CANONICAL AXES FOR OUTER AUTOMORPHISMS OF FREE GROUPS

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Let S be a closed orientable surface of genus g. The **mapping class group** of S is the group $\mathrm{Mod}(S)$ of (orientation-preserving) diffeomorphisms on S modulo isotopy.

Theorem (Dehn-Nielsen-Baer)**.** $\mathrm{Mod}(S) \cong \mathrm{Out}(\pi_1 S)$.

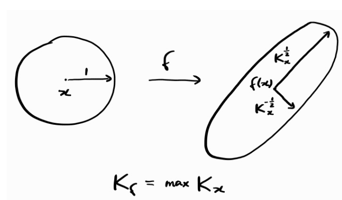
A mapping class f of S is **pseudo-Anosov** if it is infinite order and none of its powers preserve (the isotopy class of) a subsurface.

The **Teichmüller space** of S is the space $\mathrm{Teich}(S)$ of conformal structures of S modulo isotopy. More precisely, $\mathrm{Teich}(S)$ is the space of Riemann surfaces Σ along with an isomorphism $\pi_1\Sigma \to \pi_1 S$, with two elements being equivalent if there is a conformal isomorphism h such that

$$\begin{tikzcd} \pi_1\Sigma_1 \arrow[d, "h_\sharp"] \arrow[dr] & \\ \pi_1\Sigma_2 \arrow[r] & \pi_1 S \end{tikzcd}$$

commutes (up to conjugacy).

The **Teichmüller metric** on $\mathrm{Teich}(S)$ between two points Σ_1 and Σ_2 is the logarithm of the minimum quasiconformal constant over diffeomorphisms $\Sigma_1 \to \Sigma_2$ isotopic to identity.



The mapping class group $\mathrm{Mod}(S)$ acts on $\mathrm{Teich}(S)$, preserving the Teichmüller metric.

Theorem (Bers)**.** *Every pseudo-Anosov mapping class f acts on* $\mathrm{Teich}(S)$ *by translating along a unique axis A_f.*

Here, an **axis** of a isometry f on a metric space X is a geodesic on which the translation distance $\inf d(x, f(x))$ is attained.

* * *

We want to generalize this story from surface groups to free groups.

A graph G is of **rank** r if $\pi_1 G \cong F_r$.

Proposition. $\mathrm{HtpyEquiv}(G) \cong \mathrm{Out}(F_r)$.

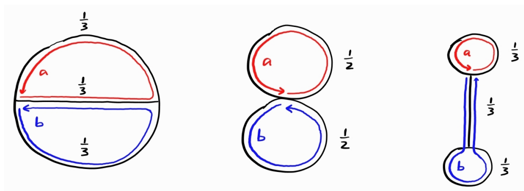
An outer automorphism ϕ of F_r is **fully irreducible** if it is infinite order and none of its powers preserve the isotopy class of a free summand. We say that ϕ is **non-geometric** if it is not induced from a homeomorphism on a (punctured) surface.

The **Culler-Vogtmann outer space** is the space CV_r of metric structures on graphs of rank r with total length 1 modulo isotopy. More precisely, CV_r is the space of metric graphs Γ with total length 1 along with a isomorphism $\pi_1\Gamma \to F_r$, with two elements being equivalent if there is an isometry h such that

$$\begin{tikzcd} \pi_1\Gamma_1 \arrow[d, "h_\sharp"] \arrow[dr] & \\ \pi_1\Gamma_2 \arrow[r] & F_r \end{tikzcd}$$

commutes.

Example. The following are points in CV_2.

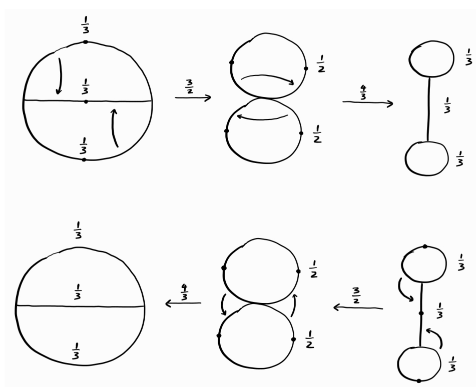


The **Lipschitz metric** on CV_r between two points Γ_1 and Γ_2 is the logarithm of the minimum Lipschitz constant over maps $\Gamma_1 \to \Gamma_2$ inducing the identity map on F_r.

