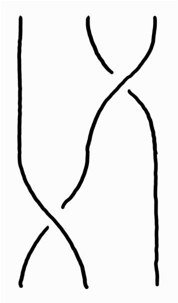
Minimum entropies of braids

Chi Cheuk Tsang (joint work with Xiangzhuo Zeng)

Slides:

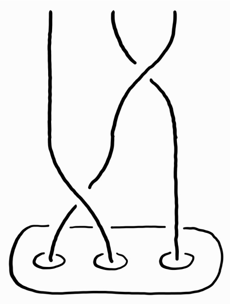
Braids and disc homeomorphisms

Braid β

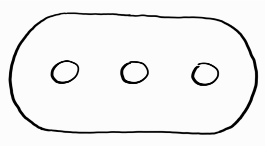


Braids and disc homeomorphisms

Braid β $\rightsquigarrow$ Disc



$\rightsquigarrow$



Braids and disc homeomorphisms

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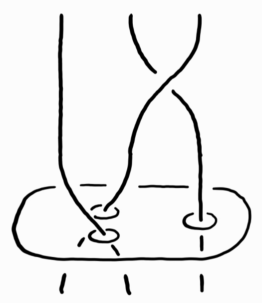

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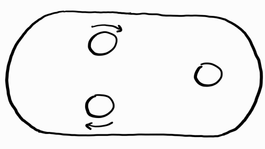
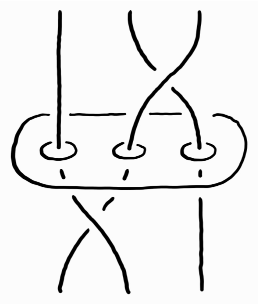
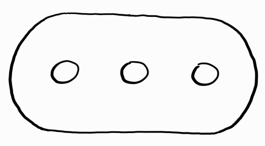


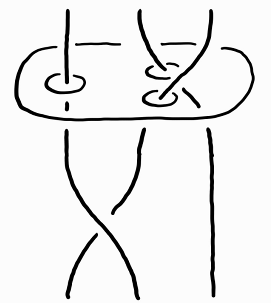
Braids and disc homeomorphisms

Braid β $\rightsquigarrow$ Disc

 $\rightsquigarrow$ 

Braids and disc homeomorphisms

Braid β $\rightsquigarrow$ Disc



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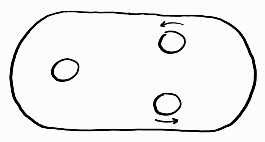
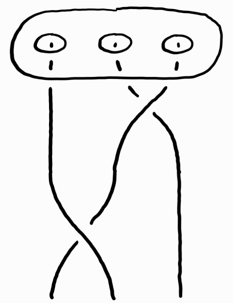


Braids and disc homeomorphisms

Braid β $\rightsquigarrow$ Disc



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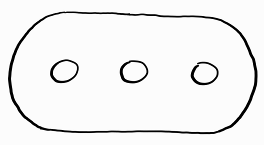
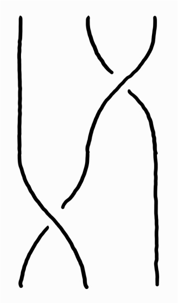


Braids and disc homeomorphisms

Braid β $\quad\leftrightsquigarrow\quad$ Disc homeomorphism f



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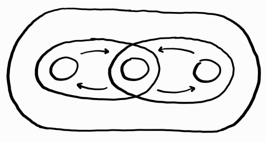
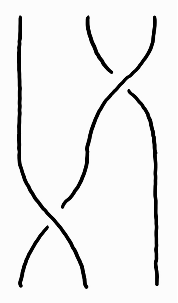


Braids and disc homeomorphisms

Braid β $\leftsquigarrow\!\!\rightsquigarrow$ Disc homeomorphism f



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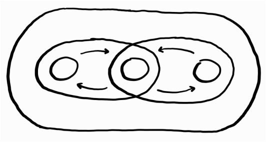
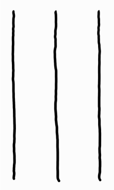


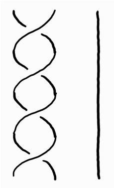
Important: f is only defined up to isotopy

