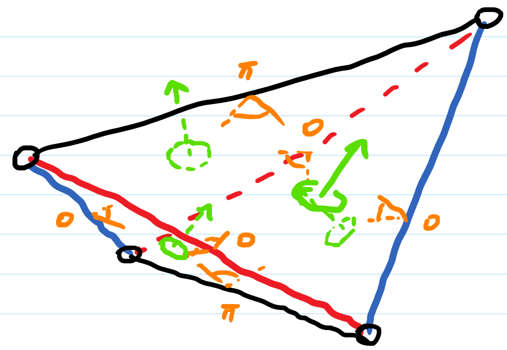
Veering branched surfaces

[Agol 2010]: Veering triangulations

M = interior of cpt ori 3-mfld with torus boundary components

Δ = ideal triangulation of M with:

- face coorientations
- taut structure
- edge coloring



[Agol - Gueritaud 2015, Schleimer - Segerman 2019]:

Veering triangulations $\longleftrightarrow$ Pseudo-Anosov flows without perfect fits

$\Rightarrow$ M hyperbolic

$\Rightarrow$ $\pi_1 M$ circular orderable

$\rightsquigarrow$ Info about Thurston norm on $H_2(M)$

E.g. Layered veering triangulations $\longleftrightarrow$ Suspension flows on mapping tori of pseudo-Anosov homeo. $\Sigma \to \Sigma$

Advantage: Veering triangulations are discrete objects!

Giannopolous - Schleimer - Segerman:

