HEEGAARD FLOER THEORY AND PSEUDO-ANOSOV FLOWS

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0. Context and motivation

Recurring theme: Dynamical systems give quantitative topological invariants of the underlying manifold.

For example:

- Let V be a vector field on a closed manifold M. Then the signed count of the zeros of V equals the Euler characteristic of M [Poincaré-Hopf].
- Let f be a real-valued function on a closed manifold M. Then the homology of the Morse complex, generated by the zeros of f, equals the homology of M [Morse]. This is a categorification of the previous fact.
- Let f be an orientation-preserving homeomorphism of a closed surface S. Then the signed count of the number of fixed points of f, or equivalently, the number of period 1 orbits of the suspension flow ϕ_f on the mapping torus M_f, equals the 'second to top' coefficient of the Alexander polynomial [Lefschetz].
- Let ϕ be a Reeb flow on a closed 3-manifold M. Then the homology of the ECH complex, generated by closed (multi)orbits of ϕ, equals the Heegaard Floer homology of M [Hutchings, Kutluhan-Lee-Taubes]. This is a categorification of the previous fact.

An important class of dynamical systems in 3-manifolds is pseudo-Anosov flows. Recall that a flow ϕ on a closed 3-manifold M is **pseudo-Anosov** if these exists a pair of transverse singular 2-dimensional foliations (Λ^s, Λ^u) on M such that:

- the leaves of Λ^s and Λ^u intersect in flow lines,
- the flow lines converges along the leaves of Λ^s, and
- the flow lines diverge along the leaves of Λ^u.

These were introduced by Thurston in the 1980s. Subsequent work by Barbot, Fenley, and others showed that the data of a pseudo-Anosov flow gives qualitative topological information of the underlying 3-manifold, for example irreducibility, atoroidality, hyperbolicity.

Motivating question: Can one extract quantitative topological invariants from a pseudo-Anosov flow?

1. From pseudo-Anosov flows to veering branched surfaces

Let ϕ be a pseudo-Anosov flow on a closed 3-manifold M. Consider the lifted flow $\widetilde{\phi}$ on $\widetilde{M}$. It can be shown that the map $\widetilde{M} \to \mathcal{O}$ quotienting each flow line to a point is a fiber bundle $\mathbb{R}^3 \to \mathbb{R}^2$. The space $\mathcal{O}$ is called the orbit space of ϕ. The pair of lifted foliations $(\widetilde{\Lambda^s}, \widetilde{\Lambda^u})$ determine a pair of transverse singular 1-dimensional foliations $(\mathcal{O}^s, \mathcal{O}^u)$ on $\mathcal{O}$.

Suppose $\mathcal{C}$ is a collection of closed orbits of ϕ. The set of all preimages $\widetilde{\mathcal{C}}$ determines a collection of points in $\mathcal{O}$. A **perfect fit rectangle** in $\mathcal{O}$ is a properly embedded copy of $[0,1]^2 \times \{(0,0)\}$. We say that ϕ has **no perfect fits** relative to $\mathcal{C}$ if every perfect fit rectangle in $\mathcal{O}$ intersects a point of $\widetilde{\mathcal{C}}$.

Proposition 1 (T. 2024)**.** *For every transitive pseudo-Anosov flow ϕ, there exists a finite collection of closed orbits $\mathcal{C}$ such that ϕ has no perfect fits relative to $\mathcal{C}$.*

Idea of proof. Pick a dense enough collection of orbits. In fact, one can just pick one dense enough orbit. □

Here, a flow is **transitive** if there is a dense orbit. A pseudo-Anosov flow is non-transitive if and only if it admits a transverse separating torus.

We now explain the Agol-Guéritaud construction: Suppose M is oriented. Suppose ϕ is transitive. Pick a collection $\mathcal{C}$ as in Proposition 1. Up to enlarging $\mathcal{C}$, suppose $\mathcal{C}$ contains all singular orbits and is nonempty.

We define:

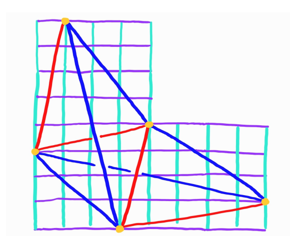
- An **edge rectangle** to be an embedded copy of $[0,1]^2$ in $\mathcal{O}$ that has a point of $\widetilde{\mathcal{C}}$ on each of two opposite corners.
- A **face rectangle** to be an embedded copy of $[0,1]^2$ in $\mathcal{O}$ that has a point of $\widetilde{\mathcal{C}}$ on one corner and two of the sides opposite to that corner.
- A **tetrahedron rectangle** to be an embedded copy of $[0,1]^2$ in $\mathcal{O}$ that has a point of $\widetilde{\mathcal{C}}$ on each of its four sides.

