Flows and foliations on 3-manifolds

Chi Cheuk Tsang

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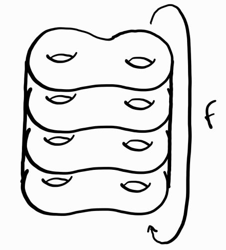
Motivating example

Let $f : S \to S$ be a homeomorphism of a (closed, oriented) surface.

The **mapping torus** of f is the 3-manifold $M_f = S \times [0,1]_t/(x,1) \sim (f(x),0)$.

The **suspension flow** is the flow generated by the vector field $\frac{\partial}{\partial t}$.

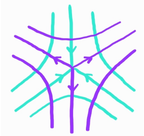
The **suspension foliation** is the partition of M_f into the surfaces $S \times \{t\}$.



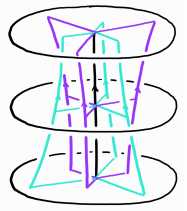
Motivating example

Theorem (Thurston 1980s, precursor to geometrization)
The 3-manifold M_f admits a hyperbolic metric if and only if f is pseudo-Anosov.

A map f is **pseudo-Anosov** if $\exists$ transverse singular 1-dimensional foliations (ℓ^s, ℓ^u) on S such that f contracts ℓ^s and expands ℓ^u.



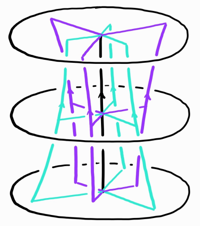
Suspending (ℓ^s, ℓ^u) gives transverse singular 2-dimensional foliations (Λ^s, Λ^u) on M_f such that flow lines contract along Λ^s and expand along Λ^u.



Motivating example

In this case,

- the foliation constraints the **dynamics** of the flow,
- the flow dictates the (coarse) **geometry** of the foliation,
- together they determine the **topology** of the 3-manifold M_f.

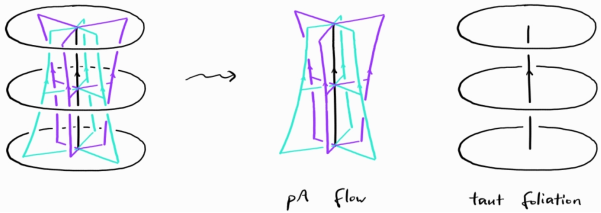


General 3-manifold

Guiding question: To what extent does this picture hold for a general (hyperbolic) 3-manifold M?

More specifically, let us define:

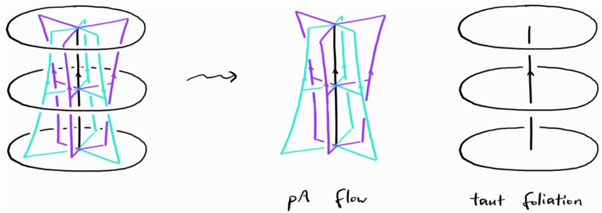
- A **pseudo-Anosov flow** to be a flow on M for which $\exists$ transverse singular foliations (Λ^s, Λ^u) on S such that leaves of Λ^s are contracted and leaves of Λ^u are expanded.
- A **taut foliation** to be a partition of M into (possibly non-compact) surfaces so that there is a closed curve passing transversely through all the surfaces.



General 3-manifold

Then one can ask:

- Which 3-manifolds admit a pseudo-Anosov flow?
- Which 3-manifolds admit a taut foliation?
- When is a taut foliation and a pseudo-Anosov flow transverse to each other?
- How can one recover topological information about the 3-manifold from a pseudo-Anosov flow and/or a taut foliation?

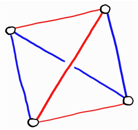


Existence of pseudo-Anosov flows

Which 3-manifolds admit a pseudo-Anosov flow?

One answer provided by veering triangulations.

A **veering triangulation** is an (ideal) triangulation of a 3-manifold M along with some additional combinatorial data.



Theorem (Agol-Guéritaud 2010, Schleimer-Segerman 2018, Landry-Minsky-Taylor 2023, Tsang 2023)

There is a 1-1 correspondence between (transitive) pseudo-Anosov flows and veering triangulations, up to suitable notions of equivalence.

