STAT 610: Discussion 1

1 Summary

- Order statistic: Let $X_1, ..., X_n$ be i.i.d sample and $X_{(1)}, ..., X_{(n)}$ as order statistics, then

find joint dis

$$F_{X_{(j)}}(x) = \sum_{k=j}^{n} \binom{n}{k} \{F_X(x)\}^k \{1 - F_X(x)\}^{n-k}.$$

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x) \{F_X(x)\}^{j-1} \{1 - F_X(x)\}^{n-j}.$$

$X_{(i)}$ and $X_{(j)}$

$$f_{X_{(i)},X_{(j)}}(u, v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f_X(u) f_X(v) \{F_X(u)\}^{i-1} \times \{F_X(v) - F_X(u)\}^{j-1-i} \{1 - F_X(v)\}^{n-j} I(u < v).$$

If $f_{\frac{X_{(i)}}{X_{(j)}}, X_{(j)}} \Leftrightarrow \frac{X_{(i)}}{X_{(j)}} \perp\!\!\!\perp X_{(j)}$

$= f_{\frac{X_{(i)}}{X_{(j)}}} \times f_{X_{(j)}}$

$$f_{X_{(1)},...X_{(n)}}(x_1, \ldots x_n) = \begin{cases} n! f_X(x_1) \cdots f_X(x_n) & \text{if } -\infty < x_1 < \cdots < x_n < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

- Sufficient statistics
  - Definition: A statistic $T(\mathbf{X})$ is sufficient for $\boldsymbol{\theta}$ if the conditional distribution of the sample $\mathbf{X}$ given the value of $T(\mathbf{X})$ does not depend on $\boldsymbol{\theta}$.
  - Factorization Theorem: $T(\mathbf{X})$ is sufficient for $\boldsymbol{\theta}$ if and only if

    to check sufficiency

$$f(\mathbf{x}|\boldsymbol{\theta}) = g(T(\mathbf{x})|\boldsymbol{\theta}) h(\mathbf{x}).$$

- Rao-Blackwell Theorem: Let X be a sample from a population indexed by $\theta \in \theta$ and statistic $T(\mathbf{X})$ is sufficient for θ. If $U(X)$ is a statistic used to estimate $\vartheta = \phi(\theta)$, and $E_\theta[U(X) - \vartheta]^2 < \infty$, then the statistics $h(T) = E[U(X)|T]$ satisfies

$$E_\theta[h(T) - \vartheta]^2 < E_\theta[U(X) - \vartheta]^2 \qquad \theta \in \theta$$

and

$$E_\theta[h(T) - \vartheta]^2 = E_\theta[U(X) - \vartheta]^2 \qquad \theta \in \theta$$

if and only if $P_\theta(U(X) = h(T)) = 1$, $\theta \in \theta$.

  - Sufficiency make sure h is a statistics.

2 Questions

1. Let $X_1, ..., X_n$ be a random sample from a population with pdf

$$f_X(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

$\sim$ Unif $(0, \theta)$

Let $X_{(1)} < ... < X_{(n)}$ be the order statistics. Show that $\frac{X_{(1)}}{X_{(n)}}$ and $X_{(n)}$ are independent.

① Find dis $X_{(1)}$ and $X_{(n)}$

② Find dis $X_{(1)}/X_{(n)}$ and $X_{(n)}$

③ Check indep

2. Show that if T is sufficient statistic and $T = \phi(S)$ where ϕ is a real function and S is another statistic, then S is sufficient.

use FT

3. Let $X_1 \ldots, X_n$ be i.i.d. random variables from the exponential distribution $E(a, \theta)$(i.e. $f_{a,\theta}(x) = \theta^{-1}e^{-(x-a)/\theta}I_{x>a}$). Find a two-dimensional sufficient statistics for (a, θ)

use FT.

4. Let X and Y be two random variables such that Y has the binomial distribution with size N and probability π and, given $Y = y$, X has the binomial distribution with size y and probability p.

   (a) Suppose that π and N are known and $p \in (0, 1)$ is unknown. Show whether X is sufficient for p and whether Y is sufficient for p.

   use. FT

Q1: $X_1, X_2, \dots, X_n \sim \text{Unif}(0, \theta)$

$X_{(1)}, X_{(2)}, \dots, X_{(n)}$ order statistics.

Check: $\frac{X_{(1)}}{X_{(n)}}$ and $X_{(n)}$ are indep

① $$f_{X_{(1)}, X_{(n)}}(a, b) = \frac{n!}{(n-2)!} f(a) f(b) \left[F(b) - F(a)\right]^{n-2} 1\{b > a\}$$

$$= n(n-1) \underline{\tfrac{1}{\theta} 1\{0 \le a \le \theta\}} \, \underline{\tfrac{1}{\theta} 1\{0 \le b \le \theta\}} \left[\frac{b-a}{\theta}\right]^{n-2} \underline{1\{b > a\}}$$

$$= n(n-1)\, \theta^{-n} (b-a)^{n-2} \, \underline{1\{0 \le a \le b \le \theta\}}.$$

② Change of variable.