The no perfect fit condition ensures that every point in $\mathcal{O} \backslash \widetilde{\mathcal{C}}$ lies in the interior of a tetrahedron rectangle.

We built an abstract triangulation $\overline{\Delta}$ as follows: Take a tetrahedron $\overline{t}_R$ for each tetrahedron rectangle R. Fix a bijection between the four vertices of $\overline{t}_R$ with the four points of $\widetilde{\mathcal{C}}$ on the boundary of R. This induces a bijection between:

- The six edges of $\overline{t}_R$ with the six edge subrectangles of R.
- The four faces of $\overline{t}_R$ with the four face subrectangles of R.

Whenever two tetrahedron rectangles R_1 and R_2 overlap in face subrectangles of each other, we glue $\overline{t}_{R_1}$ and $\overline{t}_{R_2}$ along the corresponding faces.



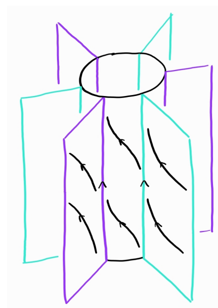
There is a (nonunique) map $\pi : \overline{\Delta} \to \mathcal{O}$ mapping each edge/face/tetrahedron within the corresponding edge/face/tetrahedron rectangle. The preimage of each point in $\widetilde{\mathcal{C}}$ is a vertex, while the preimage of each point in $\mathcal{O} \backslash \widetilde{\mathcal{C}}$ is a line. We define $\widehat{\Delta}$ to be $\overline{\Delta}$ with all the vertices removed. Then $\pi : \widehat{\Delta} \to \mathcal{O} \backslash \mathcal{C}$ is a fiber bundle.

Moreover, $\pi_1 M$ acts on $\widehat{\Delta}$ preserving π. Taking the quotient, we have a flow on a 3-manifold $\Delta = \widehat{\Delta}/\pi_1 M$. It can be shown that this flow is isotopy equivalent to the restriction of the original pseudo-Anosov flow ϕ on $M \backslash \mathcal{C}$. Here, two flows are **isotopy equivalent** if they differ by isotopy and reparametrization.

Making the identification between Δ and $M \backslash \mathcal{C}$, we now have an ideal triangulation Δ in M whose faces are positively transverse to the flow lines of ϕ.

We refer to the ideal triangulation Δ as the **veering triangulation** associated to $(\phi, \mathcal{C})$.

We caution that Δ is a triangulation of $M \backslash \mathcal{C}$ and *not* M. Therefore it makes sense for us to refine ϕ into an object on $M \backslash \mathcal{C}$ as well: We define $\phi^\sharp$ to be the **blow-up** of ϕ at each orbit of $\mathcal{C}$.



Proposition 2 (Landry-Minsky-Taylor 2023)**.** *Every closed orbit of $\phi^\sharp$ is positively transverse to the faces of Δ. Conversely, every closed curve that is transverse to the faces of Δ is homotopic to a closed orbit of $\phi^\sharp$.*

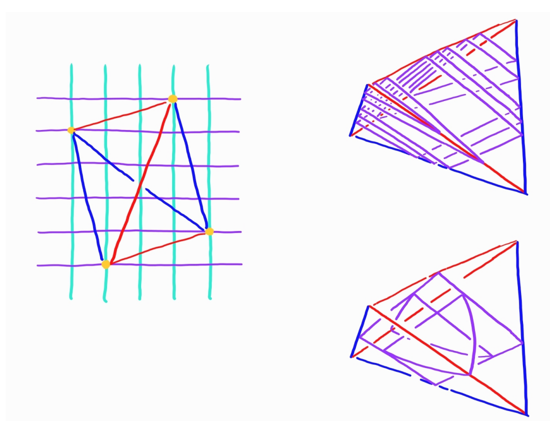
Proof. The first statement follows from the construction.

For the second statement, given such a closed curve c, we take one of its lifts $\widetilde{c}$. Suppose $\widetilde{c}$ passes through a sequence of tetrahedra $(t_{R_i})_{i \in \mathbb{Z}}$. Consider the sequence of corresponding rectangles R_i. Since $t_{R_{i+1}}$ lies above t_{R_i}, R_{i+1} is taller and narrower than R_i. This implies that the intersection $\bigcap R_i$ is a nonempty subrectangle.