Veering triangulation census

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cPcbbbdxm_10 L 1 2 N 1 [2] [2,0,0] Z/5+Z ['m003','otet02_00000']
cPcbbbiht_12 L 1 2 E 1 [4] [2,0,0] Z ['m004','4_1','K2_1','K4a1','otet02_00001']
dLQacccjsnk_200 L 1 1 N 1 [2] [2,0,1] Z ['m016','K3_1','K12n242']
dLQbccchhfo_122 L 1 2 E 1 [4] [2,0,1] Z/2+Z ['m009']
dLQbccchhsj_122 L 1 2 N 1 [2] [2,0,1] Z/6+Z ['m010']
eLAkaccddjsnak_2001 L 1 1 N 1 [2] [2,1,1] Z ['m119']
eLAkbbcdddhwqj_2102 L 1 1 N 1 [2] [2,0,2] Z ['m052']
eLAkbccddhhsqs_1220 L 1 1 N 1 [2] [2,1,1] Z/2+Z ['m146']
eLMkbcddddedde_2100 L 2 1 N 1 [2,2] [4,0,0] Z+Z ['m203','6^2_2','L6a2','otet04_00001']
eLMkbcdddhhhdu_1221 L 1 2 N 1 [2] [2,0,2] Z/7+Z ['m022']
eLMkbcdddhhhml_1221 L 1 2 E 1 [4] [2,0,2] Z/3+Z ['m023']
eLMkbcdddhhqqa_1220 L 1 2 E 1 [4] [2,1,1] Z/2+Z/2+Z ['m136']
eLMkbcdddhhqxh_1220 L 1 2 N 1 [2] [2,1,1] Z/2+Z/4+Z ['m135']
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eLMkbcdddhxqlm_1200 L 1 2 N 1 [2] [4,0,0] Z/3+Z/3+Z ['m207','otet04_00003']
eLPkaccddjnkaj_2002 L 1 1 N 1 [2] [2,0,2] Z/3+Z ['m036']
eLPkbcdddhrrcv_1200 L 1 1 E 1 [4] [2,2,0] Z ['m038']
fLAMcaccdeejsnaxk_20010 N 1 1 N 1 [2] [2,2,1] Z ['m166']
fLAMcaccdeennjaqj_10020 N 1 1 N 1 [2] [2,1,2] Z ['m276','K5_17']
fLAMcbbcdeedhwqqs_21020 N 1 1 N 1 [2] [2,0,3] Z ['m083']
fLAMcbccdeehhsqqj_12202 N 1 1 N 1 [2] [2,2,1] Z/2+Z ['m227']
fLLQcacdedejkaajk_20021 L 1 1 N 1 [2] [2,1,2] Z ['m229']
fLLQcaedeedjxahqj_20021 L 1 1 N 1 [2] [2,1,2] Z/2+Z ['m304']
fLLQcbcdeeelonlel_02211 L 1 2 N 1 [2] [4,1,0] Z ['m389','10_139','K5_22','K10n27']
fLLQcbddeeehrcdui_12000 L 1 1 E 1 [4] [4,1,0] Z ['m390']
fLLQcbecdeehhnkei_12001 L 1 1 N 1 [2] [2,1,2] Z ['m232']
fLLQcbecdeepuwsua_20102 L 2 1 N 1 [2,2] [4,1,0] Z+Z ['m367','7^2_1','L7a6']
fLLQcbeddeehhbghh_01110 L 2 1 N 1 [2,2] [2,0,3] Z+Z ['m125','L13n5885','ooct01_00000']
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fLLQcbeddeehhnkhh_21112 L 1 2 E 1 [4] [2,3,0] Z/2+Z ['m140']
fLLQccecddehqrwjj_20102 L 1 1 N 1 [2] [2,0,3] Z ['m115']
fLMPcbcdeeehhhhkn_12211 L 1 2 N 1 [2] [2,0,3] Z/8+Z ['m040']
fLMPcbcdeeehhhhvc_12211 L 1 2 E 1 [4] [2,0,3] Z/4+Z ['m039']
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fLLQcbeddeehhbghh_01110 L 2 1 N 1 [2,2] [2,0,3] Z+Z ['m125','L13n5885','ooct01_00000']
fLLQcbeddeehhcvxx_12001 L 1 1 E 1 [4] [2,2,1] Z/2+Z ['m305']
fLLQcbeddeehhnkhh_21112 L 1 2 E 1 [4] [2,3,0] Z/2+Z ['m140']
fLLQccecddehqrwjj_20102 L 1 1 N 1 [2] [2,0,3] Z ['m115']
fLMPcbcdeeehhhhkn_12211 L 1 2 N 1 [2] [2,0,3] Z/8+Z ['m040']
fLMPcbcdeeehhhhvc_12211 L 1 2 E 1 [4] [2,0,3] Z/4+Z ['m039']
fLMPcbcdeeehhhqrb_12210 L 1 2 N 1 [2] [2,1,2] Z/10+Z ['m234']
fLMPcbcdeeehhhqwg_12210 L 1 2 E 1 [4] [2,1,2] Z/6+Z ['m235']
fLMPcbcdeeehhqqnk_12200 L 1 2 E 1 [4] [4,0,1] Z/8+Z ['m370']
fLMPcbcdeeehhqqvc_12200 L 1 2 N 1 [2] [4,0,1] Z/12+Z ['m369']
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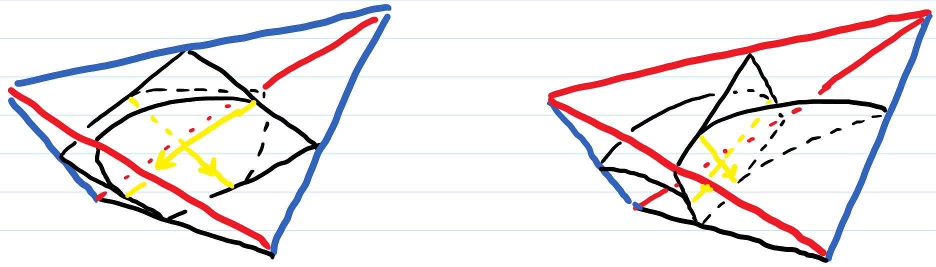
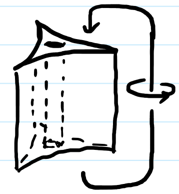
+ 87010 more

Veering triangulation $\longleftrightarrow$ Unstable branched surface



Veering triangulation $\rightsquigarrow$ branched surface B



Properties: 1. All sectors are discs &
all compl. regions are punctured cusped tori

2. Components of the branch locus can be oriented s.t. at each double point it points from side with more sectors to side with less sectors

Observation: 1 & 2 characterize these branched surfaces!

