

# **MAT344 Notes**

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# Contents

<b>1</b>	<b>Basics of Graph Theory</b>	<b>2</b>
<b>2</b>	<b>Basics in Counting</b>	<b>10</b>
<b>3</b>	<b>Counting in Graphs</b>	<b>15</b>
<b>4</b>	<b>Combinatorial and Permutation</b>	<b>19</b>
<b>5</b>	<b>Generating Functions</b>	<b>20</b>
5.1	Motivation . . . . .	20
5.2	OGF . . . . .	21
5.3	EGF . . . . .	21
<b>6</b>	<b>Counting in Structural Objects</b>	<b>22</b>
<b>7</b>	<b>Special Numbers and Estimation Techniques</b>	<b>25</b>
<b>8</b>	<b>Big List of Problems</b>	<b>26</b>

# 1 Basics of Graph Theory

1. (Vertex of G)  $V(G) = \{v_1, \dots, v_n\}$
2. (Edges of G)  $E(G) = \{\{v_i v_j\} : i, j \in \mathbb{N}\} = \{\{v_i \pi(v_i)\} : i \in \mathbb{N}\}$
3. vertices  $v_i$  and  $v_j$  are connected/ neighbourhood/ adjacent if  $v_i v_j \in E(G)$
4. In finite graph, **Order of G** :=  $|V(G)| < \infty$ ; **Size of G** :=  $|E(G)| < \infty$
5. (Neighbourhood of v)  $|N(V)| = deg(V)$ , # of vertices connected to vertex v.
6. (i)  $\Delta(G) = max(deg(v))$ ,  $\forall v \in V(G)$   
(ii)  $\delta(G) = min(deg(v))$ ,  $\forall v \in V(G)$
7. G is regular if every vertex has the same degree (i.e.  $deg(v_i) = deg(v_j) \forall i, j$ )
- 8.

## Theorem 1.1.

- (i)  $\sum_{v \in V(G)} deg(v) = 2|E|$
- (ii) # of vertices with odd degree is even

### Proof:

The proof for (i) is trivial since every edge has two vertices.

The proof for (ii) is done by contradiction.

Suppose not the case (# of odd degree is odd), since  $\sum deg(v)$  with even  $deg(v)$  can only be even, and  $\sum_{v \in V(G)} deg(v) = 2|E|$  is even, then we must have even number of vertices  $\rightarrow \leftarrow$

Concept	Def & Properties
<b>Walk</b>	$v_1, \dots, v_k \in V(G)$ with $v_i v_{i+1} \in E(G)$ ; Bunch of vertices and there edges
<b>Trail</b>	A walk that all the vertices are distinct with no repeated edge, but vertex can be repeated
<b>Path</b>	A walk that all the vertices are distinct, with no repeated vertex and no repeated edge
<b>Circuit</b>	A closed trail with no repeat edge, and starts & ends at the same point
<b>Cycle(<math>C_n</math>)</b>	A closed path, with no repeat vertex and edge, and starts & ends at the same point
<b>Tree(<math>T_n</math>)</b>	A connected graph with no cycles, a tree with n vertices has exactly $n - 1$ edges

## 10. Types of Graphs

- (1) (Regular Graph) All the vertices have the same degree
- (2) (Complete Graph( $K_n$ )) Every vertex is connected with all the vertices
- (3) (Bipartite Graph( $B_{X,Y}$ )) If  $V(G) = X \cup Y$  with  $X \cap Y = \emptyset$ , all the edges have an end in X and another in Y.
- (4) (Induced Graph) The induced graph of G with  $V'$ , is just removing the vertex that are in  $V'$
- (5) (xxx-free Graph) G doesn't include xxx as a subgraph
- (6) (Complement of G)  $\overline{G}$  is connected the vertices whose edges are not in  $E(G)$

11.

**Theorem 1.2.** Every  $u - v$  walk contains a  $u - v$  path

**Proof:** (Induction on the length of walk)

For length equals 1 is trivial.

Assume true for  $p(k - 1)$ , then the walk is  $u = w_0 - w_1 - \dots - w_{k-1} - w_k = v$

(Case1: ) each  $w_i$  is distinct, then we are done

(Case2; )  $\exists w_j = w_r$  for  $r > j$ . Then  $u = w_0 - w_1 - \dots - w_j - w_{r+1} - \dots - w_k = v$ .

By induction hypothesis, it has a  $u - v$  path.

12. G is connected if any two vertices are connected with a walk or path

13. v is a cut vertex if  $G \setminus \{v\}$  is disconnected

14.  $u - v$  is a cut edge if  $G \setminus \{uv\}$  is disconnected

15. G and H are isomorphic if there exists a bijection  $\Phi : V(G) \rightarrow V(H)$  such that if  $uv \in E(G)$ , then  $f(u)f(v) \in E(H)$  (Noticed that  $G \simeq H \implies |V(G)| = |V(H)| \& |E(G)| = |E(H)|$ )

16.

**Theorem 1.3.** G is bipartite iff G doesn't contain  $C_{2n+1}$

**Proof:**

( $\rightarrow$ ) Let  $V_1, V_2$  be two vertex sets, every step is either from  $V_1 \rightarrow V_2$  or  $V_2 \rightarrow V_1$ . To get back to the starting point, one must take even steps, therefore all the cycles in  $B_{V_1, V_2}$  must be even.

( $\leftarrow$ ) WLOG: G is connected and fix  $v \in V(G)$ . WTS: we shall construct a split in the vertex.

Take  $X = \{x \in V : \text{shortest path from } x \text{ to } v \text{ is even}\}$ , and  $Y = \{x \in V : \text{shortest path from } x \text{ to } v \text{ is odd}\}$   
Clearly  $X \cup Y = V(G)$ .

It remains to show the endpoints of every edge rest in X and Y separately.

Suppose not the case, Assume  $\exists x_1, x_2 \in X$  such that  $x_1 x_2 \in E(G)$ .

We first can construct the shortest even length path between  $x_1, x_2$  and v.

Consider

$$P_1 : v = v_0 - \dots - v_{2k} = x_1$$

and

$$P_2 : v = w_0 - \dots - w_{2l} = x_2$$

Consider the cycle  $v - x_1 - x_2 - v$ , it has length  $d(x_1, v) + d(x_2, v) + d(x_1, x_2) = d(x_1, v) + d(x_2, v) + 1$   
which is an odd cycle  $\rightarrow \leftarrow$

17.  $d(u, v) :=$  the length of the shortest walk from u to v, which is a metric satisfies 3 properties.

18. (Eccentricity)  $ecc(v) = max(d(v, u))$

19. **Radius** :=  $min(ecc(v))$  ( $\forall v \in V(G)$ ); **Diameter** :=  $max(ecc(v))$

20. (i)  $Rad(G) \leq Diam(G)$

(ii)  $Diam(G) \leq 2 \cdot Rad(G)$

21.

**Lemma 1.1.** Every graph G is isomorphic to center of some graph H

22. (Adjacent Matrix) Let  $V(G) = \{v_1, \dots, v_n\}$ ;  $A_{ij} = \begin{cases} 1 & \text{if } V_i V_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$

Observe that adjacent matrix is always square matrix, the rows of the matrix is determined by the order of G.

23.

**Lemma 1.2.**  $A, A'$  represents the same graph iff there is a permutation of vertices that takes one to the other

We can estimate that  $O(2^{c \cdot n})$  to check vertex-by-vertex which implies  $O(2^{c \cdot (\log n)^3}) \rightarrow O(n^\alpha)$  as n grows significantly large.

24.

**Lemma 1.3.**  $d(v_i, v_j) = k$ , the smallest  $k$  such that  $(A^k)_{ij} \neq 0$  but for  $k' < k$ ,  $(A^{k'})_{ij} = 0$

25.

**Lemma 1.4.**  $(A^k)_{ij} := \# \text{ paths with length } k \text{ from } v_i \text{ to } v_j$

26.

**Theorem 1.4.**  $G$  with order of  $n$  has  $n - 1$  edges iff  $G$  is a tree

**Proof:**

( $\rightarrow$ ) (By induction on # of vertices)

Base case, for  $n = 1$ , it holds clearly.

Assume  $p(n - 1)$  holds. We snip in the middle to get two separate trees  $T_1$  and  $T_2$ , where  $T_1, T_2$  has  $n_1, n_2 < n$  vertices.

Then

$$|E(T)| = |E(T_1)| + |E(T_2)| + 1 = n_1 - 1 + n_2 - 1 + 1 = n - 1$$

( $\leftarrow$ ) Suppose not the case. Suppose it contains a cycle, remove one of the edges from the cycle to get something still connected with  $n$  vertices.

Repeat the process till get no cycles with  $n$  vertices but remains connected, then we lands on attaining a tree with  $n$  vertices but has less than  $n - 1$  edges, which leads to contradiction that any tree with  $n$  vertices must has  $n - 1$  edges.  $\rightarrow \leftarrow$

27.

**Lemma 1.5.** Every tree with order  $\geq 1$  has 2 leaves (edge points) except from the trivial case with only single vertex.

**Proof:** Can be done with easy induction

28.

**Lemma 1.6.** For any  $v \in V(T)$ ,  $v$  is a leaf.

29.

**Lemma 1.7.** If a graph has  $\delta(G) \geq k$ , then every tree with  $k$  edges is a subgraph.

**Proof:**

Assume it's true for  $p(k - 1)$ ,  $T$  has  $k$  edges,  $G$  has  $\delta \geq k$ ;  $T$  has two leaves, snip one off, then  $T \setminus \{v\} \subseteq G$ . Fix  $w$  in  $G$  has  $k$  neighbours but only  $k - 1$  vertices in  $T \setminus \{v\}$ .

There must be one not included in  $T \setminus \{v\}$ .

30. G is a graph, a spanning tree  $T \subset G$ , which contains all vertices of G
31. (Minimal Spanning Tree) A spanning tree that has minimal weight. (Notice that the solution always exists by well-ordering but may not be unique)
32. (Kruskal's Algorithm)
- (1) Choose the smallest weight edge
  - (2) Choose the second smallest weighted edge that doesn't create a cycle

The complexity for it is  $O(|E(G)| \cdot \log(|V(G)|)) = O(E \cdot \log V)$

33.

**Theorem 1.5.** *kruskal's Algorithm always gives the minimal solution for spanning tree*

**Proof:**

Suppose it's not the case, it doesn't produce the smallest spanning tree, let  $e_1, \dots, e_{n-1}$  be the edges produced by the algorithm. Among all minimal weight spanning trees, there is one T such that it contains  $e_1, \dots, e_k$  with largest  $k < n - 1$ .

Then  $T' + e_{k+1}$  must contain a cycle C otherwise it won't be the largest.

So there must be an edge  $e' \notin e_1, \dots, e_{n-1} \in C$ . Remove  $e'$  from  $T' + e_{k+1}$  to get  $T''$ , a new spanning tree which contains  $e_1, \dots, e_k, e_{k+1}$  violating that k is the largest  $\rightarrow \leftarrow$

Therefore  $e_1, \dots, e_{n-1}$  is the minimal spanning tree.

