Markov partitions for geodesic flows

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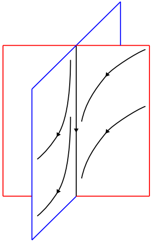
Anosov flows

Let M be a closed 3-manifold. A flow $\phi_t : M \to M$ is Anosov if:

- There are two foliations Λ^s, Λ^u intersecting transversely along flow lines
- The flow contracts exponentially along Λ^s and expands exponentially along Λ^u
- (Technical conditions about regularity or Markov partitions)



First studied by Anosov in 1960's.
Dynamically interesting: structural stability, symbolic dynamics.
Topologically interesting: left orderings, Thurston norm, generalization to pseudo-Anosov flows, etc.

Markov partitions

A flow box of an Anosov flow ϕ is a set of the form $I_s \times I_u \times [0,1]$, where

- $I_s \times \{u_0\} \times [0,1]$ lies on a stable leaf
- $\{s_0\} \times I_u \times [0,1]$ lies on an unstable leaf

A *Markov partition* is a collection of flow boxes $\{I_s^{(i)} \times I_u^{(i)} \times [0,1]\}_i$ covering N with disjoint interiors, such that

$$
\begin{aligned}
&(I_s^{(i)} \times I_u^{(i)} \times \{1\}) \cap (I_s^{(j)} \times I_u^{(j)} \times \{0\}) \\
= &\bigcup_k J_s^{(ij.k)} \times I_u^{(i)} \times \{1\} = \bigcup_k I_s^{(j)} \times J_u^{(ji.k)} \times \{0\}
\end{aligned}
$$

Markov partitions

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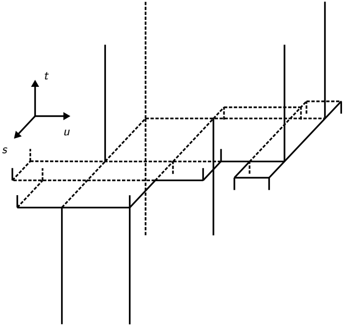
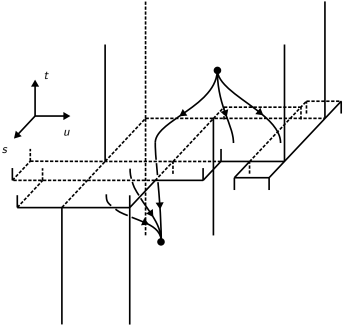
\end{aligned}
$$

Intuitively, when flowing downwards, the flow boxes stretch over multiple flow boxes in the unstable direction and contract to only cover a portion of a flow box in the stable direction.



Markov partitions

Define a directed graph G by letting the set of vertices be the flow boxes, and putting an edge from $I_s^{(j)} \times I_u^{(j)} \times [0,1]_t$ to $I_s^{(i)} \times I_u^{(i)} \times [0,1]_t$ for every $J_s^{(ij,k)}$.



We say that G encodes the Markov partition.

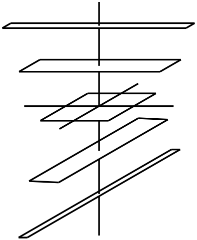
Markov partitions $\rightsquigarrow$ symbolic dynamics

Proposition

Given a cycle l of G, there is a periodic orbit c of ϕ homotopic to l. Conversely, given a periodic orbit c of ϕ, there is a cycle c of G homotopic to l.

Sketch of proof.

Given l, let $i_1, \dots, i_n$ be the sequence of vertices passed through by l. Let Z_j be the flow box $I_s^{(i_j)} \times I_u^{(i_j)} \times [0,1]_t$. The set of orbits in Z_0 passing through $Z_1, \dots, Z_N$ in forward time and $Z_{-N}, \dots, Z_{-1}$ in backward time is a decreasing sequence of flow boxes. Their intersection determines a (unique) periodic orbit c.



$\square$

Markov partitions $\rightsquigarrow$ symbolic dynamics

Proposition

Given a cycle l of G, there is a periodic orbit c of ϕ homotopic to l. Conversely, given a periodic orbit c of ϕ, there is a cycle c of G homotopic to l.

Sketch of proof.