Any corner point of $\bigcap R_i$ gives a flow line of $\widetilde{\phi}$ that is $[c]$-invariant. Thus descends to a closed orbit homotopic to c. In fact, one can show that $\bigcap R_i$ is actually just a single point. $\square$

The data of a veering triangulation can be dualized into a branched surface. More specifically,

We construct the **dual branched surface** B of Δ by placing a piece of a branched surface within each tetrahedron as follows:



Proposition 3. *We have the following properties of B:*

*(1) The branch locus of B is a 4-valent graph. We orient the edges of this graph upwards. We call the resulting directed graph the **dual graph** and denote it by G.*

(2) The sectors of B are diamonds with one top corner, one bottom corner, and two side corners.

(3) The complementary regions of B in $M \backslash \nu(\mathcal{C})$ are cusped torus shells.

(4) The orientations of each edge of the branch locus induce the maw coorientation on the other edges at each triple point.

Since B is dual to Δ, Proposition 2 has the following corollary.

Corollary 4. *Every closed orbit of ϕ is positively transverse to a directed cycle of G. Conversely, every directed cycle of G is homotopic to a closed orbit.*

Corollary 4 can be reformulated as: Let $\mathcal{F} : \{\text{directed cycles of } G\} \to \{\text{closed orbits of } \phi^\sharp\}$ be the function obtained by mapping a cycle to the closed orbit that it is homotopic to. Then $\mathcal{F}$ is surjective. However, $\mathcal{F}$ is not injective: If two cycles c_1 and c_2 of G are related by sweeping across a sector, then they are homotopic thus have the same image under $\mathcal{F}$. The following proposition states that this is the only source of non-injectivity.

Proposition 5 (Landry-Minsky-Taylor 2023)**.** *For every closed orbit γ of $\phi^\sharp$, the preimage $\mathcal{F}^{-1}(\gamma)$ is a sweep-equivalence class.*

2. From veering branched surfaces to Heegaard diagrams

Recall that a **sutured manifold** is a compact oriented 3-manifold with boundary Q, whose boundary is decomposed into two subsurfaces meeting along a collection of closed curves

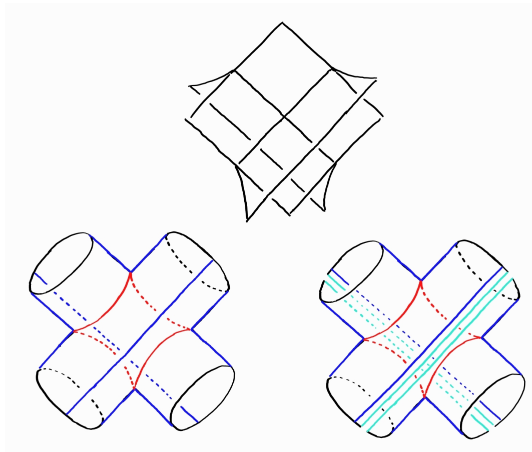
$$\partial Q = R_+ \cup_\gamma R_-$$

where γ are called the **sutures**.

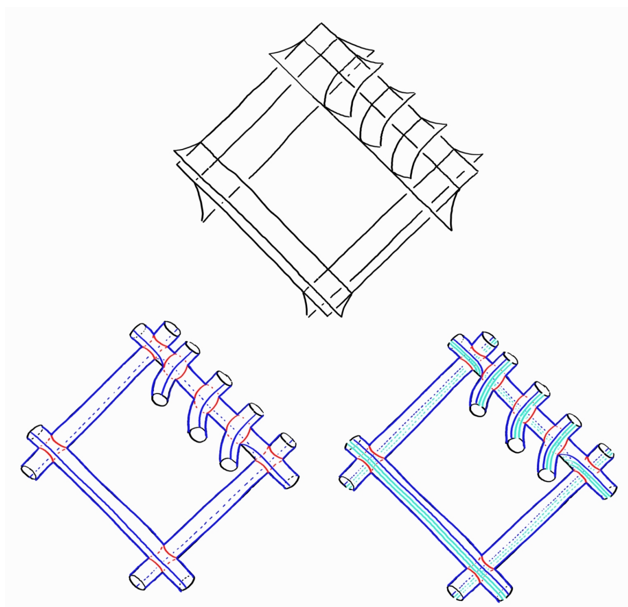
Given a compact surface with boundary Σ and two multicurves α and β on Σ, the sutured manifold $Q(\Sigma, \alpha, \beta)$ is obtained from $\Sigma \times [0,1]$ by attaching 2-handles along $\alpha \times \{0\}$ and $\beta \times \{1\}$ and declaring $\partial\Sigma \times [0,1]$ to be the sutures. A **Heegaard diagram** for a sutured manifold Q is a triple (Σ, α, β) for which $|\alpha| = |\beta|$ and $Q = Q(\Sigma, \alpha, \beta)$.

We now return to the setting of the previous subsection. We define the sutured manifold $Q(\phi, \mathcal{C})$ to be $M \backslash \nu(\mathcal{C})$ with one suture curve along each boundary closed orbit of $\phi^\sharp$. We will explain how to construct a Heegaard diagram for $Q(\phi, \mathcal{C})$ from the veering branched surface B.

