# MINIMUM ENTROPIES OF BRAIDS

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18 FEB 2025

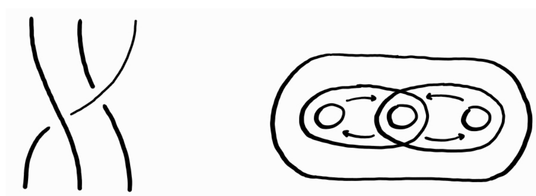
A **braid (with $n$ strands)** is a collection of $n$ properly embedded arcs in the cylinder $D^2 \times [0,1]$ where each arc projects to $[0,1]$ homeomorphically, and the endpoints of the arcs lie on a fixed set of $n$ points on $D^2$ on the top and bottom faces.

A braid $\beta$ with $n$ strands determines a homeomorphism $f$ from the $n$-holed disc to itself by sweeping the bottom face up to the top face. Importantly, note that $f$ is only well-defined up to isotopy.

**Example.** If $\beta$ is the trivial braid, then $f$ is the identity and contains trivial dynamics.

If $\beta$ consists of two strands winding around each other, then $f$ is a Dehn twist along a curve around two holes. In this case there is some dynamics, but it is still quite boring.

If $\beta$ is the **French braid**, then $f$ is the **taffy pulling map**, which is a composition of two half twists. The dynamics of this map is actually interesting, as we shall see later.



Morally, the more combinatorially complicated a braid is, the more dynamics it contains. In particular, one can obtain maps with arbitrarily large amount of dynamics by taking powers of a dynamically interesting braid.

Let us consider instead the opposite end of the spectrum:

**Question 1.** What is the least amount of dynamics that can be contained in a braid?

The answer is clearly 'none', since the trivial braid contains no dynamics at all. But more interestingly, observe that if a map contains too little dynamics, then it must be isotopic to the identity, thus correspond to the trivial braid as well. So we modify our question to:

**Question 2.** What is the least amount of dynamics that can be contained in a nontrivial braid?

We quantify this question using (topological) entropy.

**Definition.** Let $f: S \to S$ be a homeomorphism of a surface. Let $\mathcal{U}$ be an open cover of $S$. For each $n$, let $\mathcal{U}_n$ be the common refinement of $f^k(\mathcal{U})$ for $k = 0, ..., n$. Let $C_{\mathcal{U},n}$ be the minimum number of elements of $\mathcal{U}_n$ that suffices to cover $S$. Intuitively, $C_{\mathcal{U},n}$ equals the number of length $n$ itineraries of points under $f$ that are indistinguishable by $\mathcal{U}$.

The **entropy** of $f$ is defined to be

$$h(f) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log C_{\mathcal{U},n}.$$

The **entropy** of a braid is defined to be the infimum entropy of the corresponding homeomorphisms.

**Example.** Suppose $\beta$ consists of multiple strands winding around each other. Take $f$ to be a Dehn twist in a neighborhood $\nu$ of a curve $\gamma$. Then for any open cover $\mathcal{U}$, the open sets that lie away from $\nu$ contribute constant terms to $C_{\mathcal{U},n}$ while the open sets that lie in $\nu$ contribute quadratic terms to $C_{\mathcal{U},n}$. Hence $h(f) = 0$.

This forces us to modify our question again to:

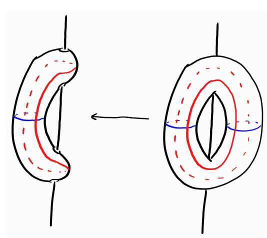
**Question 3.** What is the least amount of nontrivial dynamics that can be contained in a braid?

Or with our quantification:

**Question 4.** What is the minimum nonzero entropy over all braids?

**Example.** Suppose $\beta$ is the French braid and $f$ is the taffy-pulling map.

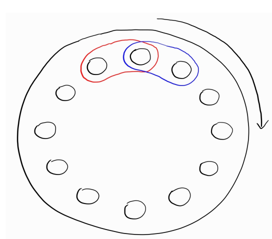
The lift of $f$ to the double cover of the 3-holed disc is isotopic to the **cat map** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on the (4-holed) torus.



