Counting

bowen

October 22, 2025

Contents

1 Set Counting

1.1 Sum-free Sets

Definition: A set $A \subseteq \{1, 2, \ldots, n\}$ is *sum-free* if there are no $x, y, z \in A$ such that $x + y = z$.

Example Problem

1. Let $A \subseteq \{1, 2, \ldots, 100\}$ have 51 elements. Show that there exist $x, y, z \in A$ with $x + y = z$.

   We first consider the complement, which is what kind of sets form a sum-free set.
   Then intuitively, we get the upper-half elements ($\{51, \ldots, 100\}$), since $50 + 50 = 100$, which is the threshold term.
   Notice such special set $\{51, \ldots, 100\}$ with size 50,
   if $|A| = 51 > 50$, there must be an element in A, which breaks the balance of this set. (By PH)

2. 6 different countries, with list of members contains 1957 names. Show that there is at least one member whose number is the sum of two members from his own country or twice as large as the number of one mumber from his own country.

   We first translate the problem, we have A_1 to A_6, 6 sets (denote countries) and we want to show among six of them, there is one with the property that

   $$\exists x, y, z \in A_i \ s.t. \ x + y = z$$

   We rewrite the two cases of the problem into one expression of the relation of the elements.
   Then we naturally want to see the complement threshold case
   i.e. for $i = 1, \ldots, 6$, A_i with the property that

   $$\forall x, y, z \in A_i, x + y \neq z$$

   Then we want to find the size for such sets, we compute $\lceil \frac{1956}{6} \rceil = 327$,
   this means there is one set with at least 327 members.
   Since all the sets are symmetric,
   WLOG: let such set be A_1, where $\{x_1, \ldots, x_{327}\}$ with $x_1 < \cdots < x_{327}$.

   Recall that we want to see how does the sets look like when each set is sum-free and with the condition that elements in A_1 is strictly increasing.
   We should try to construct other set upon what we have so far, hence we consider

   $$x_{327} - x_1, \ldots, x_{327} - x_{326}$$

   These elements are ecluded from A_1, we still want to find the size for A_2 to A_6, we compute $\lceil \frac{326}{5} \rceil = 66$, similarly we assign these into A_2 where $A_2 = \{b_1, \ldots, b_{66}\}$
   With the process above, we consider

   $$b_{66} - b_1, \ldots, b_{66} - b_{65}$$

   Similarly we have these 65 elements are in the rest 4 sets.
   Repeating what we have done above, we compute $\lceil \frac{65}{4} \rceil = 17$.
   These 17 elements are in A_3

   We have $\lceil \frac{16}{3} \rceil = 6$ elements in A_4; $\lceil \frac{5}{2} \rceil = 3$ elements in A_5, say $e_1 < e_2 < e_3$

When it comes to A_6, we see the only possible elements are $e_3 - e_2$ and $e_3 - e_1$, but it generates a new element

$$e_3 - e_1 - (e_3 - e_2) = e_2 - e_1$$

which is not in any of the six sets and hence reach a contradiction that such sum-free A_i doesn't hold for 1957 members.

3. Show that every subset of $\{1, \dots, 2n\}$ with more than n elemnts contains two numbers whose sum is $2n + 1$

   We first observe that we want the sum $2n + 1 > 2n$, the largest element in the set $S = \{1, \dots, 2n\}$. Then we consider the greatest sum-free set as the complement of the problem, we get

   $$\{n + 1, \dots, 2n\}$$

   notice the smallest sum of this set is $2n + 2 > 2n$, with the size n.

   Then any subset of S with size $> n$, there exists $x, y \in A$ such that $x + y = 2n + 1$. This is because for the largest sum-free has size n, then with size $n + 1$, the sum-free balance must be violated.

4. (**Fixed Difference sum-free**) One has 77 days to prepare for a tournament, wants to play at least 1 per day but no more than 132. Show there is a sequence of successive days on which he plays exactly 21 games.

   We first translate the problem directly, on each day he can play between 1 to 132 (inclusive) number of games. Since it's an accumulated process, as number of days goes up, the total games goes up as well.
   Naturally, we consider such sequence a_i denotes number of games has played up until day i. Consequently, we have the following strictly sequence

   $$1 \leq a_1 < a_2 < \cdots < a_{77} \leq 132$$

   By assumption, we want to show there are i,j such that $a_j = a_i + 21$ for $i < j$.
   Then it's natural to see what does the set consists of $a_i + 21$ look like and what kind of property does it preserve compared to $S = \{a_i\}$

   We construct the set $S' = \{a_1 + 21, \dots, a_{77} + 21\}$. After writing out, it obvious to see that we want to argue $S \cap S' \neq \emptyset$.
   i.e. we want to see the overlapping parts between S and S' has size > 0.

   Then we have 154 positive integers in $\{1, \dots, 153\}$, then two must take the same value, which means there exists i,j such that

   $$a_i = b_j = a_j + 21$$

   shows the desired result.

   In this type of sum-free, i regard as fix difference sum-free, we want to show there exists two elements in the set with given difference, we construct the scaling sets and argue two must have overlappings, more or less we introduce the prefix sequence to help arranging the possible values before and after the scaling.

5. **General Theorem**

6. (**Dirichlet-Type sum-free**)

Set-up

(i) **Dirichlet Approximation**

Theorem 1.1. *Let $n \geq 1$ and $x \in \mathbb{R}$, there exists a rational number $\frac{p}{q}$ such that*

$$|x - \frac{p}{q}| < \frac{1}{Nq}$$

Proof:
consider $\{x\}$, returns the fraction part for x, by taking x (mod 1), which are $\{0 \cdot x\}, \ldots, \{n \cdot x\} \in [0, 1)$. Then we derive the natural n-partitions on the unit interval, $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \ldots, [\frac{n-1}{n}, 1)$.
By PH, there are $\{m \cdot x\}$ and $\{n \cdot x\}$ such that they are in the same interval, then for some integer p, $(m - n) \cdot x - p$ is bounded by $1/n$.
Set $q = (m - n)$, we get

$$|x - \frac{p}{q}| < \frac{1}{nq}$$

(ii) **Hurwitz's Theorem**

Theorem 1.2. *For every irrational ξ, there exists rational $\frac{m}{n}$ such that*

$$|\xi - \frac{m}{n}| < \frac{1}{\sqrt{5}n^2}$$

(B2) For every $X = \{x_1, \ldots, x_n\}$ real numbers, there exists nonempty subset $S \subseteq X$ and $m \in \mathbb{N}$ such that

$$|m + \sum_{s \in S} s| \leq \frac{1}{n+1}$$

Observe the desired structure, we want to estimate the a sum with an integer, which the should be sufficiently close. Since S is a subset of X and we want to estimate the sum of S with an integer m. This drives us to constructing a list of prefix sums of x_i, denote as $s_k = \sum_{i=1}^{k} x_i$.

