BIRKHOFF SECTIONS FOR ANOSOV FLOWS

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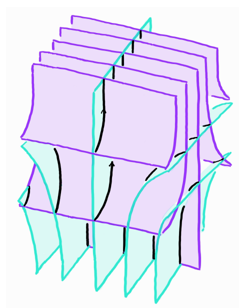
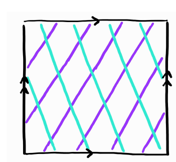
Definition. An **Anosov flow** on a closed 3-manifold M is a flow ϕ^t for which there exists a **stable foliation** Λ^s and an **unstable foliation** Λ^u intersecting transversely along flow lines, such that ϕ^t contracts along Λ^s and expands along Λ^u.



Example. Take a matrix $A \in SL_2\mathbb{Z}$ with distinct real eigenvalues $\lambda > 1, \lambda^{-1} < 1$. A acts on $\mathbb{R}^2$ preserving $\mathbb{Z}^2$, hence descends to a homeomorphism on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, which we also denote by A.

Take the mapping torus of A, i.e. take $T^2 \times [0,1]$ and glue $T^2 \times \{1\}$ to $T^2 \times \{0\}$ by A. The vertical vector field induces the **suspension flow** ϕ_A.



Let Λ^s and Λ^u be the foliations obtained by suspending the straight lines in the λ^{-1}- and λ-eigendirections respectively. ϕ_A is an Anosov flow with stable/unstable foliation $\Lambda^{s/u}$.

In this case, $T = T^2 \times \{0\}$ is said to be a **(global) section**. It satisfies the following properties:

- T is positively transverse to the flow.
- Every orbit of ϕ intersects T^2 in finite foward and finite backward time, that is, for every $x \in M$, there exists $t_1, t_2 > 0$ such that $\phi_{t_1}(x) \in T$ and $\phi_{-t_2}(x) \in T$.

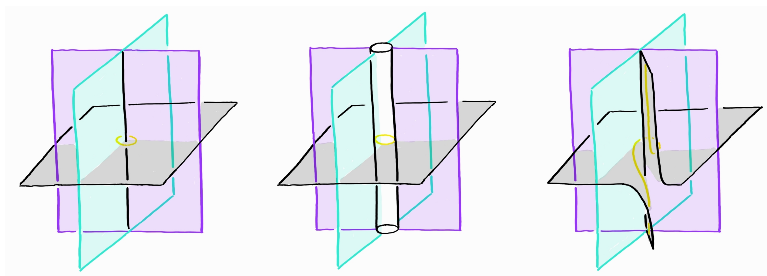
The **first return map** defined on T by following the flow lines recovers the homeomorphism A.

One can construct new Anosov flows using Goodman-Fried surgery.

Construction. Let γ be a closed orbit of an Anosov flow ϕ^t.

The blow-up of ϕ^t along γ is obtained by replacing γ with the unit normal bundle of γ. The dynamics on the blown-up torus T_γ is as follows:

- There is a closed orbit for each half-leaf of Λ^s and Λ^u that contains γ.
- Every other orbit spirals out of a closed orbit determined by a half-leaf of Λ^s and spirals into a closed orbit determined by a half-leaf of Λ^u.



Let μ be the meridian on T_γ. Let λ be the homotopy class of a closed orbit on T_γ. For every $k \in \mathbb{Z}$, we can blow down the blown-up flow by collapsing the torus along the slope $\mu + k\lambda$. The torus blows down to a closed orbit. One can show that the resulting flow $\overline{\phi}$ is Anosov. The effect on the 3-manifold is a Dehn surgery along γ.

We say that $\overline{\phi}$ is obtained from ϕ by **Goodman-Fried surgery** along γ.

Suppose we perform Goodman-Fried surgery on a suspension flow ϕ_A along a closed orbit γ. The global section T becomes a surface with boundary S satisfying the following properties:

- The interior of S is positively transverse to the orbits of ϕ
- The boundary of S is a union of closed orbits of ϕ.
- Every orbit of ϕ intersects S in finite foward and finite backward time, that is, for every $x \in M$, there exists $t_1, t_2 > 0$ such that $\phi_{t_1}(x) \in S$ and $\phi_{-t_2}(x) \in S$.

