Aircraft Structures
for engineering students
Solutions Manual

T. H. G. Megson

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Contents

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Solutions to Chapter 1 Problems

S.1.1

The principal stresses are given directly by Eqs (1.11) and (1.12) in which $\sigma_x = 80$ N/mm^2, $\sigma_y = 0$ (or vice versa) and $\tau_{xy} = 45\,\text{N/mm}^2$. Thus, from Eq. (1.11)

$$\sigma_{\text{I}} = \frac{80}{2} + \frac{1}{2}\sqrt{80^2 + 4 \times 45^2}$$

i.e.

$$\sigma_{\text{I}} = 100.2\,\text{N/mm}^2$$

From Eq. (1.12)

$$\sigma_{\text{II}} = \frac{80}{2} - \frac{1}{2}\sqrt{80^2 + 4 \times 45^2}$$

i.e.

$$\sigma_{\text{II}} = -20.2\,\text{N/mm}^2$$

The directions of the principal stresses are defined by the angle θ in Fig. 1.8(b) in which θ is given by Eq. (1.10). Hence

$$\tan 2\theta = \frac{2 \times 45}{80 - 0} = 1.125$$

which gives

$$\theta = 24^\circ\,11' \text{ and } \theta = 114^\circ\,11'$$

It is clear from the derivation of Eqs (1.11) and (1.12) that the first value of θ corresponds to σ_{I} while the second value corresponds to σ_{II}.

Finally, the maximum shear stress is obtained from either of Eqs (1.14) or (1.15). Hence from Eq. (1.15)

$$\tau_{\max} = \frac{100.2 - (-20.2)}{2} = 60.2\,\text{N/mm}^2$$

and will act on planes at 45° to the principal planes.

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S.1.2

The principal stresses are given directly by Eqs (1.11) and (1.12) in which $\sigma_x = 50\,\text{N/mm}^2$, $\sigma_y = -35\,\text{N/mm}^2$ and $\tau_{xy} = 40\,\text{N/mm}^2$. Thus, from Eq. (1.11)

$$\sigma_{\text{I}} = \frac{50 - 35}{2} + \frac{1}{2}\sqrt{(50 + 35)^2 + 4 \times 40^2}$$

i.e.

$$\sigma_{\text{I}} = 65.9\,\text{N/mm}^2$$

and from Eq. (1.12)

$$\sigma_{\text{II}} = \frac{50 - 35}{2} - \frac{1}{2}\sqrt{(50 + 35)^2 + 4 \times 40^2}$$

i.e.

$$\sigma_{\text{II}} = -50.9\,\text{N/mm}^2$$

From Fig. 1.8(b) and Eq. (1.10)

$$\tan 2\theta = \frac{2 \times 40}{50 + 35} = 0.941$$

which gives

$$\theta = 21^\circ\,38'\,(\sigma_{\text{I}}) \text{ and } \theta = 111^\circ\,38'\,(\sigma_{\text{II}})$$

The planes on which there is no direct stress may be found by considering the triangular element of unit thickness shown in Fig. S.1.2 where the plane AC represents the plane on which there is no direct stress. For equilibrium of the element in a direction perpendicular to AC

$$0 = 50\text{AB}\cos\alpha - 35\text{BC}\sin\alpha + 40\text{AB}\sin\alpha + 40\text{BC}\cos\alpha \tag{i}$$

Dividing through Eq. (i) by AB

$$0 = 50\cos\alpha - 35\tan\alpha\sin\alpha + 40\sin\alpha + 40\tan\alpha\cos\alpha$$

which, dividing through by $\cos\alpha$, simplifies to

$$0 = 50 - 35\tan^2\alpha + 80\tan\alpha$$

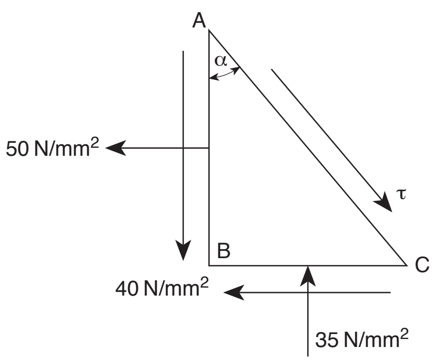

Fig. S.1.2

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from which

$$\tan \alpha = 2.797 \text{ or } -0.511$$

Hence

$$\alpha = 70^\circ\, 21' \text{ or } -27^\circ\, 5'$$

S.1.3

The construction of Mohr's circle for each stress combination follows the procedure described in Section 1.8 and is shown in Figs S.1.3(a)–(d).

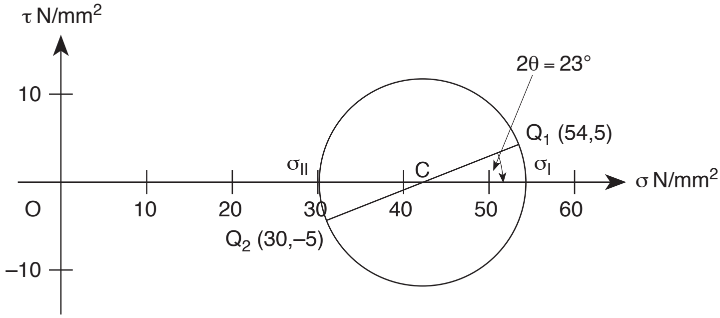

Fig. S.1.3(a)

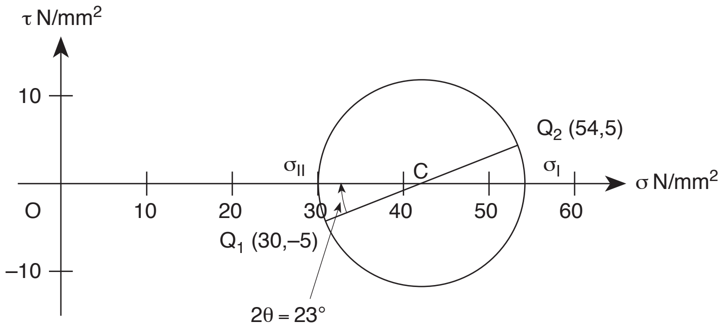

Fig. S.1.3(b)

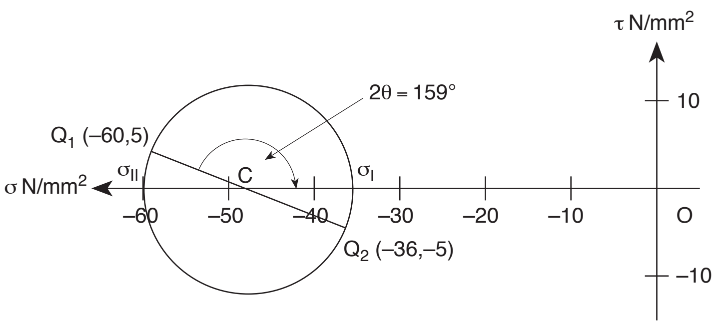

Fig. S.1.3(c)

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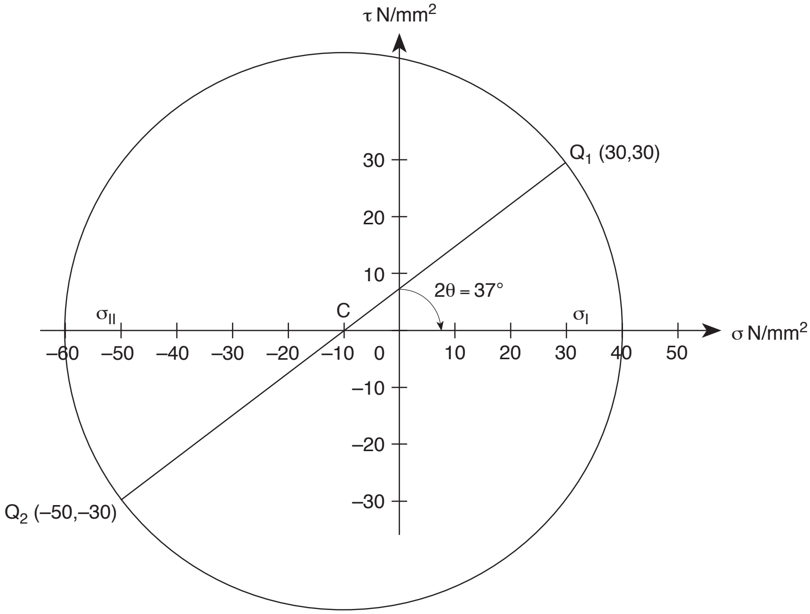

Fig. S.1.3(d)

S.1.4

The principal stresses at the point are determined, as indicated in the question, by transforming each state of stress into a σ_x, σ_y, τ_{xy} stress system. Clearly, in the first case $\sigma_x = 0$, $\sigma_y = 10\,\text{N/mm}^2$, $\tau_{xy} = 0$ (Fig. S.1.4(a)). The two remaining cases are transformed by considering the equilibrium of the triangular element ABC in Figs S.1.4(b), (c), (e) and (f). Thus, using the method described in Section 1.6 and the principle of superposition (see Section 4.9), the second stress system of Figs S.1.4(b) and (c) becomes the σ_x, σ_y, τ_{xy} system shown in Fig. S.1.4(d) while the third stress system of Figs S.1.4(e) and (f) transforms into the σ_x, σ_y, τ_{xy} system of Fig. S.1.4(g).

Finally, the states of stress shown in Figs S.1.4(a), (d) and (g) are superimposed to give the state of stress shown in Fig. S.1.4(h) from which it can be seen that $\sigma_\text{I} = \sigma_\text{II} = 15\,\text{N/mm}^2$ and that the x and y planes are principal planes.

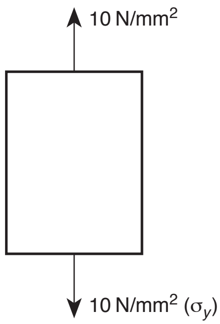

Fig. S.1.4(a)

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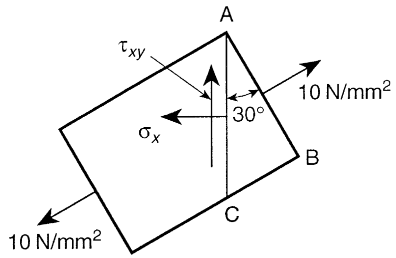

Fig. S.1.4(b)

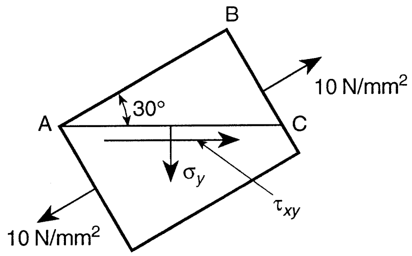

Fig. S.1.4(c)

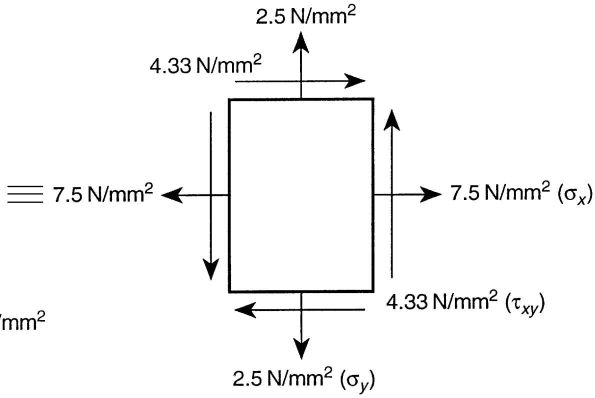

Fig. S.1.4(d)

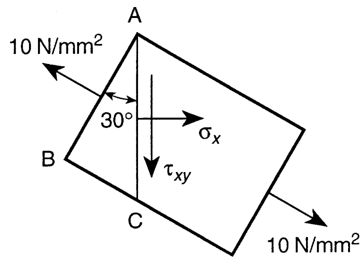

Fig. S.1.4(e)

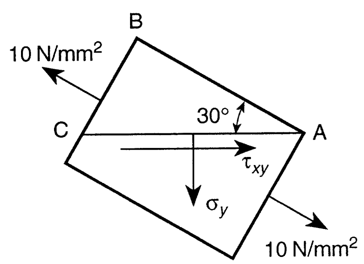

Fig. S.1.4(f)

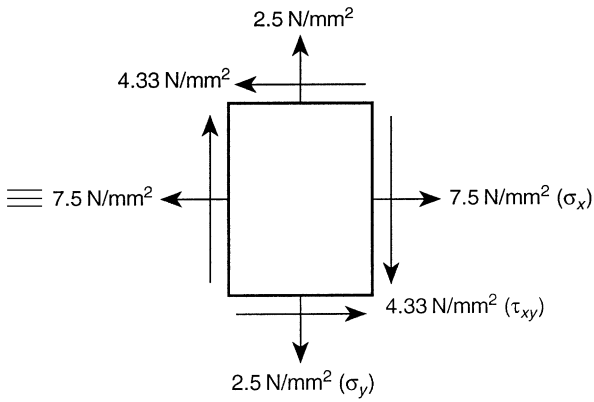

Fig. S.1.4(g)

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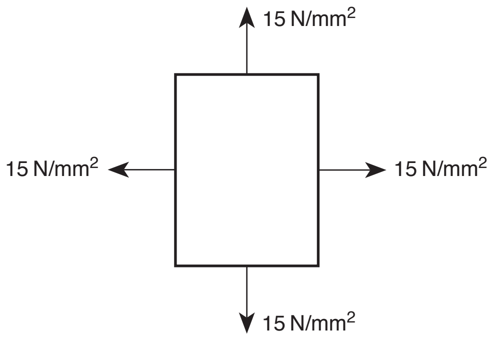

Fig. S.1.4(h)

S.1.5

The geometry of Mohr's circle of stress is shown in Fig. S.1.5 in which the circle is constructed using the method described in Section 1.8.

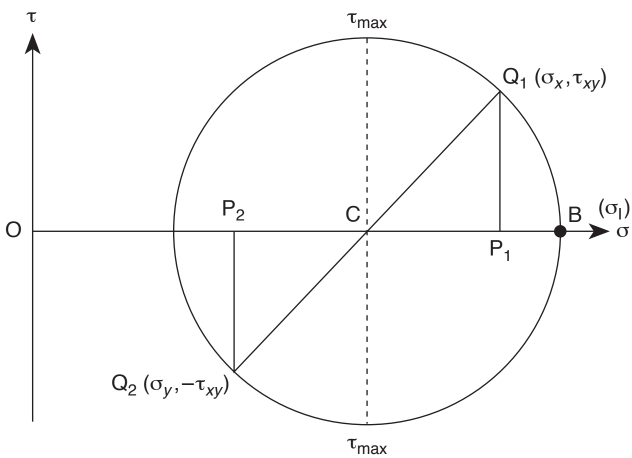

Fig. S.1.5

From Fig. S.1.5

$$\sigma_x = \mathrm{OP}_1 = \mathrm{OB} - \mathrm{BC} + \mathrm{CP}_1 \tag{i}$$

In Eq. (i) $\mathrm{OB} = \sigma_{\mathrm{I}}$, BC is the radius of the circle which is equal to $\tau_{\max}$ and $\mathrm{CP}_1 = \sqrt{\mathrm{CQ}_1^2 - \mathrm{Q}_1\mathrm{P}_1^2} = \sqrt{\tau_{\max}^2 - \tau_{xy}^2}$. Hence

$$\sigma_x = \sigma_{\mathrm{I}} - \tau_{\max} + \sqrt{\tau_{\max}^2 - \tau_{xy}^2}$$

Similarly

$$\sigma_y = \mathrm{OP}_2 = \mathrm{OB} - \mathrm{BC} - \mathrm{CP}_2 \text{ in which } \mathrm{CP}_2 = \mathrm{CP}_1$$

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Thus

$$\sigma_y = \sigma_{\mathrm{I}} - \tau_{\max} - \sqrt{\tau_{\max}^2 - \tau_{xy}^2}$$

S.1.6

From bending theory (see Eqs (9.9)) the direct stress due to bending on the upper surface of the shaft at a point in the vertical plane of symmetry is given by

$$\sigma_x = \frac{My}{I} = \frac{25 \times 10^6 \times 75}{\pi \times 150^4/64} = 75\,\mathrm{N/mm^2}$$

From the theory of the torsion of circular section shafts the shear stress at the same point is

$$\tau_{xy} = \frac{Tr}{J} = \frac{50 \times 10^6 \times 75}{\pi \times 150^4/32} = 75\,\mathrm{N/mm^2}$$

Substituting these values in Eqs (1.11) and (1.12) in turn and noting that $\sigma_y = 0$

$$\sigma_{\mathrm{I}} = \frac{75}{2} + \frac{1}{2}\sqrt{75^2 + 4 \times 75^2}$$

i.e.

$$\sigma_{\mathrm{I}} = 121.4\,\mathrm{N/mm^2}$$

$$\sigma_{\mathrm{II}} = \frac{75}{2} - \frac{1}{2}\sqrt{75^2 + 4 \times 75^2}$$

i.e.

$$\sigma_{\mathrm{II}} = -46.4\,\mathrm{N/mm^2}$$

The corresponding directions as defined by θ in Fig. 1.8(b) are given by Eq. (1.10) i.e.

$$\tan 2\theta = \frac{2 \times 75}{75 - 0} = 2$$

Hence

$$\theta = 31^\circ\, 43' \,(\sigma_{\mathrm{I}})$$

and

$$\theta = 121^\circ\, 43' \,(\sigma_{\mathrm{II}})$$

S.1.7

The direct strains are expressed in terms of the stresses using Eqs (1.42), i.e.