Recall what we did in the proof for Dirichlet Approximation, similarly we take $\{s_i\} = s_i - \lfloor s_i \rfloor$, which gives n fractions between the unit interval.
Since x_i is reals can be either positive or negative, in order to derive the partial order, we rearrange into ascending order, say

$$0 \leq \{s'_0\} \leq \cdots \leq \{s'_n\} < 1$$

Then we have two options, consider either

$$\{s'_n\} - \{s'_0\}, \ldots, \{s'_n\} - \{s'_{n-1}\}$$

or the consecutive differences

$$\{s'_1\} - \{s'_0\}, \{s'_2\} - \{s'_1\}, \ldots, \{s'_n\} - \{s'_{n-1}\}, 1 - \{s'_n\}$$

Notice:
(1) The two lists of differences have size n and $n+1$ respectively
(2) Similar to the spirit of Dirichlet, we have $[0, \frac{1}{n+1}), [\frac{1}{n+1}, \frac{2}{n+1}), \ldots, [\frac{n}{n+1}, 1)$, in total $n+1$ partitions on the unit interval.
(3) $\sum_{i=0}^{n-1} s_i + 1 - s_i = 1$, then $avg = \frac{1}{n+1}$, by PH, we find there must exist at least one consecutive

sum with difference boudned by $\frac{1}{n+1}$

(***) The third observation is driven by consecutive sums add up to 1 and what we want on RHS is $\frac{1}{n+1}$ which in this case is the average of the consecutive differences.

Let the desired consecutive sums be s_i, s_j, since $|\{s_i\} - \{s_j\}| \leq \frac{1}{n+1}$, we have $\pm(\{s_i\} - \{s_j\})$.
Now we have $|\{s_i\} - \{s_j\}| \leq \frac{1}{n+1}$, we want to connect LHS with the modulo 1 difference.

Recall that $\{s_i\} = s_i - \lfloor s_i \rfloor$, we want to pick m such that $|m + (s_j - s_i)| = |\{s_i\} - \{s_j\}|$, since $s_j - s_i = \lfloor s_j \rfloor - \lfloor s_i \rfloor + |\{s_j\} - \{s_i\}|$
Therefore we can construct $S = \{x_{i+1}, \ldots, x_j\}$

But keep in mind, there is also a case when $1 - \{s_n'\} \leq \frac{1}{n+1}$, with similar process, we take $S = \{x_1, \ldots, x_n\}$ and $m = -\lceil s_n \rceil$.

Two Estimations for the irrationals similar to Dirichlet Approximation
(i) Show that there exists some integers n such that $n\sqrt{\pi}$ differs by at most $\frac{1}{10^6}$ from its nearest integer.

This follows directly from the approximation by setting $x = \sqrt{\pi}$ and $q = n$, $N = 10^6$

(ii) α be an irrational. Show that for any a, b with $0 < a < b < 1$, there is some positive integer n such that

$$a < \{n\alpha\} < b$$

Methodology

1. General Theorems
   (1) (Vander Waerden) For natural number $k > 2$, the set $\mathbb{N}$ cannot be partitioned into finitely many $k - AP$-free sets
   (2) For a natural number $k > 2$, a $k - AP$-free set hahs density 0.
   (3) The set $\mathbb{N}$ cannot be partitioned into finitely many sum-free sets.

   Recall the Cantor Sets C. C captures all the countable binary sequence

   We define a bijection between Cantor space and the set S of all sum-free subsets of N. Given a sequence x in C, we construct S as follows:

   Consider the natural numbers in turn. When considering n, if n is the sum of two elements already put in S, then of course n is not in S. Otherwise, look at the first unused element of x; if it is 1, then put n into S, otherwise, leave n out of S. Delete this element of the sequence and continue.

2. Translate the problem directly without any extra annotation, identify the partition sets, and what free properties we have about them

3. Then we want to examine the size for each set (is it possible to measure it or compute the threshold size)

4. We try to find the relation between sets, if we have built one of the sets, how should we generate the rest from the *sum-free* property.
   This process is often done via assign partial order to each set, then we can consider $x_i - x_j$ which generates a set of elements not in the current set.

   Alternatively, we can examine and arrange the prefix sum since if x_i are positive, we get $y_j = \sum_{i=1}^{j} x_i$ a nondecreasing sequence.

5. Then we can argue the size or possible arrangement for each set along with PH to see the threshold size or if it leads to contradiction.

   In particular, if we construct consecutive differences, we may want to ask what is the sum, does this gives a nice sum (like constant), by evaluating $\frac{C}{L(n)}$ this might gives an average of the differences, then we can apply PH to show the existence of a pair of difference bounded above/below by the average.

1.2 Chain and Anitchain set

Definition: A *chain* is a poset such that every elements are comparable; an *antichain* is a subset of poset such that any two elements are incomparable

Without introducing to many topology, here we define partial order set $(\mathbb{N}, |)$ with "|" as a partial order between the naturals.
i.e. $a \preceq b$ iff $a \mid b$

We claim $(\mathbb{N}, |)$ forms a poset, and we have the classic divisor chain
Example Problem

1. The Power Tower for integers

   Theorem 1.3. *For any two positive integers a and m, the following sequence is eventually constant modulo m:*
   $$a,\ a^a,\ a^{a^a},\ a^{a^{a^a}},\ \ldots$$

   Proof. The result is trivial if $a = 1$ or $m = 1$, so we may assume that $a \geq 2$ and $m \geq 2$.

   For convenience we use Donald Knuth's arrow notation for the iterated power:
   $$a \uparrow\uparrow n = a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$$

   Its recursive definition is the following: $a \uparrow\uparrow 0 = 1$, $a \uparrow\uparrow (n+1) = a^{a\uparrow\uparrow n}$.

   So we must prove that $a \uparrow\uparrow n$ is eventually constant modulo m. The proof works by induction on m.

   (a) **Basic Step:** If $m = 2$ then obviously $a \uparrow\uparrow n \equiv a \pmod 2$ for every $n > 0$, because $a \uparrow\uparrow n$ has the same parity as a.

   (b) **Induction Step:** Assume that the result is true for every modulo up to $m - 1$. We will prove that it is also true for modulo m.

      i. **Case 1:** If $\gcd(a, m) = 1$, by Euler's theorem
      $$a \uparrow\uparrow (n+1) = a^{a\uparrow\uparrow n} \equiv a^{(a\uparrow\uparrow n) \bmod \phi(m)} \pmod m,$$
      where ϕ = Euler's phi function and $x \bmod y$ = "x reduced modulo y". Since $\phi(m) < m$, by induction hypothesis $(a \uparrow\uparrow n) \bmod \phi(m)$ is eventually constant, hence $\{a \uparrow\uparrow (n+1)\} \bmod m$ is eventually constant.

      ii. **Case 2:** If $\gcd(a, m) = g > 1$ then we write $m = m_1 m_2$, where $\gcd(m_1, m_2) = 1$ and m_1 contains exactly the same prime factors as g, perhaps raised to different exponents. Clearly $a \uparrow\uparrow n \equiv 0 \pmod{m_1}$ for n large enough. If $m_2 = 1$ then we are done, otherwise $1 < m_2 < m$ and $\gcd(a, m_2) = 1$, so by induction hypothesis $(a \uparrow\uparrow n) \bmod m_2$ is eventually constant, say $k = (a \uparrow\uparrow n) \bmod m_2$ for all n large enough. According to the Chinese Remainder Theorem, the following system of congruences
      $$\begin{cases} x \equiv 0 \pmod{m_1} \\ x \equiv k \pmod{m_2} \end{cases}$$
      has a unique solution $x = r \bmod m = m_1 m_2$, hence $a \uparrow\uparrow n \equiv r \pmod m$ for all n large enough.

