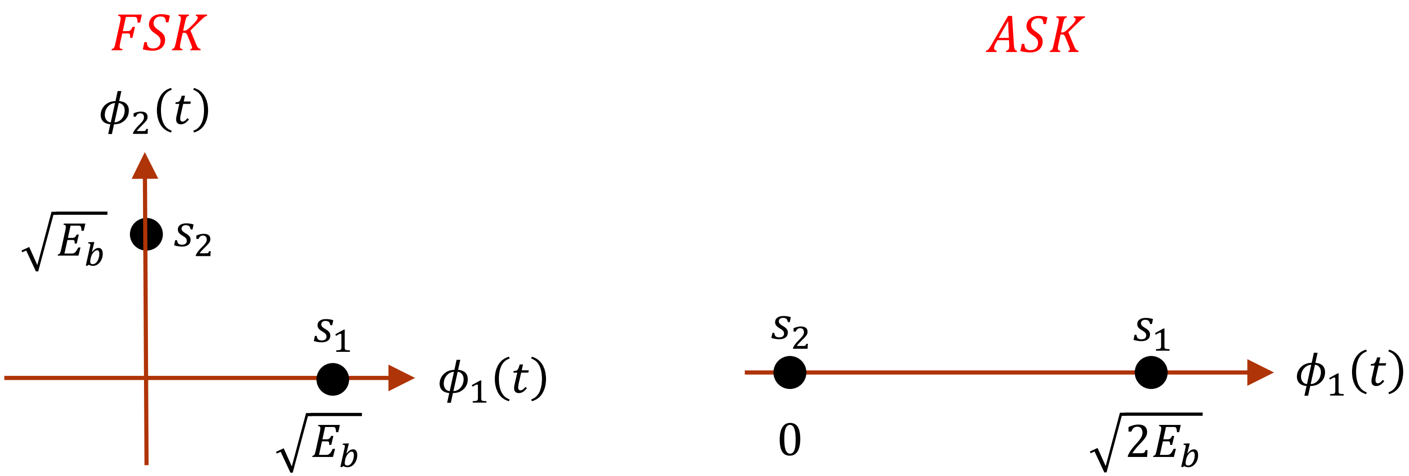
Value of Q can be found from the Q table

Exercise 3:

- Determine the bit error rate (BER) of 16-PSK modulation scheme when E_b/N_0=13dB.

BER- FSK & ASK

- Signal space representation



- The BER for FSK and ASK in the presence of additive white Gaussian noise is defined as

$$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

BER (cont...)

- Non-coherent ASK

$$\frac{1}{2}\exp(-\frac{E_b}{4N_0})$$

- Non-coherent FSK

$$\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$$

- DPSK

$$\frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$$

BER for M-ary modulation

- M-ary PSK

$$2\mathrm{Q}\left[\sqrt{\frac{2E_s}{N_0}}\,\sin(\pi/M)\right]/log_2 M$$

- M-ary FSK

$$(M/2)\,Q\left[\sqrt{\frac{E_s}{N_0}}\right]$$

- M-QAM

$$2\,\frac{1-\frac{1}{L}}{\log_2 L}\,Q\left[\sqrt{\frac{3\log_2 L}{L^2-1}\,\frac{2E_b}{N_0}}\right]$$

Exercise 4:

Perform calculations to determine which of the two systems below gives the best BER.

Non-coherently detected FSK with $E_b/N_0 = 12$ dB

Coherently detected PSK with $E_b/N_0 = 8$ dB

Summary and Further Reading

- Binary symmetric communications channel
- Shannon channel capacity
- Bit Error rate (BER)
- AWGN
- BER analysis of modulation schemes
- Further Reading
  - Digital communications – John G. Proakis, Masoud Salehi 2008
  - Digital communications: Fundamental and application – Sklar, Bernard 2001

Table of the Q Function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt$$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	5.000E-01	4.960E-01	4.920E-01	4.880E-01	4.840E-01	4.801E-01	4.761E-01	4.721E-01	4.681E-01	4.641E-01
0.1	4.602E-01	4.562E-01	4.522E-01	4.483E-01	4.443E-01	4.404E-01	4.364E-01	4.325E-01	4.286E-01	4.247E-01
0.2	4.207E-01	4.168E-01	4.129E-01	4.090E-01	4.052E-01	4.013E-01	3.974E-01	3.936E-01	3.897E-01	3.859E-01
0.3	3.821E-01	3.783E-01	3.745E-01	3.707E-01	3.669E-01	3.632E-01	3.594E-01	3.557E-01	3.520E-01	3.483E-01
0.4	3.446E-01	3.409E-01	3.372E-01	3.336E-01	3.300E-01	3.264E-01	3.228E-01	3.192E-01	3.156E-01	3.121E-01
0.5	3.085E-01	3.050E-01	3.015E-01	2.981E-01	2.946E-01	2.912E-01	2.877E-01	2.843E-01	2.810E-01	2.776E-01
0.6	2.743E-01	2.709E-01	2.676E-01	2.643E-01	2.611E-01	2.578E-01	2.546E-01	2.514E-01	2.483E-01	2.451E-01
0.7	2.420E-01	2.389E-01	2.358E-01	2.327E-01	2.296E-01	2.266E-01	2.236E-01	2.206E-01	2.177E-01	2.148E-01
0.8	2.119E-01	2.090E-01	2.061E-01	2.033E-01	2.005E-01	1.977E-01	1.949E-01	1.922E-01	1.894E-01	1.867E-01
0.9	1.841E-01	1.814E-01	1.788E-01	1.762E-01	1.736E-01	1.711E-01	1.685E-01	1.660E-01	1.635E-01	1.611E-01
1.0	1.587E-01	1.562E-01	1.539E-01	1.515E-01	1.492E-01	1.469E-01	1.446E-01	1.423E-01	1.401E-01	1.379E-01
1.1	1.357E-01	1.335E-01	1.314E-01	1.292E-01	1.271E-01	1.251E-01	1.230E-01	1.210E-01	1.190E-01	1.170E-01
1.2	1.151E-01	1.131E-01	1.112E-01	1.093E-01	1.075E-01	1.056E-01	1.038E-01	1.020E-01	1.003E-01	9.853E-02
1.3	9.680E-02	9.510E-02	9.342E-02	9.176E-02	9.012E-02	8.851E-02	8.692E-02	8.534E-02	8.379E-02	8.226E-02
1.4	8.076E-02	7.927E-02	7.780E-02	7.636E-02	7.493E-02	7.353E-02	7.215E-02	7.078E-02	6.944E-02	6.811E-02
1.5	6.681E-02	6.552E-02	6.426E-02	6.301E-02	6.178E-02	6.057E-02	5.938E-02	5.821E-02	5.705E-02	5.592E-02
1.6	5.480E-02	5.370E-02	5.262E-02	5.155E-02	5.050E-02	4.947E-02	4.846E-02	4.746E-02	4.648E-02	4.551E-02
1.7	4.457E-02	4.363E-02	4.272E-02	4.182E-02	4.093E-02	4.006E-02	3.920E-02	3.836E-02	3.754E-02	3.673E-02
1.8	3.593E-02	3.515E-02	3.438E-02	3.362E-02	3.288E-02	3.216E-02	3.144E-02	3.074E-02	3.005E-02	2.938E-02
1.9	2.872E-02	2.807E-02	2.743E-02	2.680E-02	2.619E-02	2.559E-02	2.500E-02	2.442E-02	2.385E-02	2.330E-02
2.0	2.275E-02	2.222E-02	2.169E-02	2.118E-02	2.068E-02	2.018E-02	1.970E-02	1.923E-02	1.876E-02	1.831E-02
2.1	1.786E-02	1.743E-02	1.700E-02	1.659E-02	1.618E-02	1.578E-02	1.539E-02	1.500E-02	1.463E-02	1.426E-02
2.2	1.390E-02	1.355E-02	1.321E-02	1.287E-02	1.255E-02	1.222E-02	1.191E-02	1.160E-02	1.130E-02	1.101E-02
2.3	1.072E-02	1.044E-02	1.017E-02	9.903E-03	9.642E-03	9.387E-03	9.137E-03	8.894E-03	8.656E-03	8.424E-03
2.4	8.198E-03	7.976E-03	7.760E-03	7.549E-03	7.344E-03	7.143E-03	6.947E-03	6.756E-03	6.569E-03	6.387E-03
2.5	6.210E-03	6.037E-03	5.868E-03	5.703E-03	5.543E-03	5.386E-03	5.234E-03	5.085E-03	4.940E-03	4.799E-03
2.6	4.661E-03	4.527E-03	4.397E-03	4.269E-03	4.145E-03	4.025E-03	3.907E-03	3.793E-03	3.681E-03	3.573E-03
2.7	3.467E-03	3.364E-03	3.264E-03	3.167E-03	3.072E-03	2.980E-03	2.890E-03	2.803E-03	2.718E-03	2.635E-03
2.8	2.555E-03	2.477E-03	2.401E-03	2.327E-03	2.256E-03	2.186E-03	2.118E-03	2.052E-03	1.988E-03	1.926E-03
2.9	1.866E-03	1.807E-03	1.750E-03	1.695E-03	1.641E-03	1.589E-03	1.538E-03	1.489E-03	1.441E-03	1.395E-03
3.0	1.350E-03	1.306E-03	1.264E-03	1.223E-03	1.183E-03	1.144E-03	1.107E-03	1.070E-03	1.035E-03	1.001E-03
3.1	9.676E-04	9.354E-04	9.043E-04	8.740E-04	8.447E-04	8.164E-04	7.888E-04	7.622E-04	7.364E-04	7.114E-04
3.2	6.871E-04	6.637E-04	6.410E-04	6.190E-04	5.976E-04	5.770E-04	5.571E-04	5.377E-04	5.190E-04	5.009E-04
3.3	4.834E-04	4.665E-04	4.501E-04	4.342E-04	4.189E-04	4.041E-04	3.897E-04	3.758E-04	3.624E-04	3.495E-04
3.4	3.369E-04	3.248E-04	3.131E-04	3.018E-04	2.909E-04	2.803E-04	2.701E-04	2.602E-04	2.507E-04	2.415E-04
3.5	2.326E-04	2.241E-04	2.158E-04	2.078E-04	2.001E-04	1.926E-04	1.854E-04	1.785E-04	1.718E-04	1.653E-04
3.6	1.591E-04	1.531E-04	1.473E-04	1.417E-04	1.363E-04	1.311E-04	1.261E-04	1.213E-04	1.166E-04	1.121E-04
3.7	1.078E-04	1.036E-04	9.961E-05	9.574E-05	9.201E-05	8.842E-05	8.496E-05	8.162E-05	7.841E-05	7.532E-05
3.8	7.235E-05	6.948E-05	6.673E-05	6.407E-05	6.152E-05	5.906E-05	5.669E-05	5.442E-05	5.223E-05	5.012E-05
3.9	4.810E-05	4.615E-05	4.427E-05	4.247E-05	4.074E-05	3.908E-05	3.747E-05	3.594E-05	3.446E-05	3.304E-05
4.0	3.167E-05	3.036E-05	2.910E-05	2.789E-05	2.673E-05	2.561E-05	2.454E-05	2.351E-05	2.252E-05	2.157E-05
4.1	2.066E-05	1.978E-05	1.894E-05	1.814E-05	1.737E-05	1.662E-05	1.591E-05	1.523E-05	1.458E-05	1.395E-05
4.2	1.335E-05	1.277E-05	1.222E-05	1.168E-05	1.118E-05	1.069E-05	1.022E-05	9.774E-06	9.345E-06	8.934E-06
4.3	8.540E-06	8.163E-06	7.801E-06	7.455E-06	7.124E-06	6.807E-06	6.503E-06	6.212E-06	5.934E-06	5.668E-06
4.4	5.413E-06	5.169E-06	4.935E-06	4.712E-06	4.498E-06	4.294E-06	4.098E-06	3.911E-06	3.732E-06	3.561E-06
4.5	3.398E-06	3.241E-06	3.092E-06	2.949E-06	2.813E-06	2.682E-06	2.558E-06	2.439E-06	2.325E-06	2.216E-06
4.6	2.112E-06	2.013E-06	1.919E-06	1.828E-06	1.742E-06	1.660E-06	1.581E-06	1.506E-06	1.434E-06	1.366E-06
4.7	1.301E-06	1.239E-06	1.179E-06	1.123E-06	1.069E-06	1.017E-06	9.680E-07	9.211E-07	8.765E-07	8.339E-07
4.8	7.933E-07	7.547E-07	7.178E-07	6.827E-07	6.492E-07	6.173E-07	5.869E-07	5.580E-07	5.304E-07	5.042E-07
4.9	4.792E-07	4.554E-07	4.327E-07	4.111E-07	3.906E-07	3.711E-07	3.525E-07	3.348E-07	3.179E-07	3.019E-07

Table of the Q Function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt$$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
5.0	2.867E-07	2.722E-07	2.584E-07	2.452E-07	2.328E-07	2.209E-07	2.096E-07	1.989E-07	1.887E-07	1.790E-07
5.1	1.698E-07	1.611E-07	1.528E-07	1.449E-07	1.374E-07	1.302E-07	1.235E-07	1.170E-07	1.109E-07	1.051E-07
5.2	9.964E-08	9.442E-08	8.946E-08	8.476E-08	8.029E-08	7.605E-08	7.203E-08	6.821E-08	6.459E-08	6.116E-08
5.3	5.790E-08	5.481E-08	5.188E-08	4.911E-08	4.647E-08	4.398E-08	4.161E-08	3.937E-08	3.724E-08	3.523E-08
5.4	3.332E-08	3.151E-08	2.980E-08	2.818E-08	2.664E-08	2.518E-08	2.381E-08	2.250E-08	2.127E-08	2.010E-08
5.5	1.899E-08	1.794E-08	1.695E-08	1.601E-08	1.512E-08	1.428E-08	1.349E-08	1.274E-08	1.203E-08	1.135E-08
5.6	1.072E-08	1.012E-08	9.548E-09	9.010E-09	8.503E-09	8.022E-09	7.569E-09	7.140E-09	6.735E-09	6.352E-09
5.7	5.990E-09	5.649E-09	5.326E-09	5.022E-09	4.734E-09	4.462E-09	4.206E-09	3.964E-09	3.735E-09	3.519E-09
5.8	3.316E-09	3.124E-09	2.942E-09	2.771E-09	2.610E-09	2.458E-09	2.314E-09	2.179E-09	2.051E-09	1.931E-09
5.9	1.818E-09	1.711E-09	1.610E-09	1.515E-09	1.425E-09	1.341E-09	1.261E-09	1.186E-09	1.116E-09	1.049E-09
6.0	9.866E-10	9.276E-10	8.721E-10	8.198E-10	7.706E-10	7.242E-10	6.806E-10	6.396E-10	6.009E-10	5.646E-10
6.1	5.303E-10	4.982E-10	4.679E-10	4.394E-10	4.126E-10	3.874E-10	3.637E-10	3.414E-10	3.205E-10	3.008E-10
6.2	2.823E-10	2.649E-10	2.486E-10	2.332E-10	2.188E-10	2.052E-10	1.925E-10	1.805E-10	1.693E-10	1.587E-10
6.3	1.488E-10	1.395E-10	1.308E-10	1.226E-10	1.149E-10	1.077E-10	1.009E-10	9.451E-11	8.854E-11	8.294E-11
6.4	7.769E-11	7.276E-11	6.814E-11	6.380E-11	5.974E-11	5.593E-11	5.235E-11	4.900E-11	4.586E-11	4.292E-11
6.5	4.016E-11	3.758E-11	3.515E-11	3.288E-11	3.076E-11	2.877E-11	2.690E-11	2.516E-11	2.352E-11	2.199E-11
6.6	2.056E-11	1.922E-11	1.796E-11	1.678E-11	1.568E-11	1.465E-11	1.369E-11	1.279E-11	1.195E-11	1.116E-11
6.7	1.042E-11	9.731E-12	9.086E-12	8.483E-12	7.919E-12	7.392E-12	6.900E-12	6.439E-12	6.009E-12	5.607E-12
6.8	5.231E-12	4.880E-12	4.552E-12	4.246E-12	3.960E-12	3.692E-12	3.443E-12	3.210E-12	2.993E-12	2.790E-12
6.9	2.600E-12	2.423E-12	2.258E-12	2.104E-12	1.960E-12	1.826E-12	1.701E-12	1.585E-12	1.476E-12	1.374E-12
7.0	1.280E-12	1.192E-12	1.109E-12	1.033E-12	9.612E-13	8.946E-13	8.325E-13	7.747E-13	7.208E-13	6.706E-13
7.1	6.238E-13	5.802E-13	5.396E-13	5.018E-13	4.667E-13	4.339E-13	4.034E-13	3.750E-13	3.486E-13	3.240E-13
7.2	3.011E-13	2.798E-13	2.599E-13	2.415E-13	2.243E-13	2.084E-13	1.935E-13	1.797E-13	1.669E-13	1.550E-13
7.3	1.439E-13	1.336E-13	1.240E-13	1.151E-13	1.068E-13	9.910E-14	9.196E-14	8.531E-14	7.914E-14	7.341E-14
7.4	6.809E-14	6.315E-14	5.856E-14	5.430E-14	5.034E-14	4.667E-14	4.326E-14	4.010E-14	3.716E-14	3.444E-14
7.5	3.191E-14	2.956E-14	2.739E-14	2.537E-14	2.350E-14	2.176E-14	2.015E-14	1.866E-14	1.728E-14	1.600E-14
7.6	1.481E-14	1.370E-14	1.268E-14	1.174E-14	1.086E-14	1.005E-14	9.297E-15	8.600E-15	7.954E-15	7.357E-15
7.7	6.803E-15	6.291E-15	5.816E-15	5.377E-15	4.971E-15	4.595E-15	4.246E-15	3.924E-15	3.626E-15	3.350E-15
7.8	3.095E-15	2.859E-15	2.641E-15	2.439E-15	2.253E-15	2.080E-15	1.921E-15	1.773E-15	1.637E-15	1.511E-15
7.9	1.395E-15	1.287E-15	1.188E-15	1.096E-15	1.011E-15	9.326E-16	8.602E-16	7.934E-16	7.317E-16	6.747E-16
8.0	6.221E-16	5.735E-16	5.287E-16	4.874E-16	4.492E-16	4.140E-16	3.815E-16	3.515E-16	3.238E-16	2.983E-16
8.1	2.748E-16	2.531E-16	2.331E-16	2.146E-16	1.976E-16	1.820E-16	1.675E-16	1.542E-16	1.419E-16	1.306E-16
8.2	1.202E-16	1.106E-16	1.018E-16	9.361E-17	8.611E-17	7.920E-17	7.284E-17	6.698E-17	6.159E-17	5.662E-17
8.3	5.206E-17	4.785E-17	4.398E-17	4.042E-17	3.715E-17	3.413E-17	3.136E-17	2.881E-17	2.646E-17	2.431E-17
8.4	2.232E-17	2.050E-17	1.882E-17	1.728E-17	1.587E-17	1.457E-17	1.337E-17	1.227E-17	1.126E-17	1.033E-17
8.5	9.480E-18	8.697E-18	7.978E-18	7.317E-18	6.711E-18	6.154E-18	5.643E-18	5.174E-18	4.744E-18	4.348E-18
8.6	3.986E-18	3.653E-18	3.348E-18	3.068E-18	2.811E-18	2.575E-18	2.359E-18	2.161E-18	1.979E-18	1.812E-18
8.7	1.659E-18	1.519E-18	1.391E-18	1.273E-18	1.166E-18	1.067E-18	9.763E-19	8.933E-19	8.174E-19	7.478E-19
8.8	6.841E-19	6.257E-19	5.723E-19	5.234E-19	4.786E-19	4.376E-19	4.001E-19	3.657E-19	3.343E-19	3.055E-19
8.9	2.792E-19	2.552E-19	2.331E-19	2.130E-19	1.946E-19	1.777E-19	1.623E-19	1.483E-19	1.354E-19	1.236E-19
9.0	1.129E-19	1.030E-19	9.404E-20	8.584E-20	7.834E-20	7.148E-20	6.523E-20	5.951E-20	5.429E-20	4.952E-20
9.1	4.517E-20	4.119E-20	3.756E-20	3.425E-20	3.123E-20	2.847E-20	2.595E-20	2.365E-20	2.155E-20	1.964E-20
9.2	1.790E-20	1.631E-20	1.486E-20	1.353E-20	1.232E-20	1.122E-20	1.022E-20	9.307E-21	8.474E-21	7.714E-21
9.3	7.022E-21	6.392E-21	5.817E-21	5.294E-21	4.817E-21	4.382E-21	3.987E-21	3.627E-21	3.299E-21	3.000E-21
9.4	2.728E-21	2.481E-21	2.255E-21	2.050E-21	1.864E-21	1.694E-21	1.540E-21	1.399E-21	1.271E-21	1.155E-21
9.5	1.049E-21	9.533E-22	8.659E-22	7.864E-22	7.142E-22	6.485E-22	5.888E-22	5.345E-22	4.852E-22	4.404E-22
9.6	3.997E-22	3.627E-22	3.292E-22	2.986E-22	2.709E-22	2.458E-22	2.229E-22	2.022E-22	1.834E-22	1.663E-22
9.7	1.507E-22	1.367E-22	1.239E-22	1.123E-22	1.018E-22	9.223E-23	8.358E-23	7.573E-23	6.861E-23	6.215E-23
9.8	5.629E-23	5.098E-23	4.617E-23	4.181E-23	3.786E-23	3.427E-23	3.102E-23	2.808E-23	2.542E-23	2.300E-23
9.9	2.081E-23	1.883E-23	1.704E-23	1.541E-23	1.394E-23	1.261E-23	1.140E-23	1.031E-23	9.323E-24	8.429E-24

Coventry University
7057CEM Digital Communication Systems

Tutorial: M-ary modulation

1. Show how to transmit the message m=100110001101010111 using 8-ary Amplitude Modulation. Find the corresponding amplitudes of the transmitted signal and calculate the difference between the symbols. Given that the maximum amplitude is 4 Volts.

