quantum relations

量子群の文脈出でてくる関係式のまとめ

k :field, $K = k(q)$

Def.

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$[n]! = \prod_{i=1}^{n} [i]$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}$$

Prop.

(1) $q^m[n] + q^{-n}[m] = [m+n]$

(2) $[m][m'] - [n][m'] = [m-n]\left[\frac{mm' - nn'}{m-n}\right]$

(for $m - n = m' - n'$)

(3) $\begin{bmatrix} n+1 \\ m \end{bmatrix} = q^m \begin{bmatrix} n \\ m \end{bmatrix} + q^{-(n+1-m)} \begin{bmatrix} n \\ m-1 \end{bmatrix}$

Proof

(1)

$$q^m[n] + q^{-n}[m] = \frac{q^{m+n} - q^{m-n} + q^{m-n} - q^{-m-n}}{q - q^{-1}} = [m+n]$$

(2)

(3)

$$\begin{aligned} q^m \begin{bmatrix} n \\ m \end{bmatrix} + q^{-(n+1-m)} \begin{bmatrix} n \\ m-1 \end{bmatrix} &= q^m \frac{[n]!}{[m]![n-m]!} + q^{-(n+1-m)} \frac{[n]!}{[(m-1)]![n-(m-1)]!} \\ &= \left(q^m[n+1-m] + q^{-(n+1-m)}[m]\right) \frac{[n]!}{[m]![n+1-m]!} \\ &= \frac{[n+1][n]!}{[m]![n+1-m]!} \\ &= \begin{bmatrix} n+1 \\ m \end{bmatrix} \end{aligned}$$

∎

Prop.

(1) if $xy = q^2 yx$,

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(n-k)} x^k y^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{-k(n-k)} y^{n-k} x^k$$

(2)

$$\prod_{\nu=0}^{n-1} (1+q^{1-n+2\nu}) := (1+q^{1-n}x)(1+q^{3-n}x)...(1+q^{n-1}x)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

∎

In paticular,

$$\sum_{k=0}^{n} (-1)^l q^{mk} \begin{bmatrix} n \\ k \end{bmatrix} = 0$$

for $|m| \le n-1$ and $m \equiv n-1 \bmod 2$

Proof.

(1) By induction of n . $n=1$ is obvious.

$$\begin{aligned}
(x+y)^{n+1} &= (x+y) \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(n-k)} x^k y^{n-k} \right) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(n-k)} \left(x^{k+1} y^{n-k} + y x^k y^{n-k} \right) \\
&= \sum_{k=0}^{n+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{(k-1)(n+1-k)} x^k y^{n+1-k} \\
&+ \sum_{k=9}^{n+1} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(n-k)} q^{2k} x^k y^{n+1-k} \\
&= \sum_{k=0}^{n+1} \left(q^{-(n+1-k)} \begin{bmatrix} n \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n \\ k \end{bmatrix} \right) q^{k(n+1-k)} x^k y^{n+1-k} \\
&= \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{k(n+1-k)} x^k y^{n+1-k}
\end{aligned}$$

Latter formula is obtained directly by $x^k y^{n-k} = q^{-2k(n-k)} y^{n-k} x^k$.

Since

$$(1 \otimes e)\big(e \otimes t^{-1}\big) = q^2\big(e \otimes t^{-1}\big)(1 \otimes e)$$

$$(f \otimes 1)(t \otimes f) = q^2(t \otimes f)(f \otimes 1)$$

so we get following results:

Col.

$$\Delta(e^n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} q^{-k(n-k)} e^k \otimes e^{n-k} t^{-k}$$

$$\Delta(f^n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} q^{k(n-k)} f^{n-k} t^k \otimes f^k$$

Let $e^{(n)}$, $f^{(n)}$ be

- $e^{(n)} = e^n/[n]!$, $f^{(n)} = f^n/[n]!$

, we get another description of the above formulae:

$$\Delta\big(e^{(n)}\big) = \sum_{k=0}^{n} q^{-k(n-k)} e^{(k)} \otimes e^{(n-k)} t^{-k}$$

$$\Delta\big(f^{(n)}\big) = \sum_{k=0}^{n} q^{-k(n-k)} f^{(n-k)} t^k \otimes f^{(k)}$$