A surface S satisfying these properties is called a **Birkhoff section**.

Conversely, given a Birkhoff section S for an Anosov flow, we can blow up the closed orbits that ∂S lies along then collapse along the slope determined by ∂S to get a suspension flow with section $\overline{S}$. The surface $\overline{S}$ can be obtained by collapsing each boundary component of S to a point. The first return map on $\overline{S}$ is in general some pseudo-Anosov map f. We will abuse notation slightly and say that f is the **first return map** of S.

Upshot: Birkhoff sections allow one to reduce problems about general Anosov flows to surface homeomorphisms.

Theorem 1 (Fried 1983)**.** *Every transitive Anosov flow admits a Birkhoff section.*

Here recall that an Anosov flow is **transitive** if the set of closed orbits is dense. Transitivity is necessary in this theorem.

Observe that the more 'simple' a Birkhoff section S is, the more 'simple' the resulting section $\overline{S}$ will be. Hence, given existence, the next natural problem is to ask for the 'simplest' Birkhoff section possible.

We discuss this problem under two measures of simplicity today:

(1) The number of boundary components of S.

(2) The genus of S.

The problem of obtaining Birkhoff sections with the minimum number of boundary components has been solved completely.

Theorem 2 (T. 2022)**.** *Every transitive Anosov flow admits a Birkhoff section with two boundary components.*

Moreover, it is known for which flows is it possible to do better. To discuss this classification, we have to make a definition.

Definition. Let ϕ be an Anosov flow on a closed 3-manifold M. Lift this to a flow $\widetilde{\phi}$ in the universal cover $\widetilde{M}$.

One can show that the space of orbits of $\widetilde{\phi}$ (with the quotient topology), which we denote by $\mathcal{O}$, is homeomorphic to $\mathbb{R}^2$. The lifted stable/unstable foliations $\widetilde{\Lambda^{s/u}}$ project down to 1-dimensional foliations $\mathcal{O}^{s/u}$ on $\mathcal{O}$.

The **orbit space** of ϕ refers to the data of $\mathcal{O}$ together with $\mathcal{O}^{s/u}$.

Theorem 3 (Barbot 1995)**.** *Let ϕ be an Anosov flow on a closed 3-manifold M. Exactly one of the following three cases is true:*

(1) The orbit space $\mathcal{O}$ is homeomorphic to $\mathbb{R}^2$ foliated by vertical and horizontal lines. In this case ϕ is the suspension flow of some map on the torus.

*(2) The orbit space $\mathcal{O}$ is homeomorphic to the diagonal strip $\{(x, y) \in \mathbb{R}^2 \mid x < y < x + 1\}$ foliated by vertical and horizontal lines. In this case ϕ is said to be **skew**.*

(3) The space of leaves of $\mathcal{O}^s$ and $\mathcal{O}^u$ are non-Hausdorff.

Theorem 4 (Marty 2023)**.** *A transitive Anosov flow admits a Birkhoff section with exactly one boundary component if and only if it is skew.*

Ideas in the proof of Theorem 2: Use a veering triangulation to encode the dynamics of ϕ. Construct transverse surfaces from combinatorial data of the veering triangulation. Solve the combinatorial equations which allows one to combine these surfaces into a Birkhoff section.

In comparison, much less is known about the problem of obtaining Birkhoff sections with minimum genus.

The smallest possible genus in general is one. This is because if an Anosov flow admits a genus zero Birkhoff section, then its stable and unstable foliations must be non-orientable.

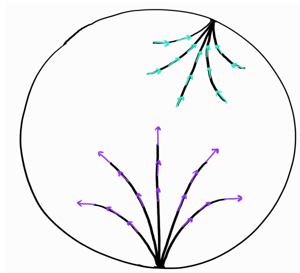
Conjecture 5 (Ghys)**.** *Every transitive Anosov flow with orientable stable and unstable foliations admits a genus one Birkhoff section.*

We survey some results.