2. A certain data source has a bit rate of 9600 bit/s. Determine the Nyquist bandwidth and null-to-null bandwidth for each of the following modulation methods:

   a) BPSK
   b) QPSK
   c) 8-PSK
   d) 256-QAM

3. Data with bit rate 9600 bit/s are transmitted using 8-FSK. The lowest frequency used is 8 MHz. Assuming orthogonal spacing, determine the frequencies used for each 8-FSK symbol.

4. A constellation diagram consists of eight equally spaced points on a circle. If the bit rate is 4800 bps, what is the symbol rate?

5. What is the bit rate for a 1000 symbol/s 16-QAM signal?

Coventry University
7057CEM Digital Communication Systems

Tutorial: M-ary modulation

1. Show how to transmit the message m=100110001101010111 using 8-ary Amplitude Modulation. Find the corresponding amplitudes of the transmitted signal and calculate the difference between the symbols. Given that the maximum amplitude is 3 Volts.

$$\delta = \frac{A}{M-1} = \frac{3}{7} = 0.429\ V$$

$V_0 = 0\ V$: 000 (S_0)

$V_1 = 0.429$: 001 (S_1)

$V_2 = 0.857$: 010 (S_2)

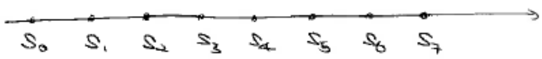
$V_3 = 1.125$: 011 (S_3)

$V_4 = 1.714$: 100 (S_4)

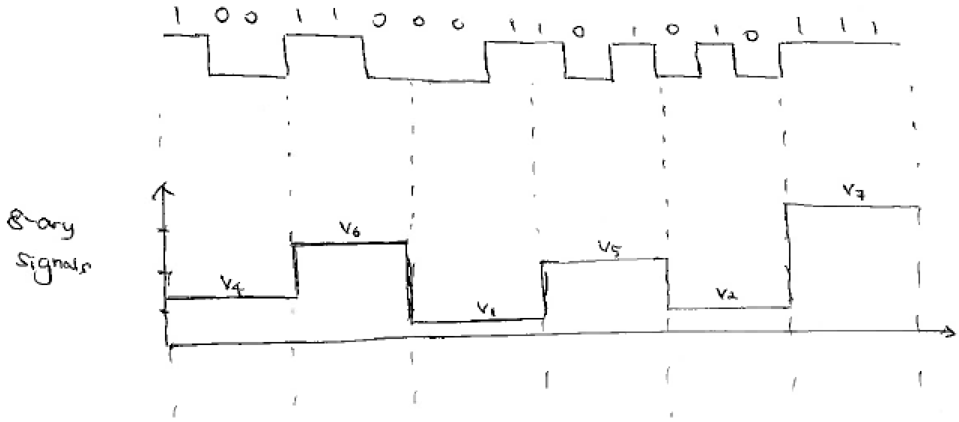
$V_5 = 2.143$: 101 (S_5)

$V_6 = 2.571$: 110 (S_6)

$V_7 = 3.0\ V$: 111 (S_7)



$m = \underset{S_4}{\underline{100}}\ \underset{S_6}{\underline{110}}\ \underset{S_1}{\underline{001}}\ \underset{S_5}{\underline{101}}\ \underset{S_2}{\underline{010}}\ \underset{S_7}{\underline{111}}$



2. A certain data source has a bit rate of 9600 bit/s. Determine the Nyquist bandwidth and null-to-null bandwidth for each of the following modulation methods:

a) BPSK
b) QPSK
c) 8-PSK
d) 256-QAM

(a) BPSK : Null-to-Null Bandwidth $B = 2R_b = 19.2$ kbps

(b) QPSK : $B = \frac{2R_b}{L}$ where $L = \log_2(M)$

$= \frac{2 \cdot 9.6K}{2} = 9.6$ kbps

(c) 8-PSK : $B = \frac{2R_b}{L} = 6.4$ kbps

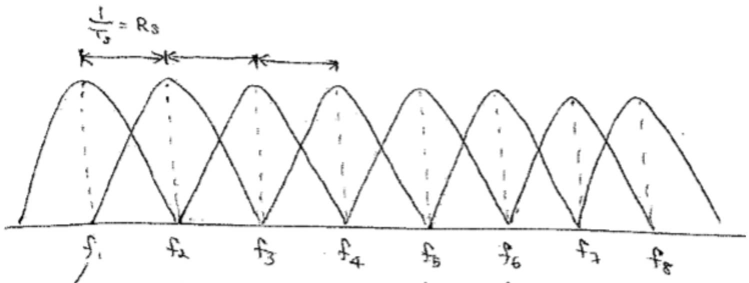
(d) 256-QAM : $B = \frac{2R_b}{L} = \frac{2 \cdot 9.6k}{8} = 2.4$ kbps

3. Data with bit rate 9600 bit/s are transmitted using 8-FSK. The lowest frequency used is 8 MHz. Assuming orthogonal spacing, determine the frequencies used for each 8-FSK symbol.

$R_b = 9600$ bps

8-FSK $\Rightarrow M = 8$, $L = \log_2(8) = 3$

$R_s = \frac{R_b}{L} = \frac{9600}{3} = 3200$ symbols/s



$f_1 = 8$ MHz
$f_2 = 8.0032$ MHz
$f_3 = 8.0064$ MHz
$f_4 = 8.0096$ MHz
$f_5 = 8.0128$ MHz
$f_6 = 8.0160$ MHz
$f_7 = 8.0192$ MHz
$f_8 = 8.0224$ MHz

4. A constellation diagram consists of eight equally spaced points on a circle. If the bit rate is 4800 bps, what is the symbol rate?

$M = 8$

$n = \log_2(M) = 3$ bits

$\therefore R_s = \frac{R_b}{n} = \frac{4800}{3} = 1600$ symbols / s

5. What is the bit rate for a 1000 symbol/s 16-QAM signal?

16-QAM : $M = 16$

$n = \log_2(M) = 4$ bits

$\therefore R_b = n \cdot R_s = 4 \cdot 1000 = 4000$ bps

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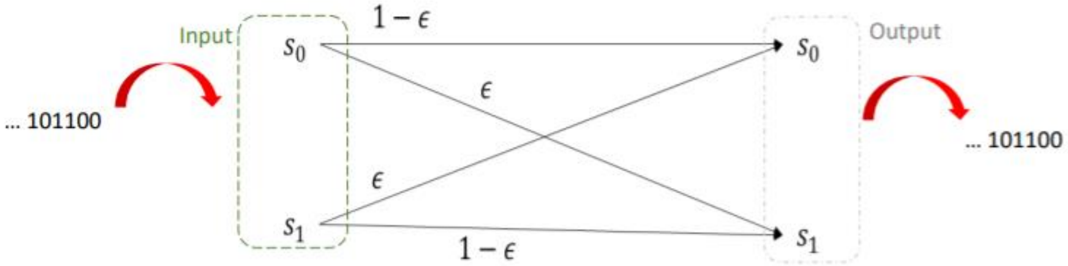
Digital Communication Systems

Tutorial: BER and Channel Capacity

1. Consider a transmission over a binary symmetric communications channel whose input is X= (s_0, s_1) and output Y= (s_0, s_1). Let p be the probability of transmitting s_0. (This implies the probability of transmitting s_1 is 1- p.)



   a) Assume that the probabilities of transmitting s_0 and s_1 are 3/4 and 1/4, respectively. Find the probability that the output symbol equals to s_0 for a given ϵ = 1/4.

   b) Let s_0 and s_1 be transmitted equally likely. Again compute the probability that the output symbol equals s_1 for a given ϵ = 1/3.

2. Consider a transmission over a binary symmetric communications channel whose input is X= (s_0, s_1) and output Y= (s_0, s_1). Let p be the probability of transmitting s_0. (This implies the probability of transmitting s_1 is 1- p.) Suppose that the output symbol is s_0. For each of the following scenarios, which event is more likely, "s_0 was transmitted" or "s_1 was transmitted"?

   [Hint] Use $p(x|y) = \frac{p(y|x)p(x)}{p(y)}$

   a) Assume ϵ = 0.3 and p = 0.1

   b) Assume ϵ = 0.3 and p = 0.5

3. Determine the bit error rate of each modulation method below at E_b/N_0 = 10 dB.

   a) Coherently detected ASK, FSK, PSK

   b) Non-coherently detected ASK, FSK, DPSK

   Use your answers to list each modulation/demodulation method in order of poorest performance in noise to best performance in noise.

> Note: when using any BER equation the value of E_b/N_0 should **not** be in dB and hence should be converted if given in dB as:
>
> $$E_b / N_0 = 10^{\frac{E_b / N_{0dB}}{10}}$$

4. Determine the bit error rate (BER) of each modulation schemes below when E_b/N_0 = 13 dB.

    a) 16-PSK

    b) 16-QAM

5. An FSK system transmits binary data at a rate of 2.5×10^6 bit/s. White Gaussian noise with power spectral density $N_0 = 2 \times 10^{-20}$ W/Hz is added to the signal. In the absence of noise, the amplitude of the received signal is 1 μV across a 1 Ω load. Determine the probability of bit error for coherent detection.

    Hint: you need E_b first.

6. Perform calculations to determine which of the two systems below gives the best BER.

- Non-coherently detected FSK with E_b/N_0 = 12 dB
- Coherently detected PSK with E_b/N_0 = 8 dB

Which of these two systems has the lower bandwidth requirement?

7. Binary FSK is used with coherent detection. The bit rate is 10^6 bit/s and the double-sided power spectral density of the noise at the receiver input is $N_0/2 = 10^{-10}$ W/Hz. Find the carrier power required to maintain a BER of less than 10^{-4}.

8. A certain broadband system uses a bandwidth of 2.2 MHz over a telephone line. Assume that additive white Gaussian noise is present. What signal-to-noise ratio is needed to achieve reliable transmission at a rate of 8 Mbit/s?

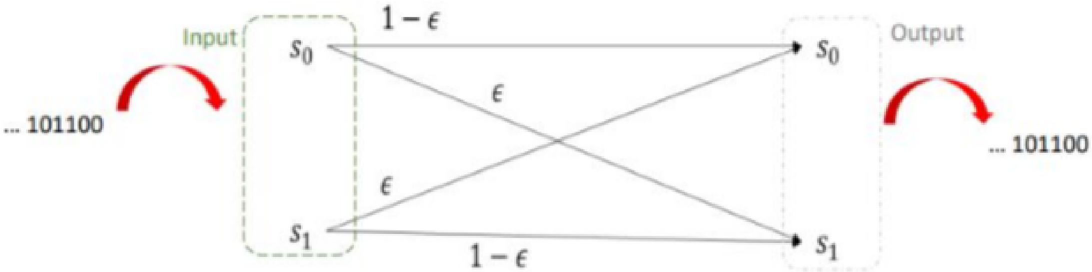
9. Answer the following equations.

    a) What is the capacity of an AWGN channel that experience a signal to noise power ratio of 30 dB and has bandwidth of 1 kHz?
    b) What is its capacity when the bandwidth is 1 MHz?
    c) Find a SNR that can reduce the capacity of the second channel to below that of the first channel.
    d) What is the minimum SNR needed to achieve 80 kbps using a bandwidth of 2 MHz?

Coventry University
Digital Communication Systems

Tutorial: BER and Channel Capacity

1. Consider a transmission over a binary symmetric communications channel whose input is X= (s_0, s_1) and output Y= (s_0, s_1). Let p be the probability of transmitting s_0. (This implies the probability of transmitting s_1 is 1- p.)



a) Assume that the probabilities of transmitting s_0 and s_1 are 3/4 and 1/4, respectively. Find the probability that the output symbol equals to s_0 for a given ϵ = 1/4.

b) Let s_0 and s_1 be transmitted equally likely. Again compute the probability that the output symbol equals s_1 for a given ϵ = 1/3.

(a) $P[y=s_0] = P[x=s_0]\underbrace{(1-\epsilon)}_{P[y=s_0|x=s_0]} + P[x=s_1]\cdot\underbrace{\epsilon}_{P[y=s_0|x=s_1]}$

$= \frac{3}{4}(1-\epsilon) + \frac{1}{4}\epsilon$

$= \frac{3}{4} - \frac{1}{2}\epsilon$ $\quad \epsilon = \frac{1}{4}$

$= \frac{3}{4} - \frac{1}{8} = \frac{5}{8} = 0.6250$

(b) $P[y=s_1] = P[x=s_0]\,\epsilon + P[x=s_1](1-\epsilon)$

$= \frac{1}{2}\epsilon + \frac{1}{2}(1-\epsilon)$

$= \frac{1}{2} = 0.5$, regardless of ϵ

2. Consider a transmission over a binary symmetric communications channel whose input is X= (s_0, s_1) and output Y= (s_0, s_1). Let p be the probability of transmitting s_0. (This implies the probability of transmitting s_1 is 1- p.) Suppose that the output symbol is s_0. For each of the following scenarios, which event is more likely, "s_0 was transmitted" or "s_1 was transmitted"?

[Hint] Use $p(x|y) = \frac{p(y|x)p(x)}{p(y)}$

a) Assume $\epsilon = 0.3$ and $p = 0.1$

b) Assume $\epsilon = 0.3$ and $p = 0.5$

We know that the output symbol is S_0.

$P[x=S_0] = p \rightarrow P[x=S_1] = 1-p$

Which event is more likely, "S_0 was transmitted" or "S_1 was transmitted"

$\underbrace{\text{"}S_0\text{ was transmitted"}}_{P[x=S_0 \mid y=S_0]}$ $\quad$ $\underbrace{\text{"}S_1\text{ was transmitted"}}_{P[x=S_1 \mid y=S_0]}$

To find out what was transmitted, we compare $P[x=S_0 \mid y=S_0]$ and $P[x=S_1 \mid y=S_0]$.

By using $P(x|y) = \frac{P(y|x)P(x)}{P(y)}$.

$$P[x=S_0 \mid y=S_0] = \frac{P[y=S_0 \mid x=S_0]\,P[x=S_0]}{P[y=S_0]} = \frac{(1-\epsilon)\cdot p}{P[y=S_0]} = \frac{p-\epsilon p}{P[y=S_0]}$$

$$P[x=S_1 \mid y=S_0] = \frac{P[y=S_0 \mid x=S_1]\,P[x=S_1]}{P[y=S_0]} = \frac{\epsilon(1-p)}{P[y=S_0]} = \frac{\epsilon-\epsilon p}{P[y=S_0]}$$

Note that both terms have "$-\epsilon p$" in the numerator and "$P[y=s_0]$" in the denominator. Therefore, we can simply compare "p" and "ϵ" parts.

(i) $\epsilon = 0.3$ and $p = 0.1$

→ We have $\epsilon > p$, i.e., $P[x=s_1 | y=s_0] > P[x=s_0 | y=s_0]$

∴ The event of $x = s_1$ is more likely conditioned on $y = s_0$.

↳ s_1 was transmitted.

(ii) $\epsilon = 0.3$ and $p = 0.5$

→ We have $\epsilon < p$, i.e. $P[x=s_1 | y=s_0] < P[x=s_0 | y=s_0]$

∴ The event of $x = s_0$ is more likely conditioned on $y = s_1$.

↳ s_1 was transmitted.

3. Determine the bit error rate of each modulation method below at $E_b/N_0 = 10$ dB.

   a) Coherently detected ASK, FSK, PSK

   b) Non-coherently detected ASK, FSK, DPSK

   Use your answers to list each modulation/demodulation method in order of poorest performance in noise to best performance in noise.