   This completes the proof.

   □

Remark: The result can be generalized to any tower of exponents with an increasing number of levels, even if the exponents are not all the same: $a_1,\ a_1^{a_2},\ a_1^{a_2^{a_3}},\ a_1^{a_2^{a_3^{a_4}}},\ \ldots$

Corollary (graduate level): $a \uparrow\uparrow n$ has a p-adic limit as $n \to \infty$ for every p.

2. Let $n \geq 1$. From the set $\{1, 2, \ldots, 2n\}$ choose $n+1$ integers. Prove that among the chosen integers there exist two, say a and b, such that $a \mid b$.

   With the same template, we first try to examine how large the set can be in which the are pairwise coprime.
   We introduce "Factor-Max-2" expression, $i = 2^i \cdot m_i$ (m_i is odd).
   Then we translate the problem with out new expression, we want to find how many are there such that

   $$i = 2^i \cdot m_i \nmid j = 2^j \cdot m_j$$

   then $m_i \nmid m_j$, since 2^{i-j} is an integer.
   Note there are exactly n odds, since we know $i = 2^i \cdot m_i$ is unique for each i, then by taking $n+1$ elements from it, there must have one odd that shows up twice.

   Then we can denote them as $i = 2^i \cdot m$ and $j = 2^j \cdot m$, then clearly $i \mid j$ for $i < j$.

3. **(A Stronger Restriction of 2)** Pick any $n+1$ elements from $\{1, \ldots, 2n\}$, there must exists two consecutive integers differ by 1.

   Since each number in the set can only be used once, we consider the disjoint union with difference 1 on the set, which are

   $$\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}$$

   in total of n holes.
   Given that we pick $n+1$ elements from it, there must be two in the same hole, which shows they are consecutive.

4. **(Any two are coprime)** Let T be the set of all positive integer divisors or 2004^{100}. Find the largest possible number of elements that a susbet S of T can have if no element of S is an integer multiple of any other element of S

   First we must observe what is T and what is subset of T. In order to get the generators of T, we must first prime factorize $2004 = 2^2 \cdot 3 \cdot 167$.
   Then we have $T = \{2^x \cdot 3^y \cdot 167^z : 0 \leq y, z \leq 100, 0 \leq x \leq 200\}$ and
   $|T| = (200+1) \cdot (100+1)^2 = (201) \cdot (101)^2$

   We now consider what kind of properties do S have, clearly S is an antichain, since every two elements in S are incomparable ($(s_i, s_j) = 1$ for $i \neq j$)
   Then we have if $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = n$, then $2^{x_1}3^{y_1}167^{z_1} \nmid 2^{x_2}3^{y_2}167^{z_2}$
   Therefore we can translate subset S of T, which is for each n,

   $$S_n = \{2^x 3^y 167^z : x + y + z = n, 0 \leq x \leq 200, 0 \leq y, z \leq 100\}$$

   And we want to find for which n, $|S_n|$ attains maximum, which gives largest subset of T.

   (1) when $n < 200$, which is clearly not the maximum, since the size is smaller than $n = 200$.
   (2) when $n = 200$, claim $|S_n|$ attains the maximum
   (3) when $n > 200$, which is also not the maximum, since the range for x must starts from 1.

To show (3), we apply the Dilworth Theorem.
Starting at each (x, y, z),
if $x + y + z \equiv 0 \pmod 3$, we increases x by 1,
if $x + y + z \equiv 1 \pmod 3$, we increases y by 1,
if $x + y + z \equiv 2 \pmod 3$, we increases z by 1

(Unclear) Here it forms a partition, and each chain will end at a triple in S_{200} uniquely, hence $|S_{200}|$ attains the maximum antichain.

Now we compute $|S_{200}|$, since $x + y + z = 200$, y and z have 101 choices respectively, therefore $|S_{200}| = (101)^2$.

5. **(4 pairwise coprime-free subset)** What is the largest subset of $\{1, 2, \ldots, 30\}$ containing no 4 pairwise coprime integers?

   Given $T = 1, 2, \ldots, 30$ and we are asked to find $S \subseteq T$ such that S has at most 3 pairwise coprime integers
   This leads to the construct of set S such that it's generated by the first 3 primes $(2, 3, 5)$.

   By Inclusion-Exclusion, we obtain $S = \{x \in T : 2 \mid x \text{ or } 3 \mid x \text{ or } 5 \mid x\}$, moreover we have
   $|S| = |\{x \in T : 2 \mid x\}| + |\{x \in T : 3 \mid x\}| + |\{x \in T : 5 \mid x\}|$
   $-|\{x \in T : 10 \mid x\}| - |\{x \in T : 6 \mid x\}| - |\{x \in T : 15 \mid x\}|$
   $+|\{x \in T : 30 \mid x\}| = 22$

   Observe that if we have one more elements added to S, it will create a 4 pairwise corpime integers, since it has another prime factor other than $2, 3, 5$.

6. **(Dual case of 4)** Let $T = \{1, 2, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains 5 numbers which are pairwise coprime.

   Notice we want to find the smallest n such that it contains 5 pairwise coprimes, we first argue the largest set that avoids containing 5 pairwise coprime integers, the first half follows the process above

   We have $S = \{x \in T : 2 \mid x \text{ or } 3 \mid x \text{ or } 5 \mid x \text{ or } 7 \mid x\}$
   By Inclusion-Exclusion, we get $|S| = 216$

   Claim: $n = 216 + 1 = 217$
   We still have to argue why 217 is reachable. We must show that such S contains 5 pairwise coprimes.
   We construct the 5 pairwise tuples

   Suppose not the case, $|S| = 217$ but it has primes ≤ 4, then S has at least 213 composite numbers.
   Since there are in total 220 composites in T, we get $220 - 213 = 7$ composites outside S at most.

   Now we consider all the possible composites that are not divisible by 2,3,5,7. In other words, they can only have prime factors ≥ 11.

   But notice $17^2 = 289 > 280$, the only possible ones are
   $11^2, 13^2, 11 \times 13, 11 \times 17, 11 \times 19, 11 \times 23, 13 \times 19, 13 \times 17$, 8 numbers in total.

   So one of it must be included in S, which creates a 5 pairwise coprime integers.

General Theorems

1. **Dilworth Theorem**

   Theorem 1.4. *if an antichain A has cardinality equal to that of a partition P of the set into chains, then that antichain is maximal.*

Methodology

1. Two frequently used tactics are $n = 2^k \cdot m$ where m is integer; the second is $(n, n+1) = 1$ for nonnegative n.

2. **(Erdos Method)** It mainly focuses on the following situation.
   Let $T = \{1, \ldots, n\}$ and we are asking to find the largest subset S of T such that no k-pairwise coprimes

   We construct $S = \{x \in T : p_i \mid x, \text{ where } p_i \in \{2, 3, \ldots, p_{k-1}\}\}$
   The maximum subset attains and equals to $|S|$

   i.e. we find the set that is generated by first (k-1)-th prime factors in T, which is the desired set, since adding a new integr will introduce new prime factors, which leads to creating a k-pairwise coprime integers.