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \qquad \text{(i)}$$

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$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \tag{ii}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \tag{iii}$$

Then

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1}{E}[\sigma_x + \sigma_y + \sigma_z - 2\nu(\sigma_x + \sigma_y + \sigma_z)]$$

i.e.

$$e = \frac{(1 - 2\nu)}{E}(\sigma_x + \sigma_y + \sigma_z)$$

whence

$$\sigma_y + \sigma_z = \frac{Ee}{(1 - 2\nu)} - \sigma_x$$

Substituting in Eq. (i)

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\frac{Ee}{1 - 2\nu} - \sigma_x\right)\right]$$

so that

$$E\varepsilon_x = \sigma_x(1 + \nu) - \frac{\nu Ee}{1 - 2\nu}$$

Thus

$$\sigma_x = \frac{\nu Ee}{(1 - 2\nu)(1 + \nu)} + \frac{E}{(1 + \nu)}\varepsilon_x$$

or, since $G = E/2(1 + \nu)$ (see Section 1.15)

$$\sigma_x = \lambda e + 2G\varepsilon_x$$

Similarly

$$\sigma_y = \lambda e + 2G\varepsilon_y$$

and

$$\sigma_z = \lambda e + 2G\varepsilon_z$$

S.1.8

The implication in this problem is that the condition of plane strain also describes the condition of plane stress. Hence, from Eqs (1.47)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{i}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{ii}$$

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$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy} \quad \text{(see Section 1.15)} \tag{iii}$$

The compatibility condition for plane strain is

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} \quad \text{(see Section 1.11)} \tag{iv}$$

Substituting in Eq. (iv) for ε_x, ε_y and γ_{xy} from Eqs (i), (ii) and (iii) respectively gives

$$2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) + \frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) \tag{v}$$

Also, from Eqs (1.6) and assuming that the body forces X and Y are zero

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0 \tag{vi}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \tag{vii}$$

Differentiating Eq. (vi) with respect to x and Eq. (vii) with respect to y and adding gives

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial y \partial x} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0$$

or

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right)$$

Substituting in Eq. (v)

$$-(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) = \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) + \frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y)$$

so that

$$-(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) = \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - \nu\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right)$$

which simplifies to

$$\frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$$

or

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

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S.1.9

The principal strains are given directly by Eqs (1.58) and (1.59). Thus

$$\varepsilon_{\mathrm{I}} = \frac{1}{2}(-0.002 + 0.002) + \frac{1}{\sqrt{2}}\sqrt{(-0.002 + 0.002)^2 + (-0.002 - 0.002)^2}$$

i.e.

$$\varepsilon_{\mathrm{I}} = +0.00283$$

Similarly

$$\varepsilon_{\mathrm{II}} = -0.00283$$

The principal directions are given by Eq. (1.60), i.e.

$$\tan 2\theta = \frac{2(-0.002) + 0.002 - 0.002}{0.002 + 0.002} = -1$$

Hence

$$2\theta = -45^\circ \text{ or } +135^\circ$$

and

$$\theta = -22.5^\circ \text{ or } +67.5^\circ$$

S.1.10

The principal strains at the point P are determined using Eqs (1.58) and (1.59). Thus

$$\varepsilon_{\mathrm{I}} = \left[\frac{1}{2}(-222 + 45) + \frac{1}{\sqrt{2}}\sqrt{(-222 + 213)^2 + (-213 - 45)^2}\right] \times 10^{-6}$$

i.e.

$$\varepsilon_{\mathrm{I}} = 94.0 \times 10^{-6}$$

Similarly

$$\varepsilon_{\mathrm{II}} = -217.0 \times 10^{-6}$$

The principal stresses follow from Eqs (1.56) and (1.57). Hence

$$\sigma_{\mathrm{I}} = \frac{31\,000}{1 - (0.2)^2}(94.0 - 0.2 \times 271.0) \times 10^{-6}$$

i.e.

$$\sigma_{\mathrm{I}} = 1.29\,\mathrm{N/mm^2}$$

Similarly

$$\sigma_{\mathrm{II}} = -8.14\,\mathrm{N/mm^2}$$

Since P lies on the neutral axis of the beam the direct stress due to bending is zero. Therefore, at P, $\sigma_x = 7\,\mathrm{N/mm^2}$ and $\sigma_y = 0$. Now subtracting Eq. (1.12) from

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Eq. (1.11)

$$\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}} = \sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

i.e.

$$1.29 + 8.14 = \sqrt{7^2 + 4\tau_{xy}^2}$$

from which $\tau_{xy} = 3.17\,\mathrm{N/mm^2}$.

The shear force at P is equal to Q so that the shear stress at P is given by

$$\tau_{xy} = 3.17 = \frac{3Q}{2 \times 150 \times 300}$$

from which

$$Q = 95\,100\,\mathrm{N} = 95.1\,\mathrm{kN}$$

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Solutions to Chapter 2 Problems

S.2.1

The stress system applied to the plate is shown in Fig. S.2.1. The origin, O, of the axes may be chosen at any point in the plate; let P be the point whose coordinates are (2, 3).

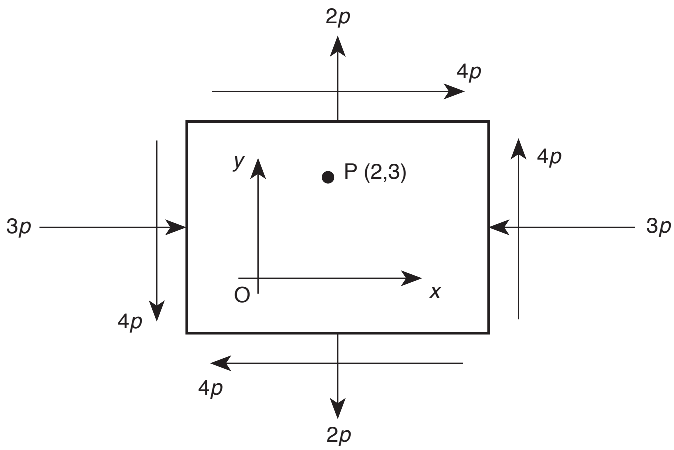

Fig. S.2.1

From Eqs (1.42) in which $\sigma_z = 0$

$$\varepsilon_x = -\frac{3p}{E} - \nu\frac{2p}{E} = -\frac{3.5p}{E} \tag{i}$$

$$\varepsilon_y = \frac{2p}{E} + \nu\frac{3p}{E} = \frac{2.75p}{E} \tag{ii}$$

Hence, from Eqs (1.27)

$$\frac{\partial u}{\partial x} = -\frac{3.5p}{E} \quad \text{so that} \quad u = -\frac{3.5p}{E}x + f_1(y) \tag{iii}$$

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where $f_1(y)$ is a function of y. Also

$$\frac{\partial v}{\partial y} = \frac{2.75p}{E} \quad \text{so that} \quad v = \frac{2.75p}{E}y + f_2(x) \tag{iv}$$

in which $f_2(x)$ is a function of x.

From the last of Eqs (1.47) and Eq. (1.28)

$$\gamma_{xy} = \frac{4p}{G} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial f_2(x)}{\partial x} + \frac{\partial f_1(y)}{\partial y} \quad \text{(from Eqs (iv) and (iii))}$$

Suppose

$$\frac{\partial f_1(y)}{\partial y} = A$$

then

$$f_1(y) = Ay + B \tag{v}$$

in which A and B are constants.

Similarly, suppose

$$\frac{\partial f_2(x)}{\partial x} = C$$

then

$$f_2(x) = Cx + D \tag{vi}$$

in which C and D are constants.

Substituting for $f_1(y)$ and $f_2(x)$ in Eqs (iii) and (iv) gives

$$u = -\frac{3.5p}{E}x + Ay + B \tag{vii}$$

and

$$v = \frac{2.75p}{E}y + Cx + D \tag{viii}$$

Since the origin of the axes is fixed in space it follows that when $x = y = 0$, $u = v = 0$. Hence, from Eqs (vii) and (viii), $B = D = 0$. Further, the direction of Ox is fixed in space so that, when $y = 0$, $\partial v/\partial x = 0$. Therefore, from Eq. (viii), $C = 0$. Thus, from Eqs (1.28) and (vii), when $x = 0$,

$$\frac{\partial u}{\partial y} = \frac{4p}{G} = A$$

Eqs (vii) and (viii) now become

$$u = -\frac{3.5p}{E}x + \frac{4p}{G}y \tag{ix}$$

$$v = \frac{2.75p}{E}y \tag{x}$$

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From Eq. (1.45), $G = E/2(1+\nu) = E/2.5$ and Eq. (ix) becomes

$$u = \frac{p}{E}(-3.5x + 10y) \tag{xi}$$

At the point (2, 3)

$$u = \frac{23p}{E} \quad \text{(from Eq. (xi))}$$

and

$$v = \frac{8.25p}{E} \quad \text{(from Eq. (x))}$$

The point P therefore moves at an angle α to the x axis given by

$$\alpha = \tan^{-1}\frac{8.25}{23} = 19.73^\circ$$

S.2.2

An Airy stress function, ϕ, is defined by the equations

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \quad \text{(Eqs (2.8))}$$

and has a final form which is determined by the boundary conditions relating to a particular problem.

Since

$$\phi = Ay^3 + By^3x + Cyx \tag{i}$$

$$\frac{\partial^4\phi}{\partial x^4} = 0, \quad \frac{\partial^4\phi}{\partial y^4} = 0, \quad \frac{\partial^4\phi}{\partial x^2\partial y^2} = 0$$

and the biharmonic equation (2.9) is satisfied. Further

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 6Ay + 6Byx \tag{ii}$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 0 \tag{iii}$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -3By^2 - C \tag{iv}$$

The distribution of shear stress in a rectangular section beam is parabolic and is zero at the upper and lower surfaces. Hence, when $y = \pm d/2$, $\tau_{xy} = 0$. Thus, from Eq. (iv)

$$B = -4C/3d^2 \tag{v}$$

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The resultant shear force at any section of the beam is $-P$. Therefore

$$\int_{-d/2}^{d/2} \tau_{xy} t \, \mathrm{d}y = -P$$

Substituting for τ_{xy} from Eq. (iv)

$$\int_{-d/2}^{d/2} (-3By^2 - C) t \, \mathrm{d}y = -P$$

which gives

$$2t\left(\frac{Bd^3}{8} + \frac{Cd}{2}\right) = P$$

Substituting for B from Eq. (v) gives

$$C = 3P/2td \qquad \text{(vi)}$$

It now follows from Eqs (v) and (vi) that

$$B = -2P/td^3 \qquad \text{(vii)}$$

At the free end of the beam where $x = l$ the bending moment is zero and thus $\sigma_x = 0$ for any value of y. Therefore, from Eq. (ii)

$$6A + 6Bl = 0$$

whence

$$A = 2Pl/td^3 \qquad \text{(viii)}$$

Then, from Eq. (ii)

$$\sigma_x = \frac{12Pl}{td^3} y - \frac{12P}{td^3} xy$$

or

$$\sigma_x = \frac{12P(l - x)}{td^3} y \qquad \text{(ix)}$$

Eq. (ix) is the direct stress distribution at any section of the beam given by simple bending theory, i.e.

$$\sigma_x = \frac{My}{I}$$

where $M = P(l - x)$ and $I = td^3/12$.

The shear stress distribution given by Eq. (iv) is

$$\tau_{xy} = \frac{6P}{td^3} y^2 - \frac{3P}{2td}$$

or

$$\tau_{xy} = \frac{6P}{td^3}\left(y^2 - \frac{d^2}{4}\right) \qquad \text{(x)}$$

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Eq. (x) is identical to that derived from simple bending theory and may be found in standard texts on stress analysis, strength of materials etc.

S.2.3

The Airy stress function is

$$\phi = \frac{p}{120d^3}[5(x^3 - l^2x)(y+d)^2(y-2d) - 3yx(y^2-d^2)^2]$$

Then

$$\frac{\partial^4\phi}{\partial x^4} = 0, \quad \frac{\partial^4\phi}{\partial y^4} = -\frac{3pxy}{d^3}, \quad \frac{\partial^4\phi}{\partial x^2\partial y^2} = \frac{3pxy}{2d^3}$$

Substituting these values in Eq. (2.9) gives

$$0 + 2 \times \frac{3pxy}{2d^3} - \frac{3pxy}{d^3} = 0$$

Therefore, the biharmonic equation (2.9) is satisfied.

The direct stress, σ_x, is given by (see Eqs (2.8))

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = \frac{px}{20d^3}[5y(x^2 - l^2) - 10y^3 + 6d^2y]$$

When $x = 0$, $\sigma_x = 0$ for all values of y. When $x = l$

$$\sigma_x = \frac{pl}{20d^3}(-10y^3 + 6d^2y)$$

and the total end load $= \int_{-d}^{d} \sigma_x 1\,\mathrm{d}y$

$$= \frac{pl}{20d^3}\int_{-d}^{d}(-10y^3 + 6d^2y)\,\mathrm{d}y = 0$$

Thus the stress function satisfies the boundary conditions for axial load in the x direction.

Also, the direct stress, σ_y, is given by (see Eqs (2.8))

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = \frac{px}{4d^3}(y^3 - 3yd^2 - 2d^3)$$

When $x = 0$, $\sigma_y = 0$ for all values of y. Also at any section x where $y = -d$

$$\sigma_y = \frac{px}{4d^3}(-d^3 + 3d^3 - 2d^3) = 0$$

and when $y = +d$

$$\sigma_y = \frac{px}{4d^3}(d^3 - 3d^3 - 2d^3) = -px$$

Thus, the stress function satisfies the boundary conditions for load in the y direction.

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The shear stress, τ_{xy}, is given by (see Eqs (2.8))

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{p}{40d^3}[5(3x^2 - l^2)(y^2 - d^2) - 5y^4 + 6y^2d^2 - d^4]$$

When $x = 0$

$$\tau_{xy} = -\frac{p}{40d^3}[-5l^2(y^2 - d^2) - 5y^4 + 6y^2d^2 - d^4]$$

so that, when $y = \pm d$, $\tau_{xy} = 0$. The resultant shear force on the plane $x = 0$ is given by

$$\int_{-d}^{d} \tau_{xy} 1\, \mathrm{d}y = -\frac{p}{40d^3}\int_{-d}^{d}[-5l^2(y^2 - d^2) - 5y^4 + 6y^2d^2 - d^4]\, \mathrm{d}y = -\frac{pl^2}{6}$$

From Fig. P.2.3 and taking moments about the plane $x = l$,

$$\tau_{xy}\,(x = 0)1dl = \frac{1}{2}lpl\frac{2}{3}l$$

i.e.

$$\tau_{xy}\,(x = 0) = \frac{pl^2}{6d}$$

and the shear force is $pl^2/6$.

Thus, although the resultant of the Airy stress function shear stress has the same magnitude as the equilibrating shear force it varies through the depth of the beam whereas the applied equilibrating shear stress is constant. A similar situation arises on the plane $x = l$.

S.2.4

From physics, the strain due to a temperature rise T in a bar of original length L_0 and final length L is given by

$$\varepsilon = \frac{L - L_0}{L_0} = \frac{L_0(1 + \alpha T) - L_0}{L_0} = \alpha T$$

Thus for the isotropic sheet, Eqs (1.47) become

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T$$

Also, from the last of Eqs (1.47) and Eq. (1.45)

$$\gamma_{xy} = \frac{2(1 + \nu)}{E}\tau_{xy}$$

Substituting in Eq. (1.21)

$$\frac{2(1 + \nu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{1}{E}\left(\frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2}\right) + \alpha\frac{\partial^2 T}{\partial x^2} + \frac{1}{E}\left(\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2}\right) + \alpha\frac{\partial^2 T}{\partial y^2}$$

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or

$$2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - \nu \frac{\partial^2 \sigma_x}{\partial x^2} - \nu \frac{\partial^2 \sigma_y}{\partial y^2} + E\alpha \nabla^2 T \tag{i}$$

From Eqs (1.6) and assuming body forces $X = Y = 0$

$$\frac{\partial^2 \tau_{xy}}{\partial y \partial x} = -\frac{\partial^2 \sigma_x}{\partial x^2}, \quad \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_y}{\partial y^2}$$

Hence

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

and

$$2\nu \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\nu \frac{\partial^2 \sigma_x}{\partial x^2} - \nu \frac{\partial^2 \sigma_y}{\partial y^2}$$

Substituting in Eq. (i)

$$-\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} = \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + E\alpha \nabla^2 T$$

Thus

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) + E\alpha \nabla^2 T = 0$$

and since

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad \text{(see Eqs (2.8))}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}\right) + E\alpha \nabla^2 T = 0$$

or

$$\nabla^2(\nabla^2 \phi + E\alpha T) = 0$$

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Solutions to Chapter 3 Problems

S.3.1

Initially the stress function, ϕ, must be expressed in terms of Cartesian coordinates. Thus, from the equation of a circle of radius, a, and having the origin of its axes at its centre,

$$\phi = k(x^2 + y^2 - a^2) \tag{i}$$

From Eqs (3.4) and (3.11)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = F = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z} \tag{ii}$$

Differentiating Eq. (i) and substituting in Eq. (ii)

$$4k = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

or

$$k = -\frac{1}{2}G\frac{\mathrm{d}\theta}{\mathrm{d}z} \tag{iii}$$

From Eq. (3.8)

$$T = 2\iint \phi \,\mathrm{d}x\,\mathrm{d}y$$

i.e.

$$T = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left[\iint_A x^2\,\mathrm{d}x\,\mathrm{d}y + \iint_A y^2\,\mathrm{d}x\,\mathrm{d}y - a^2\iint_A \mathrm{d}x\,\mathrm{d}y\right] \tag{iv}$$

where $\iint_A x^2\,\mathrm{d}x\,\mathrm{d}y = I_y$, the second moment of area of the cross-section about the y axis, $\iint_A y^2\,\mathrm{d}x\,\mathrm{d}y = I_x$, the second moment of area of the cross-section about the x axis and $\iint_A \mathrm{d}x\,\mathrm{d}y = A$, the area of the cross-section. Thus, since $I_y = \pi a^4/4$, $I_x = \pi a^4/4$ and $A = \pi a^2$ Eq. (iv) becomes

$$T = G\frac{\mathrm{d}\theta}{\mathrm{d}z}\frac{\pi a^4}{2}$$

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or

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{2T}{G\pi a^4} = \frac{T}{GI_p} \tag{v}$$

From Eqs (3.2) and Eq. (v)

$$\tau_{zy} = -\frac{\partial \phi}{\partial x} = -2kx = G\frac{\mathrm{d}\theta}{\mathrm{d}z}x = \frac{Tx}{I_p} \tag{vi}$$

and

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = 2ky = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}y = -\frac{Ty}{I_p} \tag{vii}$$

Substituting for τ_{zy} and τ_{zx} from Eqs (vi) and (vii) in the second of Eqs (3.15)

$$\tau_{zs} = \frac{T}{I_p}(xl + ym) \tag{viii}$$

in which, from Eqs (3.6)

$$l = \frac{\mathrm{d}y}{\mathrm{d}s}, \quad m = -\frac{\mathrm{d}x}{\mathrm{d}s}$$

Suppose that the bar of Fig. 3.2 is circular in cross-section and that the radius makes an angle α with the x axis. Then,

$$m = \sin\alpha \quad \text{and} \quad l = \cos\alpha$$

Also, at any radius, r

$$y = r\sin\alpha, \quad x = r\cos\alpha$$

Substituting for x, l, y and m in Eq. (viii) gives

$$\tau_{zs} = \frac{Tr}{I_p} \; (= \tau)$$

Now substituting for τ_{zx}, τ_{zy} and $\mathrm{d}\theta/\mathrm{d}z$ from Eqs (vii), (vi) and (v) in Eqs (3.10)

$$\frac{\partial w}{\partial x} = -\frac{Ty}{GI_p} + \frac{Ty}{GI_p} = 0 \tag{ix}$$

$$\frac{\partial w}{\partial y} = \frac{Tx}{GI_p} - \frac{Tx}{GI_p} = 0 \tag{x}$$

The possible solutions of Eqs (ix) and (x) are $w = 0$ and $w =$ constant. The latter solution implies a displacement of the whole bar along the z axis which, under the given loading, cannot occur. Therefore, the first solution applies, i.e. the warping is zero at all points in the cross-section.

The stress function, ϕ, defined in Eq. (i) is constant at any radius, r, in the cross-section of the bar so that there are no shear stresses acting across such a boundary. Thus, the material contained within this boundary could be removed without affecting the stress distribution in the outer portion. Therefore the stress function could be used for a hollow bar of circular cross-section.

