An Introduction to Complex Numbers

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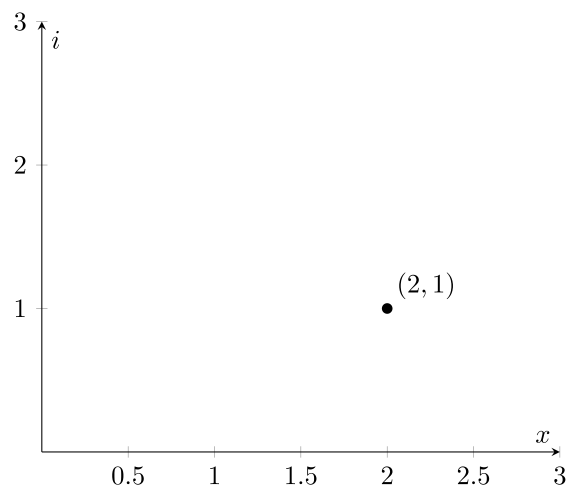
1 Introduction

Complex Analysis is quite similar to real analysis, except it works with complex numbers of the form $x + iy$, where $x, y \in \mathbb{R}, i^2 = -1$. We represent real numbers on a line, we represent complex numbers as elements on a plane.

For example:



A lot of complex analysis is really similar to real analysis, we can do the usual $+, -, \times, /$, exponentials, trignometric functions, differentiation, integration. Many of the rules for real analysis work for complex analysis. lim, series and so on...

1.1 Some Differences

1.1.1 Euler: $e^{ix} = cosx + isinx$

Trigonometric functions and exponential functions turn out to be almost the same. Using this we can write all trignometric functions in terms of exponential functions for example, $cosx = \frac{e^{ix}+e^{-ix}}{2}$. This saves a lot of labour because all the complicated identities for trigonometric functions are just special cases of identities of exponential functions. So there's a lot less to remember.

1.1.2 Differentiability

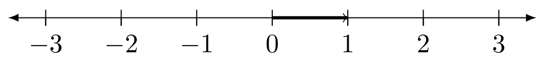
For the reals, some functions we can differentiate it once and twice, but not three times. But for a complex function $\mathbb{C} \to \mathbb{C}$, once it is differentiable once, it is automatically differentiable any number of times.

1.1.3 Integration

Suppose we want to integrate

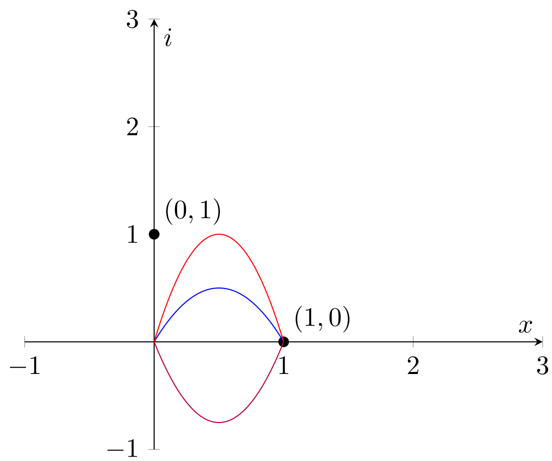
$$\int_0^1 f(x)dx$$

Suppose we want to integrate for real numbers, and there's only one way to go from 0 to 1.



So its quite clear what this integral is meant.

But if we are integrating in the complex plane,



There are many ways to go from 0 to 1. So the integral from 0 to 1 seems to depend on which path you take. Turns out that it almost doesn't. We have Cauch's theorem, that integrals are almost independent of the path we take from 0 to 1. This turns out to extremely useful. For example in real analysis, there are some equations integrals and sums that we don't learn how to evaluate in ordinary introductory calculus classes. But you can work out these using complex integration.

