# STAT 610: Discussion 6

## 1 Summary

- Under assumption (A3), $\hat{\beta} = \hat{\beta}_{V^{-1}}$, if and only if $X(X^\top X)^{-1}X^\top Var(\epsilon)$ is symmetric, where $\beta_{V^{-1}}$ is the solution of WLSE.
- Notations and propoties for GLM.
  - Y have pdf
$$\exp\left\{\frac{\eta_i y_i - \zeta(\eta_i)}{\phi}\right\} h(y_i, \phi).$$
  - $E[Y_i] = \zeta'(\eta_i) = \mu(\eta_i)$ and $Var(Y_i) = \phi\zeta''(\eta_i)$
  - $g$ is the link function. And $g \circ \mu(\eta_i) = \beta^\top x_i$.
  - Canonical link: $g = \mu^{-1}$.
  - $\psi = (g \circ \mu)^{-1}$; hence $\eta_i = \psi(\beta^\top x_i)$.

## 2 Questions

1. A linear model with $\beta$ replaced by a random vector $\boldsymbol{\beta}$ that is independent of $\epsilon$. Suppose that $Var(\epsilon) = \sigma^2 \boldsymbol{I}_n$, and $E[\boldsymbol{\beta}] = \beta$. Show that $\ell^\top \hat{\beta}$ is the BLUE for $\ell^\top \beta$.

2. Under (A3) and $X$ is full rank. Them, $\hat{\beta} = \hat{\beta}_{V^{-1}}$ if and only if $\ell^\top \hat{\beta}$ is BLUE for all $\ell$.

3. Assume that $\boldsymbol{\epsilon} \sim N(0, V)$, and $X$ is full rank. Then, $X(X^\top X)^{-1}X^\top Var(\epsilon)$ is symmetric if and only if $\hat{\beta}$ is the UMVUE for all $\ell$.

4. Let $Y_1, \dots, Y_n$ be independent Poisson random variables with

$$Y_i \sim \frac{1}{y_i!} \exp(\eta_i y_i - e^{\eta_i}) \boldsymbol{I}(y_i = 0, 1, \dots), \quad i = 1, \dots, n,$$

where $g(e^{\eta_i}) = \beta^\top x_i$, $g$ is a link function, $x_i$'s are p-dimensional covariates, and $\beta$ is a p-dimensional unknown parmeter vector. Consider a GLM for $Y_1, \dots, Y_n$ and $X_1, \dots, x_n$.

(a) With link function $g(t) = \log t$, obtain the likelihood equation for MLE of $\beta$ and show that the matrix of second order dereivative of the log likelihood function is

$$-\sum_{i=1}^{n} e^{\beta' x_i} x_i x_i^\top.$$

(b) Consider link $g(t) = 2\sqrt{t}$. Show that $\mu^{-1}(s) = \log s$ and $\psi(t) = \mu^{-1} \circ g^{-1}(t) = 2\log(t/2)$. Obtian the likelihood equation for the MLE of $\beta$.