Unfortunately, the Lipschitz metric is poorly behaved:

Firstly, the Lipschitz metric is not symmetric.

Example. The distances between the three elements in the previous example are as follows.



Secondly, the Lipschitz metric is not uniquely geodesic, i.e. there could be more than one geodesic between two points. Here by a **geodesic** we mean a path α such that $d(\alpha(t_1), \alpha(t_2)) + d(\alpha(t_2), \alpha(t_3)) = d(\alpha(t_1), \alpha(t_3))$ for all t_1, t_2, t_3.

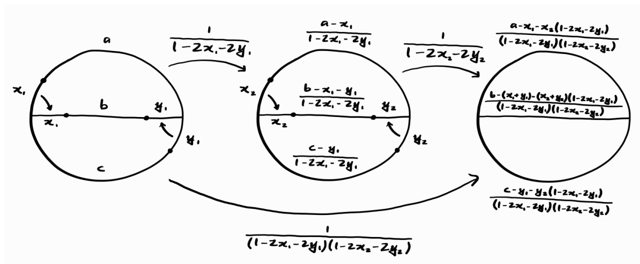
Example. Consider folding the Θ-graph with edge lengths a, b, c. The following computation shows that the path traced out by any such folding is a geodesic.

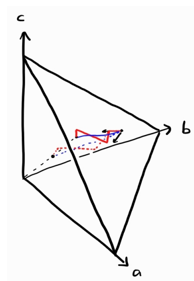
Hence there are uncountably many geodesic even locally.

Theorem (Handel-Mosher)**.** *Let ϕ be a non-geometric fully irreducible outer automorphism of F_r. The union of axes of the action of ϕ on CV_r is a subspace $\mathcal{A}_\phi$ proper homotopy equivalent to $\mathbb{R}$.*

The space $\mathcal{A}_\phi$ is called the **axis bundle** of ϕ.

Theorem 1 (Pfaff-T.)**.** *There is a finite collection of canonical axes in the axis bundle $\mathcal{A}_\phi$.*





We call these canonical axes the **arteries** of ϕ.

Corollary. *The conjugacy problem is solvable for fully irreducible outer automorphisms: Two non-geometric fully irreducible outer automorphisms are conjugate if and only if they have conjugate arteries.*

We outline some ideas in Theorem 1.

Theorem 2 (Pfaff-T.)**.** *The axis bundle $\mathcal{A}_\phi$ admits a cubist complex structure.*

Here a **cubist complex** is a cube complex except:

(1) A cubist complex is inherently directional: Every cube is equipped with a **splitting vertex** and an antipodal **folding vertex**. The **folding direction** refers to paths from the splitting vertex to the folding vertex, whereas the **splitting direction** is the opposite direction. The branched cubes in a cubist complex must intersect in a manner preserving this directionality.

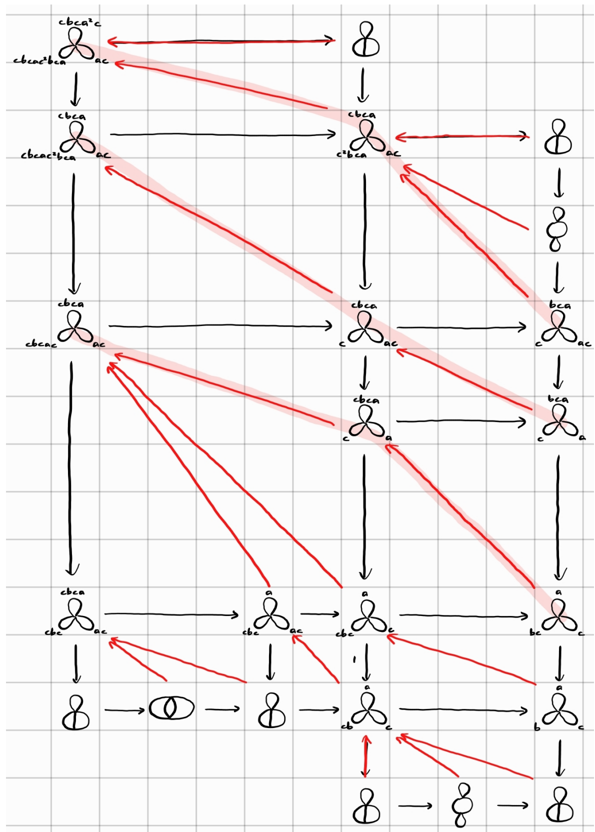
(2) When two cubes intersect in a cube complex, their intersection is a (complete) face of each of the two cubes, whereas when two cubes intersect in a cubist complex, their intersection is a (complete) face in the splitting side of one cube and a *subset* of a face in the folding side of the other cube. This causes the cubes to get finer in the folding direction.