Let U be a neighborhood of $\mathrm{brloc}(B)$. Let Σ_0 be the boundary of U. For each vertex v of $\mathrm{brloc}(B)$, let α_v be a curve on Σ_0 that bounds a disc separating $\mathrm{brloc}(B)$ into two smooth arcs near v. Let α be the union of α_v.

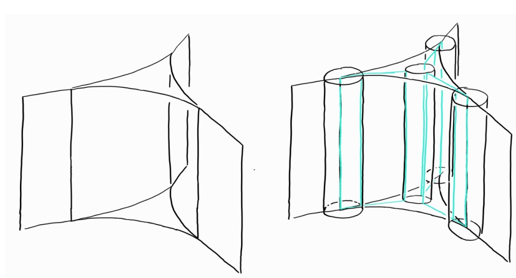


For each sector S of B, let β_S be the intersection of S with Σ_0. Let β be the union of β_S.

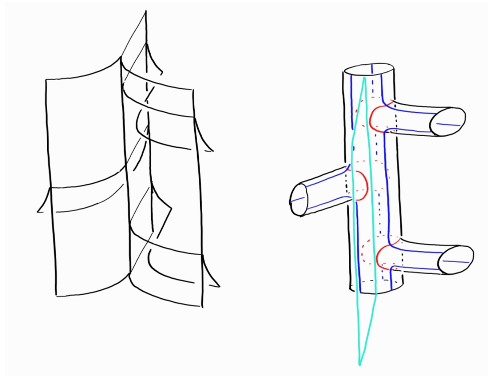


Attaching 2-handles to Σ_0 along α and β gives the 3-manifold N obtained by removing neighborhoods of $\mathcal{C}$ and neighborhoods of the branch cycle of B from M.

At each $\gamma \in \mathcal{C}$, there exists a collection of annuli each with one boundary component along γ and the other on a branch cycle.



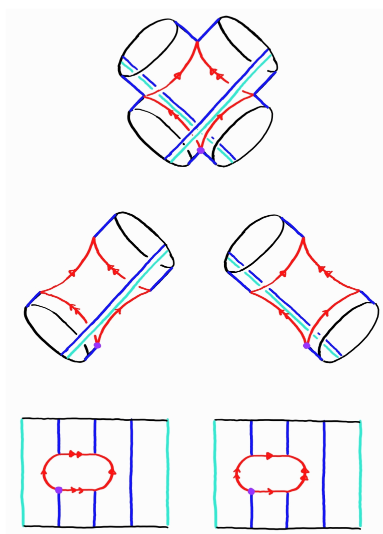
Each annulus intersects Σ_0 in a curve that is disjoint from $\alpha \cup \beta$.



Let Σ be the surface obtained by cutting Σ_0 along these curves. Then (Σ, α, β) is a Heegaard diagram for the sutured manifold obtained by cutting N along these annuli, which is exactly Q.

We can say more about the structure of the Heegaard diagram (Σ, α, β).

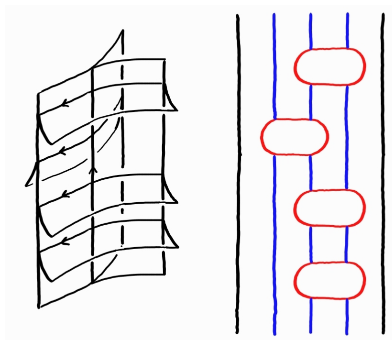
Near each vertex v, Σ_0 is a 4-holed sphere that is cut into two 1-holed cylinders by α_v. Each cylinder corresponds to an segment of a branch loop.



Gluing these cylinders together, we see that $\Sigma_0 \backslash \alpha$ is a union of tori with holes, each torus corresponding to a branch loop, with the holes corresponding to the triple points on the branch loop.

Moreover, on each local cylinder the β arcs take the form of three arcs running from one boundary component to another, with two of the arcs running through the hole cut out by α_v. When glued up, this means that on each torus the β arcs take the form of three curves running the length of the branch loop, with two of the three loops running through each hole.

The cut annuli intersect each torus in a curve that runs the length of the branch loop. Cutting along these curves, we see that $\Sigma \backslash \alpha$ is a union of cylinders with holes. The holes correspond to the triple points. Each cylinder has three curves running the length of the branch loop. Each hole is run through by either the left and middle loop, or the right and middle loop, depending on the local form of B near the corresponding triple point v.