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S.3.2

In S.3.1 it has been shown that the warping of the cross-section of the bar is everywhere zero. Thus, from Eq. (3.17) and since $\mathrm{d}\theta/\mathrm{d}z \neq 0$

$$\psi(x, y) = 0 \tag{i}$$

This warping function satisfies Eq. (3.20). Also Eq. (3.21) reduces to

$$xm - yl = 0 \tag{ii}$$

On the boundary of the bar $x = al$, $y = am$ so that Eq. (ii), i.e. Eq. (3.21), is satisfied.

Since $\psi = 0$, Eq. (3.23) for the torsion constant reduces to

$$J = \iint_A x^2\,\mathrm{d}x\,\mathrm{d}y + \iint_A y^2\,\mathrm{d}x\,\mathrm{d}y = I_p$$

Therefore, from Eq. (3.12)

$$T = GI_p \frac{\mathrm{d}\theta}{\mathrm{d}z}$$

as in S.3.1.

From Eqs (3.19)

$$\tau_{zx} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}(-y) = -\frac{Ty}{I_p}$$

and

$$\tau_{zy} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}(x) = \frac{Tx}{I_p}$$

which are identical to Eqs (vii) and (vi) in S.3.1. Hence

$$\tau_{zs} = \tau = \frac{Tr}{I_p}$$

as in S.3.1.

S.3.3

Since $\psi = kxy$, Eq. (3.20) is satisfied.

Substituting for ψ in Eq. (3.21)

$$(kx + x)m + (ky - y)l = 0$$

or, from Eqs (3.6)

$$-x(k+1)\frac{\mathrm{d}x}{\mathrm{d}s} + y(k-1)\frac{\mathrm{d}y}{\mathrm{d}s} = 0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[-\frac{x^2}{2}(k+1) + \frac{y^2}{2}(k-1)\right] = 0$$

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so that

$$-\frac{x^2}{2}(k+1)+\frac{y^2}{2}(k-1)=\text{constant on the boundary of the bar}$$

Rearranging

$$x^2+\left(\frac{1-k}{1+k}\right)y^2=\text{constant} \tag{i}$$

Also, the equation of the elliptical boundary of the bar is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or

$$x^2+\frac{a^2}{b^2}y^2=a^2 \tag{ii}$$

Comparing Eqs (i) and (ii)

$$\frac{a^2}{b^2}=\left(\frac{1-k}{1+k}\right)$$

from which

$$k=\frac{b^2-a^2}{a^2+b^2} \tag{iii}$$

and

$$\psi=\frac{b^2-a^2}{a^2+b^2}xy \tag{iv}$$

Substituting for ψ in Eq. (3.23) gives the torsion constant, J, i.e.

$$J=\iint_A\left[\left(\frac{b^2-a^2}{a^2+b^2}+1\right)x^2-\left(\frac{b^2-a^2}{a^2+b^2}-1\right)y^2\right]\mathrm{d}x\,\mathrm{d}y \tag{v}$$

Now $\iint_A x^2\,\mathrm{d}x\,\mathrm{d}y=I_y=\pi a^3b/4$ for an elliptical cross-section. Similarly $\iint_A y^2\,\mathrm{d}x\,\mathrm{d}y=I_x=\pi ab^3/4$. Eq. (v) therefore simplifies to

$$J=\frac{\pi a^3b^3}{a^2+b^2} \tag{vi}$$

which is identical to Eq. (v) of Example 3.1.

From Eq. (3.22) the rate of twist is

$$\frac{\mathrm{d}\theta}{\mathrm{d}z}=\frac{T(a^2+b^2)}{G\pi a^3b^3} \tag{vii}$$

The shear stresses are obtained from Eqs (3.19), i.e.

$$\tau_{zx}=\frac{GT(a^2+b^2)}{G\pi a^3b^3}\left[\left(\frac{b^2-a^2}{a^2+b^2}\right)y-y\right]$$

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so that

$$\tau_{zx} = -\frac{2Ty}{\pi ab^3}$$

and

$$\tau_{zy} = \frac{GT(a^2+b^2)}{G\pi a^3b^3}\left[\left(\frac{b^2-a^2}{a^2+b^2}\right)x + x\right]$$

i.e.

$$\tau_{zy} = \frac{2Tx}{\pi a^3 b}$$

which are identical to Eqs (vi) of Example 3.1.

From Eq. (3.17)

$$w = \frac{T(a^2+b^2)}{G\pi a^3b^3}\left(\frac{b^2-a^2}{a^2+b^2}\right)xy$$

i.e.

$$w = \frac{T(b^2-a^2)}{G\pi a^3b^3}xy \quad \text{(compare with Eq. (viii) of Example 3.1)}$$

S.3.4

The stress function is

$$\phi = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left[\frac{1}{2}(x^2+y^2) - \frac{1}{2a}(x^3-3xy^2) - \frac{2}{27}a^2\right] \tag{i}$$

Differentiating Eq. (i) twice with respect to x and y in turn gives

$$\frac{\partial^2\phi}{\partial x^2} = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(1-\frac{3x}{a}\right)$$

$$\frac{\partial^2\phi}{\partial y^2} = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(1+\frac{3x}{a}\right)$$

Therefore

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z} = \text{constant}$$

and Eq. (3.4) is satisfied.

Further

$$\text{on AB,}\quad x = -a/3,\quad y = y$$

$$\text{on BC,}\quad y = -x/\sqrt{3} + 2a/3\sqrt{3}$$

$$\text{on AC,}\quad y = x/\sqrt{3} - 2a/3\sqrt{3}$$

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Substituting these expressions in turn in Eq. (i) gives

$$\phi_{\mathrm{AB}} = \phi_{\mathrm{BC}} = \phi_{\mathrm{AC}} = 0$$

so that Eq. (i) satisfies the condition $\phi = 0$ on the boundary of the triangle.

From Eqs (3.2) and Eq. (i)

$$\tau_{zy} = -\frac{\partial \phi}{\partial x} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(x - \frac{3x^2}{2a} + \frac{3y^2}{2a}\right) \tag{ii}$$

and

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(y + \frac{3xy}{a}\right) \tag{iii}$$

At each corner of the triangular section $\tau_{zy} = \tau_{zx} = 0$. Also, from antisymmetry, the distribution of shear stress will be the same along each side. For AB, where $x = -a/3$ and $y = y$, Eqs (ii) and (iii) become

$$\tau_{zy} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(-\frac{a}{2} + \frac{3y^2}{2a}\right) \tag{iv}$$

and

$$\tau_{zx} = 0 \tag{v}$$

From Eq. (iv) the maximum value of τ_{zy} occurs at $y = 0$ and is

$$\tau_{zy}(\max) = -\frac{Ga}{2}\frac{\mathrm{d}\theta}{\mathrm{d}z} \tag{vi}$$

The distribution of shear stress along the x axis is obtained from Eqs (ii) and (iii) in which $x = x$, $y = 0$, i.e.

$$\tau_{zy} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(x - \frac{3x^2}{2a}\right) \tag{vii}$$

$$\tau_{zx} = 0$$

From Eq. (vii) τ_{zy} has a mathematical maximum at $x = +a/3$ which gives

$$\tau_{zy} = \frac{Ga}{6}\frac{\mathrm{d}\theta}{\mathrm{d}z} \tag{viii}$$

which is less than the value given by Eq. (vi). Thus the maximum value of shear stress in the section is $-(Ga/2)\,\mathrm{d}\theta/\mathrm{d}z$.

The rate of twist may be found by substituting for ϕ from Eq. (i) in Eq. (3.8). Thus

$$T = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z}\iint\left[\frac{1}{2}(x^2 + y^2) - \frac{1}{2a}(x^3 - 3xy^2) - \frac{2}{27}a^2\right]\mathrm{d}x\,\mathrm{d}y \tag{ix}$$

The equation of the side AC of the triangle is $y = (x - 2a/3)/\sqrt{3}$ and that of BC, $y = -(x - 2a/3)/\sqrt{3}$. Eq. (ix) then becomes

$$T = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z}\int_{-a/3}^{2a/3}\int_{(x-2a/3)/\sqrt{3}}^{-(x-2a/3)/\sqrt{3}}\left[\frac{1}{2}(x^2 + y^2) - \frac{1}{2a}(x^3 - 3xy^2) - \frac{2}{27}a^2\right]\mathrm{d}x\,\mathrm{d}y$$

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which gives

$$T = \frac{Ga^4}{15\sqrt{3}} \frac{d\theta}{dz}$$

so that

$$\frac{d\theta}{dz} = \frac{15\sqrt{3}T}{Ga^4} \tag{x}$$

From the first of Eqs (3.10)

$$\frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} + \frac{d\theta}{dz} y$$

Substituting for τ_{zx} from Eq. (iii)

$$\frac{\partial w}{\partial x} = -\frac{d\theta}{dz}\left(y + \frac{3xy}{a} - y\right)$$

i.e.

$$\frac{\partial w}{\partial x} = -\frac{3xy}{a}\frac{d\theta}{dz}$$

whence

$$w = -\frac{3x^2 y}{2a}\frac{d\theta}{dz} + f(y) \tag{xi}$$

Similarly from the second of Eqs (3.10)

$$w = -\frac{3x^2 y}{2a}\frac{d\theta}{dz} + \frac{y^3}{2a}\frac{d\theta}{dz} + f(x) \tag{xii}$$

Comparing Eqs (xi) and (xii)

$$f(x) = 0 \quad \text{and} \quad f(y) = \frac{y^3}{2a}\frac{d\theta}{dz}$$

Hence

$$w = \frac{1}{2a}\frac{d\theta}{dz}(y^3 - 3x^2 y)$$

S.3.5

The torsion constant, J, for the complete cross-section is found by summing the torsion constants of the narrow rectangular strips which form the section. Thus, from Eq. (3.29)

$$J = 2\frac{at^3}{3} + \frac{bt^3}{3} = \frac{(2a+b)t^3}{3}$$

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Therefore, from the general torsion equation, (3.12)

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{3T}{G(2a+b)t^3} \tag{i}$$

The maximum shear stress follows from Eqs (3.28) and (i), hence

$$\tau_{\max} = \pm Gt\frac{\mathrm{d}\theta}{\mathrm{d}z} = \pm\frac{3T}{(2a+b)t^2}$$

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Solutions to Chapter 4 Problems

S.4.1

This problem is most readily solved by the application of the unit load method. Thus, from Eq. (4.26), the vertical deflection of C is given by

$$\Delta_{V,C} = \sum \frac{F_0 F_{1,V} L}{AE} \tag{i}$$

and the horizontal deflection by

$$\Delta_{H,C} = \sum \frac{F_0 F_{1,H} L}{AE} \tag{ii}$$

in which $F_{1,V}$ and $F_{1,H}$ are the forces in a member due to a unit load positioned at C and acting vertically downwards and horizontally to the right, in turn, respectively. Further, the value of L/AE (=1/20 mm/N) for each member is given and may be omitted from the initial calculation. All member forces (see Table S.4.1) are found using the method of joints which is described in textbooks on structural analysis, for example, *Structural and Stress Analysis* by T. H. G. Megson (Arnold, 1996).

Table S.4.1

Member	F_0 (N)	$F_{1,V}$	$F_{1,H}$	$F_0F_{1,V}$	$F_0F_{1,H}$
DC	16.67	1.67	0	27.84	0
BC	−13.33	−1.33	1.0	17.73	−13.33
ED	13.33	1.33	0	17.73	0
DB	−10.0	−1.0	0	10.0	0
AB	−16.67	−1.67	0.8	27.84	−13.34
EB	0	0	0.6	0	0
				$\sum = 101.14$	$\sum = -26.67$

Note that the loads $F_{1,V}$ are obtained most easily by dividing the loads F by a factor of 10. Then, from Eq. (i)

$$\Delta_{V,C} = 101.14 \times \frac{1}{20} = 5.07\,\text{mm}$$

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which is positive and therefore in the same direction as the unit vertical load. Also from Eq. (ii)

$$\Delta_{\mathrm{H,C}} = -26.67 \times \frac{1}{20} = -1.33\,\mathrm{mm}$$

which is negative and therefore to the left.

The actual deflection, Δ, is then given by

$$\Delta = \sqrt{\Delta_{\mathrm{V,C}}^2 + \Delta_{\mathrm{H,C}}^2} = 5.24\,\mathrm{mm}$$

which is downwards and at an angle of $\tan^{-1}(1.33/5.07) = 14.7^\circ$ to the left of vertical.

S.4.2

Fig. S.4.2 shows a plan view of the plate. Suppose that the point of application of the load is at D, a distance x from each side of the plate. The deflection of D may be found using the unit load method so that, from Eq. (4.26), the vertical deflection of D is given by

$$\Delta_{\mathrm{D}} = \sum \frac{F_0 F_1 L}{AE} \tag{i}$$

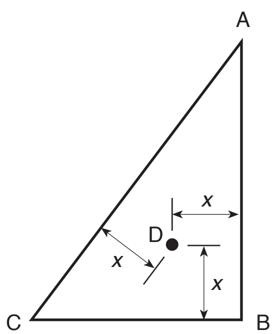

Fig. S.4.2

Initially, therefore, the forces, F_0, must be calculated. Suppose that the forces in the wires at A, B and C due to the actual load are $F_{0,\mathrm{A}}$, $F_{0,\mathrm{B}}$ and $F_{0,\mathrm{C}}$ respectively. Then resolving vertically

$$F_{0,\mathrm{A}} + F_{0,\mathrm{B}} + F_{0,\mathrm{C}} = 100 \tag{ii}$$

Taking moments about the edges BC, AC and AB in turn gives

$$F_{0,\mathrm{A}} \times 4 = 100x \tag{iii}$$

$$F_{0,\mathrm{B}} \times 4 \times \sin A = 100x$$

i.e.

$$F_{0,\mathrm{B}} \times 4 \times 0.6 = 100x \tag{iv}$$

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and

$$F_{0,\mathrm{C}} \times 3 = 100x \tag{v}$$

Thus, from Eqs (iii), (iv) and (v)

$$4F_{0,\mathrm{A}} = 2.4F_{0,\mathrm{B}} = 3F_{0,\mathrm{C}}$$

so that

$$F_{0,\mathrm{A}} = 0.6F_{0,\mathrm{B}}, \quad F_{0,\mathrm{C}} = 0.8F_{0,\mathrm{B}}$$

Substituting in Eq. (ii) gives

$$F_{0,\mathrm{B}} = 41.7\,\mathrm{N}$$

Hence

$$F_{0,\mathrm{A}} = 25.0\,\mathrm{N} \quad \text{and} \quad F_{0,\mathrm{C}} = 33.4\,\mathrm{N}$$

Now apply a unit load at D in the direction of the 100 N load. Then

$$F_{1,\mathrm{A}} = 0.25, \quad F_{1,\mathrm{B}} = 0.417, \quad F_{1,\mathrm{C}} = 0.334$$

Substituting for $F_{0,\mathrm{A}}$, $F_{1,\mathrm{A}}$ etc. in Eq. (i)

$$\Delta_{\mathrm{D}} = \frac{1440}{(\pi/4) \times 1^2 \times 196\,000}(25 \times 0.25 + 41.7 \times 0.417 + 33.4 \times 0.334)$$

i.e.

$$\Delta_{\mathrm{D}} = 0.33\,\mathrm{mm}$$

S.4.3

Suppose that joints 2 and 7 have horizontal and vertical components of displacement u_2, v_2, u_7, and v_7 respectively as shown in Fig. S.4.3. The displaced position of the member 27 is then $2'7'$. The angle α which the member 27 makes with the vertical is then given by

$$\alpha = \tan^{-1} \frac{u_7 - u_2}{3a + v_7 - v_2}$$

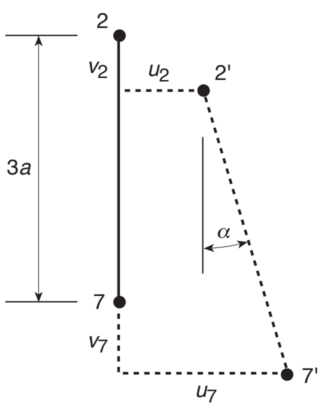

Fig. S.4.3

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Table S.4.3

Member	Length	F_0	$F_{1,2}$	$F_{1,7}$	$F_0F_{1,2}L$	$F_0F_{1,7}L$
27	$3a$	$3P$	0	0	0	0
87	$5a$	$5P/3$	0	$5/3$	0	$125Pa/9$
67	$4a$	$-4P/3$	0	$-4/3$	0	$64Pa/9$
21	$4a$	$4P$	$-4/3$	0	$-64Pa/3$	0
23	$5a$	0	$5/3$	0	0	0
26	$5a$	$-5P$	0	0	0	0
38	$3a$	0	0	0	0	0
58	$5a$	0	0	0	0	0
98	$5a$	$5P/3$	0	$5/3$	0	$125Pa/9$
68	$3a$	0	0	0	0	0
16	$3a$	$3P$	0	0	0	0
56	$4a$	$-16P/3$	0	$-4/3$	0	$256Pa/9$
13	$3a$	0	0	0	0	0
43	$5a$	0	$5/3$	0	0	0
93	$\sqrt{34}a$	0	0	0	0	0
03	$5a$	0	0	0	0	0
15	$5a$	$-5P$	0	0	0	0
10	$4a$	$8P$	$-4/3$	0	$-128Pa/3$	0
					$\sum = -192Pa/3$	$\sum = 570Pa/9$

which, since α is small and v_7 and v_2 are small compared with $3a$, may be written as

$$\alpha = \frac{u_7 - u_2}{3a} \tag{i}$$

The horizontal components u_2 and u_7 may be found using the unit load method, Eq. (4.26). Thus

$$u_2 = \sum \frac{F_0 F_{1,2} L}{AE}, \quad u_7 = \sum \frac{F_0 F_{1,7} L}{AE} \tag{ii}$$

where $F_{1,2}$ and $F_{1,7}$ are the forces in the members of the framework due to unit loads applied horizontally, in turn, at joints 2 and 7 respectively. The solution is completed in tabular form (Table S.4.3). Substituting the summation terms in Eqs (ii) gives

$$u_2 = -\frac{192Pa}{3AE}, \quad u_7 = \frac{570Pa}{9AE}$$

Now substituting for u_2 and u_7 in Eq. (i)

$$\alpha = \frac{382P}{9AE}$$

S.4.4

(a) The beam is shown in Fig. S.4.4. The principle of the stationary value of the total complementary energy may be used to determine the deflection at C. From Eq. (4.19)

$$\Delta_C = \int_L \mathrm{d}\theta \frac{\mathrm{d}M}{\mathrm{d}P} \tag{i}$$

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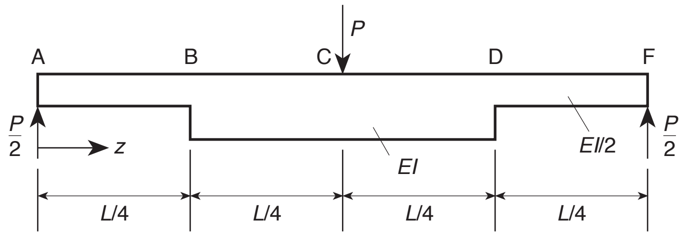

