PSEUDO-ANOSOV MAPS, DILATATIONS, AND VEERING TRIANGULATIONS

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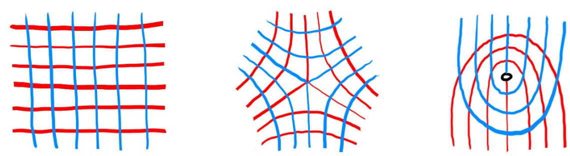
Let S be an orientable finite type surface, i.e. a closed orientable surface with finitely many points removed. An orientation preserving homeomorphism $f : S \to S$ is **pseudo-Anosov** if there exists a transverse pair of measured, singular foliations ℓ^s, ℓ^u such that f expands the measure of ℓ^u by λ and contracts the measure of ℓ^s by λ^{-1}, for some number $\lambda > 1$. The foliations ℓ^s and ℓ^u are called the **stable** and **unstable** foliations of f respectively. The number λ is called the **dilatation** of f.

Here a **foliation** is a partition of S into 1-manifolds. A **singular foliation** is a foliation except we allow finitely many **singularities**, where the foliation is locally conjugate to either

- the pull back of the foliation of $\mathbb{R}^2$ by vertical lines by the map $z \mapsto z^{\frac{n}{2}}$, for some $n \geq 3$, or
- the pull back of the foliations of $\mathbb{R}^2 \backslash \{(0,0)\}$ by vertical lines by the map $z \mapsto z^{\frac{n}{2}}$, for some $n \geq 1$.



A (singular) foliation is **measured** if it has a transverse measure, i.e. there is a way to measure the width of a band of leaves.

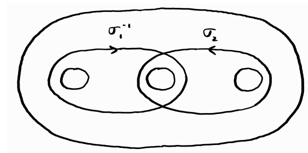
A pseudo-Anosov map is **fully-punctured** if all the singularities occur at the punctures of S, i.e. only the latter type of singularity occurs. Every pseudo-Anosov map can be made fully-punctured by puncturing at its singularities.

Example. Consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The linear transformation $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{R}^2$ descends down to a diffeomorphism f on T^2. The eigenvectors of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ are $\begin{bmatrix} \mu \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\mu^{-1} \\ 1 \end{bmatrix}$ with eigenvalues μ^2 and μ^{-2} respectively, where $\mu = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

The foliation by straight lines with tangent vector $\begin{bmatrix} \mu \\ 1 \end{bmatrix}$ descends down to a foliation on T^2, and f stretches the leaves of this foliation by μ^2. Similarly f contracts leaves of the foliation by straight lines with tangent vector $\begin{bmatrix} -\mu^{-1} \\ 1 \end{bmatrix}$ by μ^{-2}. This shows that f is pseudo-Anosov with dilatation μ^2.

Consider the involution $\iota(x, y) = (1 - x, 1 - y)$ on T^2. This has four fixed points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$, and $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. f commutes with ι, hence descends to a map $\overline{f}$ on $(T^2 \backslash \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\})/\iota \cong$ disc with 3-punctures.

One can check that $\overline{f}$ is the 3-braid $\sigma_2 \sigma_1^{-1}$. The stable/unstable foliations descend down to show that $\overline{f}$ is pseudo-Anosov with dilatation μ^2 as well.



Why should I care about pseudo-Anosov maps?

The **mapping class group** of a finite-type surface S is defined to be $\mathrm{Mod}(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$.

Theorem (Nielsen, Thurston)**.** *Every element of* $\mathrm{Mod}(S)$ *can be represented by a homeomorphism that is either*

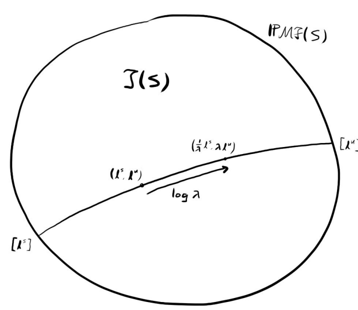
- *finite order,*
- *reducible, or*
- *pseudo-Anosov.*

Theorem (Maher)**.** *'Generic' elements of* $\mathrm{Mod}(S)$ *can be represented by pseudo-Anosov maps.*

Connection with Teichmüller theory.

