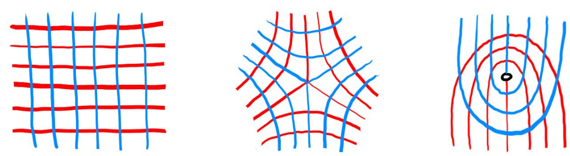
DILATATIONS OF PSEUDO-ANOSOV MAPS AND STANDARDLY EMBEDDED TRAIN TRACKS

CHI CHEUK TSANG (JOINT WORK WITH ERIKO HIRONAKA, AND WITH ERWAN LANNEAU AND LIVIO LIECHTI)
28 FEB 2024

Let S be a finite type surface, i.e. a closed surface with finitely many points removed. A homeomorphism $f : S \to S$ is **pseudo-Anosov** if there exists a transverse pair of measured, singular foliations ℓ^s, ℓ^u such that f expands the measure of ℓ^u by λ and contracts the measure of ℓ^s by λ^{-1}, for some number $\lambda > 1$. The foliations ℓ^s and ℓ^u are called the **stable** and **unstable** foliations of f respectively. The number λ is called the **dilatation** of f.

Here a **foliation** is a partition of S into 1-manifolds. A **singular foliation** is a foliation except we allow finitely many **singularities**, where the foliation is locally conjugate to either

- the pull back of the foliation of $\mathbb{R}^2$ by vertical lines by the map $z \mapsto z^{\frac{n}{2}}$, for some $n \geq 3$, or
- the pull back of the foliations of $\mathbb{R}^2 \backslash \{(0,0)\}$ by vertical lines by the map $z \mapsto z^{\frac{n}{2}}$, for some $n \geq 1$.



A (singular) foliation is **measured** if it has a transverse measure, i.e. there is a way to measure the width of a band of leaves.

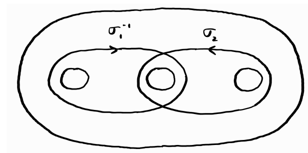
Example. Consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The linear transformation $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{R}^2$ descends down to a diffeomorphism f on T^2. The eigenvectors of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ are $\begin{bmatrix} \mu \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\mu^{-1} \\ 1 \end{bmatrix}$ with eigenvalues μ^2 and μ^{-2} respectively, where $\mu = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

The foliation by straight lines with tangent vector $\begin{bmatrix} \mu \\ 1 \end{bmatrix}$ descends down to a foliation on T^2, and f stretches the leaves of this foliation by μ^2. Similarly f contracts leaves of the foliation by straight lines with tangent vector $\begin{bmatrix} -\mu^{-1} \\ 1 \end{bmatrix}$ by μ^{-2}. This shows that f is pseudo-Anosov with dilatation μ^2.

Consider the involution $\iota(x, y) = (1-x, 1-y)$ on T^2. This has four fixed points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$, and $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. f commutes with ι, hence descends to a map $\overline{f}$ on $(T^2 \backslash \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\})/\iota \cong$ disc with 3-punctures.

One can check that $\overline{f}$ is the 3-braid $\sigma_2 \sigma_1^{-1}$. The stable/unstable foliations descend down to show that $\overline{f}$ is pseudo-Anosov with dilatation μ^2 as well.

A major motivation for studying pseudo-Anosov maps comes from the Nielsen-Thurston classification:



Theorem (Nielsen, Thurston)**.** *Every homeomorphism on a finite-type surface is isotopic to a homeomorphism that is either*

- *finite order,*
- *reducible, or*
- *pseudo-Anosov.*

Theorem (Maher)**.** *The pseudo-Anosov case is generic.*

There are many aspects of pseudo-Anosov maps that are worth studying. For today, we will focus on studying the dilatation.

Theorem (Ivanov)**.** *Let S be a finite-type surface. For any $\Lambda > 0$, there are only finitely many pseudo-Anosov maps on S that have dilatation $\leq \Lambda$.*

In particular, on any finite-type surface S, there is a minimum dilatation among all pseudo-Anosov maps defined on S.

Minimum dilatation problem. What is the value of this minimum dilatation?