$$\int_0^\infty \frac{sinx}{x}dx = \frac{\pi}{2}$$

$$\sum_{i=1}^{\infty} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

1.2 Analytic Continuation

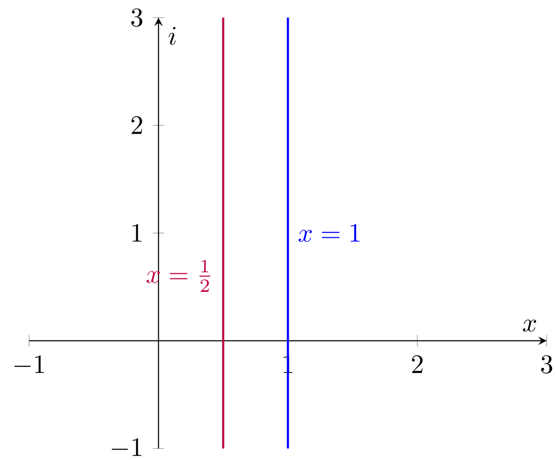
Suppose I give you a function from 0 to 1, and ask you to evaluate it at $x = -1$, that would be a completely stupid question. There's no way to evaluate it at

-1. There is no real information. However for complex functions, if I give you a function from 0 to 1, and it is differentiable, it is automatically determined in any connected region.
The function on (a, b) is determined uniquely on larger open connected set if it complex differentiable.
There is a very famous function of this. The Riemann Zeta Function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

This is probably the single most notorious function in mathematics, the Riemann hypothesis, which states that all zeros of $\zeta(s)$ are real or have real part of $\frac{1}{2}$

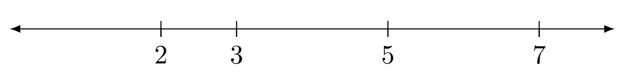
Riemann Hypothesis



Well the Riemann Hypothesis makes no sense at all because if we try to evaluate it it only converges if $\mathrm{Re}(s) > 1$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty \tag{1}$$

And the function only converges on the right side of $x = 1$. Well it turns out the Riemann Zeta function can be analytically continued. In other words, there's only one differentiable complex function that extends the Riemann Zeta function to the whole complex plane, except well, 1 because it would not converge. Why should people care about the Riemann Zeta function? Well it seems to control prime numbers, if we draw prime numbers on a real line,



You can see that they are more dense in some regions, and in some regions they are less dense. Something like a sort of compression wave. Riemann discovered that this waves in the primes happen at very precise frequencies, and the frequencies turn out to be the imaginary parts of the zeroes of the zeta function. The amplitude of each wave turn out to be the real part. So the Riemann Hypothesis seems to suggest that primes have a lot of waves going through them. And all these waves in some sense have the same loudness, or volume.

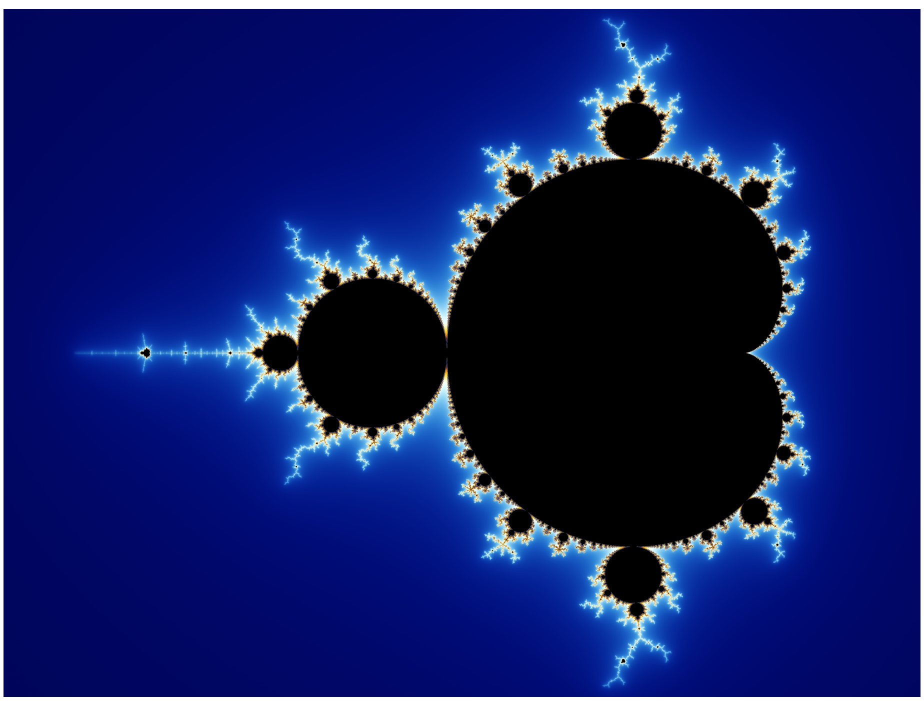
1.3 Complex Dynamics

This is related to a planar set called the Mandel Brot set. We can ask what is the Mandel Brot set?