3. From the Heegaard diagram to SFH

We recall the definition of sutured Floer homology: Let (Σ, α, β) be a Heegaard diagram for a sutured manifold Q. Consider the symmetric product $\mathrm{Sym}^g(\Sigma)$ where $g = |\alpha| = |\beta|$. The multicurves α and β determine tori $\mathbb{T}_\alpha = \mathrm{Sym}^g(\alpha)$ and $\mathbb{T}_\beta = \mathrm{Sym}^g(\beta)$. Let S be the set of intersection points between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. We refer to the elements of S as the **states**. Unwinding the definitions, the set of states can be interpreted as

$$\{(x_1, ..., x_g) \mid x_i \in \alpha_{\sigma(i)} \cap \beta_i \text{ for some permutation } \sigma\}.$$

Let $CF(\Sigma, \alpha, \beta)$ be the $\mathbb{F}_2$-chain complex generated by S. The differential $\partial : CF(\Sigma, \alpha, \beta) \to CF(\Sigma, \alpha, \beta)$ is defined by

$$\partial \mathbf{x} = \sum_{\mathbf{y}} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y})} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot \mathbf{y}$$

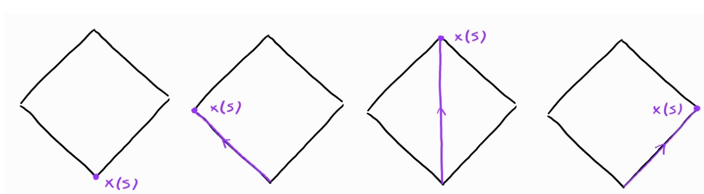
where $\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homotopy classes of holomorphic discs from $\mathbf{x}$ to $\mathbf{y}$ with Maslov index 1 and $\mathcal{M}(\phi)$ is the moduli space of holomorphic discs in the homotopy class ϕ.

Theorem 6 (Juhasz 2006)**.** *The (hat version of)* ***sutured Floer homology*** *$SFH(Q)$ to defined to be the homology of $CF(\Sigma, \alpha, \beta)$. This is independent of the choice of Heegaard diagram for Q.*

We continue the setting from the previous section. Let $CF = CF(\Sigma, \alpha, \beta)$ be the chain complex associated to the Heegaard diagram we constructed.

Let $\mathbf{x}$ be a state of (Σ, α, β). Recall that $\mathbf{x}$ is a way of assigning each β_S to a point of intersection with α in a way so that each α_v is used once. This can be reinterpreted as a way of assigning each sector to one of its corners, in a way so that each vertex is used once.

We define the **augmented dual graph** G_+ to be the union of G and the vertical diagonal in each sector. On each sector S, we let $e_{\mathbf{x},S}$ be the edge of G_+ connecting the bottom corner of S to $\mathbf{x}(S)$.



We define $\mu_{\mathbf{x}}$ to be the multicycle $\bigcup_S e_{\mathbf{x},S}$ of G_+.

Proposition 7. *The assignment $\mathbf{x} \mapsto \mu_{\mathbf{x}}$ defines a one-to-one correspondence between the generators of CF and the embedded directed multicycles of G_+.*

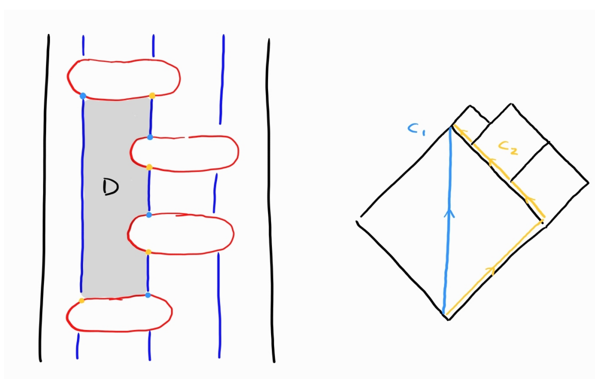
Recall the function $\mathcal{F}$: {directed cycles of G} $\to$ {closed orbits of $\phi^\sharp$}. This can be extended to $\mathcal{F}$: {directed cycles of G_+} $\to$ {closed orbits of $\phi^\sharp$} by sweeping vertical diagonals into G. With this understanding, Proposition 5 generalizes in the obvious way. We define $\gamma_{\mathbf{x}} = \mathcal{F}(\mu_{\mathbf{x}})$.

Example 8. The top state $\mathbf{x}^{\text{top}}$ is the state that assigns each sector to its top corner. The corresponding multicycle $\mu_{\mathbf{x}^{\text{top}}}$ is the union of vertical diagonals.

The bottom state $\mathbf{x}^{\text{bot}}$ is the state that assigns each sector to its bottom corner. The corresponding multicycle $\mu_{\mathbf{x}^{\text{bot}}}$ is the empty multicycle. The corresponding multiorbit $\gamma_{\mathbf{x}^{\text{bot}}}$ is the empty multiorbit.

