Working with the Solow Growth Model

Solow Growth Model

- $\Delta k/k = s\cdot(y/k) - s\delta - n$
  - $\Delta k/k$ 人均资本增长率
  - k is capital per worker
  - y is real GDP per worker
  - y/k is the average product of capital, =Y/K, 也即资本生产力
  - s is the saving rate
  - δ is the capital depreciation rate
  - n is the population (labor force) growth rate.

Solow Growth Model Steady State

- We assumed that everything on the right-hand side was constant except for *y*/*k*.
- In the transition to the steady state, the rise in *k* led to a fall in *y*/*k* and, hence, to a fall in $\Delta k/k$.
- In the steady state, *k* was constant and, therefore, *y*/*k* was constant. Hence, $\Delta k/k$ was constant and equal to zero.
- $0 = s \cdot (y^*/k^*) - s\delta - n$

Solow Growth Model

- Production function

  $y = A \cdot f(k)$

- Solow Growth Equation

  $\Delta k/k = s\, A \cdot f(k)/k - s\delta - n$

  即：$\Delta k/k$ 最终可以表达为 k 的函数。

Solow Growth Model
Change in savings rate (*s*)

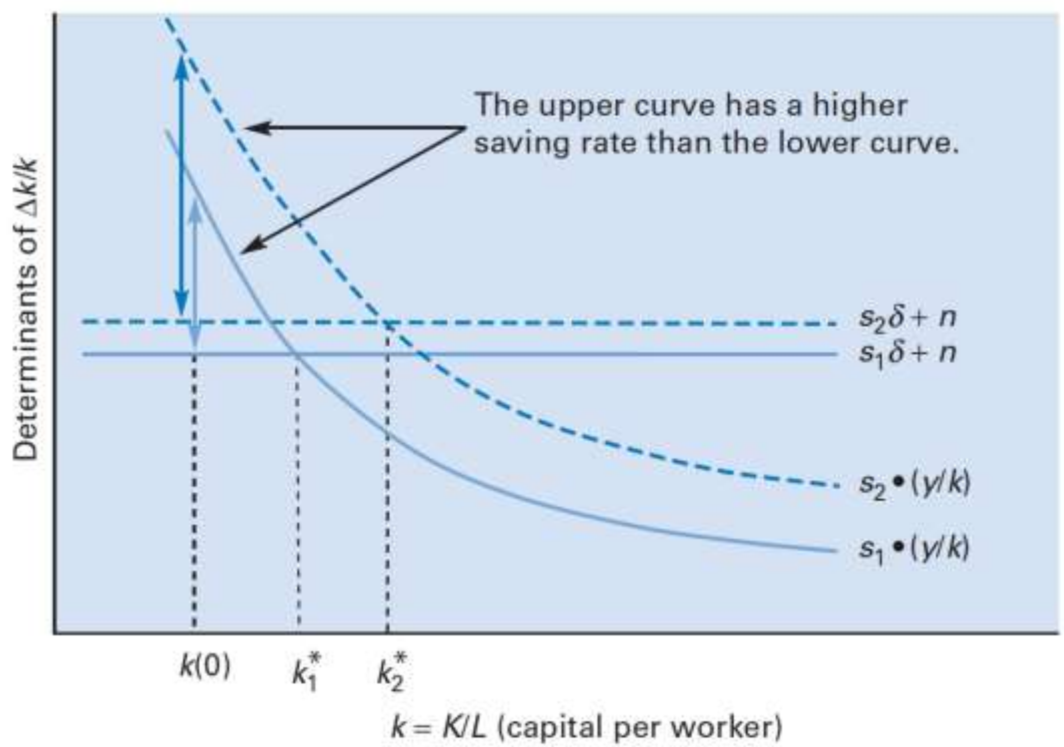


Figure 4.1 Effect of an increase in the saving rate in the Solow model

$$\Delta k/k = s\,(y/k) - s\delta - n$$

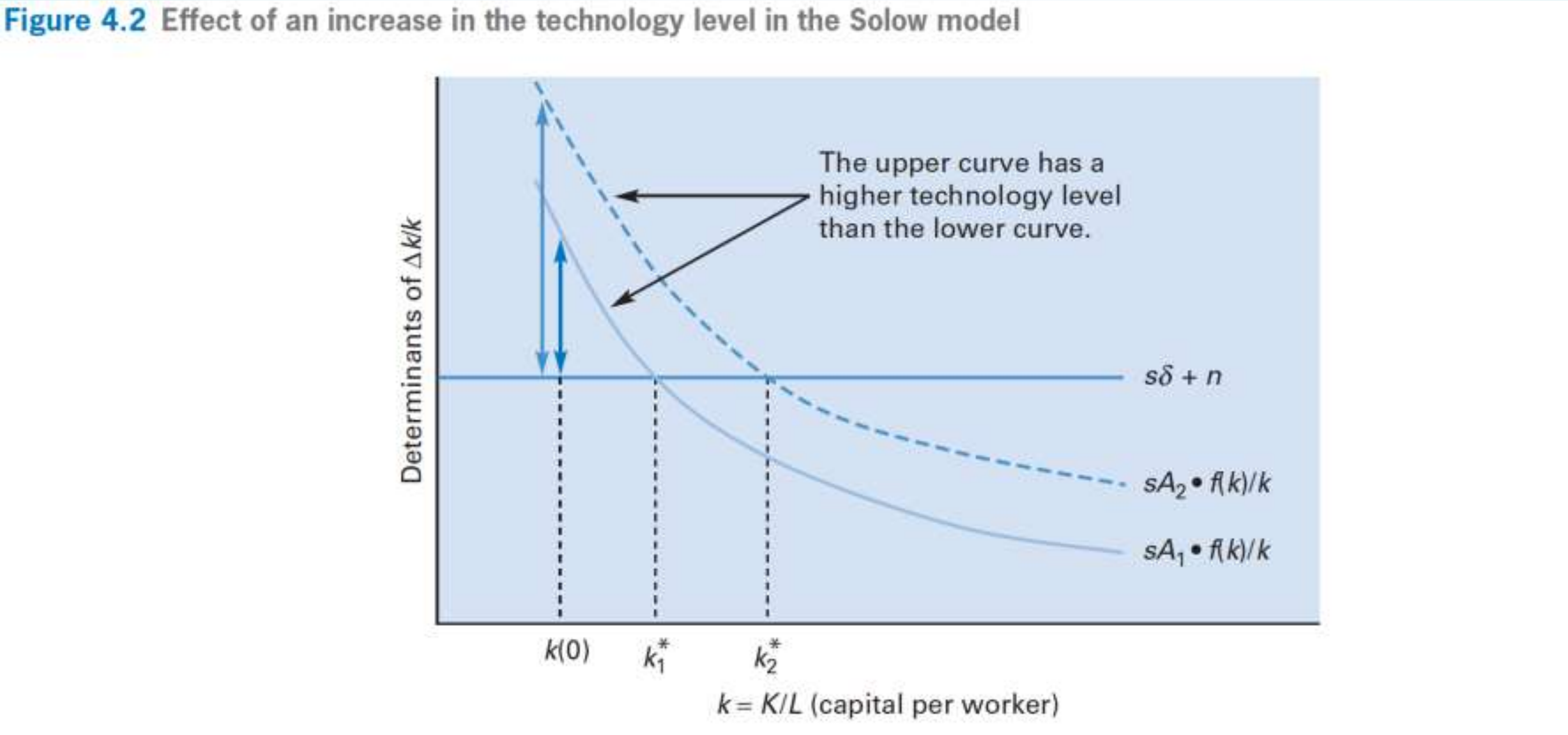
Solow Growth Model
Change in savings rate (*s*)

- In the short run, an increase in the saving rate *s* raises the growth rate of capital per worker.
- This growth rate remains higher during the transition to the steady state.

