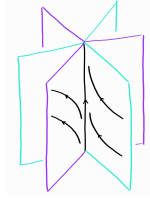


# PSEUDO-ANOSOV FLOWS, FINITE DEPTH FOLIATIONS, AND VEERING BRANCHED SURFACES

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Throughout this talk,  $M$  is a closed oriented 3-manifold.

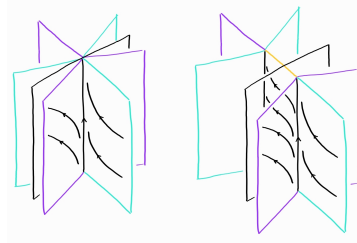
**Definition.** A **pseudo-Anosov flow** on  $M$  is a flow  $\phi$  for which there exists a pair of singular foliations  $(\Lambda^s, \Lambda^u)$  which intersect transversely along flow lines, such that the orbits of  $\phi$  contract along  $\Lambda^s$  and expand along  $\Lambda^u$ .



**Definition.** Let  $\mathcal{F}$  be a cooriented foliation on  $M$ . A leaf of  $\mathcal{F}$  is said to be **at depth 0** if it is compact. Inductively, a leaf of  $\mathcal{F}$  is said to be **at depth  $k$**  if its accumulation set is a union of leaves at depth  $\leq k - 1$ . We say that  $\mathcal{F}$  is a **finite depth foliation** if each of its leaves is at finite depth.

**Theorem 1** (Gabai, Mosher). *Let  $\mathcal{F}$  be a finite depth foliation on an irreducible, atoroidal 3-manifold  $M$ . Then there exists a pseudo-Anosov flow  $\phi$  that is almost transverse to  $\mathcal{F}$ .*

A pseudo-Anosov flow  $\phi$  is **almost transverse** to a finite depth foliation  $\mathcal{F}$  if there exists a dynamic blow-up  $\phi^\sharp$  of  $\phi$  that is transverse to  $\mathcal{F}$ .



A motivation for [Theorem 1](#) comes from the study of the Thurston norm.

**Definition.** Let  $\alpha \in H_2(M)$ . The **Thurston norm** of  $\alpha$  is defined to be

$$\|\alpha\| = \min\{|\chi(S)| \mid [S] = \alpha, S \text{ has no spherical components}\}.$$

**Fact.** When  $M$  is irreducible and atoroidal,

- $\|\cdot\|$  extends into a norm on  $H_2(M; \mathbb{R})$ , and
- the unit norm ball of  $\|\cdot\|$  is a finite-sided polyhedron.

**Theorem 2** (Mosher). *Let  $\phi$  be a pseudo-Anosov flow on an irreducible, atoroidal 3-manifold  $M$ . The cone*

$$\text{cone}(\phi) = \{[S] \in H_2(M) | S \text{ is almost transverse to } \phi\}$$

*is the cone over a subset of a face of the Thurston norm unit ball.*

When combined with results of Gabai on existence of finite depth foliations, we have

**Corollary.** *The Thurston norm unit ball of an irreducible, atoroidal 3-manifold  $M$  can be recovered from the data of all pseudo-Anosov flows on  $M$ .*

While conceptually satisfying, this corollary is impractical. It would be much more helpful if the following conjecture were true.

**Conjecture** (Mosher). *The Thurston norm unit ball of an irreducible, atoroidal 3-manifold  $M$  can be recovered from the data of finitely many pseudo-Anosov flows on  $M$ .*

An approach to this conjecture can be made by considering pseudo-Anosov flows with no perfect fits.

**Definition** (Fenley). A pseudo-Anosov flow is said to have **perfect fits** if there exists a pair of closed orbits  $\gamma_1, \gamma_2$  and a homotopically nontrivial free homotopy between  $\gamma_1$  and  $-\gamma_2$ .