Examples

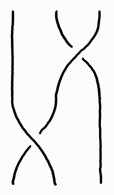
Braid



trivial

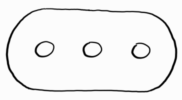


twist braid

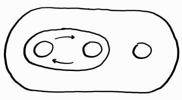


French braid
$\beta_{1,1}$

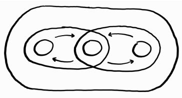
Homeo



id



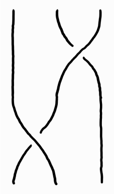
Dehn twist



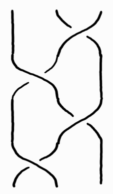
taffy-pulling
map *LR*

Examples

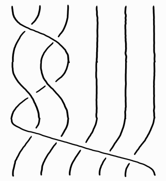
Braid



French braid
$\beta_{1,1}$

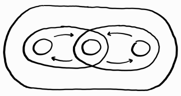


$\beta_{1,1}^2$



$\sigma_{2,3}$

Homeo



taffy-pulling
map LR

$(LR)^2$

Central question v1

Question

What is the dynamically simplest nontrivial braid?

Entropy: Definition

Let $f : S \to S$ be a homeomorphism of a surface.

Let $\mathcal{U}$ be a finite open cover of S. For each n,

- let $\mathcal{U}_{f,n}$ be the common refinement of $\mathcal{U}, f^*(\mathcal{U}), \ldots, f^{*n}(\mathcal{U})$, and
- let $C_{f,\mathcal{U},n}$ be the minimum number of elements of $\mathcal{U}_{f,n}$ that suffices to cover S.

The **entropy** of f is defined to be

$$h(f) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log C_{f,\mathcal{U},n}.$$

Entropy: Intuition

More intuitively:

- $U_0 \cap f^{-1}(U_1) \cap \cdots \cap f^{-n}(U_n)$ is an element of $\mathcal{U}_n$
  $\Leftrightarrow \exists x \in S$ such that $f^k(x) \in U_k$ for $k = 0, \ldots, n$
  $\rightarrow (U_0, \cdots, U_n)$ is the **itinerary** of x
- $C_{f,\mathcal{U},n} = \min \#$ itineraries that describes all points in S
- The entropy of f

$$h(f) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log C_{f,\mathcal{U},n}$$

is the maximum exponential growth rate of itineraries.

Central question v2

The **entropy** of a braid β

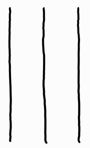
$$h(\beta) = \inf_f h(f)$$

where the infimum is taken over all homeomorphisms f representing β.

Question

What is the minimum nonzero entropy over all braids?

Example: Trivial braid



Proposition

$h(\textit{trivial braid}) = 0$.

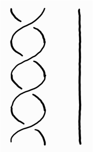
Proof.

It suffices to show that $h(id) = 0$.

Fix a finite open cover $\mathcal{U}$.

For each n, $\mathcal{U}_n$ is the common refinement of $\mathcal{U}, \ldots, \mathcal{U}$
$\Rightarrow \mathcal{U}_n \hookrightarrow 2^{\mathcal{U}}$
$\Rightarrow C_{id,\mathcal{U},n}$ is bounded. $\square$

Example: Twist braid



Proposition

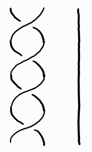
$h(\textit{twist braid}) = 0$.

Proof.

It suffices to show that $h(\text{Dehn twist } \tau \text{ in annulus}) = 0$.

$\square$

Example: Twist braid



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$h(\textit{twist braid}) = 0$.

Proof.

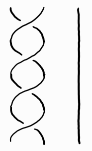
It suffices to show that $h(\text{Dehn twist } \tau \text{ in annulus}) = 0$.

Fact 1: $h(f^k) = kh(f)$.

Proof: $C_{f,\mathcal{U},kn} \leqslant C_{f^k,\mathcal{U}_k,n}$ and $C_{f^k,\mathcal{U},n} \leqslant C_{f,\mathcal{U},kn}$.

$\square$

Example: Twist braid



Proposition

$h(\textit{twist braid}) = 0$.

Proof.

It suffices to show that $h(\text{Dehn twist } \tau \text{ in annulus}) = 0$.

