STAT 610: Discussion 5

1 Summary

- Linear model: $Y = X\beta + \epsilon$.

$$MSE : \hat{\beta} = (X^\top X)^{-1} X^\top Y$$

$$SSR : \|Y - X^\top \hat{\beta}\|^2$$

- Assumptions

  (A1) $\epsilon \sim N(0, \sigma^2 I_n)$

  (A2) $E[\epsilon] = 0$, $Var(\epsilon) = \sigma^2 I_n$

  (A3) $E[\epsilon] = 0$, $Var(\epsilon) = V$

  A1 implies A2 implies A3.

- A linear models with full rank $X^\top X$ and assumption (A1).
  - $\beta \sim N(\beta, \sigma^2 (X^\top X)^{-1})$, and $\ell^\top \beta$ is the UMVUE for $\ell^\top \beta$ for any $\ell \in \mathbb{R}^p$.
  - $SSR/\sigma^2 \sim \chi_{n-p}$. $SSR/(n-p)$ is the UMVUE for σ^2
  - SSR and $\hat{\beta}$ are independent.

- Under assumption (A2), we consider (BLUE).
  - For everey $\ell \in \mathbb{R}^p$, the BLUE for $\ell^\top \beta$ is the one has the smallest variance within the following class of linear unbiased estimators

$$\{c^\top Y | \quad E[c^\top Y] = \ell^\top \beta\}.$$

  - Gauss-Markov Theorem: Given a linear models with full rank $X^\top X$ and assumption (A2). We have $\ell^\top \hat{\beta}$ is BLUE for $\ell^\top \beta$.

- Under assumption (A3), consider weighted least square.

$$\hat{\beta}_{V^{-1}} = (X^\top V^{-1} X)^{-1} X^\top V^{-1} Y.$$

2 Questions

1. Suppose that
$$Y_{ij} = \alpha_i + \beta t_{ij}, \qquad i = 1, \dots, a, \; j = 1, \dots, b.$$
   (a) Express the data in a linear model $Y = X\beta + \epsilon$ and identify X.
   (b) In what condition X is full rank?
   (c) Obtain an LSE of β.

2. Under linear models and assumption (A1), find the UMVUE's of
   (a) $(\ell^\top \beta)^2$,
   (b) $\ell^\top \beta / \sigma$,
   (c) $(\ell^\top \beta / \sigma)^2$.

3. Suppose $Y_{n\times 1} = X_{n\times p}\beta_{p\times 1} + \epsilon_{n\times 1}$, where $\epsilon_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2)$ is independent of X_i, and the design matrix $X_{n\times p}$ is of full rank.
   (a) If A is any arbitrary $p \times n$ matrix such that AX is invertible. Prove that $\tilde{\beta} = (AX)^{-1}AY$ is a linear unbiased estimator of β, and $(AX)^{-1}AA^T(X^TA^T)^{-1} \geq (X^TX)^{-1}$.
   (b) If A is any invertible symmetric matrix of size n, prove that $\tilde{\beta} = (X^TAX)^{-1}X^TAY$ is a linear unbiased estimator of β, and $(X^TAX)^{-1}X^TA^2X(X^TAX)^{-1} \geq (X^TX)^{-1}$.