FIBRATIONS, DEPTH 1 FOLIATIONS, AND BRANCHED SURFACES

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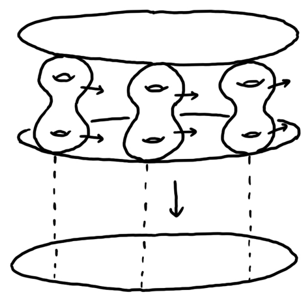
Let M be an orientable hyperbolic compact 3-manifold possibly with torus boundary components. Consider a fibration of M over S^1.



A *fiber surface* is the preimage of a point. The collection of fiber surfaces determines a *depth* 0 *foliation.*

A vector field transverse to the fiber surfaces determines a *transverse flow.* The first return map of a transverse flow on a fiber surface is the *monodromy* of the fibration.

Alternatively, we can define a fiber surface to be a cooriented surface S such that $M \backslash\backslash S \cong S \times [0, 1]$. **Then a fibration is determined by a fiber surface and vice versa (up to isotopy).**

Theorem 1 (Thurston)**.** *The collection of homology classes of fiber surfaces in* $H_2(M, \partial M; \mathbb{R})$ *are exactly the primitive points in the interior of finitely many cones, called the fibered cones.*

Fact. *A taut surface representing a class in the interior of a fibered cone must be a fiber surface.*

Remark. In fact, Thurston showed that there is a norm on $H_2(M, \partial M; \mathbb{R})$ whose unit ball B is a polyhedron, such that the fibered cones are the cones over certain top-dimensional faces of B.

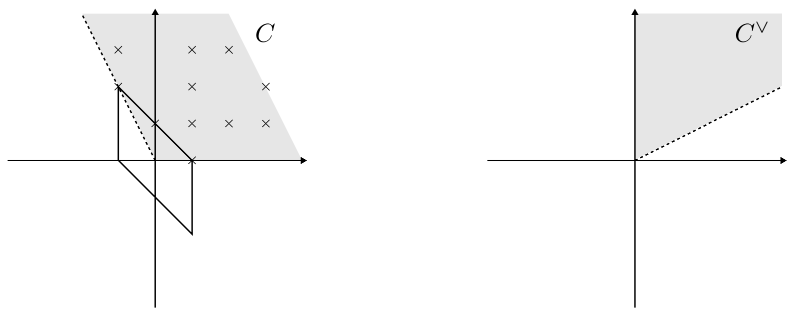
Theorem 2 (Fried)**.** *For any fibered cone C, there exists a pseudo-Anosov flow ϕ_C that is transverse to every fibration in the interior of C.*

Moreover, the cone generated by the closed orbits of ϕ_C in $H_1(M)$ is the dual cone $C^\vee = \{\alpha \in H_1(M) : \langle \alpha, \beta \rangle \geq 0$ *for every* $\beta \in C\}$

A neat way to package the information in a fibered cone is using *veering triangulations*. Here we introduce the dual notion of veering branched surfaces instead.

A *branched surface* is a subset of a 3-manifold locally modelled after a point in Figure 1.

The *branch locus* of a branched surface is a 4-valent graph. The complementary regions of the branch locus in the branched surface are called the *sectors*.



Definition 3. A branched surface in M is *veering* if:

(1) all its sectors are discs,

(2) all its complementary regions in M are *cusped solid tori* or *cusped torus shells*, and

(3) the edges of its branch locus can be oriented such that the branched surface is locally modelled after a point in Figure 1.

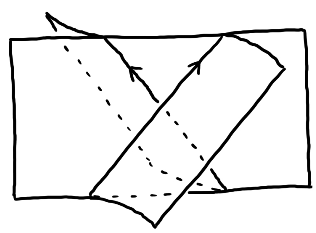


Figure 1.

Theorem 4 (Agol)**.** *For each fibered cone C, there exists a unique veering branched surface B_C carrying the unstable foliation of ϕ_C.*

Some features of the veering branched surface B_C:

- The branch locus of B_C is a directed graph Γ.

  Theorem 5 (Landry)**.** *Every closed orbit of ϕ_C is homotopic to a cycle of Γ and vice versa.*

  Theorem 6 (Agol-T.)**.** *With some more care, 'homotopic' can be replaced by 'isotopic'.*

  $\rightsquigarrow$ Can recover the fibered cone C.

