An Introduction to Complex Numbers

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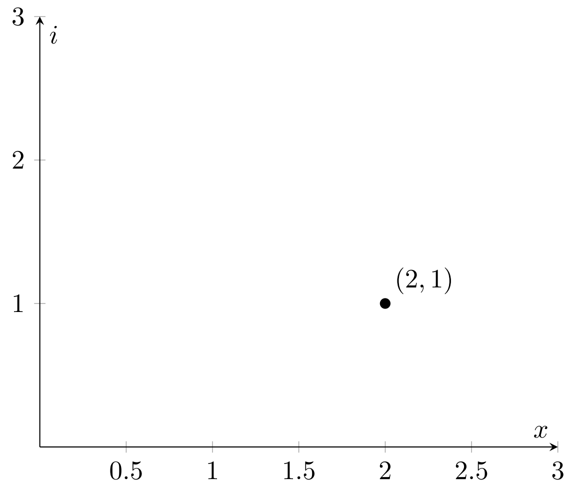
1 Introduction

Complex Analysis is quite similar to real analysis, except it works with complex numbers of the form $x+iy$, where $x, y \in \mathbb{R}, i^2 = -1$. We represent real numbers on a line, we represent complex numbers as elements on a plane.

For example:



A lot of complex analysis is really similar to real analysis, we can do the usual $+, -, \times, /$, exponentials, trignometric functions, differentiation, integration. Many of the rules for real analysis work for complex analysis. lim, series and so on...

1.1 Some Differences

1.1.1 Euler: $e^{ix} = cosx + isinx$

Trigonometric functions and exponential functions turn out to be almost the same. Using this we can write all trignometric functions in terms of exponential functions for example, $cosx = \frac{e^{ix}+e^{-ix}}{2}$. This saves a lot of labour because all the complicated identities for trigonometric functions are just special cases of identities of exponential functions. So there's a lot less to remember.

1.1.2 Differentiability

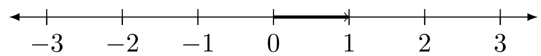
For the reals, some functions we can differentiate it once and twice, but not three times. But for a complex function $\mathbb{C} \to \mathbb{C}$, once it is differentiable once, it is automatically differentiable any number of times.

1.1.3 Integration

Suppose we want to integrate

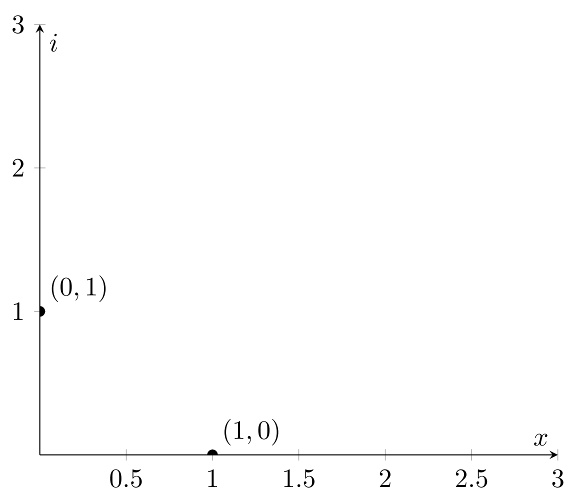
$$\int_0^1 f(x)dx$$

Suppose we want to integrate for real numbers, and there's only one way to go from 0 to 1.



So its quite clear what this integral is meant.

But if we are integrating in the complex plane,



There are many ways to go from 0 to 1. So the integral from 0 to 1 seems to depend on which path you take. Turns out that it almost doesn't. We have Cauch's theorem, that integrals are almost independent of the path we take from 0 to 1. This turns out to extremely useful. For example in real analysis

$$\int_0^{\text{inf}}$$