KAIST FALL 2025 MINICOURSE
PSEUDO-ANOSOV FLOWS AND VEERING TRIANGULATIONS

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1. Lecture 1: From pseudo-Anosov flows to veering triangulations...

A **pseudo-Anosov flow** is a flow ϕ on a 3-manifold M for which these exists a pair of transverse singular 2-dimensional foliations (Λ^s, Λ^u) on M such that:

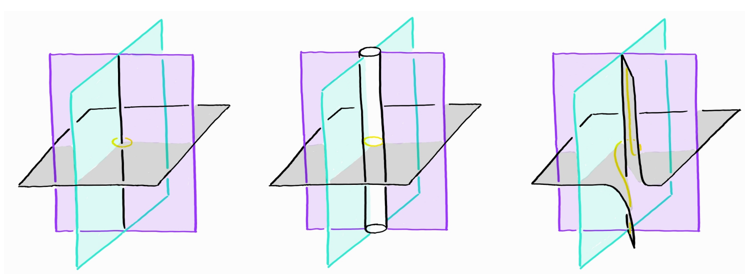
- the leaves of Λ^s and Λ^u intersect in flow lines,
- the flow lines converges along the leaves of Λ^s, and
- the flow lines diverge along the leaves of Λ^u.

Example 1.1. Let $f : S \to S$ be a **pseudo-Anosov map** on a surface. Recall that this means there exists a pair of transverse singular 1-dimensional foliations (ℓ^s, ℓ^u) on S such that:

- f contracts the leaves of ℓ^s, and
- f expands the leaves of ℓ^u.

Then the suspension flow on the mapping torus $M_f = S \times [0, 1]/(x, 1) \sim (f(x), 0)$ is a pseudo-Anosov flow.

Example 1.2. Let γ be a closed orbit of a pseudo-Anosov flow ϕ. Then one can perform **Goodman-Fried surgery** along γ by first blowing up ϕ along γ and blowing down along a slope s that intersects the stable/unstable half-leaves at γ at least 2 times.



This gives a method of constructing non-suspension pseudo-Anosov flows

One reason for caring about pseudo-Anosov flows is that they can be used as a framework for a systematic study of foliations: Thurston conjectured that every cooriented taut foliation on a hyperbolic 3-manifold admits an almost transverse foliation. Assuming the conjecture, one can then obtain the set of all foliations on a given hyperbolic 3-manifold by first classifying all pseudo-Anosov flows and studying the foliations almost transverse to a given pseudo-Anosov flow.

Another reason for caring about pseudo-Anosov flows is a program to generalize Nielsen-Thurston classification from (area-preserving/symplectic) homeomorphisms of surfaces to (volume-preserving/Reeb) flows on 3-manifolds. Morally, finite order maps should generalize to Seifert fibrations, reducing curves should generalize to reducing tori, pseudo-Anosov maps should generalize to pseudo-Anosov flows.

Given a pseudo-Anosov flow ϕ on M, consider the lifted flow $\widetilde{\phi}$ on $\widetilde{M}$. It can be shown that the map $\widetilde{M} \to \mathcal{O}$ quotienting each flow line to a point is a fiber bundle $\mathbb{R}^3 \to \mathbb{R}^2$. The space $\mathcal{O}$ is called the orbit space of ϕ. The pair of lifted foliations $(\widetilde{\Lambda^s}, \widetilde{\Lambda^u})$ determine a pair of transverse singular 1-dimensional foliations $(\mathcal{O}^s, \mathcal{O}^u)$ on $\mathcal{O}$.

Suppose $\mathcal{C}$ is a collection of closed orbits of ϕ. The set of all preimages $\widetilde{\mathcal{C}}$ determines a collection of points in $\mathcal{O}$. A **perfect fit rectangle** in $\mathcal{O}$ is a properly embedded copy of $[0,1]^2 \times \{(0,0)\}$. We say that ϕ has **no perfect fits** relative to $\mathcal{C}$ if every perfect fit rectangle in $\mathcal{O}$ intersects a point of $\widetilde{\mathcal{C}}$.