3. Speaking of example 6, we break into two main steps, step1 finding the complement case, which is using *Erdos Method* to state the largest set with no k pairwise coprime inetegrs set has size $|S|$.
   Then we should show why such size ($|S|+1$) must contain 5 coprime integers, along with the size of original set T, we list all the possible prime factors to create composites that are not in the prime-cover set.

   Mostly, we should land on an argument of number of integers generated by primes not covered in S, must have a larger number than number of composites that are in $T \setminus S$

   Here the pigeons are the composites generated by not-covered primes, the holes are the room for composites in $T \setminus S$, by PH we can see one of the composites must lie in S, which creates a k pairwise coprime integers.

1.3 Square-free or Perfect Power-Free Sets

Definition: A set is *square-free* if no element is divisible by a perfect square greater than 1.

Example Problem:

1. Find the largest subset of $\{1, 2, \ldots, 30\}$ where no element is divisible by a perfect square > 1.

   Let n be a positive integer. A number $m \le n$ is *square-free* if it is not divisible by p^2 for any prime p.

   Step 1: Define sets of multiples of squares

   Let
   $$P = \{p \text{ prime} : p^2 \le n\}.$$
   For each $p \in P$, define
   $$A_p = \{k \in \{1, 2, \ldots, n\} : p^2 \mid k\}.$$
   Then the set of numbers divisible by some perfect square > 1 is
   $$\bigcup_{p \in P} A_p.$$

   Step 2: Apply Inclusion-Exclusion Principle

   By the inclusion-exclusion principle, the number of integers divisible by at least one p^2 is
   $$\left| \bigcup_{p \in P} A_p \right| = \sum_{p \in P} \left\lfloor \frac{n}{p^2} \right\rfloor - \sum_{p<q} \left\lfloor \frac{n}{p^2 q^2} \right\rfloor + \sum_{p<q<r} \left\lfloor \frac{n}{p^2 q^2 r^2} \right\rfloor - \cdots$$

   Step 3: Count square-free numbers

   Hence, the number of square-free numbers $\le n$ is
   $$\#\{1 \le k \le n : k \text{ square-free}\} = n - \left| \bigcup_{p \in P} A_p \right|.$$

2. A set M has 48 distinct positive integers, for each element in M, its prime factors are all < 30. Show that there exists 4 distinct positive integers such that their product is a perfect square.

   First we can denote the elements in M as, $m = \Pi p_i^{\alpha_i}$, moreover $m_1 \cdot m_2 \cdot m_3 \cdot m_4 = \Pi p_i^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$ to be a perfect square requires
   $$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv 0 \pmod 2$$
   Then we can see only the degree for each prime factor really matters here.

   Hence we can naturally construct 48 binary vector, each entry is 1 if the correspond prime factor has odd degree else 0. Obviously we have $2^{10} = 1024$ possible binary vectors, since there are in total 10 primes within 30.

   Notice number of ways to pick two vectors (unordered) is $\binom{48}{2} = 1128 > 1024$, which means there are distinct vector pairs $(v_1, v_2) = (v_1', v_2')$, then clearly $v_1 + v_2 + v_1' + v_2' = 2(v_1 + v_2) \equiv 0 \pmod 2$.

   But we still have to consider what if $\{v_1, v_2\} \cap \{v_1', v_2'\} \ne \emptyset$, in this case, they can only share exactly one elements, say $v_2 = v_2'$.
   It remains to argue it's possible to get v_1, v_1' whose sum is even. Then we compute $\binom{48-2}{2} = \binom{46}{2} > 1024$,

by pigeonhole agian we get $v_1 + 2v_2 + v_1' \equiv 0 \pmod 2$.

Therefore in both cases, we show the desired.

In this problem, after we convert each element in M to be a binary string, we want to show that there are 4 vectors whose entries sum up to an even number.
We need to keep in mind binomial counts the distinct pairs, but the order can have overlappings, hence we need to take one order is the same case into consideration, and apply PH again to the rest two degree to see, they are possible to be the same.

3. A set M of 1985 distinct naturals. None of which has a prime divisor > 26. Show M contains at least one subset of 4 distinct elements whose product is the fourth power of an integer.

   We first get the list of all primes within 25, $\{2, 3, \ldots, 23\}$, in total 9 prime factors.
   After setting up the degree vector for each elements in M, since it's asking for fourth power of an integer.

   It's natural to consider the mod 4 entry vectors. But with rough estimation, 4^9 is too big compared to 1985.
   which means we have to return to considering binary entries vectors, since we compute and find out $1985 > 2^9$.

   In $\mathbb{F}_2^9$ space, there must be two vectors are equal say v, u, where the sum of entries of u,v add up to be even.

   Since the question is asking about fourth power, we consider to remove v,u and add $v + u$ to the new list M_2.
   We repeat such process until M reaches 511 elements, the threshold of 2^9.

   Now we count the size of M_2, which is $\frac{1958-511}{2} = 737$, which means there are 737 perfect square in M_2.

   We apply PH again to obtain the case when two squares are equal, say $v+u = v'+u'$, then the integers denoted by v,u,v',u' have product to be divisible by 4, since $v+u \equiv 0 \pmod 2$ and $v'+u' \equiv 0 \pmod 2$ and $v + u = u' + v' \pmod 4$

   The delete and add to a new list process makes full use of the fourth power, we build fourth power from perfect square set.

4. **(Generalization of 3)** if we know the prime divisors of all elements in M are amongst $p_1, \ldots, p_n$, and M has at least $2^n \cdot 3 + 1$ elements, then it contains at least one subset of four distinct elements whose product is a fourth power.

   According to what we have built so far, we convert each element in M to be represented by it's degree, which forms a binary n-tuple.

   Recall what we did in the previous problem, since $|M| = 2^n{\cdot}3{+}1 > 2^n{+}1$, there exists two elements in m with the same binary n-tuple, who represents $m_1, m_2 \in M$ respectively, then $m_1 \cdot m_2$ is a perfect square.

   We then want to keep track of how many elements are removed, which is $\lfloor \frac{|M|}{2} \rfloor$ number of pairs added to the M_2 list.
   Plug $n = 3 \cdot 2^n + 1$, we get $|M_2| = 2^n + 1$ pairs of perfect square in M_2.

Notice that number of binary n-tuple has in total 2^n, but we have 2^n+1 elements in M_2, by PH there must have $m_1m_2 = m_1'm_2' \pmod 2$, whose product is a fourth power of an integer.