Fig. S.4.4

in which, since the beam is linearly elastic, $\mathrm{d}\theta = (M/EI)\,\mathrm{d}z$. Also the beam is symmetrical about its mid-span so that Eq. (i) may be written

$$\Delta_{\mathrm{C}} = 2\int_0^{L/2} \frac{M}{EI}\frac{\mathrm{d}M}{\mathrm{d}P}\,\mathrm{d}z \tag{ii}$$

In AC

$$M = \frac{P}{2}z$$

so that

$$\frac{\mathrm{d}M}{\mathrm{d}P} = \frac{z}{2}$$

Eq. (ii) then becomes

$$\Delta_{\mathrm{C}} = 2\left[\int_0^{L/4} \frac{Pz^2}{4(EI/2)}\,\mathrm{d}z + \int_{L/4}^{L/2} \frac{Pz^2}{4EI}\,\mathrm{d}z \tag{iii}\right.$$

Integrating Eq. (ii) and substituting the limits gives

$$\Delta_{\mathrm{C}} = \frac{3PL^3}{128EI}$$

(b) When the beam is encastré at A and F, fixed end moments M_{A} and M_{F} are induced. From symmetry $M_{\mathrm{A}} = M_{\mathrm{F}}$. The total complementary energy of the beam is, from Eq. (4.18)

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M - P\Delta_{\mathrm{C}}$$

from which

$$\frac{\partial C}{\partial M_{\mathrm{A}}} = \int_L \mathrm{d}\theta \frac{\partial M}{\partial M_{\mathrm{A}}} = 0 \tag{iv}$$

from the principle of the stationary value. From symmetry the reactions at A and F are each $P/2$. Hence

$$M = \frac{P}{2}z - M_{\mathrm{A}} \quad \text{(assuming } M_{\mathrm{A}} \text{ is a hogging moment)}$$

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Then

$$\frac{\partial M}{\partial M_A} = -1$$

Thus, from Eq. (iv)

$$\frac{\partial C}{\partial M_A} = 2\int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial M_A}\,dz = 0$$

or

$$0 = 2\left[\int_0^{L/4} \frac{1}{(EI/2)}\left(\frac{P}{2}z - M_A\right)(-1)\,dz + \int_{L/4}^{L/2} \frac{1}{EI}\left(\frac{P}{2}z - M_A\right)(-1)\,dz\right]$$

from which

$$M_A = \frac{5PL}{48}$$

S.4.5

The unit load method, i.e. the first of Eqs (4.27), may be used to obtain a solution. Thus

$$\delta_{C,H} = \int \frac{M_0 M_1}{EI}\,dz \tag{i}$$

in which the M moments are due to a unit load applied horizontally at C. Then, referring to Fig. S.4.5, in CB

$$M_0 = W(R - R\cos\theta), \quad M_1 = 1 \times z$$

and in BA

$$M_0 = W2R, \quad M_1 = 1 \times z$$

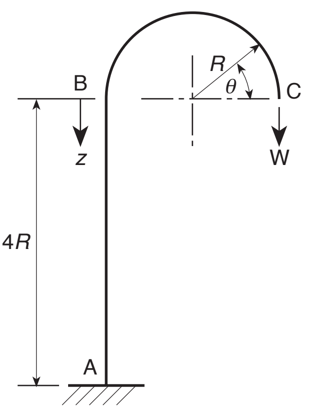

Fig. S.4.5

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Hence, substituting these expressions in Eq. (i) and noting that in CB $\mathrm{d}s = R\,\mathrm{d}\theta$ and in BA $\mathrm{d}s = \mathrm{d}z$

$$\delta_{\mathrm{C,H}} = \frac{1}{EI}\left\{\int_0^{\pi} -WR^3(1-\cos\theta)\sin\theta\,\mathrm{d}\theta + \int_0^{4R} 2WRz\,\mathrm{d}z\right\}$$

i.e.

$$\delta_{\mathrm{C,H}} = \frac{1}{EI}\left\{-WR^3\left[-\cos\theta + \frac{\cos^2\theta}{2}\right]_0^{\pi} + WR\left[z^2\right]_0^{4R}\right\}$$

so that

$$\delta_{\mathrm{C,H}} = \frac{14WR^3}{EI} \qquad \text{(ii)}$$

The second moment of area of the cross-section of the post is given by

$$I = \frac{\pi}{64}(100^4 - 94^4) = 1.076 \times 10^6\,\mathrm{mm}^4$$

Substituting the value of I and the given values of W and R in Eq. (ii) gives

$$\delta_{\mathrm{C,H}} = 53.3\,\mathrm{mm}$$

S.4.6

Either of the principles of the stationary values of the total complementary energy or the total potential energy may be used to solve this problem.

From Eq. (4.18) the total complementary energy of the system is

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M - \int_L wv\,\mathrm{d}z \qquad \text{(i)}$$

in which w is the load intensity at any point in the beam and v the vertical displacement. Eq. (i) may be written in the form

$$C = \int_L \int_0^M \frac{M}{EI}\,\mathrm{d}z\,\mathrm{d}M - \int_L wv\,\mathrm{d}z$$

since, from symmetrical bending theory

$$\delta\theta = \frac{\delta z}{R} = \frac{M}{EI}\delta z$$

Hence

$$C = \int_L \frac{M^2}{2EI}\,\mathrm{d}z - \int_L wv\,\mathrm{d}z \qquad \text{(ii)}$$

Alternatively, the total potential energy of the system is the sum of the strain energy due to bending of the beam plus the potential energy, V, of the applied load. The strain energy, U, due to bending in a beam may be shown to be given by

$$U = \int_L \frac{M^2}{2EI}\,\mathrm{d}z$$

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Hence

$$\mathrm{TPE} = U + V = \int_L \frac{M^2}{2EI}\,\mathrm{d}z - \int_L wv\,\mathrm{d}z \tag{iii}$$

Eqs (ii) and (iii) are clearly identical.

Now, from symmetrical bending theory

$$\frac{M}{EI} = -\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}$$

Therefore Eq. (ii) (or (iii)) may be rewritten

$$C = \int_0^L \frac{EI}{2}\left(\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}\right)^2 \mathrm{d}z - \int_0^L wv\,\mathrm{d}z \tag{iv}$$

Now

$$v = a_1 \sin\frac{\pi z}{L} + a_2 \sin\frac{2\pi z}{L}, \quad w = \frac{2w_0 z}{L}\left(1 - \frac{z}{2L}\right)$$

so that

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -a_1\frac{\pi^2}{L^2}\sin\frac{\pi z}{L} - a_2\frac{4\pi^2}{L^2}\sin\frac{2\pi z}{L}$$

Substituting in Eq. (iv)

$$C = \frac{EI}{2}\frac{\pi^4}{L^4}\int_0^L \left(a_1\frac{\pi^2}{L^2}\sin\frac{\pi z}{L} + a_2\frac{4\pi^2}{L^2}\sin\frac{2\pi z}{L}\right)^2 \mathrm{d}z$$
$$- \frac{2w_0}{L}\int_0^L \left(a_1 z\sin\frac{\pi z}{L} + a_2 z\sin\frac{2\pi z}{L} - a_1\frac{z^2}{2L}\sin\frac{\pi z}{L} - a_2\frac{z^2}{2L}\sin\frac{2\pi z}{L}\right)\mathrm{d}z$$

which, on expanding, gives

$$C = \frac{EI\pi^4}{2L^4}\int_0^L \left(a_1^2\sin^2\frac{\pi z}{L} + 8a_1a_2\sin\frac{\pi z}{L}\sin\frac{2\pi z}{L} + 16a_2^2\sin^2\frac{2\pi z}{L}\right)\mathrm{d}z$$
$$- \frac{2w_0}{L}\int_0^L \left(a_1 z\sin\frac{\pi z}{L} + a_2 z\sin\frac{2\pi z}{L} - a_1\frac{z^2}{2L}\sin\frac{\pi z}{L} - a_2\frac{z^2}{2L}\sin\frac{2\pi z}{L}\right)\mathrm{d}z \tag{v}$$

Eq. (v) may be integrated by a combination of direct integration and integration by parts and gives

$$C = \frac{EI\pi^4}{2L^4}\left(\frac{a_1^2 L}{2} + 8a_2^2 L\right) - a_1 w_0 L\left(\frac{1}{\pi} + \frac{4}{\pi^3}\right) + \frac{a_2 w_0 L}{2\pi} \tag{vi}$$

From the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial C}{\partial a_2} = 0$$

From Eq. (vi)

$$\frac{\partial C}{\partial a_1} = 0 = a_1\frac{EI\pi^4}{2L^3} - \frac{w_0 L}{\pi^3}(\pi^2 + 4)$$

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Hence

$$a_1 = \frac{2w_0L^4}{EI\pi^7}(\pi^2 + 4)$$

Also

$$\frac{\partial C}{\partial a_2} = 0 = a_2\frac{8EI\pi^4}{L^3} + \frac{w_0L}{2\pi}$$

whence

$$a_2 = -\frac{w_0L^4}{16EI\pi^5}$$

The deflected shape of the beam is then

$$v = \frac{w_0L^4}{EI}\left[\frac{2}{\pi^7}(\pi^2 + 4)\sin\frac{\pi z}{L} - \frac{1}{16\pi^5}\sin\frac{2\pi z}{L}\right]$$

At mid-span when $z = L/2$

$$v = 0.00918\frac{w_0L^4}{EI}$$

S.4.7

This problem is solved in a similar manner to P.4.6. Thus Eq. (iv) of S.4.6 is directly applicable, i.e.

$$C = \int_L \frac{EI}{2}\left(\frac{d^2v}{dz^2}\right)^2 dz - \int_L wv\, dz \qquad \text{(i)}$$

in which

$$v = \sum_{i=1}^{\infty} a_i \sin\frac{i\pi z}{L} \qquad \text{(ii)}$$

and w may be expressed as a function of z in the form $w = 4w_0z(L - z)/L^2$ which satisfies the boundary conditions of $w = 0$ at $z = 0$ and $z = L$ and $w = w_0$ at $z = L/2$.

From Eq. (ii)

$$\frac{d^2v}{dz^2} = -\sum_{i=1}^{\infty} a_i \frac{i^2\pi^2}{L^2}\sin\frac{i\pi z}{L}$$

Substituting in Eq. (i)

$$C = \frac{EI}{2}\int_0^L \sum_{i=1}^{\infty} a_i^2 \frac{i^4\pi^4}{L^4}\sin^2\frac{i\pi z}{L}\, dz - \frac{4w_0}{L^2}\int_0^L z(L - z)\sum_{i=1}^{\infty} a_i \sin\frac{i\pi z}{L}\, dz \qquad \text{(iii)}$$

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Now

$$\int_0^L \sin^2 \frac{i\pi z}{L}\,\mathrm{d}z = \int_0^L \frac{1}{2}\left(1 - \cos\frac{i2\pi z}{L}\right)\mathrm{d}z = \left[\frac{z}{2} - \frac{L}{i2\pi}\sin\frac{i2\pi z}{L}\right]_0^L = \frac{L}{2}$$

$$\int_0^L Lz \sin\frac{i\pi z}{L}\,\mathrm{d}z = L\left[-\frac{zL}{i\pi}\cos\frac{i\pi z}{L} + \int \frac{L}{i\pi}\cos\frac{i\pi z}{L}\,\mathrm{d}z\right]_0^L = -\frac{L^3}{i\pi}\cos i\pi$$

$$\int_0^L z^2 \sin\frac{i\pi z}{L}\,\mathrm{d}z = \left[-\frac{z^2 L}{i\pi}\cos\frac{i\pi z}{L} + \int \frac{L}{i\pi}\cos\frac{i\pi z}{L} 2z\,\mathrm{d}z\right]_0^L$$

$$= -\frac{L^3}{i\pi}\cos i\pi + \frac{2L^3}{i^3\pi^3}(\cos i\pi - 1)$$

Thus Eq. (iii) becomes

$$C = \sum_{i=1}^{\infty} \frac{EIa_i^2 i^4 \pi^4}{4L^3} - \frac{4w_0}{L^2}\sum_{i=1}^{\infty} a_i\left[-\frac{L^3}{i\pi}\cos i\pi + \frac{L^3}{i\pi}\cos i\pi - \frac{2L^3}{i^3\pi^3}(\cos i\pi - 1)\right]$$

or

$$C = \sum_{i=1}^{\infty} \frac{EIa_i^2 i^4 \pi^4}{4L^3} - \frac{4w_0}{L^2}\sum_{i=1}^{\infty} \frac{2a_i L^3}{i^3\pi^3}(1 - \cos i\pi) \tag{iv}$$

The value of $(1 - \cos i\pi)$ is zero when i is even and 2 when i is odd. Therefore Eq. (iv) may be written

$$C = \frac{EIa_i^2 i^4 \pi^4}{4L^3} - \frac{16 w_0 a_i L}{i^3\pi^3} \qquad i \text{ is odd}$$

From the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial a_i} = \frac{EIa_i i^4 \pi^4}{2L^3} - \frac{16 w_0 L}{i^3\pi^3} = 0$$

Hence

$$a_i = \frac{32 w_0 L^4}{EI i^7 \pi^7}$$

Thus

$$v = \sum_{i=1}^{\infty} \frac{32 w_0 L^4}{EI i^7 \pi^7}\sin\frac{i\pi z}{L} \qquad i \text{ is odd}$$

At the mid-span point where $z = L/2$ and using the first term only in the expression for v

$$v_{\text{m.s.}} = \frac{w_0 L^4}{94.4 EI}$$

S.4.8

The lengths of the members which are not given are:

$$L_{12} = 9\sqrt{2}a, \quad L_{13} = 15a, \quad L_{14} = 13a, \quad L_{24} = 5a$$

The force in the member 14 due to the temperature change is compressive and equal to $0.7A$. Also the change in length, Δ_{14}, of the member 14 due to a temperature change T is $L_{14}\alpha T = 13a \times 2.4 \times 10^{-6}T$. This must also be equal to the change in length produced by the force in the member corresponding to the temperature rise. Let this force be R.

From the unit load method, Eq. (4.26)

$$\Delta_{14} = \sum \frac{F_0 F_1 L}{AE} \tag{i}$$

In this case, since R and the unit load are applied at the same points, in the same direction and no other loads are applied when only the temperature change is being considered, $F_0 = RF_1$. Eq. (i) may then be written

$$\Delta_{14} = R\sum \frac{F_1^2 L}{AE} \tag{ii}$$

The method of joints may be used to determine the F_1 forces in the members. Thus

$$F_{14} = 1, \quad F_{13} = -35/13, \quad F_{12} = 16\sqrt{2}/13, \quad F_{24} = -20/13, \quad F_{23} = 28/13$$

Eq. (ii) then becomes

$$\Delta_{14} = \frac{R}{E}\left[\frac{1^2 \times 13a}{A} + \frac{35^2 \times 15a}{13^2 AE} + \frac{(16\sqrt{2})^2 \times 9\sqrt{2}a}{13^2\sqrt{2}AE} + \frac{20^2 \times 5a}{13^2 AE} + \frac{28^2 \times 3a}{13^2 AE}\right]$$

or

$$\Delta_{14} = \frac{Ra}{13^2 AE}(13^3 + 35^2 \times 15 + 16^2 \times 18 + 20^2 \times 5 + 28^2 \times 3)$$

i.e.

$$\Delta_{14} = \frac{29\,532aR}{13^2 AE}$$

Then

$$13a \times 24 \times 10^{-6}T = \frac{29\,532a(0.7A)}{13^2 AE}$$

so that

$$T = 5.6^\circ$$

S.4.9

Referring to Figs P.4.9(a), (b) and S.4.9 it can be seen that the members of 12, 24 and 23 remain unloaded until P has moved through a horizontal distance $0.25\cos\alpha$, i.e. a

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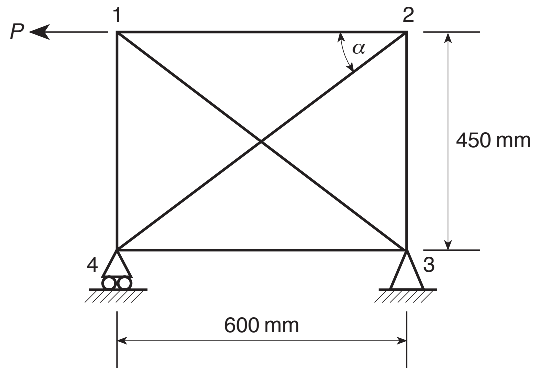

Fig. S.4.9

distance of $0.25 \times 600/750 = 0.2$ mm. Thus, until P has moved through a horizontal distance of 0.2 mm P is equilibrated solely by the forces in the members 13, 34 and 41 which therefore form a triangular framework. The method of solution is to find the value of P which causes a horizontal displacement of 0.2 mm of joint 1 in this framework.

Using the unit load method, i.e. Eq. (4.26) and solving in tabular form (see Table S.4.9(a)).