Proposition 1.3 (T. 2024)**.** *For every transitive pseudo-Anosov flow ϕ, there exists a finite collection of closed orbits $\mathcal{C}$ such that ϕ has no perfect fits relative to $\mathcal{C}$.*

Idea of proof. Pick a dense enough collection of orbits. In fact, one can just pick one dense enough orbit. $\square$

Here, a flow is **transitive** if there is a dense orbit. A pseudo-Anosov flow is non-transitive if and only if it admits a transverse separating torus.

We now explain the Agol-Guéritaud construction: Suppose ϕ is a transitive pseudo-Anosov flow on an oriented 3-manifold M. Pick a collection $\mathcal{C}$ as in Proposition 1.3. Up to enlarging $\mathcal{C}$, suppose that $\mathcal{C}$ contains all singular orbits and is nonempty.

We define:

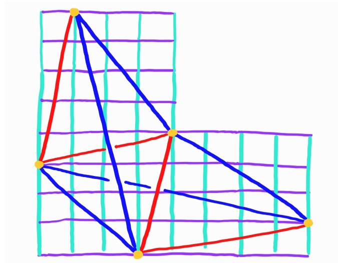
- An **edge rectangle** to be an embedded copy of $[0,1]^2$ in $\mathcal{O}$ that has a point of $\widetilde{\mathcal{C}}$ on each of two opposite corners.
- A **face rectangle** to be an embedded copy of $[0,1]^2$ in $\mathcal{O}$ that has a point of $\widetilde{\mathcal{C}}$ on one corner and two of the sides opposite to that corner.
- A **tetrahedron rectangle** to be an embedded copy of $[0,1]^2$ in $\mathcal{O}$ that has a point of $\widetilde{\mathcal{C}}$ on each of its four sides.

The no perfect fit condition ensures that every point in $\mathcal{O} \backslash \widetilde{\mathcal{C}}$ lies in the interior of a tetrahedron rectangle.

We built an abstract triangulation $\overline{\Delta}$ as follows: Take a tetrahedron $\bar{t}_R$ for each tetrahedron rectangle R. Fix a bijection between the four vertices of $\bar{t}_R$ with the four points of $\widetilde{\mathcal{C}}$ on the boundary of R. This induces a bijection between:

- The six edges of $\bar{t}_R$ with the six edge subrectangles of R.
- The four faces of $\bar{t}_R$ with the four face subrectangles of R.

Whenever two tetrahedron rectangles R_1 and R_2 overlap in face subrectangles of each other, we glue $\bar{t}_{R_1}$ and $\bar{t}_{R_2}$ along the corresponding faces.



It is possible to put a more structure on the tetrahedra $\bar{t}_R$:

- We consider the edge of $\bar{t}_R$ whose corresponding edge rectangle intersects every $\mathcal{O}^u$ leaf in R as the top edge, and the edge of $\bar{t}_R$ whose corresponding edge rectangle intersects every $\mathcal{O}^s$ leaf in R as the bottom edge. This gives a transverse taut structure to $\bar{t}_R$.
- We color the edges with positive slope red and the edges with negative slope blue. Here we use the orientation on M in order to induce an orientation on $\mathcal{O}$ and make sense of positive/negative slope.

There is a (nonunique) map $\pi : \overline{\Delta} \to \mathcal{O}$ mapping each edge/face/tetrahedron within the corresponding edge/face/tetrahedron rectangle. The preimage of each point in $\widetilde{\mathcal{C}}$ is a vertex, while the preimage of each point in $\mathcal{O}\backslash\widetilde{\mathcal{C}}$ is a line. We define Δ to be $\overline{\Delta}$ with all the vertices removed. Then $\pi : \Delta \to \mathcal{O}\backslash\mathcal{C}$ is a fiber bundle.

Moreover, $\pi_1 M$ acts on Δ preserving π. Taking the quotient, we have a flow on a 3-manifold $\Delta/\pi_1 M$. It can be shown that this flow is isotopy equivalent to the restriction of the original pseudo-Anosov flow ϕ on $M\backslash\mathcal{C}$. Here, two flows are **isotopy equivalent** if they differ by isotopy and reparametrization.