- **Theorem 7** (Landry-Minsky-Taylor, Parlak)**.** *The Alexander polynomial Δ_M and the Teichmüller polynomial Θ_C can be computed from B_C.*

  $\rightsquigarrow$ Can compute entropy of monodromies

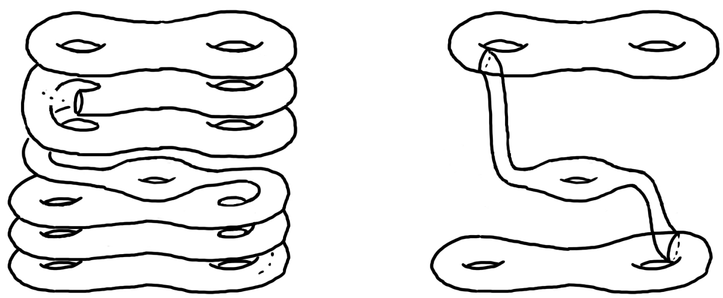
- In fact, the monodromies themselves can be determined: Each fiber surface intersects B_C positively transversely, and pushing it around gives a folding sequence of train tracks that models the fully-punctured map.

- Schleimer, Segerman, and Parlak implemented many of the computations above

Summary: Infinitely many fibrations can be encoded by finitely many flows ϕ_C (infinite-type objects), which can in turn be encoded by finitely many branched surfaces/triangulations (finite-type objects).

Remark. Veering branched surfaces in fact apply to the much more general theory of pseudo-Anosov flows, and are even in correspondence with such flows (in a suitable sense).

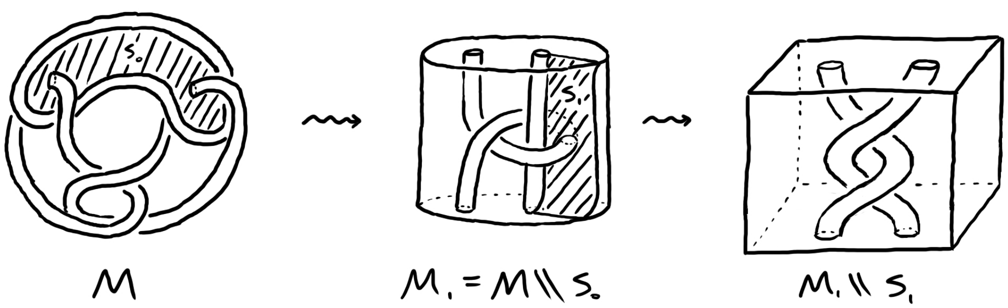
Definition 8. A *depth 1 foliaton* is a foliation having finitely many compact leaves and with all other leaves spiraling into the compact leaves.



A depth 1 foliation is determined by a cooriented compact surface (S_0 = union of compact leaves) and a cooriented compact properly embedded surface (S_1 = noncompact leaves with spiraling part truncated) in $M_1 := M \setminus\!\!\setminus S_0$ such that $M_1 \setminus\!\!\setminus S_1 \cong L \times [0,1]$, and vice versa (up to isotopy and peeling S_1 off S_0).

More precisely, one has to require that $M_1 \setminus\!\!\setminus S_1 \cong L \times [0,1]$ as *sutured manifolds*, i.e. keep track of the coorientations on the remains of S_0 and S_1.

Example. Let M be the 3-chain link exterior. Take the surfaces S_0 and S_1 as indicated below.



Note that in this example, M_1 is homeomorphic to a product as a 3-manifold but not to one as a sutured manifold.

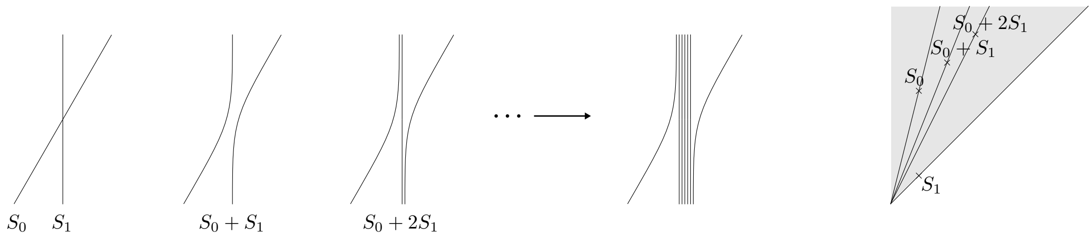
Depth 1 foliations naturally arise as limits of fibrations.

Let C be a fibered cone. Let S_0, S_1 be taut surfaces such that $[S_0] \in C^\circ, [S_1] \in \partial C$.

