STAT 610: Discussion 9

1 Summary

L-P Lemma $\begin{cases} H_0 : \theta = \theta_0 \\ H_a : \theta = \theta_1 \end{cases}$

- UMP test for one sided hypothesis:
  - Monotone likelihood ratio (MLR): $f_\theta(x)$ is MLR in $Y(X)$ if for any $\theta_1 < \theta_2$, $\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)}$ is monotone of $Y(x)$ for values of x at which at least one of $f_{\theta_1}(x)$ and $f_{\theta_2}(x)$ is positive.
  - **Karlin-Rubin Theorem**: $H_0 : \theta \le \theta_0$ v.s. $H_1 : \theta > \theta_0$. $f_\theta(x)$ has non-decreasing MLR of $Y(X)$. Then

$$T(X) = \begin{cases} 1, \text{ if } Y(X) > c \\ 0, \text{ if } Y(X) < c \end{cases},$$

    is a level-α UMP test.

- UMPU test for one parameter exp-family:
  - Unbiased test: A test with power function $\beta(\theta)$ is *unbiased* if $\beta(\theta') \ge \beta(\theta'')$ for every $\theta \in \Theta_0^c$ and $\theta'' \in \Theta_0$.
  - Suppose that U is a sufficient statistic for $\theta \in \mathbb{R}$ with pdf or pmf $g_\theta(u) = h(u)c(\theta)e^{w(\theta)u}$. Consider $H_0 : \theta = \theta_0$ v.s. $H_1 : \theta \neq \theta_0$. A UMPU test of size α satisfies

$$T(X) = \begin{cases} 1, \text{ if } U(X) < c_1 \text{ or } U(X) > c_2 \\ 0, \text{ if } c_1 < U(X) < c_2 \end{cases}$$

    for some constants c_1 and c_2 such that $\mathbb{E}_{\theta_0}(T) = \alpha$ and $\mathbb{E}_{\theta_0}(TU) = \alpha\mathbb{E}_{\theta_0}(U)$.

- p-value:
  - Valid p-value: For every $\theta \in \Theta_0$ and every $0 \le \alpha \le 1$,

$$\mathbb{P}_\theta(p(X) \le \alpha) \le \alpha.$$

  - Commonly used p-value: If large value of $W(X)$ gives evidence that H_1 is true, then

$$p(x) = \underset{\theta \in \Theta_0}{sup} \mathbb{P}_\theta(W(X) \ge W(x))$$

    is a valid p-value.
  - Another way to define p-value: The smallest possible level α at which H_0 would be rejected for the computed level α test $T_\alpha(x)$, i.e.,

$$p(x) = \inf\{\alpha \in (0,1) : T_\alpha(x) \text{ rejects } H_0\}.$$

2 Questions

use LP

1. The random variable X has pdf $f(x) = e^{-x}, x > 0$. One observation is obtained on the random variable $Y = X^{\theta}$. Construct a UMP test of size α for $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

$\theta = 1 \quad Y = X \quad \Rightarrow \quad f_Y(y) = e^{-y}$

$\theta = 2 \quad Y = X^2 \quad \Rightarrow \quad f_Y(y) = \frac{1}{2\sqrt{y}} e^{-\sqrt{y}}$

2. Consider the following distribution

$$f(x|\theta) = \frac{e^{x-\theta}}{(1 + e^{x-\theta})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

   (a) Show that this family has an MLR.

   ~~(b) Find the UMP test of size α for $H_0 : \theta = 0$ v.s. $H_1 : \theta = 1$ base on one observation X.~~

   (c) Show that the test in part (b) is UMP size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.

K-R

or

3. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be i.i.d. samples from $\mathcal{N}(\mu_1, 1)$ and $\mathcal{N}(\mu_2, 1)$. Find a UMPU test of size α for the hypothesis $H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 \neq \mu_2$.