You take a complex number, you keep on applying the transformation, while c is fixed

$$\begin{array}{lll} z & \to z^2 + c & \\ 0 & \to 0^2 + c & \to \cdots \\ z_0 & \to z_1 & \to z_2 \\ & = z_0^2 + c & = z_1^2 + c \end{array}$$

And you can ask whether this sequence is **bounded**. And this obviously depends on c. So if this sequence is bounded, it is the same as saying c is in the Mandel Brot set. The thing is that the Mandel Brot set is incredibly intricate.



Well that's the end of the summary.

2 Complex Arithmetic

Well if you were a computer, a complex number is nothing more than an ordered pair of numbers (x, y). But if you were human you would write it as $x + iy$. But as a human, writing it as an ordered pair is good too because you could understand it as a plane. So you could represent each complex number as a point in a plane.

Definition 2.1 (Addition)**.**

$$(a + ib) + (c + id) = a + c + ib + id$$

Definition 2.2 (Multiplication)**.**

$$\begin{aligned}(a, b) \cdot (c, d) &= (ac - bd, ad + bc)\\(a + ib)(c + id) &= ac + i^2bd + ibc + iad\\\textit{Since } i^2 &= -1,\\&\Rightarrow ac - bd + i(bc + ad)\end{aligned}$$

That's why we remember complex numbers as $x + iy$ form, because it makes multiplication trivial to remember.

2.1 The usual stuff

Does it satisfy the usual rules:

$$a(b + c) = ab + ac(ab)c = a(bc) \cdots$$

Well we could

1. Do a long check which we won't do because its long and tedious
2. Go to an abstract algebra course, and show that the definition of the complex numbers gives a ring with all the usual stuff

2.2 Division

Does $a + ib$ has an *inverse* if it is $\neq 0$.

2.2.1 Complex Conjugation

Denoted by:

$$x + iy \to x - iy = \overline{x + iy}$$

The reason this turns up a lot is we said $i^2 = -1$ but -1 actually has 2 squareroots because $(-i)^2 = -1$. So when we talk about a squareroot of -1 we don't really know which squareroot they are talking about and turns out, they are completely equivalent. Anything you can say about 1 squareroot you can say about the other squareroot. So you can sort of flip them around without changing anything. Complex conjugation actually *preserves* all properties of the complex numbers. For example:

$$\begin{aligned} \overline{z_1 z_2} &= \overline{z_1} \cdot \overline{z_2} \\ \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \end{aligned}$$

In fact, complex conjugation $z \to \overline{z}$ is an *AUTOMORPHISM* of the complex numbers, $\mathbb{C}$. If you've done a Galois Theory course, another way of saying is this is the Galois group of the $\mathbb{C}/\mathbb{R}$ is $\{1, \text{Complex Conjugation}\}$.

So using complex conjugation, we can find the inverse of any complex number as follows

$$\begin{aligned} z &= x + iy \\ z\overline{z} &= (x+iy)(x-iy) \\ &= x^2 + y^2 \\ \frac{1}{z} &= \frac{\overline{z}}{z\overline{z}} \\ \frac{1}{x+iy} &= \frac{x-iy}{(x+iy)(x-iy)} \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \end{aligned}$$

What you should remember is to multiply the denominator by its complex conjugate. Division by complex numbers is also reasonably easy.

2.2.2 Absolute Value

The absolute distance of x from 0 is x if $x > 0$ and $-x$ if $(x < 0)$. Similarly, for any complex number z,

$$\begin{aligned} z &= x + iy \\ |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{z\overline{z}} \end{aligned}$$

From this, then the following also follows,

$$|z_1 z_2| = |z_1||z_2|$$

All the rules of absolute value also applies,

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

2.2.3 Application for Complex Arithmetic

Which integers are sums of 2 squares? Well we know that they are closed under multiplication. Meaning that if two integers are the sum of two squares then so is their product.

Proof. Let's say we have two integers that are the some of two squares, then

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Well where on earth did this come from? It comes from complex numbers! We notice that

$$\begin{aligned} a^2 + b^2 &= |a + ib|^2 \\ c^2 + d^2 &= |c + id|^2 \\ (a^2 + b^2)(c^2 + d^2) &= |(a + ib)(c + id)|^2 \\ &= |ac - bd + i(ad + bc)|^2 \\ &= (ac - bd)^2 + (ad + bc)^2 \end{aligned}$$

$\square$

The proof with reals doesn't really explain why it exists, it is explained with complex numbers.