Table S.4.9(a)

Member	Length (mm)	F_0	F_1	F_0F_1L
13	750	$1.25P$	1.25	$1171.9P$
14	450	$-0.75P$	-0.75	$253.1P$
43	600	0	0	0
				$\sum = 1425.0P$

Then

$$0.2 = \frac{1425.0P}{300 \times 70\,000}$$

from which

$$P = 2947\,\text{N}$$

The corresponding forces in the members 13, 14 and 43 are then

$$F_{13} = 3683.8\,\text{N}, \quad F_{14} = -2210.3\,\text{N}, \quad F_{43} = 0$$

When $P = 10\,000$ N additional forces will be generated in these members corresponding to a load of $P' = 10\,000 - 2947 = 7053$ N. Also P' will now produce forces in the remaining members 12, 24 and 23 of the frame. The solution is now completed in a similar manner to that for the frame shown in Fig. 4.11 using Eq. (4.22). Suppose that R is the force in the member 24; the solution is continued in Table S.4.9(b). From Eq. (4.22)

$$2592R + 1140P' = 0$$

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Table S.4.9(b)

Member	Length (mm)	F	$\partial F/\partial R$	$FL(\partial F/\partial R)$
12	600	$-0.8R$	-0.8	$384R$
23	450	$-0.6R$	-0.6	$162R$
34	600	$-0.8R$	-0.8	$384R$
41	450	$-(0.6R + 0.75P')$	-0.6	$162R + 202.5P'$
13	750	$R + 1.25P'$	1.0	$750R + 937.5P'$
24	750	R	1.0	$750R$
				$\sum = 2592R + 1140P'$

so that

$$R = -\frac{1140 \times 7053}{2592}$$

i.e.

$$R = -3102\,\text{N}$$

Then

$$F_{12} = -0.8 \times (-3102) = 2481.6\,\text{N (tension)}$$

$$F_{23} = -0.6 \times (-3102) = 1861.2\,\text{N (tension)}$$

$$F_{34} = -0.8 \times (-3102) = 2481.6\,\text{N (tension)}$$

$$F_{41} = -0.6 \times (-3102) - 0.75 \times 7053 - 2210.3 = -5638.9\,\text{N (compression)}$$

$$F_{13} = -3102 + 1.25 \times 7053 + 3683.8 = 9398.1\,\text{N (tension)}$$

$$F_{24} = -3102.0\,\text{N (compression)}$$

S.4.10

Referring to Fig. S.4.10(a) the vertical reactions at A and D are found from statical equilibrium. Thus, taking moments about D

$$R_A \tfrac{2}{3} l + \tfrac{1}{2} lw \tfrac{2}{3} l = 0$$

i.e.

$$R_A = -\frac{wl}{2} \text{ (downwards)}$$

Hence

$$R_D = \frac{wl}{2} \text{ (upwards)}$$

Also for horizontal equilibrium

$$H_A + \frac{wl}{2} = H_D \tag{i}$$

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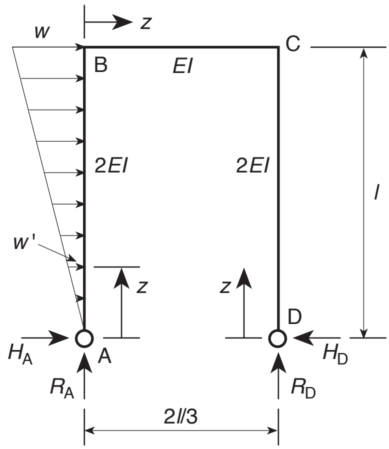

Fig. S.4.10(a)

The total complementary energy of the frame is, from Eq. (4.18)

$$C = \int_L \int_0^M \mathrm{d}\theta \, \mathrm{d}M - H_A \Delta_{A,H} - R_A \Delta_{A,V} - H_D \Delta_{D,H} - R_D \Delta_{D,V} + \int_0^l w' \Delta \, \mathrm{d}z \qquad \text{(ii)}$$

in which $\Delta_{A,H}$, $\Delta_{A,V}$, $\Delta_{D,H}$ and $\Delta_{D,V}$ are the horizontal and vertical components of the displacements at A and D respectively and Δ is the horizontal displacement of the member AB at any distance z from A. From the principle of the stationary value of the total complementary energy of the frame and selecting $\Delta_{A,H}$ as the required displacement

$$\frac{\partial C}{\partial H_A} = \int_L \mathrm{d}\theta \frac{\partial M}{\partial H_A} - \Delta_{A,H} = 0 \qquad \text{(iii)}$$

In this case $\Delta_{A,H} = 0$ so that Eq. (iii) becomes

$$\int_L \mathrm{d}\theta \frac{\partial M}{\partial H_A} = 0$$

or, since $\mathrm{d}\theta = (M/EI)\,\mathrm{d}z$

$$\int_L \frac{M}{EI} \frac{\partial M}{\partial H_A} \, \mathrm{d}z = 0 \qquad \text{(iv)}$$

In AB

$$M = -H_A z - \frac{wz^3}{6l}, \quad \frac{\partial M}{\partial H_A} = -z$$

In BC

$$M = R_A z - H_A l - \frac{wl^2}{6}, \quad \frac{\partial M}{\partial H_A} = -l$$

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In DC

$$M = -H_{\mathrm{D}}z = -\left(H_{\mathrm{A}} + \frac{wl}{2}\right)z \text{ from Eq. (i)}, \quad \frac{\partial M}{\partial H_{\mathrm{A}}} = -z$$

Substituting these expressions in Eq. (iv) gives

$$\int_0^l \frac{1}{2EI}\left(-H_{\mathrm{A}}z - \frac{wz^3}{6l}\right)(-z)\,\mathrm{d}z + \int_0^{2l/3} \frac{1}{EI}\left(-\frac{wl}{2}z - H_{\mathrm{A}}l - \frac{wl^2}{6}\right)(-l)\,\mathrm{d}z$$
$$+ \int_0^l \frac{1}{2EI}\left(-H_{\mathrm{A}} - \frac{wl}{2}\right)z(-z)\,\mathrm{d}z = 0$$

or

$$\frac{1}{2}\int_0^l \left(H_{\mathrm{A}}z^2 + \frac{wz^4}{6l}\right)\mathrm{d}z + \int_0^{2l/3}\left(\frac{wl^2}{2}z + H_{\mathrm{A}}l^2 + \frac{wl^3}{6}\right)\mathrm{d}z$$
$$+ \frac{1}{2}\int_0^l \left(H_{\mathrm{A}}z^2 + \frac{wlz^2}{2}\right)\mathrm{d}z = 0$$

from which

$$2H_{\mathrm{A}}l^3 + \frac{29}{45}wl^4 = 0$$

or

$$H_{\mathrm{A}} = -29wl/90$$

Hence, from Eq. (i)

$$H_{\mathrm{D}} = 8wl/45$$

Thus

$$M_{\mathrm{AB}} = -H_{\mathrm{A}}z - \frac{wz^3}{6l} = \frac{29wl}{90}z - \frac{w}{6l}z^3$$

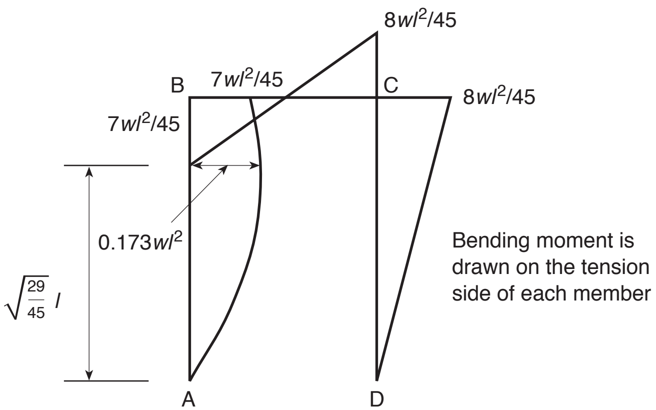

Fig. S.4.10(b)

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When $z = 0$, $M_{\mathrm{AB}} = 0$ and when $z = l$, $M_{\mathrm{AB}} = 7wl^2/45$. Also, $\mathrm{d}M_{\mathrm{AB}}/\mathrm{d}z = 0$ for a turning value, i.e.

$$\frac{\mathrm{d}M_{\mathrm{AB}}}{\mathrm{d}z} = \frac{29wl}{90} - \frac{3wz^2}{6l} = 0$$

from which $z = \sqrt{29/45}l$. Hence $M_{\mathrm{AB}}(\max) = 0.173wl^2$.

The bending moment distributions in BC and CD are linear and $M_{\mathrm{B}} = 7wl^2/45$, $M_{\mathrm{D}} = 0$ and $M_{\mathrm{C}} = H_{\mathrm{D}}l = 8wl^2/45$.

The complete bending moment diagram for the frame is shown in Fig. S.4.10(b).

S.4.11

The bracket is shown in Fig. S.4.11 in which R_{C} is the vertical reaction at C and M_{C} is the moment reaction at C in the vertical plane containing AC.

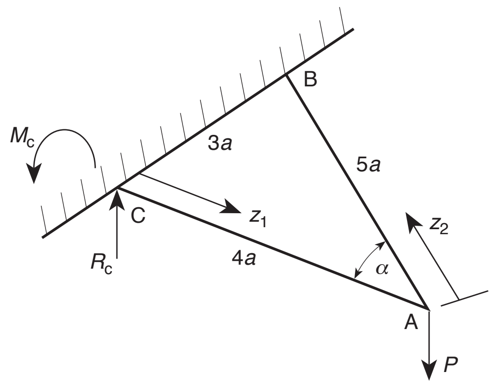

Fig. S.4.11

From Eq. (4.18) the total complementary energy of the bracket is given by

$$C = \int_L \int_0^M \mathrm{d}\theta\, \mathrm{d}M + \int_L \int_0^T \mathrm{d}\phi\, \mathrm{d}T - M_{\mathrm{C}}\theta_{\mathrm{C}} - R_{\mathrm{C}}\Delta_{\mathrm{C}} - P\Delta_{\mathrm{A}}$$

in which T is the torque in AB producing an angle of twist, ϕ, at any section and the remaining symbols have their usual meaning. Then, from the principle of the stationary value of the total complementary energy and since $\theta_{\mathrm{C}} = \Delta_{\mathrm{C}} = 0$

$$\frac{\partial C}{\partial R_{\mathrm{C}}} = \int_L \frac{M}{EI}\frac{\partial M}{\partial R_{\mathrm{C}}}\,\mathrm{d}z + \int_L \frac{T}{GJ}\frac{\partial T}{\partial R_{\mathrm{C}}}\,\mathrm{d}z = 0 \qquad \text{(i)}$$

and

$$\frac{\partial C}{\partial M_{\mathrm{C}}} = \int_L \frac{M}{EI}\frac{\partial M}{\partial M_{\mathrm{C}}}\,\mathrm{d}z + \int_L \frac{T}{GJ}\frac{\partial T}{\partial M_{\mathrm{C}}}\,\mathrm{d}z = 0 \qquad \text{(ii)}$$

From Fig. S.4.11

$$M_{\mathrm{AC}} = R_{\mathrm{C}}z_1 - M_{\mathrm{C}}, \quad T_{\mathrm{AC}} = 0$$

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so that

$$\frac{\partial M_{AC}}{\partial R_C} = z_1, \quad \frac{\partial M_{AC}}{\partial M_C} = -1, \quad \frac{\partial T_{AC}}{\partial R_C} = \frac{\partial T_{AC}}{\partial M_C} = 0$$

Also

$$M_{AB} = -Pz_2 + R_C(z_2 - 4a\cos\alpha) + M_C\cos\alpha$$

i.e.

$$M_{AB} = -Pz_2 + R_C\left(z_2 - \frac{16a}{5}\right) + \frac{4}{5}M_C$$

Hence

$$\frac{\partial M_{AB}}{\partial R_C} = z_2 - \frac{16a}{5}, \quad \frac{\partial M_{AB}}{\partial M_C} = \frac{4}{5}$$

Finally

$$T_{AB} = R_C 4a\sin\alpha - M_C\sin\alpha$$

i.e.

$$T_{AB} = \frac{12a}{5}R_C - \frac{3}{5}M_C$$

so that

$$\frac{\partial T_{AB}}{\partial R_C} = \frac{12a}{5}, \quad \frac{\partial T_{AB}}{\partial M_C} = -\frac{3}{5}$$

Substituting these expressions in Eq. (i)

$$\int_0^{4a} \frac{1}{EI}(R_C z_1 - M_C)z_1\,dz_1 + \int_0^{5a} \frac{1}{1.5EI}\left[-Pz_2 + R_C\left(z_2 - \frac{16a}{5}\right) + \frac{4}{5}M_C\right]$$
$$\times\left(z_2 - \frac{16a}{5}\right)dz_2 + \int_0^{5a} \frac{1}{3GI}\left(\frac{12a}{5}R_C - \frac{3}{5}M_C\right)\frac{12a}{5}\,dz_2 = 0 \qquad \text{(iii)}$$

Note that for the circular section tube AC the torsion constant J (i.e. the polar second moment of area) $= 2 \times 1.5I$ from the theorem of perpendicular axes.

Integrating Eq. (iii), substituting the limits and noting that $G/E = 0.38$ gives

$$55.17R_C a - 16.18M_C - 1.11Pa = 0 \qquad \text{(iv)}$$

Now substituting in Eq. (ii) for M_{AC}, $\partial M_{AC}/\partial M_C$ etc

$$\int_0^{4a} \frac{1}{EI}(R_C z_1 - M_C)(-1)\,dz_1 + \int_0^{5a} \frac{1}{1.5EI}\left[-Pz_2 + R_C\left(z_2 - \frac{16a}{5}\right) + \frac{4}{5}M_C\right]\frac{4}{5}\,dz_2$$
$$+ \int_0^{5a} \frac{1}{3GI}\left(\frac{12a}{5}R_C - \frac{3}{5}M_C\right)\left(-\frac{3}{5}\right)dz_2 = 0 \qquad \text{(v)}$$

from which

$$16.58R_C a - 7.71M_C + 6.67Pa = 0 \qquad \text{(vi)}$$

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Solving the simultaneous equations (iv) and (vi) gives

$$R_C = 0.72P$$

S.4.12

Suppose that R is the tensile force in the member 23, i.e. $R = xP_0$. Then, from Eq. (4.21)

$$\sum \lambda_i \frac{\partial F_i}{\partial R} = 0 \tag{i}$$

in which, for members 12, 23 and 34

$$\lambda_i = \varepsilon L_i = \frac{\tau_i L_i}{E}\left[1 + \left(\frac{\tau_i}{\tau_0}\right)^n\right] \tag{ii}$$

But $\tau_i = F_i/A_i$ so that Eq. (ii) may be written

$$\lambda_i = \frac{F_i L_i}{A_i E_i}\left[1 + \left(\frac{F_i}{A_i \tau_0}\right)^n\right] \tag{iii}$$

For members 15, 25, 35 and 45 which are linearly elastic

$$\lambda_i = \frac{F_i L_i}{A_i E} \tag{iv}$$

The solution is continued in Table S.4.12. Summing the final column in Table S.4.12 gives

$$\frac{4RL}{\sqrt{3}AE}\left[1 + (\alpha x)^n\right] + \frac{2\sqrt{3}RL}{AE}\left[1 + (\alpha x)^n\right] + \frac{8L}{\sqrt{3}AE}(P_0 + 2R/\sqrt{3}) + \frac{16RL}{\sqrt{3}AE} = 0 \tag{v}$$

from Eq. (1).

Table S.4.12

Member	L_i	A_i	F_i	$\partial F_i/\partial R$	λ_i	$\lambda_i\, \partial F_i/\partial R$
12	$2L$	$A/\sqrt{3}$	$R/\sqrt{3}$	$1/\sqrt{3}$	$\frac{2RL}{AE}\left[1 + \left(\frac{R}{A\tau_0}\right)^n\right]$	$\frac{2RL}{\sqrt{3}AE}[1 + (\alpha x)^n]$
23	$2L\sqrt{3}$	A	R	1	$\frac{2\sqrt{3}RL}{AE}\left[1 + \left(\frac{R}{A\tau_0}\right)^n\right]$	$\frac{2\sqrt{3}RL}{AE}\left[1 + (\alpha x)^n\right]$
34	$2L$	$A/\sqrt{3}$	$R/\sqrt{3}$	$1/\sqrt{3}$	$\frac{2RL}{AE}\left[1 + \left(\frac{R}{A\tau_0}\right)^n\right]$	$\frac{2RL}{\sqrt{3}AE}[1 + (\alpha x)^n]$
15	$2L$	A	$-P_0 - 2R/\sqrt{3}$	$-2/\sqrt{3}$	$-\frac{(P_0 + 2R/\sqrt{3})2L}{AE}$	$\frac{4L}{\sqrt{3}AE}(P_0 + 2R/\sqrt{3})$
25	$2L$	$A/\sqrt{3}$	$-2R/\sqrt{3}$	$-2/\sqrt{3}$	$-\frac{4RL}{AE}$	$\frac{8RL}{\sqrt{3}AE}$
35	$2L$	$A/\sqrt{3}$	$-2R/\sqrt{3}$	$-2/\sqrt{3}$	$-\frac{4RL}{AE}$	$\frac{8RL}{\sqrt{3}AE}$
45	$2L$	A	$-P_0 - 2R/\sqrt{3}$	$-2/\sqrt{3}$	$-\frac{2L}{AE}(P_0 + 2R/\sqrt{3})$	$\frac{4L}{\sqrt{3}AE}(P_0 + 2R/\sqrt{3})$

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Noting that $R = xP_0$, Eq. (v) simplifies to

$$4x[1 + (\alpha x)^n] + 6x[1 + (\alpha x)^n] + 8 + 16x/\sqrt{3} + 16x = 0$$

or

$$10x(\alpha x)^n + x(10 + 16/\sqrt{3} + 16) + 8 = 0$$

from which

$$\alpha^n x^{n+1} + 3.5x + 0.80 = 0$$

S.4.13

Suppose that the vertical reaction between the two beams at C is P. Then the force system acting on the beam AB is as shown in Fig. S.4.13. Taking moments about B

$$R_A \times 9.15 + P \times 6.1 - 100 \times 3.05 = 0$$

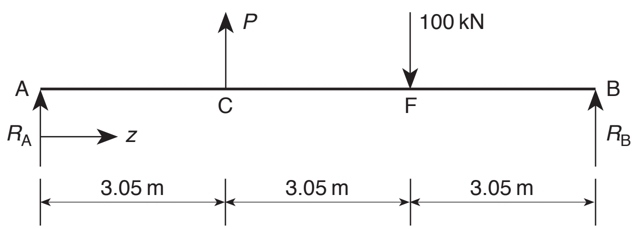

Fig. S.4.13

so that

$$R_A = 33.3 - 0.67P$$

The total complementary energy of the beam is, from Eq. (4.18)

$$C = \int_L \int_0^M d\theta\, dM - P\Delta_C - 100\Delta_F = 0$$

where Δ_C and Δ_F are the vertical displacements at C and F respectively. Then, from the principle of the stationary value of the total complementary energy of the beam

$$\frac{\partial C}{\partial P} = \int_L d\theta \frac{\partial M}{\partial P} - \Delta_C = 0$$

whence, as in previous cases

$$\Delta_C = \int_L \frac{M}{EI} \frac{\partial M}{\partial P}\, dz \qquad \text{(i)}$$

In AC

$$M_{AC} = R_A z = (33.3 - 0.67P)z$$

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so that

$$\frac{\partial M_{\mathrm{AC}}}{\partial P} = -0.67z$$

In CF

$$M_{\mathrm{CF}} = R_{\mathrm{A}}z + P(z - 3.05) = 33.3z + P(0.33z - 3.05)$$

from which

$$\frac{\partial M_{\mathrm{CF}}}{\partial P} = 0.33z - 3.05$$

In FB

$$M_{\mathrm{FB}} = R_{\mathrm{A}}z + P(z - 3.05) - 100(z - 6.1) = -66.7z + 610 + P(0.33z - 3.04)$$

which gives

$$\frac{\partial M_{\mathrm{FB}}}{\partial P} = 0.33z - 3.04$$

Substituting these expressions in Eq. (i)

$$\begin{aligned} EI\Delta_{\mathrm{C}} = &\int_0^{3.05} (33.3 - 0.67P)z(-0.67z)\,\mathrm{d}z \\ &+ \int_{3.05}^{6.1} [33.3z + P(0.33z - 3.05)](0.33z - 3.05)\,\mathrm{d}z \\ &+ \int_{6.1}^{9.15} [-66.7z + 610 + P(0.33z - 3.05)](0.33z - 3.05)\,\mathrm{d}z \end{aligned}$$

which simplifies to

$$\begin{aligned} EI\Delta_{\mathrm{C}} = &\int_0^{3.05} (-22.2z^2 + 0.44Pz^2)\,\mathrm{d}z \\ &+ \int_{3.05}^{6.1} (10.99z^2 + 0.11Pz^2 - 2.02Pz + 9.3P - 101.6z)\,\mathrm{d}z \\ &+ \int_{6.1}^{9.15} (-22.01z^2 + 404.7z + 0.11Pz^2 - 2.02Pz + 9.3P - 1860.5)\,\mathrm{d}z \end{aligned}$$

Integrating this equation and substituting the limits gives

$$EI\Delta_{\mathrm{C}} = 12.78P - 1117.8 \tag{ii}$$

From compatibility of displacement, the displacement at C in the beam AB is equal to the displacement at C in the beam ED. The displacement at the mid-span point in a fixed beam of span L which carries a central load P is $PL^3/192EI$. Hence, equating this value to Δ_{C} in Eq. (ii) and noting that Δ_{C} in Eq. (ii) is positive in the direction of P

$$-(12.78P - 1117.8) = P \times 6.1^3/192$$

which gives

$$P = 80.1\,\mathrm{kN}$$

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Thus

$$\Delta_C = \frac{80.1 \times 10^3 \times 6.1^3 \times 10^9}{192 \times 200\,000 \times 83.5 \times 10^6}$$

i.e.

$$\Delta_C = 5.6\,\text{mm}$$

Note: the use of complementary energy in this problem produces a rather lengthy solution. A quicker approach to finding the displacement Δ_C in terms of P for the beam AB would be to use Macauley's method (see, for example, *Structural and Stress Analysis* by T. H. G. Megson (Arnold, 1996)).

