STAT 610: Discussion 10

1 Summary

- Union-Intersection and Intersection Tests (UIT/IUT):
  - UIT: Suppose we want to test $H_0 : \theta \in \cap_{\gamma\in\Gamma}\Theta_\gamma$ v.s. $H_1 : \theta \in \cup_{\gamma\in\Gamma}\Theta_\gamma^c$. Suppose R_γ is the rejection region of test $H_0 : \theta \in \Theta_\gamma$ v.s. $H_1 : \theta \in \Theta_\gamma^c$. Then, UIT is $R = \cup_{\gamma\in\Gamma}R_\gamma$. If R_γ is of the form $\{x : T_\gamma(x) > c\}$, then
  $$R = \cup_{\gamma\in\Gamma}\{x : T_\gamma(x) > c\} = \{x : \sup_{\gamma\in\Gamma} T_\gamma(x) > c\}.$$
  - IUT: Suppose we want to test $H_0 : \theta \in \cup_{\gamma\in\Gamma}\Theta_\gamma$ v.s. $H_1 : \theta \in \cap_{\gamma\in\Gamma}\Theta_\gamma^c$. If each $H_{0\gamma}$ has the rejection region $R_\gamma = \{x : T_\gamma(x) > c\}$, then IUT has the rejection region
  $$R = \cap_{\gamma\in\Gamma}\{x : T_\gamma(x) > c\} = \{x : \inf_{\gamma\in\Gamma} T_\gamma(x) > c\}.$$
  - Level of UIT: If R_γ has level α_γ, then the overall level of UIT is at most $\sum_{\gamma\in\Gamma}\alpha_\gamma$.
  - Level of IUT: If R_γ has level α_γ, then the overall level of IUT is at most $\min_{\gamma\in\Gamma}\alpha_\gamma$.
  - Relationship between LRT and UIT: Refer to *Theorem 8.3.21* in the textbook.
- Confidence Interval:
  - The *coverage probability* is defined as $\mathbb{P}_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$ for an interval $[L(\mathbf{X}), U(\mathbf{X})]$.
  - The *confidence coefficient* is defined as $\inf \mathbb{P}_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$, which is the infimum of the coverage probability.
- Pivotal quantities: A random variable $Q(\mathbf{X}, \theta) = Q(X_1, \dots, X_n, \theta)$ is a *pivotal quantity* if the distribution of $Q(\mathbf{X}, \theta)$ is independent of all parameters. That is, if $\mathbf{X} \sim F(\mathbf{x} \mid \theta)$, then $Q(\mathbf{x}, \theta)$ has the same distribution for all values of θ.
  - e.g. If $X_i \overset{i.i.d}{\sim} \text{Exp}(\theta)$, then $T = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. Hence $Q(T, \theta) = 2T/\theta \sim \chi^2_{2n}$ is a pivotal quantity.
  - After figuring out a, b such that $\mathbb{P}(a \le Q(\mathbf{X}, \theta) \le a) \ge 1 - \alpha$, then $C(\mathbf{x}) = \{\theta : a \le Q(\mathbf{x}, \theta) \le b\}$ is a $1 - \alpha$ confidence set for θ.
  - If $Q(\mathbf{x}, \theta)$ is a monotone function of θ, then $C(\mathbf{x})$ will be an interval.
- Pivoting a cdf: Let T be a statistic with cdf $F_T(t \mid \theta)$. Let $\alpha_1 + \alpha_2 = \alpha$ be constants.
  - If $F_T(t \mid \theta)$ is a decreasing function of θ for each t, define $\theta_L(t)$ and $\theta_U(t)$ by
  $$F_T(t \mid \theta_U(t)) = \alpha_1, \qquad F_T(t \mid \theta_L(t)) = 1 - \alpha_2.$$
  - If $F_T(t \mid \theta)$ is a increasing function of θ for each t, define $\theta_L(t)$ and $\theta_U(t)$ by
  $$F_T(t \mid \theta_U(t)) = 1 - \alpha_2, \qquad F_T(t \mid \theta_L(t)) = \alpha_1.$$

  Then $[\theta_L(T), \theta_U(T)]$ is a $1 - \alpha$ confidence interval for θ.