Let $\mathcal{T}(S)$ be the **Teichmüller space** of S, defined as the space of conformal structures on S modulo diffeomorphisms isotopic to identity. The measured foliations ℓ^s and ℓ^u determine a conformal structure on S. Contracting and expanding the leaves of the two foliations deforms the conformal structure and determines a bi-infinite geodesic path α on $\mathcal{T}(S)$ (under the **Teichmüller metric**).

$\mathcal{T}(S)$ can be compactified with the space of projective measured (singular) foliations $\mathbb{P}\mathcal{MF}(S)$. α limits onto $[\ell^s]$ in the negative direction and onto $[\ell^u]$ in the positive direction. f acts as a translation of length $\log\lambda(f)$ along α.



Now let $\mathcal{M}(S)$ be the **moduli space** of S, defined as the space of conformal structures on S modulo diffeomorphisms. α quotients down to a closed geodesic of length $\log\lambda(f)$ on $\mathcal{M}(S)$. Conversely, every closed geodesic on $\mathcal{M}(S)$ comes from a pseudo-Anosov map.

There are many aspects of pseudo-Anosov maps that are worth studying. For this talk, we consider the dilatations of pseudo-Anosov maps.

Theorem (Ivanov)**.** *For each fixed surface* S*, the set of pseudo-Anosov maps with dilatation less than a given value is finite.*

In particular, on the surface $S_{g,n}$ with genus g and n punctures, there is a minimum dilatation $\lambda_{g,n}$ among all pseudo-Anosov maps on $S_{g,n}$. One can think of this value as the smallest amount of dynamics that can happen on $S_{g,n}$ while still doing something topologically interesting.

Minimum dilatation problem: What is this minimum dilatation?

This is a difficult problem! Currently, it is only solved for the 'small' surfaces $S_{0,4}, ..., S_{0,9}$, $S_{1,0}, S_{1,1}$, and $S_{2,0}$ with computational aid. These known minimum values have somewhat erratic patterns, adding to the confusion for predicting conjectural answers.

The problem becomes more tractible if one instead asks for the asymptotics of the minimum dilatation. In particular we have the following conjecture.

Conjecture (Hironaka)**.** $\lim_{g\to\infty} \lambda_{g,0}^{2g-2} = \mu^4 \approx 6.85$, *where* $\mu = \frac{1+\sqrt{5}}{2}$ *is the golden ratio.*

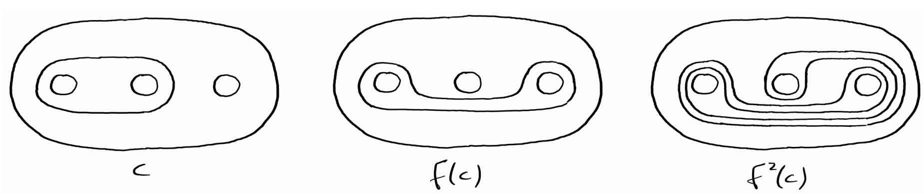
It is known via explicit examples that $\lim_{g\to\infty} \lambda_{g,0}^{2g-2} \leq \mu^4$. The difficult part is bounding $\lambda_{g,n}$ from below. The following theorem was the first general lower bound result.

Theorem (Penner)**.** *Let* $f : S \to S$ *be a pseudo-Anosov map, then* $\lambda(f)^{-\chi(S)} \geq 2^{\frac{1}{6}} \approx 1.12$.