34. Labeled Graph is G where the vertices of G are labeled with different colours or numbers
35. Chromatic Number of a graph :=  $\chi(G)$   
smallest k such that there exists a proper (two adjacent vertices have different colours) k-colouring of the graph
36. Every bipartite is 2-colourable **Proof:** It's trivial that we can label the vertices with respect to the partitions
37. Every Map is 5-colourable, but unknown whether is 4-colourable (shown by computer brutal force)
38.  $\chi(C_n) = \begin{cases} 2 & n \text{ is even} \\ 3 & n \text{ is odd} \end{cases}$
39. Exists G with no triangles with arbitrary large  $\chi(G)$
40. Greedy Algorithm
- (1) Starts with arbitrary vertex labeled as 1
  - (2) Labeled its neighbours with 2, so that adjacent vertices have different colouring

Notice that Greedy Algorithm not necessarily offers the smallest, but it ensures

$$\chi \leq \Delta(G) + 1$$

#### 41. Prüfer Code

- Get Prüfer's sequence from a given labeled tree
  - (1) Starting from the smallest labeled vertex, record the first entry as the labeled neighbour to it
  - (2) Remove the edge between the chosen vertex and the recorded vertex
  - (3) Move to the new smallest labeled vertex after removing the previous edge and repeat step 2
  - (4) Repeat the steps above until we reach  $K_2$ , which is the last entry of the prüfer sequence

Notice that the length of prüfer sequence is  $n - 2$ , if there are  $n$  vertices in the labeled tree
- Get a labeled tree from Prüfer Sequence
  - (1) we list the correspond  $n$  vertices as  $v_1, \dots, v_n$
  - (2) we write  $S = (1, 2, \dots, n)$  and  $\pi = (a_1, \dots, a_{n-2})$  be the prüfer sequence
  - (3) we list the prüfer sequence on top of the sequence  $S$
  - (4) Find the smallest index  $i$  in  $S$  which is not in  $\pi$ , then connect the  $i$ -th vertex with the first entry in the prüfer sequence, then remove those two points from  $S$  and  $\pi$
  - (5) Repeat the process until every entry in the prüfer sequence is eliminated, but we still have two index in  $S$  remaining, then connect those two vertices with the correspond index, which gives the labeled spanning tree

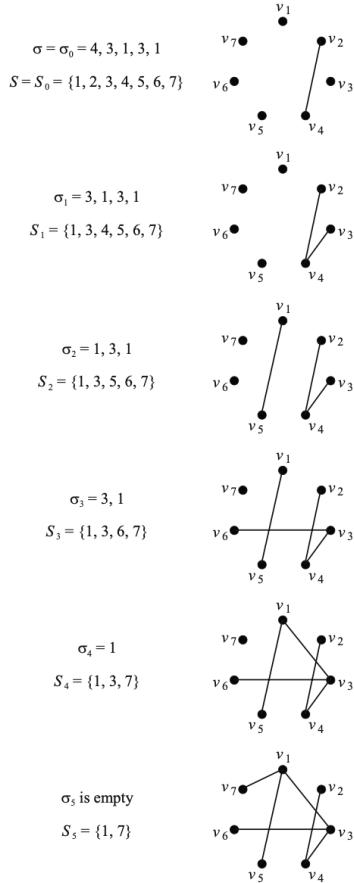


FIGURE 1.48. Building a labeled tree.

Figure 1: A Demonstration

## 42. Cayley's Tree Theorem

**Theorem 1.6.** In  $K_n$ , # of labeled spanning trees :=  $n^{n-2}$

Notice that we are not counting the labeled trees up to isomorphism (which is unsolved)

**Proof:**

The intuition behind the proof consists of two parts, first is construct a bijection between labeled trees and sequence with length  $n - 2$ ; the second part is by elementary counting we count number of sequences satisfy the condition.

- (1) To show the construction of prüfer sequence is a bijection, refer to above, we notice that we can construct one-to-one correspond between labeled trees and prüfer sequence.
- (2) Then we reframe the problem to counting number of sequence with length  $n - 2$ , where each entry takes value from  $\{1, 2, \dots, n\}$ , therefore we get

$$\# \text{ sequences} = n^{n-2}$$

## 43. Matrix Tree Theorem

- (Degree Matrix)  $D_{ij} = \begin{cases} \deg(v_i) & i = j \\ 0 & i \neq j \end{cases}$
- (Laplacian Matrix of G)  $L = D - A$ , where A is the adjacent matrix
- # labeled spanning tree =  $\det(L)$

I'm not interested in the proof of kirchhoff's Theorem

44. (Eulerian) Connected G if G contains an Eulerian circuit  
(starts & ends at same point, allows repeated vertices but travels through every edge exactly once)
45. (Hamiltonian) Connected G if G contains hamiltonian cycle  
(A walk travels through every vertex exactly once)

They are independent concepts, neither one is included in another one.

## 46.

**Lemma 1.8.** G is Eulerian iff  $\deg(v)$  is even for every vertex

**Proof:**

- (→) Clearly since every vertex can only be used once and travels through every edge exactly once, so we can always find a pair of in-out edges at every vertex, then  $\deg(v)$  is even.
- (←) Notice G is not a tree, since every vertex has even degree, then G consists at least one cycle,  $x_0 - \dots - x_l - x_0$ , where  $x_i x_{i+1} \in E(G)$ 
  - (1) If such cycle is the entrie G, then we are done
  - (2) If  $\exists y_0$  with  $y_0 x_i \in E$  for some i, and  $y_0 \notin x_0, \dots, x_l$ . Suppose not Eulerian, let C be the largest circuit made of  $E(C_{j_1}), \dots, E(C_{j_l})$ . But not everything in G. Then

$$\exists y_0 \notin \{j_1, \dots, j_l\}$$

which creates a longer cycle  $\rightarrow \leftarrow$

47. Remark  $K_{\text{odd}}$  is Eulerian but  $K_{\text{even}}$  is not  
Since in complete graph each vertex has degree  $n - 1$

48.

**Theorem 1.7.** *G contains a Eulerian Path iff it contains either two vertices with odd degree or zero*

**Proof:**

pass

49.

**Theorem 1.8.**  *$n = |V(G)|$ , if  $\delta(G) \geq \frac{n}{2}$ , then G is hamiltonian*

**Proof:**

pass

50.

**Theorem 1.9.** *G is 2-connected and  $K_{1,3}$  free and  $Z_1(K_3$  with a stack) free, then G is hamiltonian*

**Proof:**

pass

51. A graph is **planar** if we can arrange the edges so that the edges don't intersect with others in the plane

52. G is connected planar graph, then it satisfies

$$V - E + F = 2$$

**Proof:** (Induction on the edges)

Base case, for  $q = 1$ , trivial

Suppose it's true for  $1, 2, \dots, q - 1$ , where G has q edges.

If G is a tree, then  $n = q + 1$ , where  $F = 1$  holds.

Suppose G is not a tree, so it has a cycle C. Remove an edge e from cycle C, then  $G' = G - \{e\}$ , then we get  $V' = V$ ,  $E' = E - 1$ ,  $F' = F - 1$ , then

$$V' - E' + F' = 2$$

For every planar graph

53.  $K_{3,3}$  not planar

**Proof:**

Suppose not the case, where  $6 - 9 + F = 2$ , we get  $F = 2$ . A bipartite doesn't contain any odd cycle( $C_3$ ), every face should have at least 4 edges and each edge is bounded by two faces, then

$$2E \geq 4 \cdot 5 = 20$$

But we also have

$$2E \leq 2 \cdot q = 18$$

54. For planar G

- G is a bipartite, we have  $E \leq 2V - 4$
- If there's no further restriction, we have  $E \leq 3V - 6$

55.

**Lemma 1.9.**  *$K_5$  is not planar graph*

56.

**Lemma 1.10.** *If G is a planar graph, then G contains a vertex with degree at most 5*

57.

**Theorem 1.10.** Every planar has  $\chi \leq 5$

**Proof:**

Assume true for  $n - 1$ . Let  $G$  have order  $n$ , clearly holds from  $\chi(G) \leq n$ , there is a vertex with degree less or eq to 5.

Consider  $G' = G - v$ , then  $G'$  is 5-colourable by induction hypothesis  
pass

58. (SDR) System of distinct representatives for a set  $X$ , is a collection of distinct elements from the sets of  $X$ .

Given finite sets  $S_1, \dots, S_k$ ; SDR is a set  $\{a_1, \dots, a_k\}$  Defined as:

- (1)  $a_i \neq a_j$  for  $i \neq j$
- (2)  $a_i \in S_i$  for  $i = 1, 2, \dots, k$

59. Maximum/ Maximal Matching

(Maximum) A matching that has the largest possible cardinality

(Maximal) A matching that cannot be enlarged by the addition of any edge

60. Berge's Theorem

**Theorem 1.11.** Let  $M$  be a matchinh.  $M$  is maximum iff  $G$  contains no  $M$ -augmenting paths

**Proof:**

pass

61. Hall's Theorem

**Theorem 1.12.** Let  $G$  be a bipartite graph with partite sets  $X$  and  $Y$ .  $X$  can be matched into  $Y$  iff  $|N(S)| \geq |S|$  for all  $S \subseteq X$

**Proof:**

pass

62. If  $G$  is a bipartite graph that is regular of degree  $k$ , then  $G$  contains a perfect matching

63.

**Theorem 1.13.** If  $G$  is a graph of order  $2n$  such taht  $\delta(G) \geq n$ , then  $G$  has a perfect matching

64. (Tutte's Theorem)

**Theorem 1.14.** Let  $G$  be a graph of order  $n \geq 2$ .  $G$  has a perfect matching iff  $\Omega(G - S) \leq |S|$  for all  $S \subseteq V(G)$

## 2 Basics in Counting

1. (Sum Rule) Let  $S_i$  for  $i = 1, 2, \dots, n$  with  $|S_i| = n_i$  and  $S_i \cap S_j = \emptyset$  if  $i \neq j$ , then

$$\# \text{ of ways to select 1 object from any of the sets is } \sum_{i \in I} n_i$$

Notice that each set is disjoint, and they can form an event independently, in other words each set is a candidate for the event, instead all the sets are candidate all together( refers to product rule).

2. (Product Rule) Let  $S_i$  for  $i = 1, 2, \dots, n$  with  $|S_i| = n_i$  and  $S_i \cap S_j = \emptyset$  if  $i \neq j$ , then

$$\# \text{ of ways to select 1 object from each of the sets is } \prod_{i \in I} n_i$$

An example is the number of ways to form an outfit (shirts + pants + shoes), all together they form an outfit.

On the other hand, if we are asking for number of ways to pick one object from shirts, pants, shoes, then it equals to the sum of the objects.

Three Basic Problems

- (1) Number of Ways to order a collection of  $n$  different objects?  $n!$ , the intuition behind is the  $n$ -th entry has  $n + 1 - i$  candidates, therefore by the product rule, we have  $n!$
- (2) Number of Ways to order list of  $k$  objects from a collection of  $n$  different objects?  $P(n, k) = \frac{n!}{(n-k)!}$ , by the same argument, but here we only have to arrange the 1st entry til the  $k$ th entry.
- (3) Number of Ways to select  $k$  objects from a collection of  $n$  objects, if the order is irrelevant?  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ , where it's called **binomial coefficient**

3. (Binomial Coefficients)

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

4. (Binomial Theorem)

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} \cdot y^k$$

5. Some Properties

- (1) (Symmetry)

$$\binom{n}{k} = \binom{n}{n-k}$$

It can be shown either by algebra, or using the fact that to pick  $k$  elements from a set of  $n$  objects with order irrelevant, it's the same as picking those that are not picked.