Methodology

1. In this situation, we often use $n = \Pi_{i=1}^k p_i^{\alpha_i}$, then our naive approach will be defining
$$\Pi_{i=1}^k p_i^{\alpha_i} \mapsto (\alpha_i \pmod m)_{i\in I} \in (\mathbb{Z}/_m\mathbb{Z})^k$$
And compare the number of vectors in such set with the number of elements in the original set
i.e. compare m^k with $|n|$

   Hence we convert it to additive vector problems in finite abelian group, where we can consider the sum-free cases.
i.e. for given many of vectors, we want to find number of vectors whose coordinates add up to 0 $\pmod m$

2. After converting to tuples, we can regard it as a sum-free type problems on the degree.

3. After converting the problem, we deduce to zero-sum subsequence problem in G, where we can introduce PH or EGZ.

   Zero-Sum Erdos Method

   - **Zero-Sum:** We have a sequence of vectors in finite abelian group G, we want to show that for given enough elements, some susbequence sum up to $e_0 \in G$
   - **EGZ Theorem**

     Theorem 1.5. *In $\mathbb{Z}/_n\mathbb{Z}$, for multiset of size n, every subset of size $k = 2n-1$ has k elements whose sum is 0.*

   Cauchy- Davenport Theorem

   Theorem 1.6. *For any nonempty subsets A,B of $\mathbb{Z}/_p\mathbb{Z}$, we have*
$$|A+B| \geq min\{p, |A|+|B|-1\}$$
where $A+B = \{a+b \pmod p) : a \in A, b \in B\}$

   Good to know:
Restricted Sumset: $S = \{\sum_{i=1}^n a_i : a_i \in A_i \text{ and } \Pi_{1\leq i<j\leq n}(a_i - a_j) \neq 0\}$

   where $A_i \subseteq F$ finite subset of a field F.

   Davenport Constant: The smallest number such that every sequence of such length contains a nonempty subsequence add up to 0.

   Notice:
$D(\mathbb{Z}/_n\mathbb{Z}) = n$
$D((\mathbb{Z}/_p\mathbb{Z})^k) = 1 + k\cdot(p-1)$

   Then if we have more than $D(G)$ elements, such subset must exists.

4. If after $\pmod m$, the number is still too large, we can try to build up to such power from small degrees.
In the example, we can build up to x^4 via applying PH on the set for $x^2 \equiv 0 \pmod 2$

   Similarly, if we want to build up to x^6, we can apply PH on the set $x^3 \equiv 0 \pmod 3$

1.4 Extremal On Sets

Main Areas:

1. **Non-Intersection**

   Let $|S| = n$, if $F = \{A_i\}_{i \in I}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \subseteq S$.

   We may consider
   (a) How large F can be.
   (b) If no subset of S can be added to F, how small it can be.
   (c) For given n, what's the number of maximal F.
   (d) How many F are there for given size n.

   Theorems and Results:
   $S = \{1, 2, \dots, n\}$

   (a) Maximal size of F is 2^{n-1}

   (b) Maximal Families is $2^{(2^{n-1}(1+o(1)))}$

2. **Size-Limited Intersection**

   We may have the restriction such as $|A_i \cap A_j| \leq / = / \geq k$

   Theorems and Results:

   (a) **Katona**

   Theorem 1.7. *Let $S = \{1, 2, \dots, n\}$, $\forall A, B \in \mathcal{F}$ with $|A \cap B| \geq k$. The maximum possible size of $\mathcal{F}$ attains when taking*
   (1) all subsets of size $\geq \frac{n+k+1}{2}$ when $n + k$ is odd
   (2) all subsets of size $\geq \frac{n+k}{2} + 1$ and some with size $\frac{n+k}{2}$ when $n + k$ is even

   (b) **EKR**

   Theorem 1.8. *If $n \geq 2k$ and $\mathcal{F}$ is a family of k-elements subset of $\{1, 2, \dots, n\}$ with pairwise disjoint, then*
   $$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

3. **Intersection and Rank Limitations**

   We often restrict both the size of the set and the size of intersection, such as $|A_i| = k$ and $A_i \cap A_j = \emptyset$

   Packing Problem
   Given a container (graph, set) and a family of objects (subsets, shape, subgraphs), a *packing* is a collection of these objects such that pairwise disjoint or non-overlapping.
   The packing problem asks for the maximum number of objects we can pack.

   We have to distinguish a covering and a packing.
   A covering have to contain the entire space, but a packing is just a collection of objects contained in the space with pairwise disjoint property.

4. **Containment Limitations**

   No one set contains another, forms an antichain more or less

   Theorems and Results

   (a) **Sperner's Theorem**

   Theorem 1.9. *Let $\mathcal{F} \subseteq 2^{[n]}$, where $[n] = \{1, 2, \dots, n\}$ and elements in $\mathcal{F}$ satisfy $\forall A, B \in \mathcal{F}$, $A \not\subseteq B$, then*

   $$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

   In other words, $\mathcal{F}$ is inclusion-free set or forms an antichain with partial order $A \subseteq B$ iff $A \preceq B$

   Equality attains when we take all the sets of size $\lfloor \frac{n}{2} \rfloor$

   (b) **LYM**

   Theorem 1.10. *If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then*

   $$\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 1$$

   The intuition is that each of the $n!$ maximal chains intersects $\mathcal{F}$ at most one set.
   Each k-element set appears in $k!(n-k)!$ chains, giving a weight $\frac{1}{\binom{n}{k}}$
   Averaging over all the chain gives the desired.

   A consequence is that if each element of $\mathcal{F}$ has size k, then

   $$|\mathcal{F}| \leq \binom{n}{k}$$

   Related to Coding Theory:

5. **Union and Intersection Restrictions**
   – Involves Boolen Algebra –

6. **Otherwise (Miscellany)**
   Consider the 2-colouring ground set

Example Problem

1. Given $S = \{1, 2, \dots, n\}$, How many pairwise disjoint subset of n are there? (Ordered pairs)

   Here we provide 2 approaches

   (1) DP
   We loop over all the elements in S, for each element x, it has 3 possible arrangements (with equal possibilities), append to list A or append to list B or append to list C (neither in A nor B).
   Since each x is independent,we can apply the multiplicative rule,
   Hence # total $= 3 \cdot \dots \cdot 3 = 3^n$

   (2) Direct Counting
   We make the partition sets first, pick a susbet of S, then pick the second subset from the uncovered elements
   For each size m subset, number of pairwised disjoint subsets is $\binom{n}{m} \cdot 2^{n-m}$
   Then the total is $\sum_{m=0}^{n} \binom{n}{m} \cdot 2^{n-m} = 2^n \cdot \sum_{m=0}^{n} \binom{n}{m} \cdot \frac{1}{2^m} = 2^n \cdot (1 + \frac{1}{2})^n = 3^n$

2. (A Variation of 1) Given $S = \{1, 2, \dots, n\}$, what is the number of unordered pairs of disjoint subsets of S?

   Recall from (1) that # ordered pairs of disjoint subset is 3^n, since the only case for $A = B$ is $\{(\emptyset, \emptyset)\}$, then there are $3^n - 1$ pairs of disjoint sets in the form of $\{(A, B), (B, A)\}$.
   Since we want unordered pairs, we get $\frac{3^n-1}{2}$.
   Notice $\{(\emptyset, \emptyset)\}$ also counts as one case, we eventually have

   $$\frac{3^n - 1}{2} + 1 = \frac{3^n + 1}{2}$$

3. Let $\mathcal{F} \subseteq 2^{[n]}$ such that for $i \neq j$, $A_i \cap A_j \neq \emptyset$. Show that $|\mathcal{F}| \leq 2^{n-1}$.