Geodesic flows. Let Σ be a hyperbolic surface. Consider the unit tangent bundle $T^1\Sigma = \{v \in T\Sigma \mid ||v|| = 1\}$. The **geodesic flow** ϕ_Σ on $T^1\Sigma$ is defined by

$$\phi^t_\Sigma(v) = \text{time } t \text{ velocity of the geodesic with initial velocity } v.$$

ϕ_Σ is an Anosov flow; the leaves of $\Lambda^{s/u}$ are tangents of geodesics with a common forward/backward endpoint on $\partial_\infty \mathbb{H}^2$.



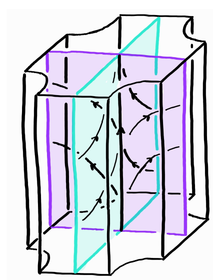
Theorem 6 (Dehornoy-Shannon 2019). *Geodesic flows of hyperbolic surfaces admit genus one Birkhoff sections.*

Totally periodic flows. A graph manifold is a closed 3-manifold that is the union of Seifert fibered spaces. An Anosov flow ϕ on a graph manifold M is totally periodic is for each Seifert fibered space P in M, the fibers in P are homotopic to closed orbits of ϕ.

Alternatively, totally periodic Anosov flows, at least those with orientable stable and unstable foliations, are those Anosov flows built out of round handles. Here a round handle is the manifold

$$\{(x, y, z) \in [-1, 1]^2 \times S^1 \mid |xy| \leq \frac{1}{4}\}$$

with the semiflow generated by $x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}$.



Each round handle has

- 4 annulus faces where the flow is tangent along the boundary,
- 2 annulus faces where the flow points out of the manifold, and
- 2 annulus faces where the flow points into the manifold.

The round handles are first glued along their tangent boundaries to give Seifert fibered spaces, then glued along their transverse boundaries to give graph manifolds.

Theorem 7 (T. 2024). *Every transitive totally periodic Anosov flow with orientable stable and unstable foliations admits a genus one Birkhoff section.*

Penner type pseudo-Anosov maps. Let α and β be two multicurves that fill a closed surface S. A **Penner type pseudo-Anosov map** is a map of the form $\sigma\tau_\alpha^{m_1}\tau_\beta^{-n_1}\tau_\alpha^{m_2}\tau_\beta^{-n_2}\ldots$, where

- each $\tau_\alpha^{m_i}$ is a product of positive Dehn twists along the curves in α,
- each $\tau_\beta^{-n_i}$ is a product of negative Dehn twists along the curves in β, and
- σ is an automorphism of S preserving α and β.

Penner type pseudo-Anosov maps are in some sense the most explicit types of pseudo-Anosov maps one can get their hands on. However, it is known that not every pseudo-Anosov map is of Penner type.

Theorem 8 (T. 2024)**.** *Suppose ϕ is an Anosov flow with orientable stable and unstable foliations. If ϕ admits a Birkhoff section whose first return map is a Penner type pseudo-Anosov map, then ϕ admits a genus one Birkhoff section.*

Dehornoy also has some other results in preparation.

Ideas in the proofs of Theorem 7 and Theorem 8: Generalize Goodman-Fried surgery to *horizontal Goodman surgery*, where one cuts along an annulus transverse to the flow then reglues to get another Anosov flow.

Show that performing horizontal Goodman surgery on a transitive Anosov flow gives an almost equivalent flow. In particular the original flow admits a genus one Birkhoff section if and only if the surgered flow admits a genus one Birkhoff section.

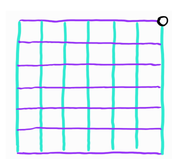
For Theorem 7, perform horizontal Goodman surgery along tori in the Seifert fibered spaces to transform them into totally periodic Anosov flows of a certain form, then show that those admit genus one Birkhoff sections.

For Theorem 8, use the correspondence theory between (pseudo-)Anosov flows and veering triangulations. Show that horizontal Goodman surgery corresponds to a certain horizontal surgery operation on veering triangulations. Use this operation to transform the veering triangulation for ϕ into that of a suspension Anosov flow.

A more in-depth sketch of proof of Theorem 2.

