An Introduction to Complex Numbers

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November 29, 2023

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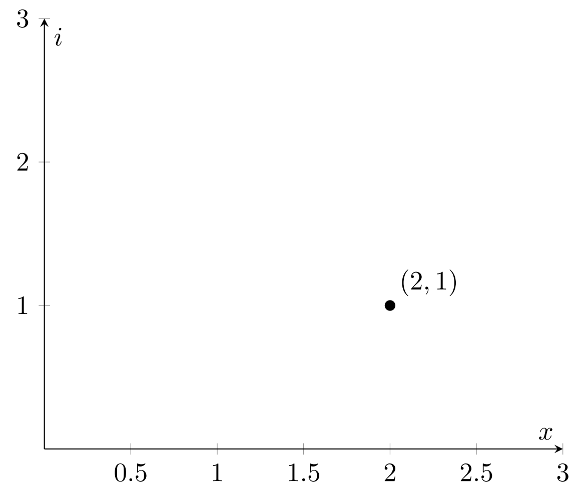
1 Introduction

Complex Analysis is quite similar to real analysis, except it works with complex numbers of the form $x+iy$, where $x, y \in \mathbb{R}, i^2 = -1$. We represent real numbers on a line, we represent complex numbers as elements on a plane.

For example:



A lot of complex analysis is really similar to real analysis, we can do the usual $+, -, \times, /$, exponentials, trignometric functions, differentiation, integration. Many of the rules for real analysis work for complex analysis. lim, series and so on...

1.1 Some Differences

1.1.1 Euler: $e^{ix} = cosx + isinx$

Trigonometric functions and exponential functions turn out to be almost the same. Using this we can write all trignometric functions in terms of exponential functions for example, $cosx = \frac{e^{ix}+e^{-ix}}{2}$. This saves a lot of labour because all the complicated identities for trigonometric functions are just special cases of identities of exponential functions. So there's a lot less to remember.

1.1.2 Differentiability

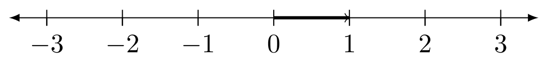
For the reals, some functions we can differentiate it once and twice, but not three times. But for a complex function $\mathbb{C} \to \mathbb{C}$, once it is differentiable once, it is automatically differentiable any number of times.

1.1.3 Integration

Suppose we want to integrate

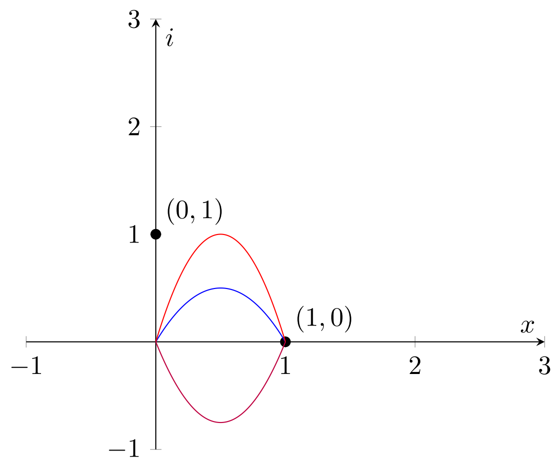
$$\int_0^1 f(x)dx$$

Suppose we want to integrate for real numbers, and there's only one way to go from 0 to 1.



So its quite clear what this integral is meant.

But if we are integrating in the complex plane,



There are many ways to go from 0 to 1. So the integral from 0 to 1 seems to depend on which path you take. Turns out that it almost doesn't. We have Cauch's theorem, that integrals are almost independent of the path we take from 0 to 1. This turns out to extremely useful. For example in real analysis, there are some equations integrals and sums that we don't learn how to evaluate in ordinary introductory calculus classes. But you can work out these using complex integration.

$$\int_0^\infty \frac{sinx}{x}dx = \frac{\pi}{2}$$

$$\sum_{i=1}^{\infty} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

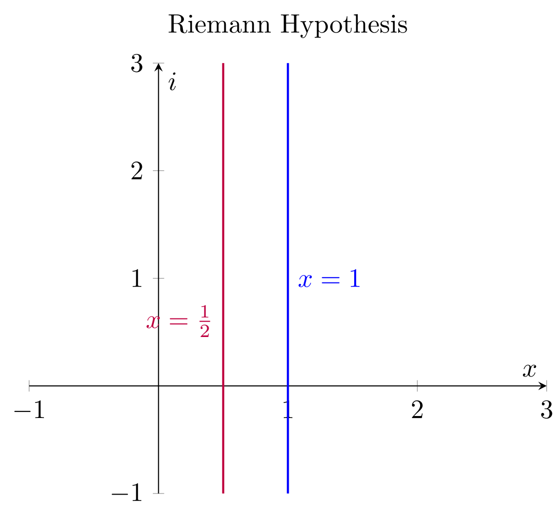
1.2 Analytic Continuation

Suppose I give you a function from 0 to 1, and ask you to evaluate it at $x = -1$, that would be a completely stupid question. There's no way to evaluate it at