Solow Growth Model
Change in savings rate (*s*)

- In the long run, the growth rate of capital per worker $\Delta k/k$ is the same—zero—for any saving rate.
- In this long-run or steady-state situation, a higher saving rate leads to higher steady state capital per worker, $k*$ (not to a change in the growth rate which remains at zero).

Solow Growth Model
Change in technology level (*A*)



$$\Delta k/k = s A \cdot f(k)/k - s\delta - n$$

Solow Growth Model
Change in technology level (*A*)

- In the short run, an increase in the technology level, *A*, raises the growth rates of capital and the growth rates of real GDP per worker.
- These growth rates remain higher during the transition to the steady state.

Solow Growth Model
Change in technology level (*A*)

- In the long run, the growth rates of capital and real GDP per worker are the same—zero—for any technology level.
- In this long-run or steady state situation, a higher technology level leads to higher steady-state capital and real GDP per worker, $k*$ and $y*$ (not to changes in the growth rates which remain at zero).

Solow Growth Model
Change in the labor input ($L0$)

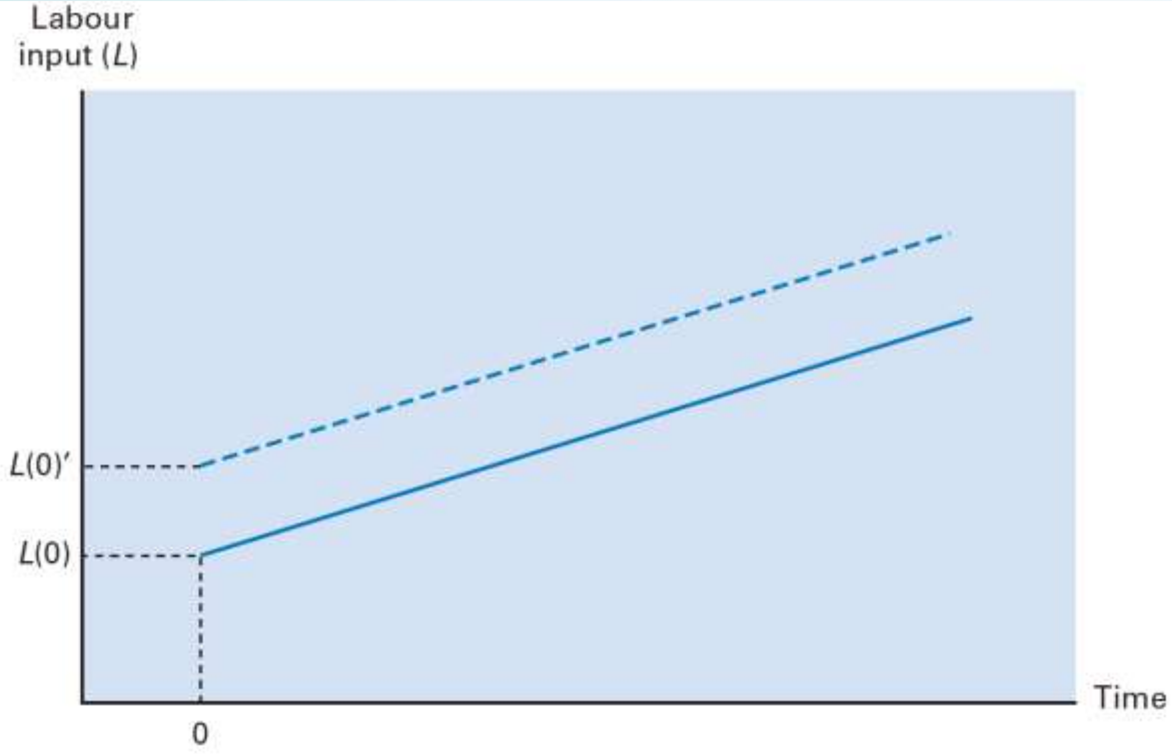


Figure 4.3 An increase in the level of labour input

In year 0, labour input jumps upward from $L(0)$ to $L(0)'$. The population growth rate, n, does not change.

Solow Growth Model
Change in the labor input ($L0$)