- (2) (Addition)

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

It can also be shown by identifying the pascal triangle, the value at the entry equals to the sum of the entries on its shoulder.

Or we can prove it by arguing that whether the last element is included in  $k$  elements, then we can break into two cases

(i) The  $n$ -th element is in the  $k$  elements, then let the  $n$ -th element fixed, we pick the rest  $n - 1$  from the previous  $1-(n - 1)$ -th entry, which is  $\binom{n-1}{k-1}$

(ii) The  $n$ -th element is not in the  $k$  elements, then we pick the  $k$  elements from the previous  $(n - 1)$ -th entry, which gives  $\binom{n-1}{k}$

By the sum-rule, they both form the event of selecting  $k$  objects without order, the total is  $\binom{n-1}{k} + \binom{n-1}{k-1}$

(3) (Vandermonde's Convolution)

$$\binom{m+n}{l+p} = \sum_k \binom{m}{p+k} \cdot \binom{n}{l-k}$$

We first show that  $\binom{m+n}{l} = \sum_k \binom{m}{l-k} \cdot \binom{n}{k}$

Here I'll provide a combinatorial proof.

Suppose there are m boys and n girls, and we want to choose l people among those boys and girls.

So LHS is counting number of ways to select l people from m boys and n girls in total

RHS is counting case by case, starting from picking 0 people among k in girls.

After getting the equality, we can substitute l by  $l + p$  and k by  $k + p$  to re-index the sum.

It's an example of counting in two-ways.

(4) (Summing on the Upper Index) If m,n are nonnegative integers, then

$$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$$

Here I'll provide a combinatorial proof

Suppose there is a set  $\{0, 1, 2, \dots, n\}$  with size  $n + 1$

RHS counts the number of subsets of size  $m + 1$ , which is  $\binom{n+1}{m+1}$

LHS we also count for the number of susbets with length  $m + 1$ , but for each subset we fix the largest element, say k, then the rest m elements can only from  $\{0, 1, 2, \dots, k - 1\}$ , a set with size k and we want to select m elements from it, which gives  $\binom{k}{m}$ .

Moreover, k can range from 0 to n, then we sum over all the possiblities, then we have

$$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$$

(5) (Extraction Identity) If n is positive integer and k is nonzero integer, then

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

Here I'll provide a combinatorial proof.

We'll try to show  $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$

LHS refers to number of ways to first pick a team of k members from n people, and then pick one of them as a team leader

RHS refers to first pick a team leader among n people, and then pick the rest  $k - 1$  team members among  $n - 1$  people

(6) (Sum of each row of Pascal Triangle)

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

(i) By binomial theorem, we can take  $x = y = 1$ , which gives  $2^n = \sum_{k=0}^n \binom{n}{k}$

(ii) Here is a combinatorial proof.

To calculate RHS, is just counting the number of subsets in total, then for each element in  $\{1, 2, \dots, n\}$ , it must either be in the subset or not in the subset, so for each element, it has 2 possible cases, by the product rule, we have  $2^n$

(7) (Cancellation Identity)

$$\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}$$

Here I'll provide a combinatorial proof.

LHS counts number of ways to select k people from n people, and then select m people from those

$k$  people. (i.e. we select  $k$  and then specify  $m$  people within those  $k$  people)  
 RHS selects those  $m$  people first, and then select the rest  $k - m$  people to construct the total  $k$  people selected from  $n - m$

(8) Power of Binomials

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

Here I'll provide a combinatorial proof.

Suppose there are  $n$  boys and  $n$  girls, RHS counts number of ways to pick  $n$  people among those  $n$  boys and  $n$  girls, which is  $\binom{2n}{n}$

LHS counts the same by the product rule, suppose we pick  $k$  boys, then the rest  $n - k$  must be selected from the  $n$  girls, which is  $\sum_{k=0}^n \binom{n}{k} \cdot \binom{n}{n-k} = \binom{n}{k}^2$

(9) (#)(Parellel Summation) If  $m, n$  are nonnegative integers, then

$$\sum_{k=0}^n \binom{m+k}{k} = \binom{m+n+1}{n}$$

Here I'll provide a combinatorial proof.

We construct set  $\{1, 2, \dots, m, \dots, m + n + 1\}$ , then RHS gives number of subsets with size  $n$

For LHS, for each  $k$ , we consider the largest element to be  $m + k + 1$  for each subset, then the rest  $k$  elements are selected from  $\{1, 2, \dots, m + k\}$

**NOT CLEAR**

6. (Multinomial) If  $n$  is nonnegative integer, and  $k_i$  are integers with  $\sum_{i \in \mathcal{I}} k_i = n$ , then we have

$$\binom{n}{k_1, \dots, k_m} = \begin{cases} \frac{n!}{k_1! \cdot k_2! \cdots k_m!} & k_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Considering about sets behind multinomial

Let  $S_i$  be sets and each  $|S_i| = k_i$  with  $\sum_i k_i = n$ , then number of ways to label those  $n$  elements is

$$\binom{n}{k_1, \dots, k_m}$$

Moreover, binomial coefficient is just a special case for multinomial

$$\binom{n}{k, n-k} = \frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}$$

Notice the intuition behind multinomial is labeled element within each group, but for the product rule, we are selecting elements from disjoint groups and then consider the szie of such group.

In multinomial, we are not selecting, but labeling instead.

So we shouldn't multiply the size of each set, but instead we can apply product rule in this way

$$\binom{n}{k_1} \cdot \binom{n - k_1}{k_2} \cdot \dots \cdot \binom{n - k_1 - \dots - k_{m-1}}{k_m}$$

In addition, multinomial can also be explained as the worst case divided by the overcounting factors. This means  $n!$  is the total permutation of  $n$  elements, but there are several groups which have the same element, so switching the position among the elements within the same group( class), there won't appear new strings or new permutation, then  $n!$  overcount,

To eliminate the overcounting ones, we need to divide by the size of each groups (equivalent class)

An Example: Find the number of 7 letter string can be constructed from 'BALLOON'

Step1: Find the groups and their correspond size    b: 1; a: 1; l: 2; o: 2, n: 1

Step2: Apply multinomial

$$\# \text{ number of ways} = \binom{7!}{2! \cdot 2!} = \frac{7!}{1! \cdot 1! \cdot 2! \cdot 2! \cdot 1!}$$

## 7. Some Properties

- (1) (Symmetry) Suppose  $\pi(1), \dots, \pi(m)$  is a permutation of  $\{1, \dots, m\}$ . Then

$$\binom{n}{k_1, \dots, k_m} = \binom{n}{k_{\pi(1)}, \dots, k_{\pi(m)}}$$

- (2) If  $n$  is a positive integer and  $\sum k_i = n$ , then

$$\binom{n}{k_1, \dots, k_m} = \binom{n-1}{k_1-1, k_2, \dots, k_m} + \binom{n-1}{k_1, k_2-1, \dots, k_m} + \dots + \binom{n-1}{k_1, k_2, \dots, k_m-1}$$

## 8. (Multinomial Theorem) If $n$ is a nonnegative integer, then

$$(x_1 + x_2 + \dots + x_m)^n = \sum_{k_1 + \dots + k_m = m} \binom{n}{k_1, \dots, k_m} x_1^{k_1} \cdot \dots \cdot x_m^{k_m}$$

Notice that multinomial theorem can also used for GF

9. (Pigeonhole Principle) If there are  $n$  containers, and distribute  $m > n$  objects, then some containers must contain more than one object

## 10. Some Examples

- (1) 400 people in a class room, then there must have 2 people with the same birthday.

Notice that we only know the existence of such day within a year (366 days), but we don't know which day it is and how many people have the same birthday.

- (2) A class of 40 people with an average of 50/100 in a math test, then there must exist at least 1 student whose mark is  $\leq 50$

The  $\leq$  is not strict, since the extreme case is when every student attains a 50.

11. (Generalized Pigeonhole Principle) Let  $m, n$  be positive integers. If more than  $mn$  objects are distributed among  $n$  containers, then at least one container must contain at least  $m + 1$  objects.

12. (Points on both sides of Mean) Suppose  $a_1, \dots, a_n$  is a sequence of real numbers with mean  $M$ , so  $M = \frac{\sum a_i}{n}$ , then there exist integers  $i$  and  $j$  with  $1 \leq i, j \leq n$ , such that  $a_i \leq M$  and  $a_j \geq M$

13. (Monotonic Subsequence) Suppose  $m$  and  $n$  are positive integers. A sequence of more than  $mn$  reals must contain either an increasing subsequence of length at least  $m + 1$  or a strictly decreasing subsequence of length at least  $n + 1$ .

Proof. Suppose not the case, where  $\{r_i\}_{i=1}^{mn+1}$  contains neither a strictly increasing subsequence of length  $m + 1$  nor a strictly decreasing subsequence of length  $n + 1$ .

Let  $a_i$  be the length of maximal increasing subsequence starting at  $r_i$ ;

$d_i$  be the length of maximal strictly decreasing subsequence starting at  $r_i$ .

Notice that  $1 \leq a_i \leq m$  and  $1 \leq d_i \leq n$  by assumption. Then for each  $i$ , there are  $mn$  possible ordered pairs of  $(a_i, d_i)$ , but we have  $mn + 1$  terms, by PHP, there must exist at least two pairs with the same value.

$\exists j < k$  such that  $(a_j, d_j) = (a_k, d_k)$ . Say  $\alpha = a_j = a_k$  and  $\delta = d_j = d_k$

Then we can construct a maximal increasing subsequence  $r_k, r_{i_2}, \dots, r_{i_\alpha}$  starting at  $r_k$ ,

and a maximal strictly decreasing subsequence  $r_k, r_{i'_2}, \dots, r_{i'_\delta}$

Compare  $r_j$  with  $r_k$ ,

if  $r_j > r_k$ , then  $r_j, r_k, r_{i'_2}, \dots, r_{i'_\delta}$  forms a longer strictly decreasing subsequence, which leads to a contradiction  $\rightarrow \leftarrow$

if  $r_j \geq r_k$ , then  $r_j, r_k, r_{i'_2}, \dots, r_{i'_\delta}$  forms a longer increasing subsequence, which leads to a contradiction as well  $\rightarrow \leftarrow$

Therefore, we show the statement.

14. (Dirichlet's Approximation Theorem) Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  and  $Q \in \mathbb{Z}_+$ . Then there exists a rational  $\frac{p}{q}$  with  $1 \leq q \leq Q$  satisfying

$$|\alpha - \frac{p}{q}| < \frac{1}{q(Q+1)}$$

The proof divides  $[0, 1]$  into  $Q + 1$  subintervals with equal length

$$[0, \frac{1}{Q+1}), [\frac{1}{Q+1}, \frac{2}{Q+1}), [\frac{2}{Q+1}, \frac{3}{Q+1}) \dots [\frac{Q-1}{Q+1}), [\frac{Q}{Q+1}, 1]$$

Since each of the  $Q + 2$  reals  $0, \{\alpha\}, \{2\alpha\}, \dots, \{Q\alpha\}, 1$  lies in  $[0, 1]$ .