-1. There is no real information. However for complex functions, if I give you a function from 0 to 1, and it is differentiable, it is automatically determined in any connected region.
The function on (a, b) is determined uniquely on larger open connected set if it complex differentiable.
There is a very famous function of this. The Riemann Zeta Function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

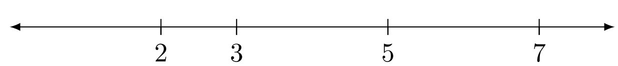
This is probably the single most notorious function in mathematics, the Riemann hypothesis, which states that all zeros of $\zeta(s)$ are real or have real part of $\frac{1}{2}$



Well the Riemann Hypothesis makes no sense at all because if we try to evaluate it it only converges if $\mathrm{Re}(s) > 1$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty \tag{1}$$

And the function only converges on the right side of $x = 1$. Well it turns out the Riemann Zeta function can be analytically continued. In other words, there's only one differentiable complex function that extends the Riemann Zeta function to the whole complex plane, except well, 1 because it would not converge. Why should people care about the Riemann Zeta function? Well it seems to control prime numbers, if we draw prime numbers on a real line,



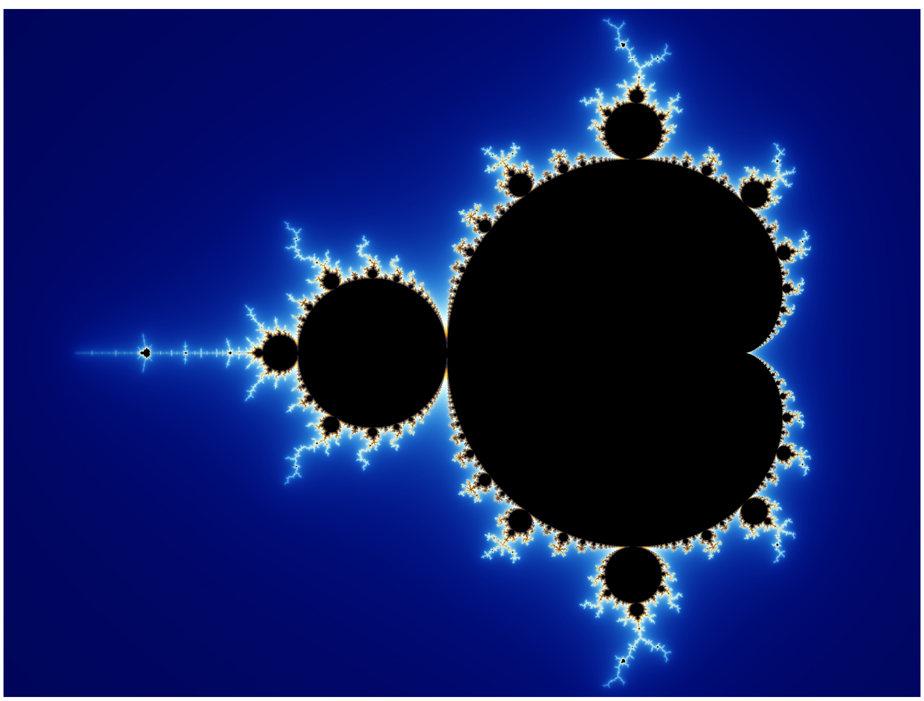
You can see that they are more dense in some regions, and in some regions they are less dense. Something like a sort of compression wave. Riemann discovered that this waves in the primes happen at very precise frequencies, and the frequencies turn out to be the imaginary parts of the zeroes of the zeta function. The amplitude of each wave turn out to be the real part. So the Riemann Hypothesis seems to suggest that primes have a lot of waves going through them. And all these waves in some sense have the same loudness, or volume.

1.3 Complex Dynamics

This is related to a planar set called the Mandel Brot set. We can ask what is the Mandel Brot set?
You take a complex number, you keep on applying the transformation, while c is fixed

$$\begin{array}{lll} z & \to z^2 + c & \\ 0 & \to 0^2 + c & \to \cdots \\ z_0 & \to z_1 & \to z_2 \\ & = z_0^2 + c & = z_1^2 + c \end{array}$$

And you can ask whether this sequence is **bounded**. And this obviously depends on c. So if this sequence is bounded, it is the same as saying c is in the Mandel Brot set. The thing is that the Mandel Brot set is incredibly intricate.



Well that's the end of the summary.

2 Arithmetic