Then $S_0 + nS_1$ is a fiber surface and $[S_0 + nS_1] \in C^\circ$ for every $n \geq 1$.

As $n \to \infty$, $S_0 + nS_1$ spirals around S_1 and converges to a leaf of a depth 1 foliation.

Remark. But not all depth 1 foliations arise this way.



There is a very similar theory for depth 1 foliations as for fibrations.

Theorem 9 (Cantwell-Conlon)**.** *For fixed S_0, the homology classes of S_1 in $H_2(M_1, \partial M_1; \mathbb{R})$ are exactly the points in the interior of finitely many cones, called the foliation cones.*

Theorem 10 (Cantwell-Conlon-Fenley)**.** *For any foliation cone C, there exists a Handel-Miller semi-flow ϕ_C which is transverse to every depth 1 foliation in the interior of F.*

Moreover, the cone generated by the closed orbits of ϕ_C in $H_1(M_1)$ is the dual cone $C^\vee$.

Question. Is there a way to generalize the theory of veering branched surfaces from fibered cones to foliation cones?

Answer. Yes!

Theorem 11 (Landry-T.)**.** *For every foliation cone C, there exists a veering branched surface B_C carrying the unstable lamination of ϕ_C. Moreover, B_C is unique up to a suitable notion of equivalence.*

Definition 12. A veering branched surface in a sutured manifold Q (e.g. $M \backslash\backslash S_0$) is a branched surface B meeting ∂Q at R_+ (e.g. $(S_0)_+$) only and satisfying:

(1) All its sectors are discs and annuli with zero index (here we calculate index by putting a cusp at corners where the maw coorientation flips and smooth out the rest of the corners)

(2) All its complementary regions are *cusped solid tori*, *cusped torus shells*, or *cusped product pieces*.

(3) The components of its branch locus and its boundary train track $B \cap R_+$ can be given source orientations such that B is locally modelled after a point in Figure 2.

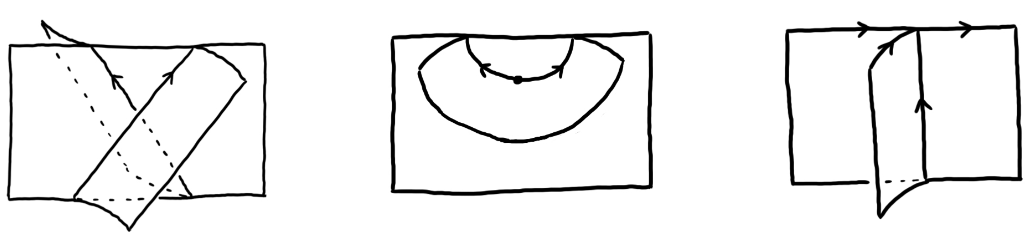


Figure 2.

Some features of the veering branched surface B_C:

- **Theorem 13** (Landry-T.)**.** *Every closed orbit of ϕ_F is homotopic to a cycle of the dual graph Γ and vice versa.*

$\rightsquigarrow$ Can recover the foliation cone C.

- Following recipe in fibered cone case

  $\rightsquigarrow$ Polynomial invariants, including Alexander polynomial (of sutured manifold).

  Questions. What information do these carry? Applications to big mapping class groups?

General idea: Translate ideas in fibered face theory to foliation cones using the combinatorial tool of veering branched surfaces.

Future directions:

- Let $M \rightsquigarrow M_0 \backslash\backslash S_0 = M_1 \rightsquigarrow M_1 \backslash\backslash S_1$ be as above. Given a map $(S_0)_+ \cong (S_0)_-$, M_1 can be glued up in a different way to a 3-manifold M'.

  Work of Gabai and Mosher shows that in 'most' cases, the Handel-Miller flow ϕ_C on M_1 can be glued up to pseudo-Anosov flow ϕ' on M.

  Question. How are the veering branched surfaces corresponding to ϕ_C and ϕ' related?

  Example. In the setting of fiber surfaces $S_0 + nS_1$ limiting to depth 1 foliation determined by (S_0, S_1), we have veering branched surfaces B_0 on M and B_1 on M_1.

  It is easy to obtain B_1 from B_0: just delete sectors.

  Obtaining B_0 from B_1 seems much more difficult!

- In general, for a *sutured hierarchy* $M \rightsquigarrow M_1 \rightsquigarrow ... \rightsquigarrow M_n$, want to extend veering branched surface on M_{n-1} inductively up to M.

  This would refine Gabai and Mosher's work, with applications to fibered face theory.