An Introduction to Complex Numbers

Joshua John Lee Shi Kai

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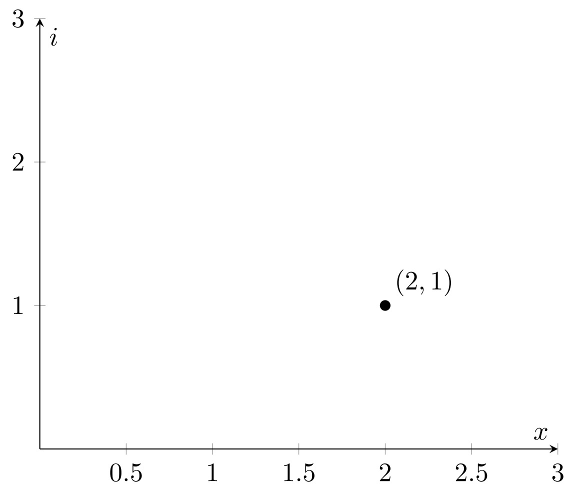
1 Introduction

Complex Analysis is quite similar to real analysis, except it works with complex numbers of the form $x+iy$, where $x, y \in \mathbb{R}, i^2 = -1$. We represent real numbers on a line, we represent complex numbers as elements on a plane.

For example:



A lot of complex analysis is really similar to real analysis, we can do the usual $+, -, \times, /$, exponentials, trignometric functions, differentiation, integration. Many of the rules for real analysis work for complex analysis. lim, series and so on...

1.1 Some Differences

1.1.1 Euler: $e^{ix} = cosx + isinx$

Trigonometric functions and exponential functions turn out to be almost the same. Using this we can write all trignometric functions in terms of exponential functions for example, $cosx = \frac{e^{ix}+e^{-ix}}{2}$. This saves a lot of labour because all the complicated identities for trigonometric functions are just special cases of identities of exponential functions. So there's a lot less to remember.

1.1.2 Differentiability

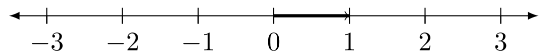
For the reals, some functions we can differentiate it once and twice, but not three times. But for a complex function $\mathbb{C} \to \mathbb{C}$, once it is differentiable once, it is automatically differentiable any number of times.

1.1.3 Integration

Suppose we want to integrate

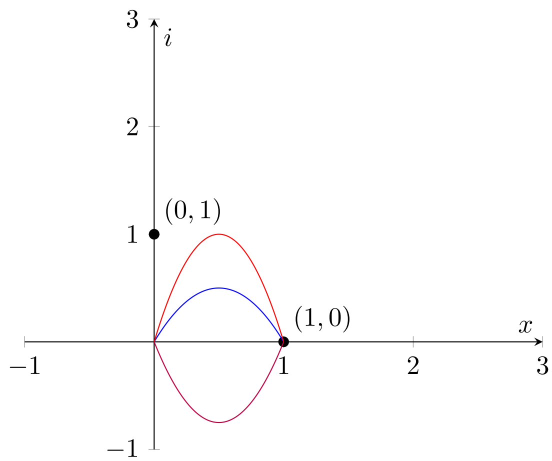
$$\int_0^1 f(x)dx$$

Suppose we want to integrate for real numbers, and there's only one way to go from 0 to 1.



So its quite clear what this integral is meant.

But if we are integrating in the complex plane,



There are many ways to go from 0 to 1. So the integral from 0 to 1 seems to depend on which path you take. Turns out that it almost doesn't. We have Cauch's theorem, that integrals are almost independent of the path we take from 0 to 1. This turns out to extremely useful. For example in real analysis, there are some equations integrals and sums that we don't learn how to evaluate in ordinary introductory calculus classes. But you can work out these using complex integration.

$$\int_0^\infty \frac{sinx}{x}dx = \frac{\pi}{2}$$

$$\sum_{i=1}^{\infty} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

1.2 Analytic Continuation

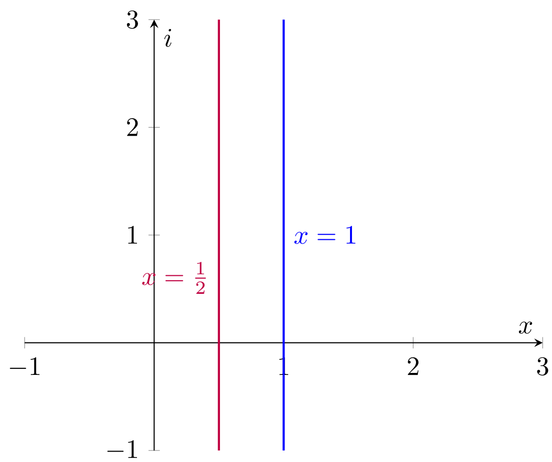
Suppose I give you a function from 0 to 1, and ask you to evaluate it at $x = -1$, that would be a completely stupid question. There's no way to evaluate it at

-1. There is no real information. However for complex functions, if I give you a function from 0 to 1, and it is differentiable, it is automatically determined in any connected region.
The function on (a, b) is determined uniquely on larger open connected set if it complex differentiable.
There is a very famous function of this. The Riemann Zeta Function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

This is probably the single most notorious function in mathematics, the Riemann hypothesis, which states that all zeros of $\zeta(s)$ are real or have real part of $\frac{1}{2}$

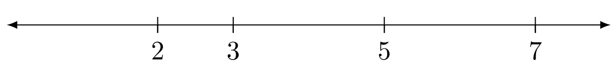
Riemann Hypothesis



Well the Riemann Hypothesis makes no sense at all because if we try to evaluate it it only converges if $\mathrm{Re}(s) > 1$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty \tag{1}$$