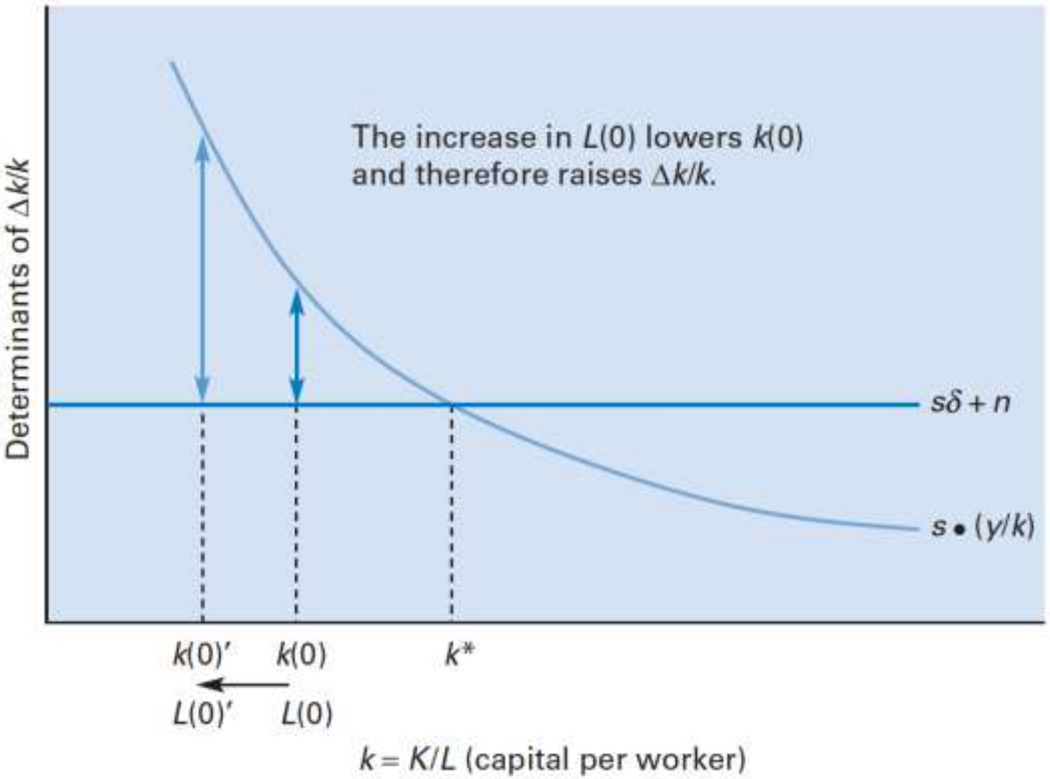


Figure 4.4 Effect of an increase in labour input in the Solow model

This graph comes from Figure 3.10. If the initial level of labour input rises from $L(0)$ to $L(0)'$, the initial capital per worker declines from $k(0) = K(0)/L(0)$ to $k(0)' = K(0)/L(0)'$. Therefore, the growth rate of capital per worker, $\Delta k/k$, rises initially. Note that the vertical distance shown by the arrows at $k(0)'$ is larger than that shown by the arrows at $k(0)$. The steady-state capital per worker, k^*, is the same for the two values of $L(0)$.

$$\Delta k/k = s\,(y/k) - s\delta - n$$

Solow Growth Model
Change in the labor input (*L*0)

- In the short run, an increase in initial labor input, $L(0)$, raises the growth rates of capital and real GDP per worker.
- These growth rates remain higher during the transition to the steady state.

Solow Growth Model
Change in the labor input (*L*0)

- In the long run, the growth rates of capital and real GDP per worker are the same—zero—for any initial level of labor input, $L(0)$.
- The steady-state capital and real GDP per worker, $k*$ and $y*$, are the same for any L.
- In the long run an economy with twice as much labor input has twice as much **total** capital and total real GDP.
- 记住：$\Delta k/k$和$\Delta y/y$为零，即人均资本和人均GDP的增速为零，不是说$\Delta K/K$ *或者* $\Delta Y/Y$ *为零。资本和GDP总量的增速还是等于人口的增速(n)，为什么？*

Solow Growth Model
Change in population growth rate (*n*)

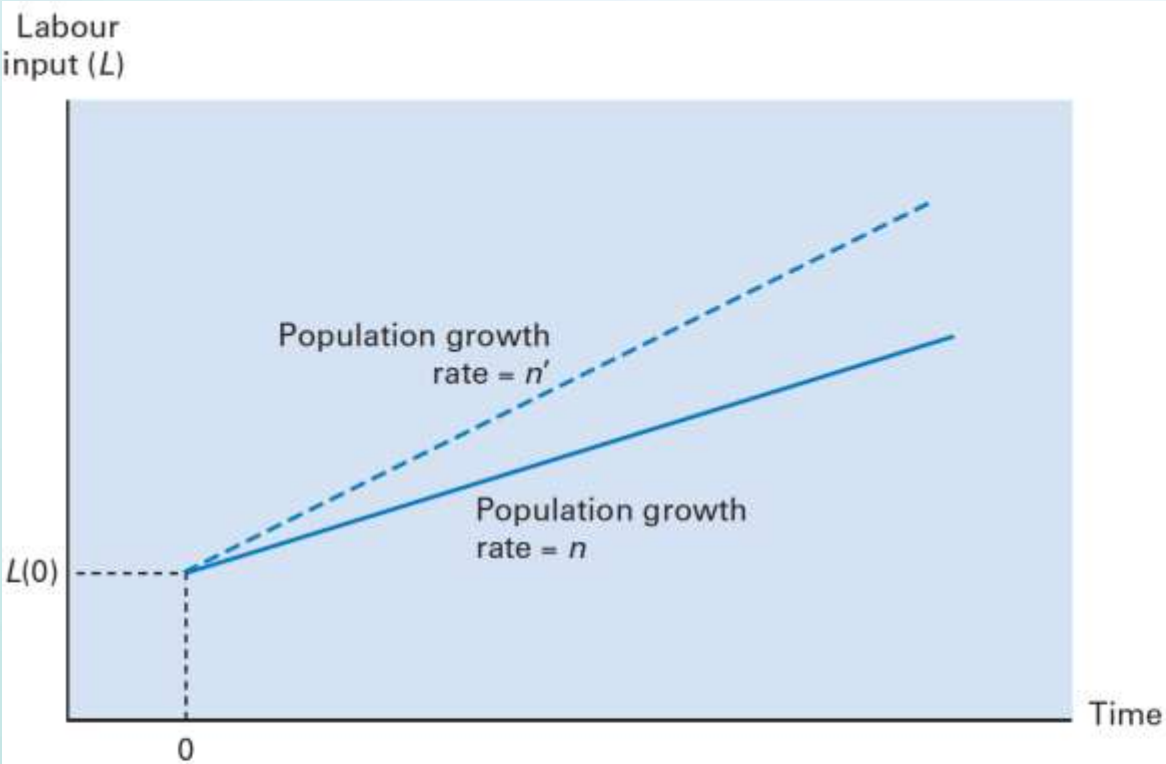


Figure 4.5 **An Increase in the population growth rate**

The population growth rate rises in year 0 from n to n'. The initial level of labour input, $L(0)$, does not change.

Solow Growth Model
Change in population growth rate (*n*)

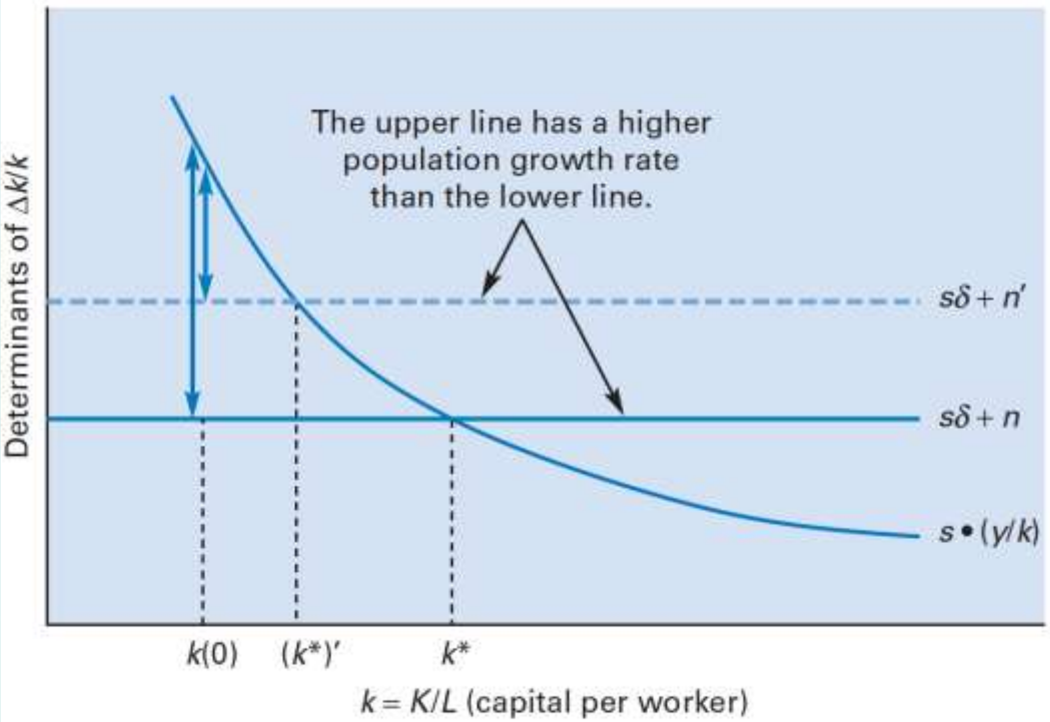


Figure 4.6 Effect of an increase in the population growth rate in the Solow model