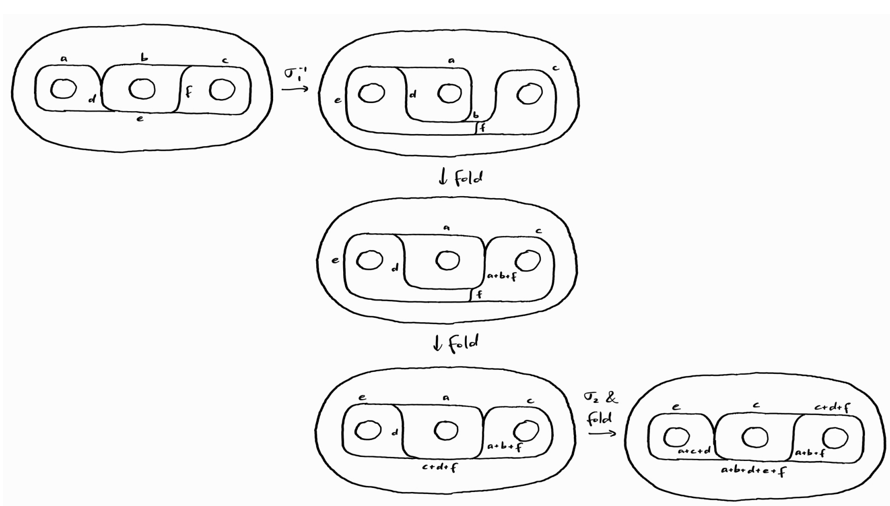
We illustrate the ideas of the proof via an example.

Example. Recall the pseudo-Anosov map $\sigma_2\sigma_1^{-1}$ considered above.

Pick some curve c, consider the iterates $f^n(c)$.



$f^n(c)$ can be approximated by a **train track** τ for n large. f acts on τ by folding up its edges.



Consider weights on the edges of τ, thought of as widths of the edges. Then the folding moves add up the weights and determines a **transition matrix** $\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ which has spectral radius μ^2.

Given any pseudo-Anosov map f, this procedure works in general to give a train track whose transition matrix is a non-negative integer matrix with spectral radius equals to the dilatation of f.

Penner showed that one can choose the train track so that the transition matrix has size $-6\chi(s)$. His theorem then follows from the following algebraic fact.

Proposition. *Let A be a n-by-n non-negative integer matrix with spectral radius $\rho(A) > 1$, where $n \geq 2$. Then $\rho(A)^n \geq 2$.*

The quantity $\lambda(f)^{-\chi(S)}$ is known as the **normalized dilatation**. To understand why this is the correct normalization, we have to move up one dimension.

Let T_f be the mapping torus of f. This is a 3-manifold that fibers over S^1 with fibers identified with S. However, in general, this is not the only way T_f can fiber over S^1.

For each fibration, say with cohomology class $\alpha \in H^1(T_f)$, there is an associated pseudo-Anosov map, hence an associated normalized dilatation.

Theorem (Thurston, Fried)**.** *The set of cohomology classes that arise from fibrations are exactly the integer points that lie within finitely many **fibered cones** in $H^1(T_f)$. Within each cone, the normalized dilatation extends to a convex function that is constant on rays passing through the origin.*

Moreover, the finiteness property on individual surfaces generalizes to the following under this normalization.

Theorem (Farb-Leininger-Margalit)**.** *The set of fully-punctured pseudo-Anosov maps with normalized dilatation less than a given value have mapping tori within a finite list of 3-manifolds.*

Note that 'fully-punctured' is necessary here.

These facts justify considering the set

$$\mathcal{D} = \{\lambda(f)^{-\chi(S)} \mid f : S \to S \text{ is a fully-punctured pseudo-Anosov map}\}$$

as a way to study the asymptotics of minimal dilatations, at least when restricted to fully-punctured pseudo-Anosov maps.

Theorem 1 (Hironaka-T., T.)**.**

- *The minimum element of $\mathcal{D}$ is μ^2.*
- *The discrete points of $\mathcal{D}$ are*

$$\tfrac{3+\sqrt{5}}{2} \approx 2.618, \quad \tfrac{4+\sqrt{12}}{2} \approx 3.732, \quad (\textit{Lehmer's number})^9 \approx 4.311,$$
$$\tfrac{5+\sqrt{21}}{2} \approx 4.791, \quad |LT_{1,2}|^3 \approx 5.107, \quad \tfrac{6+\sqrt{32}}{2} \approx 5.828,$$

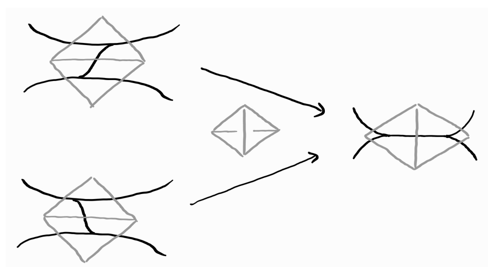
- *The minimum accumulation point of $\mathcal{D}$ is μ^4.*

The first statement has been a folklore fact among experts. The third statement can be considered as a fully-punctured version of Hironaka's conjecture.