Proof. Let's give an example of this

$$\begin{aligned} 5 &= 1^2 + 2^2 \\ 13 &= 2^2 + 3^2 \\ 5 \times 13 &= 65 \\ &= 8^2 + 1^2 \\ &= 4^2 + 7^2 \end{aligned}$$

So why are there two ways to write 65 as a sum of two squares?

$$\begin{aligned} 1^2 + 2^2 &= |1 + 2i|^2 \\ 2^2 + 3^2 &= |2 + 3i|^2 \\ (1 + 2i)(2 + 3i) &= -4 + 7i \end{aligned}$$

And you get the first solution, which is actually $(-4)^2 + 7^2$. There's another thing you can do which is change one of them to their complex conjugate

$$(1 + 2i)(2 - 3i) = 8 + i$$

And there are other things we could do like use the conjugate for the other complex number but it will just give variations of the two solutions. So we see that the two ways of writing 65 as a sum of two squares correspond to the two different products of complex numbers. $\square$

Next application, let's take a look at pythogoras' theorem, $a^2+b^2=c^2$. And we want to find solutions to this equation. How can we generate many solutions to this easily?

$$
\begin{aligned}
(a+ib) &= (x+iy)^2 \\
|a+ib| &= a^2+b^2 \\
&= |(x+iy)^2|^2 \\
&= (x^2+y^2)^2
\end{aligned}
$$

So now we can just sort of pick any random complex number for x and y,

$$
\begin{aligned}
x+iy &= 2+i \\
(x+iy)^2 &= (2+i)^2 \\
&= 3+4i \\
&= a+ib \\
|a+ib|^2 &= a^2+b^2 \\
&= 3^2+4^2 \\
|(x+iy)^2|^2 &= (|(x+iy)|^2)^2 \\
&= ((\sqrt{2^2+1^2})^2)^2 \\
&= 5^2
\end{aligned}
$$

2.2.4 Hamilton Quaternions

Quaternions means a collection of 4 things. so a quarternion is simply

$$
\begin{aligned}
a+bi+cj+dk \\
i^2 = -1 \\
j^2 = -1 \\
k^2 = -1 \\
ij = k = -ji \\
jk = i = -kj \\
ki = j = -ik
\end{aligned}
$$

Quaternions are not commutative! But they do behave very similarly to complex numbers. For example,

$$\begin{aligned} z &= a + bi + cj + dk \\ \overline{a + bi + cj + dk} &= a - bi - cj - dk \\ z\overline{z} &= a^2 + b^2 + c^2 + d^2 \end{aligned}$$

So now we can work out the inverse of the quarternion,

$$\frac{1}{a + bi + cj + dk} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

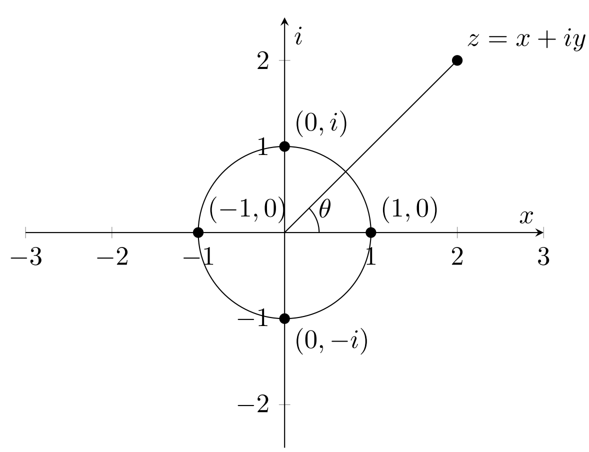
You realise that if the quarternion is non-zero, then the denominator would be non-zero, so all non-zero quarternions have inverses.

3 Roots

Do Complex Numbers have square roots? Can we find

$$\begin{gathered} \sqrt{2+i} \\ \sqrt[3]{2+i} \\ \dots \end{gathered}$$

The answer is YES! To understand this, we have to understand the geometric meaning of multiplication. To do this, we express multiplication in terms of polar coordinates.