This graph comes from Figure 3.10. An increase in the population growth rate from n to n' raises the horizontal line from $s\delta + n$ to $s\delta + n'$. The growth rate of capital per worker, $\Delta k/k$, is lower at any k when the population growth rate is higher. For example, at $k(0)$, when the population growth rate is n, $\Delta k/k$ equals the vertical distance given by the longer arrows. When the population growth rate is n', $\Delta k/k$ equals the vertical distance given by the shorter arrows. In the steady state, $\Delta k/k$ is zero, regardless of the population growth rate. A higher population growth rate yields a lower steady-state capital per worker; that is, $(k^*)'$ is less than k^*.

$$\Delta k/k = s\,(y/k) - s\delta - n$$

Solow Growth Model
Change in population growth rate (n)

- In the short run, a higher n lowers $\Delta k/k$ and $\Delta\, y/y$.
- These growth rates remain lower during the transition to the steady state.

Solow Growth Model
Change in population growth rate (n)

- In the steady state, $\Delta k/k$ and $\Delta y/y$ are zero for any n.
- A higher n leads to lower steady-state capital and real GDP per worker, $k*$ and $y*$, (not to changes in the growth rates, k/k and y/y which remain at zero).
- A change in n *does* affect the steady-state growth rates of the levels of capital and real GDP, $\Delta K/K$ $(= \Delta k/k+n)$ and $\Delta Y/Y (= \Delta y/y+n)$.

Solow Growth Model Convergence

$$k^* = k^*[\, s, \; A, \; n, \; \delta, L(0) \,]$$

$$(+) \; (+) \; (-) \; (-) \; (0)$$

Table 4.1 Effects on steady-state capital per worker, k^*

Increase in this variable	Effect on k^*
Saving rate, s	Increase
Technology level, A	Increase
Depreciation rate, δ	Decrease
Population growth rate, n	Decrease
Level of labour force, $L(0)$	No effect

Note: The right-hand column shows the effect of an increase in the variable in the left-hand column on the steady-state ratio of capital to labour, k^*. These results come from equation (4.7).

Solow Growth Model Convergence

- One of the most important questions about economic growth is "whether poor countries tend to converge or catch up to rich countries?".

Solow Growth Model
Convergence

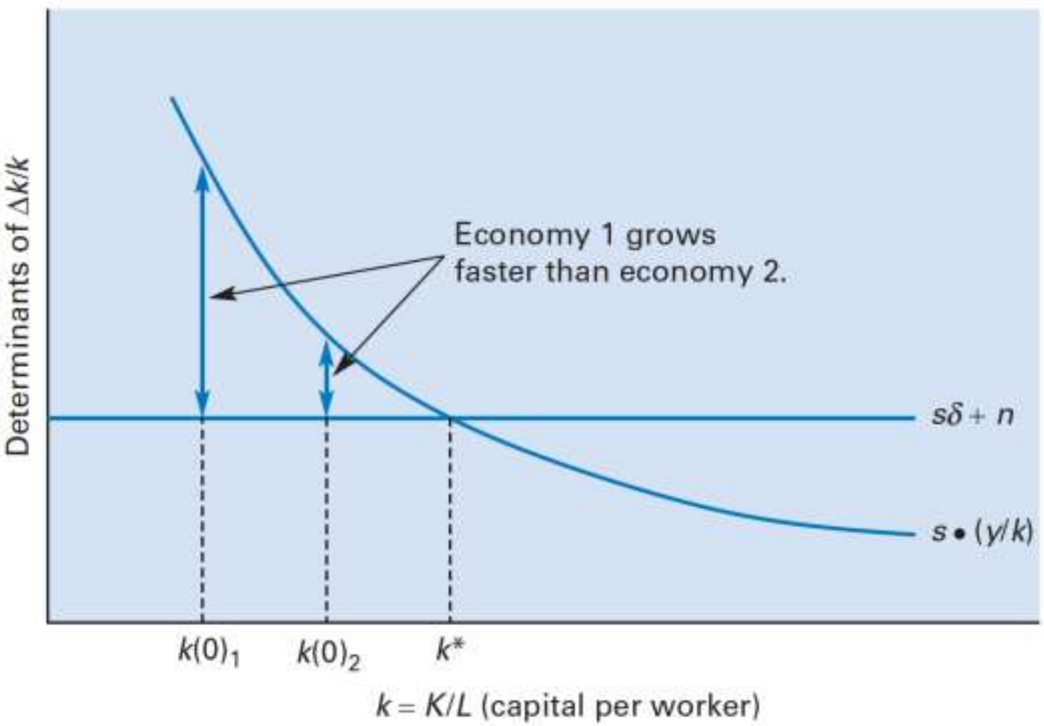


Figure 4.7 **Convergence in the Solow model**

This graph comes from Figure 3.10. Economy 1 starts with lower capital per worker than economy 2 – $k(0)_1$ is less than $k(0)_2$. Economy 1 grows faster initially because the vertical distance between the $s \bullet (y/k)$ curve and the $s\delta + n$ line is greater at $k(0)_1$ than at $k(0)_2$. That is, the distance marked by the arrows at $k(0)_1$ is greater than that marked by the arrows at $k(0)_2$. Therefore, capital per worker in economy 1, k_1, converges over time towards that in economy 2, k_2.

$\Delta k/k = s\,A \cdot f\,(k)/k - s\delta - n$

Solow Growth Model Convergence

- Economy 1 starts with lower capital per worker than economy 2—$k(0)_1$ is less than $k(0)_2$.
- Economy 1 grows faster initially because the vertical distance between the $s \cdot (y/k)$ curve and the $s\delta + n$ line is greater at $k(0)_1$ than at $k(0)_2$.
- Therefore, capital per worker in economy 1, k_1, converges over time toward that in economy 2, k_2.

