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STAT 610: Discussion 11

1 Summary

- If we have a pivot $Q(X, \theta)$, a $1-\alpha$ confidence interval involves finding a and b so that $\mathbb{P}(a < Q < b) = 1-\alpha$. Typically, the length of the interval on θ will be some function of a and b like $b-a$ or $1/a^2 - 1/b^2$. If Q has density f and the length can be expressed as $\int_a^b g(t)dt$, the shortest pivotal interval is the solution of

$$\min_C \int_C g(t)dt \quad \text{subject to} \quad \int_C f(t)dt = 1-\alpha.$$

  Then, the solution of the above constraint optimization problem is $C = \{t : g(t) < \lambda f(t)\}$, where λ is chosen so that $\int_C f(t)dt = 1-\alpha$.

- UMA confidence set: A $1-\alpha$ set is *uniformly most accurate* (UMA) if it minimizes the probability of false coverage over a class of $1-\alpha$ confidence sets.
  - UMA confidence sets are constructed by inverting the acceptance region of UMP tests.

L-P $\longrightarrow$ ① $H_0: \theta = \theta_1$ $H_1: \theta = \theta_1$

②

K-R

③ ML

$\begin{cases} H_0: \theta > \theta_0 \\ H_1: \theta < \theta_0 \end{cases}$

2 Questions

1. Let $X_1, \ldots, X_n$ be i.i.d Exp(θ).
   (a) Find a UMP size α hypothesis test of $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta < \theta_0$.
   (b) Find a UMA $1-\alpha$ confidence interval based on inverting the test in part (a). Show that the interval can be expressed as

$$C^*(x_1, \ldots, x_n) = \left\{\theta : 0 \le \theta \le \frac{\sum x_i}{G_{n,\alpha}}\right\},$$

   where $G_{n,\alpha}$ is α quantile of $Gamma(n, 1)$.
   (c) Find the expected length of $C^*(x_1, \ldots, x_n)$.

a)

Step 1: $\begin{cases} H_0: & \theta = \theta_0 \\ H_A: & \theta = \theta_1 \end{cases}$ $\theta_1 < \theta_0$.

L-P

$$R(X_1, \cdots, \underset{\sim}{X}) = \begin{cases} 1 & f_1(\underset{\sim}{x}) > c\, f_0(\underset{\sim}{x}) \\ 0 & \text{order.} \end{cases}$$

$$R(\underset{\sim}{X}) = \left\{ \theta_1^{-n} e^{-\frac{\Sigma X_i}{\theta_1}} > c\, \theta_0^{-n} e^{-\frac{\Sigma X_i}{\theta_0}} \right\}$$

$$= \left\{ \frac{1}{c}\left(\frac{\theta_0}{\theta_1}\right)^n > e^{-\Sigma X_i \boxed{\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)}} \right\}$$

$<0.$

$$= \left\{ \frac{\Sigma X_i}{\theta_0} < C^* \right\}$$

$\Sigma X_i \sim \text{Gamma}(n, 1)$

So, $C^* = G_{n,\alpha} \leftarrow \alpha$-th quantile of $\text{Gamma}(n, 1)$

$\because$ The rejection region is independent on θ,

So it can generalize to $H_A: \theta < \theta_0$

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b) $A(\underset{\sim}{X}) = \{ \sum X_i / \theta > G_{n,\alpha} \}$

$$C(\underset{\sim}{X}) = \left\{ \theta \;\middle|\; \frac{\sum X_i}{\theta} > G_{n,\alpha} \right\}$$

$$= \left\{ \theta \;\middle|\; \frac{\sum X_i}{G_{n,\alpha}} > \theta > 0 \right\}$$

c) $$E\left(\frac{\sum X_i}{G_{n,\alpha}} - \theta \right) = \frac{n\theta_0}{G_{n,\alpha}}$$

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$\frac{1}{\theta} \exp\left\{-\frac{x-\theta}{\theta}\right\} \quad x > \theta$

2. Let $X_1, \ldots, X_n$ be i.i.d. from the distribution $\boxed{E(\theta, \theta)}$, where $\theta > 0$ is unknown.
   (a) Show that both $\bar{X}/\theta$, and $X_{(1)}/\theta$ are pivotal quantities.
   (b) Find the 1-α confidence intervals based on these two pivotal quantities.
   (c) Which one is better?

a) $X_1 - \theta, \ldots X_n - \theta \sim E(0, \theta)$

$\frac{X_1}{\theta} - 1, \quad \frac{X_n}{\theta} - \theta \sim E(0, 1)$

$\Rightarrow \bar{X}/\theta$ and $\frac{X_{(1)}}{\theta}$ are pivotal quantities,

- $\frac{\sum_i X_i - n\theta}{n\theta} \sim \text{Gamma}(n, \frac{1}{n})$
- $\frac{X_{(1)} - \theta}{\theta} \sim \text{Gamma}(1, \frac{1}{n})$