2 Questions

1. Consider testing $H_0 : \theta \in \cup_{j=1}^{k} \Theta_j$. For each $j = 1, \ldots, k$, let $p_j(\mathbf{x})$ denote a valid p-value for testing $H_{0j} : \theta \in \Theta_j$. Let $p(\mathbf{x}) = \max_{1 \le j \le k} p_j(\mathbf{x})$.

   (a) Show that $p(\mathbf{X})$ is a valid p-value for testing H_0.

   (b) Show that the α level test defined by $p(\mathbf{X})$ is the same as an α level IUT defined in terms of individual tests based on the $p_j(\mathbf{x})$s.

2. Find a $1 - \alpha$ confidence interval for θ using pivots, given X with pdf

   (a) $f(x \mid \theta) = 1$, $\theta - \frac{1}{2} < x < \theta + \frac{1}{2}$.

   (b) $f(x \mid \theta) = 2x/\theta^2$, $0 < x < \theta$.

3. Let $X_1, \dots, X_n$ be i.i.d uniform$(0, \theta)$. Let Y be the largest order statistic. Prove that Y/θ is a pivotal quantity and show that $[y, y \cdot \alpha^{-1/n}]$ is the shortest $1-\alpha$ pivotal interval.

4. If $X_1, \dots, X_n$ are i.i.d from a location pdf $f(x-\theta)$. Show that the confidence set

$$C(x_1, \dots, x_n) = \{\theta : \bar{x} - k_1 \le \theta \le \bar{x} + k_2\},$$

where k_1, k_2 are constants, has constant coverage probability.

5. Let $X_1, \dots, X_n$ be a random variable with pdf $f_X(x) = \theta a^\theta x^{-(\theta+1)} I_{(a,\infty)}(x)$, where $\theta > 0$ and $a > 0$.

   (a) When θ is known, derive a confidence interval for a with confidence coefficient $1-\alpha$ by pivoting the cdf of the smallest order statistic $X_{(1)}$.

   (b) When both a and θ are unknown and $n \ge 2$, derive a confidence interval for θ with confidence coefficient $1-\alpha$ by pivoting the cdf of $T = \prod_{i=1}^n (X_i / X_{(1)})$.
   Hint: You can use the fact that $2\theta \log T \sim \chi^2_{2(n-1)}$ and then write the cdf of T in terms of the cdf of $\chi^2_{2(n-1)}$.

   (c) When both a and θ are unknown, construct a confidence set for (a, θ) with confidence coefficient $1-\alpha$ using a pivotal quantity.
   Hint: Notice that $X_{(1)}/a$ is free of a, and $X_{(1)}^\theta$ is free of θ.

1.

a) check $P_\theta\{P(X) \le p(x)\} \le \alpha, \quad \forall \theta \in \Theta_0$

pf)

$$P_\theta\{P(X) \le p(x)\}$$

$$= P_\theta\{\max_j P_j(X) \le p(x)\}$$

$$= P_\theta\{P_j(X) \le p(x) \quad \forall j\}$$

$$\le \max_j P_\theta\{P_j(X) \le p(x)\} \le \alpha \quad \#$$

b) The rejection region for $H_0: \theta \in \Theta_j$ is

$$R_j := \{x \mid P_j(x) \le \alpha\}$$

The rejection region for IUT is

$$R := \bigcap R_j = \{x \mid \max P_j(x) \le \alpha\}$$

which is the same rejection region defined by $P(X)$ #

Q2:

a) The pivot variable is $X-\theta \sim \text{Unif}(-\frac{1}{2}, \frac{1}{2})$.

So,

$$P_\theta\left(-\frac{1-\alpha}{2} \leq X-\theta \leq \frac{1-\alpha}{2}\right) = 1-\alpha$$

The CI is

$$X - \frac{1-\alpha}{2} < \theta < X + \frac{1-\alpha}{2}$$

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b) The pivot is $\frac{X}{\theta}$, which have density.