Theorem 1 is proved over two papers. In the second part of this talk we give a flavor of the tools used in the second paper, with the goal of sketching a proof of the following theorem.

Theorem 2 (T.)**.** *Let $f : S \to S$ be a fully-punctured pseudo-Anosov map with normalized dilatation $\lambda(f)^{-\chi(S)}$. Then T_f admits an ideal triangulation with at most $\frac{1}{2}\lambda(f)^{-2\chi(S)}$ tetrahedra.*

Theorem (Agol)**.** *For every fully-punctured pseudo-Anosov map $f : S \to S$, one can choose an approximating train track τ so that one only needs to perform moves of the following type to fold from $f(\tau)$ to τ.*

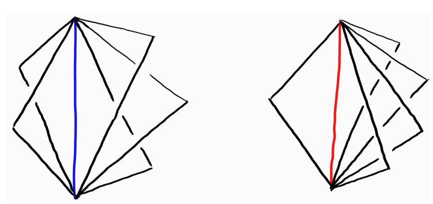


In fact, Agol showed that such a train track is essentially unique, but we do not need this fact here.

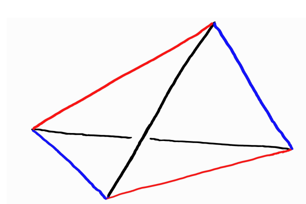
Consider the folding sequence from $f(\tau)$ to τ. Each train track in the sequence is dual to an ideal triangulation of S. Each folding move corresponds to a diagonal switch, and can be effected by layering a flat ideal tetrahedra on top.

We end up with a stack of flat ideal tetrahedra. If we glue the top of the stack to the bottom of the stack via f, we end up with an ideal triangulation Δ on T_f. This is known as a **layered veering triangulation**.

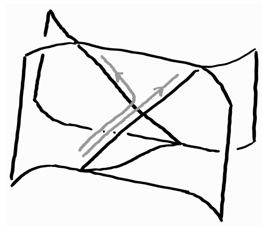
The word 'veering' comes from the following observation: For a fixed edge e, the faces to one side of e veer in a consistent direction from bottom to top.



We can color the edge e blue or red depending on whether the direction of veering is left or right. This determines an edge color of the whole triangulation so that each tetrahedron is of the following form



Meanwhile, the trace of the folding sequence determines a branched surface B that, as a 2-complex, is dual to Δ. The 2-cells of B are called sectors. The 1-skeleton of B is called the **dual graph** Γ. We orient the edges of Γ upwards.



It will also be convenient for us to consider the $\mathbb{Z}$-cover of B and Γ dual to the fibration $T_f \to S^1$. We denote these as $\widehat{B}$ and $\widehat{\Gamma}$ respectively. We let g be the generator of the deck transformation group that translates in the direction of the flow.

Sketch of proof of Theorem 2, first attempt.

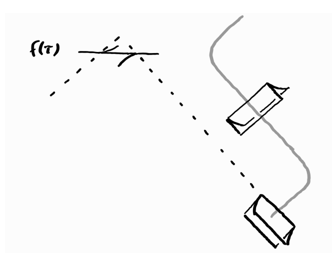
Recall that the edges of the train track in the folding sequence carry weights. This determines weights on the sectors of $\widehat{B}$. Note that for each sector $\widehat{s}$ of $\widehat{B}$, $\text{weight}(g^k \cdot s) = \lambda^k \text{weight}(s)$.

Let w be the minimum weight on the edges of $f(\tau)$. Fix an edge of $f(\tau)$ with weight w. This edge determines a sector s of B. Fix a sector $\widehat{s}_0$ of $\widehat{B}$ that covers s. We write $\widehat{s}_k = g^k \cdot \widehat{s}_0$.

Meanwhile, note that Γ is a $(2,2)$-valent directed graph. We take an Eulerian circuit c of Γ, push it upwards slightly so that it passes transversely through the 1-skeleton of B in the direction where sectors merge together.

