STAT 610: Discussion 4

1 Summary

- Cramér-Rao Lower Bound

  Suppose f_θ is differentiable as a function of θ and satisfies

  $$\frac{d}{d\theta}\int h(x)f_\theta(x)dx = \int h(x)\frac{\partial}{\partial\theta}f_\theta(x)dx$$

  for $h(x)=1$ and $h(x)=T(x)$. Then,

  $$\mathrm{Var}_\theta\{T(\mathbf{X})\} \geq \{\frac{\partial}{\partial\theta}g(\theta)\}^T\{I(\theta)\}^{-1}\{\frac{\partial}{\partial\theta}g(\theta)\},$$

  where

  $$I(\theta) = \mathbb{E}\left\{\frac{\partial}{\partial\theta}\log f_\theta(X)\left[\frac{\partial}{\partial\theta}\log f_\theta(X)\right]^T\right\}.$$

- An alternative way of calculating Fisher information matrix

  (**Lemma 7.3.11**) If $f_\theta(x)$ satisfies

  $$\frac{d}{d\theta}\mathbb{E}_\theta\left(\frac{\partial}{\partial\theta}\log f_\theta(X)\right) = \int\frac{\partial}{\partial\theta}\left\{\left(\frac{\partial}{\partial\theta}\log f_\theta(X)\right)f_\theta(X)\right\}$$

  (true for an expential family), then

  $$\mathbb{E}_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f_\theta(X)\right)^2\right\} = -\mathbb{E}_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\right).$$

- Uniform Minimum Variance Unbiased Estimator (UMVUE)
  - **Definition**: T is UMVUE of $g(\theta)$ if T has the smallest variance among all unbiased estimators of $g(\theta)$.
  - **Rao-Blackwell Theorem**: We learned it in Lecture 2.
  - **Theorem 7.3.19**: UMVUE is unique.
  - **Theorem 7.3.20**: W is UMVUE $\Leftrightarrow$ $\mathbb{E}(WU)=0$ for all U satisfying $\mathbb{E}(U)=0$.
  - **Lehmann-Scheffé Theorem**: If T is complete sufficient for θ. If $\psi(T)$ is an unbiased estimator of $g(\theta)$, then it is the unique UMVUE.
  - There are two way for finding UMVUE.
    * Find ψ.
    * Find an unbias estimator W for $g(\theta)$. Then, calculate $E[W|T]$.

2 Questions

1. Let $X_1, \ldots, X_n$ be i.i.d. $Ber(p)$. Find the UMVUE of following parameters.
   - p^m, for all $m \leq n$.
   - $P(X_1 + \cdots + X_m = k)$, where m and k are positive integers $\leq n$.
   - Find the UMVUE of $P(X_1 + \cdots + X_{n-1} > X_n)$.

2. Let $X_1, \ldots, X_n$ be i.i.d $E(a, \theta)$. Find the UMVUE of following situation.
   - Find the UMVUE of a when θ is known.
   - Find the UMVUE of θ when a is known.
   - Find the UMVUE of a and θ.

3. Suppose that T is a UMVUE of an unknown parameter θ, and for any integer $k > 0$, we have $\mathbb{E}(T^k) < \infty$. Show that T^k is a UMVUE of $\mathbb{E}(T^k)$.