   We recall that $\forall S \subseteq \{1, 2, \dots, n\}$, $S \cap \overline{S} = \emptyset$, so only one of $S, \overline{S}$ is included in $\mathcal{F}$.
   Since there are total 2^{n-1} subsets of $\{1, 2, \dots, n\}$, there are at most 2^{n-1} possible candidates in $\mathcal{F}$, which gives the statement.

4. What is the largest family of subsets of $\{1, 2, 3, 4\}$ where no set contains another?

5. For any 10 distinct elements from $S = \{1, 2, \dots, 60\}$, there exists two disjoint nonempty subsets A,B such that

   $$sum(A) = sum(B)$$

   We first consider in what range the sum of 10 distinct elements are, which gives

   $$1 + \cdots + 10 = 55 \leq S \leq 555$$

   In total 501 possible sums.

   But we have $2^10 = 1024$ possible subsets for each fixed 10 elements set.
   Since $\lfloor \frac{1024}{501} \rfloor \geq 2$, then there exists two susbets with the same sum.

6. Let $\mathcal{F} \subseteq 2^{[n]}$. Suppose for all $A, B \in \mathcal{F}$, $|A \cap B|$ is even, find the maximum size of $\mathcal{F}$.

   Recall the characteristic vector for each subset $A \subseteq \{1, 2, \dots, n\}$,
   $a_i = \begin{cases} 1 & i \in A \\ 0 & \text{otherwise} \end{cases}$
   In other words, we define a mapping $A \mapsto (a_1, \dots, a_n)$

   We try to sketch the condition $|A \cap B|$ is even, which means for any two n-dimensional vector v_A, v_B, they share even numbers of 1.
   To relate this with the two vectors, we have $v_A, v_B = 0 \pmod 2$

   Then we observe that $|\mathcal{F}|$ is self-orthogonal, which gives

   $$V \subseteq V^{\perp}$$

   Then we have $|\mathcal{F}| \leq 2^{dim(V)} \leq 2^{\lfloor \frac{n}{2} \rfloor}$

   To see such dimension is attainable.

7. How many subsets of $S = \{1, 2, \dots, n\}$ has no two successive elements?

   We first try with small numbers. Let $N(n)$ denotes number of subsets of $\{1, 2, \dots, n\}$ with no successive numbers.
   We can easily get $N(0) = 1$, since the emptyset itself satisfies.
   For $\{1\}$, we can see that both $\{1\}$ and $\emptyset$ satisfy which gives $N(1) = 2$.
   For $\{1, 2\}$, we can observe that $\emptyset, \{1\}, \{2\}$ satisfy, which gives $N(2) = 3$.

   By educational guess, we may consider cases, since $n, n+1$ cannot show up in the same subset, then we notice that for the subsets include $n+1$, we are just computing $N(n-1)$ and when the subsets don't include $n+1$ is just computing number of successive-free subsets of $\{1, 2, \dots, n\}$ which gives $N(n)$.

   Then we find a Fibonacci Type sequence, notice that $N(n+1) = N(n-1) + N(n)$, then we have

   $$N(n) = F_{n+1}$$

   Skipping the steps for OGF, we get

   $$N(n) = \frac{(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}}{\sqrt{5}}$$

8. Let $|X|$ be a finite set with subsets A_i where $i = 1, \dots, 1066$. If $|A_i| > \frac{|X|}{2}$, then there exists $x_1, \dots, x_{10}$ such that for every element at least one of A_i's contains it.

   Sol1: Probabilistics Method

   For each A_i, denote $Pr[\text{all chosen 10 elements avoid } A_i] = \frac{\binom{|X-A_i|}{10}}{\binom{|X|}{10}} < \frac{\binom{\lfloor \frac{|X|}{2} \rfloor}{10}}{\binom{|X|}{10}}$

   Now we consider $\frac{\binom{\frac{x}{2}}{10}}{\binom{x}{10}} < 1$ as $|X|$ grows.

   Then we have $Pr[\bigcup_{i=1}^{1066} A_i] \leq \sum_{i=1}^{1066} Pr[A_i] < 1066 \cdot Pr[A_i] < 1066 \cdot \frac{\binom{\rfloor \frac{|X|}{2} \rfloor}{10}}{\binom{|X|}{10}} < 1$

9. For given positive integer n. Let $X = \{1, 2, \dots, n\}$, $A_1, \dots, A_r$ are the pairwise disjoint subsets of X with size 3 such that

   - $S(A_i) \leq n$ for $1 \leq i \leq r$
   - $S(A_i) \neq S(A_j)$ for $1 \leq i < j \leq r$

   where s denotes the sum of the set.
   Find the maximum value of r.

   Obsere that by pairwise disjoint, we have $|\bigcup_{i=1}^{r} A_i| = 3r$.
   Moreover, we can bound the sum from assumption

   $$1 + 2 + \cdots + 3r \leq \sum_{i=1}^{r} S(A_i) \leq n + (n-1) + \cdots + (n - (r-1))$$

   Then we have a restriction on r, which is

   $$r \leq \lfloor \frac{1}{5}(n-1) \rfloor$$

Now we try to consider the restriction on the sum, to see if out attempt is correct
i.e. try to construct partitions on $\{1, 2, \dots, 3r\}$

To construct $S(A_i) \neq S(A_j)$ for $1 \leq i < j \leq r$, it's natural to construct $S(A_i)$ as arithmetic sequence.

In order to achieve this, we use $2r+1, 2r+2, \dots, 3r$ to create step '1' arithmetic sequence.
then we are forced to arrange $1, 2, \dots, 2r$ into r partitions but has to satisfy they all have the same sum.

The construction is natural by assigning $\{i, 2r-(i-1)\}$

Now the constrcution is clear, with $A_i = \{i, 2r-(i-1), 2r+i\}$ and the maximum sum is $5r+1 \leq (n-1)+1 = n$

Such construction shows the maximum $\lfloor \frac{1}{5}(n-1) \rfloor$ is attainbale.

Methodology

1. For $\mathcal{F} \subseteq 2^{[n]}$, if we are restricted with pairwise disjoint or intersection condition, we can try to evaluate via unordered pairs $(S, \overline{S})$, there are 2^{n-1} such pairs for nonempty subset.

   Since $S \cap \overline{S} = \emptyset$ for free, which motivates us to partition $[n]$ with 2^{n-1} possible ways.

2. Reduce numbers modulo n.

3. Apply pigeonhole principle on residues: either 3 numbers share the same residue, or one of each residue $\implies$ sum divisible by 3.

2 Lattice, Chessboard, Matrices

Definition: Mostly we consider $n \times n$ blocks and arranging numbers to each block, or doing general 'rotation', 'reflection' to some of the entries on the board.