And the function only converges on the right side of $x = 1$. Well it turns out the Riemann Zeta function can be analytically continued. In other words, there's only one differentiable complex function that extends the Riemann Zeta function to the whole complex plane, except well, 1 because it would not converge. Why should people care about the Riemann Zeta function? Well it seems to control prime numbers, if we draw prime numbers on a real line,



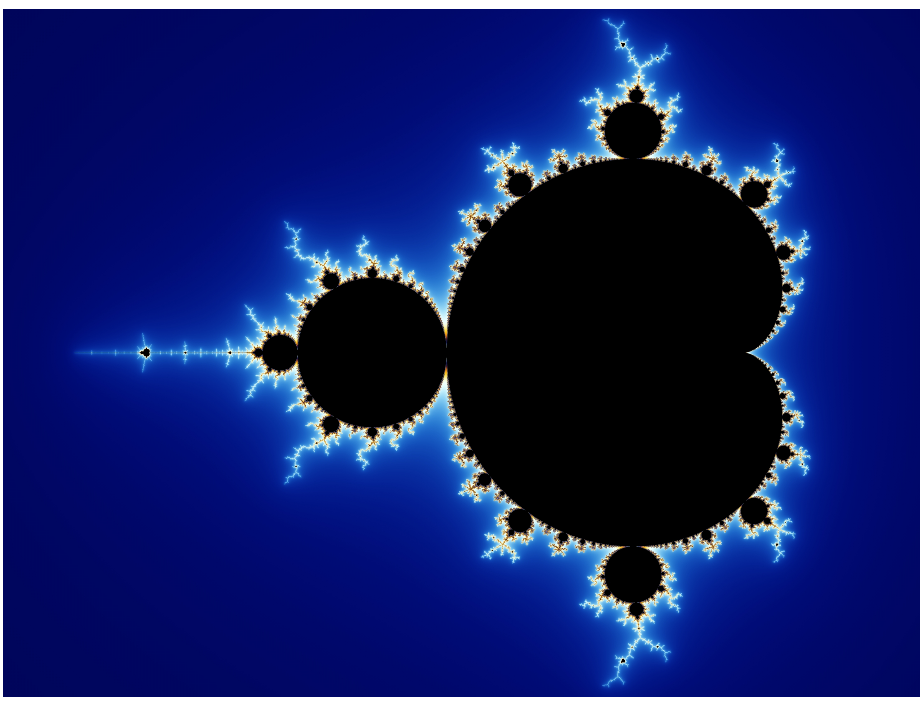
You can see that they are more dense in some regions, and in some regions they are less dense. Something like a sort of compression wave. Riemann discovered that this waves in the primes happen at very precise frequencies, and the frequencies turn out to be the imaginary parts of the zeroes of the zeta function. The amplitude of each wave turn out to be the real part. So the Riemann Hypothesis seems to suggest that primes have a lot of waves going through them. And all these waves in some sense have the same loudness, or volume.

1.3 Complex Dynamics

This is related to a planar set called the Mandel Brot set. We can ask what is the Mandel Brot set?
You take a complex number, you keep on applying the transformation, while c is fixed

$$\begin{array}{lll} z & \to z^2 + c & \\ 0 & \to 0^2 + c & \to \cdots \\ z_0 & \to z_1 & \to z_2 \\ & = z_0^2 + c & = z_1^2 + c \end{array}$$

And you can ask whether this sequence is **bounded**. And this obviously depends on c. So if this sequence is bounded, it is the same as saying c is in the Mandel Brot set. The thing is that the Mandel Brot set is incredibly intricate.



Well that's the end of the summary.

2 Complex Arithmetic

Well if you were a computer, a complex number is nothing more than an ordered pair of numbers (x, y). But if you were human you would write it as $x + iy$. But as a human, writing it as an ordered pair is good too because you could understand it as a plane. So you could represent each complex number as a point in a plane.

Definition 2.1 (Addition)**.**

$$(a + ib) + (c + id) = a + c + ib + id$$

Definition 2.2 (Multiplication)**.**

$$\begin{aligned}(a, b) \cdot (c, d) &= (ac - bd, ad + bc) \\ (a + ib)(c + id) &= ac + i^2 bd + ibc + iad \\ \textit{Since } i^2 &= -1, \\ &\Rightarrow ac - bd + i(bc + ad)\end{aligned}$$

That's why we remember complex numbers as $x + iy$ form, because it makes multiplication trivial to remember.

2.1 The usual stuff

Does it satisfy the usual rules:

$$a(b + c) = ab + ac(ab)c = a(bc) \cdots \tag{2}$$

Well we could

1. Do a long check which we won't do because its long and tedious
2. Go to an abstract algebra course, and show that the definition of the complex numbers gives a ring with all the usual stuff

2.2 Division

Does $a + ib$ has an *inverse* if it is $\neq 0$.

2.2.1 Complex Conjugation

Denoted by:

$$x + iy \to x - iy = \overline{x + iy}$$

The reason this turns up a lot is we said $i^2 = -1$ but -1 actually has 2 squareroots because $(-i)^2 = -1$. So when we talk about a squareroot of -1 we don't really know which squareroot they are talking about and turns out, they are completely equivalent. Anything you can say about 1 squareroot you can say about the other squareroot. So you can sort of flip them around without changing anything. Complex conjugation actually *preserves* all properties of the complex numbers. For example:

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2} \tag{3}$$
$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \tag{4}$$

In fact, complex conjugation $z \to \overline{z}$ is an $AUTOMORPHISM$ of the complex numbers, $\mathbb{C}$. If you've done a Galois Theory course, another way of saying is this is the Galois group of the $\mathbb{C}/\mathbb{R}$ is $\{1, -\}$.

So using complex conjugation, we can find the inverse of any complex number as follows