Solow Growth Model Convergence

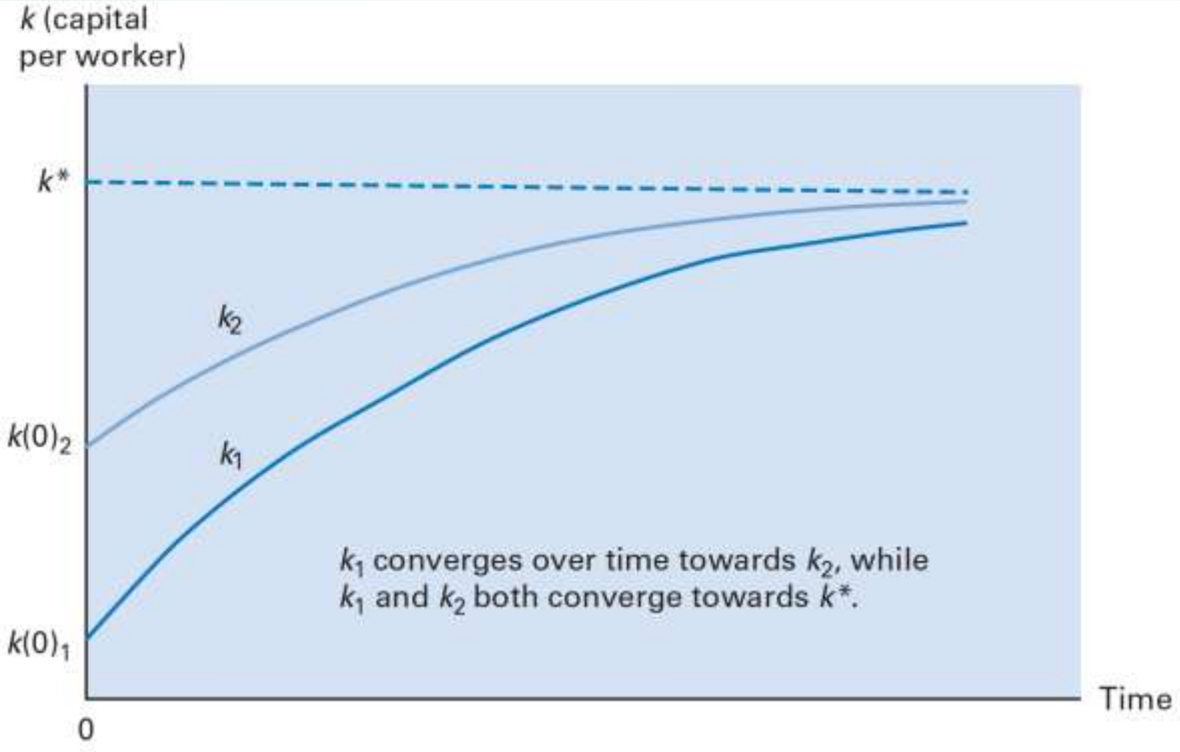


Figure 4.8 Convergence and transition paths for two economies

Economy 1 starts at capital per worker $k(0)_1$ and economy 2 starts at $k(0)_2$, where $k(0)_1$ is less than $k(0)_2$. The two economies have the same steady-state capital per worker, k^*, shown by the dashed line. In each economy, k rises over time towards k^*. However, k grows faster in economy 1 because $k(0)_1$ is less than $k(0)_2$ (see Figure 4.7). Therefore, k converges over time towards k_2.

Solow Growth Model Convergence

- The two economies have the same steady-state capital per worker, k^*, shown by the dashed blue line.
- In each economy, k rises over time toward k^*. However, k grows faster in economy 1 because $k(0)_1$ is less than $k(0)_2$.
- Therefore, k_1 converges over time toward k_2.

Solow Growth Model Convergence

- $y = A \cdot f(k)$ and $\Delta y/y = \alpha \cdot (\Delta k/k)$
- $\Delta k/k$ was higher initially in economy 1 than in economy 2.
- Therefore, $\Delta y/y$ is also higher initially in economy 1. Hence, economy 1's real GDP per worker, *y*, converges over time toward economy 2's real GDP per worker.

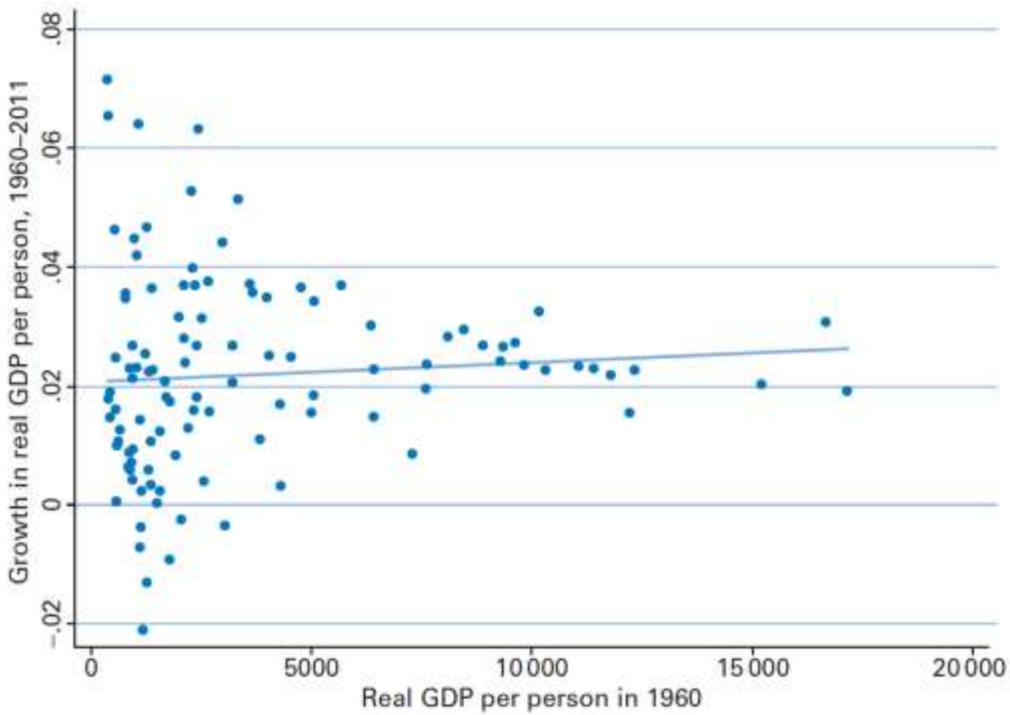
Solow Growth Model Convergence

- The Solow Model says that a poor economy—with low capital and real GDP per worker—grows faster than a rich economy. The reason is the diminishing average product of capital(y/k)
- The Solow model predicts that poorer economies tend to converge over time toward richer ones in terms of the levels of capital per worker (k^*) and real GDP per worker(y^*).

Solow Growth Model Convergence

Figure 4.9 **Growth rate versus level of real GDP per person for a broad group of countries**

The horizontal axis shows real GDP per person in 1960 in 2005 US dollars on a proportionate scale for 107 countries. The vertical axis shows the growth rate of real GDP per person for each country from 1960 to 2011. The blue line is the straight line that provides a best fit to the relation between the growth rate of real GDP per person (the variable on the vertical axis) and the level of real GDP per person (on the horizontal axis). Although this line slopes upward, the slope is – in a statistical sense – negligibly different from zero. Hence, the growth rate is virtually unrelated to the level of real GDP per person. Thus, this broad group of countries does not display convergence.



