Math Handbook

dykkos

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Contents

1 Tool Box

1. Landau Notation (O and o)

   (a) Purpose of O and o is used to express remainder, but reflects the highest degree or the dominant part in the remainder that affect the convergence or divergence of the sequence/limit/value of a function.
   More precisely, when dealing with sequences and sums and products, big O and small o provides an opportunity to **estimate term-by-term**.

   (b) Understanding limit with small o
   Def of small o:
   Let $a_n = o(f(n))$, $o(f_n)$ means:

   $$\forall \epsilon > 0, \exists N \in \mathbb{N}\, s.t. \forall n > N, |a_n| \leq \epsilon f(n)$$

   Equivalently, we can understand it as
   'For large enough n, the section grows <<<< the growing speed of f(n)'

   (c) Using small o to deal with complicated series:
   To see if

   $$S_n \to L \iff S_n = L + o(1)$$

   Our goal convert to finding such L that satisfies.

   For example $S_n = \sum_{k=0}^{\infty} sin(\frac{k}{n^2})$
   Notice $sin(\frac{k}{n^2}) = \frac{k}{n^2} + o(\frac{k}{n^2})$, we have

   $$S_n = \frac{n(n+1)}{2n^2} + o(\frac{n(n+1)}{2n^2}) = \frac{1}{2} + o(1) + o(1) = \frac{1}{2} + o(1)$$

2. Unit Root Inversion

   (1) Mostly used to solve 'Find the sum of a subsequence within a sequence where the index of subsequence satisfies specific property', such as the sum of the subsequence is divisible by some number or are primes

   (2) Preparation and Background Knowledge

   – Complex Number $i = \sqrt{-1}$ and define x to be the n-th power unit root if

   $$x^n = 1\&x \in \mathcal{C}$$

   – Euler's Formula

   $$e^{i \cdot \theta} = cos(\theta) + i \cdot sin(\theta)$$

   – The k-th root, where $1 \leq k \leq n$ is

   $$w_n^k = cos(\frac{2k\pi}{n}) + i \cdot sin(\frac{2k\pi}{n}) = exp(i \cdot \frac{k \cdot 2\pi}{n})$$

   – Except for $w_0 = 1$, the rest (k-1) roots satisfy

   $$w^{n-1} + w^{n-2} + ... + w^2 + w + 1 = 0$$

   – The geometric intuition
   The n roots evenly seperate the unit circle into n equal length sections, starting from $w_n^0 = (1, 0)$
   By the geometry and symmetry, we can see that w_n^k is equivalent to traveling along the circle $\frac{k}{n}$ laps
   Then we can get the periodic property, $\forall q \in \mathbb{Z}_+$,

   $$w_n^k = w_n^{q \cdot n + k}$$

– It also satisfies

$$w_n^x \cdot w_n^y = w_n^{x+y}$$

and

$$(w_n^x)^y = w_n^{xy}$$

and $|w| = 1$ followed from unit definition.

(3) The Basic Formula

$$[d|n] = \frac{1}{d} \cdot \sum_{i=0}^{d-1} w_d^{ni}$$

where $[d|n] = \begin{cases} \text{1\&d divides n} \\ \text{0\&otherwise} \end{cases}$

Case1, if $d|n$, then RHS

$$:= \frac{w_d^0 + w_d^n + w_d^{2n} + ... + w_d^{(d-1)n}}{d} = 1$$

since $w_d^{in} = w_d^{i(kd)} = w_d^{d^{(ik)}} = 1^{ik}$

Case2, if $n = kd + r$, then RHS

$$:= \frac{w_d^0 + w_d^n + w_d^{2n} + ... + w_d^{(d-1)n}}{d} = \frac{1}{d} \times 0 = 0$$

since $w_d^n \neq 1$ but $w_d^d = 1$, it turns into a geometric series.

Notice that the roots have to satisfy $w_d^j \neq 1$ but $w_d^d = 1$, for $1 < j < d$

(4) Introduce OGF to find the sum of the coefficients whose power is divisible by d

Consider $F(x) = \sum_{i=0}^n a_i x^i$, in other words, we are looking for

$$\sum_{i=0}^{n} a_i [d|i]$$

Then we have

$$\sum_{i=0}^{n} a_i[d|i] = \sum_{i=0}^{n} a_i \cdot \frac{1}{d} \sum_{j=0}^{d-1} w_d^{ij} = \frac{1}{d} \cdot \sum_{j=0}^{d-1} \sum_{i=0}^{n} a_i (w_d^j)^i = \frac{1}{d} \sum_{j=0}^{d-1} F(w_k^j)$$

Then we convert the problem from coutning the number of specific terms into finding the particular value of $F(x)$, where plug in special values we can get desired.

2 Number Theory

1. Basics for Number Theory

   (a) Find the last digit of
   $$9^{1003} - 7^{902} + 3^{801}$$
   Solution:
   To find the last digit, its equivalent to find the residue of mod 10.
   $$9^{1003} \equiv (-1)^{1003} \equiv (-1) \pmod{10}$$
   $$3^{801} \equiv 3 \cdot (3^4)^{200} \equiv 3 \pmod{10}$$
   $$7^{902} \equiv 49^{451} \equiv (-1)^{451} \equiv -1 \pmod{10}$$
   Then we have
   $$(-1) - (-1) + 3 = 3$$
   to be the remainder.

   (b) Show that $n \in \mathbb{N},\ 133 \mid 11^{n+2} + 12^{2n+1}$
   Solution:
   Notice
   $$12^2 \equiv 144 \equiv 11 \pmod{133}$$
   Then we have
   $$11^{n+2} + 12^{2n+1} \equiv 11^{n+2} + 12 \cdot 144^n \equiv 11^{n+2} + 12 \cdot 11^n \equiv (121 + 12) \cdot 11^n \equiv 0 \pmod{133}$$

   (c) Find the least number of the form:
   where k and l are positive integers.
      i. $|11^k - 5^l|$
      Solution:
      Since the last digit can either be 6 or 4, and $11^2 - 5^3 = -4$ the smallest is 4
      ii. $|36^k - 5^l|$
      Solution:
      We have $11 = |36|5^2|$ and we will show that this is the least number of the form $|36^k|5^l|$. Suppose that for some k,l we have $|36^k|5^l| \leq 10$. Since $36^k - 5^l \equiv 6 - 5 \equiv 1 \pmod{10}$,we deduce that $36^k|5^l = 1$ or $36^k|5^l = -9$. The first equality is impossible since it would imply that $0 - 1 \equiv -1 \pmod{4}$, impossible. The second equality is also impossible since it would yield $0 - (-1) \equiv 1 \pmod{3}$, again impossible. This finishes the proof.
      iii. $|53^k - 37^l|$

   (d)
   Lemma 2.1. *There are infinitely many primes*
   Proof:
   Let $\mathbb{P} = \{p_1, p_2, ...\}$ in ascending order and consider $\sum_{p\in\mathbb{P}} \frac{1}{p}$
   Suppose not the case, $\sum_{p\in\mathbb{P}} \frac{1}{p}$ is convergent.
   Then
   $$\exists k \text{ such that } \sum_{i\geq k+1} \frac{1}{p_i} < \frac{1}{2}$$
   We let $\{p_1, .., p_k\}$ be small primes and $\{p_{k+1}, ...\}$ be big primes, then for arbitrary $N \in \mathbb{N}$
   $$\sum_{i\geq k+1} \frac{N}{p_i} < \frac{N}{2}$$

Define N_b be the number of natural numbers n with $n \le N$ and has at least one big prime factor, N_s be the number of natural numbers n with $n \le N$ and all the prime factors are small primes. Clearly we have

$$N = N_b + N_s$$

We have

$$N_b \le \sum_{i \ge k+1} \lfloor \frac{N}{p_i} \rfloor < \frac{N}{2}$$

Then consider N_s, where we write the factors for each n as $n = a_n b_n^2$, a_n doesn't contain any perfect square factors, which is the product of coprime small primes.
Then there $\sum_{i=0}^{k} \binom{k}{i} = 2^k$
Moreover, $b_n = \sqrt{\frac{n}{a_n}} \le \sqrt{n} \le \sqrt{N}$
We have

$$N_s \le 2^k \cdot \sqrt{N}$$

Take $N = 2^{2k+2}$
then $N = N_s + N_b < \frac{N}{2} + 2^k \cdot \sqrt{N} = N$, which leads to a contradiction
Therefore

$$\sum_{p \in \mathbb{P}} \frac{1}{p}$$

diverges, which implies there are infinitely many primes.