We have the usual way to go about changing from cartesian to polar coordinates.

$$r = |z|$$
$$tan\theta = \frac{y}{x}$$
$$x = rcos\theta$$
$$y = rsin\theta$$

Now you notice if we take the values of absolute value one (the circle), these actually form a group. A **group** is a set that is closed under multiplication and inverses. For example,

$$|z_1| = 1$$
$$|z_2| = 1$$
$$|z_1 z_2| = 1$$
$$|z^{-1}| = 1$$

So we can multiply and divide points on the circle group. We also see that any complex number z, can be written as a point on the circle group, times a positive real number.

$$\text{Point on the circle group: } cos\theta + isin\theta$$
$$\text{Any Complex Number: } z = r \cdot (cos\theta + isin\theta)$$

Every non-zero complex number z, can be written *uniquely* as $z = a \cdot b$ where a is a positive real, and b is in the circle group ($|b| = 1$). Another way of writing this is $\mathbb{C}^x := \mathbb{R}_{>0} \times \mathbb{S}^1$, where $\mathbb{C}^x$ means take the complex numbers under multiplication and throw away the zero, positive reals, and the circle group where $\mathbb{S}$ means sphere and the superscript means it is one-dimension.

3.1 Multiplication in S^1

There are two ways to do multiplication in S^1, namely,

1. Mutiply them as complex numbers.
2. Multiply them by adding angles.

And we want to show that they are the same.

Proof. Complex multiplication:

$$(cos\theta_1 + isin\theta_1) \times (cos\theta_2 + isin\theta_2)$$
$$= cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2 + i(cos\theta_1 sin\theta_2 + cos\theta_2 sin\theta_1)$$

Similarly for the addition of angles,

$$cos(\theta_1 + \theta_2) = cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2$$
$$sin(\theta_1 + \theta_2) = sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2$$

So multiplying the complex numbers on the circle, is the same as adding their angles. □

This shows that complex numbers are really good for rotations in $\mathbb{R}^2$. Similarly quarternions are really good for rotations of $\mathbb{R}^3$ and $\mathbb{R}^4$. Navigation systems often make use of quarternions to calculate rotations.

3.1.1 Summary

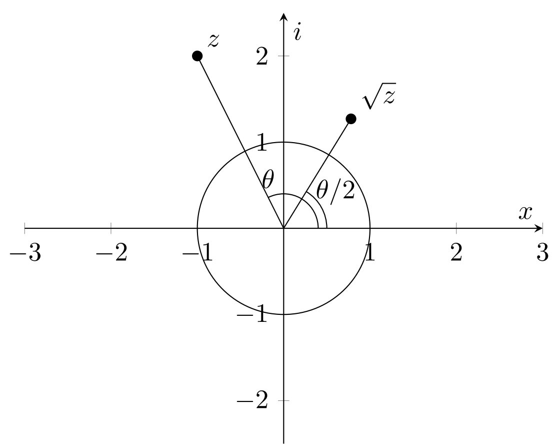
To multiply two complex numbers z_1, z_2,

1. Multiply absolute values
2. Add angles (argument)

You can multiply complex numbers by just sort of staring at them, not much algebraic stuff needed.

3.2 Square Roots

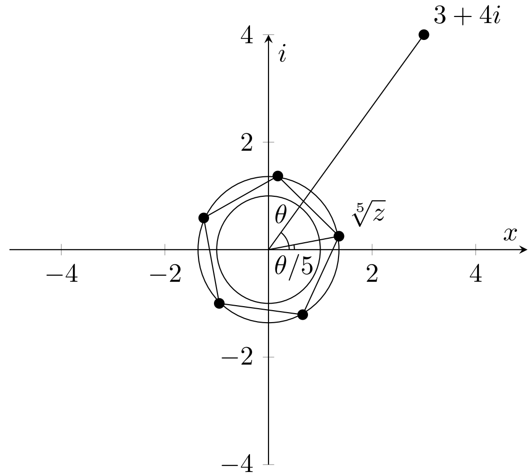
So $\sqrt{z}$ simply has half the argument, θ, and the absolute value $= \sqrt{|z|}$. This sort of shows that every complex number has a square root. Let's show this graphically.



We notice that $\sqrt{z}$ has half the argument of z, and the real part of $\sqrt{z}$ is $\sqrt{|z|}$. So the two square roots of z are $\sqrt{z}$ and $-\sqrt{z}$.