Existence of pseudo-Anosov flows

Which 3-manifolds admit a pseudo-Anosov flow?

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There is a 1-1 correspondence between (transitive) pseudo-Anosov flows and veering triangulations, up to suitable notions of equivalence.

Answer: Using a computer, one can generate a list of all veering triangulations. From the theorem, one can then produce a list of all (transitive) pseudo-Anosov flows.

Existence of ~~pseudo-~~Anosov flows

Which 3-manifolds admit a ~~pseudo-~~Anosov flow?

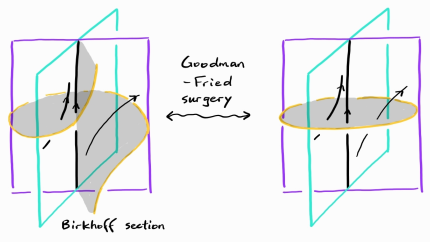
Conjecture (Fried 1983, Ghys 2000s)

Every (transitive) Anosov flow admits a genus one Birkhoff section.

If the Fried-Ghys conjecture is true...

Answer: Starting from the suspension flow of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$,

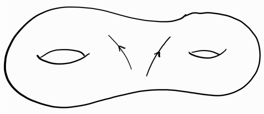
one can produce a list of all (transitive) Anosov flows by performing Goodman-Fried surgery on its closed orbits.



Progress on the Fried-Ghys conjecture

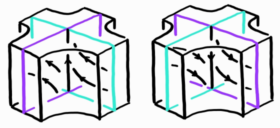
Theorem (Dehornoy-Shannon 2019)

The Fried-Ghys conjecture is true for geodesic flows on hyperbolic surfaces.



Theorem (Tsang 2024)

The Fried-Ghys conjecture is true for totally periodic Anosov flows.



First **non-explicit** construction of genus one Birkhoff sections.

Theorem (Tsang, in progress)

The Fried-Ghys conjecture is true for all Anosov flows on graph manifolds.

Progress on the Fried-Ghys conjecture

Theorem (Bonatti-Iakovoglou, Marty)

Every Anosov flow can be transformed by Goodman-Fried surgery into a contact Anosov flow.

An Anosov flow is **contact** if it is the Reeb flow of a contact form.

Conjecture (Barthelmé-Salmoiraghi-Tsang)

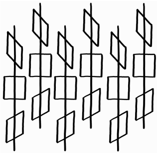
Every contact Anosov flow can be transformed by 'bicontact padding' and Goodman-Fried surgery into an Anosov flow on a graph manifold.

$$\begin{tikzcd}
& \text{general Anosov flow} \arrow[d] & \\
& \text{contact Anosov flow} \arrow[d, dashed] & \\
& \text{flow on graph manifold} \arrow[dl] \arrow[dr] & \\
\text{totally periodic flow} \arrow[dr] & & \text{geodesic flow} \arrow[dl] \\
& \text{suspension flow} &
\end{tikzcd}$$

Interlude: Contact structures

A **(positive) contact form** on an oriented 3-manifold M is a 1-form α such that $\alpha \wedge d\alpha > 0$.

A **(positive) contact structure** on M is a plane field $\xi = \ker \alpha$ determined by a (positive) contact form.



Contact topology is the 'odd-dimensional cousin' of symplectic topology: If (M, α) is a contact manifold, then $(\mathbb{R}_t \times M, d(e^t\alpha))$ is a symplectic manifold.

The **Reeb flow** of a contact form α is generated by the vector field R satisfying $\alpha(R) = 1$, $d\alpha(R, \cdot) = 0$.

Finiteness of pseudo-Anosov flows

How to classify all pseudo-Anosov flows on a 3-manifold?

Conjecture (Folklore)

Every 3-manifold has at most finitely many pseudo-Anosov flows, up to homeomorphism and reparametrization.