Example Problem

1. Let m, n be positive integers with $m < 2001$, $n < 2002$, there are 2001×2002 distinct real numbers. Arrange them into 2001×2002 chessboard such that there are exactly one in each block.
   We define a 'Bad' block if

   - The number in the block is less than at least m numbers in the same column
   - The number in the block is less than at least n numbers in the same row

   For every arrangement,find the minimum of the number of such 'Bad' blocks S.

   We first consider a natural arrangement, which assign each number from top-left to bottom-right. Since we have in total 2001×2002 distinct real numbers, this gives

   $$S \geq (2001 - m)(2002 - n)$$

   Notice that the minimum of S doesn't rely on 2001,2002, then we can try to show that on $p \times q$ board with $m < p$ and $n < q$, we have

   $$S \geq (p - m)(q - n)$$

   So far we should observe a few things, we want to show S is bounded by $(p - m)(q - n)$ for fixed m,n, and S depends on the value of p,q.

   We should consider strong induction, but on $p \cdot q$ or $p + q$?
   In the grid case, it's natural to think about doing induciton on $p \cdot q$, but then we should keep track of the extra lines and we are unsure if $pq + 1$ can be factorized.
   Moreover, we should keep the consecutive generating idea in mind, try to examine what if i have extra row or extra column, how S will be affected.

   These gives us a hint to do strong induction on $p + q$.
   i.e. we want to show $p' + q' < p + q$ holds, then $p + q$ holds.

   We first check the base case. When $p = m + 1$ and $q = n + 1$, clearly we see $S \geq 1$.

   Assume it holds for all p', q' with $p' + q' < p + q$, then for such p', q', we have

   $$S_{p',q'} \geq (p' - m)(q' - n)$$

   In strong induction problems, it's natural to think about how to build from $S_{p-1,q}, S_{p,q-1}, S_{p-1,q-1}$ to $S_{p,q}$, since the previous three terms satisfy the induction hypothesis.

   Recall that we want

   $$S_{p,q} \geq (p - m)(q - n)$$

   and to generate such inequality we need to rely on
   (1) $S_{p,q-1} \geq (p - m)(q - 1 - n)$
   (2) $S_{p-1,q} \geq (p - 1 - m)(q - n)$
   (3) $S_{p-1,q-1} \geq (p - 1 - m)(q - 1 - n)$

Observe that we need either $S_{p,q-1} + p - m$ or $S_{p-1,q} + q - n$ to get the desired.
Now we want to rigorously construct a row or a column with exactly $p - m$ or $q - n$ **Bad-blocks**, but most importantly adding such extra row or column cannot affect the **Bad-blocks** in previous grids.

In such problems, we shall boldly assume what we want is **TRUE** and try to argue.

To strip the notation, we define **Bad-row-block** as a block that is less than at least m blocks in his row, **Good-row-block** otherwise.

Similarly, we can define **Bad-column-block** and **Good-column-block**.
Then a **Bad-block** means it's a **Bad-row-block** and a **Bad-column-block** at the same time.

In order to not affecting the previous small grids, such **Bad-blocks** in such row or column must be in the same row/column as in the previous grid.
Now we claim that

Lemma 2.1. *In $p \times q$ board, either there is a row with no **Bad-row-Good-column-block** or a column with no **Bad-column-Good-row-block***

We break into cases
(1) If every block is **Bad-row-Good-column-block**-free or **Bad-column-Good-row-block**-free, then every block in such row/column is good.

(2) If on the board both, there are blocks **Bad-row-Good-column-block** and blocks are **Bad-column-Good-row-block**. Consider those blocks, and we denote the minimum as block A with x.
WLOG, assume A is **Bad-column-Good-row-block**.
If there is a **Bad-row-Good-column-block** say B with y in it. By minimality of x, we have

$$x < y$$

But A is a **Good-row** block and B is a **Bad-row** block, then

$$x > y$$

which leads to contradiction. $\rightarrow\leftarrow$

Now we consider the inductive hypothesis.
On $p \times q$ board
WLOG, There is a row has no **Bad-row-Good-column-block**.
Removing such row, we get $(p-1) \times q$, by induction assumption, we have

$$S_{p-1,q} \geq (p - 1 - m)(q - n)$$

Now adding back such row, since it's a row with no **Bad-row-Good-column-block**, the **Bad-row** block must be **Bad-column** as well, and it doesn't affect the previous **Bad-blocks**.
Then such row must have $q - n$ **Bad-blocks**

Therefore $S_{p,q} \geq (p - 1 - m)(q - n) + q - n = (p - m)(q - n)$.

2. In a $n \times n$ chessboard ($n \geq 3$). Color r blocks with red and the rest with white.
Define the following process,
if a white block is adjacent to at least 2 red blocks (share sides), then color it red. If there exists such white blocks, we continue the colouring.
If no matter which r blocks we colour to red at the beginning, we can't colour all the blocks to red.

Find the maximum value of r.

We first sketch in 3×3 or relative small case.
We have two observations

- If we place n red blocks on the diagonal, we can colour all the blocks
- If two red blocks share a side, we cannot colour any white

Then it's natural to consider the case that is slightly less than the extreme case, $r \le n-1$

Then we should argue $r = n-1$ is attainable

Sol1:
We first examine all the red boundaries for $n-1$ red blocks, which gives an upper bound $4n-4$ red boundaries.
We observe that if we gain an extra red block via two red blocks, we lose 2 red edges from the red boundary, which contributes -2, but at the same time we obtain 2 more edge from the new red blocks, which contribute $+2$.

Hence in total, the size of red boundary is an invariant via the colouring process.

But eventually if we colour all the blocks to red, we get red boundary of size $4n > 4n-4$, which shows that $r = n-1$ is attainable.

Sol2:
We define the following Grid-Adjacent Graph:
If white block A is coloured red effected by red block B, then we connect the block A with block B.

Notice there are at most $2n \cdot (n-1)$ such edges in each row/column and each white block is coloured red affected by two red blocks, hence we have at most $\frac{1}{2} \cdot 2n(n-1) = n(n-1)$ edges in the board.
To colour all the blocks red, we need at least $n^2 - n(n-1) = n-1$ edges, which gives $r \le n-1$.

3. A $m \times n$ chessboard is 2-coloured randomly, each block as prob $\frac{1}{2}$ to be coloured either red or black. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than mn/8.

   In order to show $E[|X|] > \frac{mn}{8}$
   We introduce the adjacent graph simmilar to what we did in Sol2 in the previous problem.

4. Consider $n \times n$ chessboard. Assigning $1, \ldots, n^2$ to each block. Find the minimum of r_n for aritrary arrangement, where r_n is the maximum of the absolute value of the difference of adjacent blocks for each colouring.

   It's natural to consider the arrangement in increasing order, which gives us $d \in \{1, n\}$, and $r_n = n$

   To show it's the optimal, suppose $r_n \le n-1$
   Since We want to land on a contradiction, then we should try to partition $\{1, 2, \ldots, n^2\}$.