> Note: when using any BER equation the value of E_b/N_0 should **not** be in dB and hence should be converted if given in dB as:
>
> $$E_b / N_0 = 10^{\frac{E_b/N_{0\,dB}}{10}}$$

$$Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q(\sqrt{10}) = Q(3.16) = 7.888 \times 10^{-4}$$ ASK

FSK: $Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q(\sqrt{10}) = 7.888 \times 10^{-4} \approx 8.0 \times 10^{-4}$

PSK: $Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q(\sqrt{20}) = Q(4.47) = 3.9 \times 10^{-6}$

Non-coherent ASK

$$\frac{1}{2} \exp\left(-\frac{E_b}{4N_0}\right) = \frac{1}{2}\exp\left(-\frac{10}{4}\right) = \frac{1}{2}\exp(-2.5)$$

$$= 0.04104425 \approx 4.0 \times 10^{-2}$$

Non-coherent FSK

$$\frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right) = \frac{1}{2}\exp\left(-\frac{10}{2}\right) = \frac{1}{2}\exp(-5)$$

$$= 0.00336 \approx 3.0 \times 10^{-3}$$

DPSK

$$\frac{1}{2} \exp\left(-\frac{E_b}{N_0}\right) = \frac{1}{2}\exp(-10) = 2.27 \times 10^{-5}$$

4. Determine the bit error rate (BER) of each modulation schemes below when E_b/N_0 = 13 dB.

   a) 16-PSK

   b) 16-QAM

) 16-PSK

$\rightarrow E_s = 4 \cdot E_b$

$M = 16$

$$Pe = 2Q\left[\sqrt{\frac{2E_s}{N_0}}\sin(\pi/M)\right] / \log_2 M$$

$$= 2Q\left[\sqrt{2\times4\times19.95}\ \sin\left(\frac{\pi}{16}\right)\right] / 4$$

$$= \frac{1}{2}Q(2.46) = \frac{1}{2}\times 6.947\times10^{-3} = 0.00347$$

16-QAM

$$Pe = 2\cdot\frac{1-\frac{1}{L}}{\log_2 L}\,Q\left[\sqrt{\frac{3\log_2 L}{L^2-1}\cdot\frac{2E_b}{N_0}}\right]$$

$$= 2\cdot\frac{1-\frac{1}{4}}{\log_2 4}\,Q\left[\sqrt{\frac{3\log_2 4}{4^2-1}\times2\times19.95}\right]$$

$$= 0.75\times Q(3.99) = 0.75\times3.304\times10^{-5} \approx 2.48\times10^{-5}$$

5. An FSK system transmits binary data at a rate of $2.5\text{x}10^6$ bit/s. White Gaussian noise with power spectral density $N_0 = 2\text{x}10^{-20}$ W/Hz is added to the signal. In the absence of noise, the amplitude of the received signal is 1 μV across a 1 Ω load. Determine the probability of bit error for coherent detection.

   Hint: you need E_b first.

FSK, $R_b = 2.5 \times 10^6$, $N_0 = 2 \times 10^{-20}$ J, $A = 1\mu V$

$$E_b = \frac{A^2 T_b}{2 \cdot R} = \frac{A^2}{2 \cdot R \cdot R_b} = \frac{(1 \times 10^{-6})^2}{2 \times 1 \times 2.5 \times 10^6} = 2 \times 10^{-19} \text{ J}$$

$$\therefore \frac{E_b}{N_0} = \frac{2 \times 10^{-19}}{2 \times 10^{-20}} = 10$$

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q(\sqrt{10}) = Q(3.16) = 7.888 \times 10^{-4}$$

6. Perform calculations to determine which of the two systems below gives the best BER.

- Non-coherently detected FSK with $E_b/N_0 = 12$ dB
- Coherently detected PSK with $E_b/N_0 = 8$ dB

Which of these two systems has the lower bandwidth requirement?

(a) Non-coherent FSK with $\frac{E_b}{N_0} = 12$ dB

$$12 \text{ dB} = 10^{\frac{12}{10}} = 15.8$$

$$P_e = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right) = \frac{1}{2} \exp(-7.9) = 1.85 \times 10^{-4}$$

(b) Coherent PSK with $\frac{E_b}{N_0} = 8$ dB

$$8 \text{ dB} = 10^{\frac{8}{10}} = 6.3$$

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q(\sqrt{12.6}) = Q(3.55) = 1.926 \times 10^{-4}$$

7. Binary FSK is used with coherent detection. The bit rate is 10^6 bit/s and the double-sided power spectral density of the noise at the receiver input is $N_0/2 = 10^{-10}$ W/Hz. Find the carrier power required to maintain a BER of less than 10^{-4}.

$BER < 10^{-4} \longrightarrow Q\left(\sqrt{\frac{E_b}{N_o}}\right) < 10^{-4}$

$Q(3.72) < 10^{-4}$ } from Q-function Table

$\therefore \frac{E_b}{N_o} = 3.72^2 = 13.84$

$\frac{N_o}{2} = 10^{-10} \longrightarrow N_o = 2\times10^{-10}$

$\therefore E_b = 13.84 \times 2\times10^{-10} = 2.77\times10^{-9}\ J$

$$\text{Carrier power} = \frac{E_b}{T} = E_b * \text{bitrate}$$
$$= 2.77\times10^{-9} \times 10^{6}$$
$$= 2.77\times10^{-3}\ W$$
$$= 2.77\ mW$$

8. A certain broadband system uses a bandwidth of 2.2 MHz over a telephone line. Assume that additive white Gaussian noise is present. What signal-to-noise ratio is needed to achieve reliable transmission at a rate of 8 Mbit/s?

$B = 2.2\ MHz,\quad C = 8\times10^6$

$C = B\log_2(1+\gamma)$

$$\gamma = 2^{\frac{C}{B}} - 1 = 2^{\frac{8\times10^6}{2.2\times10^6}} - 1$$

$$= 2^{3.64} - 1 = 11.467 \rightarrow \gamma_{dB} = 10.6\ dB$$

9. Answer the following equations.

a) What is the capacity of an AWGN channel that experience a signal to noise power ratio of 30 dB and has bandwidth of 1 kHz?
b) What is its capacity when the bandwidth is 1 MHz?
c) Find a SNR that can reduce the capacity of the second channel to below that of the first channel.
d) What is the minimum SNR needed to achieve 80 kbps using a bandwidth of 2 MHz?

(a) $C = B \log_2 (1+\gamma)$

$\gamma_{dB} = 30\text{ dB}$

$\gamma = 10^{\frac{30}{10}} = 1000$

$B = 1\text{ KHz}$

$\therefore C = 10^3 \log_2 (1+1000) = \underline{9.97\text{ kbps}}$

(b) $B = 1\text{ MHz}$

$C = 10^6 \log_2 (1+1000) = \underline{9.97\text{ Mbps}}$

(c) $C = B \log_2 (1+\gamma)$

$\frac{C}{B} = \log_2 (1+\gamma)$

$2^{\frac{C}{B}} = 1+\gamma \iff \gamma = 2^{\frac{C}{B}} - 1$

Where $C = 9.97$ kbps, $B = 10^6$

$\therefore \gamma = 2^{\frac{9.97\times10^3}{10^6}} - 1 = 0.0069 \rightarrow -21.59$ dB

(d) $B = 2$MHz

$C = 80$ kbps

$\gamma = 2^{\frac{C}{B}} - 1 = 2^{\frac{80\times10^3}{2\times10^6}} - 1 = 0.0281 \Rightarrow -15.51$ dB

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Study of MIMO m-CAP with Equalizer for a Band-Limited VLC System

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***Abstract*—*Optical wireless communication, in particular, Visible Light Communication (VLC) is expected to be a key part of 6G wireless network due to availability of large spectrum. However, the data rate achieved using the visible spectrum is currently limited by the low modulation bandwidth of the commercial devices. In order to increase the data rates techniques such as spectrally efficient modulation scheme, Multiple-Input-Multiple-Output (MIMO) system and equalizations are often used. In this paper, we combined MIMO with multiband carrierless amplitude and phase (m-CAP) modulation and fractionally-spaced Decision Feedback Equalizer (DFE) to improve the achievable data rates in a bandlimited VLC system.*

***Keywords*—**Visible light communication (VLC), multi-band carrier-less amplitude and phase (m-CAP) modulation, multiple-input-multiple-output (MIMO), equalization

I. INTRODUCTION

Currently, the main method used to transmit and receive information wirelessly is through Radio Frequency (RF) spectrum. However, the RF spectrum is becoming scarce as its demand exploded exponentially over the last decade. A proposed solution to "spectrum crunch" and for 6G network is to make use of higher RF frequencies such as mm Wave and optical wavelength where large bandwidth is available to support higher data rates [1]. With the recent development of Solid-State Light Emitting Diode (LED) and laser diodes (LD), the VLC system is expected to supplement the current wireless technology to provide high-date rates. Moreover, the LEDs and LDs are suitable for both illumination and communication as they have high energy efficiency, low power consumption, long lifespan, low heat generation and high tolerance to humidity as well as they can be modulated at a very high-speed beyond the flickering sensitivity of human eyes. Despite a number of potential advantages, Gallium Nitride (GaN) emitters which are the preferred method to produce white light have a slow temporal response limiting the LED modulation bandwidth to few MHz (~3MHz)[2]. Therefore, various techniques have been proposed to overcome the limited bandwidth of the phosphorescent LED such as blue filtering at the receiver to block the slow yellow light, pre or post-equalization, spectrally efficient modulation and multiple-input-multiple-output system [2].

Different modulation techniques have been considered to increase the data rate of the VLC systems over the decades. Multi-carrier modulation such as orthogonal frequency division multiplexing (OFDM) yields a high peak to average power ratio (PARP).This affects the power efficiency of the system as well as causes nonlinear distortion because of the limited dynamic range of optical sources and high-power amplifier[3]. CAP modulation is considered to be a potential solution to improve the bandwidth of the VLC system in recent times[4], [5]. CAP modulation has the advantage of being simple to implement in real-time, computationally simple and offers a high data rate. However, in the conventional CAP modulation, frequencies that are outside of the 3dB modulation bandwidth can suffer from attenuation as the band-limited VLC link is modelled as an ideal first order low pass filter (LPF)[3]. Hence, the m-CAP (often with bit and power loading) was proposed to transmit data in the attenuated spectrum as VLC tends to have a very high SNR.

In m-CAP method, the overall bandwidth of the system is equally or unequally divided into m subcarriers to reduce the effect of high-frequency attenuation. The attention to m-CAP modulation was taken when the authors [6] have experimentally proved that the conventional CAP modulation outperforms OFDM technique. The results acquired in their work have shown that CAP modulation can achieve data rate up to 3.22 Gbps and 1.32 Gbps with and without the wavelength division multiplexing (WDM) scheme, respectively. Whereas the OFDM modulation achieved data rate up to 2.93 Gbps and 1.08 Gbps with and without WDM, respectively. The relationship between the number of the subcarrier and the spectral efficiency was studied in [3], where m= {1, 2, 4, 6, 8, 10} subcarriers were used to achieve data rate of 9.04, 15.78, 23.65, 25.40, 30.88, 31.53 Mbps, respectively from a LED with bandwidth of 4.5 MHz. Further studies such as [7]–[9] considered m-CAP with equalizer, adaptive bit loading and unequally distributed subcarriers bands to further enhance the performance.

Multiple-input-multiple output (MIMO) is an attractive technique to increase the data rates in the VLC system as most buildings have multiple LEDs installed in the ceiling to fulfil the minimum illumination level in the indoor environment. These multiples LEDs can be used as multiple transmitters to send data in parallel so that a higher data rate and higher spectral efficiency can be achieved[2]. Both imaging and non-imaging MIMO has been considered in the VLC system. The non-imaging MIMO has a tendency to have an ill-conditioned channel matrix[10] which can be avoided by using the imaging MIMO system[11].

MIMO scheme can be implemented in an m-CAP system to further achieve a high data rate. By combining MIMO with m-CAP modulation scheme, a data rate of 249Mb/s was achieved using LEDs with ~4 MHz bandwidth and 20 subcarriers[12]. The performance can be further improved by using equalization (e.g. time-domain decision feedback equalizer) which can compensate the temporal inter symbol interference (ISI) due to limited bandwidth of LEDs and cross-talk among the subcarriers of the m-CAP. However, to the best of the authors' knowledge, none of the work has thoroughly studied the imaging and non-imaging MIMO with m-CAP combining with equalizer to increase the data rate of band-limited VLC

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systems. Therefore, the focus of this work is to compare the performance of different m-CAP techniques for VLC system combining fractionally spaced DFE for 4x4 imaging and non-imaging MIMO techniques. DFE has been chosen over frequency domain equalization (FDE) since authors [13] commented that the system complexity is nearly comparable to O-OFDM.

The rest of the paper is organized as follows. Section II presents the system design and section III discusses the results. The conclusion is presented in section IV.

II. System Design

In the following section, the description of m-CAP MIMO system is provided followed by the 4×4 MIMO technique and the DFE equalizer.

A. m-CAP Modulation Scheme

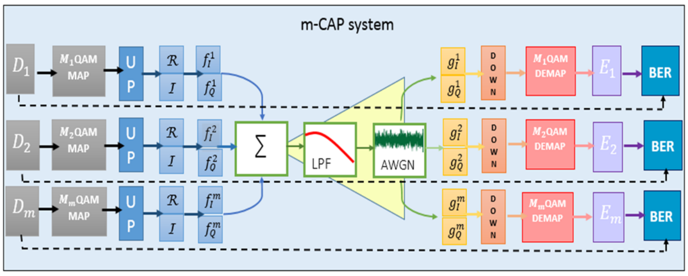


Fig. 1Block diagram of m-CAP

Fig. 1 illustrates the block diagram of m-CAP modulation scheme for VLC system. At the transmitter side, randomly generated independent data stream D_i; $i \in \{1,2,\ldots m\}$ is mapped to M-QAM constellation symbols, where $M = 2^k$and k is the bits per symbol. The QAM symbols are up-sampled by a factor N_{ss} where $N_{ss} > 1$ is the number of samples per symbol.The samples per symbol N_{ss} value can be estimated[14]:

$$N_{SS} = [2m(1+\beta)], \tag{1}$$

where the square brackets are the ceiling function and β is the roll-off factor of square-root-raised-cosine filter (SRRCF) used in CAP.

Subsequently, the real I_m and the Imaginary Q_m sequences are separated and passed through the real and imaginary pulse shaping filters with an impulse response that forms a Hilbert pair. The impulse responses of the filters are created by multiplying the SRRCF with the carrier frequency as given by [2]:

$$f_I^m(t) = \left(\frac{\sin[\gamma(1-\beta)]+4\beta\frac{t}{T_s}\cos[\gamma\delta]}{\gamma[1-(4\beta\frac{t}{T_s})^2]}\right)\cos[\gamma(2m-1)\delta] \tag{2}$$

and

$$f_Q^m(t) = \left(\frac{\sin[\gamma(1-\beta)]+4\beta\frac{t}{T_s}\cos[\gamma\delta]}{\gamma[1-(4\beta\frac{t}{T_s})^2]}\right)\sin[\gamma(2m-1)\delta] \tag{3}$$

where T_s is the symbol duration, $\gamma = \pi t/T_s$, $\delta = 1+\beta$, β is the SRRCF roll-off factor that lies in the range between 0 and 1. Hence, the minimum passband bandwidth for the CAP signal is given by $R_s(1+\beta)$ where R_s is the symbol rate (baud rate).

One criterion that must be obeyed is that the carrier frequency$f_{carrier}$ must be at least twice the SRRCF frequency. For m-CAP, the carrier frequency for i^{th} subcarrier is given by[14]:

$$f_{carrier} = B(2m_i - 1)/2 \tag{4}$$

where m_i is the i^{th} subcarrier and B is the subcarrier bandwidth. For equally spaced subcarrier, the bandwidth is calculated as

$$B = BW/m, \tag{5}$$

where BW is the signal bandwidth.

Once the impulse response filter is applied at both the real and imaginary components, the outputs are summed as given by:

$$s(t) = \sqrt{2}\sum_{n=1}^{m}\left[s_I^n(t)\otimes f_I^n(t) - s_Q^n(t)\otimes f_Q^n(t)\right] \tag{6}$$

where s_I^n and s_Q^n are the I_m and the Q_m components from the QAM modulation for the n^{th} subcarrier, the f_I^n and f_Q^n are the impulse response of the I_m and Q_m RRCF transmitter filters for the n^{th} subcarrier and the $\otimes$ is the time domain convolution.

At the receiver side, the received signal is given by:

$$y(t) = RGs(t)\otimes h(t) + n(t), \tag{7}$$

where R and G are the responsivity and gain of the photodetector, respectively, $h(t)$ is the channel impulse response and $n(t)$ is the source of noises, for instance, the background shot and the thermal noise that according to [15] can be modelled as the (AWGN). In order to recover the transmitted signal, $g_I^m$and g_Q^m filters that are matched to the transmitter filters are applied to the received signal. The receiver filter responses are given by:

$$g_I^m = f_I^m(-t) \tag{8}$$

and

$$g_Q^m = f_Q^m(-t) \tag{9}$$

The filter signals are then down-sampled and standard QAM demodulation is applied to estimate the received data. The received signal is then compared to the originally sent data to estimate the bit error rate (BER). Table I gives the values of the parameters used in the designed m-CAP system.

B. MIMO System with m-CAP modulation

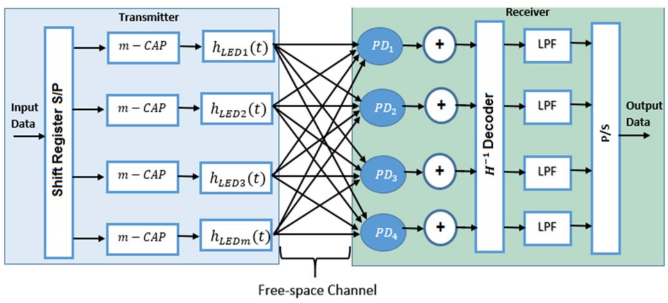


Fig. 2 Schematic diagram for a MIMO system with m-CAP modulation.

MIMO also known as spatial multiplexing is considered for this work as the data rate is increased linearly with the transmitters (LEDs). The non-imaging and imaging MIMO system are investigated in this work. A typical MIMO system is depicted in Fig. 2. The system consists of M transmitters and $N \geq M$ receivers. Each transmitter transmits independent m-CAP symbols and the system relies on optical and electrical signal processing at the receiver to successfully separate these channels for parallel transmission of the data. The received output signal at the receiver is given:

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n} \tag{10}$$

where $\mathbf{y}$ is the output signal vector, $\mathbf{H}$ is the $(M \times N)$ channel gain matrix, $\mathbf{x}$ is the transmitted signal vector and $\mathbf{n}$ is AWGN vector. The channel matrix $\mathbf{H}$ is given by [10]:

$$\mathbf{H} = \begin{bmatrix} h_{11} & \cdots & h_{1M} \\ \vdots & \ddots & \vdots \\ h_{N1} & \cdots & h_{NM} \end{bmatrix} \tag{11}$$

where h_{ij} is the channel DC gain from the i^{th} transmitter to the j^{th} receiver component. The performance of the MIMO depends on the MIMO order, constellation and H-matrix. The 4x4 MIMO order was considered in this work since its H matrix is available in the literature. However, we would expect that the trend would be similar for the others MIMO order. The H matrices used in this work are based on the work done by [16].

In order to successfully decode the transmitted signal, it is necessary to know the channel matrix $\mathbf{H}$ which is often estimated by transmitting the pilot signal [17]. The transmitted signal vector is then estimated at the receiver using various algorithms such as zero-forcing (ZF) and Vertical-Bell laboratories layered space-time (V-BLAST). It is experimentally shown in [18] that the performance of these algorithms in an optical MIMO system is similar. Since ZF is the simplest algorithm, the ZF is adopted in this paper i.e. the estimated symbol is given by:

$$\mathbf{D}_{\text{est}} = \mathbf{H}^{-1}\mathbf{y}, \tag{12}$$

where $\mathbf{H}^{-1}$ is the inverse of $\mathbf{H}$. The detail of these algorithms are given in [2] and reference therein.

C. Decision feedback equalizer

The equalizer utilized for the system was the fractionally spaced DFE with recursive least squares (RLS) adaptive algorithm. The DFE offer improved performance compared to the linear equalizer. Furthermore, it is possible to compensate for temporal and spatial ISI using the DFE (see [19] for example).