3. *(Cox's Paradox)* We are to test

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0,$$

where we observe X with distribution

$$X \sim \begin{cases} \mathcal{N}(\theta, 100), & \text{with probability } p \\ \mathcal{N}(\theta, 1), & \text{with probability } 1 - p. \end{cases}$$

$\sigma = 10$

$\sigma = 1$

   (a) Show that the test given by

$$\text{reject } H_0 \text{ if } X > \theta_0 + z_\alpha \sigma,$$

   where $\sigma = 1$ or 10 depending on which population is sampled, is a level α test.
   (b) Show that the following test of size α is given by

$$\text{reject } H_0 \text{ if } X > \theta_0 + z_{(\alpha - p)/(1-p)} \text{ and } \sigma = 1; \text{ otherwise always reject} H_0.$$

   Derive a $1 - \alpha$ confidence set by inverting the acceptance region of this test, and show that it is the empty set with positive probability.

Q2, (b)

$$\underbrace{G_{n,\alpha}}_{\text{quantiles for Gamma}(n,1\alpha)} \le \bar{X}/\theta - 1 \le \underbrace{G_{n,1-\alpha}}$$

quantiles for Gamma$(n, 1\alpha)^d$

$$\frac{\bar{X}}{G_{n,1\alpha}+1} \le \theta \le \frac{\bar{X}}{G_{n,\alpha}+1}$$

$$\frac{X_{(1)}}{E_{n,1\alpha}+1} \le \theta \le \frac{X_{(1)}}{\underbrace{E_{n,\alpha}}_{\sim\ \exp(0,\frac{1}{n})}+1}.$$

c)

$$P(X_{(1)} < c) = \int_0^c n e^{-nx}\,dx$$

$$= e^{-nx}\Big|_0^c = 1 - e^{-nc} = \alpha.$$

$$1-\alpha = e^{-nc}.$$

$$-\frac{\log(1-\alpha)}{n} = c. = E_{n,\alpha}.$$

$$-\frac{\log\alpha}{n} = E_{n,1-\alpha}$$

For $X_{(1)}$

$$\underbrace{\frac{X_{(1)}}{1-\frac{\log(1-\alpha)}{n}}}_{L(X_{(1)})} \le \theta \le \underbrace{\frac{X_{(1)}}{1-\frac{\log \alpha}{n}}}_{R(X_{(1)})}$$

$$R(X_{(1)}) - L(X_{(1)}) = \frac{1}{n}C = O\left(\frac{1}{n}\right)$$

For $\bar{X}$,

$G_{n,\alpha}$ $\quad G_{n,2\alpha}$ quantile of $\mathrm{Gamma}(n, \frac{1}{n})$

$Z_i \overset{iid}{\sim} \mathrm{Exp}(1)$

$\bar{Z} \sim \mathrm{Gamma}(n, \frac{1}{n})$

$$P\Big(\underbrace{\sqrt{n}(G_{n,\alpha}-1)}_{\to\, z_\alpha} \le \underbrace{\sqrt{n}(\bar{Z}-1)}_{\to\, N(0,1)} \le \underbrace{\sqrt{n}(G_{n,1-\alpha}-1)}_{\to\, z_{1-\alpha}}\Big) = 1-2\alpha.$$

For $\bar{X}$ the CI can be approximated by

$$\frac{\bar{X}}{\frac{1}{\sqrt{n}} z_\alpha + 2} \le \theta \le \frac{\bar{X}}{\frac{1}{\sqrt{n}} z_{1-\alpha} + 2}$$

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Q3

(a)

$$R(X) = \begin{cases} \theta = 1 & X > \theta_0 + z_\alpha \\ \theta = 10 & X > \theta_0 + z_\alpha \times 10 . \end{cases}$$

$$P(X \in R(X)) = (1-p)\,\underbrace{P(X > \theta_0 + z_\alpha)}_{\alpha}$$

$$+ \; p \; \underbrace{P(X > \theta_0 + 10 z_\alpha)}_{\alpha}$$

$$= \alpha$$

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b)

$$R(X) = \begin{cases} \theta = 1 & X > \theta_0 + z_{\frac{\alpha - p}{1-p}} \\ \theta = 10 & \text{Reject} \end{cases}$$

$$P_{\theta_0}(X \in R(X)) = (1-p)\,\underbrace{P\left(X > \theta_0 + z_{\frac{\alpha-p}{1-p}}\right)}_{\frac{\alpha-p}{1-p}}$$

$$+ \; p \times 1$$

$$= \cancel{(1-p)}\left(\frac{\alpha - p}{\cancel{1-p}}\right) + p = \alpha$$

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(b) is better than (a)

$$A(\theta) = \left\{ \theta = 1 \quad \text{and} \quad X < \theta + z_{\frac{\alpha+p}{1-p}} \right\}$$

$$\underline{C(X)} = \begin{cases} \theta = 1, & X - z_{\frac{\alpha+p}{1-p}} < \theta \\ \underbrace{\theta = 10}_{p}, & \phi \end{cases}$$

There is probability p that.

CI is an ϕ

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