STAT 610: Discussion 5

1 Summary

- Hypothesis test: after observing samples $\mathbf{X} \sim \mathbb{P}_\theta$, decide either to retain $H_0 : \theta \in \Theta_0$ (null hypothesis) or to reject H_0 and conclude $H_1 : \theta \in \Theta_0^c$ (alternative hypothesis) is true.
- Basic definitions:
  - Type I error: reject H_0 if H_0 is true.
  - Type II error: retain H_0 if H_0 is false.
  - Let R denote the rejection region for a test. Then

    $$\beta(\theta) = \mathbb{P}_\theta(\mathbf{X} \in R) = \begin{cases} \mathbb{P}(\text{Type I error}) & \text{if } \theta \in \Theta_0 \\ 1 - \mathbb{P}(\text{Type II error}) & \text{if } \theta \in \Theta_0^c \end{cases}$$

    is called the power function.
  - A test with power function $\beta(\theta)$ is a size (level) α test if $\sup_{\theta\in\Theta_0} \beta(\theta) = \alpha$ ($\leq \alpha$).
- The likelihood ratio test statistic for $H_0 : \theta \in \Theta_0$ v.s. $H_1 : \theta \in \Theta_0^c$ is

  $$\lambda(\mathbf{x}) = \frac{\sup_{\theta\in\Theta_0} L(\theta \mid \mathbf{x})}{\sup_{\theta\in\Theta} L(\theta \mid \mathbf{x})}.$$

  A *likelihood ratio test* (LRT) is any test with rejection region $\{\mathbf{x} : \lambda(\mathbf{x}) \leq c\}$, where $0 \leq c \leq 1$. Normally, we choose c such that the test

  $$T(\mathbf{X}) = \begin{cases} 1 & \text{if } \lambda(\mathbf{X}) \leq c \\ 0 & \text{if } \lambda(\mathbf{X}) > c \end{cases}$$

  has size α, that is, $\sup_{\theta\in\Theta_0} \beta(\theta) = \sup_{\theta\in\Theta_0} \mathbb{P}_\theta(\lambda(\mathbf{X}) \leq c) = \alpha$.
- **Theorem 8.2.4.** If $W(\mathbf{X})$ is a sufficient statistic for θ and $\lambda^*(w)$ and $\lambda(\mathbf{x})$ are the LRT statistic based on W and $\mathbf{X}$, then $\lambda^*(W(\mathbf{x})) = \lambda(\mathbf{x})$.
- A test T^* of size α is a *uniformly most powerful* (UMP) test if and only if $\beta_{T^*}(\theta) \geq \beta_T(\theta)$ for all $\theta \in \Theta_0^c$ and T of level α.
- **Theorem 8.3.12. (Neyman-Pearson Lemma)** Consider $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta = \theta_1$, then

  $$T^*(\mathbf{X}) = \begin{cases} 1 & f(\mathbf{X} \mid \theta_1) > cf(\mathbf{X} \mid \theta_0) \\ 0 & f(\mathbf{X} \mid \theta_1) < cf(\mathbf{X} \mid \theta_0) \end{cases},$$

  is a UMP test of size $\alpha_c := E(T^*|\theta_0)$. Note that $f(\mathbf{X} \mid \theta_1) = cf(\mathbf{X} \mid \theta_0)$ can be arbitraty.

2 Questions

1. Suppose that we observe m i.i.d Ber(θ) random variables, denoted by $Y_1, \ldots, Y_m$. Show that the LRT of $H_0 : \theta \leq \theta_0$ v.s. $H_1 : \theta > \theta_0$ will reject H_0 if $\sum_{i=1}^m Y_i > b$.

2. A random sample $X_1, \ldots, X_n$ is drawn from a Pareto distribution with pdf

$$f(x \mid \theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} I_{[\nu,\infty)}(x), \ \ \theta > 0, \nu > 0.$$

   (a) Find the MLEs of θ and ν.

   (b) Show that the LRT of

$$H_0 : \theta = 1, \quad \text{v.s.} \quad H_1 : \theta \neq 1, \ \ \nu \text{ unknown},$$

   has reject region of the form $\{\mathbf{x} : T(\mathbf{x}) \leq c_1 \text{ or } T(\mathbf{x}) \geq c_2\}$, where $0 < c_1 < c_2$ and

$$T = \log\Big[\frac{\prod_{i=1}^n X_i}{X_{(1)}^n}\Big].$$

3. Suppose X is one observation from a population with distribution Beta(θ, 1).

   (a) For testing $H_0 : \theta \leq 1$ v.s. $H_1 : \theta > 1$, find the size and the power function of the test that rejects H_0 if $X > 0.5$.

   (b) Find the most powerful level α test of $H_0 : \theta = 1$ v.s. $H_1 : \theta = 2$.