III. Results and Discussion

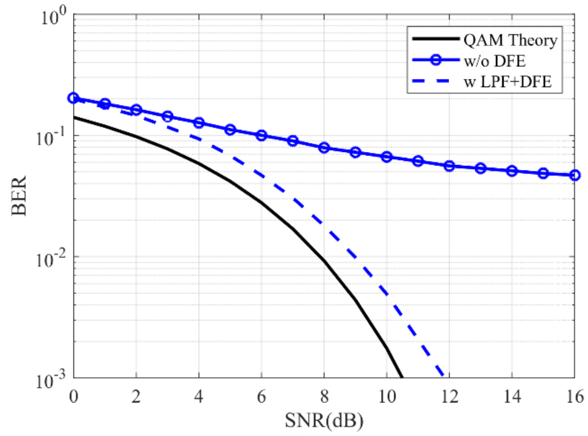
In this section, the performance of imaging and non-imaging MIMO m-CAP modulation with/without a DFE is presented. The simulation parameters are summarized in Table I. The BER for various schemes is evaluated for imaging and non-imaging 4×4 MIMO channel. Based on the ITU standard, a BER of 10^{-3} is considered as the target BER.

TABLE I. System Parameters for the study of MIMO m-CAP system.

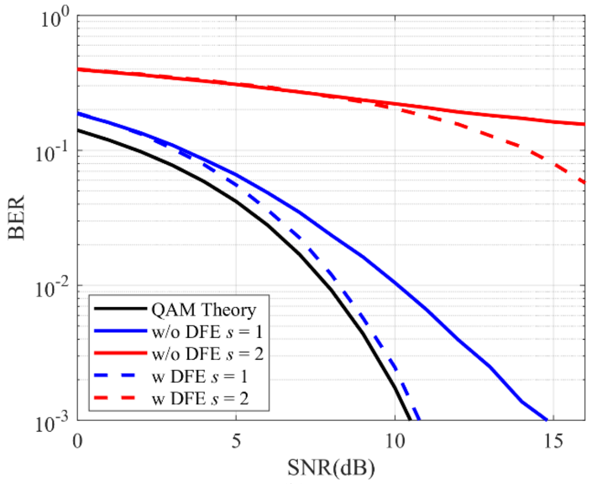
	Parameter	Value
CAP	Roll-Off β	0.15
	Number of Subcarriers m	[1,2, 5]
	Samples Per Symbol N_{ss}	2
	RRCF taps	$10N_{ss}$
	Order for QAM	16
	Normalised Signal Bandwidth W	1
	Subcarriers Bandwidth B	BW/m
	Baud Rate R_s	$BW/(1+\beta)$
	Cut Off Frequency for the LPF F_c	0.5
DFE	Number of samples per symbol (spb)	10
	Number of forwarding taps	10spb
	Number of feedback taps	15
	Forgetting factor	1
	Training length	1000
MIMO (Ideal)	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	
MIMO (Imaging)	$\begin{bmatrix} 1 & 0.3679 & 0.3679 & 0.2431 \\ 0.3679 & 1 & 0.2431 & 0.3679 \\ 0.3679 & 0.2431 & 1 & 0.3679 \\ 0.2431 & 0.3679 & 0.3679 & 1 \end{bmatrix}$	
MIMO (non-Imaging)	$\begin{bmatrix} 0.4961 & 0.4936 & 0.4906 & 0.4931 \\ 0.1995 & 0.6526 & 0.4115 & 0.0529 \\ 0 & 0.0879 & 0.5623 & 0.2001 \\ 0.0075 & 0 & 0.0538 & 0.4057 \end{bmatrix}$	

A. Single-channel m-CAP modulation

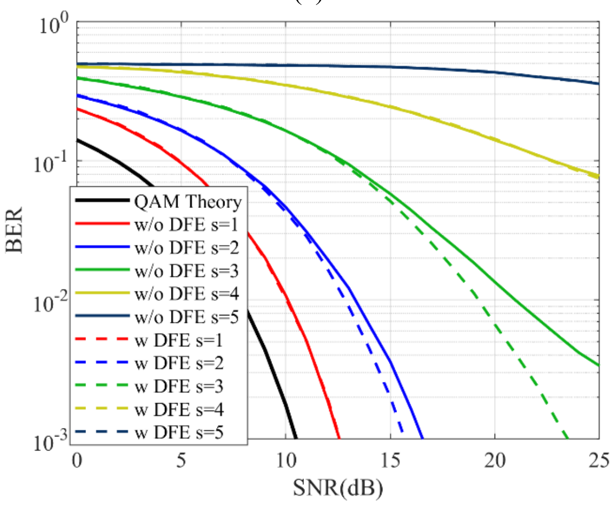
Fig.3 depicts the BER performance of 1, 2 and 5-CAP against SNR when the system has restricted bandwidth with and without DFE. For the case of without DFE, it is observed that the BER performance improves with the increase in the number of subcarriers. The improvement is clearly observed for the low order subcarriers. For example, for the 2-CAP system, the target BER of 10^{-3} is achieved at ~15 dB for the first subcarrier. However, the second subcarrier failed to meet the BER target. For the 5-CAP, the subcarrier from 1 to 3 achieved target BER at SNR of ~13dB, ~15dB and ~23dB, respectively and the other subcarriers failed to reach target BER. It is also observed that the first subcarrier of the 5-CAP has an SNR improvement of ~2dB compared with the first subcarrier of the 2-CAP. It is also observed that DFE improves the BER of all subcarriers although the performance improvement depends on the subcarrier. The improvement is more obvious for the 1-CAP and 2-CAP modulation schemes. For example, the target BER for 1-CAP is achieved at ~12dB. Similarly, the target BER for subcarrier 1 and 2 in 2-CAP is obtained at ~11dB and ~23dB, respectively. However, the 5-CAP modulation scheme with DFE does not show a significant improvement. For example, subcarriers 1, 2 and 3 demonstrate 0.1dB, 1dB and 4dB reduction in SNR to achieve the target BER. The subcarriers 4 and 5 show limited improvement. This is expected as the number of subcarriers increases, the channel is almost flat for the particular subcarrier and hence the DFE improvement is limited.



(a)



(b)



(c)

Fig. 3BER against SNR for the m-CAP system with and without DFE a) 1-CAP,b) 2-CAP andc) 5-CAP (‘*s*’ represnets the subcarrier index)

In order to improve the performance of DFE, a fractionally-space DFE system with 2-samples per symbol is also considered. The BER performance fractionally-spaced DFE system is depicted in Fig. 4. In comparison to the symbol-spaced equalizer, a noticeable improvement is observed. For example, SNRs required to meet the target BER for subcarriers 2, 3, 4 and 5 are ~16, ~17 dB, ~19.5 dB, and 21.4 dB, respectively. From these simulations, it is observed that DFE noticeably improves the performance of 1 and 2-CAP systems. A similar conclusion was reached by other researchers as well [14]. This is because for higher m-CAP orders, the bandwidth of each subcarrier is reduced, and the channel is almost flat. Hence, there is a minimal improvement.

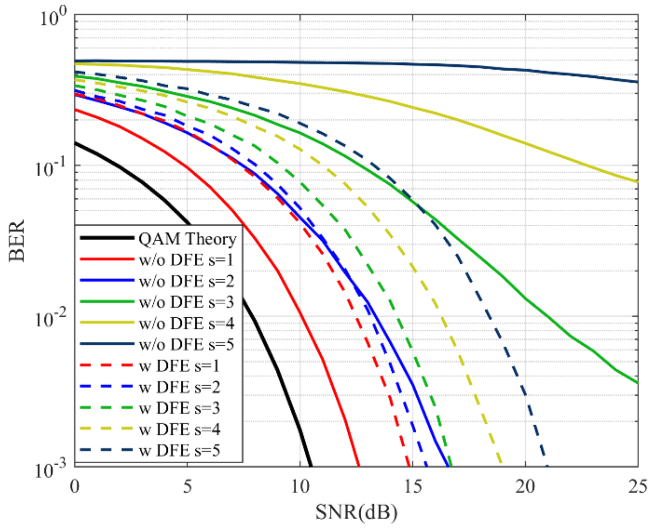


Fig. 4SNR against BER for 5-CAP system with fractionally-spaced DFE.

B. 4×4 MIMO with m-CAP Modulation

This section reports the BER performance of 4x4 imaging and non-imaging optical MIMO with m-CAP in band limited VLC system with and without DFE. In order to validate the results, the MIMO CAP system is simulated considering an ideal MIMO channel (i.e. with identity matrix for the channel).

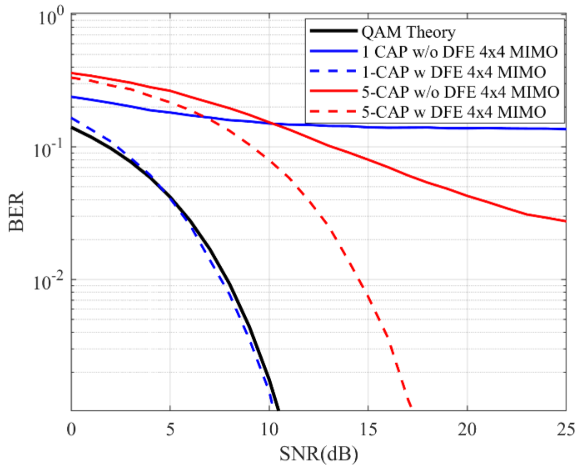
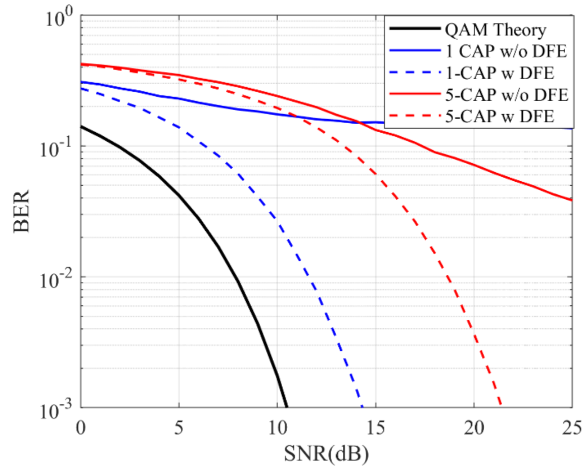


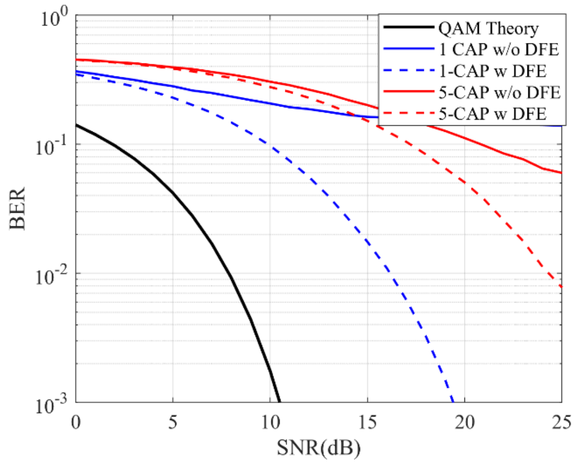
Fig. 5 BER against SNR graph for 4×4 MIMO with ideal **H** matrix.

system that has an ideal **H** matrix, with 1 and 5-CAP modulation schemes. Only fractionally-spaced DFE is considered in this section. As in the case of the single-channel, the DFE provide significant improvement. For example, the 1-CAP system without DFE has a BER floor of 0.5×10^{-1}. However, target BER is obtained at ~10dB with DFE which is very close to the theoretical SNR requirement for an ideal AWGN channel. Similar improvement was obtained for 5-CAP as well. Comparing the ideal channel MIMO with DFE results against the single channel, it is observed that the SNR values for the target BER value are very close. Both ideal channel MIMO with DFE and ideal single channel with 1-CAP and 5-CAP modulation schemes have SNR values of ~10dB and ~17dB for the target BER. However, the data throughput for ideal MIMO is increased 4-times.

The BER for imaging and non-imaging MIMO channel is shown in Fig.6. In comparison with an ideal single channel, MIMO imaging channel with DFE requires ~4 dB and ~6dB higher SNRs to obtain the same BER for 1 and 5-CAP. However, the data throughput is increased 4-times. Comparing the BER performance between the ideal imaging and non-imaging MIMO, it is clearly visible that the non-imaging configuration has the worst results. For example, the non-imaging MIMO system has ~5 dB and ~6 dB higher SNRs than imaging MIMO to achieve the target BER. This is expected as the determinant and the condition number of the imaging **H** matrix are close to the ideal one MIMO channel compared to the non-imaging **H** matrix.



(a)



(b)

Fig. 6BER performance of against SNR for 4×4 MIMO m-CAP modulation schemes a) imaging system and b) non-imaging system

IV. CONCLUSIONS

In this paper, we studied the performance of MIMO m-CAP VLC system in a bandlimited channel. The results demonstrate that m-CAP modulation is effective to improve the performance of the band-limited system. Furthermore, the fractionally-space DFE has shown noticeable improvement in the BER performance of 1-CAP and 2-CAP modulation scheme. However, DFE shows a limited performance improvement for 5-CAP modulation scheme. We also reported the performance of 4×4 MIMO CAP technique with imaging and non-imaging optics system. It is shown that imaging system offers a better BER performance compared to non-image MIMO system. This agrees with the theoretical expectation as the determinant and the condition number of the imaging channel H matrix is close to the ideal H-matrix than non-imaging MIMO system. Furthermore, MIMO system requires a higher SNR compared to a single channel. Moreover, the throughput of MIMO is higher than a single-channel by MIMO order. In this work we have not considered the non-linearity effect of the devices, if this was taken into consideration the DFE would only equalizes the inter-symbol interference and would not compensate for inter-channel interference, therefore it may not be optimum. We will consider other equalisers in our future work.

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Introduction to Channel Coding and Block code

Learning outcomes

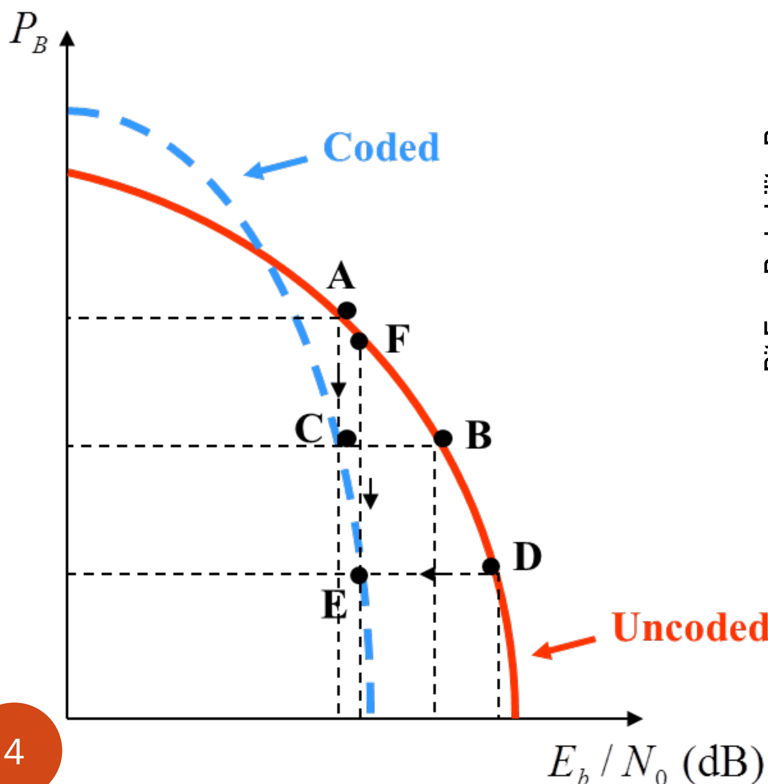
- At the end of this lecture you will:
  - Understand the channel coding techniques
  - Learn about parity check code
  - Learn about linear block code
    - Generator matrix
    - Parity check matrix
  - Error detection and correction technique

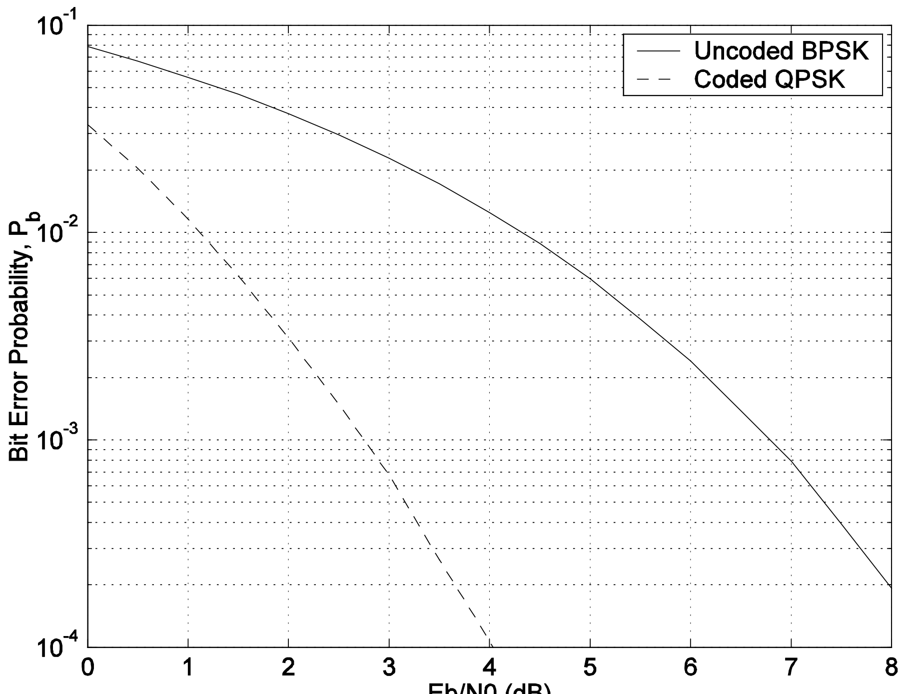
Channel Coding

- **Channel coding** techniques are used to tackle channel impairments such as noise, interference, and fading.
- **Classifications**:
  - Automatic repeat request (ARQ): consists of **error detection** and **retransmission**. A **feedback** channel is required.
  - Forward error control (FEC): introduces controlled redundancy into the transmitted signals that is exploited at the receiver to correct channel induced errors. **No need for a feedback channel**.
    - **Block code**: operates on *k*-bit **blocks** of data bits; memory is not required (**Memoryless**); used in, e.g., PDC (Personal Digital Cellular) systems.
    - **Convolutional code**: is generated by the discrete-time convolution of the input data sequence with the impulse response of the encoder; accepts a **continuous sequence** of input data bits; often preferred over block codes; **memory** is needed; used in, e.g., GSM and IS-54 systems.