$f_X(x) = 2x \Rightarrow P_X(X) = x^2 \quad x \in (0, 1)$

$$P_\theta\left(\sqrt{\alpha_1} \leq \frac{X}{\theta} \leq \sqrt{1-\alpha_2}\right) = 1-\alpha_1-\alpha_2$$

The CI is

$$\frac{X}{\sqrt{1-\alpha_2}} \leq \theta \leq \frac{X}{\sqrt{\alpha_1}}$$

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Q3: • $X_1, \dots, X_n \sim Unif(0, \theta)$

$X_1/\theta, \dots, X_n/\theta \sim Unif(0,1)$

$\Rightarrow Y/\theta = \max X_i/\theta$ which is pivot.

• The density of Y/θ is $n y^{n-1}$ $y \in (0,1)$,

which is increasing function in y.

So, the shortest $1-\alpha$ CI is.

$$P_\theta\left(Y/\theta \in [C_\alpha, 1] \right) = 1 - C_\alpha^n$$

$$\Rightarrow 1 - C_\alpha^n = 1-\alpha \quad \Rightarrow \quad C_\alpha = \alpha^{\frac{1}{n}}$$

So $Y/\theta \in [\alpha^{\frac{1}{n}}, 1]$

$$\Leftrightarrow \quad y \le \theta \le y \alpha^{\frac{1}{n}}$$

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Q4.

$\because$ $X_1 - \theta, \dots, X_n - \theta \sim f(x)$

$\bar{X} - \theta$ is pivot

$\Rightarrow C(X_1, \dots, X_n)$ has constant coverage probability #

Q5:

a) $X_1/a, \dots, X_n/a \sim f_X(x) = \theta x^{-(\theta+1)}, \; x > 1$

$$X_{(1)}/a \sim n\theta x^{-(\theta+1)} x^{-(n-1)\theta} = n\theta x^{-n\theta-1}, \quad x > 1$$

CDF is $F(x) = 1 - x^{-n\theta} \quad x \geq 1$

So, $P\{(1-\alpha_1)^{-\frac{1}{n\theta}} \leq X_{(1)}/a \leq \alpha_2^{-\frac{1}{n\theta}}\} \leq 1-\alpha_1-\alpha_2$

CI is

$$X_{(1)} \alpha_2^{\frac{1}{n\theta}} \leq a \leq X_{(1)} (1-\alpha_1)^{\frac{1}{n\theta}}$$

#

b) $\because$ $2\theta \log T \sim \chi^2_{2(n-1)}$

$\therefore$ $2\theta \log T$ is pivot.

Let $\chi^2_{\alpha_1}$ and $\chi^2_{1-\alpha_2}$ be

the α_1 and $1-\alpha_2$ quantile for $\chi^2_{2(n-1)}$

$$P_\theta\left(\chi^2_{\alpha_1} \le 2\theta \log T \le \chi^2_{1-\alpha_2} \right) = 1-\alpha_1-\alpha_2$$

CI is

$$\Rightarrow \frac{\chi^2_{1-\alpha_2}}{2\log T} \le \theta \le \frac{\chi^2_{1-\alpha_1}}{2\log T}$$

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c) $X_{(1)}/a \sim f_X(x) = n\theta\, x^{-n\theta-1}, \quad x > 1$

$\left(X_{(1)}/a\right)^\theta$;

$P\left(\{X_{(1)}/a\}^\theta \le y \right)$

$= P\left(X_{(1)}/a \le y^{\frac{1}{\theta}} \right) = n\theta\, y^{-n-\frac{1}{\theta}}$

So the density is. $n\theta\, y^{-n-\frac{1}{\theta}} \frac{1}{\theta} y^{\frac{1}{\theta}-1}$

$= n\, y^{-n-1}, \quad y \ge 1$

$\Rightarrow$ CDF is $1-y^{-n}$

So, $P\left((1-\alpha_1)^{\frac{-1}{n}} \le \left(X_{(1)}/a\right)^\theta \le \alpha_2^{\frac{-1}{n}} \right) = 1-\alpha_1-\alpha_2$

CS is.

$$\left\{ (1-\alpha_1)^{-\frac{1}{n}} \le \left(X_{(1)}/a\right)^\theta \le \alpha_2^{\frac{-1}{n}} \right\}$$

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