Ordinary Differential Equation

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1 First Order Differential Equation

1.1 Linear First Order

$$y' + p(t)y = g(t) \tag{1}$$

Solution :

$$y = \frac{1}{u(t)} \int g(t)u(t)\,dt + c \tag{2}$$

where $u = e^{\int p(t)\,dt}$

!!! coefficient of y' has to be 1 !!!

1.2 Separable Equations

$$\frac{dy}{dx} = p(x)q(y) \tag{3}$$

Take Integral on both sides, then try to express y either implicitly or explicitly

1.3 Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \tag{4}$$

Step 1 : Test Exactness
$\frac{\partial M}{\partial y} \quad =? = \quad \frac{\partial N}{\partial x}$

If it's not exact : $\frac{\partial M(\Phi)}{\partial y} = \frac{\partial N(\Phi)}{\partial x} \qquad \Phi = x^m y^n$
Step 2 :

$$\phi(x,y) = \int M(x,y)\,dx \tag{5}$$

Step 3 :

$$\frac{\partial \phi}{\partial y} = N(x,y) \tag{6}$$

Get h'(y) from equation(6)
Step 4 :

$$h(y) = \int h'(y)\, dy \tag{7}$$

Step 5 :
Express the solution

$$\phi(x, y) = F(x) + h(y) \tag{8}$$

With given initial value, $\phi(x, y)$ is constant

1.4 Bernoulli Equation

$$y'(x) + p(x)y(x) = q(x)y^n(x) \tag{9}$$

Step 1 :
$y^{-n}(x)y'(x) + p(x)y^{1-n} = q(x)$
Step 2 :
$v = y^{1-n}(x)$
$v' = \frac{1}{1-n}y^{-n}(x)y'(x)$
Step 3 :

$$v' + \frac{p(x)}{1-n}v(x) = \frac{q(x)}{1-n} \tag{10}$$

Step 4 :
Solve the linear equation

1.5 Homogeneous Equation

$$y' = F(\frac{y}{x}) \tag{11}$$

$v = \frac{y}{x}$

$$y' = v'x + v \tag{12}$$

Simplify til Separable Equation

1.6 Linear Equations

$$y' = F(ax + by) \tag{13}$$

Assume v(x) = ax + by
Then Simplify til Separable Form

2 Two Ways of numerically Generate Functions

2.1 Newtons Method & Picard Iteration

Step 1 :
$y' = f(x, y)$

Step 2 :
Input Initial Value

$$y(x_0) = y_0 \tag{14}$$

Step 3 :

$$y^{n+1}(t) = y_0 + \int_0^t f(s, y^n(s))\, ds \tag{15}$$

2.2 Euler's Method

Step 1 :
$y' = f(x, y)$

Step 2 :

$$y_{n+1} = y_n + \Delta x f(x_n, y_n) \tag{16}$$

3 Second Order Differential Equations

3.1 Homogeneous

3.1.1 Constant Coefficient

Set

$$y = e^{rt} \tag{17}$$

After substitute back, we can get :

$$\alpha r^2 + \beta r + \gamma = C \tag{18}$$

Then solve for r :

$$y(t) \;=\; c_1 e^{r_1 t} \;+\; c_2 e^{r_2 t} \tag{19}$$

When coming across complex roots :

$$y \;=\; e^{(\lambda \,\pm \mu)t} \tag{20}$$

Then we can get the homogeneous solution :

$$y(t) \;=\; c_1 e^{\lambda t} cos(\mu t) \;+\; c_2 e^{\lambda t} sin(\mu t) \tag{21}$$

3.2 Non-constant Coefficient

3.2.1 Reduction of Order

Known $y_1(t)$, find $y_2(t)$:

$$y_2(t) \;=\; v_1(t)\, y_1(t) \tag{22}$$

3.2.2 Abel's Theorem

Use the Wronskian $W = y_1 y_2' - y_1' y_2$ to find y_2 (where $p(t)$ is the coefficient of y')

$$W' + p(t)W = 0 \tag{23}$$

With wronskian, we can no whether the two given functions are solutions to the same ODE. In addition, we can also find out whether the two solutions are linearly independent.

$$\frac{y_1}{y_2} = \begin{cases} \text{constant} \quad W(y_1, y_2) = 0 \\ \text{non-constant} \quad W(y_1, y_2) \neq 0 \end{cases}$$

3.3 Non-homogeneous

3.3.1 Constant Coefficient

Undetermined Coefficients : Find homogeneous solution ($y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$), then assume particular solution to generate general solution. Then assume particular solution $y_p(t)$ and solve for $y_p(t)$. Therefore get the general solution $y_(t) = y_h(t) + y_p(t)$

$$R.H.S = \begin{cases} e^{\alpha t} \quad \text{assume } y_p(t) \;= Ae^{\alpha t} \\ sin(\alpha t)/cos(\alpha t) \quad \text{assume } y_p(t) \;= Acos(\alpha t) + Bcos(\alpha t) \\ Polynomials \quad \text{assume } y_p(t) \;= \sum a_n t^n \quad \text{(n corresponds with highest degree of R.H.S)} \end{cases}$$

If particular solution is part of the homogeneous solution, set $y_p(t) = y_p(t)\, t^n$

3.4 Non-constant Coefficients

3.4.1 Euler-Cauchy Equation

$Deg(x^p) \;=\; y^{[p]}(x)$ Assume $y = x^m$, get $y_h(x) = c_1 x_1^m + c_2 x_2^m$. After getting homogeneous solution, use "Variation of Parameters" get particular solutions.

