

STAT 610: Discussion 10

1 Summary

- Union-Intersection and Intersection Tests (UIT/IUT):

- UIT: Suppose we want to test $H_0 : \theta \in \cap_{\gamma \in \Gamma} \Theta_\gamma$ v.s. $H_1 : \theta \in \cup_{\gamma \in \Gamma} \Theta_\gamma^c$. Suppose R_γ is the rejection region of test $H_0 : \theta \in \Theta_\gamma$ v.s. $H_1 : \theta \in \Theta_\gamma^c$. Then, UIT is $R = \cup_{\gamma \in \Gamma} R_\gamma$. If R_γ is of the form $\{x : T_\gamma(x) > c\}$, then

$$R = \cup_{\gamma \in \Gamma} \{x : T_\gamma(x) > c\} = \{x : \sup_{\gamma \in \Gamma} T_\gamma(x) > c\}.$$

- IUT: Suppose we want to test $H_0 : \theta \in \cup_{\gamma \in \Gamma} \Theta_\gamma$ v.s. $H_1 : \theta \in \cap_{\gamma \in \Gamma} \Theta_\gamma^c$. If each $H_{0\gamma}$ has the rejection region $R_\gamma = \{x : T_\gamma(x) > c\}$, then IUT has the rejection region

$$R = \cap_{\gamma \in \Gamma} \{x : T_\gamma(x) > c\} = \{x : \inf_{\gamma \in \Gamma} T_\gamma(x) > c\}.$$

- Level of UIT: If R_γ has level α_γ , then the overall level of UIT is at most $\sum_{\gamma \in \Gamma} \alpha_\gamma$.
- Level of IUT: If R_γ has level α_γ , then the overall level of IUT is at most $\min_{\gamma \in \Gamma} \alpha_\gamma$.
- Relationship between LRT and UIT: Refer to *Theorem 8.3.21* in the textbook.

- Confidence Interval:

- The *coverage probability* is defined as $\mathbb{P}_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$ for an interval $[L(\mathbf{X}), U(\mathbf{X})]$.
- The *confidence coefficient* is defined as $\inf \mathbb{P}_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$, which is the infimum of the coverage probability.

- Pivotal quantities: A random variable $Q(\mathbf{X}, \theta) = Q(X_1, \dots, X_n, \theta)$ is a *pivotal quantity* if the distribution of $Q(\mathbf{X}, \theta)$ is independent of all parameters. That is, if $\mathbf{X} \sim F(\mathbf{x} \mid \theta)$, then $Q(\mathbf{x}, \theta)$ has the same distribution for all values of θ .

- e.g. If $X_i \stackrel{i.i.d.}{\sim} \text{Exp}(\theta)$, then $T = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. Hence $Q(T, \theta) = 2T/\theta \sim \chi_{2n}^2$ is a pivotal quantity.
- After figuring out a, b such that $\mathbb{P}(a \leq Q(\mathbf{X}, \theta) \leq b) \geq 1 - \alpha$, then $C(\mathbf{x}) = \{\theta : a \leq Q(\mathbf{x}, \theta) \leq b\}$ is a $1 - \alpha$ confidence set for θ .
- If $Q(\mathbf{x}, \theta)$ is a monotone function of θ , then $C(\mathbf{x})$ will be an interval.

- Pivoting a cdf: Let T be a statistic with cdf $F_T(t \mid \theta)$. Let $\alpha_1 + \alpha_2 = \alpha$ be constants.

- If $F_T(t \mid \theta)$ is a decreasing function of θ for each t , define $\theta_L(t)$ and $\theta_U(t)$ by

$$F_T(t \mid \theta_U(t)) = \alpha_1, \quad F_T(t \mid \theta_L(t)) = 1 - \alpha_2.$$

- If $F_T(t \mid \theta)$ is an increasing function of θ for each t , define $\theta_L(t)$ and $\theta_U(t)$ by

$$F_T(t \mid \theta_U(t)) = 1 - \alpha_2, \quad F_T(t \mid \theta_L(t)) = \alpha_1.$$

Then $[\theta_L(T), \theta_U(T)]$ is a $1 - \alpha$ confidence interval for θ .

2 Questions

1. Consider testing $H_0 : \theta \in \cup_{j=1}^k \Theta_j$. For each $j = 1, \dots, k$, let $p_j(\mathbf{x})$ denote a valid p-value for testing $H_{0j} : \theta \in \Theta_j$. Let $p(\mathbf{x}) = \max_{1 \leq j \leq k} p_j(\mathbf{x})$.
 - (a) Show that $p(\mathbf{X})$ is a valid p-value for testing H_0 .
 - (b) Show that the α level test defined by $p(\mathbf{X})$ is the same as an α level IUT defined in terms of individual tests based on the $p_j(\mathbf{x})$ s.

2. Find a $1 - \alpha$ confidence interval for θ using pivots, given X with pdf
 - (a) $f(x | \theta) = 1, \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$.
 - (b) $f(x | \theta) = 2x/\theta^2, 0 < x < \theta$.

3. Let X_1, \dots, X_n be i.i.d uniform $(0, \theta)$. Let Y be the largest order statistic. Prove that Y/θ is a pivotal quantity and show that $[y, y \cdot \alpha^{-1/n}]$ is the shortest $1 - \alpha$ pivotal interval.

4. If X_1, \dots, X_n are i.i.d from a location pdf $f(x - \theta)$. Show that the confidence set

$$C(x_1, \dots, x_n) = \{\theta : \bar{x} - k_1 \leq \theta \leq \bar{x} + k_2\},$$

where k_1, k_2 are constants, has constant coverage probability.

5. Let X_1, \dots, X_n be a random variable with pdf $f_X(x) = \theta a^\theta x^{-(\theta+1)} I_{(a, \infty)}(x)$, where $\theta > 0$ and $a > 0$.

- (a) When θ is known, derive a confidence interval for a with confidence coefficient $1 - \alpha$ by pivoting the cdf of the smallest order statistic $X_{(1)}$.
- (b) When both a and θ are unknown and $n \geq 2$, derive a confidence interval for θ with confidence coefficient $1 - \alpha$ by pivoting the cdf of $T = \prod_{i=1}^n (X_i/X_{(1)})$.

Hint: You can use the fact that $2\theta \log T \sim \chi_{2(n-1)}^2$ and then write the cdf of T in terms of the cdf of $\chi_{2(n-1)}^2$.

- (c) When both a and θ are unknown, construct a confidence set for (a, θ) with confidence coefficient $1 - \alpha$ using a pivotal quantity.

Hint: Notice that $X_{(1)}/a$ is free of a , and $X_{(1)}^\theta$ is free of θ .