2. This is an example problem using Infinite Descent taken from IMO 1988:

   If $a, b \in \mathbb{Z}$ such that $\frac{a^2+b^2}{1+ab} \in \mathbb{Z}$, then $\frac{a^2+b^2}{1+ab}$ is a perfect square.

   Proof: Suppose $\frac{a^2+b^2}{1+ab} = k \in \mathbb{Z}$ and k is not a perfect square. Choose a and b such that $\max(a, b)$ is as small as possible.

   Without loss of generality, assume $a < b$. If $a = b$, then $k = 1$, which is a perfect square. Consider the equation:

$$a^2 + b^2 - k(ab + 1) = 0.$$

   This can be rewritten as a quadratic in b:

$$b^2 - (ka)b + (a^2 - k) = 0.$$

   Let b_1 and b_2 be the roots of this quadratic. By Vieta's formulas:

$$b_1 + b_2 = ka \quad \text{and} \quad b_1 b_2 = a^2 - k.$$

   If $b_1 < 0$, then it is incompatible with $a^2 + b_1^2 = k(ab_1 + 1)$. If k is not a perfect square, then $b_1 = 0$ is also incompatible with $a^2 + 0^2 = k(0 \cdot a + 1)$.

   Now, observe that:

$$b_1 = \frac{a^2 - k}{b} < \frac{b^2 - k}{b} < b.$$

   This implies that $b_1 < \max(a, b)$, which contradicts the minimality of $\max(a, b)$. Therefore, k must be a perfect square.

3. Find all integer solutions of

$$a^3 + 2b^3 = 4c^3$$

   Since x^3 is only congruent to $0, 1, 8 \pmod 9$, the possible values for the left-hand side (LHS) are

$$0, 1, 2, 3, 6, 7, 8 \pmod 9,$$

while the right-hand side (RHS) can only be

$$0, 4, 5 \pmod 9.$$

For the equation to hold, we must have LHS $\equiv$ RHS $\pmod 9$, which forces a, b, c to be divisible by 3.

Now, suppose (a, b, c) is a solution with the smallest a. Then we can write:

$$a = 3x, \quad b = 3y, \quad c = 3z.$$

Substituting into the original equation, we obtain a new equation in terms of (x, y, z), which represents a smaller solution, leading to a contradiction.

4. Prove that the equality $x^2 + y^2 + z^2 = 2xyz$ can hold for whole numbers x,y,z only when x = y = z = 0.

   Lemma 1. The sum of the squares of odd numbers can never be a multiple of 4. Therefore, for the sum to be a multiple of 4, all numbers involved must be even.

   Suppose that (x, y, z) is a solution. An even number of these must be odd. If two are odd, say x and y, then their squares satisfy:

   $$x^2 + y^2 \equiv 4k + 2 \pmod 4.$$

   Since z^2 is divisible by 4, adding it to the sum preserves the form:

   $$x^2 + y^2 + z^2 \equiv 4k + 2 \pmod 4.$$

   On the other hand, the term $2xyz$ satisfies:

   $$2xyz \equiv 4k \pmod 4.$$

   Since these two expressions cannot be equal modulo 4, we conclude that x, y, z must all be even.

   Thus, we write:

   $$x = 2u, \quad y = 2v, \quad z = 2w.$$

   Substituting these into the original equation yields:

   $$u^2 + v^2 + w^2 = 4uvw.$$

   By applying the same reasoning, u, v, w must also be even.

   Repeating this process indefinitely, we conclude that x, y, z are divisible by every power of 2. The only possibility is:

   $$x = y = z = 0.$$

   Remark: The same argument applies to the equation:

   $$x^2 + y^2 + z^2 = 2axyz.$$

5. **(USAMO 1978)** An integer n will be called good if we can write

   $$n = a_1 + a_2 + \cdots + a_k,$$

   where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying

   $$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

   Given the information that the integers 33 through 73 are good, prove that every integer ≥ 33 is good.

   Solution: The key observation here is

   $$73 = 2 \cdot 33 + 7$$

We first prove that if n is good, then $2n+8$ and $2n+9$ are good. For assume that

$$n = a_1 + a_2 + \cdots + a_k,$$

and

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}.$$

Then

$$2n+8 = 2a_1 + 2a_2 + \cdots + 2a_k + 4 + 4$$

and

$$\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.$$

Also,

$$2n+9 = 2a_1 + 2a_2 + \cdots + 2a_k + 3 + 6$$

and

$$\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

Therefore, if n is good, both $2n+8$ and $2n+9$ are good. (1.1)

We now establish the truth of the assertion of the problem by induction on n. Let $P(n)$ be the proposition "all the integers $n, n+1, n+2, \ldots, 2n+7$" are good. By the statement of the problem, we see that $P(33)$ is true. But (1.1) implies the truth of $P(n+1)$ whenever $P(n)$ is true. The assertion is thus proved by induction.

6. Let s be a positive integer. Prove that every interval $[s,\, 2s]$ contains a power of 2.

   Solution:
   If s is a power of 2, then there is nothing to prove. If s is not a power of 2 then it must lie between two consecutive powers of 2, i.e., there is an integer r for which $2r < s < 2r + 1$. This yields $2r + 1 < 2s$. Hence $s < 2r + 1 < 2s$, which gives the required result.

7. Prove that

$$\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right),$$

   where there are n square roots in the expression.

   Solution:
   For $n = 1$, the statement is trivial. Assume $P(n)$ holds, i.e.,

$$\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right),$$

   where there are n square roots in the expression. For $P(n+1)$, we have:

$$2\cos\left(\frac{\pi}{2^{n+2}}\right) = 2\cos\left(\frac{1}{2} \cdot \frac{\pi}{2^{n+1}}\right).$$

   Using the half-angle formula for cosine, this becomes:

$$2\sqrt{\frac{1 + \cos\left(\frac{\pi}{2^{n+1}}\right)}{2}} = \sqrt{2 + 2\cos\left(\frac{\pi}{2^{n+1}}\right)}.$$

   By the induction hypothesis, this simplifies to:

$$\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}},$$

   where there are $n+1$ square roots in the expression. Thus, $P(n+1)$ holds.

8. **(IMO 1977) functional equations** Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

$$f(n+1) > f(f(n))$$

for each positive integer n. Prove that $f(n) = n$ for each n.

Solution:
The first step is to show that $f(1) < f(2) < f(3) < \cdots$. This is done by induction on n. We take S_n to be the statement that $f(n)$ is the unique smallest element of $\{f(n), f(n+1), f(n+2), \ldots\}$.

For $m > 1$, $f(m) > f(s)$ where $s = f(m-1)$, so $f(m)$ is not the smallest member of the set $\{f(1), f(2), f(3), \ldots\}$. But the set is bounded below by zero, so it must have a smallest member. Hence the unique smallest member is $f(1)$. So S_1 is true.

Suppose S_n is true. Take $m > n+1$. Then $m-1 > n$, so by S_n, $f(m-1) > f(n)$. But S_n also tells us that $f(n) > f(n-1) > \cdots > f(1)$, so $f(n) \geq n-1+f(1) \geq n$. Hence $f(m-1) \geq n+1$. So $f(m-1)$ belongs to $\{n+1, n+2, n+3, \ldots\}$. But we are given that $f(m) > f(f(m-1))$, so $f(m)$ is not the smallest element of $\{f(n+1), f(n+2), f(n+3), \ldots\}$. But there must be a smallest element, so $f(n+1)$ must be the unique smallest member, which establishes S_{n+1}. So, S_n is true for all n.

So $n \leq m$ implies $f(n) \leq f(m)$. Suppose for some m, $f(m) \geq m+1$, then $f(f(m)) \geq f(m+1)$. Contradiction. Hence $f(m) \leq m$ for all m. But since $f(1) \geq 1$ and $f(m) > f(m-1) > \cdots > f(1)$, we also have $f(m) \geq m$. Hence $f(m) = m$ for all m.

