STAT 610: Discussion 3

1 Summary

- Know how to find Method of moments estimate.
- Know how to find MLE.
- Know how to find Bayes estimators.
  - Risk(MSE): The risk of an estimator $T(X)$ of $g(\theta)$ is athe function of θ defined by
  $$R_T(\theta) = E_\theta[T(X) - g(\theta)]^2.$$
  - Bayes risk: Let $\pi(\theta)$ be a pdf (or pmf) on Θ. The Bayes risk is the averaged MSE
  $$r_T = \int_\Theta R_T(\theta)\pi(\theta)d\theta.$$
  - Bayes estimators is the estimators minimize Bayes risk.
  $$T_\pi = \frac{\int g(\theta) f_\theta \pi(\theta) d\theta}{\int f_\theta \pi(\theta) d\theta} = E[g(\theta)|X].$$
  - Sufficient statistics can reduce the calculation of bayes estimator. Let S be the sufficient statistics for θ, then
  $$T_\pi = E[g(\theta)|S].$$

2 Questions

1. Let $X_1, ..., X_n$ be iid with pdf
$$f(x|\theta) = \theta x^{\theta-1},\ 0 \leq x \leq 1,\ 0 < \theta < \infty$$
   (a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$.
   (b) Find the method of moments estimator of θ.

2. Let $X_1, \ldots, X_n$ be i.i.d. binary random variables with $P(X_i = 1) = \theta \in (0, 1)$. Consider estimating θ with the squared error loss. Calculate the risks of the following estimators:

   (a) $\bar{X}$

   (b)

$$T_0(X) = \begin{cases} 0 & \text{if more than half of } X_i \text{ are } 0 \\ 1 & \text{if more than half of } X_i \text{ are } 1 \\ \frac{1}{2} & \text{if exactly half of } X_i \text{ are } 0 \end{cases}$$

3. Let $X_1, \ldots, X_n$ be i.i.d. $Exp(\theta)$, $\theta \in (0, \infty)$. Calculate the MSE of the sample mean $\bar{X}$ and $cX_{(1)}$ for some contant c. Is $\bar{X}$ better than $cX_{(1)}$?

4. Let $\bar{X}$ be the sample mean of n i.i.d. observations from $N(\theta, \sigma^2)$ with a known $\sigma > 0$, and unknown $\theta \in \mathbb{R}$. Let π be a piror p.d.f on $\mathbb{R}$. Show that the Bayes estimator of θ, given $\bar{X} = x$, is of the form

$$T(x) = x + \frac{\sigma^2}{n} \frac{d \log(x))}{dx},$$

   where $p(x)$ is the marginal p.d.f. of $\bar{X}$.