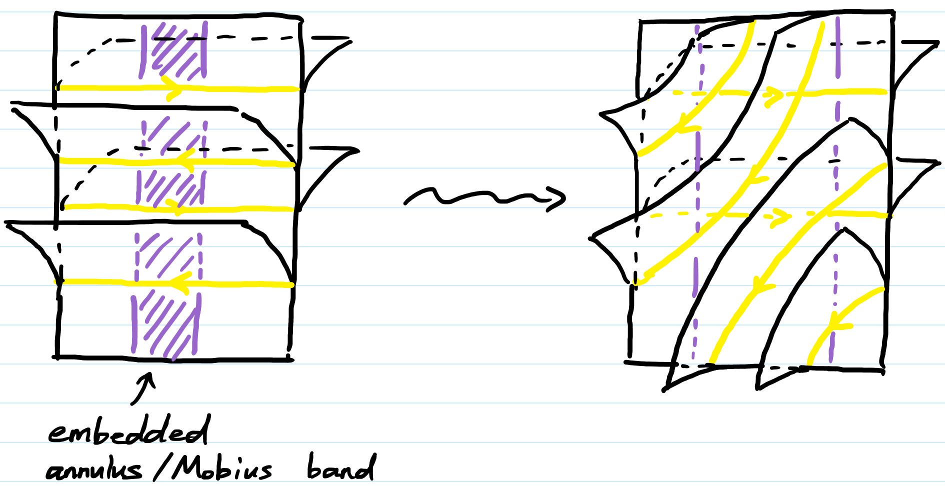
$\Rightarrow$ Veering triangulations $\longleftrightarrow$ 'Veering branched surfaces'
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\parallel$
Branched surfaces satisfying 1 & 2

Surgery of veering triangulations

Idea: Veering branched surface on original mfld

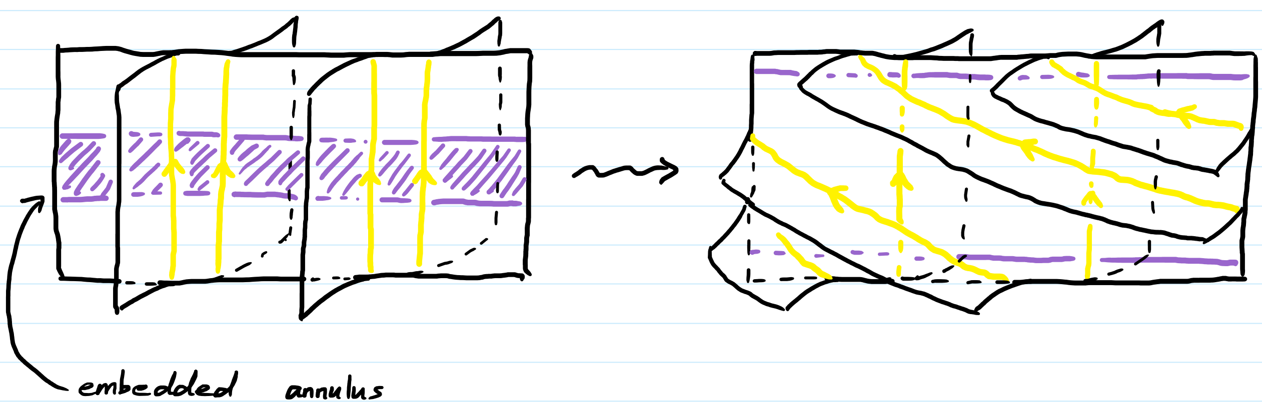
⇝ Veering branched surface on (some) Dehn surgeries

① Vertical surgery



e.g. when the flow graph is not strongly connected (Agol - T.)