**Theorem 3** (Mosher). *Let  $\phi$  be a pseudo-Anosov flow on an irreducible, atoroidal 3-manifold  $M$  that has no perfect fits. The cone*

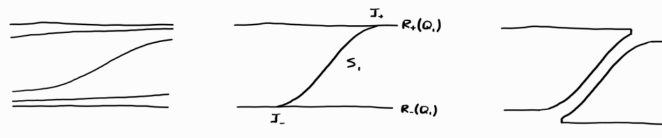
$$\text{cone}(\phi) = \{[S] \in H_2(M) | S \text{ is almost transverse to } \phi\}$$

*is the cone over a face of the Thurston norm unit ball.*

**Question.** When do Gabai and Mosher's pseudo-Anosov flows have no perfect fits?

We tackle this question using the language of sutured hierarchies and veering branched surfaces.

Suppose  $\mathcal{F}$  is a finite depth foliation on  $M$ . We set  $Q_0 = M$ , and let  $S_0$  be the union of depth 0 leaves. Let  $Q_1$  be the result of cutting  $Q_0$  along  $S_0$ . Let  $S_1$  be the union of depth 1 leaves with the portions where they spiral into  $S_0$  truncated. Inductively, let  $Q_2$  be the result of cutting  $Q_1$  along  $S_1$  and so on.



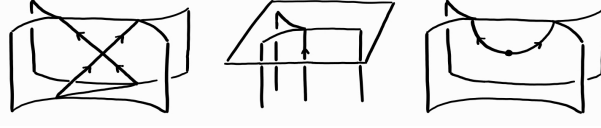
Each  $Q_k$  is a sutured manifold. Intuitively, this means that  $\partial Q_k = R_+(Q_k) \cup \gamma(Q_k) \cup R_-(Q_k)$  where  $\partial Q_k$  is positively/negatively tangent to  $\mathcal{F}$  along  $R_\pm(Q_k)$  and transverse to  $\mathcal{F}$  along  $\gamma(Q_k)$ .

Since  $\mathcal{F}$  is of finite depth, eventually some  $Q_n$  will be a product sutured manifold, i.e.  $Q_n \cong S_n \times [0, 1]$ , at which point the process terminates. The sequence  $Q_0 \rightsquigarrow \dots \rightsquigarrow Q_n$  is known as a **sutured hierarchy**.

**Definition.** A **veering branched surface** on a sutured manifold  $Q$  is a subset  $B$  locally modelled on one of the following pictures

such that

- there is a nonvanishing vector field tangent to  $B$  that is positively transverse to the branch locus,
- there is a source orientation on the branch locus as indicated above,



- all cusped solid torus complementary regions have negative index, i.e. are  $\geq 3$ -pronged, and
- some more technical conditions.

**Theorem 4** (Schleimer-Segerman, T.). *Let  $B$  be a veering branched surface on a closed, oriented, irreducible, atoroidal 3-manifold  $M$ . Then  $M$  admits a pseudo-Anosov flow  $\phi$  whose unstable foliation is carried by  $B$ .*

**Idea of the project.** Build veering branched surfaces inductively up the sutured hierarchy.

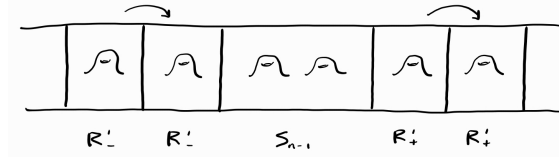
**Theorem 5** (Landry-T.). *Let  $Q_0 \rightsquigarrow \dots \rightsquigarrow Q_n$  be a sutured hierarchy of an irreducible and atoroidal 3-manifold  $M$ , then  $Q_{n-1}$  admits a veering branched surface.*

*Moreover, the veering branched surface is essentially canonical given the data of the sutured hierarchy.*