By PHP,  $\exists r_1, r_2$  and  $s_1, s_2$  such that  $|r_2\alpha - s_2| - |r_1\alpha - s_1| < \frac{1}{Q+1}$   
Then take  $q = r_2 - r_1$ ,  $p = s_2 - s_1$ . The

$$|q\alpha - p| < \frac{1}{Q+1}$$

Then we have

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2 + q}$$

15. (Inclusion & Exclusion)

### 3 Counting in Graphs

#### 1. Ramsey Number

- Problems in Ramsey theory typically ask a question of the form: "how big must some structure be to guarantee that a particular property holds?"

(1) Let  $p, q \in \mathbb{Z}_+$ .

Classical Ramsey number,  $R(p, q)$  defines as the smallest integer such that every 2-coloring of the edges of  $K_n$  either contains a red  $K_p$  or a blue  $K_q$  as a subgraph

Notice we should find the solution that works for every 2-coloring, instead of finding only one fixed coloring

(2) A clique is a complete subgraph of  $G$

(3) Examples of Ramsey Number

(i)  $R(1, 3) = 1$ , since we are looking for the smallest  $n$  such that  $K_n$  contains either a  $K_1$  or  $K_3$ (triangle)

(ii)  $\mathbf{R(1,n) = R(n,1) = 1}$  for all  $n \geq 1$ , since we can always construct a singleton vertex with one colour

(iii)  $R(2, 4) = 4$

Notice that  $R(2, 4) > 3$ , since  $K_3$  doesn't meet the solution (if we color  $K_2$  as red, then for the all-blue  $K_3$ , there's no solution).

Furthermore, we show  $R(2, 4) = 4$  can be attained.

If  $K_4$  includes any red edge, then we found  $K_2$  as a subgraph. Otherwise, no red edge exists in  $K_4$ , all the edges are blue, then it's a subgraph of  $K_4$ .

(iv)  $\mathbf{R(2,n) = n}$  for all positive integers  $n$ .

**Proof:** (Show through the two inequality)

(1)  $(R(2, n) \leq n)$  Since  $K_n$  holds, then

$$R(2, n) \leq n$$

(2)  $(R(2, n) \geq n)$  It's suffices to show that there is a coloring on  $K_{n-1}$  doesn't hold. Clearly, if we consider  $K_2, K_n$  have red and blue edges respectively. Take the coloring that  $K_{n-1}$  is monochromatic, coloring all the edges with blue, then neither  $K_2$  nor  $K_n$  is in such coloring of  $K_{n-1}$ . Then

$$R(2, n) \geq n$$

(v)  $\mathbf{R(3,3) = 6}$

**Proof:**

(1)  $(R(3, 3) \geq 6)$  It's sufficient to check  $K_5$  has a coloring that doesn't meet the condition. Color the outer side of the pentagon with blue, and then color the inner sides with red. Then

$$R(3, 3) \geq 6$$

(2)  $(R(3, 3) \leq 6)$  Pick arbitrary vertex  $v$  in  $K_6$ , since we are coloring it with 2 colours, and it has exactly 5 adjacent points, then there are at least 3 red or at least 3 blue edges induced at such point.

WLOG: Assume there are 3 blue edges and 2 red edges, say  $vx, vy, vz$ . If any  $x, y, z$  is connected with blue edges, then it will create a blue  $K_3$ .

Then we must connect  $x, y, z$  with red edges which offers a red  $K_3$ . In both cases we get a monochromatic  $K_3$ . Therefore

$$R(3, 3) \leq 6$$

(vi)  $\mathbf{R(3,4) = 9}$

**Proof:**

(1) it can be shown that  $K_8$  has a colouring that doesn't meet the condition. Then

$$R(3, 4) \geq 9$$

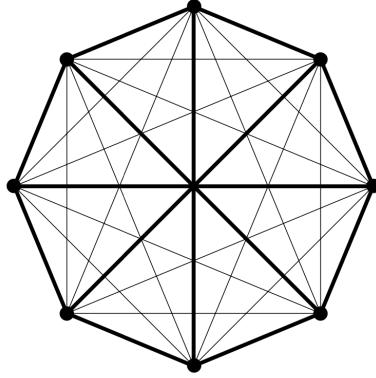


FIGURE 1.126. A 2-coloring of the edges in  $K_8$ .

Figure 2: From the book

(2) We split into three cases

The intuition behind the splitting is from partition on the 8 neighbours, but considering the asymmetry of the arbitrary colouring of blue and red.

- v has at least 4 red edges

Since we know  $R(2, 4) = 4$ , then under this case where  $vx, vy, vz, vw$  are red, adding one more edge will create a  $K_3$ , but in  $K_4$  it contains either a red  $K_2$  or a blue  $K_4$ , then creating red  $K_3$  or blue  $K_4$  is inevitable

- v has at least 6 blue edges

By the similar argument, under the case where  $vx_1, \dots, vx_6$  are blue edges. Since  $R(3, 3) = 6$ , then it will create either a red  $K_3$  or a blue  $K_3$ . If it creates a red  $K_3$ , then we found a  $K_3$ . Else if it creates a blue  $K_3$ , along with the six blue induced edges, creates a blue  $K_4$ .

- v has edges less than 4 and less than 6

The only cases is 3 red edges and 5 blue edges, which is impossible, since the sum of the blue edges are  $5 \cdot 9$  an odd number  $\rightarrow \leftarrow$

(vii) List of all known Ramsey Numbers

$$\begin{aligned} R(3, 5) &= 14; R(3, 6) = 18; R(3, 7) = 23; R(3, 8) = 28; R(3, 9) = 36; R(4, 4) = 18; \\ R(4, 5) &= 25 \end{aligned}$$

(4) Bounds on Ramsey

In general, it's extremely difficult to determine the exact numbers, but there are various ways to estimate them as  $n$  is sufficiently large.

(i)

**Theorem 3.1.** *For positive integers  $p, q$ ,*

$$R(p, q) \leq \frac{(p + q - 2)!}{(p - 1)! \cdot (q - 1)!}$$

(ii)

**Theorem 3.2.** If  $p \geq 3$ , then

$$R(p, p) > \left\lfloor 2^{\frac{n}{2}} \right\rfloor$$

**Proof:** This involves probability pass

(iii)

**Theorem 3.3.** For  $q \geq 3$ ,

$$R(3, q) \leq \frac{q^2 + 3}{2}$$

(iv)

**Theorem 3.4.**  $R(m, n) \leq R(m - 1, n) + R(m, n - 1)$

**Proof:**

Let  $N = R(m - 1, n) + R(m, n - 1)$ , fix a vertex  $v$ , then consider the induced edges at  $v$ , they are either blue or red.

Take  $V_R = \{x : vx \text{ is a red edge}\}$  and similar for  $V_B$ .

Since  $|V_R| + |V_B| = N - 1 = R(m - 1, n) + R(m, n - 1)$ , then we have either

$$|V_R| \geq R(m - 1, n)$$

or

$$|V_B| \geq R(m, n - 1)$$

Notice the statement  $a + b = x + y - 1 \implies a \geq x$  or  $b \geq y$ , can be shown by contradiction. Suppose  $|V_R| \geq R(m - 1, n)$ , then either  $V_R$  contains a  $K_n$  or it contains a red  $K_{m-1}$ , but then  $K_{m-1} \cup \{v\}$  gives a red  $K_m$ .

Similar for blue edges.

Furthermore, if both terms on the right are even, then the inequality is strict.

(v)

**Theorem 3.5.**  $R(m, n) \leq \binom{m+n-2}{m-1}$

**Proof:**

We can rewrite the statement as

$$R(m, n) \leq \binom{m+n-3}{m-2} + \binom{m+n-3}{n-2}$$

By the theorem above, we know  $R(m, n) \leq R(m - 1, n) + R(m, n - 1)$ .

By induction on  $m + n$ , we can show the inequality holds.

Notice that there is a special case for  $R(m, n) \leq \binom{m+n-2}{m-1}$ , take  $m = n$ , we have

$$R(m, n) \leq \binom{2n-2}{n-1} \leq 4^n$$

(vi) Lower bound on  $R(n, n)$

Take  $\Omega := \# \text{ labeled graphs of order } k := 2^{\binom{k}{2}}$  and

$$A = \binom{k}{n} \cdot 2^{\binom{k-n}{2}} \cdot 2^{n \cdot (k-n)}$$

be the number of labeled graphs of order  $k$  with  $K_n$ . And  $B$  be the number of labeled graphs of order  $k$  with an independent set.

If  $|A| + |B| < |\Omega|$ , then  $R(n, n) \geq k$ . ( $\binom{k}{n} \leq \frac{k^n}{n!}$  is frequently used below). Then we have

$$2^{\binom{k}{n}} 2^{n(k-n)} \cdot 2^{\binom{k-n}{2}} < 2^{\binom{k}{2}}$$

Then  $k^n \cdot n^{-(n+\frac{1}{2})} e^n < 2^{\frac{-n^2}{2} - \frac{n}{2} - 1}$   
Set  $\hat{k} = 2^{\frac{-n}{2}} \cdot k$ . We can rewrite as

$$\hat{k}^n n^{\frac{-1}{2}} e^n < 2^{\frac{-n}{2} - 1} n^n$$

Then we estimate  $k \sim 2^{\frac{n}{2}}$

Therefore eventually, we have

$$2^{\frac{n}{2}} < R(n, n) < 4^n$$

## (5) Ramsey In Graphs

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**Theorem 3.6.** *If  $G$  is a graph of order  $p$  and  $H$  is a graph of order  $q$ , then*

$$R(G, H) \leq R(p, q)$$

### Proof:

Let  $n = R(p, q)$  and consider arbitrary 2-coloring of  $K_n$ . Then it contains either a red  $K_p$  or a blue  $K_q$ .

Since  $G \subseteq K_p$  and  $H \subseteq K_q$ , there must either be a red  $G$  or a blue  $H$  in  $K_n$ .

---

**Theorem 3.7.**  *$R(G, H) \geq (\chi(G) - 1) \cdot (C(H) - 1) + 1$*

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**Theorem 3.8.** *If  $T_m$  is a tree with  $m$  vertices, then*

$$R(T_m, K_n) = (m - 1)(n - 1) + 1$$

## 2. In Eulerian and Hamiltonian Graphs

- Number of walks in a Hamiltonian Graph

## **4 Combinatorial and Permutation**

1. Inclusion and Exclusion
2. Derangement Problem
3. Pigeonhole and Generalized Pigeonhole Principle
4. Combinatorial Proof and Counting in Two ways
5. Topics on Random Walk

## 5 Generating Functions

### 5.1 Motivation

1. Generating Function can be used in complicated counting problems and finding the formula for a given recursion, or used in computing probability and randomness.

2. Several types of GF:

(i) OGF and EGF on sequence

Mostly the generating functions on recursively defined sequences are formal power series.

Here we can regard G as an operator, defined as  $G : \{a_k\}_{k \geq 0} \rightarrow x$ , in physics it can be interpreted as a transformation from time (k) to frequency of a wave (x).

Moreover, such transformation has proper algebraic structure (<https://en.wikipedia.org/wiki/Z-transform>)

– (Additive)  $G(a_k + b_k) = G(a_k) + G(b_k)$

– (Convolution)  $G(a_k * b_k) = G(a_k) \times G(b_k)$ , where '\*' denotes convolution product.