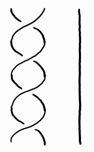
Fact 1: $h(f^k) = kh(f)$.

Fact 2: $\widehat{f}$ (branch) covers $f \Rightarrow h(\widehat{f}) = h(f)$.

Proof: $C_{f,\mathcal{U},n} \leqslant C_{\widehat{f},\widehat{\mathcal{U}},n}$ and $C_{\widehat{f},\mathcal{U},n} \leqslant [f : \widehat{f}]C_{f,\overline{\mathcal{U}},n}$

$\square$

Example: Twist braid



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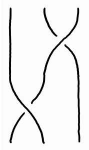
τ covers $\tau^k \Rightarrow h(\tau) = h(\widehat{\tau^k}) = h(\tau^k) = kh(\tau)$

$\square$

Example: French braid

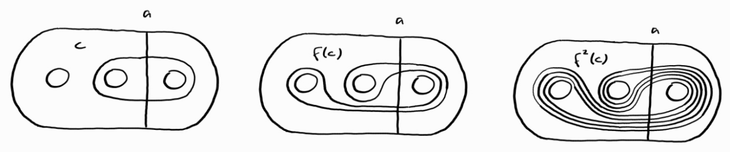


Proposition

h(French braid) $= 2\log\mu$*, where* $\mu = \frac{1+\sqrt{5}}{2}$ *is the golden ratio.*

Proof that $h(\text{French braid}) \geqslant 2\log\mu$.

Suppose f represents the French braid $\beta_{1,1}$.



Observe that a and $f^n(c)$ have $2F_{2n+1}$ essential intersections, where (F_k) is the Fibonacci sequence

Example: French braid

Claim: $2F_{2n+1}$ such intersection points have distinct itineraries, provided that $\mathcal{U}$ is fine enough.

$\Rightarrow h(f) \geqslant \lim_{n\to\infty} \frac{1}{n} \log(2F_{2n+1}) = 2\log\mu$

$\Rightarrow h(\beta_{1,1}) \geqslant 2\log\mu$

Example: French braid

Claim: $2F_{2n+1}$ such intersection points have distinct itineraries, provided that $\mathcal{U}$ is fine enough, so that:

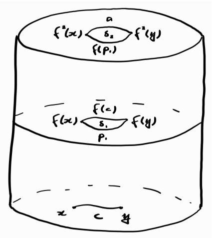
- For every $U_i \in \mathcal{U}$, U_i is contractible.
- For every $U_i, U_j \in \mathcal{U}$, $U_i \cap f^{-1}(U_j)$ is contractible.
- For every $U_i \in \mathcal{U}$, $U_i \cap a$ and $U_i \cap c$ are contractible.

Proof: Suppose $x, y \in c$ have the same itinerary.

For $k = 0, \ldots, n$, connect $f^k(x)$ to $f^k(y)$ by a path p_k within the common element of $\mathcal{U}$ that they lie in.

Next, connect $f(p_k)$ and p_{k+1} by a disc δ_k.

Then $\bigcup_{k=0}^{n} f^k(\delta_k)$ is a disc connecting an arc on a and an arc on $f^n(c)$ between x and y. $\rightarrow\!\leftarrow$



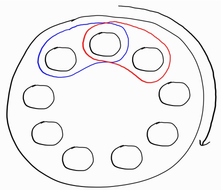
Example: More strands

Proposition

There exists a sequence of n-strand braids β_n such that $\lim_{n\to\infty} h(\beta_n) = 0$

Proof.

Let S_n be the $3n$-holed disc and let f_n be the composition of a taffy-pulling map f_1 with a rotation of S_n by three holes.



f_n^n is a branched cover of f_1
$\Rightarrow h(f_n) = \frac{1}{n} h(f_1)$.

Isotope f_1 to attain $h(\text{French braid}) = 2\log\mu$
$\Rightarrow h(\beta_n) = \frac{2}{n}\log\mu \to 0$. $\square$

Central question v3

Question

For each n, what is the minimum nonzero entropy among all n-strand braids?