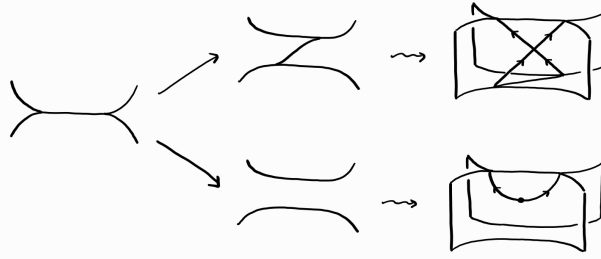
*Overview of the proof.* The interior of  $Q_{n-1}$  fibers over  $S^1$  where the fibers are the restrictions of the depth  $n-1$  leaves. In the notation above, each fiber can be written as

$$L = \bigcup_{i=-\infty}^{-1} R'_-(Q_{n-1}) \cup_{J_-} S_{n-1} \cup_{J_+} \bigcup_{i=1}^{\infty} R'_+(Q_{n-1})$$

where  $J_{\pm} = S_{n-1} \cap R_{\pm}(Q_{n-1})$  and  $R'_{\pm}(Q_{n-1}) = R_{\pm}(Q_{n-1}) \setminus J_{\pm}$ . The monodromy on the fiber is an endperiodic map  $f : L \rightarrow L$  which translates the  $i^{\text{th}}$  copy of  $R'_{\pm}(Q_{n-1})$  to the  $(i+1)^{\text{th}}$  copy of  $R'_{\pm}(Q_{n-1})$ .



Handel and Miller showed that there exists a pair of laminations  $(\Lambda_+, \Lambda_-)$  on  $L$  such that  $f$  stretches the leaves of  $\Lambda_+$  and contracts the leaves of  $\Lambda_-$ . We consider endperiodic train tracks that carry  $\Lambda_+$ . Generalizing work of Agol, we show that there is an essentially unique periodic splitting sequence  $\tau_0 \rightarrow \dots \rightarrow \tau_p = f(\tau_0)$ , where by a splitting, we refer to one of the following moves.



By suspending the splitting sequence, we obtain a veering branched surface. Conversely, every veering branched surface that carries the unstable Handel-Miller lamination can be isotoped to intersect  $S_{n-1}$  transversely, thus comes from suspending a periodic splitting sequence, and is essentially unique.

To be more precise about this essential uniqueness:

- The periodic splitting sequence is unique up to commutation if the endperiodic part of the train tracks is fixed. This means that the veering branched surface is unique up to isotopy if its boundary train track is fixed.
- The possible options for the boundary train track differ from each other by shifting moves. The corresponding veering branched surfaces differ by some prescribed move.

□

One can use these veering branched surfaces to study foliation cones.

**Theorem 6** (Cantwell-Conlon). *Let  $Q$  be a sutured manifold. The set*

$$\{[S] \in H_2(Q, \partial Q) \mid Q \setminus S \text{ is a product sutured manifold}\}$$

*is exactly the set of integer points in the interior of finitely many **foliation cones**.*

*For every foliation cone  $C$ , there exists a semiflow  $\phi_C$  on  $Q$  such that  $C$  is the dual cone to*

$$\{[\gamma] \in H_1(Q) \mid \gamma \text{ is a closed orbit of } \phi_C\}.$$

**Theorem 7** (Landry-T.). *Let  $Q$  be an atoroidal sutured manifold. The veering branched surfaces associated to  $Q \rightsquigarrow Q \setminus S$ , as  $[S]$  varies over the interior of a foliation cone  $C$ , are essentially isotopic. We denote this common veering branched surface as  $B_C$ .*

*Every closed orbit of  $\phi_C$  is homotopic to a curve on  $B_C$  that is positively transverse to  $\text{brloc}(B_C)$ , and conversely every curve on  $B_C$  that is positively transverse to  $\text{brloc}(B_C)$  is homotopic to a closed orbit of  $\phi_C$ .*

*In particular, one can recover  $C$  as the dual cone to*

$$\{[c] \in H_1(Q; \mathbb{R}) \mid c \subset B \text{ is positively transverse to } \text{brloc}(B)\}.$$

More quantitatively, one can define a **flow graph**  $\Phi$  on  $B$ , which allows one to compute the entropy function on  $C$  (recently defined by Landry-Minsky-Taylor) as the growth rate of directed cycles in  $\Phi$ .