From the second property, we can see that generating function is a nice way to handle product of sophisticated sequence besides Cauchy product.

(ii) Probability GF

Here we consider more on the discrete random variable X. Define  $P(X = i) = p_i$ , then the PGF of X is

$$G(X) = \sum_{i=0}^n p_i \cdot t^i = E(t^X)$$

Notice that it has the property  $G(X + Y) = G(X) \times G(Y)$ , it holds since product rule, we have

$$\{p_{X+Y} = i\} = \{p_X = i\} * \{p_Y = i\}$$

(iii) Matrix GF

To investigate more on the behavior, we are aiming to find a well-defined matrix. The MGF of X is

$$G(X) = E(e^{tX}) = \begin{cases} \sum_{i=0}^n e^{ti} \cdot p_i & \text{if } X \text{ is discrete} \\ \int_x e^{tx} \cdot p(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

3. The Transformation Behind

The OGF/EGF on sequence, it's Z-transform. In other words, such G is a bijection between vector space of sequences and the ring of formal power series. (ring is easily identified through the two properties)

Since differentiate and integration are well-defined operators on the ring, we are allowed to treat the series as objects in terms of calculus 'stuff'.

Observe that in the continuous X case, it's exactly the **Two-Sided Laplace Transform**, but here t is a real number instead of complex, but if getting rid off the real part, Laplace Transform is converted into Fourier Transform.

Only including the real part allows us to weaken the condition of the convergence of the integral in Fourier Transformation.

4. Other Related Topics

Recall that when solving linear differential equations, we are introduced generating function to do the approximation or numerical solution, and introducing Laplace Transform to solve the tricky ones, here we can see the motivation behind, due to the nice bijection, we can study the sequence on a continuous ring.

It links discrete world with continuous formal power series.

## 5.2 OGF

- Given a sequence  $\{a_k\}_{k=0}^n$ , the generating function of such sequence is

$$G(x) = \sum_{k \geq 0} a_k \cdot x^k$$

If  $k < \infty$ , then  $G(x)$  is a polynomial, otherwise it's a power series, where test of convergence is required.

## 5.3 EGF

- Given a sequence  $\{a_k\}_{k=0}^n$ , the exponential generating function of such sequence is

$$G(x) = \sum_{k \geq 0} \frac{a_k}{k!} \cdot x^k$$

If  $k < \infty$ , then  $G(x)$  is a polynomial, otherwise it's a power series, where test of convergence is required.

## 6 Counting in Structural Objects

1. Structural Objects refer to counting on special objects when symmetry is involved, so it's crucial to avoid overcounting under such cases
2. Group G equipped with the binary operator o is defined as,

- (Closue)  $\forall a, b \in G, a \circ b \in G$
- (Associativity)  $\forall a, b, c \in G, (a \circ b) \circ c = a \circ (b \circ c)$
- (Inverses)  $\forall a \in G, \exists b \in G \text{ s.t. } a \circ b = b \circ a = e$

In addition, G is **abelian** if  $\forall a, b \in G, a \circ b = b \circ a$

3. (Group Permutations)  $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ , a bijective function  
And define  $\pi_0$  as  $\pi_0(k) = k$  for each k, which is the identity map.

### 4. Some Examples

- (1) (Symmetric Group)  $\forall \pi \in S_n, \pi \circ \pi_0 = \pi_0 \circ \pi = \pi$   
Notice  $S_n$  is not abelian group since  $\pi_1 = (13)(254)$  and  $\pi_2 = (15423)$ , where  $\pi_1 \circ \pi_2 = (1452)$  but  $\pi_2 \circ \pi_1 = (2435)$
- (2) (Cyclic Group)  $\langle \pi \rangle = \{\pi^m : m \geq 0\}$   
Notice  $C_n$  is a subgroup of  $S_n$ , generated by  $(12\dots n)$ , we can also write as

$$C_n = \langle (12\dots n) \rangle$$

Clearly  $|C_n| = n$

Take  $(1234)$  as an example, which generates  $C_4$

$$(1234)^1 = (1234); (1234)^2 = (1234)(1234) = (13)(24); (1234)^3 = (1234)(13)(24) = (1432); (1234)^4 = (1)$$

Furthermore,  $C_n$  can be interpreted as rotation symmetries of a regular polygon.

- (3) (Diheral Group)  $D_n$  is the group of symmetries of **regular** polygon with n sides, including rotation and reflection symmetry  
 $|D_n| = 2n$ , since there are n rotations and n reflections  
Consider  $D_4$ , apart from  $C_4$ , we should also include the reflection symmetries (4), which are  $(12)(34), (14)(23), (1)(24)(3), (13)(2)(4)$ .  
By obervation, we can figure out the reflection symetry when n is odd, can only be happennd through the vertices, unlike even n's where reflections can take place at the edges.
- (4) (Alternating Group)  $A_n$  consists of even permutations of  $S_n$ , which is saying it includes all the permutations with even number of transpositions.  
 $|A_n| = \frac{n!}{2}$ , it can be shown by constructing a bijection  $T : A_n \rightarrow B_n$ , defined by  $T(\pi) = T \circ \pi$ , where  $B_n$  denotes the set of odd permutations in  $S_n$

### 5. Preparations

- (1) C is the group consisting the entire permutation, with the size  $n!$
- (2) (Invariant Set of  $\pi$  in C)  $C_\pi = \{c \in C : \pi^*(c) = c\}$   
It captures all the elements that remains unchanges/fixed under at least one of the permutations.  
(Capture the fixed points)
- (3) (The Stabilizer)  $G_c = \{\pi \in G : \pi^*(c) = c\}$   
It captures all the permutations that makes at least one element in G fixed-point. (Capture the permutations)
- (4) (Orbit of c)  $\bar{c} = \{\pi^*(c) : \pi \in G\}$   
For fixed c, the orbit captures all the image of different permutations  
It also contains all the equivalent classes (permutations) applied to element c  
Orbit of a colouring can be explained as a set contains the colouring that can be attained by applying all the permutations within the group G

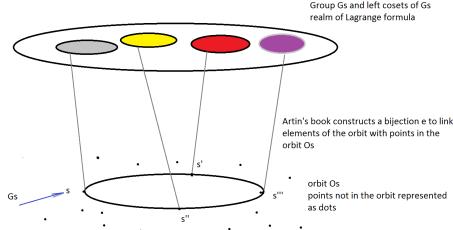


Figure 3: Orbit of A Colouring

#### 6. An Example of orbit/stabilizer

Let  $C$  be the set of colouring 4 vertices with 2 colours.

Take  $G$  to be  $D_4 = \{e, (13)(24), (1234), (1423), (14)(23), (12)(34), (13), (24)\}$

Then  $\overline{ggrr} = \{rggg, grgg, gggr, ggrg\}$ , it captures all the possible output of such colouring.

Then  $G_{gggr} = \{e, (13)\}$ , the identity map and  $(13)$  are the only two permutations that act as fix point for such colouring.

#### 7. Orbit-Stabilizer Theorem (Lagrange) $\leftarrow$ (he's smart)

**Theorem 6.1.**  $|G_c| \cdot |\bar{c}| = |G|$

##### Proof:

Let  $P$  with  $|P| = |\bar{c}|$ , every element of  $G$  can be represented as a product of something in  $G_c$  and something in  $P$ .

Then  $\bar{c} = \{c_1, \dots, c_m\} = \{\pi_1^*(c), \dots, \pi_m^*(c)\}$  and  $P = \{\pi_1, \dots, \pi_m\}$

Take arbitrary  $\pi \in G$ , then  $\pi^*(c) = c_i = \pi_i^*(c)$ , for some  $i$ . (then  $c = (\pi_i^{-1} \circ \pi)^*(c)$ )

So we have  $(\pi_i^{-1} \circ \pi)^* \in G_c$ .

Now we have  $\pi = \pi_i \circ \pi_i^{-1} \circ \pi$ , where  $\pi \in G$  and  $\pi_i \in P$  and  $\pi_i^{-1} \circ \pi \in G_c$

Then everything in  $G$  can be expressed as the product from  $P$  and  $G_c$ , then

$$|G| \leq |G_c| |P|$$

It remains to show such product expression is unique.

Let  $\pi_i \circ \sigma = \pi_j \circ \tau$ . WTS  $i = j$ ;  $\sigma = \tau$

we have  $c_i = c_j$ , while

$$c_i = \pi_i^*(\sigma^*(c)) = \pi_i^*(c)$$

and

$$c_j = \pi_j^*(\tau^*(c)) = \pi_j^*(c)$$

then  $i = j$ , which implies  $\pi_i \circ \sigma = \pi = \pi_i \circ \tau$ , therefore

$$\sigma = \tau$$

#### 8. Burnside's Lemma

**Lemma 6.1.** # equiv classes :=  $\frac{1}{|G|} \cdot \sum_{\pi \in G} |C_\pi|$

##### Proof:

We directly evaluate the number of equivalent classes

$$\sum_{\pi \in G} |C_\pi|$$

$$= \sum_{c \in C} |G_c|$$

$$= \sum_{c \in C} \frac{|G|}{|\bar{c}|}$$

$$\begin{aligned}
&= |G| \sum_{c \in G} \frac{1}{c} \\
&= |G| \cdot \sum_{\text{orbits}} \sum_{c \in \bar{c}} \frac{1}{(\bar{c})} \\
&= |G| \cdot |\# \text{ equiv classes}|
\end{aligned}$$

- The first equality holds since it counts the stabilizer in two-ways, one is counting the permutationsl the other is counting the elements, but since the map is bijective, they should agree.
- The second equality holds by Lagrange Theorem
- The forth equality holds since we partition C into the orbits, so we sum over the orbit and each element in it, instead of summer through each element in G
- The last equality holds since the inner sum sums through  $\frac{\# \text{ elements in } \bar{c}}{|\bar{c}|}$ , which gives 1, then summing through the orbits gives the cardinality of the orbit, i.e. the number of equivalent classses.