Known results

Theorem

For small values of n, the minimum entropy η_n among n-strand braids is given by:

$$\begin{aligned}
&& \eta_3 &= 2\log\mu \\
&& &= \log|x^3 - 2x^2 - 2x + 1| \\
&\textit{[Ko-Los-Song 2002]} & \eta_4 &= \log|x^4 - 2x^3 - 2x + 1| \\
&\textit{[Ham-Song 2006]} & \eta_5 &= \log|x^5 - 2x^3 - 2x^2 + 1| \\
&& \eta_6 &= \tfrac{1}{2}\log|x^3 - 2x^2 - 2x + 1| \\
&\textit{[Lanneau-Thiffeault 2011]} & \eta_7 &= \log|x^7 - 2x^4 - 2x^3 + 1| \\
&& \eta_8 &= \log|x^8 - 2x^5 - 2x^3 + 1|
\end{aligned}$$

where $|P(x)|$ denotes the largest positive root of a polynomial $P(x)$.

Main theorem

Theorem (T.-Zeng 2024)

For $n \geqslant 9, n \neq 10, 12, 14$, the minimum entropy η_n among n-strand braids is given by:

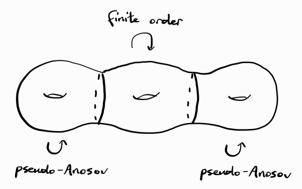
$$\begin{cases} \log |x^{2k+1} - 2x^{k+1} - 2x^k + 1| & \textit{if } n = 2k+1 \\ \log |x^{4k} - 2x^{2k+1} - 2x^{2k-1} + 1| & \textit{if } n = 4k \\ \frac{1}{2} \log |x^{2k+1} - 2x^{k+1} - 2x^k + 1| & \textit{if } n = 4k+2 \end{cases}$$

Conjecturally, the same expression hold for $n = 10, 12, 14$ as well.

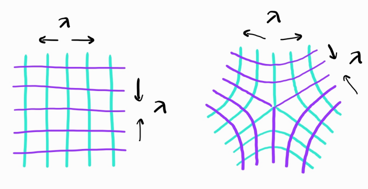
Nielsen-Thurston classification

Theorem (Nielsen, Thurston)

Up to isotopy, every surface homeomorphism can be decomposed into finite-order and pseudo-Anosov pieces.



A homeomorphism f is **pseudo-Anosov** if $\exists$ transverse pair of singular foliations (ℓ^s, ℓ^u) on S such that leaves of ℓ^s are contracted by λ and leaves of ℓ^u are expanded by λ.



The number $\lambda > 1$ is the **dilatation** of f.

Nielsen-Thurston classification

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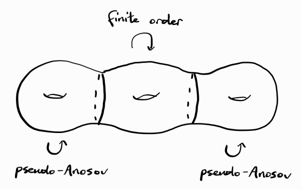


Fact 1: The entropy of a pseudo-Anosov map is log(dilatation).

Fact 2: The entropy of an isotopy class is the maximum of the entropies over all pseudo-Anosov pieces.

Central question v4

This reduces our central question to:

Question

For each n, what is the minimum dilatation among all n-strand pseudo-Anosov braids?

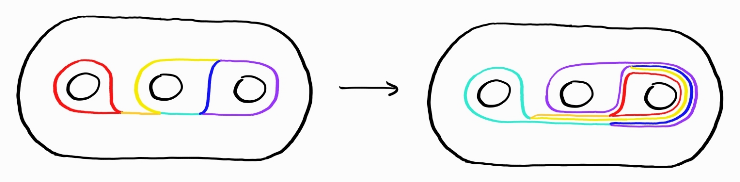
Theorem (T.-Zeng 2024)

For $n \geqslant 9, n \neq 10, 12, 14, 18, 22, 26$, the minimum dilatation among n-strand pseudo-Anosov braids is given by

$$\begin{cases} |x^{2k+1} - 2x^{k+1} - 2x^{k} + 1| & \text{if } n = 2k+1 \\ |x^{4k} - 2x^{2k+1} - 2x^{2k-1} + 1| & \text{if } n = 4k \\ |x^{4k+2} - 2x^{2k+3} - 2x^{2k-1} + 1| & \text{if } n = 4k+2 \end{cases}$$

Train tracks

Every pseudo-Anosov map f can be homotoped into a train track map.