### Problems.

- Generalize the taut and veering polynomials from the closed manifold setting to the sutured manifold setting.
- Interpret the generalized taut polynomial as a twisted Alexander polynomial. Use this to understand the relation between foliation cones and sutured Thurston norm unit balls.
- One can repeat the construction using the negative Handel-Miller lamination  $\Lambda_-$  instead. Can one isotope the stable and unstable veering branched surfaces into a dynamic pair?
- Given a foliation cone  $C$  on a sutured manifold  $Q$ , a result of Landry-Minsky-Taylor states that one can embed  $Q$  in a closed 3-manifold  $M$  such that  $C$  is the restriction of a fibered cone  $F$  on  $M$ . What is the relation between the veering branched surface  $B_C$  on  $Q$  and the veering branched surface  $B_F$  on  $M$ ? What is the relation between their polynomial invariants, etc?
- Generalize the dual notion of veering triangulations from the closed manifold setting to the sutured manifold setting. Generate some sort of census for these combinatorial objects.

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**Next stage in the project.** For a sutured manifold decomposition  $Q \rightsquigarrow Q'$ , investigate how to take a veering branched surface  $B'$  in  $Q'$  and build one in  $Q$ .

We do not expect this to be always possible. Suppose there is a pair of curves  $c_1, c_2 \subset B'$  that are positively transverse to  $\text{brloc}(B')$ , such that  $c_1$  becomes homotopic to  $-c_2$  after recomposing  $Q$  from  $Q'$ , then intuitively, the induced semiflow on  $Q$  would contain a pair of closed orbits  $\gamma_1, \gamma_2$  where  $\gamma_1$  is

homotopic to  $-\gamma_2$ . This pair of closed orbits will persist into the closed manifold  $M$  and give rise to perfect fits.

A priori, this condition seems difficult to certify, since there are infinitely many curves to check. However, we believe that only the boundary parallel orbits matter, and that instead of being homotopic, one can just check whether these orbits are parallel. Intuitively, this is because the topology of the stable/unstable foliations forbid the homotopy from interfering with the core dynamics.

We say that the decomposition  $Q \rightsquigarrow Q'$  satisfies the **No Oppositely Oriented Parallel Orbits (NOOPO) condition** if there does not exist a pair of boundary parallel curves  $c_1, c_2 \subset B'$  that are positively transverse to  $\text{brloc}(B')$ , such that  $c_1$  becomes parallel to  $-c_2$  after recomposing  $Q$  from  $Q'$ . This is now a condition that can be checked in finite time.

**Conjecture.** *Let  $B' \subset Q'$  be a veering branched surface on a sutured manifold. Suppose  $Q \rightsquigarrow Q'$  is a sutured manifold decomposition that satisfies the NOOPO condition. Then  $Q$  admits a veering branched surface  $B$ .*

#### Strategy.

- Extend  $B'$  into a dynamic pair  $(B^s, B^u)$  where  $B^u = B'$ .
- Inductively, transfer the boundary train track of  $B^u$  from  $R_+$  to  $R_-$  and ‘flow it forward’ in each complementary region of  $B^s \cup B^u$ . Take care to ensure that the source orientation condition is preserved. This process stabilizes eventually.
- Similarly, transfer the boundary train track of  $B^s$  from  $R_-$  to  $R_+$  and ‘flow it backward’ in each complementary region of  $B^s \cup B^u$ . This process stabilizes eventually.
- Perform the recomposition. We now have a pair of branched surfaces in  $Q$ . Analyze the complementary regions of their union. Subdivide these regions further if necessary.