Calculus II Final Review

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1 Infinite Integrals

Two useful Improper Integral:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1; \\ \infty, & \text{if } p \leq 1; \end{cases}$$

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p}, & \text{if } p < 1; \\ \infty, & \text{if } p \geq 1 \end{cases}$$

2 Area

$$A = \int_a^b f(x) - g(x)\,dx \tag{1}$$

(a,b are two intersect points)

3 Two Ways of Finding Volumes

3.1 Washer

$$\begin{aligned} V &= \int_a^b \pi(r_out^2 - r_in^2)\,dx \quad \text{(Rotate about X-axis)} \\ V &= \int_a^b \pi(r_out^2 - r_in^2)\,dy \quad \text{(Rotate about Y-axis)} \end{aligned} \tag{2}$$

3.2 Cylinder Shell

$$\begin{aligned} V &= \int_a^b 2(x)H(x)\,dx \quad \text{(Rotate about Y-axis)} \\ V &= \int_a^b 2(y)H(y)\,dy \quad \text{(Rotate about X-axis)} \end{aligned} \tag{3}$$

4 Center of Mass

Compute M_x and M_y

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2}(f^2 - g^2)\, dx \\ M_y &= \rho \int_a bx(f-g)\, dx \end{aligned} \tag{4}$$

Compute Total Mass

$$m = \rho \int_a^b (f-g)\, dx \tag{5}$$

Get the coordinate of the center of mass(x, y)

$$\begin{aligned} \overline{x} &= \frac{M_y}{m} \\ \overline{y} &= \frac{M_x}{m} \end{aligned} \tag{6}$$

5 Work

5.1 Pumping the Water

$$W = \rho g \int_{lowest-water-level}^{highest-water-level} V(H_{Target} - x)\, dx \tag{7}$$

5.2 Lifting Things

$$W = \int_0^{Length-of-chain} \rho(N/m)x + G \ \ dx \tag{8}$$

5.3 Spring(Hook's Law)

$$\begin{aligned} W &= \int_{start-point}^{endpoint} kx\, dx \\ k &= \frac{F}{\Delta x} \end{aligned} \tag{9}$$

6 Hydrostatic-Force

$$F = \int_a^b \rho g x L\, dx \tag{10}$$

7 Application of ODE

7.1 Population Growth

$$\frac{dN}{dt} = kN \qquad \text{(k is the relative growth rate)} \tag{11}$$

$m(t) = m_0(\frac{1}{2})^{\frac{t}{\text{Half-Life}}}$

7.2 Newton's Law of Cooling/Heating

$$\frac{dT}{dt} = k(T - T_s) \qquad \text{(Cooling)} \tag{12}$$

$$\frac{dT}{dt} = k(T_s - T) \qquad \text{(Heating)} \tag{13}$$

7.3 Mixing problem

$$\frac{dQ}{dt} = V_{Salt-Entering} - V_{Salt-Leaving} \quad \text{(kg/day)} \tag{14}$$

8 Euler's Method

Given $y' = f(x, y)$ and $y(x_0) = y_0$:

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x \tag{15}$$

9 Sequence

$$\lim_{n\to\infty} a_n = \begin{cases} c \quad (c \in R) \qquad \text{Convergent} \\ \text{D.N.E} \qquad \text{Divergent} \end{cases}$$

10 Series

Series are the sum of sequence ($S_n = \sum_{n=1}^{\infty} a_n$)

10.1 Two Special Series

10.1.1 Geometric Series

$$S_n = \begin{cases} \frac{a_1}{1-r} & \text{if} \quad |r| < 1 \\ \infty & \text{if} \quad |r| \geq 1 \end{cases} \tag{16}$$

10.1.2 Telescope Series

Use partial fraction to create $\sum(A_n - B_n)$

$$S_n = \sum_{n=1}^{\infty} \frac{1}{(x-a)(x-b)} \tag{17}$$

10.2 Ways to Find Convergence/Divergence of A Series

10.2.1 Test for Divergence

$(lim_{n\to\infty} a_n \neq 0) \implies (\sum_{n=1}^{\infty} a_n$ is divergent)
The converse is not True

10.2.2 The Integral Test

Three Conditions Before using:
1) f is continuous 2) f is decreasing 3) f is positive on the interval

$[\sum_{n=1}^{\infty} a_n$is convergent] $\iff$ $[\int_1^{\infty} f(x)\,dx$ limit exists]

10.2.3 Comparison Test

If $0 \leq \sum b_n \leq \sum a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