One can compute that the entropy of the cat map is $\log \frac{3+\sqrt{5}}{2}$.

Intuitively, this is because the number of period $n$ points is $\operatorname{tr} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^n - 2 \sim (\frac{3+\sqrt{5}}{2})^n$. Moreover, each periodic point is hyperbolic thus cannot be 'perturbed away'.

**Example.** Let $S_n$ be the $3n$-holed disc and let $f_n$ be the composition of the taffy-pulling map with a rotation of $S$ by three holes. We call this construction the **$n$-fold Ferris wheel** over the taffy-pulling map $f_1$.



One can compute that the entropy of this map is $\frac{1}{n} \log \frac{3+\sqrt{5}}{2}$. Morally, because this is because the itineraries of $f_n$ are exactly $n$-fold lifts of those of the taffy-pulling map $f_1$.

In particular, $\lim_{n\to\infty} h([f_n]) = 0$.

Perhaps it is not surprising that the minimum entropy would decrease to 0 as the number of strands increases, since one can do more delicate movements with more strands. In any case, we modify our question yet again:

**Question 5.** For fixed $n$, what is the minimum nonzero entropy over all $n$-strand braids?

Finally, we end up with a meaningful question, as evidenced by the following nontrivial answers.

**Theorem.** *For small values of $n$, the minimum entropy $\eta_n$ among $n$-strand braids is given by:*

$$\eta_3 = \log \frac{3+\sqrt{5}}{2} = \log |x^3 - 2x^2 - 2x + 1|$$

$$\text{[Ko-Los-Song 2002] } \eta_4 = \log |x^4 - 2x^3 - 2x + 1|$$

$$\text{[Ham-Song 2006] } \eta_5 = \log |x^5 - 2x^3 - 2x^2 + 1|$$

$$\text{[Lanneau-Thiffeault 2011] } \begin{cases} \eta_6 = \frac{1}{2} \log |x^3 - 2x^2 - 2x + 1| \\ \eta_7 = \log |x^7 - 2x^4 - 2x^3 + 1| \\ \eta_8 = \log |x^8 - 2x^5 - 2x^3 + 1| \end{cases}$$

*where we write $|P(x)|$ for the largest positive root of a polynomial $P(x)$.*

Xiangzhuo Zeng and myself resolved Question 5 for all but three of the values of $n$.

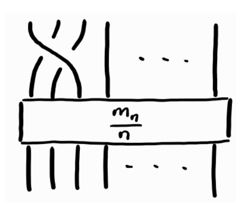
**Theorem** (T.-Zeng 2024)**.** *For $n \geq 9, n \neq 10, 12, 14$, the minimum entropy $\eta_n$ among $n$-strand braids is given by:*

$$\begin{cases} \log |x^{2k+1} - 2x^{k+1} - 2x^k + 1| & \text{if } n = 2k+1 \\ \log |x^{4k} - 2x^{2k+1} - 2x^{2k-1} + 1| & \text{if } n = 4k \\ \frac{1}{2} \log |x^{2k+1} - 2x^{k+1} - 2x^k + 1| & \text{if } n = 4k+2 \end{cases}$$

Conjecturally, the same expression hold for $n = 10, 12, 14$ as well.

The minimum entropies are attained (possibly non-uniquely) by the following braids.

**Example** (Hironaka-Kin, Venzke)**.** For each $n$, consider the following braid $\beta_n$.




Here the box labelled $\frac{m_n}{n}$ means we do $m_n$ many positive $\frac{1}{n}$-twists, where $m_n = \begin{cases} 2 & \text{if } n = 2k+1 \\ 2k+1 & \text{if } n = 4k \\ 2k+1 & \text{if } n = 8k+2 \\ 2k+1 & \text{if } n = 8k+6 \end{cases}$

One can compute that

$$h(\beta_n) = \begin{cases} \log |x^{2k+1} - 2x^{k+1} - 2x^k + 1| & \text{if } n = 2k+1, \\ \log |x^{4k} - 2x^{2k+1} - 2x^{2k-1} + 1| & \text{if } n = 4k, \\ \log |x^{4k+2} - 2x^{2k+3} - 2x^{2k-1} + 1| & \text{if } n = 4k+2. \end{cases}$$

where we write $|P(x)|$ for the largest positive root of a polynomial $P(x)$.

