HEEGAARD FLOER THEORY AND PERIODIC POINTS OF PSEUDO-ANOSOV MAPS

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Recall that a **sutured manifold** is a compact oriented 3-manifold M whose boundary is divided into two subsurfaces R_+ and R_- by a multi-curve $\Gamma \subset \partial M$. It is **balanced** if each boundary component has some suture curve and $\chi(R_+) = \chi(R_-)$.

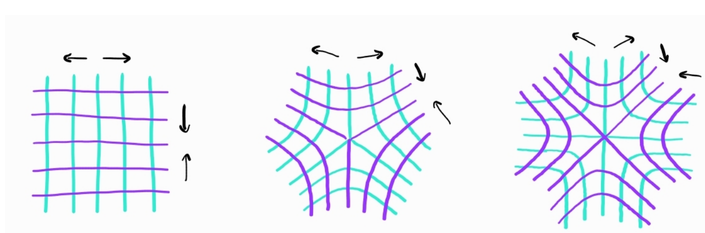
Example 1. Let L be a link in a closed oriented 3-manifold Y. The link exterior $Y \backslash \nu(L)$ can be endowed with a sutured manifold structure by placing two meridional sutures on each boundary component.

All sutured manifolds considered in this talk have torus boundary components as in Example 1, but some boundary components may have more than two sutures.

Juhász (2006) generalized Heegaard Floer theory to sutured manifolds. Associated to each balanced sutured manifold (M, Γ), there is a sutured Floer homology group $SFH(M, \Gamma)$. This is a finite dimensional vector space over $\mathbb{F} = \mathbb{F}_2$ with a relative $H_1(M)$ spin^c grading and a Maslov grading.

- If Y is a closed oriented 3-manifold, let $Y(1)$ be the sutured manifold obtained by removing a ball from Y and placing two sutures on the resulting boundary sphere. Then $SFH(Y(1)) \cong \widehat{HF}(Y)$.
- If L is a link in a closed oriented 3-manifold Y, then $SFH(Y \backslash \nu(L)) \cong \widehat{HFL}(Y, L)$, where $Y \backslash \nu(L)$ is endowed with the sutured manifold structure in Example 1.
- If (M, Γ) has a torus boundary component with $2n$ sutures, let Γ' be the same sutured manifold structure but removing all but 2 sutures on that boundary component, then $SFH(M, \Gamma) \cong SFH(M, \Gamma') \otimes (\mathbb{F} \oplus \mathbb{F})^{\otimes(n-1)}$. Here the two $\mathbb{F}$ summands differ by the homology class of one suture in the spin^c grading and have the same Maslov grading.

Recall that an orientation preserving homeomorphism f of a closed oriented surface S is **pseudo-Anosov** if there exists a transverse pair of singular foliations (ℓ^s, ℓ^u) such that f contracts the leaves of ℓ^s and expands the leaves of ℓ^u with respect to some transverse measures.



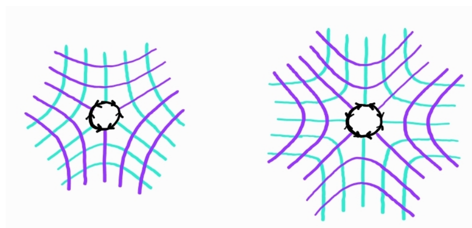
Note that f possibly rotates the leaves at the singular points.

Some facts about pseudo-Anosov maps:

- Nielsen-Thurston classification states that up to isotopy, every surface homeomorphism f can be decomposed along a (possibly empty) collection of curves into subsurfaces on which f is either finite order or pseudo-Anosov.

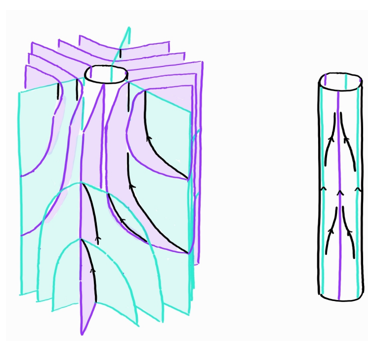
- Thurston showed that the mapping torus of f is hyperbolic if and only if f is pseudo-Anosov.
- Maher showed that 'most' surface homeomorphisms are pseudo-Anosov.