9. (a) Prove that the sum of three consecutive positive integers is divisible by 9
Solution:
We want to show that:

$$n^3 + (n+1)^3 + (n+2)^3 \text{ is divisible by } 9 \iff \delta \text{ is divisible by } 9,$$

where:

$$\delta = \left[(n+1)^3 + (n+2)^3 + (n+3)^3\right] - \left[n^3 + (n+1)^3 + (n+2)^3\right].$$

Expanding δ, we get:

$$\delta = (n+3)^3 - n^3.$$

Using the binomial expansion for $(n+3)^3$:

$$(n+3)^3 = n^3 + 9n^2 + 27n + 27.$$

Thus:

$$\delta = (n^3 + 9n^2 + 27n + 27) - n^3 = 9n^2 + 27n + 27.$$

Factoring out 9:

$$\delta = 9(n^2 + 3n + 3).$$

Since $n^2 + 3n + 3$ is an integer, δ is always divisible by 9. Therefore:

$$n^3 + (n+1)^3 + (n+2)^3 \text{ is divisible by } 9 \iff \delta \text{ is divisible by } 9.$$

(b) Prove the sum of three consecutive positive integers is divisible by 3
Solution:
Notice that $x^3 \equiv 0, 1, 2 \pmod 3$,
also good to recap $x^3 \equiv 0, 1, 8 \pmod 9$.
Then, modulo 3:

$$0^3 + 1^3 + 2^3 \equiv 0 + 1 + 8 \equiv 9 \equiv 0 \pmod 3.$$

Therefore:

$$3 \mid n^3 + (n+1)^3 + (n+2)^3.$$

10. Prove that if n is a natural number, then

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

is always an integer.

(a) **Solution:** Recall Faulhaber's formula:

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{r=0}^{p} \binom{p+1}{r} B_r n^{p+1-r},$$

where B_r are the Bernoulli numbers.
A hand-wavy form of the formula is:

$$\sum_{m=1}^{n} m^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + O(n^{k-1}).$$

The given sum $:= \sum_{k=1}^{n} k^4$ which is obviously an integer.

(b) **Solution:** To show that the sum is an integer is equivalent to showing:

$$30 \mid 6n^5 + 15n^4 + 10n^3 - n \iff 2, 3, 5 \mid 6n^5 + 15n^4 + 10n^3 - n.$$

Case 1: Modulo 2

$$6n^5 + 15n^4 + 10n^3 - n \equiv 0 + n^4 + 0 - n \equiv n^4 - n \equiv n - n \equiv 0 \pmod{2}.$$

Case 2: Modulo 3

$$6n^5 + 15n^4 + 10n^3 - n \equiv 0 + 0 + n^3 - n \equiv n^3 - n \equiv n - n \equiv 0 \pmod{3}.$$

Case 3: Modulo 5

$$6n^5 + 15n^4 + 10n^3 - n \equiv n^5 + 0 + 0 - n \equiv n^5 - n \equiv n - n \equiv 0 \pmod{5}.$$

Therefore, the sum is divisible by 2, 3, and 5, which implies it is divisible by 30 as well.

Faulhaber's formula for some usual power p, especially notice the relation between coefficients and power of each term.

It can also be expressed as

$$\sum_{k=1}^{n} n^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + O(n^{k-1})$$

The first seven examples of Faulhaber's formula are

$$\sum_{k=1}^{n} k^0 = n$$

$$\sum_{k=1}^{n} k^1 = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$$\sum_{k=1}^{n} k^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

$$\sum_{k=1}^{n} k^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$$

$$\sum_{k=1}^{n} k^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}$$

11. Summing from $k = 0$ to $k = n$:

$$\sum_{k=0}^{n} \frac{2^k - 2^k x^{2^k}}{1 - x^{2^{k+1}}} = \frac{1}{x-1} + \frac{2^{n+1}}{1 - x^{2^{n+1}}}$$

equals

$$\frac{1}{x-1} + \frac{2^{n+1}}{1 - x^{2^{n+1}}}.$$

Solution:
For the case $n = 2$, it clearly holds.

Now, assume that $P(n)$ holds, i.e.,

$$S_n = \frac{1}{x-1} + \frac{2^{n+1}}{1 - x^{2^{n+1}}}.$$

We prove that $P(n+1)$ also holds:

$$S_{n+1} = S_n + \frac{2^{n+1}}{1 + x^{2^n}}.$$

Substituting the assumption for S_n:

$$S_{n+1} = \frac{1}{x-1} + \frac{2^{n+1}}{1 - x^{2^{n+1}}} + \frac{2^{n+1}}{1 + x^{2^n}}.$$

After simplification, we obtain:

$$S_{n+1} = \frac{1}{x-1} + \frac{2^{n+2}}{1 - x^{2^{n+2}}}.$$

Thus, by induction, the formula holds for all n.

12. Some inequalities that can be easily proved by induction but handful for estimation

(a) *Prove that if n is a natural number, then*

$$1 \cdot 2 + 2 \cdot 5 + \cdots + n \cdot (3n-1) = n^2(n+1).$$

(b) *Prove that if n is a natural number, then*

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}.$$

(c) *Prove that*

$$\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$$

for all natural numbers $n > 1$.

(d) *Let $a_1, a_2, \ldots, a_n$ be positive real numbers with*

$$a_1 \cdot a_2 \cdots a_n = 1.$$

(e) *Use induction to prove that*

$$a_1 + a_2 + \cdots + a_n \geq n,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n = 1$.

13. Another Example of **Functional Equations:**

If $p(x)\,, q(x)$ are polynomials, satisfy $p(q(x)) = q(p(x))\ \forall x \in \mathbb{R}$. If $p(x) = q(x)$ has no real solutions, show that $p(p(x)) = q(q(x))$ has no real solutions
Solution:
since $p - q$ has no real solutions, WLOG assume $p > q$ for all x, then $p(p(x)) > q(p(x)) = p(q(x)) > q(q(x))\ \ \forall x \in \mathbb{R}$

14. Let $F_0(x) = x$, $F(x) = 4x(1-x)$, and define the recursive sequence:

$$F_{n+1}(x) = F(F_n(x)), \quad n = 0, 1, \ldots$$

Prove that

$$\int_0^1 F_n(x)\,dx = \frac{2^{2n-1}}{2^{2n}-1}.$$

It's good to notice recursively defined

$$f_n(x) = ax(1-x)$$

try trig-substitution like

$$x = sin^2 u \ or \ x = tan^2 u$$

Solution:
To evaluate the integral:

$$\int_0^1 F_n(x)\,dx,$$

where $F_n(x)$ is defined recursively by:

$$F_0(x) = x, \quad F(x) = 4x(1-x), \quad F_{n+1}(x) = F(F_n(x)),$$

we use the substitution $x = \sin^2(u)$. Then:

$$dx = 2\sin(u)\cos(u)\,du,$$

and the limits of integration change to $u = 0$ and $u = \frac{\pi}{2}$. Thus:

$$\int_0^1 F_n(x)\,dx = \int_0^{\frac{\pi}{2}} F_n(\sin^2(u)) \cdot 2\sin(u)\cos(u)\,du.$$

Using the recursive structure of $F_n(x)$, we have:

$$F_n(\sin^2(u)) = \sin^2(2^n u).$$

Substituting this into the integral:

$$\int_0^{\frac{\pi}{2}} \sin^2(2^n u) \cdot 2\sin(u)\cos(u)\,du.$$

Simplify the integrand using $2\sin(u)\cos(u) = \sin(2u)$:

$$\int_0^{\frac{\pi}{2}} \sin^2(2^n u) \cdot \sin(2u)\,du.$$

Using the identity $\sin^2(A) = \frac{1-\cos(2A)}{2}$, we rewrite the integrand:

$$\sin^2(2^n u) = \frac{1-\cos(2^{n+1}u)}{2}.$$

Thus, the integral becomes:

$$\int_0^{\frac{\pi}{2}} \frac{1-\cos(2^{n+1}u)}{2} \cdot \sin(2u)\,du.$$

Split the integral into two parts:

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2u)\,du - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2^{n+1}u)\cdot\sin(2u)\,du.$$

Evaluate the first integral:

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2u)\,du = \frac{1}{2}\left[-\frac{1}{2}\cos(2u)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}.$$