This is a difficult problem! Currently, it is only solved for the 'small' surfaces $S_{0,4}, ..., S_{0,9}$, $S_{1,0}, S_{1,1}$, and $S_{2,0}$ with computational aid.

The problem somehow becomes simpler if one restricts to fully-punctured pseudo-Anosov maps. Here, we say that a pseudo-Anosov map is **fully-punctured** if all the singularities occur at the punctures of S. Every pseudo-Anosov map can be made fully-punctured by puncturing at its singularities.

A homeomorphism on a punctured surface acts on the set of punctures. A **puncture orbit** is an orbit of this action.

Theorem 1 (Hironaka-T.)**.** *Let $f : S \to S$ be an orientation-preserving fully-punctured pseudo-Anosov map with at least two puncture orbits. Then $\lambda(f)^{-\chi(S)} \geq \mu^4 \approx 6.85$, where $\mu = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Moreover, this bound is asymptotically sharp in the sense that there exists a sequence of orientation-preserving fully-punctured pseudo-Anosov maps $f_i : S_i \to S_i$ with at least two puncture orbits such that $\lambda(f_i)^{-\chi(S_i)} \to \mu^4$.*

Theorem 2 (Lanneau-Liechti-T.)**.** *Let $f : S \to S$ be an orientation-reversing fully-punctured pseudo-Anosov map with at least two puncture orbits, where $-\chi(S) \geq 4$. Then $\lambda(f)^{-\chi(S)} \geq \sigma^2 \approx 5.83$, where $\sigma = 1 + \sqrt{2}$ is the silver ratio. Moreover, this bound is asymptotically sharp in the sense that there exists a sequence of orientation-reversing fully-punctured pseudo-Anosov maps $f_i : S_i \to S_i$ with at least two puncture orbits such that $\lambda(f_i)^{-\chi(S_i)} \to \sigma^2$.*

Remark. One can remove 'with at least two puncture orbits' from Theorem 1 by another result of myself. Erwan, Livio, and myself conjecture that one can remove this from Theorem 2 as well.

The assumption that $-\chi(S) \geq 4$ is necessary in Theorem 2. There is an orientation-reversing fully-punctured pseudo-Anosov map f with two puncture orbits on the 4-punctured sphere such that $\lambda(f)^{-\chi(S)} = \mu^2 \approx 2.62$.

The proof of these theorems uses the tool of standardly embedded train tracks.

A **train track** on a surface S is an embedded graph with a smoothing of the half-edges at each vertex, such that:

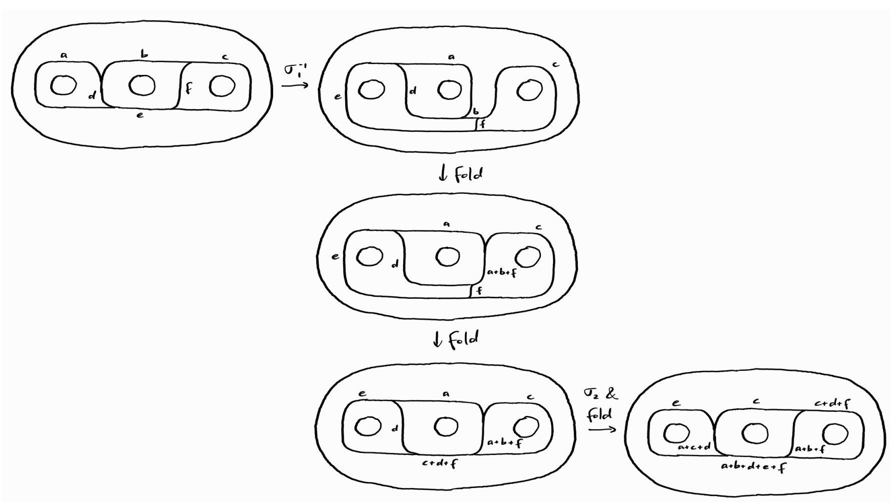
- there is at least one half-edge on each side of each vertex, and
- the complementary regions of τ are either discs with ≥ 3 cusps or once-punctured discs with ≥ 1 cusps.