If $0 \leq \sum a_n \leq \sum b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

10.2.4 Limit Comparison Test

If $lim_{n\to\infty}(\frac{a_n}{b_n}) = c$ (where $c > 0$ and $c \neq \infty$), then $\sum b_n$ behaves similar to $\sum a_n$
Sometimes use 10.2.3 first, then use 10.2.4

10.2.5 Alternating Series Test

Two Conditions: $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} \leq b_n$ (where $b_n = |a_n|$)

Then $\sum_{n=1}^{\infty}(-1)^n b_n$ is convergent

10.2.6 Conditionally & Absolutely Convergent

Conditionally Convergent : $\sum a_n$ is convergent and $\sum |a_n|$ is divergent

Absolutely Convergent : $\sum |a_n|$ is convergent

Theorem : $(\sum_{n\geq 1} |a_n|$ is convergent$) \implies (\sum_{n\geq 1} a_n$ is convergent$)$

10.2.7 Ratio & Root Test

Ratio Test : $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \begin{cases} > 1 \text{ divergent} \\ = 1 \text{ uncertain} \\ < 1 \text{ convergent} \end{cases}$

Root Test : $\lim_{n\to\infty} \sqrt[n]{|a_n|} \begin{cases} > 1 \text{ divergent} \\ = 1 \text{ uncertain} \\ < 1 \text{ convergent} \end{cases}$

10.2.8 Power Series

$\sum_{n=0}^{\infty} a_n(x-c)^n$ (c is the center point)

Use Ratio Test to find Convergence : $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \infty \quad \text{Interval is c,} \quad \text{R} = 0 \\ > 1 \quad \text{divergent} \\ = 1 \quad \text{uncertain} \\ = 0 \quad \text{interval is } (-\infty, \infty) \quad \text{R} = \infty \\ < 1 \quad \text{written in } \frac{1}{R}|x-c| \end{cases}$

10.3 Error Bounds

$\int_{n+1}^{\infty} f(x)\,dx \;\leq |R_n| \;\leq \int_{n}^{\infty} f(x)\,dx$

Error bounds for Alternating Series : $|R_n| \leq b_{n+1}$

11 Taylor Series & Maclaurin Series

Taylor Series :

$$f(x) = C_n(x-a)^n = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!} \tag{18}$$

Maclaurin Series :

$$f(x) = C_n(x)^n = \sum_{n=0}^{\infty} \frac{f^n(0)(x)^n}{n!} \tag{19}$$

The Basic Two :

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{20}$$

$$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \tag{21}$$

Maclaurin Series of Basic Functions :

$$sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!} \tag{22}$$

$$cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n)}}{(2n)!} \tag{23}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{24}$$

$$ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(n+1)}}{n+1} \tag{25}$$

$$arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{2n+1} \tag{26}$$

Binomial Theorem :

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k \tag{27}$$

For the combination Part : $\binom{n}{k} = \frac{n..(n-(k-1))}{k!}$

12 Functions With Several Variables

When taking partial derivatives, always regard the rest variables as constant.

$$f_{xy} = \frac{\partial \frac{\partial f}{\partial x}}{\partial y} \tag{28}$$

$$f_{xyz} = f_{yxz} = f_{zxy} \tag{29}$$

Alternating Approach To Implicit Differentiation :

$$F(x, y, z(x, y)) = 0$$
$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \tag{30}$$

13 Tangent Plane

Equation for the tangent plane at point (a,b) :

$$Z = f(a,b) \, + \, \frac{\partial f}{\partial x}_{(a,b)}(x-a) \, + \, \frac{\partial f}{\partial y}_{(a,b)}(y-b) \tag{31}$$

The Linearization of f(x,y) near (a,b) :

$$f(x,y) = f(a,b) \, + \, \frac{\partial f}{\partial x}_{(a,b)}(x-a) \, + \, \frac{\partial f}{\partial y}_{(a,b)}(y-b) \tag{32}$$

14 Directional Derivatives & Gradient Vectors

The Directional Derivatives of $f(x,y)$ at point (x_0, y_0) in the direction of the vector $\hat{u} = ai + bj$:

$$D_{\hat{u}}(x_0, y_0) = (\frac{\partial f}{\partial x}_{(x_0,y_0)}, \frac{\partial f}{\partial y}_{(x_0,y_0)}) \, \cdot \, (\frac{a}{|\hat{u}|}, \frac{b}{|\hat{u}|}) \tag{33}$$