Evaluate the second integral using the product-to-sum identity:

$$\cos(2^{n+1}u)\cdot\sin(2u) = \frac{1}{2}\left(\sin((2^{n+1}+2)u) - \sin((2^{n+1}-2)u)\right).$$

Thus:

$$-\frac{1}{4}\int_0^{\frac{\pi}{2}} \left(\sin((2^{n+1}+2)u) - \sin((2^{n+1}-2)u)\right)\,du.$$

Integrate term by term:

$$-\frac{1}{4}\left[-\frac{\cos((2^{n+1}+2)u)}{2^{n+1}+2} + \frac{\cos((2^{n+1}-2)u)}{2^{n+1}-2}\right]_0^{\frac{\pi}{2}}.$$

Evaluate at the limits:

$$-\frac{1}{4}\left(\frac{2}{2^{n+1}+2} - \frac{2}{2^{n+1}-2}\right).$$

Simplify:

$$-\frac{1}{2}\left(\frac{1}{2^{n+1}+2} - \frac{1}{2^{n+1}-2}\right).$$

Combining the results:

$$\int_0^1 F_n(x)\,dx = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2^{n+1}+2} - \frac{1}{2^{n+1}-2}\right).$$

Simplify further to match the desired result:

$$\int_0^1 F_n(x)\,dx = \frac{2^{2n-1}}{2^{2n}-1}.$$

15. The maximum number of m^2+n^2 if $m,n \in \{1,2,3,\ldots,1981\}$, and it is known that

$$(n^2-mn-n^2)^2=1?$$

Solution:

Call a pair (n,m) *admissible* if $m,n \in \{1,2,\ldots,1981\}$ and

$$(n^2-mn-m^2)^2=1.$$

If $m=1$, then $(1,1)$ and $(2,1)$ are the only admissible pairs. Suppose now that the pair (n_1,n_2) is admissible, with $n_2>1$.

As $n_1(n_1-n_2)=n_2^2 \pm 1>0$, we must have $n_1>n_2$.

Let now $n_3=n_1-n_2$. Then

$$1=(n_1^2-n_1n_2-n_2^2)^2=(n_2^2-n_2n_3-n_3^2)^2,$$

making (n_2,n_3) also admissible. If $n_3>1$, in the same way we conclude that $n_2>n_3$ and we can let $n_4=n_2-n_3$ making (n_3,n_4) an admissible pair.

We have a sequence of positive integers $n_1>n_2>\ldots$, which must necessarily terminate. This terminates when $n_k=1$ for some k. Since $(n_{k-1},1)$ is admissible, we must have $n_{k-1}=2$.

The sequence follows as $1,2,3,5,8,\ldots,987,1597$, i.e., a truncated Fibonacci sequence. The largest admissible pair is thus $(1597,987)$ and so the maximum sought is

$$1597^2+987^2.$$

Using **Pell Equation**:

The problem involves finding the maximum value of m^2+n^2, where m,n are positive integers in the set $\{1,2,3,\ldots,1981\}$ and satisfy the equation:

$$(n^2-mn-m^2)^2=1.$$

The equation $(n^2-mn-m^2)^2=1$ implies:

$$n^2-mn-m^2=\pm 1.$$

Case 1: $n^2-mn-m^2=1$, then

$$n^2-mn-m^2-1=0.$$

This is a quadratic equation in n. Solving for n using the quadratic formula:

$$n=\frac{m \pm \sqrt{m^2+4(m^2+1)}}{2}=\frac{m \pm \sqrt{5m^2+4}}{2}.$$

For n to be an integer, $\sqrt{5m^2+4}$ must be an integer. Let $k=\sqrt{5m^2+4}$. Then:

$$k^2=5m^2+4.$$

Rearranging:

$$k^2-5m^2=4.$$

Case 2: $n^2 - mn - m^2 = -1$, then

$$n^2 - mn - m^2 + 1 = 0.$$

Solving for n using the quadratic formula:

$$n = \frac{m \pm \sqrt{m^2 + 4(m^2 - 1)}}{2} = \frac{m \pm \sqrt{5m^2 - 4}}{2}.$$

For n to be an integer, $\sqrt{5m^2 - 4}$ must be an integer. Let $k = \sqrt{5m^2 - 4}$. Then:

$$k^2 = 5m^2 - 4.$$

Rearranging:

$$k^2 - 5m^2 = -4.$$

The two Pell-like equations are:

1. $k^2 - 5m^2 = 4$, 2. $k^2 - 5m^2 = -4$.

The fundamental solution for the standard Pell equation $k^2 - 5m^2 = 1$ is $(k, m) = (9, 4)$. Using this, we can generate solutions for the two cases.

The solutions to the Pell-like equations can be generated using the recurrence relations:

$$k_{n+1} + m_{n+1}\sqrt{5} = (k_1 + m_1\sqrt{5})(9 + 4\sqrt{5})^n,$$

where (k_1, m_1) is a fundamental solution for the respective equation.

For each solution (k, m) to the Pell-like equations, we can recover n using the relationship $k = 2n - m$. This gives:

$$n = \frac{k + m}{2}.$$

We ensure that m and n are positive integers and lie in the range $\{1, 2, 3, \ldots, 1981\}$.

For each valid pair (m, n), compute $m^2 + n^2$. The maximum value of $m^2 + n^2$ occurs for the largest m and n in the given range.

The largest solution (m, n) within the range $\{1, 2, 3, \ldots, 1981\}$ is $(m, n) = (987, 1597)$. This pair satisfies:

$$n^2 - mn - m^2 = 1,$$

and lies within the given range. The value of $m^2 + n^2$ for this pair is:

$$m^2 + n^2 = 987^2 + 1597^2 = 974169 + 2550409 = 3524578.$$

The maximum value of $m^2 + n^2$ is:

$$\boxed{3524578}.$$

This corresponds to the solution $(m, n) = (987, 1597)$, which satisfies the given constraints.

16. **Cassini's Identity** and its converse:

    Cassini's Identity states that for the Fibonacci sequence $\{f_n\}$, the following holds:

$$f_{n+1}^2 - f_n f_{n+2} = (-1)^n.$$

    Proof:
    Consider the Fibonacci matrix, where

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix},$$

    where $\{f_n\}$ is the Fibonacci sequence.
    We get the identity through computing the determinant of both sides.

    Converse of Cassini's Identity

    If k and m are positive integers such that:

$$|m^2 - km - k^2| = 1,$$

    then there exists an integer n such that:

$$k = \pm f_n \quad \text{and} \quad m = \pm f_{n+1}.$$

17. **Binet's Formula** Binet's Formula expresses the n-th Fibonacci number f_n as:

$$f_n = \frac{r^n - (1-r)^n}{\sqrt{5}},$$

    where $r = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and $1 - r = \frac{1-\sqrt{5}}{2}$.

18. Condition for n to be in the Fibonacci Sequence:
    A number n is a Fibonacci number if and only if one or both of $5n^2 + 4$ or $5n^2 - 4$ is a perfect square.
    (derives from converse cassini's identity)

19. Two Handy Identity:

$$\sum \binom{n}{k} 2^k f_k = f_{3n}$$

$$f_n^2 + f_{n-1}^2 = f_{2n+1}$$

20. Find all nonnegative integer solutions to the equation

$$4^a + 5^b + 6^c = 7^d$$

    Solution:
    We first consider mod 3 to both sides, then we have
    $4^a \equiv 1 \pmod 3$; $5^b \equiv (1, -1) \pmod 3$; $6^c \equiv (1, 0) \pmod 3$; $7^d \equiv 1 \pmod 3$
    For 5^b and 6^c part, they can't be congruent to 1, otherwise the equality won't hold. Then we know

$$4^a \equiv 1 \pmod 3 \text{ and } 5^b \equiv (-1) \pmod 3 \text{ and } 6^c \equiv 1 \pmod 3 \text{ and } 7^d \equiv 1 \pmod 3$$

    Then we know b is odd and c is 0.
    Rewrite the equation :

$$4^a + 5^b + 1 = 7^d$$

where b is odd.
Observe that RHS always odd, and the second and third term on the left are always odd, then 4^a must be odd, then we get

$$a = 0$$

Then we get

$$5^b + 2 = 7^d$$

where b is odd
We get $(b, d) = (1, 1)$, with more attempts, we try to show it's the only solution.
When $b \geq 2$, $5^b \equiv 0 \pmod{25}$, to make the equality holds, we must have $7^d \equiv 2 \pmod{25}$. Notice that(with some try)

$$7^d \equiv (1, 7, -7, -1) \pmod{25}$$

then b can only be either 0 or 1. Clearly $b = 0$ doesn't work, then $b = d = 1$.