Known results:

When the 3-manifold is

- an Anosov mapping torus (Plante 1981)
- a graph manifold (Barbot-Fenley 2013 - 2021)
- surgery of the figure-eight knot (Yu 2023)

When one restricts the flow to

- suspension flows (Thurston, Fried 1980s)
- contact Anosov flows (Barthelmé-Bowden-Mann 2024)

Finiteness of pseudo-Anosov flows

Conjecture (Barthelmé-Tsang-Zung)

Every 3-manifold has at most finitely many pseudo-Anosov flows without perfect fits.

Here a pseudo-Anosov flow is said to have **perfect fits** if there exists a pair of closed orbits γ_1, γ_2 so that γ_1 is homotopic to γ_2^{-1}.

Strategy:

- Find a Birkhoff section whose only negative boundary orbits are along singular orbits (Tsang 2024)
- 'Blow up' the pseudo-Anosov flow into a Reeb flow using the Birkhoff section (Zung 2024)
- Use a cylindrical contact homology argument similarly as Barthelmé-Bowden-Mann 2024

Existence of taut foliations

Which 3-manifolds admit a taut foliation?

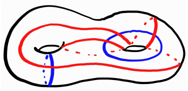
Conjecture (Boyer-Gordon-Watson 2013, Juhász 2015)

Let M be a closed orientable irreducible 3-manifold. TFAE:

1. *M admits a taut foliation.*
2. *$\pi_1 M$ admits a left order.*
3. *The Heegaard Floer homology group $HF_{red}(M)$ is nontrivial.*

A **left order** on a group is a total order such that $h < k \Rightarrow gh < gk$.

The **(reduced) Heegaard Floer homology group** $HF_{\mathrm{red}}(M)$ is computed by representing M by a Heegaard diagram.



Progress on the L-space conjecture

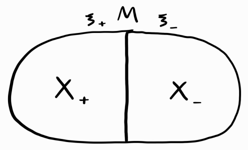
The only known direction is taut foliation $\Rightarrow HF_{\text{red}}(M) \neq 0$.

Idea of proof: Given a taut foliation $\mathcal{F}$.

Deform $\mathcal{F}$ into a positive contact structure ξ_+ and a negative contact structure ξ_- (Eliashberg-Thurston 1998, Bowden 2016, Kazez-Roberts 2017)

The contact structures $\xi_\pm$ determine contact invariants $c(\xi_\pm)$ in *HF* (Ozsváth-Szabó 2005)

Fill (ξ_+, ξ_-) and realize M as a separating 3-manifold in a symplectic 4-manifold (X, ω) (Etnyre-Honda 2002)



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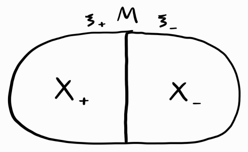


TQFT properties of *HF* and Seiberg-Witten invariants
$\Rightarrow \langle c(\xi_+), c(\xi_-) \rangle = 1$ (Kronheimer-Mrowka-Ozsváth-Szabó 2007, Kutluhan–Lee–Taubes 2020, Colin–Ghiggini–Honda 2011)

Conclusion: The contact invariant $c(\xi_+)$ certifies that $HF_{\mathrm{red}}(M) \neq 0$

Interaction between flows and foliations

Which taut foliations admit a transverse pA flow?

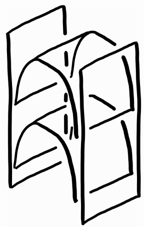
Conjecture (Thurston 1980s)

Every taut foliation on a hyperbolic 3-manifold admits an almost transverse pseudo-Anosov flow.

Known results:

When the taut foliation is

- has 'no branching' (Calegari, Fenley)
- has 'only one-sided branching' (Calegari)
- finite depth (Gabai, Mosher)



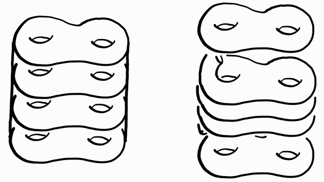
Interaction between flows and foliations

Which taut foliations admit a transverse pA flow?

Theorem (Gabai 1980s, Mosher 1990s)

Every finite depth foliation on a hyperbolic 3-manifold admits an almost transverse pseudo-Anosov flow.