Making the identification between $\Delta/\pi_1 M$ and $M\backslash\mathcal{C}$, we now have an ideal triangulation Δ in M whose faces are positively transverse to the flow lines of ϕ.

We refer to the ideal triangulation Δ, along with the data of the transverse taut structure and the edge colorings, as the veering triangulation associated to $(\phi, \mathcal{C})$.

Theorem 1.4 (Schleimer-Segerman 2018)**.** *Let $\mathcal{F}$ be the set of $(\phi, \mathcal{C})$ where:*

- *ϕ is a pseudo-Anosov flow,*
- *$\mathcal{C}$ is a finite nonempty collection of closed orbits of ϕ, containing the singular orbits, and*
- *ϕ has no perfect fits relative to $\mathcal{C}$,*

modulo isotopy equivalence.

Let $\mathcal{T}$ be the set of (Δ, s) where:

- *Δ is a veering triangulation,*
- *s is a collection of slopes on the cusps of Δ, and*
- *s intersects the ladderpole slope of Δ at least 2 times on each cusp,*

modulo isotopy.

Then the Agol-Guéritaud construction determines a bijection $\mathsf{F} \cong \mathsf{V}$.

2. Lecture 2: … and back again

Let ϕ be a pseudo-Anosov flow and let Δ be the veering triangulation associated to $(\phi, \mathcal{C})$ for some $\mathcal{C}$.

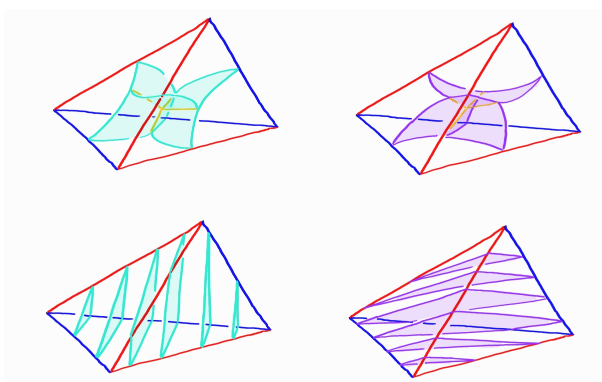
Proposition 2.1 (Landry-Minsky-Taylor 2023)**.** *Every closed orbit of ϕ is positively transverse to the faces of Δ. Conversely, every closed curve that is transverse to the faces of Δ is homotopic to a closed orbit.*

Proof. The first statement follows from the Agol-Guéritaud construction.

For the second statement, given such a closed curve c, we take one of its lifts $\widetilde{c}$. Suppose $\widetilde{c}$ passes through a sequence of tetrahedra $(t_{R_i})_{i\in\mathbb{Z}}$. Consider the sequence of corresponding rectangles R_i. Since $t_{R_{i+1}}$ lies above t_{R_i}, R_{i+1} is taller and narrower than R_i. This implies that the intersection $\bigcap R_i$ is a nonempty subrectangle.

Any corner point of $\bigcap R_i$ gives a flow line of $\widetilde{\phi}$ that is $[c]$-invariant. Thus descends to a closed orbit homotopic to c. In fact, one can show that $\bigcap R_i$ is actually just a single point. □

We next explain how to recover the foliations $\Lambda^{s/u}$. We construct the **stable/unstable branched surface** $B^{s/u}$ of Δ by placing a piece of a branched surface within each tetrahedron as follows:



Proposition 2.2. *The branched surface $B^{s/u}$ is uniquely laminar and carries the lamination obtained by blowing air into the singularities of the singular foliation $\Lambda^{s/u}$.*

Using these two propositions, we outline some ideas for why Theorem 1.4 is true.

Sketch proof of injectivity. Suppose ϕ_1 and ϕ_2 have the same associated veering triangulation. Then Proposition 2.1 implies that the set of homotopy classes of closed orbits of ϕ_1 and ϕ_2 agree.