## 7 Special Numbers and Estimation Techniques

1. Stirling Formula
2. Ramsey Number
3. Catalan Number
4. Bell Number
5. Fibonacci Number
6. Eulerian Number

## 8 Big List of Problems

1. Let  $x, y, z$  be primes, then the number of positive integer solutions for  $x^2 + y^2 + z^2 = 2019$ ?  
 WLOG:  $x \leq y \leq z$ , then  $3x^2 \leq 2019$ , then  $x \leq 25$ , then  $x = 2, 3, 5, 7, 11, 13, 17, 19, 23$   
 If  $x = 2$ , then  $y^2 + z^2 = 2015$  which is odd, then  $y = 2 \implies z^2 = 2011$  Not satisfied  
 If  $x = 3$ , then  $y^2 + z^2 = 2010$ , then  $y \leq 31$ , then  $y = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$ , but they don't imply  $z$  to be perfect square  
 If  $x = 5$ , then  $y^2 + z^2 = 1994$ , then  $5 \leq y \leq 31$   
 If  $x = 7$ , then  $y^2 + z^2 = 1970$ , then  $7 \leq y \leq 31$ , then we get  $y = 11, 17$  satisfy, then  $(11, 43)$  and  $(17, 41)$  are the two solutions  
 If  $x = 11$ , then  $y^2 + z^2 = 1898$ , then  $11 \leq y \leq 30$ , then  $(23, 37)$   
 If  $x = 13$ , then  $y^2 + z^2 = 1850$ , then  $13 \leq y \leq 30$ , then  $(13, 41)$   
 If  $x = 17$ , we have  $(19, 37)$ ; if  $x = 19$ , there's no solution; if  $x = 23$ ,  $y^2 + z^2 = 1490$ , then  $(23, 31)$   
 When we fix the order, we have  $(x, y, z) = (7, 11, 43), (7, 17, 41), (11, 23, 37), (13, 13, 41), (17, 19, 37), (23, 23, 31)$ . We also have to consider permutation along the entries.  
 If three entries are all distinct, then we have  $4 \times 3!$ ; the rest two they have two entries are identical, then we just need to count the appearance of single element, which is  $2 \times 3$ , so the total is

$$4 \times 3! + 2 \times 3 = 30$$

2. Assume we have 9 ropes with length 1, 2, ..., 9 respectively. Number of ways to pick sides from the nine ropes so that they form a square. Notice the ropes can not be bent or overlapped except at the endpoints.  
 Reframe the problem, find 4 groups of numbers from  $\{1, 2, \dots, 9\}$ ,  $4S \leq 1 + 2 + \dots + 45$ , then  $S \leq 11$   
 If  $S = 11$ , then  $11 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6$   
 If  $S = 10$ , then  $10 = 1 + 9 = 2 + 8 = 3 + 7 = 4 + 6$   
 If  $S = 9$ , then  $9 = 9 = 1 + 8 = 2 + 7 = 3 + 6 = 4 + 5$ , then we have  $\binom{5}{4} = 5$   
 If  $S = 8$ , then  $8 = 8 = 1 + 7 = 2 + 6 = 3 + 5$   
 If  $S = 7$ , then  $7 = 7 = 1 + 6 = 2 + 5 = 3 + 4$   
 For  $S \leq 6$ , no solution.
3. Let a three digit number  $n = \overline{abc}$ , if  $a, b, c$  form an isosceles triangle, including equilateral triangle, how many three digit numbers are there?  
 If  $a = b = c$ , we take every number between 1 to 9  
 If  $a, b, c$  forms an isosceles triangle, we split into two cases ((i) the bottom side is the smallest side; (ii) the bottom is the greatest)  
 For the first case,  $\sum_{i=1}^8 (9 - i) = 36$ . Here 36 is the total number of value for fixed bottom side can take. But also need to consider permutation, there are three position can take such  $i$ , so  $36 \cdot 3 = 108$   
 For the second case, we enumerate  $(2, 2, 3), (3, 3, 5), (3, 3, 4), (4, 4, 7), (4, 4, 6), (4, 4, 5), (5, 5, 6/7/8/9), (6, 6, 7/8/9), (7, 7, 8/9)$   
 So the total number is  $3 \times (1 + 2 + 3 + 4 + 3 + 2 + 1) = 3 \cdot 4^2 = 48$   
 Therefore

$$9 + 108 + 48 = 165$$

4. A seven digit number, with seven numbers 1 to 7, and every even digit position equals to the absolute value of the difference of the adjacent numbers, find how many such 7-digit numbers are there?  
 The total sum of every digit is 28, we can write it as abcdefg, and from def we have  $f = |e - g|$  and  $b = |a - c|$  and  $d = |c - e|$ , then we have  $2|e + f + g|$  and  $2|a + b + c|$ , then  $2|d| \implies d = 2, 4, 6$   
 If  $d = 2$ , then we have  $\{1, 5, 6\}, \{3, 4, 7\}$ , then we have  $7432 - \begin{cases} 516 \\ 156 \end{cases}$   
 or  $73426 - (51/15)$  or 4372416  
 or 3472516. In total  $6 \times 2 = 12$ , since the order can be reverse.  
 If  $d = 4$ , we have  $\{2, 3, 5\}\{1, 6, 7\}$ , then we have  $7 - \begin{cases} 6145 - \begin{cases} 32 \\ 23 \end{cases} \\ 164235 \end{cases}$

or 6174325 or 1674325. In total  $5 \times 2 = 10$

If  $d = 6$ , we have  $\{1, 3, 4\}, \{2, 5, 7\}$ , then we have 5276134 or 2576134. In total 4.

## 5. Mapping in Combinatorics

- (1)  $f : X \rightarrow Y$  is injective, then  $f(x_1) = f(x_2) \implies x_1 = x_2$  and  $|X| \leq |Y|$ ; surjective if  $\forall y \in Y, \exists x \in X$  s.t.  $f(x) = y$ , and we have  $|X| \geq |Y|$

- (2) Randomly pick 4 integers at most 10, so that 4 of them with arithmetic can output 24. We're allowed to repeat picking. Without considering about the order of the numbers, how many cases are there in total?

First Approach (Enumeration and Combinatorics)

Case1(All numbers are same): 10;

Case2(Two types of numbers): (i) AAAB:  $A_{10}^2 = 90$ , (ii) AABB:  $\binom{10}{2} = 45$

Case3(Theoryhree types of numbers): (i) AABC:  $\binom{10}{1} \cdot \binom{9}{2} = 360$

Case4(Four types of numbers):  $\binom{10}{4} = 210$

Second Approach (Constructing Mappings)

WLOG:  $1 \leq a \leq b \leq c \leq d \leq 10$

Set  $x = a, y = b + 1, z = c + 2, w = d + 3$ , then

$$1 \leq x < y < z < w \leq 13$$

Then we have  $\binom{13}{4} = 715$

- (3) Let  $n$  people stand in a line, pick  $k$  people out of it, so that all  $k$  people don't stand around the people in previous line. Find the number of ways to pick  $k$  people?

$1 \leq a_1 < a_2 < \dots < a_k \leq n$  such that  $a_{i+1} - a_i \geq 2$  for  $i = 1, 2, \dots, k-1$ .

Let  $b_1 = a_1, \dots, b_k = a_k - (k-1)$ ,

Then  $1 \leq b_1 < b_2 < \dots < b_k \leq n-k+1$ . Then we compute

$$\# = \binom{n-k+1}{k}$$

- (3)  $I = \{1, 2, \dots, n\}$  and  $M$  be arbitrary nonempty subset, if we rearrange the elements in descending order as  $i_1 > \dots > i_k$ , and define  $i_1 - i_2 + i_3 - \dots + (-1)^{k-1} \cdot i_k$  as the 'Alternating Sum'. Find the sum of all the 'Alternating Sum'

Notice there are  $2^n$  subset, if we define  $\emptyset$  has alternating set 0. (To include  $\emptyset$ , so that we can construct pairwise)

We define two types of sets, one is  $A := \{a_1, \dots, a_k, n\}$  and  $B := \{a_1, \dots, a_k\}$ , both arranging in increasing order. In total, there are  $2^{n-1}$  pairs.

Adding them up, we have  $n - a_k + a_{k-1} - \dots + (-1)^k \cdot a_1 + a_k - a_{k-1} + a_{k-2} - \dots + (-1)^{k-1} \cdot a_1$ . Then the total sum is  $n2^{n-1}$ .

- (4) There are  $n(n \geq 6)$  points on the circle, and connect any two points with a cord. Any arbitrary three cords don't have an intersection point within the circle. Find the number of triangles, such cords intersect can form?

First case is all the points of the triangle are on the circle, where  $\binom{n}{3}$

Second case is 2 points of the triangle are on the circle, and 1 point is in the circle, any 4 points, we can create 4 triangles whose vertex is in the circle, then  $4 \cdot \binom{n}{4}$

Third case is 1 point on the circle and 2 points in the circle, then  $5 \cdot \binom{n}{5}$

Forth case is all the points are in the circle, then we have  $\binom{n}{6}$

In total, we have  $\binom{n}{3} + 4\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}$

- (5) A set  $M$  consists of 48 different positive numbers, and the prime factors of every positive integers are less than 30. Show that there are 4 distinct positive integers in  $M$ , such that their product is perfect square.

Notice there are 10 primes less than 30,  $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

Set  $Y = \{p_1^{a_1} \cdot \dots \cdot p_{10}^{a_{10}} : a_1 = 0 \text{ or } 1\}$ , Y doesn't produce any perfect square.

X is the collection of all the two element subset of M, then  $\forall \{a, b\} \in X$ ,  $a \cdot b$  can be written uniquely as  $ab = K_{ab}^2 \cdot m_{ab}$ , where  $K_{ab} \in \mathbb{Z}_+$  and  $m_{ab} \in Y$ .

Let  $\{a, b\}$  maps with  $m_{ab}$ , defined as  $f : X \rightarrow Y$ , where  $|X| = \binom{48}{2} = 1128$  and  $|Y| = 2^{10} = 1024$ , then f is not injection.

Then  $\exists \{a, b\} \neq \{c, d\} \in X$ , such that  $m_{ab} = f(\{a, b\}) = f(\{c, d\}) = m_{cd}$

Then  $abcd = K_{ab}^2 \cdot m_{ab} \cdot K_{cd}^2 \cdot m_{cd} = (K_{ab} \cdot K_{cd} \cdot m_{ab})^2$  is a perfect square.

Case1 a,b,c,d are distinct, then we're done

Case2  $\{a, b\}$  has exactly one point in  $\{c, d\}$ , WLOG:  $b = d$ ,  $a \neq c$ ,  $ac$  is a perfect square since  $abcd = acb^2$ .

After removing the identical a,c, we have  $|X \setminus \{a, c\}| \binom{46}{2} = 1035 > 1024 = |Y|$ , it follows from the argument above, we can find  $a', b', c', d'$  are perfect square.

- If  $a', b', c', d'$  are distinct, then we're done.
- Otherwise, WLOG: assume  $a' \neq c'$ ,  $b' = d'$ , then  $a' \cdot c'$  is a perfect square, then we found  $\{a, c\}$  and  $\{a', c'\}$  they are distinct but the product is perfect square.

- (6) A sequence of symbols consists of '+' and '-', How many string with length of m, has exactly n times symbol change? ( $n < m$  and a change of symbol refers to change from '+' to '-' or '-' to '+')

Let  $X = \{() : \text{length of m with n times symbol change}\}$ ,  $X_1 = \{x \in X : \text{first symbol is positive}\}$ ,  $X_2 = \{x \in X : \text{first symbol is negative}\}$

Since for any x in X, if we change the symbol (from '+' to '-' or '-' to '+'), we can get an element in another set, clearly  $|X_1| = |X_2| = \frac{|X|}{2}$

$$|2 \cdot \binom{m-1}{n}|$$

- (7) Given positive integer n, a finite positive integer sequence  $(a_1, \dots, a_m)$  is a 'n - Even Sequence', if  $n = a_1 + \dots + a_m$ , and there are even number of  $(i, j)$  satisfy  $1 \leq i < j \leq m$  and  $a_i > a_j$ . Find number of n - even sequence.

For Example, there are 6 4-even sequence : (4), (1, 3), (2, 2), (1, 1, 2), (2, 1, 1), (1, 1, 1, 1)

Take A be the set of n-even sequence, B be the set of n-odd sequence.

