STAT 610: Discussion 2

1 Summary

- Minimal sufficient statistics
  - Definition: A statistic $T(\mathbf{X})$ is minimal sufficient if for any other sufficient statistics $T'(\mathbf{X})$, $T(\mathbf{x})$ is a function of $T'(\mathbf{x})$.
  - Theorem 6.2.13. $T(\mathbf{X})$ is minimal sufficient for $\boldsymbol{\theta}$ if T has the property that

$$T(\mathbf{x}) = T(\mathbf{y}) \Leftrightarrow \frac{f(\mathbf{x}|\boldsymbol{\theta})}{f(\mathbf{y}|\boldsymbol{\theta})} \text{ does not depend on } \boldsymbol{\theta}.$$

  - Theorem 6.6.5. See lecture 3.
- In the case of exponential families

  Let $X_1, \ldots, X_n$ be i.i.d. from an exponential family with pdf

$$f_{\boldsymbol{\theta}}(x) = h(x)c(\boldsymbol{\theta}) \exp\{\boldsymbol{\eta}(\boldsymbol{\theta})^T \mathbf{T}(x)\},$$

  where $\boldsymbol{\eta}(\boldsymbol{\theta})^T = (w_1(\boldsymbol{\theta}), \ldots, w_k(\boldsymbol{\theta}))$, and $\mathbf{T}(x)^T = (t_1(x), \ldots, t_k(x))$. Then,
  - $T(\mathbf{X}) = (\sum_{i=1}^n t_1(X_i), \ldots, \sum_{i=1}^n t_k(X_i))$ is a sufficient statistic for $\boldsymbol{\theta}$.
  - $T(\mathbf{X}) = (\sum_{i=1}^n t_1(X_i), \ldots, \sum_{i=1}^n t_k(X_i))$ is also minimal sufficient if the parameter space contains an open set in $\mathbb{R}^k$.
- Ancillary statistics
  - Definition: $U(\mathbf{X})$ is ancillary of θ if the distribution of U does not depend on θ.
- Complete statistics
  - Definition: $T(\mathbf{X})$ is complete if T has the property that

$$\mathbb{E}_\theta g(T) = 0 \text{ for any measurable } g \text{ and } \theta \Rightarrow g(T) = 0 \text{ a.s.}$$

  - Basu's Theorem: Complete sufficient statistics and ancillary statistics for parameter θ are independent for all θ.

2 Questions

1. Let $(X_1, \ldots, X_n)$ be a random sample from density $\theta^{-1}e^{-(x-\theta)/\theta}I_{(\theta,\infty)}(x)$, where $\theta > 0$ is an unknown parameter.

   (a) Find a statistic that is minimal sufficient for θ.

   (b) Show whether the minimal sufficient statistic in (a) is complete.

2. Let X be a discrete random variable with

$$\mathbb{P}_\theta(X = x) = \frac{\binom{\theta}{x}\binom{N-\theta}{n-x}}{\binom{N}{n}}, \quad x = 0, 1, 2, \ldots, \min\{\theta, n\}, n - x \leq N - \theta$$

   where n and N are known positive integers, $N \geq n$, and $\theta = 0, 1, \ldots, N$. Show that X is complete.

3. (a) If $\frac{X}{Y}$ and Y are independent random variables, show that

$$\mathbb{E}\left(\frac{X}{Y}\right)^k = \frac{\mathbb{E}(X^k)}{\mathbb{E}(Y^k)}.$$

   (b) Use Basu's theorem to show that if $X_1, ..., X_n$ are iid gamma(α, β), where α is known, then for $T = \sum_{j=1}^{i} X_j$

$$\mathbb{E}(X_{(i)}|T) = \mathbb{E}\left(\frac{X_{(i)}}{T}T|T\right) = T\frac{\mathbb{E}(X_{(i)})}{\mathbb{E}T}$$

4. Let $(X_1, \ldots, X_n)$, $n \geq 2$, be a random sample from a distribution having density $f_{\theta,j}$, where $\theta > 0$, $j = 1, 2$, $f_{\theta,1}$ is the density of $\mathcal{N}(0, \theta^2)$, and $f_{\theta,2}(x) = (2\theta)^{-1}e^{-|x|/\theta}$. Show that $T = (T_1, T_2)$ is minimal sufficient for (θ, j), where $T_1 = \sum_{i=1}^{n} X_i^2$ and $T_2 = \sum_{i=1}^{n} |X_i|$.