3.4.2 Variation of Parameters

When R.H.S is not in special forms (i.e exponential , trigonometric , polynomial)

$$\begin{aligned} u_1 \;&=\; -\int \frac{y_2 g}{w}\, dx \\ u_2 \;&=\; \int \frac{y_1 g}{w}\, dx \end{aligned} \tag{24}$$

Then we can obtain the general solution : $y(x) \;=\; c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$

3.4.3 Power Series

Mostly Power Series is a way of approximation rather than obtaining the precise solution.

$$y \;=\; \sum_{0}^{\infty} a_n\, x^n \tag{25}$$

Get $y' \;=\; \sum_1^\infty a_n\, n\, x^{n-1}$ & $y'' \;=\; \sum_2^\infty a_n\, n\,(n-1)\, x^{n-2}$ and substitute back into the equation solve for a_n and the recursive formula for a_{n+1} using $a_n = 0$

4 Laplace Transform Solving ODEs

For t ¿ 0 :

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\, dt \tag{26}$$

Frequently-Used Formula :
$\mathscr{L}\{k\} = \frac{k}{s}$
$\mathscr{L}\{t^m\} = \frac{m!}{s^{m+1}}$
$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$

Shift-Theorem :

$$\mathscr{L}^{-1}\{f(s-h)\}(t) \;=\; e^h \mathscr{L}^{-1}\{f(s)\}(t) \tag{27}$$

Step 1 : Take Laplace Transform on both sides
$\alpha(s^2Y - sy(0) - y'(0)) \; + \; \beta(sY - y(0)) + \gamma Y(s) = \mathscr{L}\{R.H.S\}(s)$
Step 2 : Isolate Y(s)
Step 3 : Take $\mathscr{L}^{-1}\{f(t)\}$
Case1 :
Denominator can be written as $(s-k)^2 + k$, use the "shift-theorem"
Case2 :
Denominator can be factorized, then use partial fraction
Case3 :
R.H.S is piecewise function or spike function, try to construct 't-c' on R.H.S

$$\mathscr{L}^{-1}\{e^{-cs}F(s)\} \; = \; u_c(t)f(t-c) \tag{28}$$

Formula for spike function :

$$\mathscr{L}\{\delta(t-c)\} = e^{-cs} \tag{29}$$

A brief summary of expressing piecewise function :
When f is in the following expression : $f(t) \; = \begin{cases} p(t) & t < m \\ q(t) & m \leq t \leq n \\ r(t) & t > s \end{cases}$

We can rewrite it with spike function :
$f(t) = p(t)[1 - u_m(t)] \; + \; q(t)[u_m(t) - u_n(t)] \; + \; r(t)[u_s(t)]$

5 Solving 2x2 Linear Systems

5.1 Variation of Parameters

$$y(t) \; = \; c_1 e^{r_1 t}\vec{v_1} \; + \; c_2 e^{r_2 t}\vec{v_2} + X\int X^{-1}g(t)\,dt \tag{30}$$

where $X = [c_1 e^{r_1 t}[\vec{v_1}], c_2 e^{r_2 t}[\vec{v_2}]]$; r_1 & r_2 are eigenvalues ; $\vec{v_1}$ & $\vec{v_2}$ are corresponded eigenvectors

5.2 Matrix Exponentials

Basic equations for matrix exponentials :
1)
$exp(\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}) \; = \; \begin{bmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{bmatrix}$

2)
$exp(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) \; = \; \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

3)
When A, B are commute, $exp(\big[A\big] + \big[B\big]) = exp(A) \cdot exp(B)$

4)
$exp(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}) = e^a \cdot \begin{bmatrix} cos(b) & sin(b) \\ -sin(b) & cos(b) \end{bmatrix}$

5.3 Steps for Matrix Exponential solving systems

Step 1 :
$e^{At} = p \cdot e^{Dt} \cdot p^{-1}$
$e^{Dt} = exp([P^{-1}AP]t)$

Step 2 :

$$X(t) = e^{At} \cdot X_0 + e^{At} \int_0^t e^{-As} \cdot g(s)\, ds \tag{31}$$

5.4 Steps for Polynomials solving systems

Based on the equation :

$$e^{At} = \alpha_1 At + \alpha_0 I \tag{32}$$

Then we can get two equations :

$$\begin{aligned} e^{\lambda_1 t} &= \alpha_1 \lambda_1 t + \alpha_0 \\ e^{\lambda_2 t} &= \alpha_1 \lambda_2 t + \alpha_0 \end{aligned} \tag{33}$$

Therefore, we solve for $\alpha_1 \,\&\, \alpha_0$.