STAT 610: Discussion 11

1 Summary

- If we have a pivot $Q(X, \theta)$, a $1-\alpha$ confidence interval involves finding a and b so that $\mathbb{P}(a < Q < b) = 1-\alpha$. Typically, the length of the interval on θ will be some function of a and b like $b-a$ or $1/a^2 - 1/b^2$. If Q has density f and the length can be expressed as $\int_a^b g(t)dt$, the shortest pivotal interval is the solution of

$$\min_C \int_C g(t)dt \quad \text{subject to} \quad \int_C f(t)dt = 1-\alpha.$$

  Then, the solution of the above constraint optimization problem is $C = \{t : g(t) < \lambda f(t)\}$, where λ is chosen so that $\int_C f(t)dt = 1-\alpha$.

- UMA confidence set: A $1-\alpha$ set is *uniformly most accurate* (UMA) if it minimizes the probability of false coverage over a class of $1-\alpha$ confidence sets.
  - UMA confidence sets are constructed by inverting the acceptance region of UMP tests.

2 Questions

1. Let $X_1, \ldots, X_n$ be i.i.d Exp(θ).
   (a) Find a UMP size α hypothesis test of $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta < \theta_0$.
   (b) Find a UMA $1-\alpha$ confidence interval based on inverting the test in part (a). Show that the interval can be expressed as

$$C^*(x_1, \ldots, x_n) = \left\{\theta : 0 \le \theta \le \frac{\sum x_i}{G_{n,\alpha}}\right\},$$

   where $G_{n,\alpha}$ is α quantile of $Gamma(n, 1)$.
   (c) Find the expected length of $C^*(x_1, \ldots, x_n)$.

2. Let $X_1, \dots, X_n$ be i.i.d. from the distribution $E(\theta, \theta)$, where $\theta > 0$ is unknown.

   (a) Show that both $\bar{X}/\theta$, and $X_{(1)}/\theta$ are pivotal quantities.

   (b) Find the 1-α confidence intervals based on these two pivotal quantities.

   (c) Which one is better?

3. *(Cox's Paradox)* We are to test

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0,$$

   where we observe X with distribution

$$X \sim \begin{cases} \mathcal{N}(\theta, 100), & \text{with probability } p \\ \mathcal{N}(\theta, 1), & \text{with probability } 1-p. \end{cases}$$

   (a) Show that the test given by

$$\text{reject } H_0 \text{ if } X > \theta_0 + z_\alpha \sigma,$$

   where $\sigma = 1$ or 10 depending on which population is sampled, is a level α test.

   (b) Show that the following test of size α is given by

$$\text{reject } H_0 \text{ if } X > \theta_0 + z_{(\alpha-p)/(1-p)} \text{ and } \sigma = 1; \text{ otherwise always reject} H_0.$$

   Derive a $1-\alpha$ confidence set by inverting the acceptance region of this test, and show that it is the empty set with positive probability.