Solow Growth Model Convergence

Figure 4.10 **Growth rate versus level of real GDP per person for OECD countries**

The horizontal axis shows real GDP per person in 1960 in 2005 US dollars on a proportionate scale for 28 of the current 34 members of OECD. The abbreviation identifies each country. The vertical axis shows the growth rate of real GDP per person for each country from 1960 to 2011. The blue line is the straight line that provides a best fit to the relation between the growth rate of real GDP per person (the variable on the vertical axis) and the level of real GDP per person (on the horizontal axis). The line has a clear negative slope – therefore, a lower level of real GDP per person in 1960 matches up with a higher growth rate of real GDP per person from 1960 to 2011. Thus, the group of OECD countries exhibit convergence.

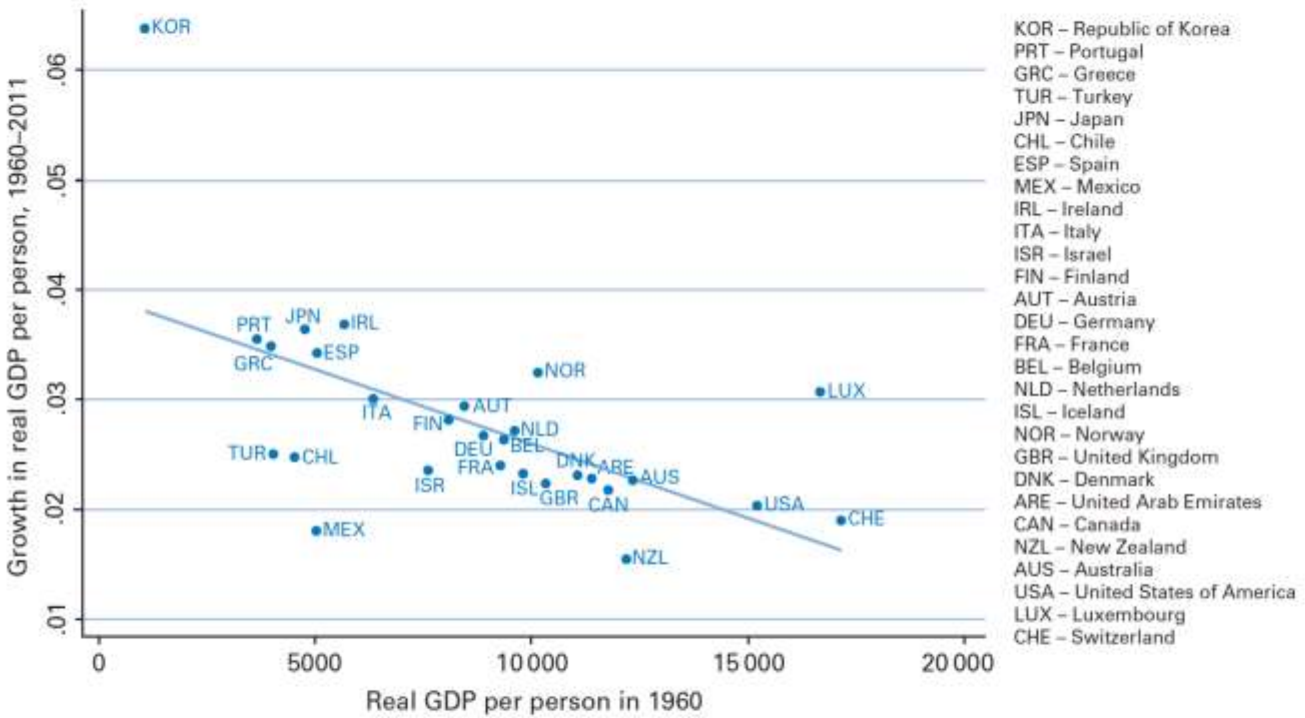
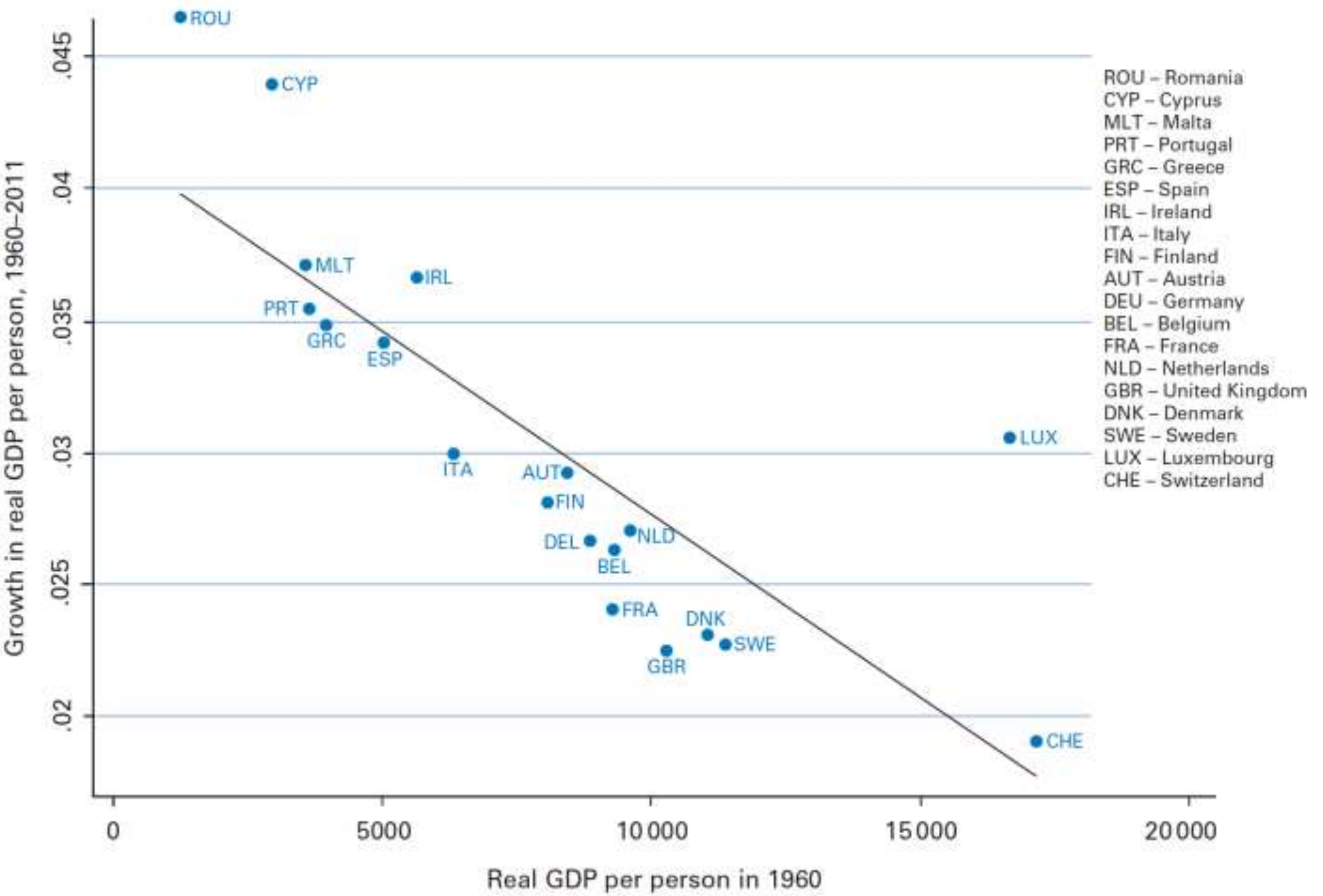