Let ϕ be an Anosov flow on an oriented closed 3-manifold M. Let $\mathcal{C}$ be a finite collection of closed orbits of ϕ which includes the singular orbits. The lift of $\mathcal{C}$ to $\widetilde{M}$ determines a collection of points $\widetilde{\mathcal{C}}$ on $\mathcal{O}$. Suppose that ϕ has no perfect fits relative to $\mathcal{C}$.

Here, a perfect fit rectangle is a properly embedded copy of $([0,1] \times [0,1]) \backslash \{(1,1)\}$ foliated by vertical and horizontal lines. We say that ϕ has **no perfect fits relative to** $\mathcal{C}$ if every perfect fit rectangle in $\mathcal{O}$ contains a point of $\widetilde{\mathcal{C}}$.



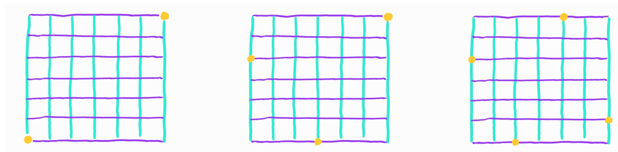
Proposition 9 (T. 2022)**.** *Such a collection $\mathcal{C}$ exists if and only if ϕ is transitive.*

Definition 10. An **edge rectangle** in $\mathcal{O}$ is a rectangle that meets $\widetilde{\mathcal{C}}$ in two opposite vertices.

A **face rectangle** in $\mathcal{O}$ is a rectangle that meets $\widetilde{\mathcal{C}}$ in one vertex and in one point in the interior of each of the two opposite sides.

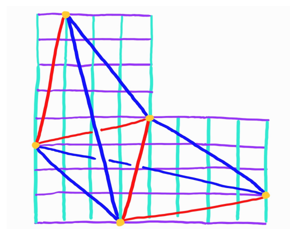
A **tetrahedron rectangle** in $\mathcal{O}$ is a rectangle that meets $\widetilde{\mathcal{C}}$ in one point in the interior of each of the four sides.

To each tetrahedron rectangle R we associate an ideal tetrahedron t_R. The ideal vertices of t_R correspond to the points of $\widetilde{\mathcal{C}}$ on R. The edges of t_R correspond to the edge rectangles contained in R. The faces of t_R correspond to the face rectangles contained in R.



We glue the tetrahedra t_{R_1} and t_{R_2} along a face if R_1 and R_2 intersect in the corresponding face rectangle.

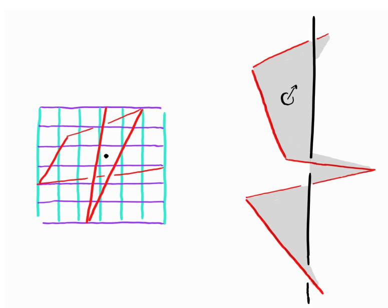
Finally, we color an edge red/blue if it has positive/negative slope, respectively.



This defines an ideal triangulation $\widetilde{\Delta}$ of $\widetilde{M}\backslash\widetilde{\mathcal{C}}$ that is invariant under the $\pi_1 M$ action. Taking the quotient gives a *veering triangulation* Δ on $M\backslash\mathcal{C}$.

Meanwhile, we define a **winding red/blue edge path** to be a path of red/blue edges of $\widetilde{\Delta}$ that winds around the lift of a closed orbit γ. Notice that a cyclic sequence of red/blue edges in Δ can be lifted to a winding red/blue edge path. Moreover, there is a freedom on 'how much we follow' each element of $\mathcal{C}$ when we lift.

Each winding edge path bounds a helicoidal transverse surface in $\widetilde{M}$. If the winding edge path comes from the lift of a cyclic sequence of edges in Δ, then the helicoid descends down to an immersed helicoid bounded by the sequence of edges and a closed orbit.



We now take a linear combination of faces of Δ so that their union determines a surface with two boundary components, where one boundary component is a cyclic sequence of red edges while the other boundary components is a cyclic sequence of blue edges. We take the union of this surface with helicoids associated to the two cyclic sequences as constructed above. Finally, we resolve the self-intersections.