S.4.14

The internal force system in the framework and beam is statically determinate so that the unit load method may be used directly to determine the vertical displacement of D. Hence, from the first of Eqs (4.27) and Eq. (4.26)

$$\Delta_{D,V} = \int_L \frac{M_0 M_1}{EI}\,dz + \sum_{i=1}^{k} \frac{F_{i,0}F_{i,1}L_i}{A_i E_i} \tag{i}$$

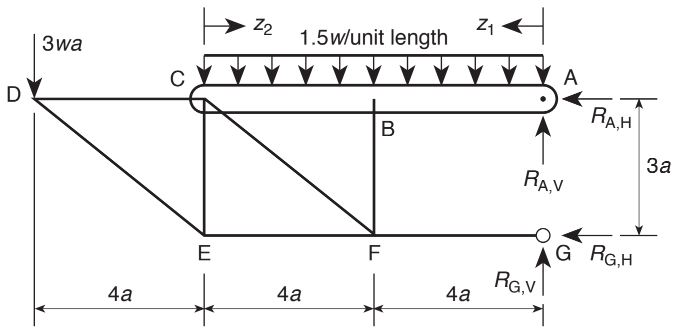

Fig. S.4.14

Referring to Fig. S.4.14 and taking moments about A

$$R_{G,H}3a - 1.5w\frac{(8a)^2}{2} - 3wa12a = 0$$

from which

$$R_{G,H} = 28wa$$

Hence

$$R_{A,H} = -28wa$$

From the vertical equilibrium of the support G, $R_{G,V} = 0$, so that, resolving vertically

$$R_{A,V} - 1.5w8a - 3a = 0$$

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i.e.

$$R_{\mathrm{A,V}} = 15wa$$

With a unit vertical load at D

$$R_{\mathrm{G,H}} = 4, \quad R_{\mathrm{A,H}} = -4, \quad R_{\mathrm{A,V}} = 1, \quad R_{\mathrm{G,V}} = 0$$

For the beam ABC, in AB

$$M_0 = R_{\mathrm{A,V}} z_1 - \frac{1.5wz^2}{2} = 15waz_1 - 0.75wz_1^2, \quad M_1 = 1 \times z_1$$

and in BC

$$M_0 = 15waz_2 - 0.75wz_2^2, \quad M_1 = 1 \times z_2$$

Hence

$$\int_L \frac{M_0 M_1}{EI}\,\mathrm{d}z = \frac{16}{Aa^2E}\left[\int_0^{4a}(15waz_1^2 - 0.75wz_1^3)\,\mathrm{d}z_1 + \int_0^{4a}(15waz_2 - 0.75wz_2^3)\,\mathrm{d}z_2\right]$$

Suppose $z_1 = z_2 = z$ say, then

$$\int_L \frac{M_0 M_1}{EI}\,\mathrm{d}z = \frac{16}{Aa^2E}2\int_0^{4a}(15waz^2 - 0.75wz^3)\,\mathrm{d}z = \frac{32w}{Aa^2E}\left[5az^3 - \frac{0.75}{4}z^4\right]_0^{4a}$$

i.e.

$$\int_L \frac{M_0 M_1}{EI}\,\mathrm{d}z = \frac{8704wa^2}{AE}$$

The solution is continued in Table S.4.14.

Table S.4.14

Member	L	A	F_0	F_1	F_0F_1L/A
AB	$4a$	$4A$	$28wa$	4	$112wa^2/A$
BC	$4a$	$4A$	$28wa$	4	$112wa^2/A$
CD	$4a$	A	$4wa$	4/3	$64wa^2/3A$
DE	$5a$	A	$-5wa$	$-5/3$	$125wa^2/3A$
EF	$4a$	A	$-4wa$	$-4/3$	$64wa^2/3A$
FG	$4a$	A	$-28wa$	-4	$448wa^2/A$
CE	$3a$	A	$3wa$	1	$9wa^2/A$
CF	$5a$	A	$-30wa$	$-10/3$	$500wa^2/A$
BF	$3a$	A	$18wa$	2	$108wa^2/A$
					$\sum = 4120wa^2/3A$

Thus

$$\Delta_{\mathrm{D}} = \frac{8704wa^2}{AE} + \frac{4120wa^2}{3AE}$$

i.e.

$$\Delta_{\mathrm{D}} = \frac{30\,232wa^2}{3AE}$$

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S.4.15

The internal force systems at C and D in the ring frame are shown in Fig. S.4.15. The total complementary energy of the half-frame is, from Eq. (4.18)

$$C = \int_L \int_0^M \mathrm{d}\theta \, \mathrm{d}M - F\Delta_B$$

in which Δ_B is the horizontal displacement of the joint B. Note that, from symmetry, the translational and rotational displacements at C and D are zero. Hence, from the principle of the stationary value of the total complementary energy and choosing the horizontal displacement at C (= 0) as the unknown

$$\frac{\partial C}{\partial N_C} = \int_L \frac{M}{EI} \frac{\partial M}{\partial N_C} \, \mathrm{d}z = 0 \tag{i}$$

In CB

$$M_{CB} = M_C - N_C(r - r\cos\theta_1) \tag{ii}$$

At B, $M_{CB} = 0$. Thus

$$M_C = N_C(r + r\sin 30^\circ) = 1.5N_C r \tag{iii}$$

Eq. (ii) then becomes

$$M_{CB} = N_C r(0.5 + \cos\theta_1) \tag{iv}$$

Then

$$\frac{\partial M_{CB}}{\partial N_C} = r(0.5 + \cos\theta_1) \tag{v}$$

In DB

$$M_{DB} = M_D - N_D(r - r\cos\theta_2) \tag{vi}$$

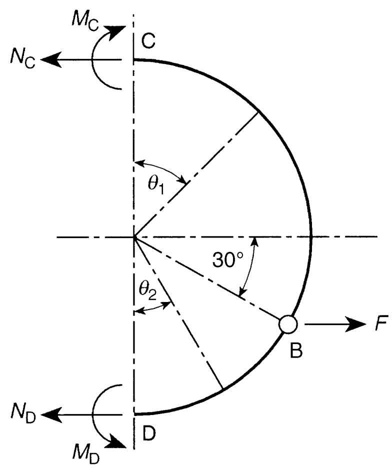

Fig. S.4.15

Again the internal moment at B is zero so that

$$M_D = N_D(r - r\sin 30^\circ) = 0.5N_D r \tag{vii}$$

Hence

$$M_{DB} = N_D r(\cos\theta_2 - 0.5) \tag{viii}$$

Also, from horizontal equilibrium

$$N_D + N_C = F$$

so that

$$N_D = F - N_C$$

and Eq. (viii) may be written

$$M_{DB} = (F - N_C)r(\cos\theta_2 - 0.5) \tag{ix}$$

whence

$$\frac{\partial M_{DB}}{\partial N_C} = -r(\cos\theta_2 - 0.5) \tag{x}$$

Substituting from Eqs (iv), (v), (ix) and (x) in Eq. (i)

$$\int_0^{120^\circ} \frac{1}{EI} N_C r^3 (0.5 + \cos\theta_1)^2 \, d\theta_1 - \int_0^{60^\circ} \frac{(F - N_C)}{xEI} r^3 (\cos\theta_2 - 0.5)^2 \, d\theta_2 = 0$$

i.e.

$$N_C \int_0^{120^\circ} (0.25 + \cos\theta_1 + \cos\theta_1^2)\, d\theta_1 - \frac{(F - N_C)}{x} \int_0^{60^\circ} (\cos^2\theta_2 - \cos\theta_2 + 0.25)\, d\theta_2 = 0$$

which, when expanded becomes

$$N_C \int_0^{120^\circ} \left(0.75 + \cos\theta_1 + \frac{\cos 2\theta_1}{2}\right) d\theta_1 - \frac{(F - N_C)}{x} \times \int_0^{60^\circ} \left(\frac{\cos 2\theta_2}{2} - \cos\theta_2 + 0.75\right) d\theta_2 = 0$$

Hence

$$N_C \left[0.75\theta_1 + \sin\theta_1 + \frac{\sin 2\theta_1}{4}\right]_0^{120^\circ} - \frac{(F - N_C)}{x} \left[\frac{\sin 2\theta_2}{4} - \sin\theta_2 + 0.75\theta_2\right]_0^{60^\circ} = 0$$

from which

$$2.22N_C - 0.136\frac{(F - N_C)}{x} = 0 \tag{xi}$$

The maximum bending moment in ADB is equal to half the maximum bending moment in ACB. Thus

$$M_D = \tfrac{1}{2} M_C$$

Then, from Eqs (vii) and (iii)

$$0.5N_D r = 0.75 N_C r$$

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so that

$$0.5(F - N_C) = 0.75N_C$$

i.e.

$$F - N_C = 1.5N_C$$

Substituting for $F - N_C$ in Eq. (xi)

$$2.22N_C - 0.136 \times \frac{1.5N_C}{x} = 0$$

whence

$$x = 0.092$$

S.4.16

From symmetry the shear force in the tank wall at the lowest point is zero. Let the normal force and bending moment at this point be N_O and M_O respectively as shown in Fig. S.4.16.

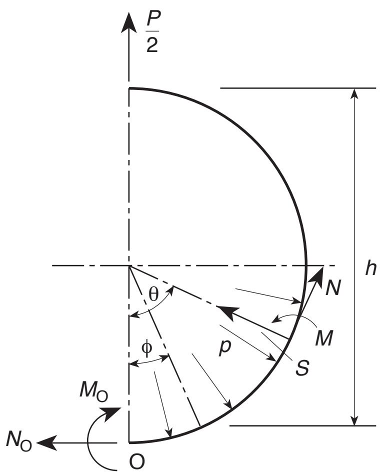

Fig. S.4.16

The total complementary energy of the half tank is, from Eq. (4.18)

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M - \frac{P}{2}\Delta_P$$

where Δ_P is the vertical displacement at the point of application of P. Since the rotation and translation at O are zero from symmetry then, from the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial M_O} = \int_L \frac{M}{EI}\frac{\partial M}{\partial M_O}\,\mathrm{d}z = 0 \tag{i}$$

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and

$$\frac{\partial C}{\partial N_{\rm O}} = \int_L \frac{M}{EI}\frac{\partial M}{\partial N_{\rm O}}\,{\rm d}z = 0 \tag{ii}$$

At any point in the tank wall

$$M = M_{\rm O} + N_{\rm O}(r - r\cos\theta) - \int_0^\theta pr^2\sin(\theta-\phi)\,{\rm d}\phi \tag{iii}$$

For unit length of tank

$$p = \pi r^2\rho$$

where ρ is the density of the fuel.

At the position θ,

$$p = \rho h = \rho(r + r\cos\phi)$$

Hence

$$p = \frac{P}{\pi r}(1+\cos\phi)$$

and the last term in Eq. (iii) becomes

$$\int_0^\theta \frac{Pr}{\pi}(1+\cos\phi)\sin(\theta-\phi)\,{\rm d}\phi = \frac{Pr}{\pi}\int_0^\theta (1+\cos\phi)(\sin\theta\cos\phi - \cos\theta\sin\phi)\,{\rm d}\phi$$

Expanding the expression on the right-hand side gives

$$\frac{Pr}{\pi}\int_0^\theta(\sin\theta\cos\phi - \cos\theta\sin\phi + \sin\theta\cos^2\phi - \cos\theta\sin\phi\cos\phi)\,{\rm d}\phi$$
$$= \frac{Pr}{\pi}\left(1 + \frac{\theta}{2}\sin\theta - \cos\theta\right)$$

Hence Eq. (iii) becomes

$$M = M_{\rm O} + N_{\rm O}r(1-\cos\theta) - \frac{Pr}{\pi}\left(1 + \frac{\theta}{2}\sin\theta - \cos\theta\right) \tag{iv}$$

so that

$$\frac{\partial M}{\partial M_{\rm O}} = 1 \quad \text{and} \quad \frac{\partial M}{\partial N_{\rm O}} = r(1-\cos\theta)$$

Substituting for M and $\partial M/\partial M_{\rm O}$ in Eq. (i) and noting that EI = constant,

$$\int_0^\pi \left[M_{\rm O} + N_{\rm O}r(1-\cos\theta) - \frac{Pr}{\pi}\left(1 + \frac{\theta}{2}\sin\theta - \cos\theta\right)\right]{\rm d}\theta = 0 \tag{v}$$

from which

$$M_{\rm O} + N_{\rm O}r - \frac{3Pr}{2\pi} = 0 \tag{vi}$$

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Now substituting for M and $\partial M/\partial N_{\rm O}$ in Eq. (ii)

$$\int_0^\pi \left[M_{\rm O} + N_{\rm O}r(1-\cos\theta) - \frac{Pr}{\pi}\left(1+\frac{\theta}{2}\sin\theta - \cos\theta\right)\right] r(1-\cos\theta)\,{\rm d}\theta = 0$$

The first part of this integral is identical to that in Eq. (v) and is therefore zero. The remaining integral is then

$$\int_0^\pi \left[M_{\rm O} + N_{\rm O}r(1-\cos\theta) - \frac{Pr}{\pi}\left(1+\frac{\theta}{2}\sin\theta - \cos\theta\right)\right] \cos\theta\,{\rm d}\theta = 0$$

which gives

$$\frac{N_{\rm O}}{2} - \frac{5}{8}\frac{Pr}{\pi} = 0$$

Hence

$$N_{\rm O} = 0.398P$$

and from Eq. (vi)

$$M_{\rm O} = 0.080Pr$$

Substituting these values in Eq. (iv)

$$M = Pr(0.160 - 0.080\cos\theta - 0.159\theta\sin\theta)$$

S.4.17

The internal force systems at A and B are shown in Fig. S.4.17; from symmetry the shear forces at these points are zero as are the translations and rotations. It follows that the total complementary energy of the half frame is, from Eq. (4.18)

$$C = \int_L \int_0^M {\rm d}\theta\,{\rm d}M$$

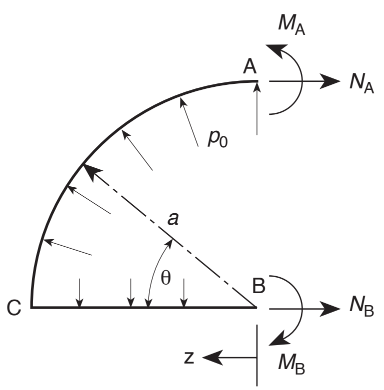

Fig. S.4.17

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From the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial M_{\mathrm{B}}} = \int_L \frac{M}{EI}\frac{\partial M}{\partial M_{\mathrm{B}}}\,\mathrm{d}z = 0 \tag{i}$$

and

$$\frac{\partial C}{\partial N_{\mathrm{B}}} = \int_L \frac{M}{EI}\frac{\partial M}{\partial N_{\mathrm{B}}}\,\mathrm{d}z = 0 \tag{ii}$$

In BC,

$$M = M_{\mathrm{B}} + \frac{p_0 z^2}{2} \tag{iii}$$

so that

$$\frac{\partial M}{\partial M_{\mathrm{B}}} = 1, \quad \frac{\partial M}{\partial N_{\mathrm{B}}} = 0$$

In CA

$$M = M_{\mathrm{B}} - N_{\mathrm{B}} a\sin\theta + p_0 a\left(a\cos\theta - \frac{a}{2}\right) + p_0\frac{(a\sin\theta)^2}{2} + \frac{p_0}{2}(a - a\cos\theta)^2$$

which simplifies to

$$M = M_{\mathrm{B}} - N_{\mathrm{B}} a\sin\theta + \frac{p_0 a^2}{2} \tag{iv}$$

Hence

$$\frac{\partial M}{\partial M_{\mathrm{B}}} = 1, \quad \frac{\partial M}{\partial N_{\mathrm{B}}} = -a\sin\theta$$

Substituting for M and $\partial M/\partial M_{\mathrm{B}}$ in Eq. (i)

$$\int_0^a \frac{1}{2EI}\left(M_{\mathrm{B}} + \frac{p_0 z^2}{2}\right)\mathrm{d}z + \int_0^{\pi/2}\frac{1}{EI}\left(M_{\mathrm{B}} - N_{\mathrm{B}} a\sin\theta + \frac{p_0 a^2}{2}\right)a\,\mathrm{d}\theta = 0$$

i.e.

$$\frac{1}{2}\left[M_{\mathrm{B}} z + \frac{p_0 z^3}{6}\right]_0^a + a\left[M_{\mathrm{B}}\theta + N_{\mathrm{B}} a\cos\theta + \frac{p_0 a^2}{2}\right]_0^{\pi/2} = 0$$

which simplifies to

$$2.071 M_{\mathrm{B}} - N_{\mathrm{B}} a + 0.869 p_0 a^2 = 0$$

Thus

$$M_{\mathrm{B}} - 0.483 N_{\mathrm{B}} a + 0.420 p_0 a^2 = 0 \tag{v}$$

Now substituting for M and $\partial M/\partial N_{\mathrm{B}}$ in Eq. (ii)

$$\int_0^{\pi/2}\frac{1}{EI}\left(M_{\mathrm{B}} - N_{\mathrm{B}} a\sin\theta + \frac{p_0 a^2}{2}\right)(-a\sin\theta)a\,\mathrm{d}\theta = 0$$

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or

$$\int_0^{\pi/2} \left(M_\mathrm{B} \sin\theta - N_\mathrm{B} a \sin^2\theta + \frac{p_0 a^2}{2} \sin\theta \right) \mathrm{d}\theta = 0$$

which gives

$$M_\mathrm{B} - 0.785 N_\mathrm{B} a + 0.5 p_0 a^2 = 0 \tag{vi}$$

Subtracting Eq. (vi) from Eq. (v)

$$0.302 N_\mathrm{B} a - 0.08 p_0 a^2 = 0$$

so that

$$N_\mathrm{B} = 0.265 p_0 a$$

Substituting for N_B in Eq. (v) gives

$$M_\mathrm{B} = -0.292 p_0 a^2$$

Therefore, from Eq. (iii)

$$M_\mathrm{C} = M_\mathrm{B} + \frac{p_0 a^2}{2} = -0.292 p_0 a^2 + \frac{p_0 a^2}{2}$$

i.e.

$$M_\mathrm{C} = 0.208 p_0 a^2$$

and from Eq. (iv)

$$M_\mathrm{A} = -0.292 p_0 a^2 - 0.265 p_0 a^2 + \frac{p_0 a^2}{2}$$

i.e.

$$M_\mathrm{A} = -0.057 p_0 a^2$$

Also, from Eq. (iii)

$$M_\mathrm{BC} = -0.292 p_0 a^2 + \frac{p_0}{2} z^2 \tag{vii}$$

At a point of contraflexure $M_\mathrm{BC} = 0$. Thus, from Eq. (vii), a point of contraflexure occurs in BC when $z^2 = 0.584a^2$, i.e. when $z = 0.764a$. Also, from Eq. (iv), $M_\mathrm{CA} = 0$ when $\sin\theta = 0.208/0.265 = 0.785$, i.e. when $\theta = 51.7^\circ$.