(3) The cubes in a cubist complex are allowed to be branched cubes in general. These are unions of cubes glued along certain affine slices.

Example 3. Let $\phi \in \mathrm{Out}(F_3)$ be defined by

$$\phi(a) = cbca$$
$$\phi(b) = cbc$$
$$\phi(c) = ac.$$

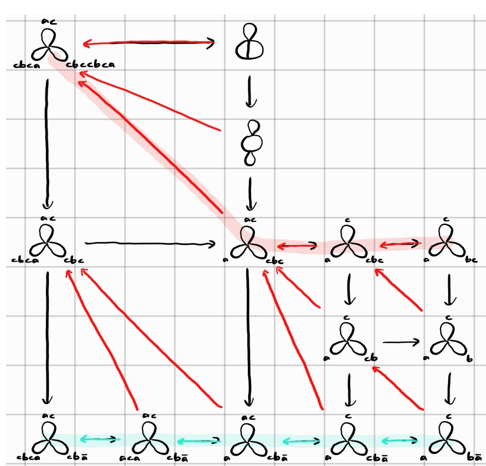
The cubist complex structure on $\mathcal{A}_\phi$ is as follows.



Example 4. Let $\phi \in \mathrm{Out}(F_3)$ be defined by

$$\phi(a) = ac$$
$$\phi(b) = cbc$$
$$\phi(c) = cbca.$$

The cubist complex structure on $\mathcal{A}_\phi$ is as follows.



We can now define a directed graph $\mathfrak{c}$, called the **cardiovascular system**, by connecting each vertex v to the next splitting vertex. Since every vertex has exactly one successor, there is a unique directed ray r_v starting at each vertex v.

Proposition 5. *Each directed ray r_v is eventually periodic.*

This implies that periodic directed lines exist. These are the arteries.

It follows from the combinatorics of the cubist complexes and cocompactness of the action of ϕ on $\mathcal{A}_\phi$ that

(1) there are finitely many arteries, and

(2) different arteries are related by sweeping across 2-dimensional cubes.

This situation should be compared with the theory of train tracks on surfaces. Let S be a closed orientable surface. The train track complex $\mathcal{TT}(S)$, defined by Hamenstädt, is the cube complex where

- the vertices are the trivalent train tracks on S (up to isotopy),
- the edges are the splitting moves, and
- the higher dimensional cubes are commuting splitting moves.

Let f be a pseudo-Anosov mapping class on S. The diffeomorphism that realizes the minimum quasiconformal constant is a canonical representative of f that contracts a stable foliation ℓ^s and expands an unstable foliation ℓ^u on S. Alternatively, the stable and unstable foliations can be defined as the endpoints of the axis $A_f \subset \mathrm{Teich}(S)$.

Theorem (Hamenstädt)**.** *The subcomplex of train tracks that carry the unstable foliation ℓ^u is a CAT(0) sub-complex $\mathcal{A}_f$ of $\mathcal{TT}(S)$.*

Theorem (Agol)**.** *There is a canonical axis in $\mathcal{A}_f$.*

Future direction: Study the geodesic flow of CV_r.

We expect the cubist complex structure to hold for generic geodesics, not just those periodic under a fully irreducible.

By assembling together all the canonical geodesics, one can make sense of the 'geodesic flow'. We would then like to extend the dynamic properties of the geodesic flow of Teichmüller space to this setting, for example ergodicity.