Advantage of Channel Coding

- The same BER can be achieved with a lower SNR in a coded system than in an uncoded system. This allows us to **save power**.
- **Coding gain G(dB)**: the reduction in E_b/N_0 in a coded sytem compared with a comparable uncoded sytem for a given BER for the same data rate.





$$G\,[\mathrm{dB}] = \left(\frac{E_b}{N_0}\right)_{\mathrm{u}} [\mathrm{dB}] - \left(\frac{E_b}{N_0}\right)_{\mathrm{c}} [\mathrm{dB}]$$

Binary Block Codes

- **(n, k) block code**: A block of k data bits is appended by c redundant parity bits, thereby producing a **codeword** consisting of n code bits. There are 2^k **codewords**.

k information bits	$c=n-k$ check bits

n

Code Rate

$$R_c = k/n$$

- Example:

(7,4) block code 1010110

1 0 1 0	1 1 0

$$R_c = 4/7$$

Single Parity Check Codes

- Number of information bits transmitted = k
- Number of bits actually transmitted = n = k+1
- Code Rate R = k/n = k/(k+1)

1 0 0 1 0 0	Even number of 1s
0 1 0 0 0 1	Odd number of 1s
1 0 0 1 0 0	Even number of 1s
1 1 0 1 1 0	Even number of 1s

what's if the received signal for the first sent information is 1 1 1 1 0 0

errors cannot be detected

- Error detecting capability = 1
  - If two errors occur the receiver cannot detect an error
- Error correcting capability = 0
  - Retransmission is needed, e.g. ARQ system

Linear Block codes

- Consider the single error detecting code with $n = 3$

k_1	k_2	c
0	0	0
0	1	1
1	0	1
1	1	0

- There are two information digits and one check digit
  - Information digits, $k = 2$
  - Block length, $n = 3$
  - Check digits, $n\text{-}k = 1$
  - $R_c = 2/3$
- The code is **linear** since any two code words can be added (modulo-2), i.e., exclusive-ORed, to give another codeword.
- This **single parity check** code is described by the parity check equations.

$$c = k_1 + k_2 \qquad \text{or} \qquad k_1 + k_2 + c = 0$$

Generator Matrix

- A linear block code can be described by either of a generator matrix, **G**, or a parity check matrix, **H**.
- Consider the single parity check code

$$c = k_1 + k_2 \qquad \text{or} \qquad k_1 + k_2 + c = 0$$

  - A parity check matrix can be formed as: $\mathbf{H} = (1\ 1\ 1)$

- **Generator matrix**: $\mathbf{G}=[g_{ij}]_{k\times n}$, a $\boldsymbol{k\times n}$ matrix used to generate a **length-*n* codeword x** from a **length-*k* input vector k** through the linear mapping $\mathbf{x} = \mathbf{k}\,\mathbf{G}$.
  - $\mathbf{x}=(x_1, x_2, \ldots, x_n)$ where $x_i \in \{0, 1\}$.
  - $\mathbf{k}=(k_1, k_2, \ldots, k_k)$ where $k_i \in \{0, 1\}$.

⇒ **Task of designing a block code**: to find the generator matrices that yield codes that are both **powerful** and **easy to decode**

Parity Check Matrix

- For any block code x with generator matrix **G**, there exists an **$(n\text{-}k)\times n$ parity check matrix H**$=[h_{ij}]_{(n\text{-}k)\times n}$ such that $\mathbf{GH}^T=\mathbf{0}_{k\times(n\text{-}k)}$.
  - The matrix **H** is orthogonal to all code words, i.e., $\mathbf{xH}^T=\mathbf{0}_{1\times(n\text{-}k)}$.
  - The matrix **H** is generator matrix of a **dual code x^T**, consisting of $2^{n\text{-}k}$ code words.

  $\Rightarrow$ The parity check matrix of x^T is the matrix **G**, i.e., $\mathbf{HG}^T=\mathbf{0}_{(n\text{-}k)\times k}$.

- **Systematic block code**: has generator matrix **G** of the form

  $\mathbf{G} = [\mathbf{I}_{k\times k}\,|\,\mathbf{g}]$, where **g** is a $k \times (n\text{-}k)$ matrix.

  - The first k coordinates of each codeword are equal to the k-bit input vector **k**, while the last n-k coordinates are the parity check bits.
  - The parity check matrix **H** has the form $\mathbf{H} = [\mathbf{g}^T\,|\,\mathbf{I}_{(n\text{-}k)\times(n\text{-}k)}]$.

Examples

- **Example 1**: Consider a (5, 2) block code, where a block of k=2 information bits is appended by n–k=3 parity bits. The **code rate**: R_c=2/5=0.4. The resulting $\mathbf{2^k=4}$ **codewords** are $\mathbf{x}_0$=00000, $\mathbf{x}_1$=01110, $\mathbf{x}_2$=10011, $\mathbf{x}_3$=11101. The resulting **code** x: {00000, 01110, 10011, 11101}.

Information bits	Parity bits
0 0	0 0 0
0 1	1 1 0
1 0	0 1 1
1 1	1 0 1

- **Example 2**: Consider a systematic (7, 4) Hamming code, k=4, n=7, n–k=3.
  - Generator matrix: $\mathbf{G} = \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{\mathbf{I}_{4\times 4}} & \begin{matrix} : \\ : \\ : \\ : \end{matrix} & \underbrace{\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}}_{\mathbf{g}_{4\times 3}} \end{bmatrix}$
  - Parity matrix: $\mathbf{H} = \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{matrix}}_{\mathbf{g}^T} & \begin{matrix} : \\ : \\ : \end{matrix} & \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\mathbf{I}_{3\times 3}} \end{bmatrix}$

Free Hamming Distance

- **Hamming distance** between the codewords $\mathbf{x}_1$ and $\mathbf{x}_2$: $d(\mathbf{x}_1, \mathbf{x}_2)$, the number of coordinates in which $\mathbf{x}_1$ and $\mathbf{x}_2$ differ.

- For linear block codes, $d(\mathbf{x}_1, \mathbf{x}_2) = w(\mathbf{x}_1 + \mathbf{x}_2)$: the weight of $\mathbf{x}_1 + \mathbf{x}_2$, equal to the number of of non-zero coordinates of $\mathbf{x}_1 + \mathbf{x}_2$.

- **Free Hamming distance** d_{free} of a linear block code: the **minimum number** of coordinates in which any two codewords differ; also equals to the **minimum weight** of non-zero codewords.
  - $d_{free} = \min d(\mathbf{x}_1, \mathbf{x}_2) = \min w(\mathbf{x}_1 + \mathbf{x}_2) = \min w(\mathbf{x}) = \min d(\mathbf{x}, \mathbf{0})$.
- **Singleton bound**: an upper bound for d_{free} given by $d_{free} \leq n - k + 1$.

Error Detection

- A linear block code can detect all error patterns of $\boldsymbol{d_{free}} - \mathbf{1}$ or fewer errors.

- Number of undetecable error patterns: $2^k - 1$.
  Number of possible non-zero error patterns: $2^n - 1$.
  Number of detectable error patterns: $2^n - 1 - (2^k - 1) =$
  $2^n - 2^k$.
  - Normally, $2^k - 1$ is a small fraction of $2^n - 2^k$.

- **Example 3**: For the (7,4) Hamming code, k=4 and n=7.
  $\Rightarrow 2^4 - 1 = 15$ undetectable error patterns
  $\Rightarrow 2^7 - 1 = 127$ possible non-zero error patterns
  $\Rightarrow 2^7 - 2^4 = 112$ detectable error patterns
  see that $(2^4 - 1)/(2^7 - 2^4) = 15/112$.

Error Correction

- A linear block code **x** with free Hamming distance d_{free} **can** correct **all** error patterns of $\boldsymbol{t}$ or fewer errors, where

$$t \le \left\lfloor \frac{d_{free} - 1}{2} \right\rfloor,$$

and $\lfloor x \rfloor$ is the largest integer contained in x.

- **Example 4**: For the (7, 4) Hamming code in Example 2, $d_{free} = 3$.

$\Rightarrow$ $t \le \left\lfloor \frac{3-1}{2} \right\rfloor = 1.$

$\Rightarrow$The (7, 4) Hamming code can correct all error pattern of 1 error.

Syndromes

- **Error vector e**: $\mathbf{y} = \mathbf{x} + \mathbf{e}$.
  - **x**: transmited codeword.
  - **y**: received vector.
- **Syndrome of the received vector**: $\mathbf{s} = \mathbf{y}\mathbf{H}^T$
  - $\mathbf{s} \neq \mathbf{0} \Leftrightarrow \mathbf{y}$ is not a code word, an error must have occured.
  - $\mathbf{s} = \mathbf{0} \Leftrightarrow \mathbf{y}$ is a code word, no errors or errors are undetectable.
    - The number of undetectable error patterns is equal to $\mathbf{2^k - 1}$, which equals to the number of non-zero codewords.
- The syndrome **only** depends on the error vector **e**:

$$\mathbf{s} = \mathbf{y}\mathbf{H}^T = \mathbf{x}\mathbf{H}^T + \mathbf{e}\mathbf{H}^T = 0 + \mathbf{e}\mathbf{H}^T = \mathbf{e}\mathbf{H}^T.$$

Syndrome Decoding

- Can be used for **any linear** block code.
- **Principle**: all 2^k n-tuples in the same coset of the standard array have the same syndrome since the syndrome only depends on the coset leader.
- **Decoding procedure**:
  - **Step 1**: Compute the syndrome $\mathbf{s} = \mathbf{y}\mathbf{H}^T$.
  - **Step 2**: Locate the coset leader $\mathbf{e}_l$ where $\mathbf{e}_l\mathbf{H}^T = \mathbf{s}$.
  - **Step 3**: Decode $\mathbf{y}$ into $\dot{\mathbf{x}} = \mathbf{y} + \mathbf{e}_l$.

  - **Disadvantage**: For large $n-k$, it becomes impractical because 2^{n-k} syndromes and 2^{n-k} error patterns must be stored.

Syndrome Decoding: Example

Consider a $(7, 4)$ code with $\mathbf{H}$,

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The received code word is (1 0 1 0 0 1 0)

What is the Hamming decoder output?

Example (Solution)

We find the syndrome using $\boldsymbol{s} = \boldsymbol{y}\mathbf{H}^{\mathrm{T}}$

s = (1 1 1)

This can be found in the fourth column of **H**. The error is assumed to be in the fourth bit i.e., **e** = 0 0 0 1 0 0 0.

The corrected word is $\widehat{\boldsymbol{x}} = \boldsymbol{y} + \boldsymbol{e} =$ **1 0 1 1** 0 1 0

The decoder outputs the information bits 1 0 1 1

Hamming Codes

- A family of codes such that for an integer, m ($m \geq 3$), the block length, n, is equal to 2^m-1 and the number of check bits, c, is equal to m.
- Hamming codes have **a minimum distance of 3**.
- A common Hamming code has a block length of seven. It has three check bits. We refer to this as the (7, 4) Hamming code.

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Hamming and Cyclic Code

Learning outcomes of today's lecture

- At the end of this lecture you will:
  - Know Hamming and cyclic code
  - Learn how to encode and decode using cyclic code
  - Be able to detect and correct error using hamming and cyclic code

Hamming Codes

- A family of codes such that for an integer, m ($m \geq 3$), the block length, n, is equal to 2^m-1 and the number of check bits, c, is equal to m.
- Hamming codes have **a minimum distance of 3**.
- A common Hamming code has a block length of seven. It has three check bits. We refer to this as the (7, 4) Hamming code.

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Cyclic codes

- Cyclic codes belong to a sub-class of linear block codes which are simple to encode and decode.
- A code is cyclic if all cyclic shifts of a codeword are also codewords.
- The single parity check code is cyclic

000 has no distinct cyclic shifts

$011 \rightarrow 101 \rightarrow 110$

$101 \rightarrow 110 \rightarrow 011$

$110 \rightarrow 011 \rightarrow 101$

Cyclic codes (Contd.)

- Any cyclic codeword can be represented by a polynomial generator, e.g., 011 can have the generator polynomial

$$G(x) = x + 1.$$

according to

$$\begin{array}{ccc} 0 & 1 & 1 \\ x^2 & x^1 & x^0 \end{array}$$

- This is the codeword polynomial with the smallest degree and x is a dummy variable.

- The degree of G(x) is equal to the number of check digits n-k.

Generator Matrix of a Cyclic Code (1/2)

- To form a generator matrix, consider g(x) in binary form, as the lowest row of **G**, **then** xg(x) as the next row up and so on until k rows have been filled.

$$\mathbf{G} = \begin{pmatrix} x^{k-1}G(x) \\ \vdots \\ \vdots \\ \vdots \\ xG(x) \\ G(x) \end{pmatrix} = \begin{pmatrix} G_0 & G_1 & G_2 & \ldots & G_{k-1} & 0 & 0 & 0 & \ldots & 0 \\ 0 & G_0 & G_1 & G_2 & \ldots & G_{k-1} & 0 & 0 & \ldots & 0 \\ 0 & 0 & G_0 & G_1 & G_2 & \ldots & G_{k-1} & 0 & \ldots & 0 \\ .. & .. & & & & & & & & .. \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & G_0 & \ldots & G_{k-1} \end{pmatrix}$$

- **Example 2**: A (7,4) cyclic code has the generator polynomial g(x) = $x^3 + x + 1$, find the generator matrix in the systematic format

Generator Matrix of a Cyclic Code (2/2)

- Solution:

$$\mathbf{G} = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} & \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{matrix} \end{array}$$

- To convert it to a systematic form change row 2 and 1 according to $\mathbf{2} \rightarrow \mathbf{2} + \mathbf{4}$

  $\mathbf{1} \rightarrow \mathbf{1} + \mathbf{3} + \mathbf{4}$ then

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Exercise

A (7,4) cyclic code has the generator polynomial $g(x) = x^3 + x^2 + 1$, find the generator matrix in the systematic format (Standard Echelon Format) then find *H*

Answer:

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Cyclic Encoding

- To **generate an (*n*, *k*) Cyclic Code**, the following procedure can be applied:

  1) **Multiply** the information bits polynomial $K(x)$ by x^c

  2) **Divide** $x^c\, K(x)$ by the generator polynomial $g(x)$ obtaining a remainder called $b(x)$

  3) **Add** the remainder $b(x)$ to $x^c\, K(x)$ to obtain the codeword polynomial

Example

A (7,4) cyclic code has the generator polynomial g(x) = $x^3 + x + 1$, encode the information bits 1101

Solution: $c = 3$, $K = 1101$ ➔ $K(x) = x^3 + x^2 + 1$

1) $x^c K(x) = x^3 (x^3 + x^2 + 1) = x^6 + x^5 + x^3$

2) division

$$
\begin{array}{r|l}
 & x^3 + x^2 + x + 1 \\
\hline
x^3 + x + 1 & x^6 + x^5 + x^3 \\
 & \underline{x^6 + x^4 + x^3} \\
 & x^5 + x^4 \\
 & \underline{x^5 + x^3 + x^2} \\
 & x^4 + x^3 + x^2 \\
 & \underline{x^4 + x^2 + x} \\
 & x^3 + x \\
 & \underline{x^3 + x + 1} \\
 & 1
\end{array}
$$

g(x)

$x^c K(x)$

$b(x)$

$b(x) = 1$ ➔ $x^c K(x) + b(x) = x^6 + x^5 + x^3 + 1$ ➔ codeword 1101001

Exercise

A (7,4) cyclic code has the generator polynomial $g(x) = x^3 + x^2 + 1$, encode the information bits 1010

Answer: 1010001

Syndrome of cyclic code

- The syndrome is used to detect errors
- To find the syndrome for cyclic codes

Divide the received word polynomial y(x) by the generator polynomial g(x) and the **remainder** is the **syndrome polynomial**

- Compare the binary syndrome (**remainder word**) with the columns of **H** to find the position of the error then correct the error

Syndrome of cyclic code

A (7,4) cyclic code has the generator polynomial
$g(x) = x^3 + x^2 + 1$, decode the received codeword 1101101

Solution:

$y = 1101101 \rightarrow y(\mathrm{x}) = x^6 + x^5 + x^3 + x^2 + 1$

$$
\begin{array}{r|l}
 & x^3 \\
\hline
\underbrace{x^3 + x^2 + 1}_{g(x)} & \underbrace{x^6 + x^5 + x^3 + x^2 + 1}_{y(x)} \\
 & x^6 + x^5 + x^3 \\
\hline
 & \qquad\qquad\qquad x^2 + 1
\end{array}
$$

$s(x) = x^2 + 1 \rightarrow \mathbf{s} = 101$

The fourth column of **H** is 101 $\rightarrow$ the fourth bit is in error $\rightarrow$ the correct word is 1100101 $\rightarrow$ the information bits are 1100

Commonly Used CRC Codes and Corresponding Generator Polynomials

Code	Number of Bits in FCS	Generator Polynomial $g(x)$
CRC-8	8	$x^8 + x^7 + x^6 + x^4 + x^2 + 1$
CRC-12	12	$x^{12} + x^{11} + x^3 + x^2 + x + 1$
CRC-16	16	$x^{16} + x^{15} + x^2 + 1$
CRC-CCITT	16	$x^{16} + x^{12} + x^5 + 1$
CRC-32	32	$x^{32} + x^{26} + x^{23} + x^{22} + x^{16} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^5 + x^4 + x^2 + x + 1$

Source: B.P. Lathi and Z. Ding, 'Modern digital and analog communication systems' 2009

Further reading

- B. Sklar, *Digital Communications: Fundamentals and Application*. 2002.
- T. K. Moon, *Error correction coding: Mathematical methods and algorithms* . New Jersey: Wiley-Interscience, 2005.
- B. A. Forouzan, *Data Communications and Networking*. 2001.

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Tutorial- Channle coding

1) A (7, 3) cyclic code has the generator word 10111.

    a) Determine the codeword for the information bits 101.

    b) Determine the generator matrix and parity check matrix in standard echelon form.

    c) If the received word is 1110101. Identify whether any errors have occurred and, if so, find the position of the error and the corrected codeword then the decoder output.

2) An alternative generator polynomial for the (7, 4) Hamming code is $x^3 + x + 1$. What is the output of the decoder if the received word is 0110001?

3) Consider a (7, 4) code with the generator matrix G:

$$G = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

    a) Find all the codewords of the code.

    b) Find the parity check matrix H of the code.

    c) What is the output of the decoder if the received word is 0110011?

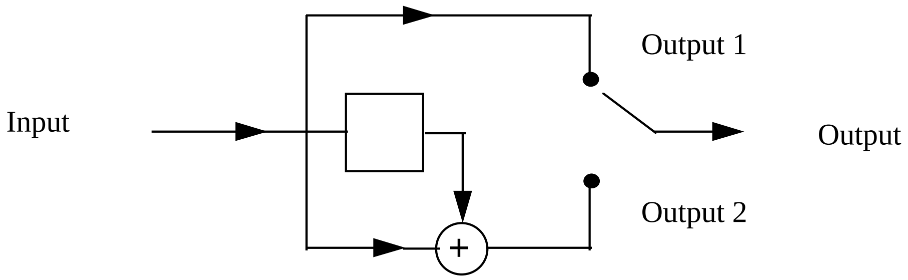
4) A (15, 11) cyclic code has the generator polynomial $x^4 + x + 1$.

    a) What is the transmitted codeword for an encoder input of 10101010101.

    b) What is the syndrome if an error occurs in the sixth bit?

    c) By using any other cyclic codeword of this code (i.e., a shifted version of the codeword above), show that the same syndrome occurs when the sixth bit is received in error.

5) For the following convolutional encoder



The code rate = ½, and the constraint length L = 2

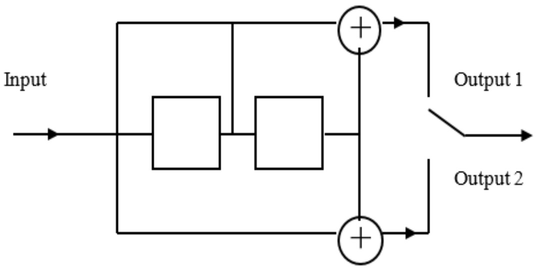
a. Determine the encoder output for the input information bits 10101

b. Apply Viterbi algorithm to decode the received bits

i) 11 01 01 01 11

ii) 11 01 00 01 11

6) A convolutional encoder is shown in figure below



a) What are the code rate and constraint length for this encoder?

b) Write down a state table for this code.

c) Construct the state diagram for this code.

d) Construct a four-stage trellis diagram.

e) Use the Viterbi code to determine the decoder output if the received word is

i) 11 10 01 00.

ii) 11 11 10 01

7) A rate ½ convolutional code has the generator words 111 and 011

- Draw a diagram of the encoder.
- Produce a state table and trellis diagram for this code.
- Perform Viterbi decoding on the received sequence 01 00 00 00.