WTS:  $|A|$ , we know  $A \cap B = \emptyset$  and  $|A| + |B| = |C|$

Set  $S(n) = |A| - |B|$ , then  $S(0) = 1$

If it's singleton sequence, then  $S(1) = 1$

If length  $\geq 2$ :

- (i) Notice  $(a_1, a_2, \dots, a_m)$  has opposite property with  $(a_2, a_1, \dots, a_m)$ , one is even, the rest must be odd, which implies the map between them is bijection
- (ii) If  $a_1 = a_2$ , assume  $a_1 = a_2 = k$ ,  $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ , then  
want to find a relation between  $(a_1, \dots, a_m)$  and  $(a_3, \dots, a_m)$ , where we have

$$S(n) = 1 + 0 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} S(n-2k) = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} S(n-2k)$$

$$\text{Consider } S(n) - S(n-2) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} S(n-2k) - \sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} S(n-2-2k)$$

$$\text{If } n = 2m, \text{ even, then } S(n) - S(n-2) = S(n-2) + S(n-4) + \dots + S(n-2m) - [S(n-4) + S(n-6) + \dots + S(n-2m)] = S(n-2)$$

$$\text{We get } S(n) = 2 \cdot S(n-2)$$

$$\text{If } n = 2m+1, \text{ then } S(n) - S(n-2) = S(n-2), \text{ then } S(n) = 2 \cdot S(n-2).$$

$$\text{If } n \text{ is even, } k = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}, \text{ then}$$

$$S(n) = 1 + S(0) + S(2) + \dots + S(n-2) = 1 + 1 + 2 + 2^2 + \dots + 2^{\frac{n-2}{2}} = 2^{\frac{n-2}{2}+1} = 2^{\frac{n}{2}}$$

$$\text{If } n \text{ is odd, } k = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}, \text{ then}$$

$$S(n) = 1 + S(1) + S(3) + S(5) + \dots + S(n-2) = 1 + 1 + 2 + 2^2 + \dots + 2^{\frac{n-3}{2}} = 2^{\frac{n-3}{2}+1} = 2^{\frac{n-1}{2}}$$

Therefore we get  $S(n) = 2^{[\frac{n}{2}]} = |A| - |B|$ , then

$$|A| = \frac{2^{n-1} + 2^{[\frac{n}{2}]}}{2}$$

## 6. Menage Problem

We consider the Menage problem where there are  $2n$  seats and  $n$  couples, with the constraint that no one is allowed to sit beside their spouse.

$$\begin{aligned} M_n &= \sum_{k=0}^n (-1)^k (2n)! \frac{2n}{2n-k} \frac{(2n-k)!}{(n-k)!(n-k)!} \\ &= \sum (-1)^k (2n)! \frac{2n}{2n-k} \frac{(2n-k)!}{(2n-2k)!k!} \end{aligned}$$

still unknown it converges or not

**7. Prime Distribution** The number of integers  $m$  less than  $n$  and coprime to  $n$  is given by:

$$\varphi(n) = \#\{m < n \mid \gcd(m, n) = 1\}$$

This can be computed using:

$$\begin{aligned} \varphi(n) &= n - \sum_{i=1}^s \text{number divisible by } p_i \leq n + \dots \\ &= n \prod_{i=1}^s \left(1 - \frac{1}{p_i}\right) \end{aligned}$$

Number of Primes  $\leq n$

The prime counting function  $\pi(n)$  satisfies:

$$\pi(n) = n - \sum_{p_i \leq n} \# \text{ divisible by } p_i + \sum_{p_i p_j \leq n} \# \text{ divisible by } p_i p_j + \dots$$

Using an approximation:

$$\sum \frac{n}{p_i} - \frac{n}{p_i p_j} \approx \frac{n}{p_1} - \frac{n}{p_1 p_2}$$

By the prime number theorem:

$$\pi(n) \sim \frac{n}{\log n}$$

## Estimation Difficulty

The term  $(-1)^k$  makes the difference hard to estimate.

## 8. Deranged Twins

Suppose  $n+2$  people are seated behind along table facing an audience to staff a panel discussion. Two of the people are identical twins, wearing identical clothing. At intermission, the panelists decide to rearrange themselves so that it will be apparent to the audience that everyone has moved to a different seat when the panel reconvenes. Each twin can therefore take neither her own former place, nor her twin's. Let  $T_n$  denote the number of different ways to derange the panel in this way.

**Solution:**

Let  $!(n)$  be the subfactorial-derangement function. There are  $!(n+2)$  ways to derange the  $n+2$  persons, however, some of these derangements send one of the twins to the position of the other twin. We classify these cases into three types, which need to be subtracted:

1. If twin 1 gets sent to twin 2's position but twin 2 does not get sent back to twin 1's position, then for each of the remaining  $n$  persons, there will be exactly one forbidden seat choice. This results in  $!(n+1)$  valid derangements.
2. If twin 2 gets sent to twin 1's position but twin 1 does not get sent back to twin 2's position, then again, for each of the remaining  $n$  persons, there will be exactly one forbidden seat choice, giving another  $!(n+1)$  derangements.
3. If twin 1 gets sent to twin 2's seat and twin 2 gets sent to twin 1's seat, there are exactly  $!n$  ways to arrange the remaining people.

Thus, the total valid arrangements are:

$$t(n) = !(n+2) - 2!(n+1) - !n.$$

However, the two twins are identical, meaning that swapping them does not result in a new arrangement. Since we have counted each such swap twice in our calculations, we must divide by 2 to obtain the correct count:

$$t(n) = \frac{!(n+2) - 2!(n+1) - !n}{2}.$$

Checking for  $t(10)$ , we obtain 72,755,370 as expected.

9. **Putnam 1978** Let  $A$  be any set of twenty integers chosen from the arithmetic progression  $1, 4, \dots, 100$ . Prove that there must be two distinct integers in  $A$  whose sum is 104.

**Solution:**

We partition the thirty-four elements of this progression into nineteen groups:

$$\{1\}, \{52\}, \{4, 100\}, \{7, 97\}, \{10, 94\}, \{49, 55\}, \dots$$

Since we are choosing twenty integers and we have nineteen sets, by the Pigeonhole Principle, there must be two integers that belong to one of the pairs, which add to 104.

10. Show that amongst any seven distinct positive integers not exceeding 126, one can find two of them, say  $a$  and  $b$ , which satisfy

$$b < a \leq 2b.$$

**Solution:**

Split the numbers  $\{1, 2, 3, \dots, 126\}$  into the six sets:

$$\begin{aligned} &\{1, 2\}, \{3, 4, 5, 6\}, \{7, 8, \dots, 13, 14\}, \{15, 16, \dots, 29, 30\}, \\ &\{31, 32, \dots, 61, 62\}, \{63, 64, \dots, 126\}. \end{aligned}$$

By the Pigeonhole Principle, two of the seven numbers must lie in one of the six sets, and obviously, any such two will satisfy the stated inequality.

11. No matter which fifty five integers may be selected from  $\{1, 2, \dots, 100\}$ , prove that one must select some two that differ by 10.

**Solution:**

First observe that if we choose  $n+1$  integers from any string of  $2n$  consecutive integers, there will always

be some two that differ by  $n$ . This is because we can pair the  $2n$  consecutive integers  $\{a+1, a+2, a+3, \dots, a+2n\}$  into the  $n$  pairs  $\{a+1, a+n+1\}, \{a+2, a+n+2\}, \dots, \{a+n, a+2n\}$ , and if  $n+1$  integers are chosen from this, there must be two that belong to the same group. So now group the one hundred integers as follows: and  $\{1, 2, \dots, 20\}, \{21, 22, \dots, 40\}, \{41, 42, \dots, 60\}, \{61, 62, \dots, 80\}, \{81, 82, \dots, 100\}$ . If we select fifty five integers, we must force choose eleven from some group. From that group, by the above observation (let  $n = 10$ ), there must be two that differ by 10.

12. Let  $a_1, a_2, \dots, a_n$  be a sequence of integers. Show that there exist integers  $j$  and  $k$  with  $1 \leq j \leq k \leq n$  such that the sum

$$\sum_{i=j}^k a_i$$

is a multiple of  $n$ .

**Solution:**

Consider the prefix sums:

$$S_1 = a_1, \quad S_2 = a_1 + a_2, \quad \dots, \quad S_n = a_1 + a_2 + \dots + a_n.$$

**Case I: All Prefix Sums are Distinct (mod  $n$ )**

If all  $S_1, S_2, \dots, S_n$  are distinct modulo  $n$ , then they must cover all  $n$  residue classes modulo  $n$ , including 0. Therefore, there exists some  $k$  such that:

$$S_k \equiv 0 \pmod{n}.$$

This means the sum  $\sum_{i=1}^k a_i$  is a multiple of  $n$ .

**Case II: Two Prefix Sums are Congruent (mod  $n$ )**

If two prefix sums are congruent modulo  $n$ , say  $S_k \equiv S_j \pmod{n}$  with  $k > j$ , then:

$$S_k - S_{j-1} \equiv 0 \pmod{n}.$$

But  $S_k - S_{j-1} = a_j + a_{j+1} + \dots + a_k$ , so:

$$\sum_{i=l+1}^k a_i \equiv 0 \pmod{n}.$$

This means the sum  $\sum_{i=j}^k a_i$  is a multiple of  $n$ .

In both cases, there exist integers  $j$  and  $k$  with  $1 \leq j \leq k \leq n$  such that the sum  $\sum_{i=j}^k a_i$  is a multiple of  $n$ .

13. **PH dealing with averages:**

- (a) Suppose  $A = (a_1, a_2, \dots, a_n)$  is a sequence of positive real numbers. Let  $H(A)$  denote the *harmonic mean* of  $A$ , defined by

$$H(A) = n \left( \sum_{i=1}^n \frac{1}{a_i} \right)^{-1}.$$

Show there exist integers  $i$  and  $j$ , with  $1 \leq i, j \leq n$ , satisfying

$$a_i \leq H(A) \leq a_j.$$

**Solution:**

By simply considering the upper-bound and lower-bound of  $H(A)$

- (b) Suppose the integers from 1 to  $n$  are arranged in some order around a circle, and let  $k$  be an integer with  $1 \leq k \leq n$ . Show that there must exist a sequence of  $k$  adjacent numbers in the arrangement whose sum is at least  $\lceil k(n+1)/2 \rceil$ .

**Solution:**

Adding up all the  $k$  sums  $n$  times we get

$$\sum_k \frac{kn(n+1)}{2}$$

Then take the average, we show the existence of such  $k$ .

- (c) Suppose the integers from 1 to  $n$  are arranged in some order around a circle, and let  $k$  be an integer with  $1 \leq k \leq n$ . Show that there must exist a sequence of  $k$  adjacent numbers in the arrangement whose product is at least  $\lceil (n!)^{k/n} \rceil$ .

**Solution:**

Similar process, we take the product to the power of  $k$ , then take the geometric-mean of it, gives us the "average" of the  $k$ -product, we show the existence of such  $k$ .