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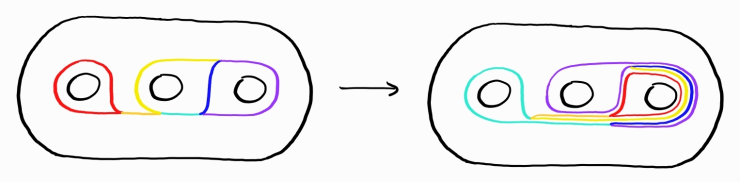


The **transition matrix** records the number of times branches map over each other.

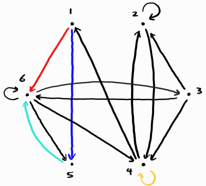
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The dilatation of f is the spectral radius of the transition matrix.

Growth rate of directed graphs

Matrix $A \in M_{d\times d}(\mathbb{Z}_{\geqslant 0}) \rightsquigarrow$ Directed graph G

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow$$



Fact: $\rho(A)$ = growth rate of $G = \lim_{n\to\infty}(\text{length } n \text{ cycles in G})^{\frac{1}{n}}$.

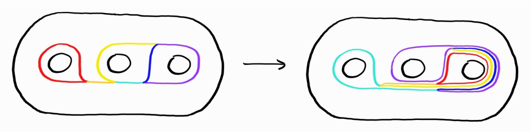
Upshot: $\dim A \downarrow$, entries of $A \uparrow$

$\Rightarrow$ length of cycles in $G \downarrow$, $\#$ cycles in $G \uparrow$

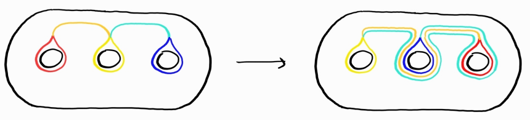
$\Rightarrow$ growth rate $\uparrow$

Standardly embedded train track: Example

A pseudo-Anosov map can represented by many train track maps. Some are better than others.

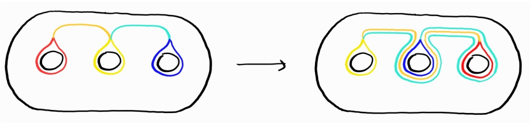


$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Standardly embedded train track: Definition



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

A train track map is **standardly embedded** if

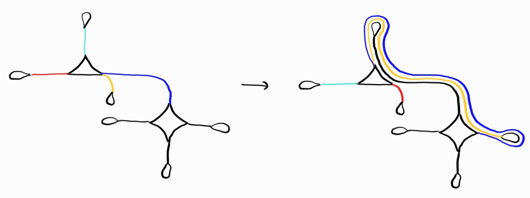
- the train track consists of **infinitesimal polygons** and **real edges**,
- the map permutes the infinitesimal polygons and folds the real edges.

Proposition (Hironaka-T. 2022)

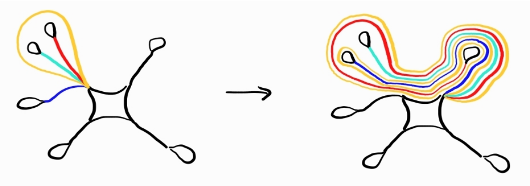
Every pseudo-Anosov braid is represented by a standardly embedded train track map.

Floral train track: Example

In our work with Zeng, we found even better representatives

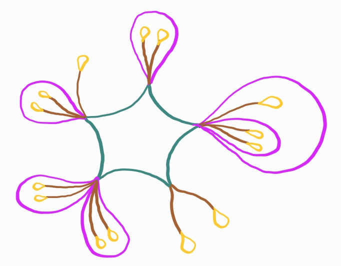


$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 6 & 2 & 2 & 0 & 0 & 0 \end{bmatrix}$$

Floral train track: Definition



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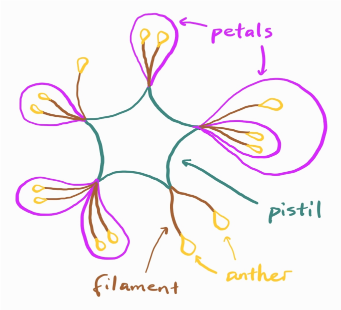