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Tutorial- Channle coding solution

1) A (7, 3) cyclic code has the generator word 10111.

   a) Determine the codeword for the information bits 101.

Answer:

c = 7-3 = 4

$K = 101 \Rightarrow K(x) = x^2 + 1$

$x^c K(x) = x^4(x^2 + 1) = x^6 + x^4$

$g(x) = x^4 + x^2 + x + 1$

$$\begin{array}{r|l} & \quad\quad\quad\quad x^2 \\ \hline x^4 + x^2 + x + 1 & x^6 + x^4 \\ & \underline{x^6 + x^4 + x^3 + x^2} \\ & \quad\quad x^3 + x^2 \end{array}$$

$b(x) = x^3 + x^2$

$x^c K(x) + b(x) = x^6 + x^4 + x^3 + x^2 \Rightarrow$ Codeword is 1011100

   b) Determine the generator matrix and parity check matrix in standard echelon form.

Answer:

$$\mathbf{G} = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

1 → 1 + 3

$$\mathbf{G}_{SEF} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

c) If the received word is 1110101. Identify whether any errors have occurred and, if so, find the position of the error and the corrected codeword then the decoder output.

Answer:

y = 1110101 ➔ $y(x) = x^6 + x^5 + x^4 + x^2 + 1$
$s(x)$ = the remainder of $(y(x)/g(x))$

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x^4 + x^2 + x + 1 & x^6 + x^5 + x^4 + x^2 + 1 \\
 & x^6 + x^4 + x^3 + x^2 \\
\cline{2-2}
 & \quad x^5 + x^3 + 1 \\
 & \quad x^5 + x^3 + x^2 + x \\
\cline{2-2}
 & \qquad\quad x^2 + x + 1
\end{array}
$$

$s(x) = x^2 + x + 1$ ➔ **s** = 0111 (the length of **s** should be equivalent to the length of each column in **H**)
0111 is located in the third column of H ➔ the third bit is en error ➔
codeword is 1100101➔ the estimated information bits are **110**

2) An alternative generator polynomial for the (7, 4) Hamming code is $x^3 + x + 1$. What is the output of the decoder if the received word is 0110001?

Answer:

$s(x)$ = the remainder of $(y(x)/g(x))$

$$
\begin{array}{r|l}
 & x^2 + x + 1 \\
\hline
x^3 + x + 1 & x^5 + x^4 + 1 \\
 & x^5 + x^3 + x^2 \\
\cline{2-2}
 & \quad x^4 + x^3 + x^2 + 1 \\
 & \quad x^4 + x^2 + x \\
\cline{2-2}
 & \qquad x^3 + x + 1 \\
 & \qquad x^3 + x + 1 \\
\cline{2-2}
 & \qquad\quad 0
\end{array}
$$

s = 000 ➔ no errors ➔ estimated information bits are 0110

3) Consider a (7, 4) code with the generator matrix G:

$$
G = \begin{pmatrix}
\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0}
\end{pmatrix}
$$

a) Find all the codewords of the code.

b) **Answer:**

Message	Code
0000	0000000
0001	0001110
0010	0010011
0011	0011101
0100	0100101
0101	0101011
0110	0110110
0111	0111000
1000	1000111
1001	1001001
1010	1010100
1011	1011010
1100	1100010
1101	1101100
1110	1110001
1111	1111111

c) Find the parity check matrix H of the code.

Answer:

$$\mathbf{H} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

d) What is the output of the decoder if the received word is 0110011?

Answer:

$$S = yH^T = [0110011]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1 \quad 0 \quad 1]$$

S in second column of H

2nd bit in error

Corrected word 0010011

Decoder output 0010

4) A (15, 11) cyclic code has the generator polynomial $x^4 + x + 1$.

a) What is the transmitted codeword for an encoder input of 10101010101.

Answer:

$x^c K(x) = x^4(x^{10} + x^8 + x^6 + x^4 + x^2 + 1) = x^{14} + x^{12} + x^{10} + x^8 + x^6 + x^4$

$$
\begin{array}{r}
x^{10}+x^{8}+x^{7}+x^{5}+x^{4}+x^{3}+1 \\
x^{4}+x+1 \overline{\big) x^{14}+x^{12}+x^{10}+x^{8}+x^{6}+x^{4}} \\
\underline{x^{14}+x^{11}+x^{10}} \\
x^{12}+x^{11}+x^{8}+x^{6}+x^{4} \\
\underline{x^{12}+x^{9}+x^{8}} \\
x^{11}+x^{9}+x^{6}+x^{4} \\
\underline{x^{11}+x^{8}+x^{7}} \\
x^{9}+x^{8}+x^{7}+x^{6}+x^{4} \\
\underline{x^{9}+x^{6}+x^{5}} \\
x^{8}+x^{7}+x^{5}+x^{4} \\
\underline{x^{8}+x^{5}+x^{4}} \\
x^{7} \\
\underline{x^{7}+x^{4}+x^{3}} \\
x^{4}+x^{3} \\
\underline{x^{4}+x+1} \\
x^{3}+x+1
\end{array}
$$

$b(x) = x^3 + x + 1$

$x^c K(x) + b(x) = x^{14} + x^{12} + x^{10} + x^8 + x^6 + x^4 + x^3 + x + 1$

➔ Codeword is 101010101011011

b) What is the syndrome if an error occurs in the sixth bit?

Error in the sixth bit ➔ **y** = 101011101011011

$y(x) = x^{14} + x^{12} + x^{10} + x^9 + x^8 + x^6 + x^4 + x^3 + x + 1$

$s(x)$ is the remainder of $(y(x)/g(x))$

$$
\begin{array}{r}
x^{10}+x^{8}+x^{7}+x^{4}+x^{3}+x^{2}+x+1 \\
x^{4}+x+1 \overline{\big) x^{14}+x^{12}+x^{10}+x^{9}+x^{8}+x^{6}+x^{4}+x^{3}+x+1} \\
\underline{x^{14}+x^{11}+x^{10}} \\
x^{12}+x^{11}+x^{9}+x^{8}+x^{6}+x^{4}+x^{3}+x+1 \\
\underline{x^{12}+x^{9}+x^{8}} \\
x^{11}+x^{6}+x^{4}+x^{3}+x+1 \\
\underline{x^{11}+x^{8}+x^{7}} \\
x^{8}+x^{7}+x^{6}+x^{4}+x^{3}+x+1 \\
\underline{x^{8}+x^{5}+x^{4}} \\
x^{7}+x^{6}+x^{5}+x^{3}+x+1 \\
\underline{x^{7}+x^{4}+x^{3}} \\
x^{6}+x^{5}+x^{4}+x+1 \\
\underline{x^{6}+x^{3}+x^{2}} \\
x^{5}+x^{4}+x^{3}+x^{2}+x+1 \\
\underline{x^{5}+x^{2}+x} \\
x^{4}+x^{3}+1 \\
\underline{x^{4}+x+1} \\
x^{3}+x
\end{array}
$$

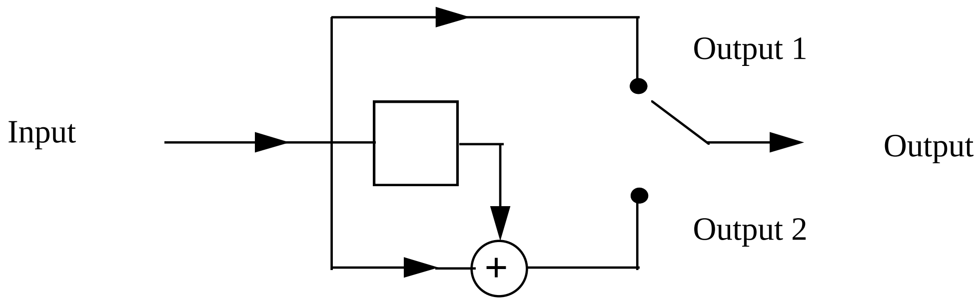
$s(x) = x^3 + x$ ➔ **s** = 1010

c) By using any other cyclic codeword of this code (i.e., a shifted version of the codeword above), show that the same syndrome occurs when the sixth bit is received in error.

Answer:

You can obtain any new codeword by cyclic shifting the codeword 101010101011011, e.g., 110101010101101 or 111010101010110, etc.
Flip the sixth bit and repeat the same process of solving this question to find the syndrome. The syndrome should be the same because $\mathbf{s} = \mathbf{e}\mathbf{H}^T$. The syndrome is then governed by $\mathbf{e}$ as $\mathbf{H}^T$ is fixed. In this case $\mathbf{e}$ = 000001000000000 (the sixth bit is in error).

5) For the following convolutional encoder



The code rate = ½, and the constraint length *L* = 2

a. Determine the encoder output for the input information bits 10101

Solution:

Using the encoder directly, or using the path through the trellis diagram:
Encoder o/p is 11 01 11 01 11

b. Apply Viterbi algorithm to decode the received bits

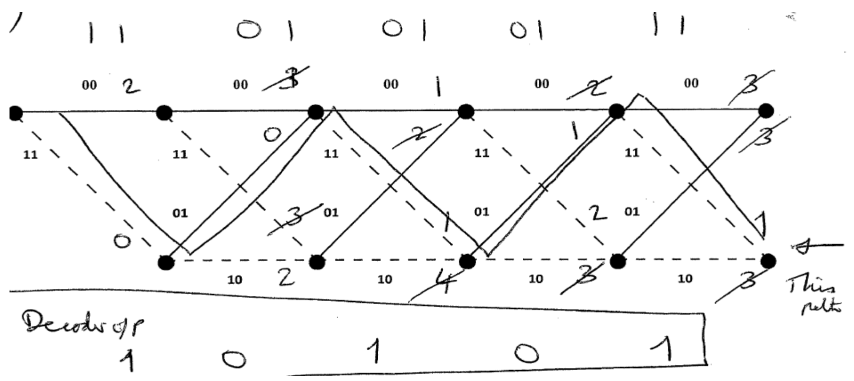
i) 11 01 01 01 11

ii) 11 01 00 01 11

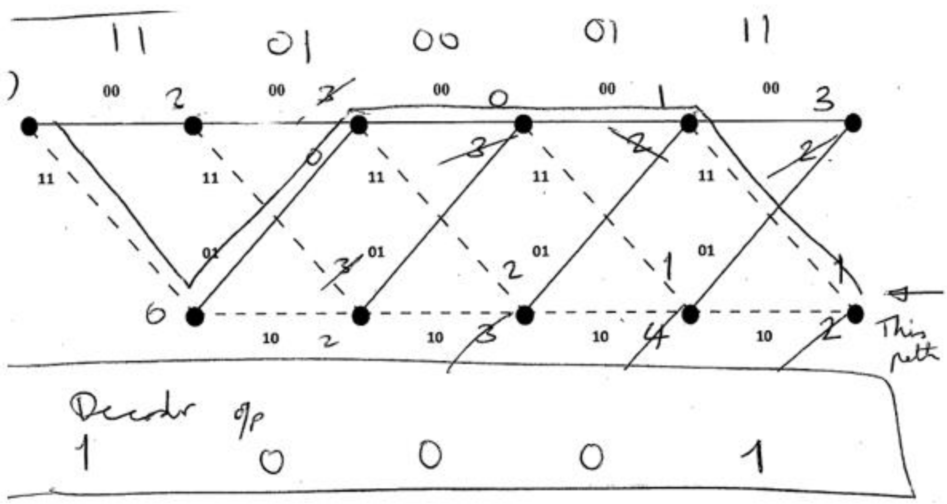
Assuming that the transmitted bits were the encoder output of question a above, comment on the success or failure for each decoding of (i) & (ii) and why this happened?

Solution:

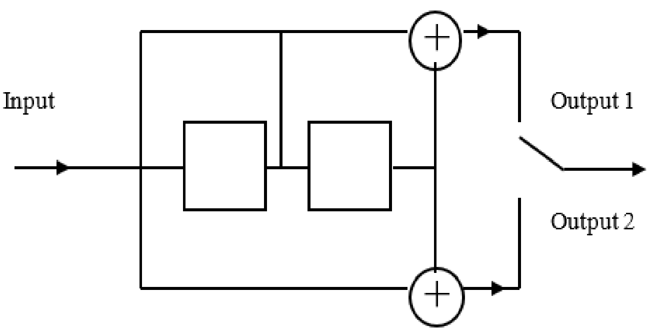
i)



ii)



6) A convolutional encoder is shown in figure below



a) What are the code rate and constraint length for this encoder?

Solution:

R=1/2. L=3

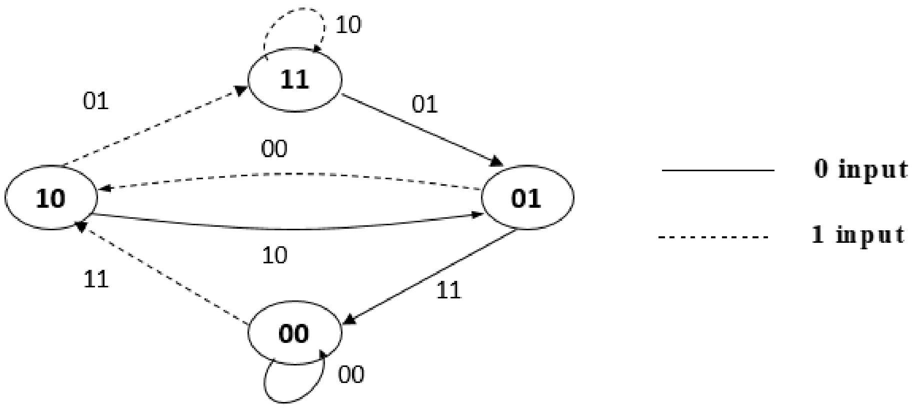
b) Write down a state table for this code.

Solution:

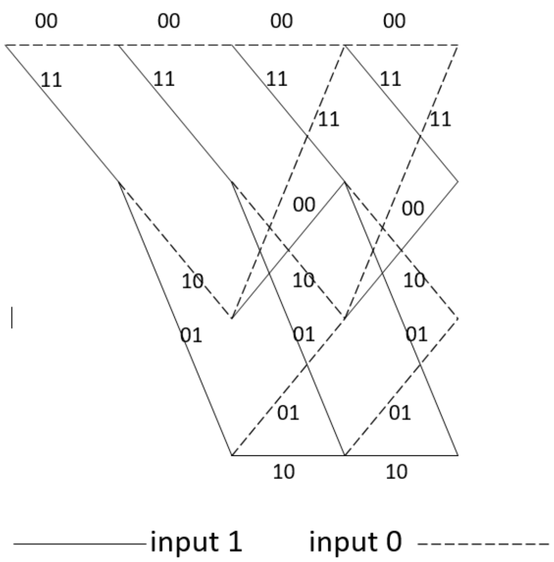
Input	Present State FF1 FF2	Next State FF1 FF2	Output 1 2
0	0 0	0 0	0 0
0	0 1	0 0	1 1
0	1 0	0 1	1 0
0	1 1	0 1	0 1
1	0 0	1 0	1 1
1	0 1	1 0	0 0
1	1 0	1 1	0 1
1	1 1	1 1	1 0

c) Construct the state diagram for this code.

Solution:

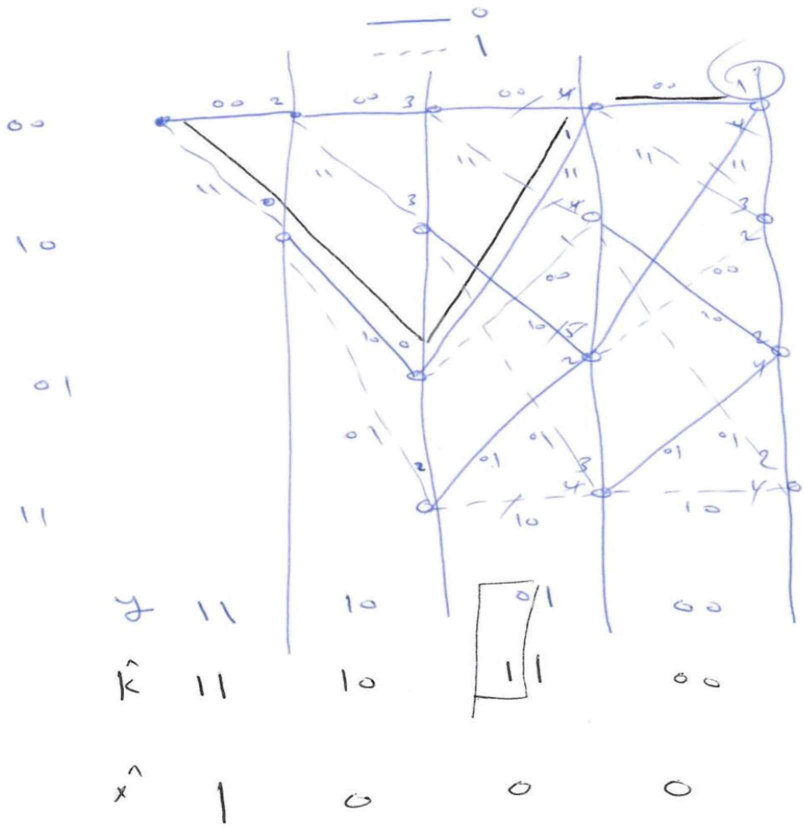


d) Construct a four stage trellis diagram

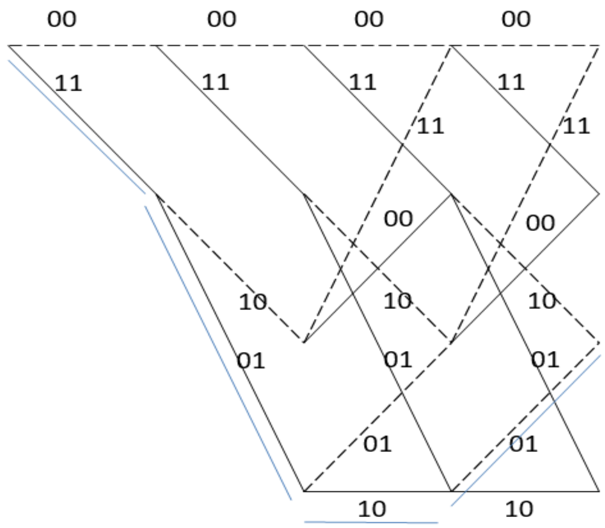


e) Use the Viterbi code to determine the decoder output if the received word is

i) 11 10 01 00



ii) 11 11 10 01



input 1

input 0

Corrected word 11 01 10 01
Output of decoder 1 1 1 0

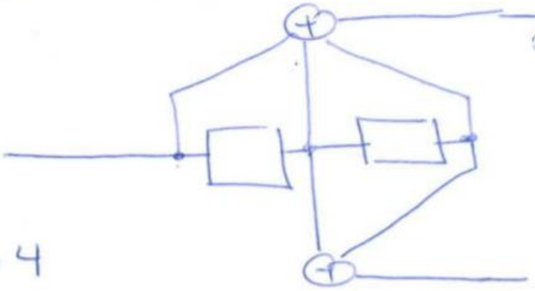
7) A rate ½ convolutional code has the generator words 111 and 011

