STAT 610: Discussion 9

1 Summary

- UMP test for one sided hypothesis:
  - Monotone likelihood ratio (MLR): $f_\theta(x)$ is MLR in $Y(X)$ if for any $\theta_1 < \theta_2$, $\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)}$ is monotone of $Y(x)$ for values of x at which at least one of $f_{\theta_1}(x)$ and $f_{\theta_2}(x)$ is positive.
  - **Karlin-Rubin Theorem**: $H_0 : \theta \leq \theta_0$ v.s. $H_1 : \theta > \theta_0$. $f_\theta(x)$ has non-decreasing MLR of $Y(X)$. Then

$$T(X) = \begin{cases} 1, \text{ if } Y(X) > c \\ 0, \text{ if } Y(X) < c \end{cases},$$

    is a level-α UMP test.

- UMPU test for one parameter exp-family:
  - Unbiased test: A test with power function $\beta(\theta)$ is *unbiased* if $\beta(\theta') \geq \beta(\theta'')$ for every $\theta \in \Theta_0^c$ and $\theta'' \in \Theta_0$.
  - Suppose that U is a sufficient statistic for $\theta \in \mathbb{R}$ with pdf or pmf $g_\theta(u) = h(u)c(\theta)e^{w(\theta)u}$. Consider $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta \neq \theta_0$. A UMPU test of size α satisfies

$$T(X) = \begin{cases} 1, \text{ if } U(X) < c_1 \text{ or } U(X) > c_2 \\ 0, \text{ if } c_1 < U(X) < c_2 \end{cases}$$

    for some constants c_1 and c_2 such that $\mathbb{E}_{\theta_0}(T) = \alpha$ and $\mathbb{E}_{\theta_0}(TU) = \alpha\mathbb{E}_{\theta_0}(U)$.

- p-value:
  - Valid p-value: For every $\theta \in \Theta_0$ and every $0 \leq \alpha \leq 1$,

$$\mathbb{P}_\theta(p(X) \leq \alpha) \leq \alpha.$$

  - Commonly used p-value: If large value of $W(X)$ gives evidence that H_1 is true, then

$$p(x) = \underset{\theta \in \Theta_0}{sup} \mathbb{P}_\theta(W(X) \geq W(x))$$

    is a valid p-value.
  - Another way to define p-value: The smallest possible level α at which H_0 would be rejected for the computed level α test $T_\alpha(x)$, i.e.,

$$p(x) = \inf\{\alpha \in (0, 1) : T_\alpha(x) \text{ rejects } H_0\}.$$

2 Questions

1. The random variable X has pdf $f(x) = e^{-x}, x > 0$. One observation is obtained on the random variable $Y = X^\theta$. Construct a UMP test of size α for $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

2. Consider the following distribution

$$f(x|\theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

   (a) Show that this family has an MLR.

   (b) Find the UMP test of size α for $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$. base on one observation X.

3. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be i.i.d. samples from $\mathcal{N}(\mu_1, 1)$ and $\mathcal{N}(\mu_2, 1)$. Find a UMPU test of size α for the hypothesis $H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 \neq \mu_2$.

4. Prove that $p(x) = \inf\{\alpha \in (0, 1) : T_\alpha(x) \text{ rejects } H_0\}$ is a valid p-value.