3.3 Other Roots

Just as numbers have more than one square root, they have more than one 5th root, so if we take $\frac{\theta}{5}$, we can also take $\frac{\theta+2\pi}{5}$, $\frac{\theta+4\pi}{5}$ and so on...



The five roots lie on a regular pentagon. And the same happens if you take any nth roots of any non-zero complex number, you find that there are going to be n roots, and they lie on the vertices of a nice regular, n-sided polygon. And you can find out these values explicitly by working out the argument and absolute value of z.

So any complex number, z, has an n-th root. If z is non-zero, it is going to have n distinct, n-th roots. So polynomials of the form $z^n - a$ always has root. The same happens for more complex polynomials, and will be shown later to always have a complex root. We sometimes express this by saying that $\mathbb{C}$ is algebraically closed. "algebraically closed" just means that whenever you have any non constant polynomial, you can always find a root of that polynomial.

3.4 Multiple Angle Formulas

Everyone is familiar with the old trusty angle formulas

$$cos2\theta = cos^2\theta - sin^2\theta$$
$$cos3\theta = cos^3\theta - 3cos\theta sin^2\theta = 4cos^3\theta - 3cos\theta$$

You *could* prove these formulas by doing some tedious algebra, but with complex numbers, the work becomes a lot less. We get it from the binomial theorem,

$$\begin{aligned} cosn\theta + isinn\theta &= (\cos\theta + i\sin\theta)^n \\ &= \binom{n}{0}(\cos\theta)^n + i\binom{n}{1}(cos\theta)^{n-1}\sin\theta - \binom{n}{2}(cos\theta)^{n-2}(sin\theta)^2 \end{aligned}$$

And then just take real parts of both sides.

$$\cos n\theta = (\cos\theta)^n - \binom{n}{2}(\cos\theta)^{n-2}(\sin\theta)^2 + \cdots$$

4 Elementary Transcendental Functions

This section will discuss $e^x, \log, \sin, \cos, \tan$, and so on... We will start with the exponential function, and see that all the others are just special cases of it.

4.1 Exponential

We can define the exponential function using the normal power series, as we do for over the reals,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

So we can ask the question, does this converge? Turns out it is absolutely convergent. And if a series of complex numbers are absolutely convergent, then so is the series of real parts, and the same for the imaginary parts.

$$\begin{aligned}
e^{z_1+z_2} &= e^{z_1}e^{z_2} \\
\sum_n \frac{(z_1+z_2)^n}{n!} &= \sum_{n,m} \frac{\binom{n}{m} z_1^m z_2^{n-m}}{n!} \\
&= \sum_{n,m} \frac{z_1^m}{m!}\frac{z_2^{n-m}}{(n-m)!} \\
&= \sum_{n,m} \frac{z_1^m}{m!}\frac{z_2^n}{n!} \\
&= e^{z_1}e^{z_2}
\end{aligned}$$

You realise we did some rearrangement, but that's fine because everything is absolutely convergent. So let's try now to calculate it,

$$\begin{aligned}
z &= x + iy \\
e^z &= e^x e^{iy} \\
e^{iy} &= 1 + iy + \frac{i^2y^2}{2!} + \frac{i^3y^3}{3!} + \cdots \\
&= 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} + \cdots
\end{aligned}$$

We realise that

$$1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots = \cos y$$
$$iy - \frac{iy^3}{3!} + \frac{iy^5}{5!} + \cdots = i \sin y$$

This brings us to Euler's identity, where $e^{iy} = \cos y + i \sin y$. Which allows us to work out the exponential for all complex z. Also it follows that,

$$\begin{aligned} y &= \pi \\ e^{\pi i} &= -1 \\ y &= 2\pi \\ e^{2\pi i} &= 1 \\ e^{2n\pi i} &= 1, \text{ for } n \in \mathbb{Z} \end{aligned}$$

We can also say that the exponential function is a homomorphism of groups. We know that

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$$

Therefore, we see from algebra courses that this is just a homomorphism from the complex numbers under addition, to the complex numbers under multplication. $\mathbb{C} \to \mathbb{C}^{\times}$. Saying its a homomorphism just says that it turns addition into multiplication. We also notice that this exponential map is onto. So, provided that r is non-zero,

$$r(\cos\theta + i \sin\theta) = \exp(\log r + i\theta)$$

We can also ask what is the kernel? Where $\exp(x + iy) = 1$?

$$\exp(x + iy) = \exp(x)(\cos y + i \sin y)$$

Thus we can see that $x = 0$, and y has to be some multiple of 2π.

