STAT 610: Discussion 12

1 Summary

- Types of convergence
  - Converge almost surely: $\mathbf{X}_n \xrightarrow{a.s.} \mathbf{X}$.
  - Converge in probability: $\mathbf{X}_n \xrightarrow{p} \mathbf{X}$.
  - Converge in distribution (weak convergence): $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$.
- (**Continuous mapping theorem**) Let $\mathbf{X}_1$, $\mathbf{X}_2$,... be random k-vectors and g be measurable and continuous function from $\mathbb{R}^k$ to $\mathbb{R}^l$. Then $\mathbf{X}_n \to \mathbf{X}$ (a.s. / in prob. / in dist.) implies $g(\mathbf{X}_n) \to g(\mathbf{X})$ (a.s. / in prob. / in dist.).
- (**Slutsky's theorem**) Suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ for a constant c, then
  - $X_n + Y_n \xrightarrow{d} X + c$,
  - $X_n Y_n \xrightarrow{d} cX$,
  - $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}$ if $c \neq 0$.
- (**First order delta method**) If $\sqrt{n}(X_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ and g is a function satisfying that $g'(\theta)$ exists and is not 0, then

$$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{d} \mathcal{N}(0, \sigma^2[g'(\theta)]^2)$$

- Consistency: A sequence of estimators T_n is consistent w.r.t θ if $T_n \xrightarrow{p} \theta$.
- (**Consistency of MLEs**) Let $X_1, \dots, X_n \overset{i.i.d}{\sim} f_\theta(x)$. Let $\hat{\theta}_n$ denote the MLE of θ. Under certain regularity conditions, $\hat{\theta}_n$ is a consistent estimator of θ_n.
- Suppose that $k_n(T_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$. Then σ^2 is called the *asymptotic variance* of T_n.
- A sequence T_n is *asymptotic efficient* for $\tau(\theta)$ if $\sqrt{n}\{T_n - \tau(\theta)\} \xrightarrow{d} \mathcal{N}(0, \nu(\theta))$, where

$$\nu(\theta) = \frac{\{\tau'(\theta)^2\}}{\mathbb{E}\left[\left\{\frac{\partial}{\partial\theta} \log f_\theta(X)\right\}^2\right]}.$$

- Under regularity conditions, MLEs are asymptotic efficient.
- Suppose $\sqrt{n}(T_n - \theta) \xrightarrow{d} (0, \sigma_T^2)$ and $\sqrt{n}(W_n - \theta) \xrightarrow{d} (0, \sigma_W^2)$. The *asymptotic relative efficiency* (ARE) of W_n w.r.t. T_n is $\text{ARE}(W_n, T_n) = \sigma_T^2 / \sigma_W^2$

2 Questions

1. A random sample $X_1, \ldots, X_n$ is drawn from the following pdf

$$f_\theta(x) = \frac{1}{2}(1 + \theta x), \quad -1 < x < 1, \quad -1 < \theta < 1.$$

   Find a consistent estimator of θ and show it is consistent.

2. Let $X_1, \ldots, X_n$ be a random sample of random variables with unknown mean $\mu \in \mathbb{R}$ and unknown variance $\sigma^2 > 0$ and $\mathbb{E}(X^4) < \infty$. Consider the estimation of μ^2 and the following three estimators:

   - $T_{1n} = \bar{X}^2$,
   - $T_{2n} = \bar{X}^2 - S^2/n$,
   - $T_{3n} = \max\{0, T_{2n}\}$.

   (a) Show that the asymptotic variance of $T_{jn}, j = 1, 2, 3$, are the same when $\mu \neq 0$.

   (b) Find the asymptotic distribution when $\mu \neq 0$.

   (c) Find the asymptotic distribution when $\mu = 0$.

   (d) Which one is better?

3. Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{N}(0, \sigma^2)$. Consider the estimation of σ. Consider these two estimates $T_{1n} = \sqrt{\pi/2} \sum_{i=1}^n |X_i|/n$ and $T_{2n} = (\sum_{i=1}^n X_i^2/n)^{1/2}$.

   (a) Find the asymptotic distributions of T_{1n} and T_{2n}.

   (b) Which one is better?

   Hint: You can use the fact that $\mathbb{E}(\sqrt{\pi/2}|X_1|) = \sigma$ *and* $Var(\sqrt{\pi/2}|X_1|) = (\frac{\pi}{2} - 1)\sigma^2$