A result of Barthelmé-Frankel-Mann (2025) states that this set is *almost* a complete invariant of pseudo-Anosov flows, up to isotopy equivalence. For example, it is a complete invariant when the 3-manifold M is hyperbolic. In general, one has to observe that in Proposition 2.1, the homotopy can be taken to lie away from $\mathcal{C}$, in order to upgrade [BFM25] and get isotopy equivalence. □

Sketch proof of surjectivity. Mosher (1996) has a notion of a dynamic pair. This is a transverse pair of branched surfaces satisfying certain combinatorial conditions, modeled after a pair of branched surfaces carrying the stable and unstable foliations of a pseudo-Anosov flow. Mosher showed that conversely, every dynamic pair induces a pseudo-Anosov flow.

Given a veering triangulation Δ, Schleimer-Segerman (2023) show that B^s and B^u can be isotoped into a dynamic pair. Working harder, one can identify edge/face/tetrahedron rectangles for the corresponding pseudo-Anosov flow ϕ and show that Δ is a veering triangulation associated to ϕ. $\square$

Application 1: Dilatation bounds

Let $f : S \to S$ be a pseudo-Anosov map on a surface. Recall that this means there exists a pair of transverse singular 1-dimensional foliations (ℓ^s, ℓ^u) on S such that:

- f contracts the leaves of ℓ^s, and
- f expands the leaves of ℓ^u.

The **dilatation** of f is the factor of contraction/expansion $\lambda > 1$ on the leaves of $\ell^{s/u}$. It is a measure of the dynamical complexity of f.

Question 2.3. For a given surface S, what is the minimum dilatation among all pseudo-Anosov maps on S?

Morally, this is asking: What is the smallest amount of dynamics one can have while still doing something topologically interesting?

Conjecture 2.4 (Hironaka 2010)**.** *Let δ_g be the minimum dilatation among all orientation-preserving pseudo-Anosov maps on the closed oriented genus g surface. Then $\lim_{g\to\infty} \delta_g^g = \mu^4$, where $\mu = \frac{1+\sqrt{5}}{2}$ is the golden ratio.*

We know from examples of Hironaka that $\limsup_{g\to\infty} \delta_g^g \leq \mu^4$, so it remains to prove good lower bounds.

Theorem 2.5 (Hironaka-T. 2022, T. 2023)**.** *For $g \geq 6$, we have $\delta_g^g \geq \mu^{\frac{4}{3}}$.*

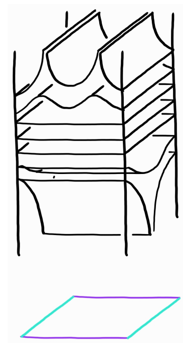
Sketch of proof. Let $f : S \to S$ be a pseudo-Anosov map. It can be shown that the suspension flow ϕ on the mapping torus M_f has no perfect fits relative to the collection of singular orbits $\mathcal{C}$. Let Δ be the associated veering triangulation.

Now, the dilatation of a pseudo-Anosov map can be characterized as the growth rate of closed orbits of the suspension flow. If Δ is combinatorially complicated, then there will be many closed curves positively transverse to the faces, forcing the dilatation to go up. Thus, given a bound on the dilatation of f, one can list out all possibilities for Δ, thus list out all possibilities for f itself. $\square$

Application 2: Foliations transverse to a pseudo-Anosov flow

Theorem 2.6 (Zung 2024)**.** *Let ϕ be a pseudo-Anosov flow and let Δ be a veering triangulation associated to ϕ. Then every foliation $\mathcal{F}$ transverse to ϕ is carried by the 2-skeleton of Δ.*

Sketch of proof. For each tetrahedron rectangle R, consider the portion of $\widetilde{\mathcal{F}}$ lying above R. In the forward direction, leaves possibly branch along stable leaves, while in the backward direction, leaves possibly branch along unstable leaves. Thus there must be some middle leaf that is mapped homeomorphically to R. We split this leaf along a horizontal quadrilateral of t_R.



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Modulo the difference between transverse and almost transverse, this solves the second step in our program to classify all foliations on a given 3-manifold.