Let $\mathrm{sing}(f)$ be the collection of singular points of f on S. The **blow-up** of f (at $\mathrm{sing}(f)$) is the homeomorphism $f^\sharp$ of the compact surface $S^\sharp = S \backslash \nu(\mathrm{sing}(f))$ obtained by modifying f as follows.



Note that the periodic points of $f^\sharp$ in the interior of $S^\sharp$ are exactly the non-singular periodic points of f. Meanwhile, the number and periods of the boundary periodic points are determined by the period and rotation of f on $\mathrm{sing}(f)$.

The mapping torus of $f^\sharp$ is the compact oriented 3-manifold $Y^\sharp = S^\sharp \times [0,1]/(x,1) \sim (f^\sharp(x),1)$. The suspension of the stable/unstable foliations induce curves on each boundary component of $Y^\sharp$. We call these the **degeneracy curves**. Their slope is usually referred to the fractional Dehn twist coefficient in the literature.



Let $\pi : H_1(Y^\sharp) \to \mathbb{Z}$ be the epimorphism given by the intersection number with the fiber $[S^\sharp]$. Given any sutured structure Γ on $Y^\sharp$, we write $SFH(Y^\sharp, \Gamma, n) = \oplus_{\pi(\mathfrak{s})=n} SFH(Y^\sharp, \Gamma, \mathfrak{s})$. Strictly speaking, this is only well-defined up to a shift in n, corresponding to identifying $\mathrm{Spin}^c(Y^\sharp)$ with $H_1(Y^\sharp)$ in various ways, but at least it makes sense to say 'top grading', 'second-to-top grading', etc.

Theorem 2 (Ni, Ghiggini 2006)**.** *Let Γ be a sutured manifold structure on $Y^\sharp$ consisting of curves that have total intersection number k with $[S^\sharp]$. Then the dimensions of the top grading $SFH(Y^\sharp, \Gamma, |\chi(S^\sharp)| + \frac{k}{2})$ and the bottom grading $SFH(Y^\sharp, \Gamma, 0)$ are 1.*

Theorem 3 (Ni, Ghiggini-Spano 2022)**.** *Suppose $\mathrm{sing}(f)$ only has one element. Then for any sutured manifold structure Γ consisting of 2 sutures each intersecting the fiber surface $S^\sharp$ once, the dimension of the second-to-top grading $SFH(Y^\sharp, \Gamma, |\chi(S^\sharp)|)$ is # fixed points of $f^\sharp - (4g-3)$.*

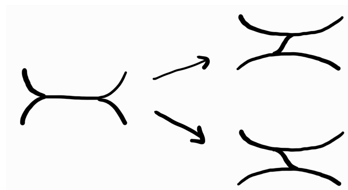
We remark that Theorem 3 is much more general than the version we stated here: It applies to certain non-pseudo-Anosov f, and any blow-up of f at a fixed point.

Theorem 3 is proved using ideas related to the equivalence between Heegaard Floer homology and embedded contact homology. Using more hands-on methods, Alfieri and myself showed the following theorem.

Theorem 4 (Alfieri-T. 2025)**.** *Suppose the minimum period of periodic points of $f^\sharp$ is P. Then for $\Gamma = 2 \cdot$ degeneracy curves, the dimensions of the next-to-top gradings $SFH(Y^\sharp, \Gamma, \frac{3}{2}|\chi(S^\sharp)| - n)$ are $\frac{1}{n}\#$ period n points of $f^\sharp$ for $n = 1, ..., 2P - 1$.*

For the rest of the talk, we outline some ideas in proving Theorem 4.

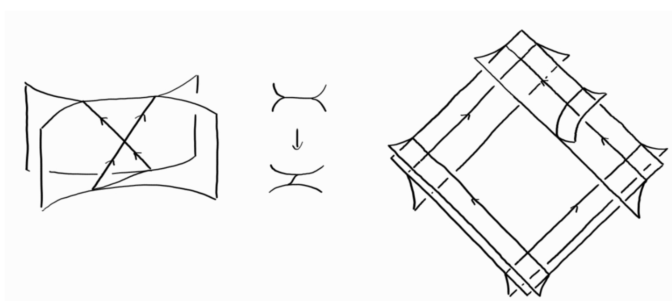
A **train track** on a surface S is an embedded trivalent graph such that the half-edges at each vertex are tangent to a common line. A **split** on a train track is an operation of the following form.