4. Prove that $p(x) = \inf\{\alpha \in (0, 1) : T_\alpha(x) \text{ rejects } H_0\}$ is a valid p-value.

Q1: $\theta = 1$ $\quad f_1(y) = e^{-y}$

$\theta = 2$ $\quad f_2(y) = \frac{1}{2\sqrt{y}} e^{-\sqrt{y}}$

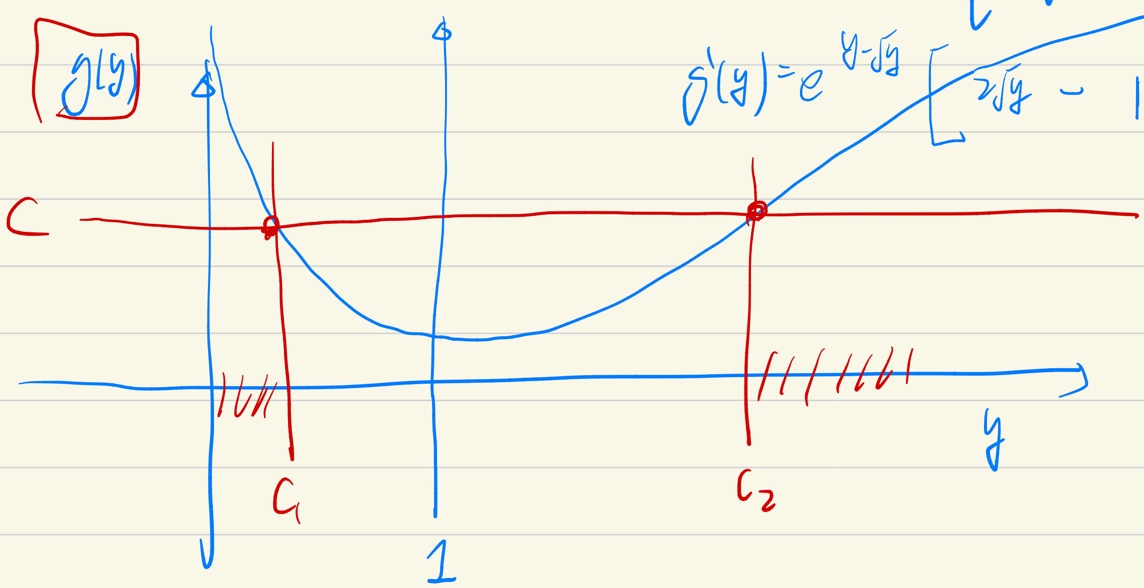
By the CP Lemma, the UMP test should have the following form.

$$\boxed{T_c(y) = \begin{cases} 1 & f_2(y) > c\, f_1(y) \\ 0 & f_2(y) < c\, f_1(y) \end{cases}}$$

$$\frac{f_2(y)}{f_1(y)} = \frac{e^{y-\sqrt{y}}}{2\sqrt{y}} = \boxed{g(y)}$$

$$\begin{cases} \lim_{y \to \infty} g(y) = \infty \\ \lim_{y \to 0} g(y) = \infty \\ g'(1) = 0 \end{cases}$$

$$g'(y) = e^{y-\sqrt{y}} \left[2\sqrt{y} - 1 - y^{-\frac{1}{2}} \right]$$



$$\boxed{T_{c_1,c_2}(y)} = \begin{cases} \boxed{1} & y < c_1 \text{ or } y > c_2 \\ \boxed{0} & c_1 < y < c_2 \end{cases}$$

Find c_1 , c_2

- $E\left[T_{c_1,c_2}(y) \mid \theta=1\right] = \alpha$

- $g(c_1) = g(c_2)$

Q2:

$$f(x|\theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2}$$

a) MLR.

$\theta_2 > \theta_1$ check: $\frac{f(x|\theta_2)}{f(x|\theta_1)}$ $\nearrow$ or $\searrow$

pf)

$$\frac{e^{\cancel{x}-\theta_2}(1+e^{x-\theta_1})^2}{(1+e^{x-\theta_2})^2 \; e^{\cancel{x}-\theta_1}}$$

$$= \underbrace{e^{\theta_1-\theta_2}}_{\text{Constant}} \left[\underbrace{\boxed{\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}}}_{= g(x)} \right]^2$$

check $g(x)$ is $\nearrow$ or $\searrow$

$$g'(x) = \frac{(1+e^{x-\theta_2})e^{x-\theta_1} - (1+e^{x-\theta_1})e^{x-\theta_2}}{(1+e^{x-\theta_2})^2}$$

$$= \frac{\boxed{e^{x-\theta_1} - e^{x-\theta_2}}}{(1+e^{x-\theta_2})^2}$$

$\because$ e^x $\nearrow$ and $x-\theta_1 > x-\theta_2$ $(\theta_2 > \theta_1)$

so, $e^{x-\theta_1} > e^{x-\theta_2}$ $\Rightarrow$ $g'(x) > 0$ $\Rightarrow$ g is $\nearrow$