For the first step, one has to classify all pseudo-Anosov flows on a given 3-manifold. A good first step would be to prove the following conjecture.

Conjecture 2.7. *Every hyperbolic 3-manifold admits at most finitely many pseudo-Anosov flows up to isotopy equivalence.*

Application 3: Connection to Heegaard Floer homology

Theorem 2.8 (Alfieri-T. 2025)**.** *Let ϕ be a pseudo-Anosov flow on a 3-manifold M without perfect fits relative to $\mathcal{C}$. Then the (sutured) Heegaard Floer homology of $M \backslash \nu(\mathcal{C})$ can be computed from a chain complex generated by the closed orbits of ϕ.*

Sketch of proof. One can build a Heegaard diagram for $M \backslash \nu(\mathcal{C})$ from the veering triangulation associated to $(\phi, \mathcal{C})$.

One then interprets the generators of the defining chain complex for HF in terms of closed curves positively transverse to the faces of Δ, which by Proposition 2.1, correspond to closed orbits of ϕ. ∎

This theorem should be compared with the HF=ECH theorem.

Theorem 2.9 (Kutluhan–Lee–Taubes 2020, Colin–Ghiggini–Honda 2011)**.** *Let ϕ be a Reeb flow on a 3-manifold M. Then the Heegaard Floer homology of M can be computed from a chain complex generated by the closed orbits of ϕ.*

This ties into a potential Nielsen-Thurston classification of 3-dimensional Reeb flows. Morally, as one 'pulls tight' a Reeb flow into a pseudo-Anosov flow, the chain complex in Theorem 2.9 is homotoped to the chain complex in Theorem 2.8.

Application 4: Birkhoff sections

Let ϕ be a pseudo-Anosov flow A **partial section** to ϕ is a surface-with-boundary S whose interior is transverse to ϕ and whose boundary is tangent to ϕ. A **Birkhoff section** is a partial section that intersects every orbit.

For example, if ϕ is obtained by Goodman-Fried surgery on a suspension pseudo-Anosov flow, then the image of the fiber surface is a Birkhoff section. Conversely, a Birkhoff section specifies a way of performing Goodman-Fried surgery that turns ϕ into a suspension flow. In other words, a Birkhoff section allows one to reduce 3-dimensional flows into 2-dimensional maps.

Theorem 2.10 (T. 2024)**.** *Every transitive pseudo-Anosov flow admits a Birkhoff section with two boundary components.*

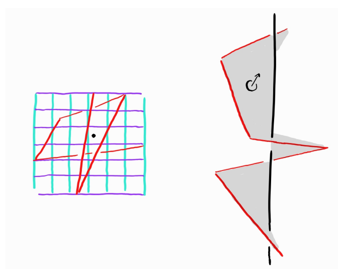
Proof. Pick $\mathcal{C}$ as in Proposition 1.3. Let Δ be the veering triangulation associated to $(\phi, \mathcal{C})$. We will construct a Birkhoff section out of two parts:

(1) A part that is transverse to the flow everywhere, and intersects every flow line.

(2) A part that supplies the tangent boundary components.

Part (1) can be constructed by just taking the union of horizontal quadrilaterals in the tetrahedra.

For part (2), we show that the set of red edges of the surface in (1) can be concatenated into the outer boundary component of a helicoid.



Similarly, the set of blue edges can be concatenated into the outer boundary component of a helicoid.

Taking the union of the surface in (1) and the two helicoids gives the desired Birkhoff section. □

It can be shown that the number 'two' in Theorem 2.10 is sharp. That is, there exists pseudo-Anosov flows that cannot admit Birkhoff sections with less than two boundary components.

Thus in terms of number of boundary components, Theorem 2.10 solves the question of minimizing the complexity of a Birkhoff section. More interestingly, we do not yet know the minimum complexity in terms of the genus, even for Anosov flows.

Conjecture 2.11 (Fried 1983, Ghys 2000s)**.** *Every transitive Anosov flow admits a genus one Birkhoff section.*