Figure 4.11 **Growth rate versus level of real GDP per person for countries of europe**

The horizontal axis shows real GDP per person in 1960 for 19 of the current 28 members of the European Union. The vertical axis shows the growth rate of real personal income per person for each country from 1960 to 2011. The solid line is the straight line that provides a best fit to the relation between the growth rate of real GDP per person (the variable on the vertical axis) and the level of real GDP per person (on the horizontal axis). The line has a clear negative slope – therefore, a lower level of real GDP per person in 1960 matches up with a higher growth rate of income per person from 1960 to 2011. Thus, the members of the European Union exhibit convergence.



Solow Growth Model Convergence

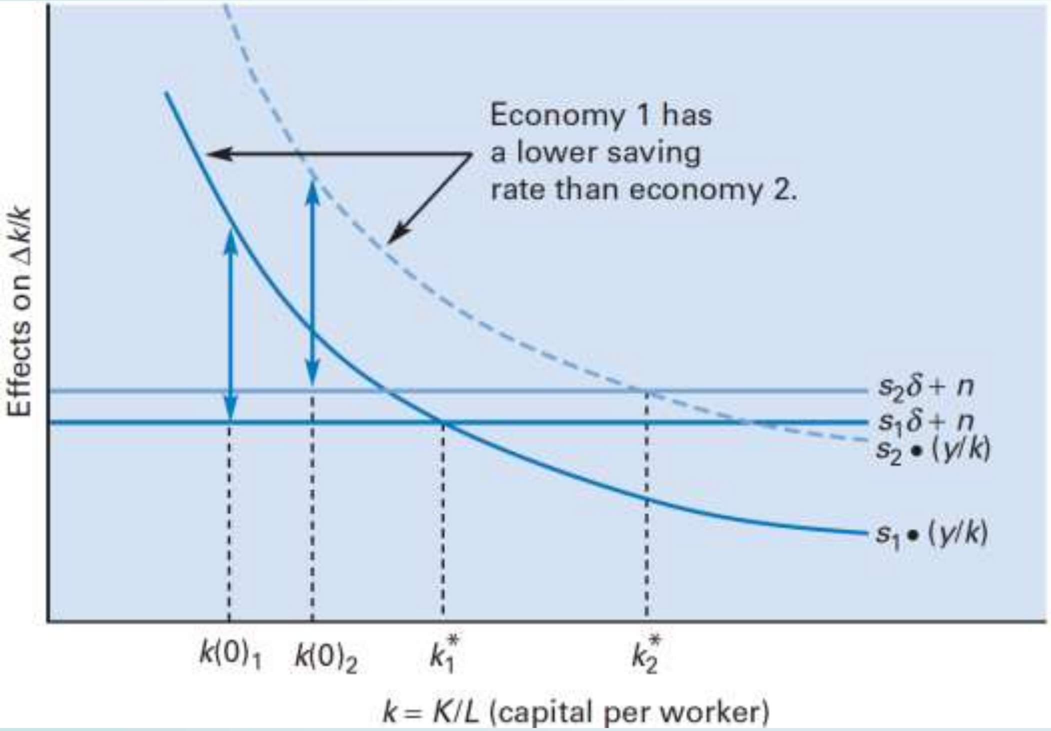


Figure 4.12 Failure of convergence in the Solow model: differences in saving rates

As in Figure 4.7, economy 1 starts with lower capital per worker than economy 2; i.e., $k(0)_1$ is less than $k(0)_2$. However, we now assume that economy 1 also has a lower saving rate; that is, s_1 is less than s_2. The two economies have the same technology levels, A, and population growth rates, n. Therefore, k_1^* is less than k_2^* In this case, it is uncertain which economy grows faster initially. The vertical distance marked with the arrows at $k(0)_1$ may be larger or smaller than the one marked with the arrows at $k(0)_2$.

$$\Delta k/k = s A \cdot f(k)/k - s\delta - n$$

Solow Growth Model Convergence

- Economy 1 starts with lower capital per worker than economy 2, （maybe $L(0)_1$ is bigger）

  $k(0)_1 < k(0)_2$.
- Assume that economy 1 also has a lower saving rate;

  $s_1 < s_2$.
- The two economies have the same technology levels, A, and population growth rates, n.
- Therefore, k^*_1 is less than k^*_2 .
- It is uncertain which economy grows faster initially. The vertical distance marked with the blue arrows cannot tell which one is longer.

Solow Growth Model Convergence

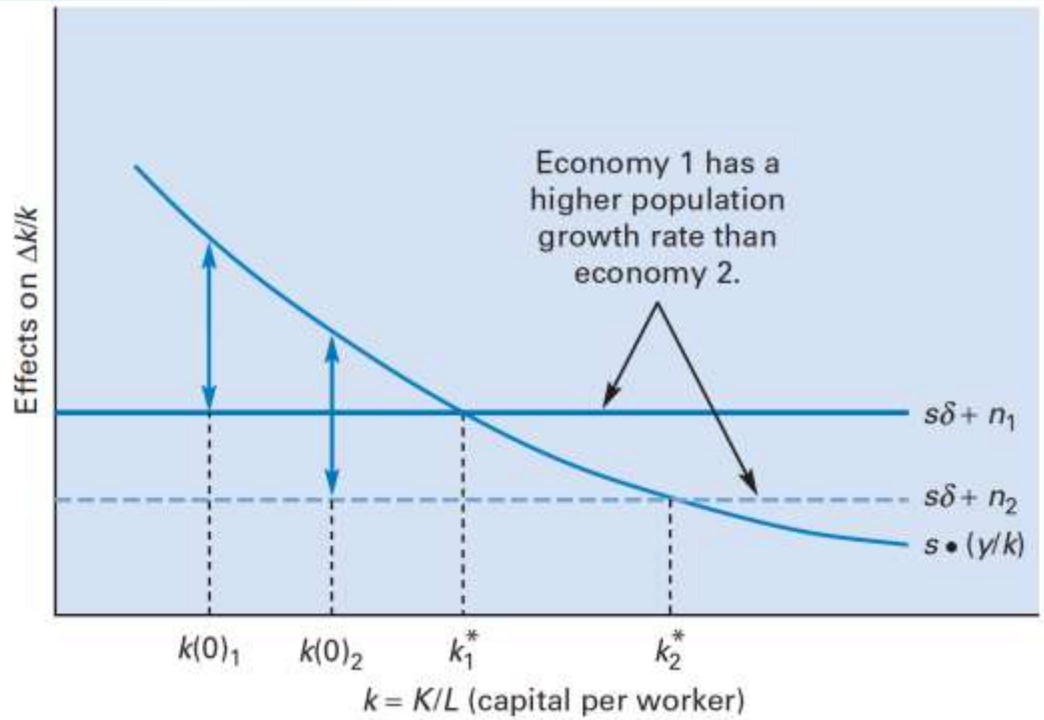


Figure 4.13 **Failure of convergence in the Solow model: Differences in population growth rates**

