STAT 610: Discussion 13

1 Summary

- Asymptotic tests:
  - **LRT statistic**: Let $X_1, \ldots, X_n$ from $f_\theta(x)$. Under regularity conditions in §10.6.2, if $\theta \in \Theta_0$, then the distribution of the statistic $-2\log\lambda(\mathbf{X}) \sim \chi^2_r$ as $n \to \infty$, where r is the difference between the dimension of θ and number of free parameters in Θ_0.
  - Based on the above asymptotic result above, we can construct an asymptotic α-level test as $R = \{\mathbf{X} : -2\log\lambda(\mathbf{X}) \geq \chi^2_{r,\alpha}\}$.
  - **Wald test**: Suppose H_0 is equivalent to a set of equations $R(\theta) = 0$ where R is a continuous function from $\mathbb{R}^k \to \mathbb{R}^l$. Wald (1943) introduces the following statistic:

    $$W_n = R(\hat\theta)^\top \{c(\hat\theta)^\top I_n(\hat\theta)^{-1} c(\hat\theta)\}^{-1} R(\hat\theta),$$

    where $C(\theta) = \partial R(\theta)/\partial\theta$, $I_n(\hat\theta)$ is the fisher information matrix based on $X_1, \ldots, X_n$ and $\hat\theta$ is the MLE of θ. Then, under regularity conditions, $W_n \xrightarrow{d} \chi^2_r$, where r equals the difference of number of all free parameters and free parameters in Θ_0.
  - For testing $H_0 : \theta = \theta_0$ with θ_0 known, $R(\theta) = \theta - \theta_0$ and W_n simplifies to

    $$W_n = (\hat\theta - \theta_0)^\top I_n(\hat\theta)(\hat\theta - \theta_0).$$

  - **Score test**: Rao (1947) proposed the following statistic:

    $$R_n = s_n(\hat\theta_0)^\top I_n(\hat\theta_0)^{-1} s_n(\hat\theta_0),$$

    where $\hat\theta_0$ is the MLE under H_0 and $s_n(\theta) = \partial \log f_\theta(x)/\partial\theta$ is called the score function. Under regularity conditions, $W_n \xrightarrow{d} \chi^2_r$.
  - LRT, Wald test and score test are asymptotic equivalent.
- Asymptotic confidence sets:
  - **Asymptotic pivots**: A known function $q_n(\mathbf{X}, \theta)$ is a *Asymptotic pivot* if the asymptotic distribution of $q_n(\mathbf{X}, \theta)$ is free of the unknown parameter θ.
  - We can also obtain asymptotic confidence sets by inverting asymptotic tests.

2 Questions

1. Let $X_1, \ldots, X_n$ be i.i.d. samples from $\mathcal{N}(\mu, \sigma^2)$. Derive a score test statistic for testing $H_0 : \sigma = \sigma_0$ if μ is known.

2. Let $X_1, \dots, X_n$ be a random sample from $\mathcal{N}(\mu, \varphi)$ with unknown $\theta = (\mu, \varphi)$. Obtain $1-\alpha$ asymptotically correct confidence sets for μ by inverting acceptance regions of LR tests, Wald's tests and score test.

3. Let $X_1, \dots, X_n$ be i.i.d. negative binomial(r, p). Consider $Y = \sum_{i=1}^n X_i \sim$nb$(nr, p)$.

   (a) Prove that $2pY \xrightarrow{d} \chi^2_{2nr}$ as $p \to 0$.

   (b) Show that for small p, the interval

   $$\left\{ p : \frac{\chi^2_{2nr,1-\alpha/2}}{2Y} \leq p \leq \frac{\chi^2_{2nr,\alpha/2}}{2Y} \right\}$$

   is an approximate $1-\alpha$ confidence interval.

   (c) Obtain a minimum length $1-\alpha$ CI based on the asymptotic pivot $2pY$.

   Hint: mgf of χ^2_r is $(1-2t)^{-k/2}$ for $t < \frac{1}{2}$, and mgf of $nb(r,p)$ is $[\frac{p}{1-(1-p)e^t}]^r$ for $t < \log(1-p)$.