Lift this to a path $\widehat{c}$ on $\widehat{B}$ that starts on $\widehat{s}_0$. Then $\widehat{c}$ ends on $\widehat{s}_k$ for some $k > 0$.

We claim that $k = -2\chi(S)$. First notice that k equals the number of cusps of $f(\tau)$ in S. We then compute the latter using an Euler characteristic argument. Each complementary region of $f(\tau)$ is a once-punctured p-gon, which has index $-\frac{p}{2}$. These add up to $\chi(S)$, which implies the claim.



In particular, the difference in weights between the starting and ending points of $\widehat{c}$ is $(\lambda^{-2\chi(S)} - 1)w$.

On the other hand, since c is an Eulerian circuit, it has $2N$ vertices, where N is the number of vertices of B, which also equals the number of tetrahedra of the triangulation. This implies that $\widehat{c}$ passes through the 1-skeleton of $\widehat{B}$ for $2N$ times, and $(\lambda^{-2\chi(S)} - 1)w$ is the sum of the weights of the $2N$ sectors that get merged into $\widehat{c}$.

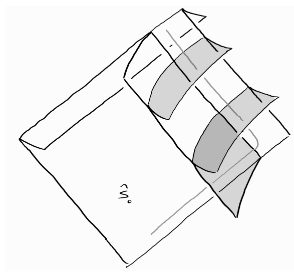
But recall that w is the lowest weight of $f(\tau)$. Each sector that gets merged into $\widehat{c}$ lies higher up than $\widehat{s}_0$, hence has weight $\geq w$. We conclude that $(\lambda^{-2\chi(S)} - 1)w \geq 2Nw$.

Gap in this proof: Here when we say 'a sector $\widehat{t}$ is higher up than $\widehat{s}_0$', we just mean that there is an upwards path from $\widehat{s}_0$ to $\widehat{s}$, which is insufficient to conclude that $\widehat{t}$ corresponds to an edge of a train track further along the folding sequence than $f(\tau)$. What is needed for this conclusion is that there is an upwards path from the topmost point of $\widehat{s}_0$ to $\widehat{t}$.

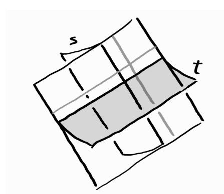
Sketch of proof of Theorem 2, second attempt.

We define the **height** of a sector $\widehat{t}$ of $\widehat{B}$ to be the largest integer k such that there is an upwards path from the topmost point of $\widehat{s}_k$ to $\widehat{t}$. Then the problematic sectors are the ones that merge into $\widehat{c}$ yet have negative height.

We suppose that the path $\widehat{c}$ 'hooks around' $\widehat{s}_0$.



Then the problematic sectors are the shaded ones. We discard the first shaded sector. Let $\widehat{t}$ be a subsequent shaded sector and suppose it has height $-k < 0$. This sector contributes a weight of $\geq \lambda^{-k} w$ to $\widehat{c}$. Let t be the image of $\widehat{t}$ in B. Recall that c is an Eulerian circuit, so it must pass through the vertex where t merges into c again at some point. At the corresponding vertex of $\widehat{c}$, the lift of t must have height $h \geq 0$, since it is either $\widehat{s}_0$ or there is an upwards path from the topmost point of $\widehat{s}_0$ to it, namely $\widehat{c}$. Thus the sector that merges into $\widehat{c}$ is $\widehat{s}_{h+k}$, hence contributes a weight of $\lambda^{h+k} w \geq \lambda^k w$. The sum of these two terms is $\geq \lambda^{-k} w + \lambda^k w > 2w$, so our argument is salvaged.



