Problem 1. *Consider samples $X_1, \dots, X_n$ are from Simple Random Sampling(sample without replacement) with population size N. From the Lecture, we know that there is an unbias estimator for σ^2,*

$$\frac{S^2}{n}(1 - \frac{n}{N}),$$

where

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$

Please proof the following statements.

(i) If $X_1, \cdots, X_n$ are either 1 or 0, then $\hat{\sigma^2}(\bar{X}_n) := \frac{\bar{X}_n(1-\bar{X}_n)}{n-1}\left(1 - \frac{n}{N}\right)$ is an unbias estimator for σ^2.

(ii) $\sup_p \hat{\sigma^2}(p) = \frac{1}{4(n-1)}\left(1 - \frac{n}{N}\right)$.

(iii) Apply CLT, then $\bar{X}_n \pm z_{\alpha/2}\frac{1}{2}\sqrt{\frac{1}{n-1}\left(1 - \frac{n}{N}\right)}$ is a $(1-\alpha)$ confidence interval for all $\mu = E[X]$.

Problem 2. *Under Simple Random Sampling, find an unbias estimator for μ^2.*