$Y = X_{(1)}/X_{(n)}$, $Z = X_{(n)}$

- Find Jacobian matrix.

$$\frac{\partial Y \, \partial Z}{\partial X_{(1)} \partial X_{(n)}} = \begin{bmatrix} 1/X_{(n)} & -X_{(1)}/X_{(n)}^2 \\ 0 & 1 \end{bmatrix}$$

$Y = a/b$ $\quad Z = b$

$$\left| \frac{\partial Y \, \partial Z}{\partial X_{(1)} \partial X_{(n)}} \right| = 1/X_{(n)}$$

so $|b-a| = Z(1-Y)$

- $$f_{Y,Z}(y, z) = f_{X_{(1)}, X_{(n)}}(a,b) \times \left| \cancel{\frac{\partial X_{(1)} \, \partial X_{(n)}}{\partial Y \, \partial Z}} \right| \quad X_{(n)}$$

$$= n(n-1)\, \theta^{-n} (b-a)^{n-2} b \quad \underline{1\{0 \le a \le b \le \theta\}}$$

$$= n(n-1)\, \theta^{-n} \left[z(1-y)\right]^{n-2} z \quad 1\{0 \le z \le \theta\} \, 1\{0 \le y \le 1\}$$

$(\because 0 \le \frac{X_{(1)}}{X_{(n)}} \le 1)$

$$= n(n-1)\,\theta^{-n}\ \boxed{z^{n-1}\,\mathbb{1}_{\{0\le z\le\theta\}}}\ \boxed{(1-y)^{n-2}\,\mathbb{1}_{\{0\le y\le 1\}}}$$

$\because$ It is seperable $\Rightarrow z \perp\!\!\!\perp y$.

~~Change of variables~~

If $Y = g(X)$

$$f_Y(y) = \frac{d}{dy} P(Y \le y)$$

$$= \frac{d}{dy} P(g(X) \le y)$$

$$= \frac{d}{dy} P(X \le g^{-1}(y)) = f_X(g^{-1}(y))\,\frac{d}{dy} g^{-1}(y)$$

Q_2: T is sufficient.

$T = \phi(S) \overset{?}{\Rightarrow} S$ is sufficient.

Pf) By FT

$$f_X(x|\theta) = g(T(X)\,|\,\theta)\ h(X)$$

$$= g(\phi \circ \boxed{S(X)}\,|\,\theta)\ h(X)$$

$$= g'(S(X)\,|\,\theta)\ h(X)$$

a function of S and θ

a function only depend on X

By FT $\Rightarrow S$ is sufficient.

Q_3: $X_1, X_2, X_3, \cdots, X_n \sim Exp(a, \theta)$

$$f_X(x) = \frac{1}{\theta} \exp\left\{-\frac{x-a}{\theta}\right\} 1\{x>a\}$$

Find sufficient statistics

pf)

$$f(\underset{\sim}{X}) := f(X_1, \cdots, X_n) = \prod_{i=1}^{n} f(X_i) = \prod_{i=1}^{n} \frac{1}{\theta} \exp\left\{-\frac{X_i - a}{\theta}\right\} 1\{X_i > a\}$$

$$= \theta^{-n} \exp\left\{-\frac{\Sigma X_i - na}{\theta}\right\} \overset{1\{X_{(1)}>a\}}{\cancel{\prod 1\{X_i>a\}}}$$

$$= \theta^{-n} \exp\left\{-\frac{\Sigma X_i - na}{\theta}\right\} 1\{X_{(1)} > a\}$$

$$= g(\Sigma X_i, X_{(1)} \mid \theta, a) \overset{1}{\cancel{h(\underset{\sim}{X})}}$$

By FT

$(\Sigma X_i, X_{(1)})$ are sufficient for θ and a.

A statistic cannot depend on parameters. #

Q4: $Y \sim Bin(N, \pi)$

only p is unknown.

$X \mid Y=y \sim \underline{Bin(y, p)}$

check whether X or Y are sufficient.

pf) $f_{Y,X}(y,x) = f_Y(y) \times \underline{f_{X|Y}(x|y)}$

$$= \underbrace{\binom{N}{y} \pi^{y} (1-\pi)^{N-y} \binom{y}{x}}_{h(x,y)} p^{x} (1-p)^{y-x}$$

$$= h(x,y) \underbrace{\left(\frac{p}{1-p}\right)^{x}}_{} \underbrace{(1-p)^{y}} \longrightarrow y \text{ and } p.$$

we cannot seperate X and p.

By FT X is not sufficient for p.

Y is also not sufficient for p.

But (X, Y) is sufficient for p.

#