   Let $1 \le k \le n(n-1)$. Define $A_k = \{1, \ldots, k\}$, $B_k = \{k+1, \ldots, k+(n-1)\}$, $C_k = \{k+n, \ldots, n^2\}$
   Notice $|A_k| = k, |B_K| = n-1, |C_K| = n^2 - (k+n) + 1$

5. 40 unique numbers from 0 to 99 are written into each of the first 4 rows of a 10×10 grid so that in every row and column no two last digits are the same. Show that we can write the remaining 60 numbers into the remaining 60 empty grids so that in every row and column, no two last digits are the same.

   We first list.

2.1 Geometry Division and Tiling

Definition: We want to find the property for arbitrary partition on the object or using particular geometry to cover all the object.

2.2 Non-Attacking Rook Type

Definition: We want to find the number of arrangements to arrange the objects on the board such that the satisfies some rules (non-attacking queen or rook).

2.3 Traveling Problems

Definition: For any given plane ((lattice) plane, chessboard, euclidean geometry object), most of the time will define the moving rule for the object. Such rule can be each time it can move $NWSE, NW$ etc, or Zig-zaging along x-axis or in local nbhd.

2.4 Colouring

Definition: We will define a way of colouring such that we want to show the existence of same property for arbitrary colouring or we will define a restriction to the colouring to show extremal

Example Problem

1. Find the smallest natural number n such that for any 2-colouring on the edges of K_n, there exists 2 nonadjacent monochromatic triangles with the same colour.

   In order to find two nonadjacent monochromatic triangles with the same colour, the sufficient condition is to find 3 monochromatic triangles pairwise nonadjacent.
   By pigeonhole, there must exist 2 chromatic triangles with the same colour.

2.5 General Graph Approach

Definition: Sometimes we can convert a chessboard or a matrix into a bipartite graph or other graphs (Hamilton, Eulerian, Cycle, K_n, Tree, Path ,etc)

Basics

1. Maximal and Maximum
2. Vertex & Edge Connectivity
3. Matching Maximum Matching
   Maximum Matching
   Perfect Matching
4. Induced Subgraph

Theorem

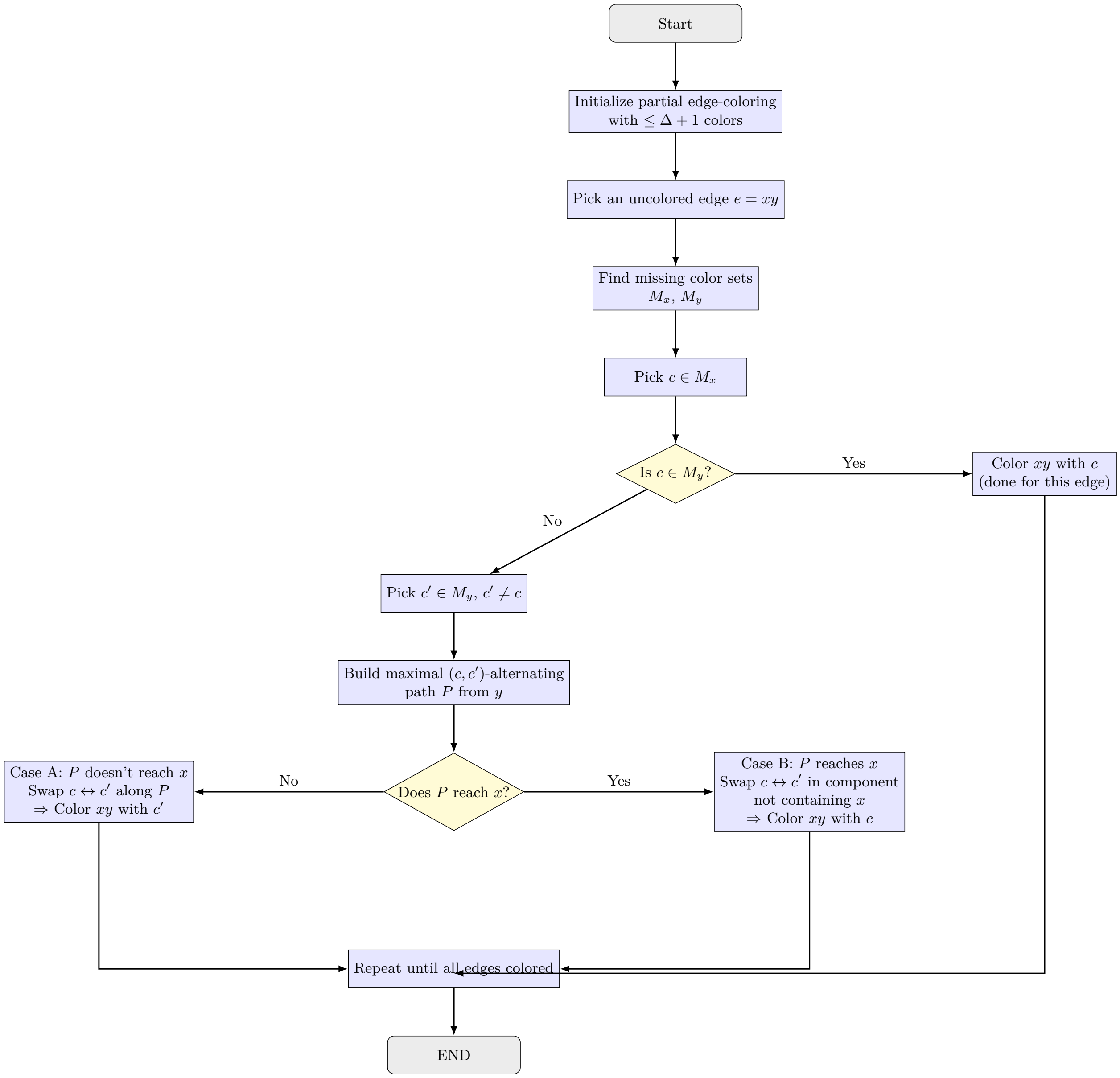
1. **Handshaking Lemma**

   Theorem 2.1. *On connected graph G,*

   $$\sum_{v \in V} deg(v) = 2|E|$$

2. **Hall's Marriage Theorem**
3. **Konig's Theorem**
4. **Max-Flow-Min-Cut Theorem**
5. **Menger's Theorem**

6. Vizing's Theorem



Example Problems

1. The minimum number of edges in a n vertices graph is $n-1$

   Every n vertices k edges graph has at least $n-k$ connected component. Then every n vertices graph with $n-1$ edges has at least two connected components and are disconnected.

   Consider the contrapositive, we have any n vertices graph, it has at least $n-1$ edges. Equality attains when G is P_n.

2. If G is a simple n vertex graph and $\delta(G) \geq \frac{n-1}{2}$, then G is connected.

   It suffices to show that if u,v are not adjacent, then the are not connected.
   Pick $u, v \in V(G)$, in simple connected graph G, $|N(v)| \geq \delta(G) \geq \frac{n-1}{2}$, and similar for u.

   When u,v are disconnected, then by inclusion-exclusion,

$$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq 2 \cdot \frac{n-1}{2} - (n-2) = 1$$

3 Invariant

3.1 Round Table Arrangement

3.2 Erasing Numbers