The minimum entropies for $n = 2k+1$ and $n = 4k$ are attained by $\beta_n$. The minimum entropies for $n = 4k+2$ are attained by the 2-fold Ferris wheel over $\beta_{2k+1}$.

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For the rest of this talk, we describe some underlying ideas.

**Theorem** (Nielsen, Thurston)**.** *Let $f : S \to S$ be a homeomorphism of a compact surface. Then $f$ is isotopic to a homeomorphism that is either*

- *finite order,*
- ***reducible**, i.e. there is an essential multicurve $c$ such that $f(c) = c$, or*
- ***pseudo-Anosov**, i.e. there exists a transverse pair of measured, singular foliations $\ell^s, \ell^u$ such that $f$ expands the measure of $\ell^u$ by $\lambda$ and contracts the measure of $\ell^s$ by $\lambda^{-1}$, for some number $\lambda > 1$.*

The dynamics of a reducible mapping class can be reduced into its restrictions in the complement of the reducing multicurve $c$. In particular, the entropy of a reducible mapping class is the maximum of the entropies on its restrictions.

Meanwhile, one can show that the entropy of a finite order mapping class is 0, while the entropy of a pseudo-Anosov mapping class is $\exp \lambda$ for the number $\lambda$ appearing in the definition.

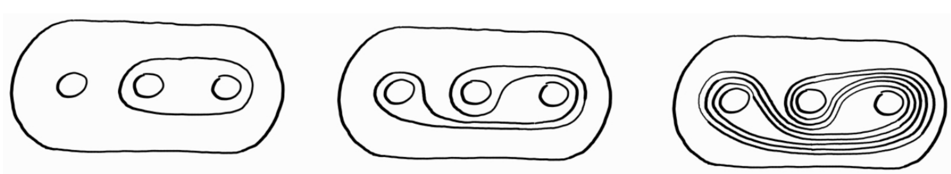
From this, we can reduce to studying pseudo-Anosov braids.

**Theorem** (T.-Zeng 2024)**.** *For $n \geq 9, n \neq 10, 12, 14, 18, 22, 26$, the minimum dilatation among $n$-strand pseudo-Anosov braids is given by:*

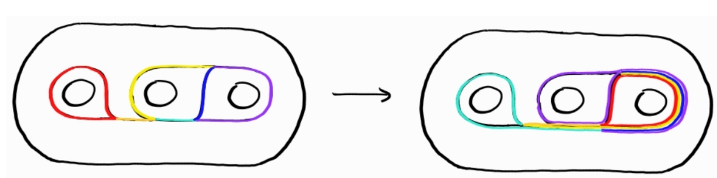
$$\begin{cases} |x^{2k+1} - 2x^{k+1} - 2x^k + 1| & \text{if } n = 2k+1 \\ |x^{4k} - 2x^{2k+1} - 2x^{2k-1} + 1| & \text{if } n = 4k \\ |x^{4k+2} - 2x^{2k+3} - 2x^{2k-1} + 1| & \text{if } n = 4k+2 \end{cases}$$

It is an observation of Thurston that if $f$ is a pseudo-Anosov map, then $f^n(\alpha)$ can be approximated by a **train track** $\tau$ for large $n$. The map $f$ induces an edge map on $\tau$ and the dilatation $\lambda$ of $f$ can be computed as the spectral radius of the **transition matrix** of $\tau$.

**Example.** Recall the taffy-pulling map $f$. Let $\alpha$ be the curve on the left. Its first two iterates under $f$ are as shown.