② Horizontal surgery (previously discovered by Schleimer - Segerman & called 'veering surgery')

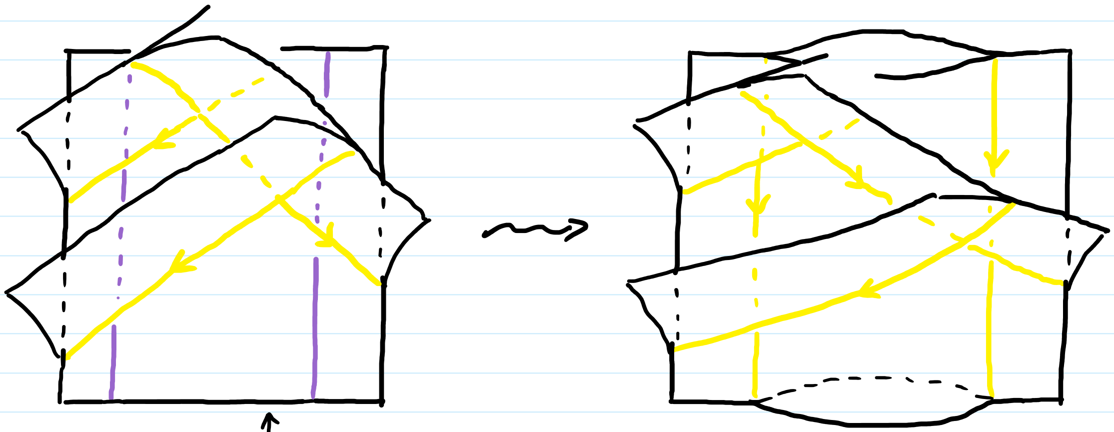


① & ② good for producing non-layered (even non-measurable) v.t.

Related conjecture (Giannopolous - Scheimer - Segerman):

Proportion of non-measurable v.t. $\to 0$ (# tetrahedra $\to \infty$)

③ Vertical drilling



Isotopy class of core can be chosen to be any cycle carried by the flow graph

Theorem (Fried, T.): Any v.t. can drilled to become layered

④ Vertical filling

Reverse of vertical drilling

Takes a 2-cusped torus compl. region & flattens it

Much more subtle: not all 2-cusped torus compl. regions can be filled

Question: Is there a combinatorial condition that governs which 2-cusped torus compl. regions can be filled?

Future directions

① How to construct veering branched surfaces from scratch?

- Sutured hierarchies (ongoing project with Landry)

  [Mosher 1996]: Sutured hierarchy on atoroidal 3-mfld

  $\rightsquigarrow$ almost transverse pseudo-Anosov flow

- $\mathbb{R}$-covered foliations

  [Calegari 2000, Fenley 2002, 2012]:

  $\mathbb{R}$-covered foliation on atoroidal 3-mfld

  $\rightsquigarrow$ transverse pseudo-Anosov flow without perfect fits

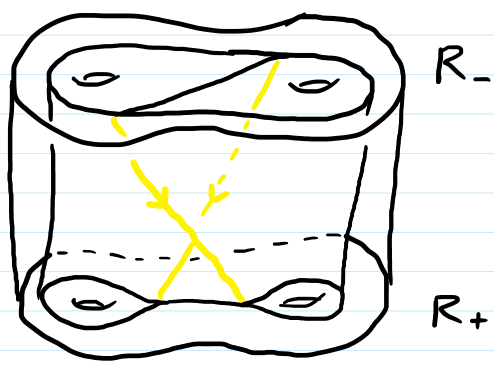
② Relative version?

Veering branched surface in a sutured manifold

:= 1. All sectors are discs &
all compl. regions are punctured cusped tori / drums

2. Components of branch locus can be oriented s.t.
at each double point, goes from side with more sectors to side with less sectors &
points inwards along R_- & outwards along R_+

Category:



Objects = surfaces with train tracks

Morphisms = sutured manifolds with veering branched surfaces

Trace $\overset{!}{\rightsquigarrow}$ veering triangulations

'TQFTs' ??? $\rightsquigarrow$ invariants of veering triangulations

$\overset{!}{\rightsquigarrow}$ invariants of pseudo-Anosov flows without perfect fits