A **periodic splitting sequence** for a pseudo-Anosov map $f : S \to S$ is a sequence of train tracks $\tau_0 \to ... \to \tau_p = f(\tau_0)$ where τ_{n+1} is a split of τ_n for every n.

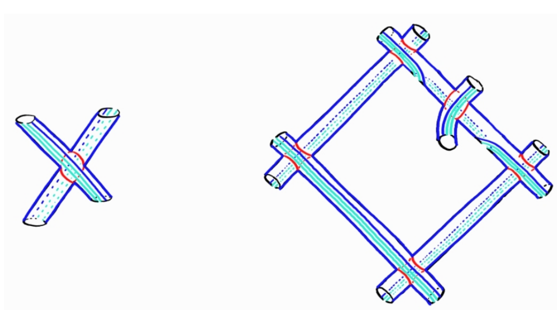
Theorem 5 (Agol 2011)**.** *Every pseudo-Anosov map has a periodic splitting sequence.*

Given a periodic splitting sequence $\tau_0 \to ... \to \tau_p = f(\tau_0)$, we can suspend it to get a **branched surface** B where the **vertices** and **sectors** are of the following form.



We also orient the edges of B upwards.

We can then build a Heegaard diagram by locally replacing the branched surface B near each vertex and sector as follows.



One can check that the sutured manifold described by this Heegaard diagram is ($Y^\sharp$, 2·degeneracy curves).

Also, observe that:

- Each α-curve corresponds to a vertex.
- Each β-curve corresponds to a sector.
- An α-curve and a β-curve intersect if and only if the corresponding vertex is a corner of the corresponding sector.

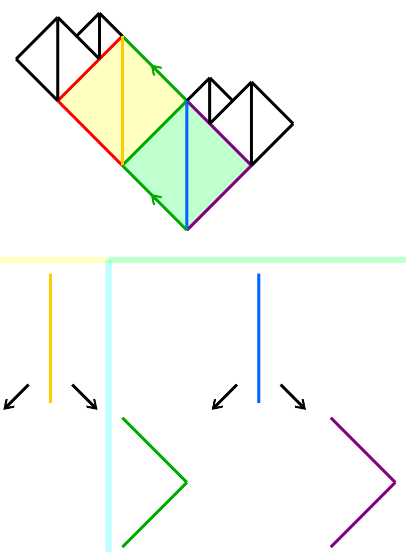
So the states of the diagram are exactly methods of assigning a corner to each sector such that each vertex is picked once.

Moreover, we can associate to any state $\mathbf{x}$ a multi-loop $\mu_{\mathbf{x}}$ in the 1-skeleton $B^{(1)}$ by connecting the bottom corner of each sector to the corner specified by $\mathbf{x}$. This determines a bijection between the states and embedded multi-loops of the augmented dual graph Γ_+, defined as the union of the 1-skeleton $B^{(1)}$ and an upwards edge in each sector.

Theorem 6 (Landry-Minsky-Taylor 2023)**.** *There is a surjective map $\mathcal{F}$ from the set of directed loops of Γ_+ to the closed orbits of the suspension flow of $f^\sharp$.*

Moreover, each fiber of $\mathcal{F}$ is the sweep equivalence of a loop.

In the context of Theorem 4, suppose γ is a closed orbit of period $n = P, ..., 2P-1$. Then every loop in $\mathcal{F}^{-1}(\gamma)$ is embedded. We show that $\oplus_{\mathcal{F}(\mu_{\mathbf{x}})=\gamma}\mathbb{F}\cdot\mathbf{x}$ is a summand of the whole Heegaard Floer chain complex, and that its homology is 1-dimensional.



If γ is a closed orbit of even higher period, loops in $\mathcal{F}^{-1}(\gamma)$ can fail to be non-embedded. This corresponds to a pair of pants connecting γ to two closed orbits γ_1 and γ_2 whose periods add up to that of γ. These pairs of pants are counted by Zung's pair of pants differential.