- **Draw a diagram of the encoder.**
- **Produce a state table and trellis diagram for this code.**
- **Perform Viterbi decoding on the received sequence 01 00 00 00.**

Solution:

$g_1 = 111$

$g_2 = 011$

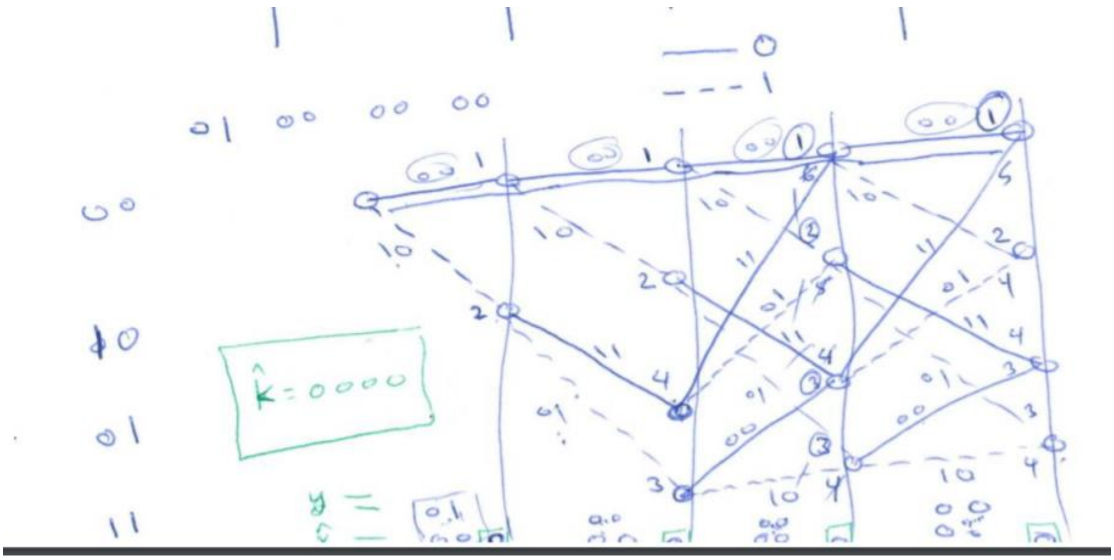


$x_1 = \text{Input} + FF_1 + FF_2$

$N_s = 2^M = 2^2 = 4$

$x_2 = FF_1 + FF_2$

Input	CS FF_1	CS FF_2	NS FF_1	NS FF_2	output x_1	output x_2
0	0	0	0	0	0	0
0	0	1	0	0	1	1
0	1	0	0	1	1	1
0	1	1	0	1	0	0
1	0	0	1	0	1	0
1	0	1	1	0	0	1
1	1	0	1	1	0	1
1	1	1	1	1	1	0


k̂ = 0000

Convolution Code

Lecture overview

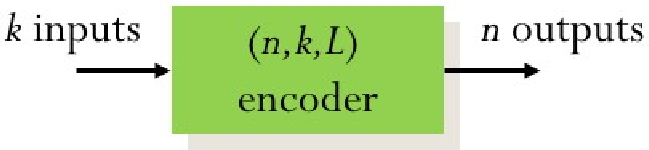
- Introduction to **Convolutional codes**
- Convolutional encoders
- Construct the state diagram and trellis diagram

Convolutional codes

- **Block codes:** *k* symbols is used to produce **blocks** of *n* symbols (where $n > k$).

- **Convolutional code:**
  - Work on **code streams** rather than blocks.
  - Have **memory** that makes use of **previous bits to encode or decode the following bits**.
    - **Block codes are memoryless.**
  - **Superiority performance** in comparison to blocks code → used in applications that **require good performance.**
  - **soft decoding algorithms** is possible

Convolutional codes: Encoder



- **Encoder** denoted by (n,k,L), where L is the **memory depth or constraint length (number of shift register stages)**
  - Output depends on L previous input blocks
- ***k*-bit input sequence applied to an encoder produces an *n*-bit output sequence**
- **code rate *r* = *k*/*n***
- **Better performance** can be achieved by increasing the memory depth but at the **cost of complexity**

Convolutional Encoder

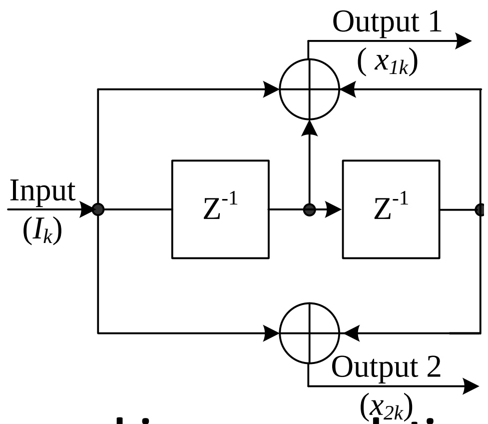


Figure: binary convolutional encoder

- **Rate-1/*n* binary convolutional encoder**: can be viewed as a **finite-state machine** (FSM) that consists of
  - ***M*** shift register ($M = L$-1)
  - Maximum ***n*** modulo-2 adders (***n*** outputs)
  - transmitted bits sequence $\{x_{1k}\}$ and $\{x_{2k}\}$ are not independent but depend on a number of the *n*-bit input sequence
  - an output multiplexer that converts the adder output to serial codewords.
- **Constraint length**: $L = M + 1$
  - the number of shifts over which a single data bit can affect the encoder output.

Convolutional encoder : Representation

- Convolutional encoder can be **represented** by :
  - Generator representation
  - Tree diagram representation
  - State diagram representation
  - Trellis diagram representation

Generator Sequence

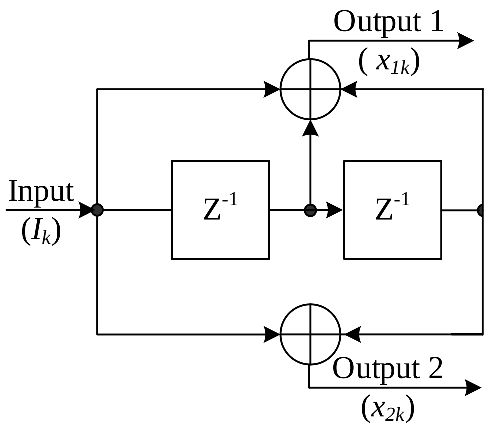
- **Generator representation:**
  - A convolutional encoder can be described by the set of **impulse responses** or **generator sequences** g_i where $i = 1,\ldots,n$
  - shows the **hardware connection** of the shift register to the modulo-2 adder.
  - **A "1" represents a connection and a "0" represents no connection**.

  Example: $g_1 = [111]$, $g_2 = [101]$

- **The length of each generator sequence g_i is L**

Example

- Generator sequences for the encoder :

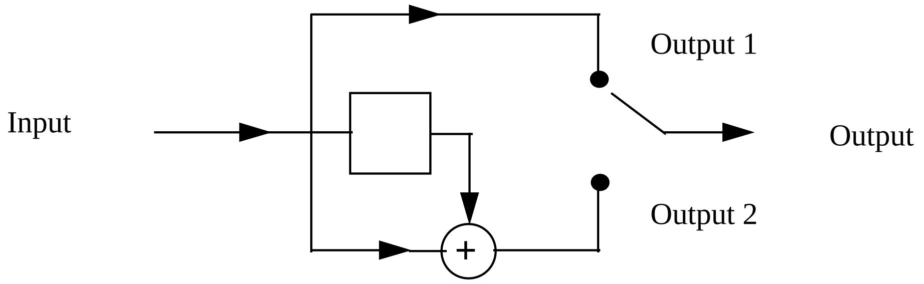


$L = 3$

$g_1 = (111)$, $g_2 = (101)$.

- **Exercise:**

  What is the generator sequences for the encoder



State Diagram

- **State of the encoder**: defined by the shift register contents.
  - For a rate-1/*n* code:

    Total number of encoder states: $N_s = 2^M$ with ***M*** being the **number of Flip Flops**.

- **State diagram** for a rate-1/*n* code:
  - **States**: labelled using all binary combinations of N_s
  - **Branches**: labelled with the **output, *x***
  - **State transitions**: means that the input *K* to the encoder with the current state gives the output *x* and transites the encoder to th next state.

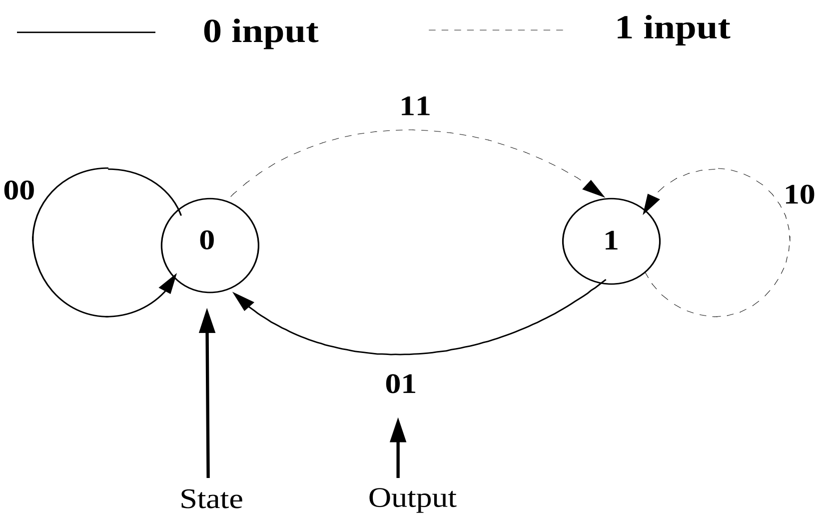
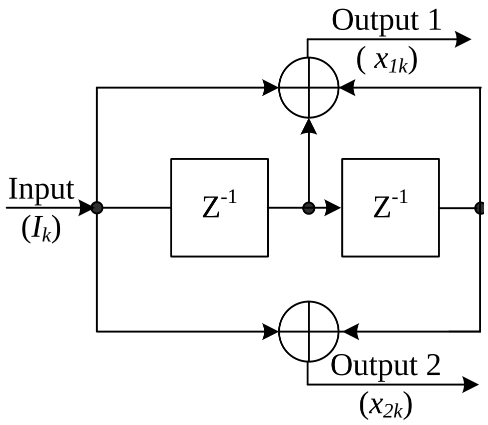
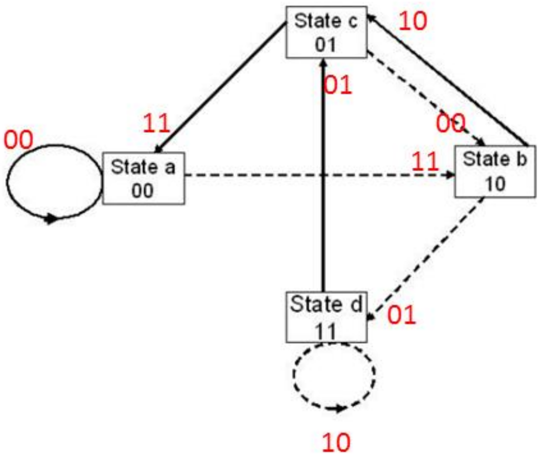


Figure: State diagram for the binary convolutional encoder with **1 Flip Flop**.

State Diagram: Example

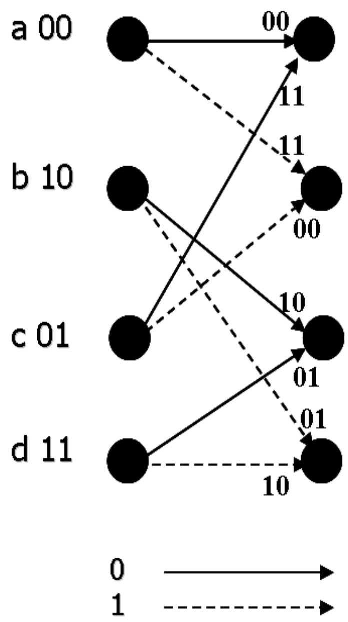
State	Input	Current State	Next state	Output
a	0	00	00(a)	00
	1	00	10(b)	11
b	0	10	01(c)	10
	1	10	11(d)	01
c	0	01	00(a)	11
	1	01	10(b)	00
d	0	11	01(c)	01
	1	11	11(d)	10





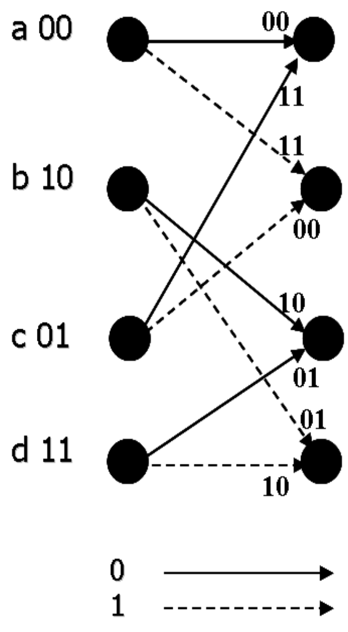
Trellis Diagram

- The **branches** are labelled with the encoder output bits that correspond to various state transitions.
- State transitions with a **solid line** correspond to an input "**0**" and **dashed line** correspond to an input "**1**".
- At a particular node, the **branching rule** is to follow the **upper branch** if the next input bit is a "0" and the **lower branch** if the bit is a "1".
- The trellis diagram repeats itself after the ***L*th stage**. This behaviour is consistent with the fact that the constraint length is ***L***.



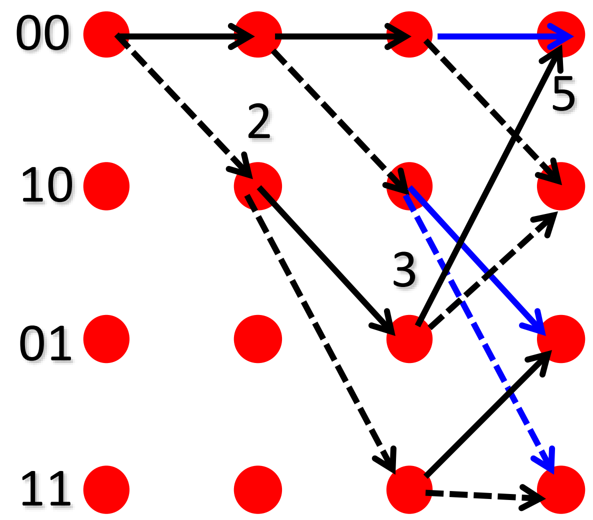
Trellis Diagram: Example

State	Input	Current State	Next state	Output
a	0	00	00(a)	00
	1	00	10(b)	11
b	0	10	01(c)	10
	1	10	11(d)	01
c	0	01	00(a)	11
	1	01	10(b)	00
d	0	11	01(c)	01
	1	11	11(d)	10



Free Hamming distance

- **Free Hamming distance d_{free}:** the minimum weight of non-zero code sequences from the all-zero path that merges with the all-zero path at a given node.
  - Convolutional codes are designed to have the largest possible d_{free} for a given code rate and total encoder memory.



- $d_{free} = 5$ and $t = \lfloor (5-1)/2 \rfloor = 2$. Therefore, **one error** can be corrected.
- Difficult to ascertain the **exact error detection and correction capabilities** → depend on the decoding method and on the Hamming distance properties.

Viterbi Decoding

Learning outcome

- Apply Viterbi algorithm to decode received data
- Design more complicated convolutional codes

Viterbi Decoding

- Originally devised by Viterbi in 1967 for **maximum likelihood (ML)** decoding of convolutional codes
- This is a **minimum distance decoding method** applicable to convolutional codes. The most likely transmitted sequence is the one with the **smallest Hamming distance** to the received sequence.
    1. Begin decoding assuming that the encoder began in the all zero state.
    2. Compute the Hamming distances for all paths entering each state, between these paths and the corresponding received digits.
    3. Select the path with the lowest 'score' arriving at each node and discard the other one(s). Store the score and its corresponding path. If the scores are equal select one at random.
    4. Repeat 2 and add the new scores.
    5. Repeat 3 storing the new scores and paths (survivor sequences).
    6. Continue until all digits have been received.
    7. Output the sequence which has the lowest running score.

Convolutional Encoder with L =3

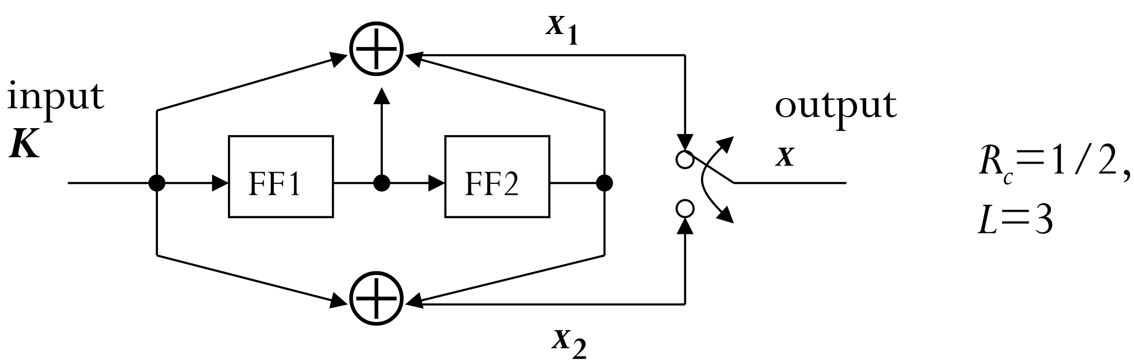


Figure: binary convolutional encoder

Code Rate, R_c = no of inputs / no of outputs = ½

Number of Flip Flops, M = 2

Constraint length, L = M+1=3

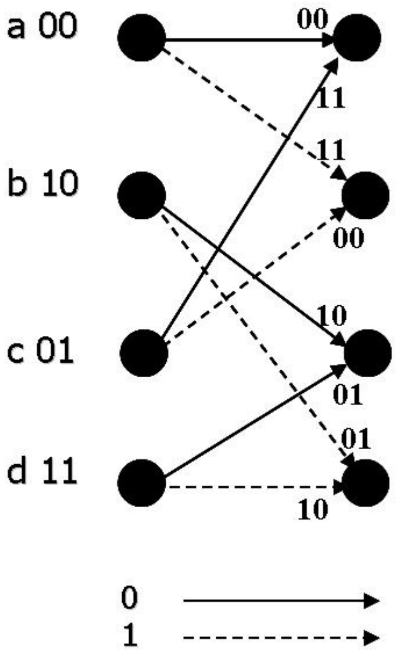
Generator Sequences (Words) : (g_1 = 111 , g_2 = 101)