As in Figure 4.12, economy 1 starts with lower capital per worker than economy 2; i.e., $k(0)_1$ is less than $k(0)_2$. The two economies now have the same saving rates, *s*, and technology levels, *A*, but economy 1 has a higher population growth rate, *n*; that is, n_1 is greater than n_2. Therefore, as in Figure 4.12, k_1^* is less than k_2^* It is again uncertain which economy grows faster initially. The vertical distance marked with the arrows at $k(0)_1$ may be larger or smaller than the one marked with the arrows at $k(0)_2$.

$$\Delta k/k = s\,A \cdot f\,(k)/k - s\delta - n$$

Solow Growth Model Convergence

- Economy 1 starts with lower capital per worker than economy 2, （maybe $L(0)_1$ is bigger）:

  $k(0)_1 < k(0)_2$.
- The two economies now have the same saving rates, s, and technology levels, A, but economy 1 has a higher population growth rate, n;

  $n_1 > n_2$.
- Therefore, k^*_1 is less than k^*_2 .
- It is again uncertain which economy grows faster initially. The vertical distance marked with the blue arrows again cannot tell which one is longer.

Solow Growth Model Convergence

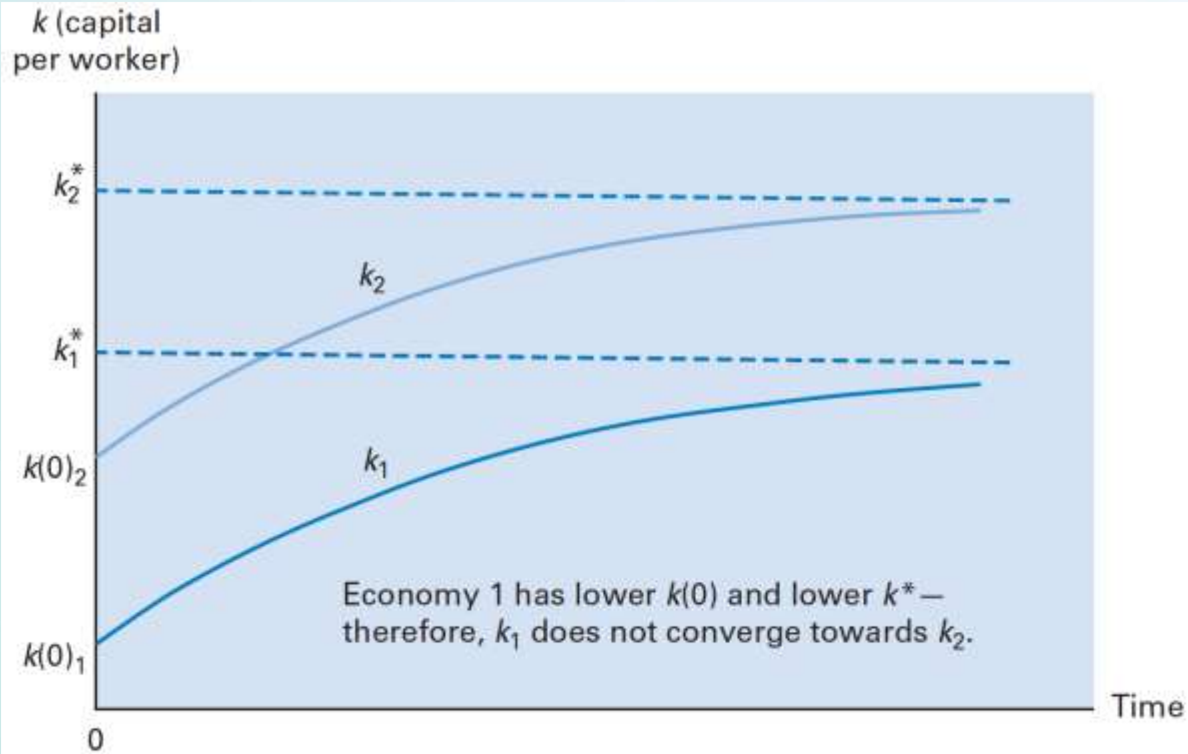


Figure 4.14 Failure of convergence and transition paths for two economies

As in Figures 4.12 and 4.13, economy 1 has a lower starting capital per worker (i.e., $k(0)_1$ is less than $k(0)_2$) and also has a lower steady-state capital per worker; i.e., k_1^* is less than k_2^*. Each capital per worker converges over time towards its own steady-state value: k_1 towards k_1^*, and k_2 towards k_2^*. However, since k_1^* is less than k_2^*, k_1 does not converge towards k_2.

Solow Growth Model Convergence

- Economy 1 has a lower starting capital per worker—$k(0)_1 < k(0)_2$—and also has a lower steady-state capital per worker— k^*_1 is less than k^*_2 .
- Each capital per worker converges over time toward its own steady-state value: k_1 toward k^*_1 ,and k_2 toward k^*_2 .
- However, since k^*_1 is less than k^*_2 , k_1 does not converge toward k_2.

Solow Growth Model Convergence

- Key Results
  - ✓ $k^* = k^*[\ s,\ A,\ n,\ \delta,\ L(0)\ ]$

    $(+)\ (+)\ (-)\ (-)\ (0)$
  - ✓ $\Delta k/k = \phi[\ k(0)\ ,\ k^*]$

    $(-)\quad (+)$
  - ✓ 推断：$k(0)$ 和 k^*之间的距离越远，$\Delta k/k$就越大。

Solow Growth Model Convergence

- Conditional convergence: 条件收敛
  - ✓a lower $k(0)$ predicts a higher $\Delta k/k$, conditional on k^*.
  - ✓两个经济体的稳态k^*不同，则$k^*/k(0)$越大的国家，就说明这个经济体的当下水平距离其稳态水平越远，因此这个经济体和人均资本和人均产出的增长率越高。

Solow Growth Model Convergence

- Absolute convergence:绝对收敛
  - ✓the prediction that a lower $k(0)$ raises $\Delta k/k$ without any conditioning.
  - ✓两个经济体的k^*相同，则$k(0)$低的国家，其经济增长率更高，因此预期两个经济体会收敛于相同的稳态。
- 绝对还是相对收敛？认为不同经济体拥有相同的稳态k^*，就是绝对收敛；认为两个经济体的稳态不相同，就是相对收敛。