21. Find all positive integers n such that there exist integers a,b and c satisfying

$$2a^n + 3b^n = 4c^n$$

Solution:
By observation, b^n is even.
For $n = 1$, clearly we have

$$(a, b, c) = (1, 2, 2)$$

Now consider $n \geq 2$, for $n = 2$, we have $2a^2 + 3b^2 = 4c^2$.
First consider $\pmod 4$ on both sides, we get $4c^2 \equiv 0 \pmod 4$ and $3b^2 \equiv 0 \pmod 4$, then the equation under $\pmod 4$ is

$$2a^2 + 0 = 0$$

Then a is even.
Now consider $\pmod 3$, we have

$$2a^2 + 0 = c^2$$

Then c is even.
Assume $a = 2k_1$, $b = 2k_2$, $c = 2k_3$, where $k_1, k_2, k_3 \in \mathbb{Z}$, then we get

$$2k_1^2 + 3k_2^2 = 4k_3^2$$

We can keep taking mod 3, by infinite descent, we will eventually reach a contradiction that $5 = 4$.
For $n \geq 3$, by taking the equation on $\pmod 3$, we get c is even. By taking the equation over $\pmod 8$, we get a is even, since $3b^n \equiv 0 \pmod 8$ and $4c^n \equiv 0 \pmod 8$.
By similar arguement above (infinite descent), we reach an contradiction.
Then we show that for $n \geq 2$, there are no solutions.

3 Algebra

1. Show that for positive reals a, b, c,

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2.$$

Solution:
Substituting into the inequality, we obtain:

$$(a^2b + b^2c + c^2a)(c^2b + a^2c + b^2a) \geq (a\sqrt{b} \cdot c\sqrt{b} + b\sqrt{c} \cdot a\sqrt{c} + c\sqrt{a} \cdot b\sqrt{a})^2.$$

Simplifying the right-hand side:

$$(a\sqrt{b}c\sqrt{b} + b\sqrt{c}a\sqrt{c} + c\sqrt{a}b\sqrt{a})^2 = (abc + abc + abc)^2 = (3abc)^2 = 9a^2b^2c^2.$$

Thus, we have proven:

$$(a^2b + b^2c + c^2a)(c^2b + a^2c + b^2a) \geq 9a^2b^2c^2.$$

2. Let a, b, c be positive reals such that $abc = 1$. Prove that

$$a + b + c \leq a^2 + b^2 + c^2.$$

Solution:
Since $abc = 1$, by the AM-GM inequality, we have:

$$a + b + c \geq 3\sqrt[3]{abc} = 3.$$

Using the identity:

$$(a^2 + b^2 + c^2)(1^2 + 1^2 + 1^2) = (a + b + c)^2,$$

we obtain:

$$a^2 + b^2 + c^2 \geq \frac{(a + b + c)^3}{3}.$$

Rewriting:

$$a^2 + b^2 + c^2 \geq \frac{a + b + c}{3} \cdot (a + b + c).$$

Since $a + b + c \geq 3$, we also have:

$$\frac{a + b + c}{3} \geq 1.$$

Thus,

$$\frac{(a + b + c)^2}{3} \geq a + b + c.$$

Therefore, we have shown the desired inequality.

3. Let $P(x)$ be a polynomial with positive coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for $x = 1$, then it holds for all $x > 0$.

4. Show that for all positive reals a, b, c, d,

$$\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

Solution:
Rewriting the inequality, we want to show that:

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d}\right) \geq \left(\sqrt{a}\cdot\frac{1}{\sqrt{a}}+\sqrt{b}\cdot\frac{1}{\sqrt{b}}+\sqrt{c}\cdot\frac{2}{\sqrt{c}}+\sqrt{d}\cdot\frac{4}{\sqrt{d}}\right)^2.$$

Simplifying the right-hand side:

$$(1+1+2+4)^2 = 64.$$

Thus, we have proven the desired inequality.

5. (USAMO 1980/5) Show that for all non-negative reals $a, b, c \leq 1$,

$$\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}+(1-a)(1-b)(1-c) \leq 1.$$

Solution:
By taking the second partial derivatives with respect to a, b, c:

$$\frac{\partial^2 f}{\partial a^2}, \quad \frac{\partial^2 f}{\partial b^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial c^2},$$

we observe that all three are > 0. This implies that the maximum of f is attained on the boundary. By checking the following eight points:

$$(0,0,0), \quad (0,0,1), \quad (0,1,0), \quad (1,0,0), \quad (1,1,0), \quad (1,0,1), \quad (0,1,1), \quad (1,1,1),$$

we find that the maximum value satisfies:

$$\text{Max} \leq 1.$$

6. (USAMO 1977/5) If a, b, c, d, e are positive reals bounded by p and q with $0 < p \leq q$, prove that

$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \leq 25+6\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2.$$

Determine when equality holds.

7. Let a, b, c be non-negative reals such that $a+b+c=1$. Prove that

$$a^3+b^3+c^3+6abc \geq \frac{1}{4}.$$

8. (IMO 1995/2) a, b, c are positive reals with $abc=1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

9. Let a, b, c be positive reals such that $abc=1$. Show that

$$\frac{2}{(a+1)^2+b^2+1}+\frac{2}{(b+1)^2+c^2+1}+\frac{2}{(c+1)^2+a^2+1} \leq 1.$$

10. (USAMO 1998/3) Let $a_0, \ldots, a_n$ be real numbers in the interval $(0, \frac{\pi}{2})$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1.$$

Prove that

$$\tan(a_0)\tan(a_1)\cdots\tan(a_n) \geq n^{n+1}.$$

11. Show that $\frac{1\cdot 3\cdot \ldots (2n-1)}{2\cdot 4\cdot \ldots (2n)} < \frac{1}{\sqrt{3n+1}}$

Solution:
It can be easily proved by induction

12. Show that $1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} < 3$

(a) **Solution:**
Firstly, we are going to prove the following fact:

$$2^{n-1} \leq n! \quad \forall n \in \mathbb{N}, \quad (1)$$

which is pretty trivial. You can obviously prove it using mathematical induction, but here is an easier way to see this is true:

$$n! = 1 \cdot 2 \cdot 3 \cdots n, \quad 2^{n-1} = 2 \cdot 2 \cdots 2.$$

Now we can prove the stricter inequality:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots < 1 + \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 1 + 2 = 3.$$

So we have proved a stricter inequality, and obviously the inequality in the question is also true.

(b) **Solution:**
Notice

$$LHS := exp(1)$$

since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Sometimes **Stirling Formula** might be helpful:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

13. Consider complex number z_i, satistfy $|z_2| = 1$ and

$$|z_1 + iz_2^2| + |z_2 + z + 3^2| \leq 3$$

Find Maximum of $|z_1^2 + z_2| + |z_2^2 + z_3|$

14. Let integer $n \geq 2$, $x_i \in \mathbb{R}$. Show that

$$(max_{1\leq i\leq n}\, x_i)^2 + 4 \cdot \sum_{i=1}^{n-1}(x_{i+1} - x_i) \cdot max_{1\leq j\leq i}\, x_j \leq 4x_n^2$$

Solution:

Set $y_i = max_{1\le j\le i} x_j$, clearly y_i is monotonic increasing.
Notice $y_i \ge x_{i+1}$ then $y_i = y_{i+1}$. Reconstruct y_{i_k}, it's sufficient to show

$$y_n^2 + 4\cdot\sum_{j=1}^{k-1}(x_{i_{j+1}} - x_{i_j})y_{i_j} + 4(x_n - x_{i_k})y_{i_k} \le 4x_n^2$$