Given c, write down a sequence of flow boxes which c passes through (might not be unique). The corresponding vertices of G form a cycle.

$\square$

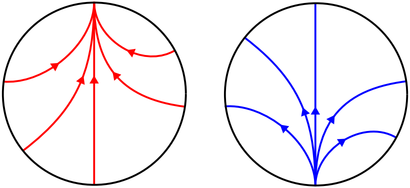
Geodesic flows

Let (S, g) be a Riemannian manifold.
Let $T^1S = \{v \in TS : ||v||_g = 1\}$ be its unit tangent bundle.
The geodesic flow on T^1S is defined by $\phi_t(v) = c'(t)$ for the unit speed geodesic $c(t)$ with $c'(0) = v$.

Geodesic flows

Fact: If g is negatively curved (and $\dim S = 2$), then the geodesic flow is Anosov.
Proof when (S, g) is hyperbolic:
Leaves of the stable/unstable foliation are

$$\{c'(t) : c \text{ has forward/backward limit point at a fixed } \xi \in \partial_\infty \mathbb{H}^2\}_\xi$$



Geodesic flows

(Still assuming g is negatively curved and $\dim S = 2$)

$$\begin{aligned} &\text{Closed orbits of geodesic flow} \\ \leftrightarrow\ &\text{Closed geodesics on } (S, g) \\ \leftrightarrow\ &\text{Isotopy classes of closed curves on } S \end{aligned}$$

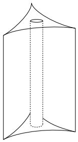
Goal: Find Markov partitions for geodesic flows

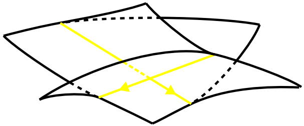
Previous work by Series, Adler-Flatto, Katok-Ugarcovici, etc.:
Symbolic dynamics found but not clear how the system sits inside the 3-manifold, also relies on geometry.
Rest of the talk: New approach by veering branched surfaces.

Veering branched surfaces

Let B be a branched surface in M. B along with a choice of orientations on the components of its branch locus is *veering* if:

1. Each sector of B is homeomorphic to a disc
2. Each component of $M \backslash\backslash B$ is a cusped solid torus or a cusped punctured solid torus.
3. At each triple point, the orientation of each component induces the maw coorientation on the other component.





Veering branched surfaces

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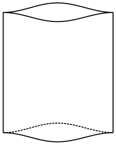
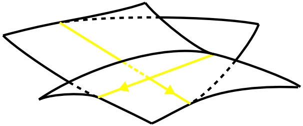
1. Each sector of B is homeomorphic to a disc
2. Each component of $M \backslash\backslash B$ is a ~~cusped solid torus or a cusped punctured solid torus~~ 2-cusped solid torus.
3. At each triple point, the orientation of each component induces the maw coorientation on the other component.





Veering branched surfaces $\rightsquigarrow$ Anosov flows

On an oriented closed 3-manifold M,

$$
\begin{aligned}
&\text{Veering branched surfaces in } M \\
\overset{\text{dual}}{\leftrightarrow}\ &\text{Veering triangulations on } M \text{ with curves drilled out} \\
\overset{!}{\rightsquigarrow}\ &\text{Pseudo-Anosov flow on } M \text{ without perfect fits rel filled orbits}
\end{aligned}
$$

Remark: Agol-Gueritaud showed a converse of the last arrow.

Veering branched surfaces $\rightsquigarrow$ Anosov flows

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Veering branched surfaces in M

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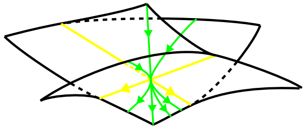
$\rightsquigarrow$ ~~Pseudo-~~Anosov flow on M without perfect fits rel filled orbits

Remark: Agol-Gueritaud showed a converse of the last arrow.

Flow graph

Given a veering branched surface B, construct its flow graph Φ by:

- Take set of vertices to be set of sectors of B
- Add 3 edges for each triple point, according to the folding action



There is a natural embedding of Φ in B
Remove the infinitesimal cycles to get the reduced flow graph Φ_{red}.

Flow graph $\rightsquigarrow$ Markov partition

Proposition (Agol-T.)