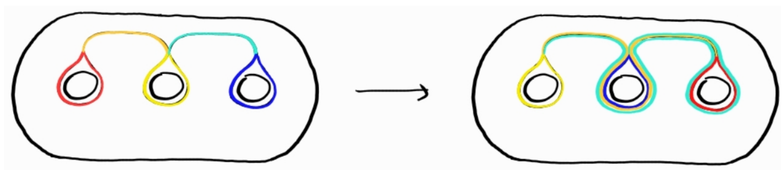
For large $n$, $f^n(\alpha)$ can be approximated by the following train track $\tau$. The map $f$ induces the following edge map on $\tau$.



The transition matrix of this edge map is $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ which has spectral radius $\mu^2$.

The approximating train track $\tau$ is highly non-unique. In previous work with Hironaka, we showed that as long as $f$ has at least two singularity orbits, we can choose $\tau$ to be a standardly embedded train track. In this case the transition matrix encapsulates the dynamics of $f$ in a much more efficient way.

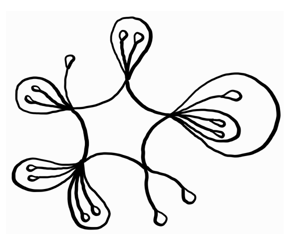
**Example.** In our ongoing example, we can instead choose the train track



By following through the weights, we compute the transition matrix $A$ to be $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

The **real transition matrix** $A^{\text{real}}$ is $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ which again has spectral radius $\mu^2$.

In the setting of homeomorphisms on punctured discs, we show that one can choose $\tau$ to be a **floral train track**, which is a train track of the form below.



We refer to the real edges that meet the 1-pronged singularities as the **filaments**, and the rest of the real edges as the **petals**.

In this case, the real transition matrix enjoys some special properties.

**Proposition 6.** *Let $g : \tau \to \tau$ be a floral train track map.*

*(1) Each filament is folded onto by an odd number of filaments + petals.*

*(2) Each petal folds onto an odd number of $\frac{1}{2}$filaments + petals.*

We then apply the machinery of clique polynomials to estimate the spectral radii of these matrices.

In more detail, let $A$ be a $n \times n$ Perron-Frobenius matrix. Construct a directed graph $\Gamma$ with vertex set $\{1, ..., n\}$ and with $A_{ji}$ edges from $i$ to $j$. In other words, $\Gamma$ is the directed graph whose adjacency matrix is $A$.

A **curve** in $\Gamma$ is an embedded directed edge cycle. Construct a (simple, undirected) graph $G$ with vertex set equal to the set of curves in $\Gamma$ and with an edge between two curves if they are disjoint. Let $w : V(G) \to \mathbb{R}_+$ be the function that sends each curve to its length. The weighted graph $(G, w)$ is called the **curve complex** of $\Gamma$.

The **right-angled Artin semi-group** associated to $G$ is defined to be

$$A_+(G) = \langle v \in V(G) \mid [v_1, v_2] = 1 \text{ if there is an edge between } v_1 \text{ and } v_2 \rangle.$$

The function $w : V(G) \to \mathbb{R}_+$ extends to a semi-group homomorphism $w : A_+(G) \to \mathbb{R}_+$. The **growth rate** of $(G, w)$ is defined to be

$$\lambda(G, w) = \lim_{T \to \infty} (\text{number of elements } g \in A_+(G) \text{ with } w(g) \leq T)^{\frac{1}{T}}.$$

A **clique** of $G$ is an induced subgraph $K$ that is a complete graph. The empty subgraph is a clique by convention. The **clique polynomial** of $G$ is

$$Q(t) = \sum_K (-1)^{|K|} t^{w(K)}.$$

**Theorem 7** (McMullen)**.**

*(1) The growth rate of $(G, w)$ equals $\rho(A)$.*

*(2) The smallest positive root of $Q(t)$ is the reciprocal of the growth rate of $(G, w)$.*

Morally, the larger the entries of $A$ are, the more connected $\Gamma$ is, i.e. we get shorter curves in larger quantities, which leads to a large growth rate, thus a large value for $\rho(A)$.

We use the properties of the real transition matrix associated to a floral train track map, such as those mentioned in Proposition 6, to locate specific curves, which gives a lower bound on the growth rate, thus for the spectral radius, which equals the dilatation of the pseudo-Anosov braid.