A **train track map** is a map $f : \tau_1 \to \tau_2$ that sends the switches of τ_1 to switches of τ_2 and smooth edge paths of τ_1 to smooth edge paths of τ_2. Intuitively, a train track map folds up the edges of τ_1 to get to τ_2.

The **transition matrix** of a train track map $f : \tau_1 \to \tau_2$ is the matrix $f_* \in \mathrm{Hom}(\mathbb{R}^{E(\tau_1)}, \mathbb{R}^{E(\tau_2)})$ whose (e_2, e_1)-entry is the number of times $f(e_1)$ passes through e_2.

Fact. If a pseudo-Anosov map is homotopic to a train track map $f : \tau \to \tau$ then the dilatation of the pseudo-Anosov map is equal to the spectral radius of the transition matrix f_*.

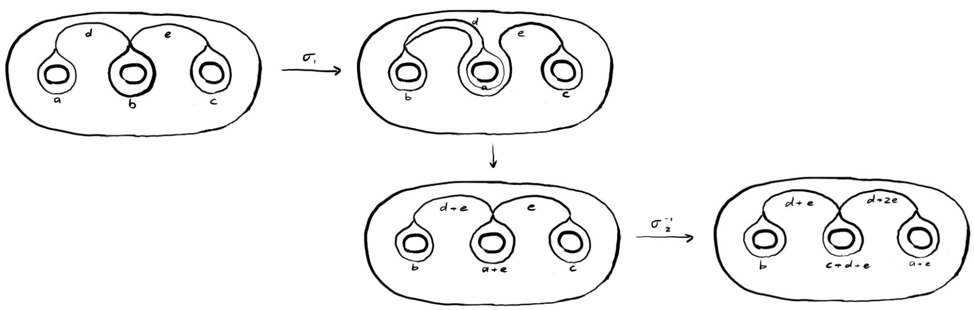
Example. Recall the pseudo-Anosov map $\sigma_2\sigma_1^{-1}$ considered above. This pseudo-Anosov map is homotopic to a train track map f on a train track τ as follows.



One computes $f_* = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$. The spectral radius of f_* is μ^2, which agrees with the dilatation of $\sigma_2\sigma_1^{-1}$.

One can always compute the dilatation by a train track map. However, the choice of the train track τ is far from unique. Some choices could be better than others. In some sense, standardly embedded train tracks offer most efficient choices of τ.

Example. The pseudo-Anosov map $\sigma_2\sigma_1^{-1}$ is also homotopic to the following train track map f.

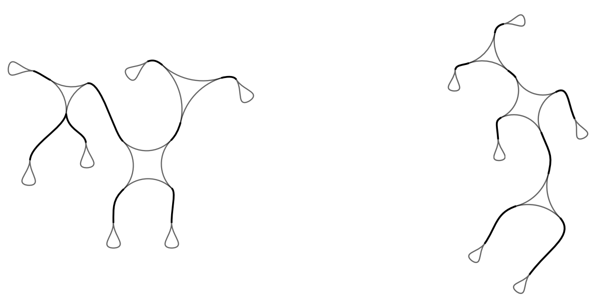


One computes $f_* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. The top left block is a permutation matrix thus is inconsequential for computing the spectral radius. The important part is the lower right block, which we call the **real transition matrix** f_*^{real}. In this example, this is $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ which again has spectral radius μ^2.

Definition. A train track τ is said to be **standardly embedded** if its set of edges E can be partitioned into a set of **infinitesimal edges** E_{inf} and a set of **real edges** E_{real}, such that:

- The union of infinitesimal edges is a disjoint union of cycles, which we call the **infinitesimal polygons**.
- The real edges connect up the vertices of the infinitesimal polygons.
- The smoothing at each vertex v is defined by separating the infinitesimal edges and the real edges.

Here are two more examples of standardly embedded train tracks.



The efficiency of standardly embedded train tracks is quantified by the following lemma.