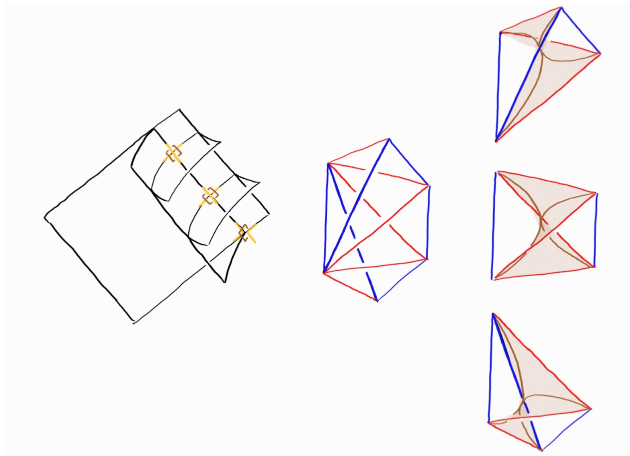
The difficulty now is showing that we can arrange for $\widehat{c}$ to hook around $\widehat{s}_0$. This is equivalent to showing that the dual graph Γ remains connected after we resolve a particular set of vertices.

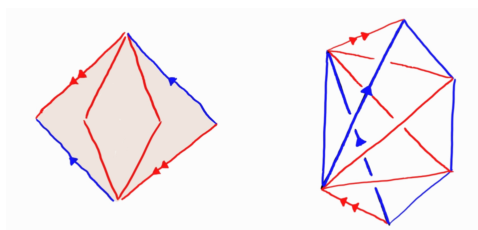
Each way to resolve a vertex v is dual to a quadrilateral in the tetrahedron dual to v. One can show by Alexander duality that the resolved dual graph is connected if and only if the quadrilaterals form a separating surface.

In our setting, the union of quadrilaterals piece up to a large quadrilateral. For this large quadrilateral to form a separating surface, the red and blue edges on its boundary must be identified. This imposes a combinatorial condition on the triangulation near the sector s.

Another key observation is that we can attempt to run the whole hook circuit argument on the intermediate terms of the folding sequence from $f(\tau)$ to τ. The above argument shows that either we find a hook circuit, or we have the same condition on a different sector. Piecing these conditions together poses a very strong restriction on the triangulation, implying the following proposition.

Proposition 3. *Either the triangulation* $\Delta =$ `cPcbbbdxm_10`*, or one can pick some hook circuit.*





We illustrate how one might use Theorem 2 to show the first statement of Theorem 1.

We want to show that there are no fully-punctured pseudo-Anosov map $f : S \to S$ whose normalized dilatation is less than μ^2. Suppose otherwise, then by Theorem 2, T_f would admit an ideal triangulation with $\leq \frac{1}{2}\mu^4 \approx 3.43$ tetrahedra.

For each N, there are at most finitely many ideal triangulations with $\leq N$ tetrahedra. In this case, according to SnapPea, there are 11 triangulations with ≤ 3 tetrahedra. In fact, not all of these triangulations admit the type of edge coloring that veering triangulations come with. We can restrict to only those triangulations that have such a coloring, of which there are 5.

Using the tool of the Teichmüller polynomial, introduced by McMullen in 2000, one can then compute the normalized dilatations of the pseudo-Anosov maps arising from fibrations of these 3-manifolds over S^1. One checks that none of these normalized dilatations are actually less than μ^2. Contradiction.

The strategy for showing the second statement of Theorem 1 is more complicated.

We could theoretically just use Theorem 2 again. But that would involve looking at all veering triangulations with ≤ 23 tetrahedra. There are way too many of these for this task to be computationally feasible with current technology.

Instead, we first use the following theorem to reduce to the case where T_f has only one boundary component.

Theorem 4 (Hironaka-T.)**.** *Let $f : S \to S$ be a fully-punctured pseudo-Anosov map so that T_f has at least two boundary components. Then $\lambda(f)^{-\chi(S)} \geq \mu^4$.*

The ideas used in this proof are completely different from those of Theorem 2.

Meanwhile, the argument used to show Theorem 2 can be sharpened in the one boundary component case, so that we have

Theorem 5 (T.). *Let $f : S \to S$ be a fully-punctured pseudo-Anosov map with normalized dilatation $\lambda(f)^{-\chi(S)} \leq 6.86$ and so that T_f has only one boundary component. Then T_f admits an ideal triangulation with at most 16 tetrahedra.*

This means that we 'only' have to look at all veering triangulations of 3-manifolds with one boundary component and with ≤ 16 tetrahedra. There are still 30079 such triangulations. Using code based on those written by Parlak, Schleimer, and Segerman, we carried out this computation in SageMath within 4 hours.