Veering triangulations and pseudo-Anosov flows

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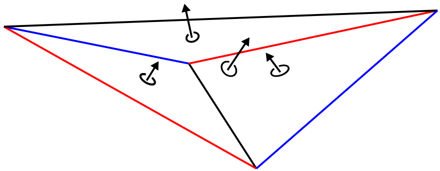
Veering triangulations

Let $\overline{M}$ be an oriented 3-manifold with torus boundary components and $M = int\overline{M}$.

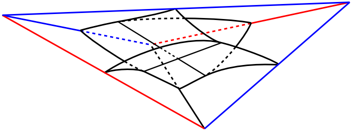
A *veering triangulation* on M is an ideal triangulation along with:

- face coorientations
- taut structure
- edge coloring

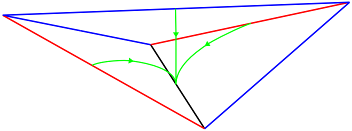


Veering triangulations

Associated to a veering triangulation:



Unstable branched surface B



Flow graph Φ

Veering triangulations

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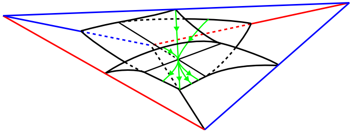

Unstable branched surface $B \supset$ Flow graph Φ

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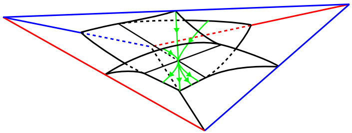
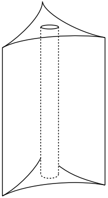


Unstable branched surface $B \supset$ Flow graph Φ

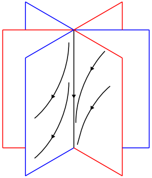
Complementary regions of B are $T^2 \times [0, \infty)$. Isotopy class of branch locus on $T^2 \times \{0\}$ are called *ladderpole curves*.



Pseudo-Anosov flows

A pseudo-Anosov flow on a closed 3-manifold N is a flow ϕ_t such that:

- There are two singular foliations Λ^s, Λ^u intersecting transversely along flow lines
- The flow contracts exponentially along Λ^s and expands exponentially along Λ^u
- (Technical conditions about Markov partitions & behavior along singular orbits)



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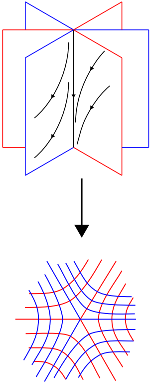

Orbit space of $\widetilde{\phi}_t$ on universal cover $\widetilde{N}$ is homeomorphic to $\mathbb{R}^2$ with two singular foliations.

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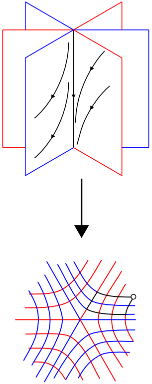


Orbit space of $\widetilde{\phi}_t$ on universal cover $\widetilde{N}$ is homeomorphic to $\mathbb{R}^2$ with two singular foliations.
ϕ_t is said to be *without perfect fits* if there are no *perfect fit rectangles* on this orbit space.

Correspondence

For $|\langle s, \text{ladderpoles}\rangle| \geqslant 3$

Veering triangulation on M $\rightsquigarrow$ Pseudo-Anosov flow without perfect fits on $\overline{M}(s)$

[Schleimer-Segerman, Agol-T.]

For $N^\circ = N \backslash \{\text{singular orbits}\}$

Veering triangulation on N° $\leftsquigarrow$ Pseudo-Anosov flow without perfect fits on N

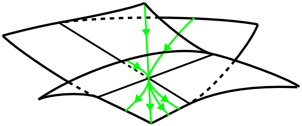
[Agol-Gueritaud, Landry-Minsky-Taylor]

Moreover, in both constructions:

- The unstable branched surface B carries the unstable foliation
- The flow graph Φ encodes the orbits

Sketch of $\rightsquigarrow$

Consider $\Phi \subset B$



Sketch of $\rightsquigarrow$

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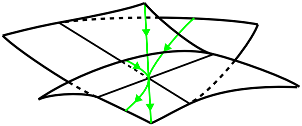

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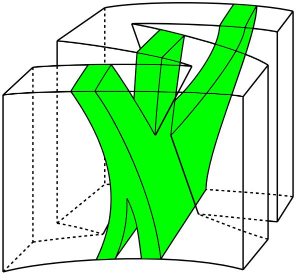

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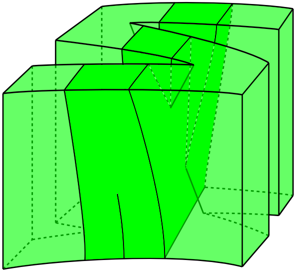

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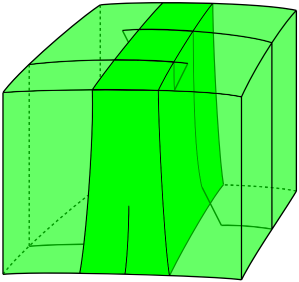

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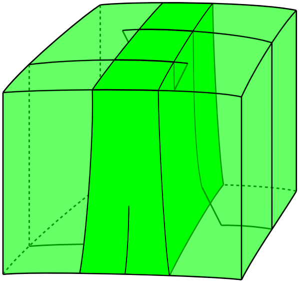
4. Glue $N(B)$ across $\overline{M}(s) \backslash\backslash N(B)$.
5. Parametrize the resulting (honest) 1-dimensional foliation on $\overline{M}(s)$.



Future directions

Veering triangulations	$\leftrightsquigarrow$	Pseudo-Anosov flows
Combinatorial invariants (e.g. top-to-bottom loops)	$\rightsquigarrow$	???
Veering surgery	$\rightsquigarrow$	???
???	$\leftsquigarrow$	Construction from finite-depth foliations [Gabai, Mosher] (Ongoing project with Michael Landry)
???	$\leftsquigarrow$	Construction from $\mathbb{R}$-covered foliations [Calegari, Fenley]