A CONNECTION BETWEEN PSEUDO-ANOSOV FLOWS AND SUTURED FLOER HOMOLOGY

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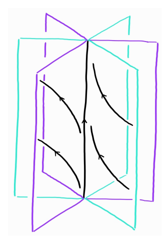
A **pseudo-Anosov flow** is a flow ϕ^t on a closed orientable 3-manifold for which there exists a pair of transverse singular foliations Λ^s and Λ^u such that:

- The leaves of Λ^s and Λ^u intersect transversely along orbits of ϕ.
- The orbits of ϕ^t are forward asymptotic along the leaves of Λ^s.
- The orbits of ϕ^t are bacakward asymptotic along the leaves of Λ^u.



Example. Let $f : S \to S$ be a pseudo-Anosov homeomorphism on a closed orientable surface. The **mapping torus** of f is the 3-manifold M_f obtained from $S \times [0,1]$ by identifying $(x,1) \sim (f(x),0)$. The suspension flow ϕ^t_f on M_f is a pseudo-Anosov flow.

Question. Can one define a Floer homology using the closed orbits of a pseudo-Anosov flow? More specifically, we wish for a homology theory that both

(1) captures the dynamics of the pseudo-Anosov flow and
(2) be related to the topology of the underlying 3-manifold.

Previous work.

- Periodic Floer homology PFH(f) of a symplectomorphism f
  - Chain complex generated by sets of closed orbits of the suspension flow ϕ^t_f
  - Differential counts pseudo-holomorphic curves between closed orbits
- Embedded contact homology ECH(α) of a contact form α
  - Chain complex generated by sets of closed orbits of the Reeb flow R^t_α
  - Differential counts pseudo-holomorphic curves between closed orbits
- Zung's pair of pants complex associated to an Anosov flow ϕ^t
  - Chain complex generated by sets of closed orbits of ϕ^t
  - Differential counts pairs of pants between closed orbits that are transverse to ϕ^t in their interior

In work in progress with Antonio Alfieri, we found a way of computing sutured Floer homology using a chain complex that is generated by closed orbits of a pseudo-Anosov flow.

We first briefly sketch the definition of sutured Floer homology.

Let Σ be a compact surface with boundary. Let α and β be multicurves on Σ. The sutured manifold $Q(\Sigma, \alpha, \beta)$ is obtained from $\Sigma \times [0,1]$ by attaching 2-handles along $\alpha \times \{0\}$ and $\beta \times \{1\}$ and declaring $\partial\Sigma \times [0,1]$ to be the sutures. A **Heegaard diagram** for a sutured manifold Q is a triple (Σ, α, β) for which $|\alpha| = |\beta|$ and $Q = Q(\Sigma, \alpha, \beta)$.

Consider the symmetric product $\mathrm{Sym}^g(\Sigma)$ where $g = |\alpha| = |\beta|$. The multicurves α and β determine tori $\mathbb{T}_\alpha = \mathrm{Sym}^g(\alpha)$ and $\mathbb{T}_\beta = \mathrm{Sym}^g(\beta)$. Let S be the set of intersection points between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. We refer to the element of S as the **states**. Each state is a collection of g intersection points between α and β such that each component of α and each component of β contains exactly one point.

Let $\mathrm{CF}(\Sigma, \alpha, \beta)$ be the $\mathbb{F}_2$ chain complex generated by S. The differential $\partial : \mathrm{CF}(\Sigma, \alpha, \beta) \to \mathrm{CF}(\Sigma, \alpha, \beta)$ is defined by counting holomorphic discs.

Theorem (Juhasz)**.** *The homology of* $\mathrm{CF}(\Sigma, \alpha, \beta)$ *is independent of the choice of Heegaard diagram for* Q

The homology of $\mathrm{CF}(\Sigma, \alpha, \beta)$ is referred to as the (hat version of) **sutured Floer homology** of Q and denoted by $\mathrm{SFH}(Q)$.

Next, we need some definitions from pseudo-Anosov flows.

Definition. Let ϕ^t be a pseudo-Anosov flow. Let $\mathcal{C}$ be a nonempty finite set of closed orbits of ϕ^t. For each $\gamma \in \mathcal{C}$, we let ν_γ be a small tubular neighborhood of γ. The local stable leaf that contains γ intersects ν_γ in a multicurve d_γ, which we call the **degeneracy locus**. The sutured manifold $Q(\phi^t, \mathcal{C})$ is obtained by removing ν_γ and putting two annular sutures along each component of d_γ, for every $\gamma \in \mathcal{C}$.