A leaf of a foliation has **depth** 0 if it is compact, and has **depth** N if its accumulation set is a union of leaves of depth $\leqslant N-1$.
A foliation has **depth** N if all of its leaves are of depth $\leqslant N$.



However, the theorem has been unpublished for the past 30+ years!

PA flows transverse to finite depth foliations

Ongoing project to reprove and upgrade the theorem with Landry using the machinery of veering triangulations.

Strategy: Given a depth N foliation on a 3-manifold M.

The complement of the depth $\leqslant k$ leaves is a 3-manifold with boundary M_{k+1}, for $k = 0, ..., N$. The sequence

$$M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_N \rightsquigarrow M_{N+1}$$

is known as a **sutured hierarchy**. Each M_k can be recovered from M_{k+1} from gluing up part of its boundary.

Base step: Build a 'pseudo-Anosov flow' on M_N.

Gluing step: Given a 'pseudo-Anosov flow' on M_{k+1}, build a 'pseudo-Anosov flow' on M_k.

PA flows transverse to finite depth foliations

Base step: Build a 'pseudo-Anosov flow' on M_N.

Gluing step: Given a 'pseudo-Anosov flow' on M_{k+1}, build a 'pseudo-Anosov flow' on M_k.

$$M = M_0 \rightsquigarrow M_1 \rightsquigarrow \cdots \rightsquigarrow M_N \rightsquigarrow M_{N+1}$$

So far we have completed the base step.

Theorem (Landry-Tsang 2023)

Suppose M admits a depth N foliation, then M_N admits a 'pseudo-Anosov flow'.

Topological information from flows

How to extract topological information of 3-manifold from dynamical information of pA flow?

Theorem (Alfieri-Tsang 2025)

Suppose ϕ is a pseudo-Anosov flow without perfect fits on M. Then the Heegaard Floer homology of $M \backslash \nu(\text{singular orbits})$ can be computed from a chain complex generated by closed orbits of ϕ.

Compare with the $HF = ECH$ theorem:

Theorem (Kutluhan–Lee–Taubes 2020, Colin–Ghiggini–Honda 2011)

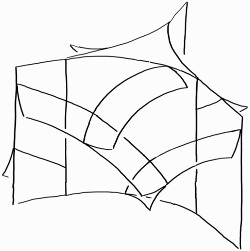
Suppose ϕ is a Reeb flow on M. Then the Heegaard Floer homology of M can be computed from a chain complex generated by closed orbits of ϕ.

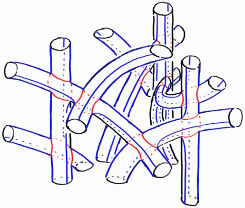
Topological information from flows

Important feature: Our proof is completely combinatorial, while $HF = ECH$ involves a lot of analysis.

Strategy of proof:

- Represent the pseudo-Anosov flow by a veering triangulation.
- Build a Heegaard diagram from the veering triangulation.
- Interpret the generators of the defining chain complex for HF in terms of closed orbits of ϕ.





Dynamical information from manifold

How to extract dynamical information of pA flow from topological information of 3-manifold?

Theorem (Alfieri-Tsang 2025)

Let f be a pseudo-Anosov map on a surface with boundary with no interior singularities. Then for $n = 1, \ldots, 2p-1$, the dimension of the n^{th} grading in the Heegaard Floer homology group

$$\dim SFH(M \backslash \nu(\textit{singular orbits})) = \frac{1}{n}(\#\ \textit{period } n \textit{ points})$$

where p is the minimum period of the periodic points.

This generalizes work of Ni and Ghiggini-Spano from 2022 which shows the case $p = 1$.

Again, our methods are completely combinatorial while previous approaches are analytic.

Dynamical information from manifold

Ongoing project with Alfieri and Alishahi to generalize the contact invariant in *HF* to pseudo-Anosov flows.

Expected applications in problems such as:

- Which 3-manifolds admit pseudo-Anosov flows?
- Which pseudo-Anosov flows admit a transverse taut foliation?
- When are two pseudo-Anosov flows related by surgery?
- When can a pseudo-Anosov flow be 'filled'?

Thanks for listening!