We turn to analyzing the differential. From the definition, this would require counting holomorphic discs, but this is difficult, so we will study shadows of holomorphic discs instead.

An **(effective) domain** of (Σ, α, β) connecting states $\mathbf{x}$ and $\mathbf{y}$ is an immersed subsurface $D \looparrowright \Sigma$ lying away from $\partial\Sigma$ and with boundary $\partial D = \partial_\alpha D \cup \partial_\beta D$ lying along α and β respectively, so that $\partial\partial_\alpha D = \mathbf{x} - \mathbf{y}$ and $\partial\partial_\beta D = \mathbf{y} - \mathbf{x}$.



Each homotopy class of a holomorphic disc ϕ from $\mathbf{x}$ to $\mathbf{y}$ projects down to a domain $D(\phi)$ from $\mathbf{x}$ to $\mathbf{y}$. In particular, if there are no domains from $\mathbf{x}$ to $\mathbf{y}$, then the coefficient $\sum_{\phi \in \pi_2(\mathbf{x},\mathbf{y})} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right)$ is zero.

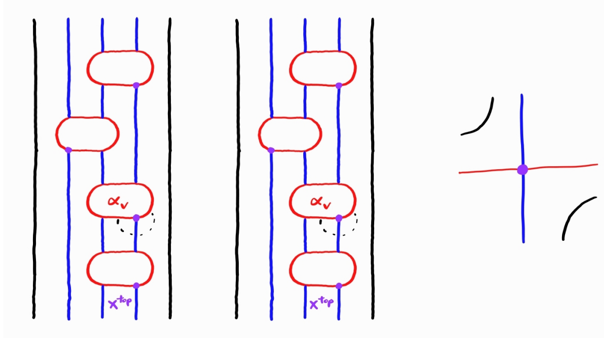
There is a formula to compute the Maslov index of ϕ from the combinatorics of $D(\phi)$, but we will not state it here.

On the other hand, there is no general way to compute $\# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right)$ from $D(\phi)$. However, one important special case is if $D(\phi)$ is an (embedded) polygon. In this case, $\# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right) = 1$. In particular, the domain D in the example above gives a differential $\mathbf{x}_1 \to \mathbf{x}_2$. In general, there are differentials coming from sweeping vertical diagonals across sectors.

The special combinatorics of our Heegaard diagram allows for easy analysis of the domains.

Proposition 9. *There are no domains from $\mathbf{x}^{\text{top}}$ to any state, nor from any state to $\mathbf{x}^{\text{top}}$.*

Proof. Assume there is a domain D from $\mathbf{x}^{\text{top}}$ to some state $\mathbf{y}$. Consider a point of $\mathbf{x}^{\text{top}}$ near a left hole. For $\partial\partial_\beta D = \mathbf{y} - \mathbf{x}^{\text{top}}$ to hold, D cannot pass through the left regions. So D must be a union of right regions.



But then we do not have $\partial\partial_\beta D = \mathbf{y} - \mathbf{x}^{\text{top}}$ as well.

A similar argument shows that there is no domain from any state $\mathbf{y}$ to $\mathbf{x}^{\text{top}}$. $\square$

Corollary 10. *The top state* $\mathbf{x}^{\text{top}}$ *determines a nonzero class in* $SFH(Q)$.

Similarly, one can prove the following.

Proposition 11. *There are no domains from* $\mathbf{x}^{\text{bot}}$ *to any state, nor from any state to* $\mathbf{x}^{\text{bot}}$.

Corollary 12. *The bottom state* $\mathbf{x}^{\text{bot}}$ *determines a nonzero class in* $SFH(Q)$.

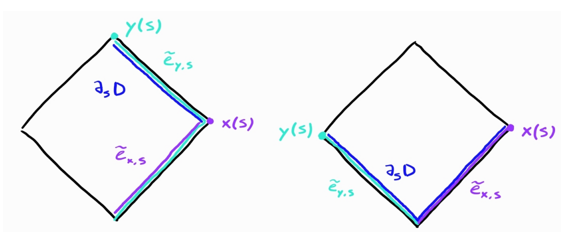
Taking the ideas of these arguments further, we can define a grading $\widetilde{\varepsilon}(\mathbf{x})$ for each state $\mathbf{x}$.

Recall the multicycle $\mu_{\mathbf{x}}$. We define $\widetilde{\mu}_{\mathbf{x}}$ to be the set containing all multicycles of G obtained by sweeping vertical diagonals of $\mu_{\mathbf{x}}$ across a sector. In other words, $\widetilde{\mu}_{\mathbf{x}}$ is the set containing all multicycles of G of the form $\bigcup_S \widetilde{e}_{\mathbf{x},S}$, where $\widetilde{e}_{\mathbf{x},S}$ is a path in G connecting the bottom corner of S to $\mathbf{x}(S)$.