Definition. Let ϕ^t be a pseudo-Anosov flow. Let $\mathcal{C}$ be a nonempty finite set of closed orbits of ϕ^t. We say that ϕ^t has **no perfect fits** relative to $\mathcal{C}$ if there does not exist a pair of closed orbits γ_1, γ_2 such that γ_1 is homotopic to γ_2^{-1} in the complement of $\mathcal{C}$.

Theorem 1 (Alfieri-T.)**.** *Let ϕ^t be a pseudo-Anosov flow. Let $\mathcal{C}$ be a nonempty finite set of closed orbits of ϕ^t. Suppose $\mathcal{C}$ contains all singular orbits of ϕ^t and suppose ϕ^t has no perfect fits relative to $\mathcal{C}$.*

Then the sutured Floer homology of $Q(\phi^t, \mathcal{C})$ can be computed as the homology of a chain complex that is generated by certain finite sets of closed orbits of ϕ^t.

Theorem 1 is obtained using the tool of veering branched surfaces.

A **branched surface** in a closed orientable 3-manifold M is a 2-complex $B \subset M$ where every point in B has a neighborhood smoothly modeled on a point in Figure 1. The **branch locus** $\mathrm{brloc}(B)$ is the set of non-manifold points of B.

A branched surface B in a closed orientable 3-manifold M is **veering** if:

(1) Each sector of B is homeomorphic to a disc.

(2) Each component of $M \backslash\backslash B$ is a **cusped solid torus**.

(3) There is an orientation on each edge of $\mathrm{brloc}(B)$ so that each vertex is of the form in Figure 1.

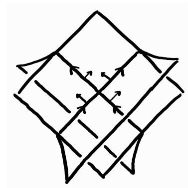


FIGURE 1.

One can show that (1) and (3) imply that each sector is a diamond as in Figure 2. In particular, each sector has a unique top vertex, and each vertex is the top vertex of a unique sector. Hence the number of vertices equals the number of sectors.

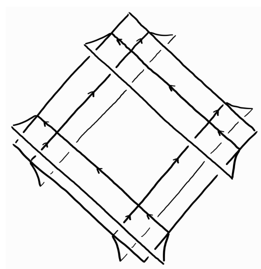


FIGURE 2.

The orientations in (3) endow $\mathrm{brloc}(B)$ with a structure of a directed graph.

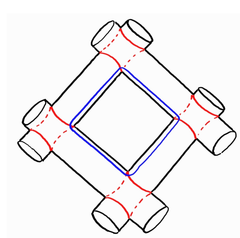
Theorem 2 (Agol-Guéritaud, Landry-Minsky-Taylor, T.)**.** *Let ϕ^t be a pseudo-Anosov flow. Let $\mathcal{C}$ be a nonempty finite set of closed orbits of ϕ^t. Suppose $\mathcal{C}$ contains all singular orbits of ϕ^t and suppose ϕ^t has no perfect fits relative to $\mathcal{C}$. Then M admits a unique veering branched surface $B(\phi^t, \mathcal{C})$ satisfying the property that every directed cycle of $\mathrm{brloc}(B(\phi^t, \mathcal{C}))$ is homotopic to a unique closed orbit of ϕ^t in $Q(\phi^t, \mathcal{C})$.*

Proof of Theorem 1. The strategy is to define a Heegaard diagram for $Q = Q(\phi^t, \mathcal{C})$ from $B = B(\phi^t, \mathcal{C})$.

Let U be a neighborhood of $\mathrm{brloc}(B)$. Let $\widehat{\Sigma}$ be the boundary of U.

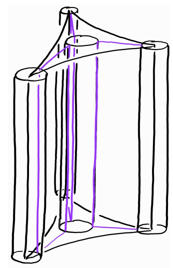
For each vertex v of $\mathrm{brloc}(B)$, let α_v be a curve on $\widehat{\Sigma}$ that bounds a disc separating $\mathrm{brloc}(B)$ into two smooth arcs near v. Let α be the union of α_v.