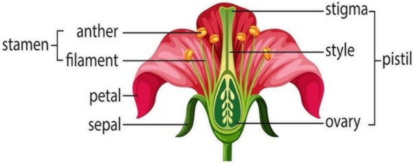

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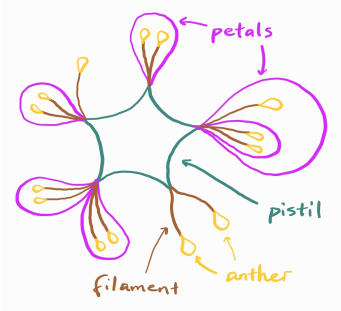


A standardly embedded train track map is **floral** if it consists of

- one fixed infinitesimal polygon (**pistil**),
- multiple 1-cusped infinitesimal polygons (**anthers**),
- some real edges connecting the anthers to the pistil (**filaments**),
- remaining real edges having endpoints on the same vertex of the pistil (**petals**).

Proposition (T.-Zeng)

Every pseudo-Anosov braid is represented by a floral train track map.

Floral train track: Construction

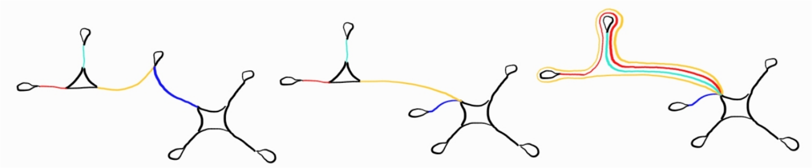
Floral train tracks are specific to braids.

Lefschetz fixed point theorem $\Rightarrow$ every homeomorphism of a disc has at least one fixed point b

Choose a standardly embedded train track with all inner singularities (including b) as infinitesimal polygons

Split switches incident to 1-pronged infinitesimal polygons (c.f. Farber-Reinoso-Wang)

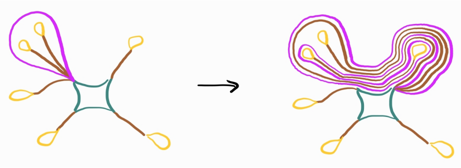
Split switches of infinitesimal polygons that are non-1-pronged or not b

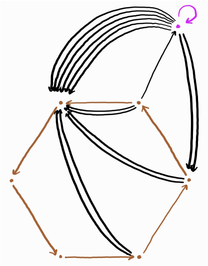


Floral train track: Properties

Sample properties of directed graphs associated to real transition matrix of floral train track maps.

- Each filament lies on one and only one **filament curve**.
- Each petal lies on one and only one **petal curve**.
- Edges enter filament curves in pairs.
- Edges leave petal curves in pairs.





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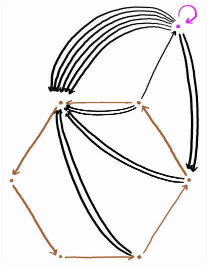
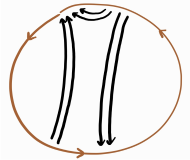


Braids attaining the lower bound

Properties + Applications of Lefschetz fixed point theorem to powers of f + Computations $\rightsquigarrow$ Lower bounds for dilatation

In fact, the directed graph for an n-strand braid β that attain the lower bound must be of the form:



This implies that β must have the following properties.

- The closure of the braid is a knot.
- Every strand is a punctured 1-pronged singularity.
- Aside from the strands, there are exactly two other singularities.

Future directions

- Determine the remaining values of minimum dilatation/entropy for braids

  See computational approaches of Lanneau-Thiffeault

- Classify all braids that attain the minimum dilatation/entropy

  Work backwards: directed graph $\rightsquigarrow$ floral train track $\rightsquigarrow$ braid

- Other minimum dilatation problems

  Golden ratio conjecture (Hironaka): The minimum dilatation λ_g among pseudo-Anosov maps on the closed oriented genus g surface satisfies $\lim_{g\to\infty} \lambda_g^g = \mu^2$

- Galois conjugates of the dilatation

  What information do the other eigenvalues of the transition matrix carry?

  Related to Lehmer's conjecture that there is a universal lower bound to the Mahler measure of an algebraic integer