S.4.18

Consider the half frame shown in Fig. S.4.18(a). On the plane of antisymmetry through the points 7, 8 and 9 only shear forces S_7, S_8 and S_9 are present. Thus from the horizontal equilibrium of the frame

$$S_7 + S_8 + S_9 - 6aq = 0 \tag{i}$$

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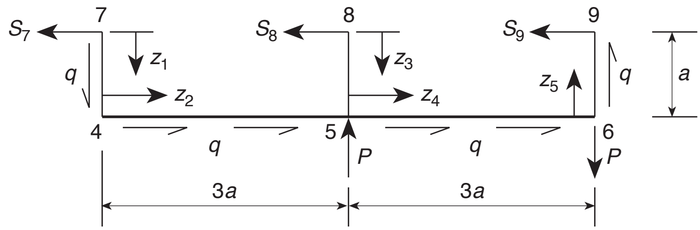

Fig. S.4.18(a)

Also, from the overall equilibrium of the complete frame and taking moments about the corner 6

$$2aq6a + 6aq2a - 2P3a = 0$$

which gives

$$q = P/4a$$

The total complementary energy of the half frame is, from Eq. (4.18)

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M - P\Delta_5 - P\Delta_6 = 0$$

Noting that the horizontal displacements at 7, 8 and 9 are zero from antisymmetry, then

$$\frac{\partial C}{\partial S_7} = \int_L \frac{M}{EI}\frac{\partial M}{\partial S_7}\,\mathrm{d}z = 0 \qquad \text{(ii)}$$

and

$$\frac{\partial C}{\partial S_8} = \int_L \frac{M}{EI}\frac{\partial M}{\partial S_8}\,\mathrm{d}z = 0 \qquad \text{(iii)}$$

In 74

$$M = S_7 z_1 \quad \text{and} \quad \partial M/\partial S_7 = z_1, \quad \partial M/\partial S_8 = 0$$

In 45

$$M = S_7 a + qaz_2 \quad \text{and} \quad \partial M/\partial S_7 = a, \quad \partial M/\partial S_8 = 0$$

In 85

$$M = S_8 z_3 \quad \text{and} \quad \partial M/\partial S_7 = 0, \quad \partial M/\partial S_8 = z_3$$

In 56

$$M = S_7 a + S_8 a + qa(3a + z_4) - Pz_4 \quad \text{and} \quad \partial M/\partial S_7 = a, \quad \partial M/\partial S_8 = a$$

In 69

$$M = S_7(a - z_5) + S_8(a - z_5) + 6a^2 q - 3Pa + 6aqz_5$$

and

$$\partial M/\partial S_7 = (a - z_5), \quad \partial M/\partial S_8 = (a - z_5)$$

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Substituting the relevant expressions in Eq. (ii) gives

$$\int_0^a S_7 z_1^2 \, \mathrm{d}z_1 + \int_0^{3a} (S_7 a^2 + qa^2 z_2) \, \mathrm{d}z_2 + \int_0^{3a} [S_7 a + S_8 a + qa(3a + z_4) - Pz_4] a \, \mathrm{d}z_4$$

$$+ \int_0^a [S_7(a - z_5) + S_8(a - z_5) + 6a^2 q - 3Pa + 6aqz_5](a - z_5) \, \mathrm{d}z_5 = 0 \qquad \text{(iv)}$$

from which

$$20S_7 + 10S_8 + 66aq - 18P = 0 \qquad \text{(v)}$$

Now substituting for M and $\partial M / \partial S_8$ in Eq. (iii)

$$\int_0^a S_8 z_3^2 \, \mathrm{d}z_3 + \int_0^{3a} [S_7 a + S_8 a + qa(3a + z_4) - Pz_4] a \, \mathrm{d}z_4$$

$$+ \int_0^a [S_7(a - z_5) + S_8(a - z_5) + 6a^2 q - 3Pa + 6aqz_5](a - z_5) \, \mathrm{d}z_5 = 0 \qquad \text{(vi)}$$

The last two integrals in Eq. (vi) are identical to the last two integrals in Eq. (iv). Thus, Eq. (vi) becomes

$$10S_7 + 11S_8 + 52.5aq - 18P = 0 \qquad \text{(vii)}$$

The simultaneous solution of Eqs (v) and (vii) gives

$$S_8 = -\frac{39}{12} aq + \frac{3}{2} P$$

whence, since $q = P/4a$

$$S_8 = 0.69P$$

Substituting for S_8 in either of Eqs (v) or (vii) gives

$$S_7 = -0.27P$$

Then, from Eq. (i)

$$S_9 = 1.08P$$

The bending moment diagram is shown in Fig. S.4.18(b) in which the bending moments are drawn on the tension side of each member.

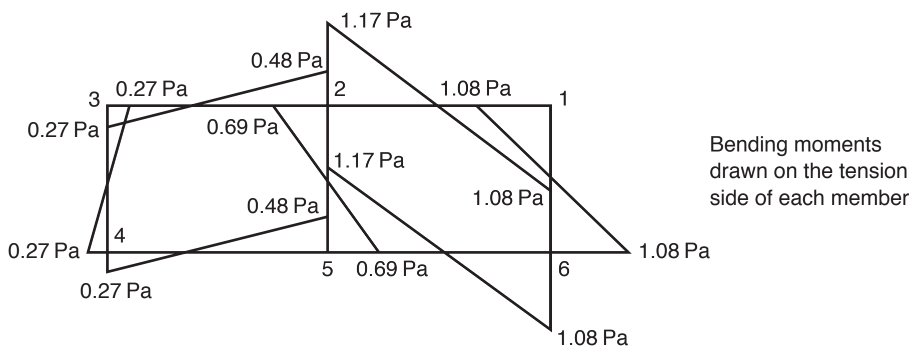

Fig. S.4.18(b)

S.4.19

From the overall equilibrium of the complete frame

$$\int_0^{2\pi r} qr\,\mathrm{d}s = T$$

which gives

$$2\pi r^2 q = T$$

i.e.

$$q = \frac{T}{2\pi r^2} \tag{i}$$

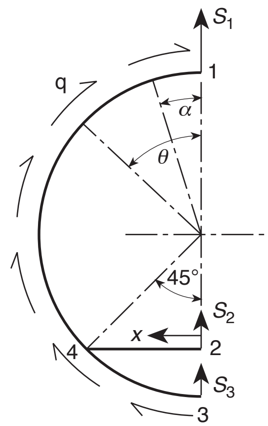

Fig. S.4.19

Considering the half frame shown in Fig. S.4.19 there are only internal shear forces on the vertical plane of antisymmetry. From the vertical equilibrium of the half frame

$$S_1 + S_2 + S_3 + \int_0^{\pi} q \sin \alpha r\,\mathrm{d}\alpha = 0$$

Substituting for q from Eq. (i) and integrating

$$S_1 + S_2 + S_3 + \frac{T}{2\pi r}\Big[-\cos\alpha\Big]_0^{\pi} = 0$$

which gives

$$S_1 + S_2 + S_3 = -\frac{T}{\pi r} \tag{ii}$$

The vertical displacements at the points 1, 2 and 3 are zero from antisymmetry so that, from Eq. (4.18), the total complementary energy of the half frame is given by

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M$$

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Then, from the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial S_1} = \int_L \frac{M}{EI}\frac{\partial M}{\partial S_1}\,\mathrm{d}z \tag{iii}$$

and

$$\frac{\partial C}{\partial S_2} = \int_L \frac{M}{EI}\frac{\partial M}{\partial S_2}\,\mathrm{d}z \tag{iv}$$

In the wall 14

$$M = S_1 r\sin\theta - \int_0^\theta q[r - r\cos(\theta-\alpha)]r\,\mathrm{d}\alpha$$

i.e.

$$M = S_1 r\sin\theta - \frac{T}{2\pi}\left[\alpha - \sin(\alpha-\theta)\right]_0^\theta$$

which gives

$$M = S_1 r\sin\theta - \frac{T}{2\pi}(\theta - \sin\theta) \tag{v}$$

whence

$$\frac{\partial M}{\partial S_1} = r\sin\theta, \quad \frac{\partial M}{\partial S_2} = 0$$

In the wall 24

$$M = S_2 x \tag{vi}$$

and

$$\frac{\partial M}{\partial S_1} = 0, \quad \frac{\partial M}{\partial S_2} = x$$

In the wall 43

$$M = S_1 r\sin\theta - \frac{T}{2\pi}(\theta - \sin\theta) + S_2 r\sin\theta \tag{vii}$$

and

$$\frac{\partial M}{\partial S_1} = r\sin\theta, \quad \frac{\partial M}{\partial S_2} = r\sin\theta$$

Substituting for M and $\partial M/\partial S_1$ in Eq. (iii)

$$\int_0^{3\pi/4}\left[S_1 r\sin\theta - \frac{T}{2\pi}(\theta-\sin\theta)\right] r\sin\theta\, r\,\mathrm{d}\theta + \int_{3\pi/4}^{\pi}\left[S_1 r\sin\theta - \frac{T}{2\pi}(\theta-\sin\theta) + S_2 r\sin\theta\right] r\sin\theta\, r\,\mathrm{d}\theta = 0$$

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which simplifies to

$$\int_0^{\pi}\left[S_1 r\sin\theta-\frac{T}{2\pi}(\theta-\sin\theta)\right]r^2\sin\theta\,\mathrm{d}\theta+\int_{3\pi/4}^{\pi}S_2 r^3\sin\theta\,\mathrm{d}\theta=0$$

Integrating and simplifying gives

$$S_1 r-0.16T+0.09S_2 r=0 \tag{viii}$$

Now substituting for M and $\partial M/\partial S_2$ in Eq. (iv)

$$\int_{3\pi/4}^{\pi}\left[S_1 r\sin\theta-\frac{T}{2\pi}(\theta-\sin\theta)+S_2 r\sin\theta\right]r\sin\theta\, r\,\mathrm{d}\theta+\int_0^{r/\sqrt{2}}S_2 x^2\,\mathrm{d}x=0$$

Integrating and simplifying gives

$$S_1 r-0.69T+1.83S_2 r=0 \tag{ix}$$

Subtracting Eq. (ix) from Eq. (viii)

$$0.53T-1.74S_2 r=0$$

whence

$$S_2=0.30T/r$$

From Eq. (viii)

$$S_1=0.13T/r$$

and from Eq. (ii)

$$S_3=-0.75T/r$$

Hence, from Eqs (v), (vi) and (vii)

$$M_{14}=T(0.29\sin\theta-0.16\theta)$$
$$M_{24}=0.30Tx/r$$
$$M_{43}=T(0.59\sin\theta-0.16\theta)$$

S.4.20

Initially the vertical reaction at C, R_C, must be found. From Eq. (4.18) the total complementary energy of the member is given by

$$C=\int_L\int_0^M \mathrm{d}\theta\,\mathrm{d}M-R_C\Delta_C-F\Delta_B$$

From the principle of the stationary value of the total complementary energy and since $\Delta_C=0$

$$\frac{\partial C}{\partial R_C}=\int_L\frac{M}{EI}\frac{\partial M}{\partial R_C}\,\mathrm{d}s=0 \tag{i}$$

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Referring to Fig. S.4.20

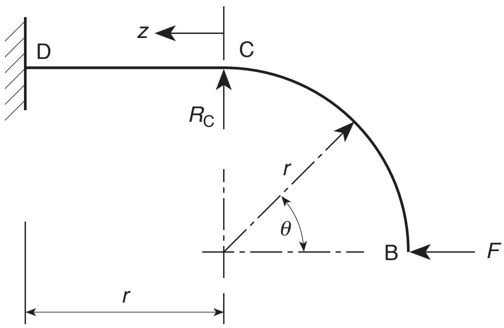

Fig. S.4.20

In BC

$$M = Fr\sin\theta \quad \text{and} \quad \partial M/\partial R_C = 0$$

In CD

$$M = Fr - R_C z \quad \text{and} \quad \partial M/\partial R_C = -z$$

Substituting these expressions in Eq. (i) gives

$$\int_0^r (Fr - R_C z)(-z)\,dz = 0$$

from which

$$R_C = 1.5F$$

Note that Eq. (i) does not include the effects of shear and axial force. If these had been included the value of R_C would be $1.4F$; the above is therefore a reasonable approximation. Also, from Eq. (1.45), $G = 3E/8$.

The unit load method may now be used to complete the solution. Thus, from the first of Eqs (4.27), Eq. (4.26) and Eq. (9.87)

$$\delta_{B,H} = \int_L \frac{M_0 M_1}{EI}\,ds + \int_L \frac{F_0 F_1}{AE}\,ds + \int_L \frac{S_0 S_1}{GA'}\,ds \qquad \text{(ii)}$$

In BC

$$M_0 = Fr\sin\theta, \quad M_1 = r\sin\theta$$
$$F_0 = F\sin\theta, \quad F_1 = \sin\theta$$
$$S_0 = F\cos\theta, \quad S_1 = \cos\theta$$

In CD

$$M_0 = F(r - 1.5z), \quad M_1 = (r - 1.5z)$$
$$F_0 = F, \quad F_1 = 1$$
$$S_0 = 1.5F, \quad S_1 = 1.5$$

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Substituting these expressions in Eq. (ii) gives

$$\delta_{B,H} = \int_0^{\pi/2} \frac{Fr^3 \sin^2\theta}{EI}\,d\theta + \int_0^{\pi/2} \frac{Fr\sin^2\theta}{AE}\,d\theta + \int_0^{\pi/2} \frac{Fr\cos^2\theta}{GA'}\,d\theta + \int_0^r \frac{F}{EI}(r-1.5z)^2\,dz$$
$$+ \int_0^r \frac{F}{AE}\,dz + \int_0^r \frac{2.25F}{GA'}\,dz$$

or

$$\delta_{B,H} = \frac{400Fr}{AE}\int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)\,d\theta + \frac{Fr}{AE}\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\,d\theta$$
$$+ \frac{32Fr}{3AE}\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta + \frac{400F}{Ar^2E}\int_0^r (r^2 - 3rz + 2.25z^2)\,dz$$
$$+ \frac{F}{AE}\int_0^r dz + \frac{24F}{AE}\int_0^r dz$$

from which

$$\delta_{B,H} = \frac{448.3Fr}{AE}$$

S.4.21

From Clerk-Maxwell's reciprocal theorem the deflection at A due to W at B is equal to the deflection at B due to W at A, i.e. δ_2.

What is now required is the deflection at B due to W at B.

Since the deflection at A with W at A and the spring removed is δ_3, the load in the spring at A with W at B is $(\delta_2/\delta_3)W$ which must equal the load in the spring at B with W at B. Thus, the resultant load at B with W at B is

$$W - \left(\frac{\delta_2}{\delta_3}\right)W = W\left(1 - \frac{\delta_2}{\delta_3}\right) \tag{i}$$

Now the load W at A with the spring in place produces a deflection of δ_1 at A. Thus, the resultant load at A is $(\delta_1/\delta_3)W$ so that, if the load in the spring at A with W at A is F, then $W - F = (\delta_1/\delta_3)W$, i.e.

$$F = W\left(1 - \frac{\delta_1}{\delta_3}\right) \tag{ii}$$

This then is the load at B with W at A and it produces a deflection δ_2. Therefore, from Eqs (i) and (ii) the deflection at B due to W at B is

$$\frac{W(1-\delta_2/\delta_3)}{W(1-\delta_1/\delta_3)}\delta_2$$

Thus the extension of the spring with W at B is

$$\frac{(1-\delta_2/\delta_3)}{(1-\delta_1/\delta_3)}\delta_2 - \delta_2$$

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i.e.

$$\delta_2\left(\frac{\delta_1-\delta_2}{\delta_3-\delta_1}\right)$$

S.4.22

Referring to Fig. S.4.22

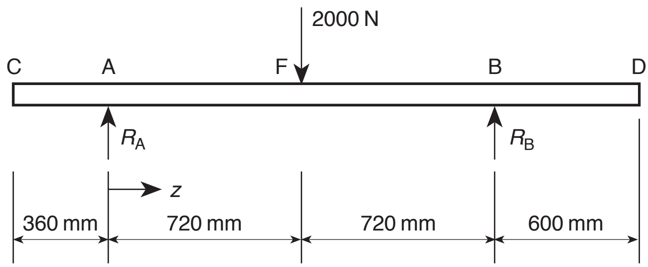

Fig. S.4.22

$R_A = R_B = 1000$ N from symmetry.

The slope of the beam at A and B may be obtained from the second of Eqs (9.20), i.e.

$$v'' = -\frac{M}{EI}$$

where, for the half-span AF, $M = R_A z = 1000z$. Thus

$$v'' = -\frac{1000}{EI}z$$

and

$$v' = -\frac{500}{EI}z^2 + C_1$$

When $z = 720$ mm, $v' = 0$ from symmetry and hence $C_1 = 2.59 \times 10^8/EI$. Hence

$$v' = \frac{1}{EI}(-500z^2 + 2.59 \times 10^8)$$

Thus v' (at A) $= 0.011$ rads $= v'$ (at B). The deflection at C then $= 360 \times 0.011 = 3.96$ mm and the deflection at D $= 600 \times 0.011 = 6.6$ mm.

From the reciprocal theorem the deflection at F due to a load of 3000N at C $= 3.96 \times 3000/2000 = 5.94$ mm and the deflection at F due to a load of 3000N at D $= 6.6 \times 3000/2000 = 9.9$ mm. Therefore the total deflection at F due to loads of 3000N acting simultaneously at C and D is $5.94 + 9.9 = 15.84$ mm.

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S.4.23

Since the frame is symmetrical about a vertical plane through its centre only half need be considered. Also, due to symmetry the frame will act as though fixed at C (Fig. S.4.23).