Lemma. *Let τ be a standardly embedded train track on a surface S where the complementary regions of τ are all once-punctured discs. Then the number of real edges of τ equals $-\chi(S)$.*

Proof. It follows from the definition that $|E_{\text{inf}}| = |V|$, so τ has $|V| + |E_{\text{real}}|$ edges in total. Hence $\chi(S) = \chi(\tau) = |V| - (|V| + |E_{\text{real}}|) = -|E_{\text{real}}|$. $\square$

Theorem 3 (Hironaka-T.)**.** *If a fully-punctured pseudo-Anosov map f has at least two puncture orbits, then it is homotopic to a train track map on a standardly embedded train track whose real transition matrix is Perron-Frobenius, i.e. $(f_*^{\text{real}})^N$ has positive entries for large $N > 0$.*

Sketch of proof of Theorem 1. Let $f : S \to S$ be an orientation-preserving fully-punctured pseudo-Anosov map with at least two puncture orbits. By Theorem 3, f is homotopic to a train track map on a standardly embedded train track whose real transition matrix f_*^{real} is Perron-Frobenius.

Furthermore, we claim that f_*^{real} is reciprocal, i.e. the set of its eigenvalues (with multiplicity) is invariant under $\lambda \mapsto \lambda^{-1}$. This is essentially because f_* preserves the **Thurston symplectic form** on $\mathbb{R}^{E(\tau)}$ defined by

$$\omega(w_1, w_2) = \sum_{v \in V(\tau)} \sum_{e_1 \text{ left of } e_2} (w_1(e_1)w_2(e_2) - w_1(e_2)w_2(e_1))$$

where the second summation is taken over all pairs of half-edges e_1, e_2 for which e_1 is on the left of e_2.

There are some technical details here. The Thurston symplectic form is not actually a symplectic form in general, because it can be degenerate. However, we show that f_* acts cyclotomically, thus reciprocally on its radical, so combining reciprocity on the radical and on the nondegenerate part gives reciprocity overall.

We then apply the following result of McMullen.

Theorem (McMullen)**.** *Let A be a n-by-n Perron-Frobenius reciprocal matrix, where $n \geq 2$. Then $\rho(A)^n \geq \mu^4$.*

□

Sketch of proof of Theorem 2. Let $f : S \to S$ be an orientation-reversing fully-punctured pseudo-Anosov map with at least two puncture orbits. By Theorem 3, f is homotopic to a train track map on a standardly embedded train track whose real transition matrix f_*^{real} is Perron-Frobenius.

Here, instead of preserving the Thurston symplectic form ω, f_* sends ω to $-\omega$. One might think that this implies f_*^{real} is skew-reciprocal, i.e. the set of its eigenvalues (with multiplicity) is invariant under $\lambda \mapsto -\lambda^{-1}$. However, this is not true in general because of the cyclotomic action on the radical of ω, which may or may not be skew-reciprocal. Instead, f_*^{real} is skew-reciprocal up to cyclotomic factors, i.e. the set of its eigenvalues (with multiplicity) is invariant under $\lambda \mapsto -\lambda^{-1}$ except possibly on the unit circle.

We then show an analogue of McMullen's result.

Theorem (Lanneau-Liechti-T.)**.** *Let A be a n-by-n Perron-Frobenius skew-reciprocal up to cyclotomic factors matrix, where $n \geq 4$. Then $\rho(A)^n \geq \sigma^2$.*

□

It might be interesting to follow this strategy to explore other variants of the minimum dilatation problem. We formulate two conjectures along these lines:

Conjecture 4. *Let $f : S \to S$ be an orientation-preserving fully-punctured pseudo-Anosov map with at least three puncture orbits. Then $\lambda(f)^{-\chi(S)} \geq (2+\sqrt{3})^2$.*

Conjecture 5. *Let $f : S \to S$ be an orientation-preserving fully-punctured pseudo-Anosov map on a planar surface S with at least two puncture orbits. Then $\lambda(f)^{-\chi(S)} \geq (2+\sqrt{3})^2$.*

Question 6. Let $f : S \to S$ be an orientation-preserving fully-punctured pseudo-Anosov map with at least two puncture orbits. Suppose f acts trivially on $H_1(S)$. What is a good lower bound for $\lambda(f)$?

A result of Farb-Leininger-Margalit suggests that the lower bound should be some constant (as opposed to Theorem 1 and Theorem 2), but what is the best possible value?