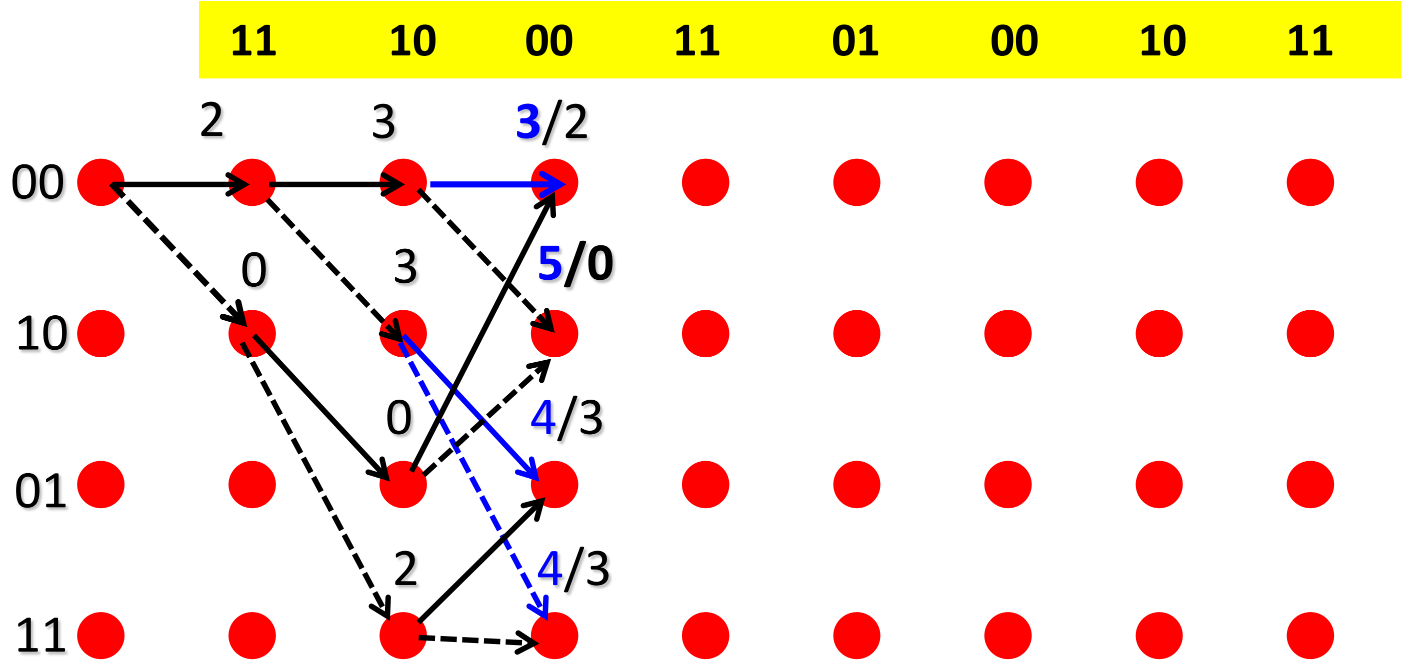
Number of Sates, N_s = 2^M = 4

Viterbi Decoding Example

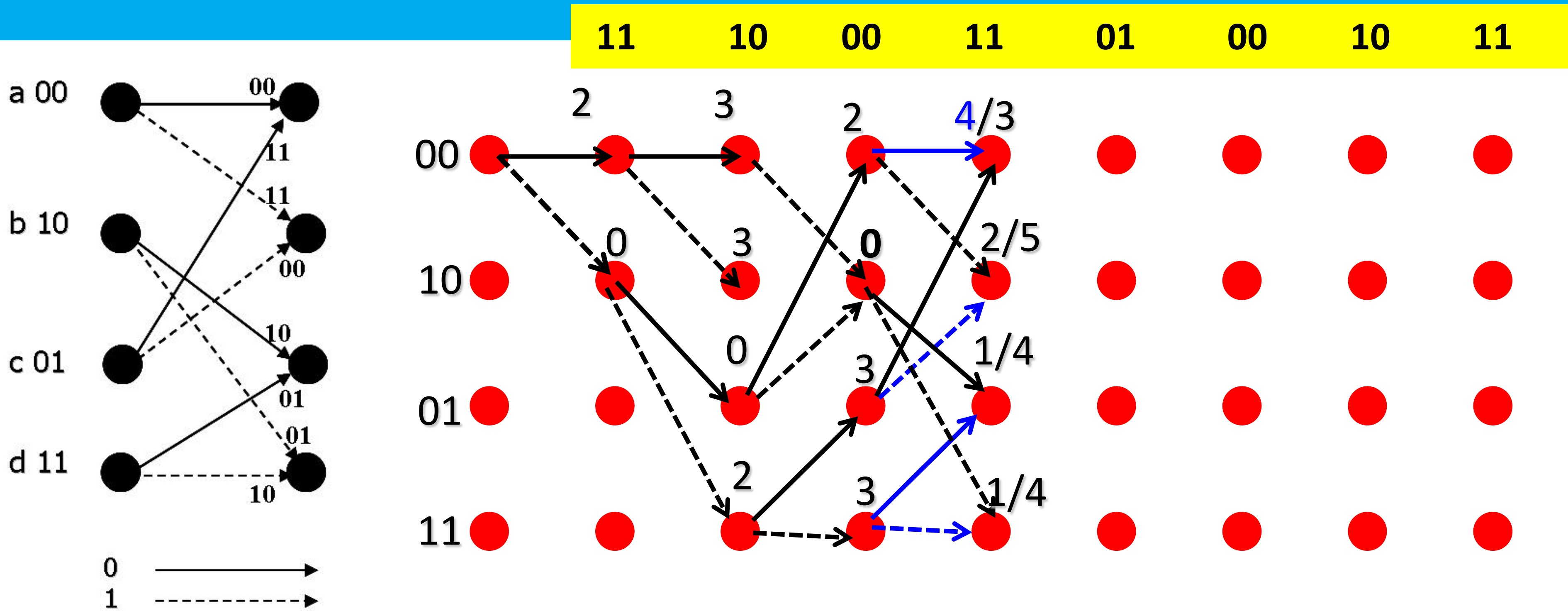
If the received sequence **y** = 11 10 00 11 01 00 10 11, what is the output of the decoder

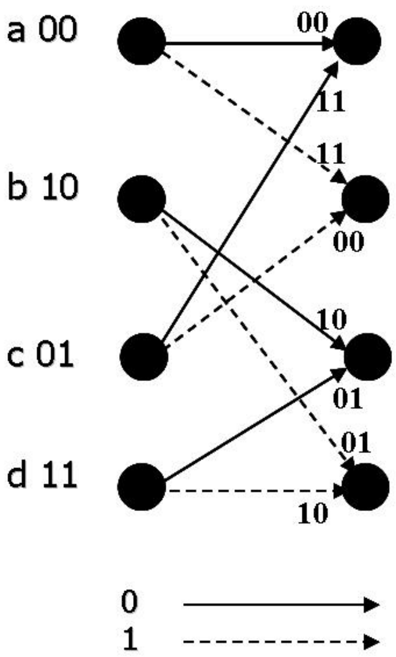
Viterbi Decoding (cont...)

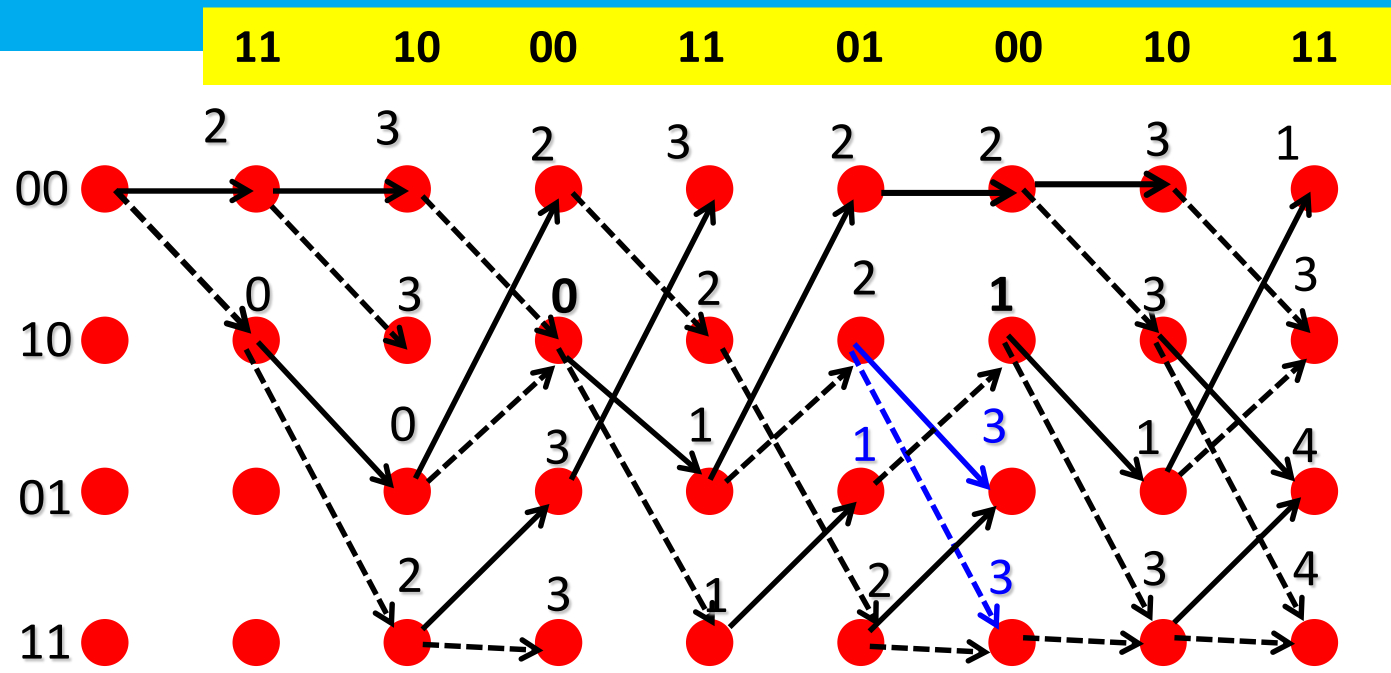


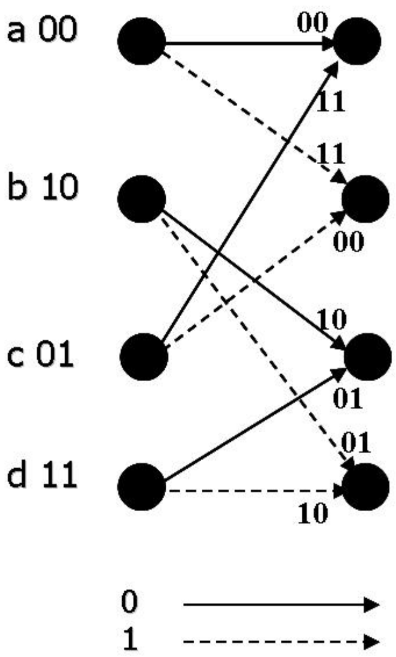


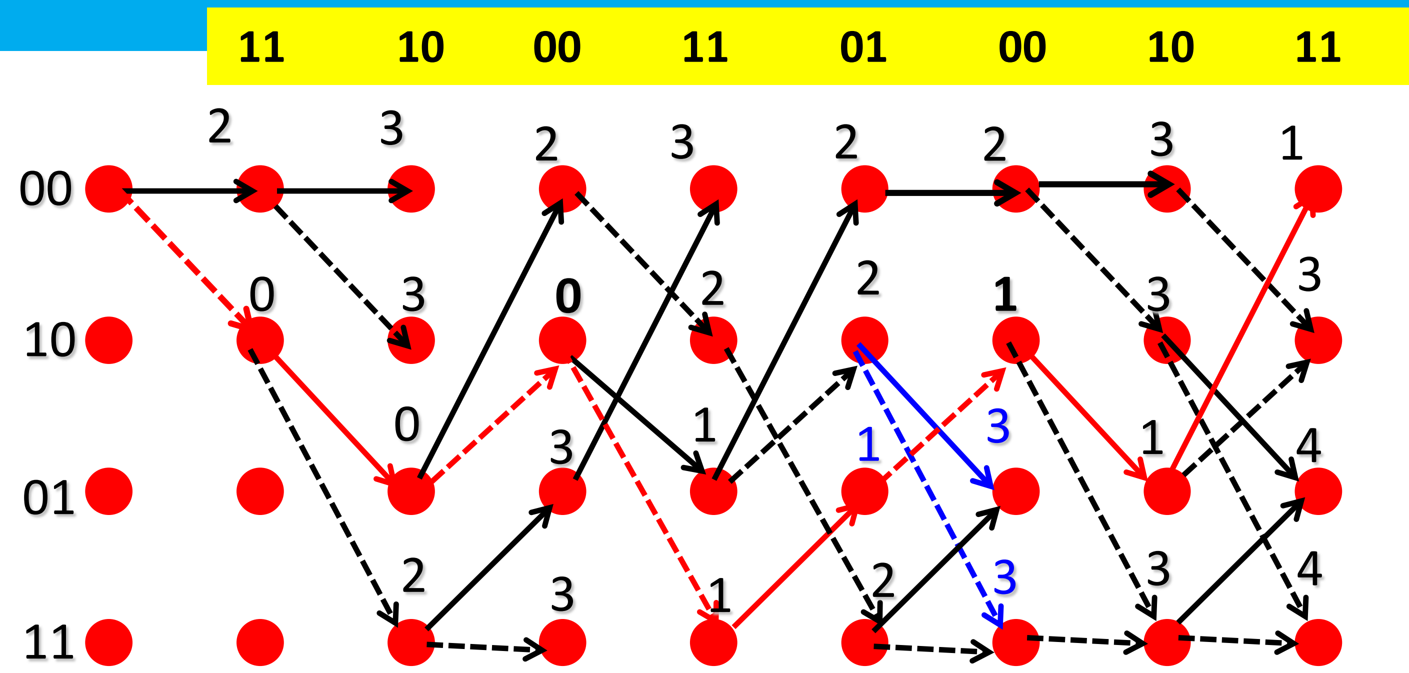
Viterbi Decoding (cont...)




a 00
b 10
c 01
d 11
00
11
11
00
10
01
01
10
0
1


11 10 00 11 01 00 10 11
00
10
01
11

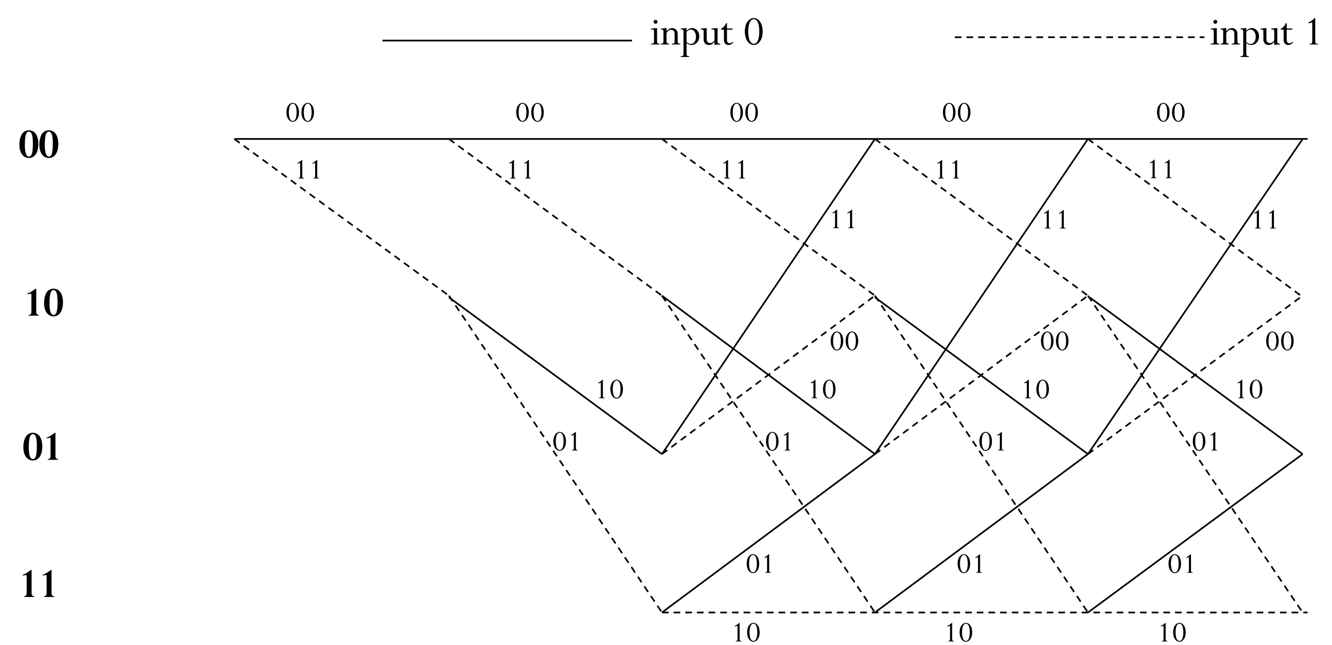




- Estimated encoder output sequence: = 11 10 00 01 01 00 10 11
- Estimated encoder input sequence: = 1 0 1 1 0 1 0 0

Exercise

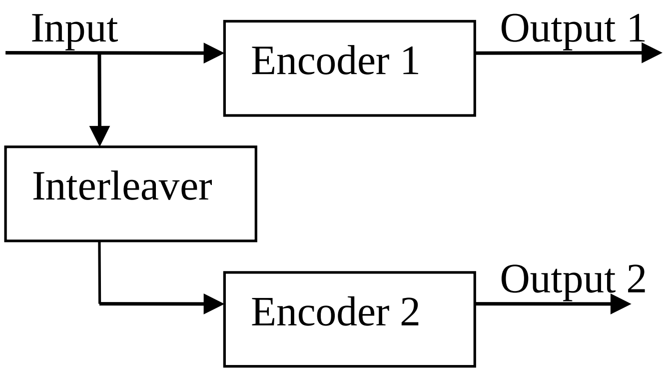
If the received sequence **y** = 11 11 10 01 00, what is the output of the decode



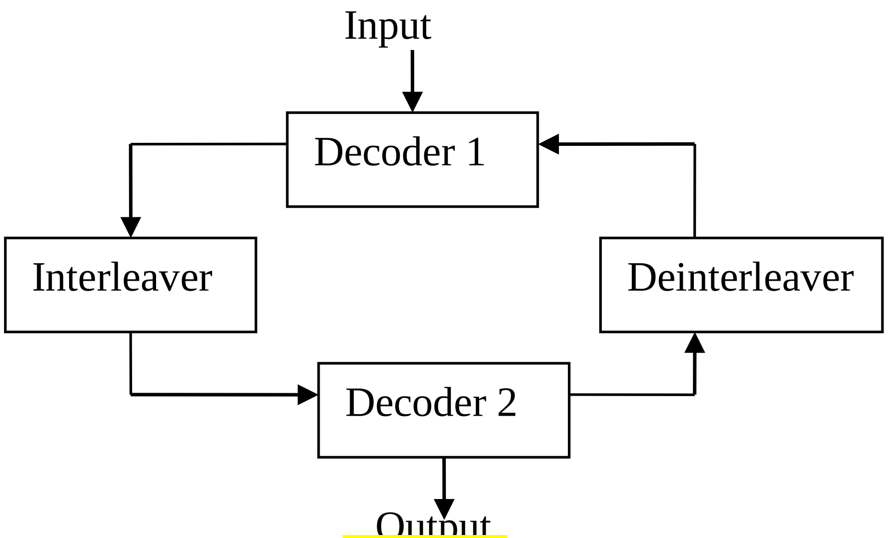
- Estimated encoder output sequence: $\hat{\mathbf{x}}$ = 11 01 10 01 00
- Estimated encoder input sequence: $\hat{\mathbf{k}}$ = 1 1 1 0 1

Turbo Code

- Introduced by Berrou, Glavieux, and Thitimajshima in 1993
- Parallel concatenation of codes (commonly Convolutional Codes)
- Produce high weight code words
- Interleaver shuffles the input sequence
- Iterative decoding



Encoder



Decoder