Suppose veering branched surface $B \rightsquigarrow$ Anosov flow ϕ.
Then the reduced flow graph of B encodes a Markov partition of ϕ.

Theorem (Barbot-Fenley)

An Anosov flow on a unit tangent bundle must be (conjugate via a homeomorphism isotopic to identity to) the geodesic flow of the underlying surface.

Strategy: Construct a veering branched surface on $T^1 S$ and read off its (reduced) flow graph.

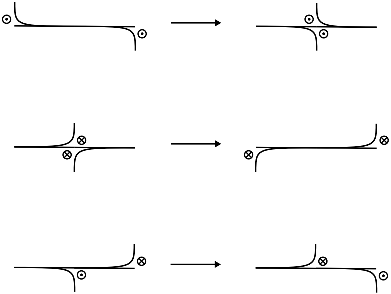
Veering branched surfaces for geodesic flows

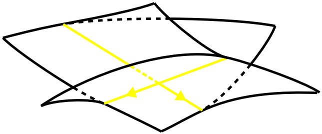
Setup: Let S be a closed surface, c be a multi-curve with no triple intersection points, such that $S \backslash\backslash c$ are $n \geq 4$-gons.
Strategy:

1. Construct a 'train track' on $T^1S|_c$.
2. Consider $T^1S|_Q =: T$ for each component Q of $S \backslash\backslash c$. There is a train track on ∂T from (1). Extend into a branched surface in T by prescribing a movie of train tracks on tori = intersection with tori sweeping inwards from ∂T.
3. Keep track of transverse orientations of switches to get orientation on components of branch locus.
4. Cap off final frame of the movie with some portion of branched surface.

Veering branched surfaces for geodesic flows

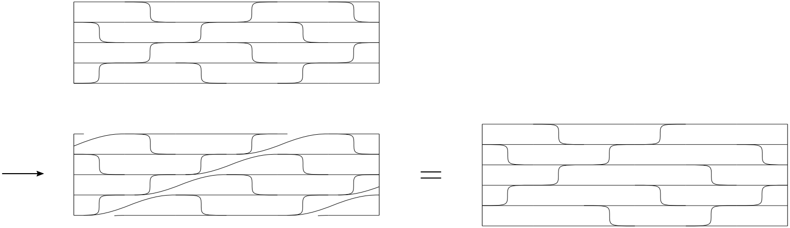
Only certain moves are allowed in the movie of train tracks, in order for condition (3) to be satisfied at each triple point.





Veering branched surfaces for geodesic flows

The movie for $Q =$ hexagon:



Veering branched surfaces for geodesic flows

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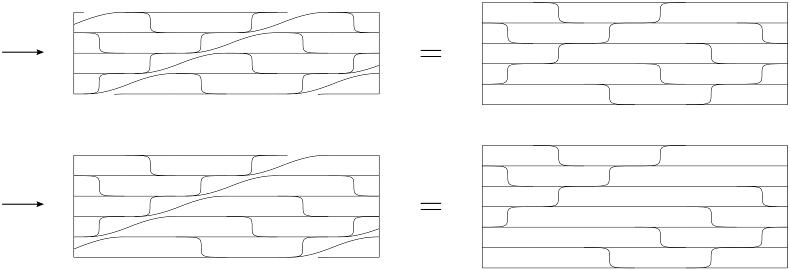

Veering branched surfaces for geodesic flows

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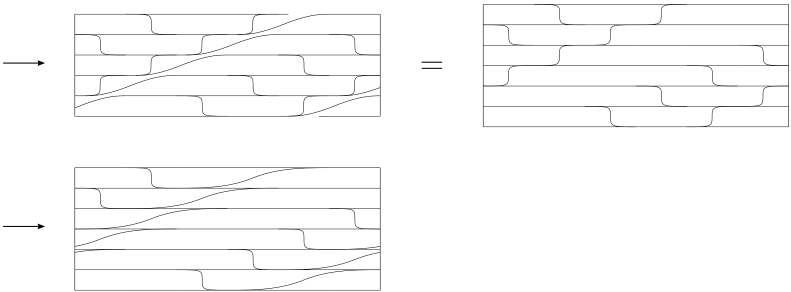
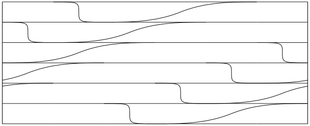