#### 14. Catalan Numbers and Dyck Path:

Consider a path from  $(0, 0)$  to  $(n, n)$  consisting of  $n$  right steps (R) and  $n$  up steps (U). The total number of such paths is given by the binomial coefficient:

$$\# \text{ Total Paths} = \binom{2n}{n}$$

A path is called a Dyck path if it never crosses above the diagonal  $y = x$ . To count the number of Dyck paths, we use a bijection that maps paths that go above the diagonal to paths from  $(0, 0)$  to  $(n-1, n+1)$ . This leads to:

$$\# \text{ Dyck Paths} = \binom{2n}{n} - \binom{2n}{n-1}$$

Using factorial notation:

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}, \quad \binom{2n}{n-1} = \frac{(2n)!}{(n-1)!(n+1)!}$$

Thus, the number of Dyck paths simplifies to:

$$\frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{1}{n+1} \binom{2n}{n} = C(n)$$

where  $C(n)$  denotes the  $n$ th Catalan number.

The probability that a randomly chosen path does not go above the diagonal (i.e., is a Dyck path) is:

$$P(\text{not above diagonal}) = \frac{\# \text{ Dyck Paths}}{\# \text{ Total Paths}} = \frac{1}{n+1}$$

#### 15. Integer Partitions

- (a) (Using Generating Functions)

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

(b) An estimation of  $p(n)$  as  $n \rightarrow \infty$

$$p(n) \sim \frac{1}{4\pi\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{2}}\right)$$

16. Let  $n$  be a positive integer and  $S$  a set of  $n^2 + 1$  positive integers with the property that every  $(n+1)$ -element subset of  $S$  contains two numbers one of which is divisible by the other. Show that  $S$  contains  $n+1$  different numbers  $a_1, a_2, \dots, a_{n+1}$  such that  $a_i | a_{i+1}$  for each  $i = 1, 2, \dots, n$ .

**Solutions:**

Let  $n$  be a positive integer and  $S$  a set of  $n^2 + 1$  positive integers with the property that every  $(n+1)$ -element subset of  $S$  contains two numbers one of which is divisible by the other. Show that  $S$  contains  $n+1$  different numbers  $a_1, a_2, \dots, a_{n+1}$  such that  $a_i | a_{i+1}$  for each  $i = 1, 2, \dots, n$ .

Use the divisibility relation to obtain a poset on  $S$  (that is,  $x \leq y$  iff  $x | y$ ). Check that this makes a poset. The condition that there does not exist an  $n+1$ -element subset of  $S$  where no element divides another translates into the condition that there does not exist an antichain of length  $n+1$  in  $S$ .

Thus, the longest antichain in  $S$  has length at most  $n$ , and by Dilworth's theorem,  $S$  can be written as the union of at most  $n$  chains. Since  $S$  has  $n^2 + 1$  elements, this implies that one of these chains has length at least  $n+1$ . This proves the result.

17. Show that Every natural can be devided by a Fibonacci number (apart from 1).

**Proof:**

Define  $F_0 = 0$

Fix a positive integer  $n$ , now consider the modulo residue, there are  $n$  equivalent classes.

**Lemma 8.1.** *For any natural number  $n$ , there exists distinct natural numbers  $k$  and  $k'$  such that*

$$(F_k, F_{k'}) \equiv (F_{k-1}, F_{k'-1}) \pmod{n}$$

**Proof:**

Since  $(F_k, F_{k-1})$  can only have  $n^2$  candidates, since we took the modulo of  $n$  over the natural numbers. There are  $n^2$  candidate pairs but only  $n$  equivalent classes, there must exists  $k$  and  $k'$  satisfy

$$(F_k, F_{k'}) \equiv (F_{k-1}, F_{k'-1}) \pmod{n}$$

which proves the lemma.

Given  $F_n = F_{n-1} + F_{n-2}$ , we have

$$F_{k-2} \equiv F_{k'-2} \pmod{n}$$

(We can subtract the index to get a new pairs that satisfies the congruences are modulo  $n$ )

By induction, if  $k$  and  $k'$  satisfy the lemma, so does  $k-1 \in \mathbb{N}$  and  $k'-1 \in \mathbb{N}$ , plug them into the lemma, we get

$$F_{|k-k'|} \equiv F_0 \equiv 0 \pmod{0}$$

18. Let  $v_i \in \mathbb{R}^3$  and  $\|v_i\| = 1$ . Show that there exists a permutation  $u_i$  such that

$$\sum_{i=1}^{n-1} \|u_{i+1} - u_i\| \leq 8\sqrt{n}$$

19. A Hard Problem (Infinite Partitions with Ramsey Theory)

If the union of disjoint nonempty sets  $A_1, \dots, A_n$  is  $[1, N] \cap \mathbb{Z}$ , and  $\forall m : 1 \leq m \leq n$ ,  $a \in A_m$ ,  $b \in A_m$ , we have

$$a + b \notin A_m$$

Then we say the n sets are 'Sume-free n-partition' for  $[1, N] \cap \mathbb{Z}$   
If  $\{x_n\}$  satisfies  $x_2 = 5$ ,  $x_n = nx_{n-1} + 1$  ( $n \geq 3$ ). Show that

$$\forall n \geq 2, \#(n, x_n)$$

sum-free partition

**Solution:**

pass

20. Bijection And Reflection

For given  $n \geq 2$ ,  $(a_1, a_2, \dots, a_n)$  is a 'good' tuple if

- $a_i \neq 0 \quad \forall i$
- $a_1 = 1$  and  $\forall 1 \leq k \leq n-1$ ,  $(a_{k+1} + a_k)(a_{k+1} - a_k - 1) = 0$

Find number of such 'good' tuples?

First observe, the recursive process offers the next term is  $+1$  or takes the opposite. We can attempt to draw the binary tree, without the noteq 0 condition, the answer clearly is  $2^{n-1}$ .

Second for n relatively small, we can get the total numbers are 2, 3, 6, 10, 20..., which seems to be the center of each row in Paascal Triangle. **Solution:**

For every  $(a_i)$ , we . define  $(b_i)_{i=1}^n$ , with

$$b_i = \begin{cases} a_i & a_i < 0 \\ -a_i - 1 & a_i > 0 \end{cases}$$

We want to show such relation is a bijection

21. Let  $n \geq 2$ , there are n 1's on the board. Every step we erase two numbers on the board a,b, and then write  $a+b$  or  $\min(a^2, b^2)$ . After  $n-1$  steps, there is only one number on the board, say the max is  $f(n)$ . Show that

$$2^{\frac{n+1}{3}} < f(n) \leq 3^{\frac{n}{3}}$$

**Solution:**

Assume  $p(n-1)$  holds

Let  $f(x)$  be the value after x steps and  $f(y)$  be the correspond value after y steps.

WLOG:  $f(x) \leq f(y)$  we estimate  $2f(x+y) \leq f(x) + f(y) \leq 2f^2(x) < 2 \cdot 3^{\frac{n-1}{3}}$   
for the first inequality, we have  $f(k) \geq f(\frac{k}{2}) + f(\frac{k}{2}) \geq 2^k$

22. Let

$$G(x) = \frac{1 - (1 - 4x)^{1/2}}{2x} = \sum_{n=0}^{\infty} g_n x^n.$$

Give a combinatorial problem for which the answer is  $g_n$ .

23. Any six people there are either 3 people know each other or 3 people don't know each other

**Solution:**

We can convert the problem into 2-colouring  $K_6$ , if they know each other colour the edge with red, otherwise colour the edge with blue.

Then we converge the problem into  $R(3, 3) = 6$ .

The following proof relies on Pigeonhole Principle.

Fix a vertex  $v_1$ , it has 5 edges induced at  $v_1$ , since it's 2-colouring, it has at least 3 edges with the same colour.

WLOG: Assume  $v_1$  has 3 induced red edges and 2 blue induced edges. Consider the three red edge end points are x, y, z

Then if one the edges that connect x,y,z is red, then we are done.

So the worst case is such  $K_3$  has all blue edges, which forms a blue  $K_3$  as well.

Therefore we show that  $R(3, 3) = 6$

24. 3-colouring complete graph  $K_n$ , there must exists a monochromatic  $K_3$

**Solution:**

With rough calculation  $\frac{16}{3} = 5.33\dots$ , so there are at least 6 edges with the same colouring

Fix a vertex  $v_1$  WLOG: Assume 6 edges with the same colouring,  $v_2, \dots, v_7$

Then we want to colour those six vertices  $v_2, \dots, v_7$  with two colours (Since if any one of the edges is red, then we are done immediately)

So the problem converts to  $R(3, 3) = 6$ , which means there must exists a monochromatic  $K_3$  within the vertices  $v_2, \dots, v_7$ , then we have shown the existence of a chromatic  $K_3$

Therefore we have shown the statement

25. In a 2-colouring  $K_6$ , there exists a monochromatic cycle with length 4

**Solution:**

WLOG: Assume  $v_1 - v_2 - v_3 - v_1$  forms a monochromatic  $K_3$ , with red colouring the edges

Consider  $v_4$ , connect with  $v_1, v_2, v_3$ , it's impossible to have  $v_1v_4$  and  $v_2v_4$  with blue, since it creates a blue  $C_4$ , so  $v_1v_4$  and  $v_2v_4$  must be both red or 1 blue 1 red

Assume  $v_2v_4$  is blue, then  $v_1v_4$  and  $v_3v_4$  must be red

Similarly consider  $v_5$ , the edges between  $v_1, v_2, v_3$  must follow the same pattern. Otherwise if  $v_1v_5$  and  $v_3v_5$  are both red, then  $v_1v_5v_3v_4$  forms a red  $C_4$ .

So  $v_5v_1$  and  $v_5v_3$  must be 1 blue 1 red. Then wlog we can set  $v_3v_5$  be blue, so  $v_2v_5$  and  $v_1v_5$  must be red

Consider  $v_6$ , by the similar argument, the edges between  $v_6$  and  $v_1, v_2, v_3$  must be either 3 red or 1 blue 2 red.

If  $v_6v_1$  and  $v_6v_2$  and  $v_6v_3$  are red, then it must create a  $C_4$ ,  $v_1v_5v_2v_6$ .

So it must be 1 blue and 2 red. The only possible case is  $v_1v_6$  is blue and the rest two are red  $v_2v_6$  and  $v_3v_6$

By obeservation,  $v_5v_6$  must be red, otherwise  $v_1v_3v_5v_6$  forms a  $C_4$

Similarly,  $v_6v_4$  must be blue, otherwise  $v_1v_5v_6v_4$  forms a  $C_4$

And  $v_4v_5$  must be red, otherwise  $v_2v_3v_5v_4$  forms a blue  $C_4$

But the second last step creates a contradiction, since  $v_1v_6v_4v_2$  forms a blue  $C_4$

Therefore no matter which colouring, we can always attain a monochromatic  $C_4$

26. Colouring complete graph  $K_n$  with n colouring. For which value of n colouring, there exists for any 3 colours among n, there exists a  $K_3$  consists of edges with those 3 colours.

**Solution:**

There are  $\binom{n}{3}$  ways to pick 3 colours out of n colours. In addition, it's the same as picking a  $K_3$  out of  $K_n$

Let  $1, \dots, n$  be the n colours, and vertices  $v_1, \dots, v_n$

Consider  $\{v_i, v_j, v_k\}$  and  $\{i, j, k : 1 \leq i < j < k \leq n\}$  forms an injection

Clearly triangle  $v_iv_jv_k$  with the colouring i,j,k is unique in the complete graph