We have $y_{i_j} = x_{i_j}$ and $y_n = y_{i_k} = x_{i_k}$.
Then we have $4(x_{i_{j+1}} - x_{i_j}) \le 2(y_{i_{j+1}}^2 - y_{i_j}^2)$; and $4(x_n - x_{i_k})y_{i_k} = 4(x_n - y_n)y_n$. It's sufficient to show

$$-2y_{i_1}^2 - (2x_n - y_n)^2 \le 0$$

15. Given $a, b, c \in \mathbb{R}_+$, Show that

$$abc \ge \frac{a+b+c}{a^{-2}+b^{-2}+c^{-2}} \ge (a+b-c)(b+c-a)(c+a-b)$$

Solution:

- The second inequality, we split into cases,
  - a,b,c form a triangle, we rearrange as $2x = b+c-a$, $2y = c+a-b$, $2z = a+b-c$, which is equivalent to

$$2(x+y+z) \ge 8xyz\sum_{cyc}\frac{1}{(x+y)^2}$$

  - a,b,c don't form a triangle, RHS less than 0, clearly holds.
- The first inequality is equivalent to $\sum_{cyc}(a^2b^2) \ge \sum_{cyc}(a^2bc)$

16. Let A,B,C be the inner angle of a triangle, find the maximum of

$$sin^2A + sin^2B + sin^2C$$

Solution:
By $|AB|^2 + |BC|^2 + |CA|^2 + 9|OG|^2 = 3(|OA|^2 + |OB|^2 + |OC|^2)$, we get the max is 9/4.
O is arbitrary point and G is the centriod point of the triangle.

17. $\{a_n\}$ with $a_1 = 1$, $a_{n+1} = \frac{1}{a_1+\ldots+a_n}$. Show that for $n \ge 2$,

$$\frac{1}{\sqrt{2n}+1} < a_{n+1} \le \frac{1}{\sqrt{2n}}$$

Set $S_n = \sum_i a_i$, rewrite the recursion we get $S_{n+1} = S_n + \frac{1}{S_n}$, and $S_2 = 2$. We can get $S_{n+1}^2 - S_n^2 \ge 2$ and $S_n^2 \ge 2n$ for ($n \ge 2$), we get the second inequality.
For the first inequality, we use $(S_{n+1}-1)^2 = ((S_n-1)+S_n^{-1})^2 \le (S_n-1)^2+2$

18. Let positive reals a,b,c satisfy $abc > 1$, then the minimum of $\frac{abc(a+b+c+8)}{abc-1} = 16$
Solution:
Fix a,b,c, we get $a+b+c \ge 3(abc)^{\frac{1}{3}}$. Set $x = (abc)^{\frac{1}{3}} > 1$, $\frac{x^3(3x+8)}{x^3-1}$ attains minimum, taking derivatives, we get the minimum.

19. Let $a, b, c, d > 0$, Show that

$$\sum_{cyc}\frac{1}{a^4+b^4+c^4+abcd} \le \frac{1}{abcd}$$

Proof:
We rearrange the inequality partially,

$$\frac{1}{a^4+b^4+c^4+abcd} \le \frac{1}{abc(a+b+c)+abcd} = \frac{d}{abcd(a+b+c+d)}$$

4 Combinatorics

1. Some Uesful combinatorial coefficients and interpretation

   (a) If m and n are nonnegative integers, then

   $$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

   (b) **The Vandermonde identity:**

   $$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

   (c) **Cancellation Identity:**

   $$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

   (d) **Absorption (Extraction) Identity:**

   $$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$

   (e) **Parallel Summation Identity:**

   $$\sum_{k} \binom{m+k}{k} = \binom{m+n+1}{n}$$

   (f) **Hexagon Identity:**

   $$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

   In symbols, the identity is

   $$\binom{n-1}{m-1}\binom{n}{m+1}\binom{n+1}{m} = \binom{n}{m-1}\binom{n-1}{m}\binom{n+1}{m+1}.$$

   The usual combinatorial interpretation of a binomial coefficient $\binom{n}{m}$ is that it counts subsets of size m from a set of size n. Multiplication is usually interpreted as mutually exclusive choice ($f(n)g(n)$ counts the process of picking $f(n)$ configurations, then picking (independently) $g(n)$ items).

   Putting this together, the LHS counts subsets of size $m-1$ from a set of size $n-1$, then subsets of size m from an (independent) set of size $n+1$, then (again independently) subsets of size $m+1$ from a set of size n. This corresponds one-to-one with the RHS because the things counted by the LHS can be counted in a different way by the RHS: For the RHS distinguish an element of the n set and one of the $n+1$ set. What's left over for those two sets can be chosen by $\binom{n-1}{m-1}$ and

$\binom{(n+1)-1}{m-1}$ respectively, and then the two distinguished elements can be included to be (possibly) chosen in the $n-1$ set to account for $\binom{(n-1)+2}{(m-1)+2}$.

To be clearer about the combinatorial interpretation, there are three sets, of size $n-1$, n, and $n+1$, from which you choose subsets of size $m-1$, $m+1$, and m, respectively. Another way to count this situation is to take 1 item each out of the n and $n+1$ sets, and add them to the $n-1$ set. So now you're counting out of sets of size $n+1$, $n-1$, and n, from which you choose subsets of size $m+1$, m, and $m-1$, respectively.

Let there be m men and n women in a committee. We want to form a subcommittee consisting of $m+p$ members. Counting directly, we find that there are

$$\binom{m+n}{m+p}$$

ways to form this subcommittee, which is the RHS of the equation.

(g) Square of the binomial coefficients:

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

(h) $\sum_{k\leq m}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m}$

There is a nice combinatorial proof of this identity using a sign-reversing involution. The summation counts subsets of $\{1, 2, \ldots, n\}$ of size m or fewer, except subsets with even size are counted positively and those with odd size are counted negatively.

For each set S which does not contain 1, pair it with the set $S \cup \{1\}$. Note that the sizes of S and $S \cup \{1\}$ have opposite parities, so they cancel each other in your sum and can be ignored.

Which sets are not paired with anything? The only reason $S\cup\{1\}$ would not exist was if $|S| = m$, in which case $S \cup \{1\}$ would be too big and would not be counted. Therefore, the number of unpaired sets is $\binom{n-1}{m}$, and these sets all have parity $(-1)^m$ in your sum, so the sum is

$$(-1)^m\binom{n-1}{m}.$$

2. Some examples on **Pigeonhole**

(a) No matter which fifty five integers may be selected from $\{1, 2, ..., 100\}$, prove that one must select some two that differ by 10.

Solution:

First observe that if we choose $n+1$ integers from any string of $2n$ consecutive integers, there will always be some two that differ by n. This is because we can pair the 2n consecutive integers $\{a+1, a+2, a+3, ..., a+2n\}$ into the n pairs ,$\{a+1, a+n+1\}, \{a+2, a+n+2\}, ..., \{a+n, a+2n\}$, and if $n+1$ integers are chosen from this, there must be two that belong to the same group. So now group the one hundred integers as follows: and $\{1, 2, ...20\}, \{21, 22, ..., 40\}, \{41, 42, ..., 60\}, \{61, 62, ..., 80\}\{81, 82, ..., 100\}$. If we select fifty five integers, we must perforce choose eleven from some group. From that group, by the above observation (let n = 10), there must be two that differ by 10.

(b) **Putnam 1978** Let A be any set of twenty integers chosen from the arithmetic progression $1, 4, \dots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

Solution:
We partition the thirty-four elements of this progression into nineteen groups:

$$\{1\}, \{52\}, \{4, 100\}, \{7, 97\}, \{10, 94\}, \{49, 55\}, \dots$$

Since we are choosing twenty integers and we have nineteen sets, by the Pigeonhole Principle, there must be two integers that belong to one of the pairs, which add to 104.

(c) Show that amongst any seven distinct positive integers not exceeding 126, one can find two of them, say a and b, which satisfy

$$b < a \leq 2b.$$

Solution:
Split the numbers $\{1, 2, 3, \dots, 126\}$ into the six sets:

$$\{1, 2\}, \{3, 4, 5, 6\}, \{7, 8, \dots, 13, 14\}, \{15, 16, \dots, 29, 30\},$$

$$\{31, 32, \dots, 61, 62\}, \{63, 64, \dots, 126\}.$$

By the Pigeonhole Principle, two of the seven numbers must lie in one of the six sets, and obviously, any such two will satisfy the stated inequality.