We write $[\widetilde{\mu}_{\mathbf{x}}] \subset H_1(G)$ for the set containing the homology classes of the elements of $\widetilde{\mu}_{\mathbf{x}}$.

Proposition 13. *If there is a domain* D *from* $\mathbf{x}$ *to* $\mathbf{y}$, *then* $[\widetilde{\mu}_{\mathbf{x}}] \cap [\widetilde{\mu}_{\mathbf{y}}] = \varnothing$ *in* $H_1(G)$.

Proof. We write $\partial_\beta D = \sum_S \partial_S D$ where $\partial_S D \subset \beta_S$. By the same reasoning as in the proof of Proposition 9, $\partial_S D$ does not pass through the top corner of S, so we can always choose $\widetilde{e}_{\mathbf{x},S}$ and $\widetilde{e}_{\mathbf{y},S}$ so that $\partial_S D = \widetilde{e}_{\mathbf{x},S} - \widetilde{e}_{\mathbf{y},S}$.



Hence $\partial_\beta D = \sum_S \widetilde{e}_{\mathbf{x},S} - \sum_S \widetilde{e}_{\mathbf{y},S} \in C_1(\Sigma)$.

Meanwhile, note that if we collapse each α-curve on Σ to a point, we get a space homotopy equivalent to G, so $C_1(\Sigma)/C_1(\alpha) = C_1(G)$. Hence $\partial D = \partial_\beta D = \sum_S \widetilde{e}_{\mathbf{x},S} - \sum_S \widetilde{e}_{\mathbf{y},S}$ in $C_1(G)$. $\square$

We declare two states $\mathbf{x}$ and $\mathbf{y}$ to be $\widetilde{\varepsilon}$**-equivalent** if $[\widetilde{\mu}_{\mathbf{x}}] \cap [\widetilde{\mu}_{\mathbf{y}}] = \varnothing$, and we take the transitive closure.

Corollary 14. *The chain complex CF splits into subcomplexes $CF(\widetilde{\mathfrak{s}}) = \bigoplus_{\widetilde{\varepsilon}(\mathbf{x})=\widetilde{\mathfrak{s}}} \mathbb{F}_2 \cdot \mathbf{x}$ where $\widetilde{\mathfrak{s}}$ ranges over all $\widetilde{\varepsilon}$-equivalence classes.*

Note that if $\widetilde{\varepsilon}(\mathbf{x}) = \widetilde{\varepsilon}(\mathbf{y})$, then $[\mu_{\mathbf{x}}] = [\mu_{\mathbf{y}}] \in H_1(Q)$. We define $\varepsilon(\mathbf{x}) = [\mu_{\mathbf{x}}] \in H_1(Q)$. Then the $\widetilde{\varepsilon}$-grading is finer than the ε-grading. The latter is sometimes referred to as the **spin$^{\mathbf{c}}$-grading**.

Corollary 15. *The homology $SFH(Q(\phi, \mathcal{C}))$ splits into subspaces $SFH(Q(\phi, \mathcal{C}), \mathfrak{s})$ where $\mathfrak{s}$ ranges over all ε-equivalence classes.*

Recall the function $\mathcal{F} : \{\text{directed cycles of } G_+\} \to \{\text{closed orbits of } \phi^\sharp\}$, whose preimages are sweep equivalence classes. We say that an orbit γ is **sleek** if are cycles in $\mathcal{F}^{-1}(\gamma)$ are embedded.

From the definitions, if γ is sleek, then the set $\{\mathbf{x} \mid \gamma_{\mathbf{x}} = \gamma\}$ is a $\widetilde{\varepsilon}$-equivalence class $\widetilde{\mathfrak{s}}_\gamma$. Thus we can consider the subcomplex $CF(\gamma) := CF(\widetilde{\mathfrak{s}}_\gamma)$.

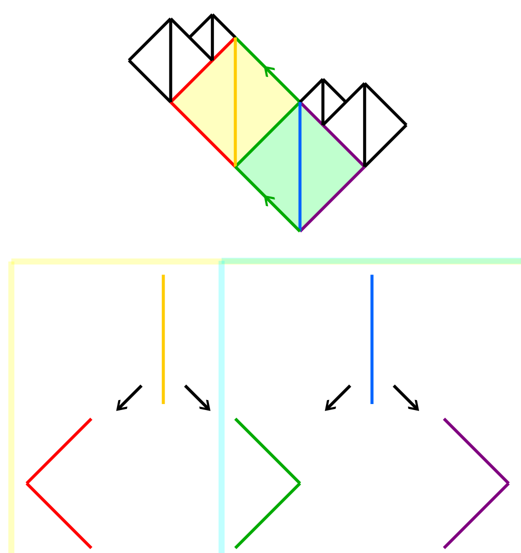
Meanwhile, recall that sweeping vertical diagonals across sectors gives differentials. Let $CC(\gamma)$ be the subcomplex of $CF(\gamma)$ obtained by restricting to these sweeping differentials. A careful inspection of Proposition 13 shows the following proposition.