Note: For the reals, we also get a map from the reals under addition to the reals under multiplication, $\mathbb{R} \to \mathbb{R}^{\times}$. Except this time, there's no kernel and the map is not unto. It is injective but surjective. Whereas the complex exponentials is surjective but not injective.

4.2 Logarithm

Solve: $e^{a+bi} = z$

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ a + bi &= \log z \\ e^a &= r \\ &= |z| \\ a &= \log|z|, z \neq 0 \\ \theta &= \arg(z) \end{aligned}$$

The problem is that $arg(z)$ is not unique, so θ is only defined up to multiples of 2π. $\log(z)$ NOT unique, only defined up to multiples of $2\pi i$.

We can also ask if,

$$z_1^{z_2} = \exp(z_2 \log(z_1))$$

Is well defined, since $\log(z_1)$ is ambiguous, there are only 2 cases where this function is well defined.

1. $z_1 > 0$ and REAL, take the real part of $\log(z_1)$
2. z_2 is an integer, $\exp(n\log(z_1)) = \exp(n2\pi i) = 1$

4.3 Trigonometric Functions

$$\begin{aligned} e^{iz} &= \cos z + i\sin z \\ e^{-iz} &= \cos z - i\sin z \\ \frac{e^{iz} + e^{-iz}}{2} &= \cos z \\ \frac{e^{iz} - e^{-iz}}{2i} &= \sin z \end{aligned}$$

An amazing unification of exponential functions and trigonometric functions. In real analysis, they seem so different. Over the complex numbers, they seem like almost the same function. All identities of trigonometric functions also follow from the exponential functions.

4.4 Applications of Trigonometric Functions

Suppose you want to solve a simple linear differential equation.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

If you've been to a real analysis course, you know you tend to be able to find the solution,

$$y = e^{\alpha} \cos \beta x$$

and its really a bit of a mess checking what α, β has to be for this to be a solution. With complex numbers this becomes a lot easier to solve, you just substitute in $y = e^{\lambda x}$.

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$
$$a\lambda^2 + b\lambda + c = 0$$

So now λ is given by the root of some quadratic equation. We get a much simpler form for the solution. As an example,

$$\begin{aligned} \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0 \\ \Rightarrow \lambda^2 + 2\lambda + 2 = 0 \\ \Rightarrow \lambda = -1 \pm i \\ y = e^{(1+i)x}, e^{(i-1)x} \\ \Rightarrow y = e^{x} \cos x, e^{x} \sin x \end{aligned}$$

Another way complex numbers make things simplier is if you're doing Fourier series. If you have a function that's periodic,

$$f(x) = f(2\pi + x)$$
$$\Rightarrow f(x) = \sum_{n>0} a_n \sin(nx) + \sum_{n\geq 0} b_n \cos(nx)$$

Now if you use complex numbers you can very easily simplify this, because sin and cos can be written as exponentials.

$$f(x) = \sum_{n \in z} c_n e^{inx}$$

This makes computing the fourier series a lot easier and neater.

5 Complex Derivatives

We are not going to go straight into complex derivatives, but we are going to look at real derivatives of complex functions.

5.1 Real Derivatives of Complex Functions

Let's take $w = u + iv$ is a function of $z = x + iy$, where $u, v, x, y \in \mathbb{R}$. Such that u, v are functions of x, y. First we need to write u, v as a vector.

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \underbrace{\begin{bmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}_{\text{linear function}} + \epsilon$$

If you compare it with the real derivative, you see that it is much the same. Except that now we are doing it with two variables. Again, this error should be less than any linear function such that

$$\text{As } (x, y) \to (x_0, y_0)$$
$$\frac{|\epsilon|}{\text{dist}(x, y), (x_0, y_0)} \to 0$$

Saying that w is differentiable at the point z_0 just means that we can approximate the function w by this linear function with a small error term, ϵ. The matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

can also be written as the partial derivates,

$$\begin{bmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} \end{bmatrix}$$

So far we haven't done anything in complex analysis, all we have done is discussed a function from the real plane to the real plane, and discussed whether its differentiable. What we have defined here is just real differentiability.