Veering branched surfaces for geodesic flows

The movie for $Q =$ hexagon:

$\longrightarrow$



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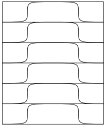

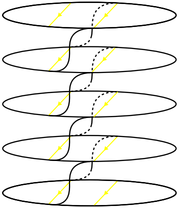

Veering branched surfaces for geodesic flows

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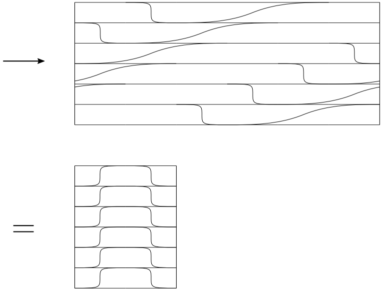
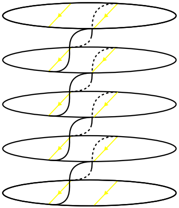
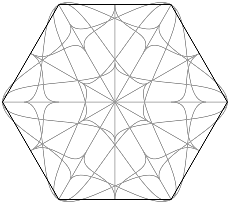


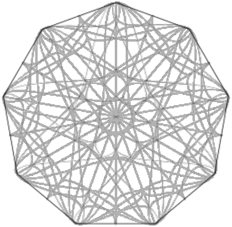


Remark: The same strategy can be used to construct laminar branched surfaces on other Seifert fibered spaces.

Markov partitions for geodesic flows

Trace through the construction to obtain the flow graph within each $T^1S|_Q$.
E.g. for $Q =$ hexagon, nonagon, the projection of the flow graph onto Q:



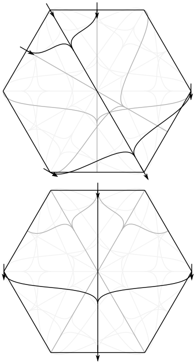


Markov partitions for geodesic flows

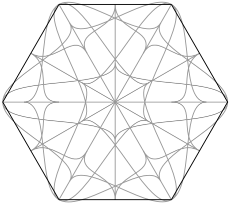
General description of the projections:
Label vertices $d_1, \dots, d_n$ and edges c_i going from d_i to d_{i+1}.

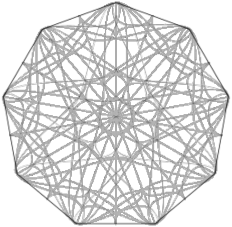
Take paths from d_i to $\begin{cases} d_{i+\frac{n}{2}}, n \text{ even} \\ c_{i+\frac{n-1}{2}}, n \text{ odd} \end{cases}$
with incoming branches from c_{i-1}, c_i, d_{i-2}, d_{i+2}, d_{i-3}, d_{i+3}, ..., $d_{i-\lceil \frac{n-1}{2} \rceil+1}, d_{i+\lceil \frac{n-1}{2} \rceil-1}$

Take paths from c_i to $\begin{cases} c_{i+\frac{n}{2}}, n \text{ even} \\ d_{i+\frac{n+1}{2}}, n \text{ odd} \end{cases}$
with incoming branches from d_{i-1}, d_{i+2}, d_{i-2}, d_{i+3}, ..., $d_{i-\lceil \frac{n}{2} \rceil+2}, d_{i+\lceil \frac{n}{2} \rceil-1}$



Markov partitions for geodesic flows





Piece together the flow graphs in each $T^1S|_Q$ to get the flow graph of the whole veering branched surface.
If complementary regions of c are all $n \geq 5$-gons, then $\Phi_{red} = \Phi$ provides a Markov partition for the geodesic flow on T^1S.

Future directions

- Use the Markov partitions to study surface-theoretic questions

  e.g. growth rates of homotopy classes/lengths of curves
- Try to find 'simplest' Markov partitions for geodesic flows

  e.g. smallest number of flow boxes
- Understand what happens when there are triangle complementary regions

  Veering branched surfaces still exists, since the drilled 3-manifolds are fibered by a classical argument of Brunella. But can they be made explicit?
- Understand geometricity of dual veering triangulations

  $\Longrightarrow$ volume bounds for full lift complements