Proposition 16. *Let γ be a sleek orbit. Then $CC(\gamma) = CF(\gamma)$.*

We can then compute the homology by a combinatorial argument.

Proposition 17. *Let γ be a sleek orbit. Then $H_*(CC(\gamma)) \cong \mathbb{F}_2$.*

Idea of proof. The chain complex $CC(\gamma)$ is a zig-zag. An induction argument shows that its homology has dimension 1.



□

We give one application of this in the case of suspension flows. Let $f : S \to S$ be a pseudo-Anosov map. Let ϕ_f be the suspension flow on the mapping torus M_f. Let $\mathcal{C}$ be the set of singular orbits. It can be shown that ϕ_f has no perfect fits relative to $\mathcal{C}$. Let $\phi_f^\sharp$ be the blow-up of ϕ_f. This is the suspension flow on the mapping torus $M_{f^\sharp}$ of the blown-up map $f^\sharp$. Let P be the minimum period of periodic points of $f^\sharp$.

Corollary 18. *Let P be the minimum period of periodic points of $f^\sharp$. Then the dimension of the k^{th} grading $\bigoplus_{\langle \mathfrak{s},[S^\sharp]\rangle=k} SFH(Q(\phi_f,\mathcal{C}),\mathfrak{s})$ equals the number of period k closed orbits of $\phi_f^\sharp$, for $k=1,...,2P-1$.*

Proof. It suffices to show that for $k=1,...,2P-1$, all period k closed orbits are sleek. If not, then there is a period k cycle c of G_+ that passes through a vertex twice. We can write c as a concatenation of cycles c_1 and c_2, with periods k_1 and k_2 respectively. But we have a contradiction between the facts $k=k_1+k_2$, $k_1,k_2\geq P$, and $k\leq 2P-1$. □

4. Future directions

- Understand the relation between CF and the pair of pants complex CA defined by Zung.
  - CA is generated by all orbits of $\phi^\sharp$, with differential counting (embedded) Fried pants.
  - Zung conjectures that there is a chain complex $C\Omega$ generated by all multi-cycles of G_+, with differential counting sweeping and resolution, so that $CF\simeq C\Omega\simeq CA$.
- Define an invariant of pseudo-Anosov flow in Heegaard Floer homology.
  - There exists homomorphisms $SFH(Q(\phi,\mathcal{C}))\to\widehat{HF}(M)$. One can attempt to define such an invariant by taking the image of $\mathbf{x}^{\text{top}}$ or $\mathbf{x}^{\text{bot}}$ under such a homomorphism.
  - For this to be an invariant of pseudo-Anosov flows, one has to show that the definition is independent of the collection of closed orbits $\mathcal{C}$.
  - The difficulty is that there are many such homomorphisms, e.g. there is one for each contact structure on the filling solid tori. One needs to choose the 'correct' homomorphism, otherwise it might lead to an identically zero invariant.
- Mitsumatsu showed that associated to an Anosov flow ϕ with orientable foliations, there is a pair (ξ_+,ξ_-) of positive/negative contact structures. Bowden-Massoni showed that $\xi_\pm$ are uniquely determined by ϕ. Investigate the relation between the dynamics of the Anosov flow ϕ and the Reeb flows $R_\pm$ of $\xi_\pm$.
  - Is $CF/C\Omega/CA$ of ϕ homotopy equivalent to ECC of $R_\pm$?
  - Is the pseudo-Anosov flow invariant of ϕ equal to the contact invariant of $\xi_\pm$?
- Develop some version of this theory in the partially punctured/closed setting.
  - One can also define Heegaard diagrams from Markov partitions. These work in the partially punctured/closed setting.
  - In general, a common strategy is to represent the pseudo-Anosov flow using some combinatorial device and build a Heegaard diagram from it. Some combinatorial devices other than veering triangulations and Markov partitions include: dynamic pairs, geometric type.
- Come up with a good notion of cobordance of pseudo-Anosov flows for which there is an induced homomorphism on $CF/C\Omega/CA$.
  - There is an operation of Goodman-Fried surgery along closed orbits, which is topologically a Dehn surgery. However, these surgeries are integer-reciprocal, hence fit poorly with the usual 2-handle attachment operation.