For each sector S of B, let β_S be the intersection of S with $\widehat{\Sigma}$. Let β be the union of β_S.



Attaching 2-handles to $\widehat{\Sigma}$ along α and β gives the 3-manifold N obtained by removing neighborhoods of $\mathcal{C}$ and neighborhoods of the branch curves of B from M.

At each $\gamma \in \mathcal{C}$, there exists a collection of annuli each with one boundary component along γ and the other on a branch curve.



Each annulus intersects $\widehat{\Sigma}$ in a curve that is disjoint from $\alpha \cup \beta$. Let Σ be the surface obtained by cutting $\widehat{\Sigma}$ along these curves. Then (Σ, α, β) is a Heegaard diagram for the sutured manifold obtained by cutting N along these annuli, which is exactly Q.

Finally, we show how to associate a set of closed orbits to each state.

Each β_S contains 4 intersection points, corresponding to the 4 corners of S. Each state can be thought of as an assignment of each sector S to one of its corners, so that no vertex is being assigned to twice.

Given a state x, we take paths connecting the bottom corner of each sector S to the assigned corner $\mathrm{x}(S)$. That x is a state means that these connecting paths form a union of closed curves. Moreover, each curve is homotopic to a directed cycle in $\mathrm{brloc}(B)$, thus is homotopic to a closed orbit by Theorem 2. $\square$

Further properties.

- The **bottom state** $\mathrm{x}^{\mathrm{bot}}$ that assigns each sector S to its bottom corner, which corresponds to the empty set of closed orbits, determines a nonvanishing class in $\mathrm{SFH}(Q(\phi^t, \mathcal{C}))$.
- The **top state** $\mathrm{x}^{\mathrm{top}}$ that assigns each sector S to its top corner also determines a nonvanishing class in $\mathrm{SFH}(Q(\phi^t, \mathcal{C}))$.
- The Spin^c grading on $\mathrm{SFH}(Q(\phi^t, \mathcal{C}))$ is given by the homology class of closed (multi-)orbits.
- Every effective domain in the Heegaard diagram (Σ, α, β) that we constructed is a subsurface of Σ.

Future directions.

- Describe the differential in terms of the flow ϕ^t.
  - Each effective domain connecting a state x to a state y can be intepreted as a surface connecting the corresponding closed orbits.
  - However, we don't know how to characterize the surfaces that contribute to the differential purely in terms of the pseudo-Anosov flow.
- Define an invariant of pseudo-Anosov flow in Heegaard Floer homology.
  - There exists homomorphisms $\mathrm{SFH}(Q(\phi^t, \mathcal{C})) \to \widehat{\mathrm{HF}}(M)$. One can attempt to define such an invariant by taking the image of the bottom or top state under such a homomorphism.
  - However, such a definition is not meaningful unless one can show some non-vanishing results.
  - Also, for this to be an invariant of pseudo-Anosov flows, one has to show that the definition is independent of the collection of closed orbits $\mathcal{C}$.
- Understand the relation between the contact invariant and potential pseudo-Anosov flow invariants for **contact Anosov flows**, i.e. Anosov flows that are also Reeb flows.

- For a suspension pseudo-Anosov flow ϕ_f^t, compute the dimension of $\mathrm{SFH}(Q(\phi_f^t, \mathcal{C}))$ in terms of the number of periodic points of f.
  - This is inspired by results of Ni and Ghiggini-Spano that for a fibered knot $K \subset Y$, the dimension of the second-to-top grading of $\widehat{\mathrm{HFK}}(Y, K)$ essentially equals the number of fixed points of the monodromy.
- Investigate the dynamical information contained in the decategorification of $\mathrm{SFH}(Q(\phi_f^t, \mathcal{C}))$.
- Come up with a good notion of cobordance of pseudo-Anosov flows for which there is an induced homomorphism $\mathrm{SFH}(Q(\phi_0^t, \mathcal{C}_0)) \to \mathrm{SFH}(Q(\phi_1^t, \mathcal{C}_1))$.
  - One can perform surgery along certain closed orbits to stay within the category of a pseudo-Anosov flow ϕ^t with a collection of closed orbits $\mathcal{C}$ satisfying the appropriate properties.
  - However, these surgeries are integer-reciprocal, hence fit poorly with the usual 2-handle attachment operation.