$\Rightarrow \dfrac{f(x|\theta_2)}{f(x|\theta_1)}$ is $\boxed{\nearrow}$ #

If X is large. $\Rightarrow$ tend to believe θ is large

(b) UMP test $\begin{cases} H_0: & \theta \le 0 \\ H_1: & \theta > 0 \end{cases}$

By K-R MLR is $\nearrow$

$$T_c := \begin{cases} 1 & X > c \\ 0 & X < c \end{cases}$$

Find c s.t. $E[T_c \mid \theta=0] = \alpha$.

$$E[T_c \mid \theta=0] = \int_c^{\infty} \frac{e^y}{(1+e^y)^2}\, dy = \alpha.$$

logistic distribution.

$$= 1 - F_0(c) = \alpha.$$

$$c = F_0^{-1}(1-\alpha)$$

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Q 3:
$$\begin{cases} H_0: & \mu_1 = \mu_2 \\ H_A: & \mu_1 \neq \mu_2 \end{cases} \equiv \begin{cases} H_0: & \boxed{\mu_1 - \mu_2} = 0 \\ H_A: & \mu_1 - \mu_2 \neq 0 \end{cases}$$

- $Z = \bar{X} - \bar{Y}$ is sufficient for $\mu_1 - \mu_2$
- $Z \sim N(0, \frac{2}{n}) \Rightarrow$ It statisfies the thm

By the Thm, the UMPU test should look like

$$T_{c_1, c_2}(Z) = \begin{cases} 1 & Z > c_2 \text{ or } Z < c_1 \\ 0 & c_1 < Z < c_2 \end{cases}$$

And it should satisfy

1. $\alpha = E\left[T_{c_1c_2}(Z) \mid \mu_1 = \mu_2\right]$
2. $E\left[T_{c_1c_2}(Z) \times Z \mid \mu_1 = \mu_2\right] = \alpha \underbrace{\cancel{E\left[Z \mid \mu_1 = \mu_2\right]}}_{0}$

$$E\left[T_{c_1c_2}(Z) \times Z \mid \mu_1 = \mu_2\right] = 0$$

$$\parallel$$

$$\int_{c_2}^{\infty} z\, f_Z(z)\, dz + \int_{-\infty}^{c_1} z\, f_Z(z)\, dz$$

$\because$ Z is symetric

$$= \underbrace{\int_{c_2}^{\infty} z f_Z(z)\, dz}_{} - \underbrace{\int_{-c_1}^{\infty} z\, f_Z(z)\, dz}_{} = 0$$

$$\Rightarrow -C_1 = C_2$$

① $\alpha = E\left[T_{c_1,c_2}(Z) \mid \mu_1 = \mu_2\right]$

$Z \sim N(0, \frac{2}{n})$

$$= \underbrace{2 \times \int_c^{\infty} f_Z(z)\, dz}$$

$P(N \le c) = \Phi(c)$

$\alpha = 2 \times \left[1 - \Phi\left(\sqrt{\frac{n}{2}}\, c\right)\right]$

$P\left(\underbrace{\boxed{\sqrt{\frac{n}{2}}\, N}}^{Z} \le c\right)$

So $\left(1 - \frac{\alpha}{2}\right) = \Phi\left(\sqrt{\frac{n}{2}}\, c\right)$

So. $c = \sqrt{\frac{2}{n}}\, \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$

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Q4: Check

$$P_\theta\{P(X) \le \alpha\} \le \alpha \quad \text{for all } \theta \in H_0 .$$

$$T_{\boxed{\alpha}}(X) = 1 \implies T_{\boxed{\alpha'}}(X) = 1 \quad \text{when } \alpha' > \boxed{\alpha}$$

$$\{P(X) \le \alpha\} = \{\alpha \in \{\alpha' \mid T_{\alpha'} = 1\}\}$$

$$= \{T_\alpha = 1\}$$

$$P_\theta\{P(X) \le \alpha\} = P_\theta\{T_\alpha = 1\} \le \alpha$$

$\because T_\alpha$ is a level α test

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