# Inequality

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# Contents

# 1 Basics

## 1.1 Methodology

1. scaling
2. Algebraic Approach
   - Identity
   - Straight Computation
   - Rotation-Symmetric series pairing

   (a) $|a| \geq a \geq -|a|$
   (b) $|a| - |b| \leq |a \pm b| \leq |a| + |b|$
   (c) $a \geq b$, $b \geq c$ then $a \geq c$
   (d) $a, b \geq 0$, then $a \cdot b \geq 0$
   (e) $a \geq b > 0$, $c > 0$, then $\frac{a}{c} \geq \frac{b}{c}$
   (f) (Lagrange Identity) $\sum_{1 \leq i < j \leq n}(a_i b_j - a_j b_i)^2 \geq 0$
3. Function Aspects
   - Monotone
   - Convexity
   - Tangent-Line scaling
   - Critical Points

   (a) Continuous f on $[a, b]$,if $f'(x) \geq 0$, then $f(a) \geq f(b)$, equality attains when f is constant function.
   (b) $f''(x) \geq 0$ on $[a, b]$, then $r \cdot f(a) + (1-r)f(b) \geq f(r \cdot a + (1-r) \cdot b)$

   Take $s \in [0, 1]$, let $g(s) = rf((1-s)a + s(ra + (1-r)b)) + (1-r)f((1-s)b + s(rb + (1-r)a))$
   Then $g(0) = rf(a) + (1-r)f(b) = LHS$; $g(1) = f(ra + (1-r)b) = RHS$
   And $g'(r) \leq 0$, then $g(0) \geq g(1)$.

   (c) (Jensen) $\sum w_i = 1$, f is convex, then

$$f(\sum w_i \cdot x_i) \leq \sum w_i f(x_i)$$

4. Combinatorial Methods
   - Adjustment/ Polishing
   - Inclusion-Exclusion
   - Method of Area
   - Probabilistics
5. Analysis Methods
   - Taylor Expansion
   - Scaling Integrals
   - Maximum Modulo Principle
6. Adv Algebra Methods
   - Canonical Form
   - Eigenvalues and Rayleigh Quotient

## 1.2 Examples

1. $xyz = 1$, show

$$1 < \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} < 2$$

Notice $S = \frac{3+2(x+y+z)+(xy+yz+zx)}{2+x+y+z+xy+yz+zx} > 1$ is trivial and less than 2 is also trivial by comparing each term.

## 2 Convex-Concave

# 3 Mean Inequality Chain

## 3.1 QM-AM-GM-HM

1. For nonnegative $x_i$,
$$\frac{\sum x_i}{n} \geq (\Pi x_i)^{\frac{1}{n}}$$
2. (Weighted) For nonnegative $x_i$ and $a_i > 0$ with $\sum a_i = 1$, then
$$\sum a_i \cdot x_i \geq \Pi x_i^{a_i}$$

## 3.2 Power Mean

1. Define **Power-Mean** as, for $x_i > 0$, $r \in \mathbb{R}$, $A_r = (\frac{1}{n} \sum (x_i)^r)^{\frac{1}{r}}$

   If $s < r$, then $A_s \leq A_r$, equality attains when $x_i$ are equal.
2. Two Variations
   - $a_i, b_i > 0$, we have
$$\sum \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}$$
   - For $m > 0$, we have
$$\sum \frac{a_i^{m+1}}{b_i^m} \geq \frac{(\sum a_i)^{m+1}}{(\sum b_i)^m}$$

## 3.3 Maclaurin's Inequality

# 4 Cauchy-Schwarz

## 4.1 Cauchy-Schwarz

For real $a_i, b_i$ with $1 \leq i \leq n$, we have

$$(\sum a_i^2)(\sum b_i^2) \geq (\sum a_i b_i)^2$$

## 4.2 Aczel

If $A, B > 0$, $A^2 \geq \sum a_i^2$, $B^2 \geq \sum b_i^2$ then

$$\sqrt{(A^2 - \sum a_i^2)(B^2 - \sum b_i^2)} \leq AB - \sum a_i b_i$$

Sol1: We can apply on $n = 1$ case and convert to the norm and then apply Cauchy.
Sol2: Consider the $\Delta \geq 0$ of $(AX + B)^2 - \sum (a_i X + b_i)^2$

## 4.3 Titu's

For real number $a_i$ and positive $b_i$, we have

$$\sum \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}$$

Equality attains when $\frac{a_i}{b_i} = \frac{\lambda}{\mu}$

## 4.4 Holder

## 4.5 Carlson Inequality

## 4.6 Minkowski Inequality

## 4.7 Examples

1. $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c$ for $a, b, c > 0$
   Directly from Titu

2. $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}$ for $a, b, c > 0$

# 5 Order Inequality

## 5.1 Rearrangement

Given $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, we have

$$\sum_i a_i b_{n+1-i} \leq \sum_i a_{\sigma(i)} b_{\sigma i} \leq \sum_i a_i b_i$$

## 5.2 Chebyshev

If sequence $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_n$ are similarly sorted, we have

$$\frac{1}{n} \sum a_i b_i \leq (\frac{1}{n} \sum_i a_i) \cdot (\frac{1}{n} \sum_i b_i)$$

If one is reversed, the inequality sign reversed.

## 5.3 Majorization

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,
$x \succ y$ if

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i$$

for $k = 1, 2, \ldots, n-1$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

If $x \succ y$ and f is convex, we have

$$\sum_i f(x_i) \geq \sum_i f(y_i)$$

## 6 Bernoulli Inequality

# 7 Schur Inequality

Notice that Schur only works for 3 variable inequalities

## 7.1 Basic Schur

1. For $x, y, z \geq 0$ and $r \in \mathbb{R}$, we have

$$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0$$

2. Two Variations(Algebraic Manipulation)

   - $\sum_{cyc} x^3 - \sum_{cyc}[x^2(y+z)) + 3xyz \geq 0$
   - $(\sum_{cyc} x)^3 - 4\sum_{cyc} x \sum_{cyc} xy + 9xyz \geq 0$

## 7.2 pqr Inequality

Let $p = x + y + z$, $q = xy + yz + zx$, $r = xyz$ where $x, y, z \in \mathbb{R}_+$, we have

- $p \geq \sqrt[3]{r}$
- $q \geq 3\sqrt[3]{r^2}$
- $p^2 \geq 3q$
- $q^2 \geq 3pr$
- $p^3 - 4pq + 9r \geq 0$

# 8 Lagrange Multiplier

# 9 Merge and Adjustment

Given a constraint on sum of $x_i$, we want to find the extremum of $\sum_i f(x_i)$
We first have to identify the convexity-concavity of f, take the second derivative identify the intervals on $I$ where is concave and convex.