(d) Given any set of ten natural numbers between 1 and 99 inclusive, prove that there are two disjoint nonempty subsets of the set with equal sums of their elements.

Solution:
There are $2^{10} - 1 = 1023$ non-empty subsets that one can form with a given 10-element set. To each of these subsets, we associate the sum of its elements. The maximum value that any such sum can achieve is:

$$90 + 91 + \cdots + 99 = 945 < 1023.$$

Therefore, there must be at least two different subsets that have the same sum.

(e) Let $a_1, a_2, \dots, a_n$ be a sequence of integers. Show that there exist integers j and k with $1 \leq j \leq k \leq n$ such that the sum

$$\sum_{i=j}^{k} a_i$$

is a multiple of n.

Solution:
Consider the prefix sums:

$$S_1 = a_1, \quad S_2 = a_1 + a_2, \quad \dots, \quad S_n = a_1 + a_2 + \cdots + a_n.$$

Case I: All Prefix Sums are Distinct (mod n)

If all $S_1, S_2, \dots, S_n$ are distinct modulo n, then they must cover all n residue classes modulo n, including 0. Therefore, there exists some k such that:

$$S_k \equiv 0 \pmod{n}.$$

This means the sum $\sum_{i=1}^{k} a_i$ is a multiple of n.

Case II: Two Prefix Sums are Congruent (mod n)

If two prefix sums are congruent modulo n, say $S_k \equiv S_j \pmod{n}$ with $k > j$, then:

$$S_k - S_{j-1} \equiv 0 \pmod{n}.$$

But $S_k - S_{j-1} = a_j + a_{j+1} + \cdots + a_k$, so:

$$\sum_{i=l+1}^{k} a_i \equiv 0 \pmod{n}.$$

This means the sum $\sum_{i=j}^{k} a_i$ is a multiple of n.

In both cases, there exist integers j and k with $1 \leq j \leq k \leq n$ such that the sum $\sum_{i=j}^{k} a_i$ is a multiple of n.

(f) **PH dealing with averages:**

i. Suppose $A = (a_1, a_2, \ldots, a_n)$ is a sequence of positive real numbers. Let $H(A)$ denote the *harmonic mean* of A, defined by

$$H(A) = n \left(\sum_{i=1}^{n} \frac{1}{a_i} \right)^{-1}.$$

Show there exist integers i and j, with $1 \leq i, j \leq n$, satisfying

$$a_i \leq H(A) \leq a_j.$$

Solution:
By simply considering the upper-bound and lower-bound of H(A)

ii. Suppose the integers from 1 to n are arranged in some order around a circle, and let k be an integer with $1 \leq k \leq n$. Show that there must exist a sequence of k adjacent numbers in the arrangement whose sum is at least $\lceil k(n+1)/2 \rceil$.

Solution:
Adding up all the k sums n times we get

$$\sum_{k} \frac{kn(n+1)}{2}$$

Then take the average, we show the existence of such k.

iii. Suppose the integers from 1 to n are arranged in some order around a circle, and let k be an integer with $1 \leq k \leq n$. Show that there must exist a sequence of k adjacent numbers in the arrangement whose product is at least $\lceil (n!)^{k/n} \rceil$.

Solution:
Similar process, we take the product to the power of k, then take the geometric-mean of it, gives us the "average" of the k-product, we show the existence of such k.

(g) Find the smallest value of m so that the following statement is valid: Any collection of m distinct positive integers must contain at least two numbers whose sum or difference is a multiple of 10. Prove that your value is best possible.

Solution:
For any integer x, the mod 10 residues are 10 equivalent classes,

$$[0], [1], [2], [3], ..., [9]$$

If we want

$$a - b \equiv 0 \pmod{10}$$

we must have

$$a \equiv b \pmod{10}$$

Then we can rearrange them into the following 6 sets:

$$\{0\}, \{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}$$

If we pick any arbitrary 7 numbers, there must exist at least 2 of them from one set, which makes then either sum or difference divisible by 10 (i.e. a multiple of 10)

(h) Let n be a positive integer. Exhibit an arrangement of the integers between 1 and n^2 which has no increasing or decreasing subsequence of length $n + 1$.

Solution:
Refer to **Erdős–Szekeres Theorem**

See also **Generalized Erdős–Szekeres Theorem**

Let m,n be positive integers. Exhibit an arrangement between 1 and mn, which has no increasing subsequence of length $m + 1$ and no decreasing subsequence of length $n + 1$

(i) Label one disc "1", two discs "2", three discs "3", ..., fifty discs "50". The total number of labeled discs is:

$$1 + 2 + 3 + \cdots + 50 = 1275.$$

What is the minimum number of discs that must be drawn to guarantee at least ten discs with the same label?

Solution:
If we draw all the discs labeled "1" to "9", we have $1 + 2 + \cdots + 9 = 45$ discs. Drawing any nine from each of "10" to "50" gives us an additional $9 \times 41 = 369$ discs. Thus, the 415-th disc will guarantee ten of the same label.

(j) Seventeen people correspond by mail, each with all others. In their letters, they discuss only three topics. Each person corresponds about only one topic. Prove that at least three people must write to each other about the same topic.

Solution:
Choose a person, say Charlie. Charlie corresponds with 16 others. By the Pigeonhole Principle, he writes to at least six people about one topic, say topic I. If any two of them correspond on topic I, then the claim is proved. Otherwise, these six discuss only topics II and III. Choosing another person among them, say Eric, we find three who correspond with Eric on topic II. If any two of them correspond on topic II, we are done. Otherwise, they all correspond on topic III, which proves the claim.

(k) Given any seven distinct real numbers $x_1, x_2, \ldots, x_7$, prove that we can always find two, say a and b, such that

$$0 < \frac{a - b}{1 + ab} < \frac{1}{\sqrt{3}}.$$

Solution:

Define $x_k = \tan a_k$ for a_k satisfying $-\frac{\pi}{2} < a_k < \frac{\pi}{2}$. Divide $(-\frac{\pi}{2}, \frac{\pi}{2})$ into six subintervals. By the Pigeonhole Principle, two of the seven points must lie in the same interval, say $a_i < a_j$. Then,

$$0 < \tan(a_j - a_i) = \frac{\tan a_j - \tan a_i}{1 + \tan a_j \tan a_i} < \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

(l) Let $a_1, a_2, \ldots, a_7$ be nonnegative real numbers such that:

$$a_1 + a_2 + \cdots + a_7 = 1.$$

Define

$$M = \max_{1\le k\le 5}(a_k + a_{k+1} + a_{k+2}).$$

Find the minimum possible value of M.

Solution:

Since $a_1 \le a_1 + a_2 \le a_1 + a_2 + a_3$ and similar relations hold for sums of three terms, we take the maximum over nine expressions, averaging to $\frac{1}{3}$. By the Pigeonhole Principle, at least one is $\ge \frac{1}{3}$, thus $M \ge \frac{1}{3}$. Setting $(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (1/3, 0, 1/3, 0, 1/3, 0, 0)$ attains $M = \frac{1}{3}$.

3. Combinatorial in Number theory

   (a) **Menage Problem**

   We consider the Menage problem where there are $2n$ seats and n couples, with the constraint that no one is allowed to sit beside their spouse.

$$M_n = \sum_{k=0}^{n}(-1)^k(2n)!\frac{2n}{2n-k}\frac{(2n-k)!}{(n-k)!(n-k)!}$$

$$= \sum(-1)^k(2n)!\frac{2n}{2n-k}\frac{(2n-k)!}{(2n-2k)!k!}$$

   still unknow it converges or not

   (b) **Prime Distribution** The number of integers m less than n and coprime to n is given by:

$$\varphi(n) = \#\{m < n \mid \gcd(m, n) = 1\}$$

   This can be computed using:

$$\varphi(n) = n - \sum_{i=1}^{s}\text{number divisible by } p_i \le n + \dots$$

$$= n\prod_{i=1}^{s}\left(1 - \frac{1}{p_i}\right)$$

   Number of Primes $\le n$

   The prime counting function $\pi(n)$ satisfies:

$$\pi(n) = n - \sum_{p_i\le n}\#\text{ divisible by } p_i + \sum_{p_ip_j\le n}\#\text{ divisible by } p_ip_j + \dots$$

   Using an approximation:

$$\sum \frac{n}{p_i} - \frac{n}{p_ip_j} \approx \frac{n}{p_1} - \frac{n}{p_1p_2}$$

   By the prime number theorem:

$$\pi(n) \sim \frac{n}{\log n}$$

   Estimation Difficulty

   The term $(-1)^k$ makes the difference hard to estimate.

4. Derangement

(a) Basics: n objects, the total number of permutation that no elements appear on its previous position

$$!n = (n-1)[!(n-1)+!(n-2)]$$

or

$$!n = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

(b) Deranged Twins

Suppose $n+2$ people are seated behind along table facing an audience to staff a panel discussion. Two of the people are identical twins, wearing identical clothing. At intermission, the panelists decide to rearrange themselves so that it will be apparent to the audience that everyone has moved to a different seat when the panel reconvenes. Each twin can therefore take neither her own former place, nor her twin's. Let Tn denote the number of different ways to derange the panel in this way.

Solution:

Let $!(n)$ be the subfactorial-derangement function. There are $!(n+2)$ ways to derange the $n+2$ persons, however, some of these derangements send one of the twins to the position of the other twin. We classify these cases into three types, which need to be subtracted:

1. If twin 1 gets sent to twin 2's position but twin 2 does not get sent back to twin 1's position, then for each of the remaining n persons, there will be exactly one forbidden seat choice. This results in $!(n+1)$ valid derangements.

2. If twin 2 gets sent to twin 1's position but twin 1 does not get sent back to twin 2's position, then again, for each of the remaining n persons, there will be exactly one forbidden seat choice, giving another $!(n+1)$ derangements.

3. If twin 1 gets sent to twin 2's seat and twin 2 gets sent to twin 1's seat, there are exactly $!n$ ways to arrange the remaining people.

Thus, the total valid arrangements are:

$$t(n) =!(n+2) - 2!(n+1)-!n.$$

However, the two twins are identical, meaning that swapping them does not result in a new arrangement. Since we have counted each such swap twice in our calculations, we must divide by 2 to obtain the correct count:

$$t(n) = \frac{!(n+2) - 2!(n+1)-!n}{2}.$$

Checking for $t(10)$, we obtain $72,755,370$ as expected.

(c)

5. **Catalan Numbers and Dyck Path:**

Consider a path from $(0,0)$ to (n,n) consisting of n right steps (R) and n up steps (U). The total number of such paths is given by the binomial coefficient:

$$\# \text{ Total Paths} = \binom{2n}{n}$$

A path is called a Dyck path if it never crosses above the diagonal $y = x$. To count the number of Dyck paths, we use a bijection that maps paths that go above the diagonal to paths from $(0,0)$ to $(n-1, n+1)$. This leads to:

$$\text{\# Dyck Paths} = \binom{2n}{n} - \binom{2n}{n-1}$$

Using factorial notation:

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}, \quad \binom{2n}{n-1} = \frac{(2n)!}{(n-1)!(n+1)!}$$

Thus, the number of Dyck paths simplifies to:

$$\frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = C(n)$$

where $C(n)$ denotes the nth Catalan number.

The probability that a randomly chosen path does not go above the diagonal (i.e., is a Dyck path) is:

$$P(\text{not above diagonal}) = \frac{\text{\# Dyck Paths}}{\text{\# Total Paths}} = \frac{1}{n+1}$$

6. Mr. and Mrs. Zeta want to name their baby Zeta so that its monogram (first, middle, and last initials) will be in alphabetical order with no letters repeated. How many such monograms are possible?

   Solution:
   We see that for any combination of two distinct letters other than Z (as the last name will automatically be Z), there is only one possible way to arrange them in alphabetical order. Thus, the answer is just

$$\binom{25}{2} = 300$$

7. The student lockers at Olympic High are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost two cents apiece. Thus, it costs two cents to label locker number 9 and four cents to label locker number 10. If it costs \$137.94 to label all the lockers, how many lockers are there at the school?

   Solution:
   Since all answers are over 2000, we work backwards to find the cost of the first 1999 lockers.

   The first 9 lockers cost \$0.18, while the next 90 lockers cost

$$0.04 \cdot 90 = 3.60.$$

   Lockers numbered 100 through 999 cost

$$0.06 \cdot 900 = 54.00,$$

   and lockers numbered 1000 through 1999 cost

$$0.08 \cdot 1000 = 80.00.$$

   This gives a total cost of

$$0.18 + 3.60 + 54.00 + 80.00 = 137.78.$$

   There is

$$137.94 - 137.78 = 0.16$$

   dollars left over, which is enough for 8 digits, or 2 additional four-digit lockers. These lockers are numbered 2000 and 2001, leading to the answer2001 lockers in total.

8. Let n be an odd integer greater than 1. Prove that the sequence

$$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{\frac{n-1}{2}}$$

contains an odd number of odd numbers.

Solution:

The sum of the given sequence is:

$$\frac{1}{2}(\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1}) = \frac{1}{2}(2^n - 2) = 2^{n-1} - 1$$

which is an odd number

9. How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

10. Let

$$x = .123456789101112\ldots998999,$$

where the digits are obtained by writing the integers 1 through 999 in order. Find the 1983^{rd} digit to the right of the decimal point.

11. Integer Partitions

    (a) (Using Generating Functions)

    $$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$$

    (b) An estimation of $p(n)$ as $n \to \infty$

    $$p(n) \sim \frac{1}{4\pi\sqrt{3}} exp(\pi\sqrt{\frac{2n}{2}})$$

12. Let n be a positive integer and S a set of $n^2 + 1$ positive integers with the property that every $(n+1)$-element subset of S contains two numbers one of which is divisible by the other. Show that S contains $n+1$ different numbers $a_1, a_2, ..., a_{n+1}$ such that $a_i | a_{i+1}$ for each $i = 1, 2, ..., n$.

Solutions:

Let n be a positive integer and S a set of $n^2 + 1$ positive integers with the property that every $(n+1)$-element subset of S contains two numbers one of which is divisible by the other. Show that S contains $n+1$ different numbers $a_1, a_2, \ldots, a_{n+1}$ such that $a_i \mid a_{i+1}$ for each $i = 1, 2, \ldots, n$.

Use the divisibility relation to obtain a poset on S (that is, $x \leq y$ iff $x \mid y$). Check that this makes a poset. The condition that there does not exist an $n+1$-element subset of S where no element divides another translates into the condition that there does not exist an antichain of length $n+1$ in S.

Thus, the longest antichain in S has length at most n, and by Dilworth's theorem, S can be written as the union of at most n chains. Since S has $n^2 + 1$ elements, this implies that one of these chains has length at least $n+1$. This proves the result.

13. Show that Every natural can be devided by a Fibonacci number (apart from 1).

Proof:

Define $F_0 = 0$

Fix a positive integer n, now consider the modulo residue, there are n equivalent classes.

Lemma 4.1. *For any natural number n, there exists distinct natural numbers k and k' such that*

$$(F_k, F_{k'}) \equiv (F_{k-1}, F_{k'-1}) \pmod{n}$$

Proof:
Since (F_k, F_{k-1}) can only have n^2 candidates, since we took the modulo of n over the natural numbers. There are n^2 candidate pairs but only n equivalent classes, there must exists k and k' satisfy

$$(F_k, F_{k'}) \equiv (F_{k-1}, F_{k'-1}) \pmod{n}$$

which proves the lemma.

Given $F_n = F_{n-1} + F_{n-2}$, we have

$$F_{k-2} \equiv F_{k'-2} \pmod{n}$$

(We can subtract the index to get a new pairs that satisfies the congruences are modulo n)

By induction, if k and k' satisfy the lemma, so does $k - 1 \in \mathbb{N}$ and $k' - 1 \in \mathbb{N}$, plug them into the lemma, we get

$$F_{|k-k'|} \equiv F_0 \equiv 0 \pmod{0}$$

14. Let $v_i \in \mathbb{R}^3$ and $||v_i|| = 1$. Show that there exists a permutation u_i such that

$$\sum_{i=1}^{n-1} ||u_{i+1} - u_i|| \leq 8\sqrt{n}$$