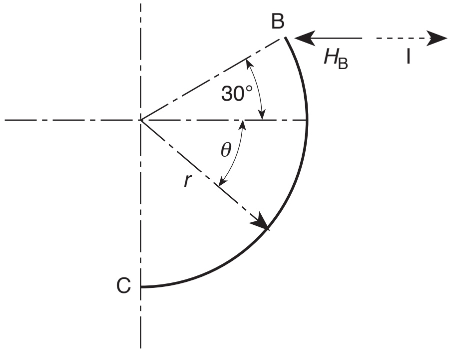

Fig. S.4.23

If the frame were unsupported at B the horizontal displacement at B, $\Delta_{B,T}$, due to the temperature rise may be obtained using Eq. (4.35) in which, due to a unit load acting horizontally at B, $M_1 = 1 \times (r \sin 30^\circ + r \sin\theta)$. Hence

$$\Delta_{B,T} = \int_{-\pi/6}^{\pi/2} (0.5r + r\sin\theta) \frac{2\alpha T}{d} r \, d\theta$$

i.e.

$$\Delta_{B,T} = \frac{2\alpha T r^2}{d} \left[0.5\theta - \cos\theta\right]_{-\pi/6}^{\pi/2}$$

which gives

$$\Delta_{B,T} = \frac{3.83\alpha T r^2}{d} \quad \text{(to the right)} \tag{i}$$

Suppose that in the actual frame the horizontal reaction at B is H_B. Since B is not displaced, the 'displacement' $\Delta_{B,H}$ produced by H_B must be equal and opposite to $\Delta_{B,T}$ in Eq. (i). Then, from the first of Eqs (4.27) and noting that $M_0 = -H_B(0.5r + r\sin\theta)$

$$\Delta_{B,H} = -\frac{1}{EI} \int_{-\pi/6}^{\pi/2} H_B (0.5r + r\sin\theta)^2 r \, d\theta$$

i.e.

$$\Delta_{B,H} = -\frac{H_B r^3}{EI} \int_{-\pi/6}^{\pi/2} (0.25 + \sin\theta + \sin^2\theta) \, d\theta$$

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Hence

$$\Delta_{\mathrm{B,H}} = -\frac{H_{\mathrm{B}} r^3}{EI}\left[0.75\theta - \cos\theta - \frac{\sin 2\theta}{4}\right]_{-\pi/6}^{\pi/2}$$

so that

$$\Delta_{\mathrm{B,H}} = -\frac{2.22 H_{\mathrm{B}} r^3}{EI} \quad \text{(to the left)} \tag{ii}$$

Then, since

$$\Delta_{\mathrm{B,H}} + \Delta_{\mathrm{B,T}} = 0$$

$$-\frac{2.22 H_{\mathrm{B}} r^3}{EI} + \frac{3.83\alpha T r^2}{d} = 0$$

from which

$$H_{\mathrm{B}} = \frac{1.73 EIT\alpha}{d} \tag{iii}$$

The maximum bending moment in the frame will occur at C and is given by

$$M(\max) = H_{\mathrm{B}} \times 1.5r$$

Then, from symmetrical bending theory the direct stress through the depth of the frame section is given by

$$\sigma = \frac{My}{I} \quad \text{(see Eqs (9.9))}$$

and

$$\sigma_{\max} = \frac{M(\max)y(\max)}{I}$$

i.e.

$$\sigma_{\max} = \frac{H_{\mathrm{B}} \times 1.5r \times 0.5d}{I}$$

or, substituting for H_{B} from Eq. (iii)

$$\sigma_{\max} = 1.30 ET\alpha$$

S.4.24

The solution is similar to that for P.4.23 in that the horizontal displacement of B due to the temperature gradient is equal and opposite in direction to the 'displacement' produced by the horizontal reaction at B, H_{B}. Again only half the frame need be considered from symmetry.

Referring to Fig. S.4.24

$$M_1 = r\cos\psi \text{ in BC and CD}$$

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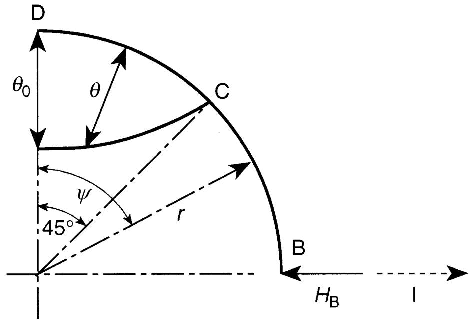

Fig. S.4.24

Then, from Eq. (4.35)

$$\Delta_{\mathrm{B,T}} = \int_0^{\pi/4} (r\cos\psi)\alpha \frac{\theta_0 \cos 2\psi}{h} r\,\mathrm{d}\psi + \int_{\pi/4}^{\pi/2} (r\cos\psi)\alpha\left(\frac{0}{h}\right) r\,\mathrm{d}\psi$$

i.e.

$$\Delta_{\mathrm{B,T}} = \frac{r^2\alpha\theta_0}{h}\int_0^{\pi/4} \cos\psi\cos 2\psi\,\mathrm{d}\psi$$

or

$$\Delta_{\mathrm{B,T}} = \frac{r^2\alpha\theta_0}{h}\int_0^{\pi/4} (\cos\psi - 2\sin^2\psi\cos\psi)\,\mathrm{d}\psi$$

Hence

$$\Delta_{\mathrm{B,T}} = \frac{r^2\alpha\theta_0}{h}\left[\sin\psi - \frac{2}{3}\sin^3\psi\right]_0^{\pi/4}$$

which gives

$$\Delta_{\mathrm{B,T}} = \frac{0.47r^2\alpha\theta_0}{h} \quad \text{(to the right)} \tag{i}$$

From the first of Eqs (4.27) in which $M_0 = -H_{\mathrm{B}} r\cos\psi$

$$\Delta_{\mathrm{B,H}} = \int_0^{\pi/2} -\frac{H_{\mathrm{B}} r\cos\psi\, r\cos\psi}{EI} r\,\mathrm{d}\psi$$

i.e.

$$\Delta_{\mathrm{B,H}} = -\frac{H_{\mathrm{B}} r^3}{EI}\int_0^{\pi/2} \cos^2\psi\,\mathrm{d}\psi$$

or

$$\Delta_{\mathrm{B,H}} = -\frac{H_{\mathrm{B}} r^3}{EI}\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\psi)\,\mathrm{d}\psi$$

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whence

$$\Delta_{\mathrm{B,H}} = -\frac{0.79H_{\mathrm{B}}r^3}{EI} \quad \text{(to the left)} \tag{ii}$$

Then, since $\Delta_{\mathrm{B,H}} + \Delta_{\mathrm{B,T}} = 0$, from Eqs (i) and (ii)

$$-\frac{0.79H_{\mathrm{B}}r^3}{EI} + \frac{0.47r^2\alpha\theta_0}{h} = 0$$

from which

$$H_{\mathrm{B}} = \frac{0.59EI\alpha\theta_0}{rh}$$

Then

$$M = H_{\mathrm{B}}r\cos\psi$$

so that

$$M = \frac{0.59EI\alpha\theta_0\cos\psi}{h}$$

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Solutions to Chapter 5 Problems

S.5.1

Substituting for $\left(\dfrac{1}{\rho_x}+\dfrac{\nu}{\rho_y}\right)$ and $\left(\dfrac{1}{\rho_y}+\dfrac{\nu}{\rho_x}\right)$ from Eqs (5.5) and (5.6) respectively in Eqs (5.3)

$$\sigma_x = \frac{Ez}{1-\nu^2}\frac{M_x}{D} \quad \text{and} \quad \sigma_y = \frac{Ez}{1-\nu^2}\frac{M_y}{D} \tag{i}$$

Hence, since, from Eq. (5.4), $D = Et^3/12(1-\nu^2)$, Eqs (i) become

$$\sigma_x = \frac{12zM_x}{t^3}, \quad \sigma_y = \frac{12zM_y}{t^3} \tag{ii}$$

The maximum values of σ_x and σ_y will occur when $z = \pm t/2$. Hence

$$\sigma_x(\max) = \pm\frac{6M_x}{t^2}, \quad \sigma_y(\max) = \pm\frac{6M_y}{t^2} \tag{iii}$$

Thus

$$\sigma_x(\max) = \pm\frac{6\times 10\times 10^3}{10^2} = \pm 600\,\text{N/mm}^2$$

$$\sigma_y(\max) = \pm\frac{6\times 5\times 10^3}{10^2} = \pm 300\,\text{N/mm}^2$$

S.5.2

From Eq. (5.11) and since $M_{xy} = 0$

$$M_t = \frac{M_x - M_y}{2}\sin 2\alpha \tag{i}$$

M_t will be a maximum when $2\alpha = \pi/2$, i.e. $\alpha = \pi/4\,(45^\circ)$. Thus, from Eq. (i)

$$M_t(\max) = \frac{10-5}{2} = 2.5\,\text{Nm/mm}$$

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S.5.3

The relationship between M_n and M_x, M_y and M_{xy} in Eq. (5.10) and between M_t and M_x, M_y and M_{xy} in Eq. (5.11) are identical in form to the stress relationships in Eqs (1.8) and (1.9). Therefore, by deduction from Eqs (1.11) and (1.12)

$$M_{\mathrm{I}} = \frac{M_x + M_y}{2} + \frac{1}{2}\sqrt{(M_x - M_y)^2 + 4M_{xy}^2} \tag{i}$$

and

$$M_{\mathrm{II}} = \frac{M_x + M_y}{2} - \frac{1}{2}\sqrt{(M_x - M_y)^2 + 4M_{xy}^2} \tag{ii}$$

Further, Eq. (5.11) gives the inclination of the planes on which the principal moments occur, i.e. when $M_t = 0$. Thus

$$\tan 2\alpha = -\frac{2M_{xy}}{M_x - M_y} \tag{iii}$$

Substituting the values $M_x = 10\,\mathrm{Nm/mm}$, $M_y = 5\,\mathrm{Nm/mm}$ and $M_{xy} = 5\,\mathrm{Nm/mm}$ in Eqs (i), (ii) and (iii) gives

$$M_{\mathrm{I}} = 13.1\,\mathrm{Nm/mm}$$

$$M_{\mathrm{II}} = 1.9\,\mathrm{Nm/mm}$$

and

$$\alpha = -31.7^\circ \quad \text{or} \quad 58.3^\circ$$

The corresponding principal stresses are obtained directly from Eqs (iii) of S.5.1. Hence

$$\sigma_{\mathrm{I}} = \pm\frac{6 \times 13.1 \times 10^3}{10^2} = \pm 786\,\mathrm{N/mm^2}$$

$$\sigma_{\mathrm{II}} = \pm\frac{6 \times 1.9 \times 10^3}{10^2} = \pm 114\,\mathrm{N/mm^2}$$

S.5.4

Substituting $q(x, y) = q_0 x/a$ in Eq. (5.29) and noting that the plate is square and of side a

$$a_{mn} = \frac{4}{a^2}\int_0^a\int_0^a q_0\frac{x}{a}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}\,\mathrm{d}x\,\mathrm{d}y$$

i.e.

$$a_{mn} = \frac{4q_0}{a^3}\int_0^a x\sin\frac{m\pi x}{a}\left[-\frac{a}{n\pi}\cos\frac{n\pi y}{a}\right]_0^a\mathrm{d}x$$

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Hence

$$a_{mn} = -\frac{4q_0}{a^2 n\pi}\int_0^a x \sin\frac{m\pi x}{a}(\cos n\pi - 1)\,\mathrm{d}x$$

The term in brackets is zero when n is even and equal to -2 when n is odd. Thus

$$a_{mn} = \frac{8q_0}{a^2 n\pi}\int_0^a x \sin\frac{m\pi x}{a}\,\mathrm{d}x \quad (n \text{ odd}) \tag{i}$$

Integrating Eq. (i) by parts

$$a_{mn} = \frac{8q_0}{a^2 n\pi}\left[-x\frac{a}{m\pi}\cos\frac{m\pi x}{a} + \int\frac{a}{m\pi}\cos\frac{m\pi x}{a}\,\mathrm{d}x\right]_0^a$$

i.e.

$$a_{mn} = \frac{8q_0}{amn\pi^2}\left[-x\cos\frac{m\pi x}{a} + \frac{a}{m\pi}\sin\frac{m\pi x}{a}\right]_0^a$$

The second term in square brackets is zero for all integer values of m. Thus

$$a_{mn} = \frac{8q_0}{amn\pi^2}(-a\cos m\pi)$$

The term in brackets is positive when m is odd and negative when m is even. Thus

$$a_{mn} = \frac{8q_0}{mn\pi^2}(-1)^{m+1}$$

Substituting for a_{mn} in Eq. (5.30) gives the displaced shape of the plate, i.e.

$$w = \frac{1}{\pi^4 D}\sum_{m=1,2,3}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{8q_0(-1)^{m+1}}{mn\pi^2[(m^2/a^2) + (n^2/a^2)]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

or

$$w = \frac{8q_0 a^4}{\pi^6 D}\sum_{m=1,2,3}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{m+1}}{mn(m^2+n^2)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

S.5.5

The boundary conditions which must be satisfied by the equation for the displaced shape of the plate are $w = 0$ and $\partial w/\partial n = 0$ at all points on the boundary; n is a direction normal to the boundary at any point.

The equation of the ellipse representing the boundary is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{i}$$

Substituting for $x^2/a^2 + y^2/b^2$ in the equation for the displaced shape clearly gives $w = 0$ for all values of x and y on the boundary of the plate. Also

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial n} \tag{ii}$$

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Now

$$w = w_0\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2$$

so that

$$\frac{\partial w}{\partial x} = -\frac{4w_0 x}{a^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{iii}$$

and

$$\frac{\partial w}{\partial y} = -\frac{4w_0 y}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{iv}$$

From Eqs (i), (ii) and (iv) it can be seen that $\partial w/\partial x$ and $\partial w/\partial y$ are zero for all values of x and y on the boundary of the plate. It follows from Eq. (ii) that $\partial w/\partial n = 0$ at all points on the boundary of the plate. Thus the equation for the displaced shape satisfies the boundary conditions.

From Eqs (iii) and (iv)

$$\frac{\partial^4 w}{\partial x^4} = \frac{24w_0}{a^4}, \quad \frac{\partial^4 w}{\partial y^4} = \frac{24w_0}{b^4}, \quad \frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{8w_0}{a^2 b^2}$$

Substituting these values in Eq. (5.20)

$$w_0\left(\frac{24}{a^4} + \frac{16}{a^2 b^2} + \frac{24}{b^4}\right) = \frac{p}{D}$$

whence

$$w_0 = \frac{p}{8D\left(\dfrac{3}{a^4} + \dfrac{2}{a^2 b^2} + \dfrac{3}{b^4}\right)}$$

Now substituting for D from Eq. (5.4)

$$w_0 = \frac{3p(1 - \nu^2)}{2Et^3\left(\dfrac{3}{a^4} + \dfrac{2}{a^2 b^2} + \dfrac{3}{b^4}\right)} \tag{v}$$

From Eqs (5.3), (5.5) and (5.7)

$$\sigma_x = -\frac{Ez}{1 - \nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{vi}$$

and from Eqs (5.3), (5.6) and (5.8)

$$\sigma_y = -\frac{Ez}{1 - \nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{vii}$$

From Eqs (iii) and (iv)

$$\frac{\partial^2 w}{\partial x^2} = -\frac{4w_0}{a^2}\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2}\right), \quad \frac{\partial^2 w}{\partial y^2} = -\frac{4w_0}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2}\right)$$

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Substituting these expressions in Eq. (vi) and noting that the maximum values of direct stress occur at $z = \pm t/2$

$$\sigma_x(\max) = \pm\frac{Et}{2(1-\nu^2)}\left[-\frac{4w_0}{a^2}\left(1-\frac{3x^2}{a^2}-\frac{y^2}{b^2}\right)-\frac{4w_0\nu}{b^2}\left(1-\frac{x^2}{a^2}-\frac{3y^2}{b^2}\right)\right] \quad \text{(viii)}$$

At the centre of the plate, $x = y = 0$. Then

$$\sigma_x(\max) = \pm\frac{2Etw_0}{(1-\nu^2)}\left(\frac{1}{a^2}+\frac{\nu}{b^2}\right) \quad \text{(ix)}$$

Substituting for w_0 in Eq. (ix) from Eq. (v) gives

$$\sigma_x(\max) = \pm\frac{3pa^2b^2(b^2+\nu a^2)}{t^2(3b^4+2a^2b^2+3a^4)} \quad \text{(x)}$$

Similarly

$$\sigma_y(\max) = \pm\frac{3pa^2b^2(a^2+\nu b^2)}{t^2(3b^4+2a^2b^2+3a^4)} \quad \text{(xi)}$$

At the ends of the minor axis, $x = 0$, $y = b$. Thus, from Eq. (viii)

$$\sigma_x(\max) = \pm\frac{2Etw_0}{(1-\nu^2)}\left(\frac{1}{a^2}-\frac{1}{a^2}+\frac{\nu}{b^2}-\frac{3\nu}{b^2}\right)$$

i.e.

$$\sigma_x(\max) = \pm\frac{4Etw_0\nu}{b^2(1-\nu^2)} \quad \text{(xii)}$$

Again substituting for w_0 from Eq. (v) in Eq. (xii)

$$\sigma_x(\max) = \pm\frac{6pa^4b^2}{t^2(3b^4+2a^2b^2+3a^4)}$$

Similarly

$$\sigma_y(\max) = \pm\frac{6pb^4a^2}{t^2(3b^4+2a^2b^2+3a^4)}$$

S.5.6

The potential energy, V, of the load W is given by

$$V = -Ww$$

i.e.

$$V = -W\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

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Therefore, it may be deduced from Eq. (5.47) that the total potential energy, $U + V$, of the plate is

$$U + V = \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \frac{\pi^4 ab}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - W\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U+V)}{\partial A_{mn}} = DA_{mn}\frac{\pi^4 ab}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - W\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b} = 0$$

Hence

$$A_{mn} = \frac{4W\sin\dfrac{m\pi\xi}{a}\sin\dfrac{n\pi\eta}{b}}{\pi^4 Dab[(m^2/a^2) + (n^2/b^2)]^2}$$

so that the deflected shape is obtained.

S.5.7

From Eq. (5.45) the potential energy of the in-plane load, N_x, is

$$-\frac{1}{2}\int_0^a\int_0^b N_x\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

The combined potential energy of the in-plane load, N_x, and the load, W, is then, from S.5.6

$$V = -\frac{1}{2}\int_0^a\int_0^b N_x\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}y - W\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

or, since

$$\frac{\partial w}{\partial x} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$V = -\frac{1}{2}\int_0^a\int_0^b N_x\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\frac{m^2\pi^2}{a^2}\cos^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y$$
$$- W\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

i.e.

$$V = -\frac{ab}{8}N_x\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\frac{m^2\pi^2}{a^2} - W\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

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Thus, from Eq. (5.47), the total potential energy of the plate is

$$U + V = \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \frac{\pi^4 ab}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{ab}{8}N_x \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \frac{m^2\pi^2}{a^2}$$

$$- W\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

Then, from the principle of the stationary value of the total potential energy

$$\frac{\partial(U+V)}{\partial A_{mn}} = DA_{mn}\frac{\pi^4 ab}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{ab}{4}N_x A_{mn}\frac{m^2\pi^2}{a^2} - W\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b} = 0$$

from which

$$A_{mn} = \frac{4W\sin\dfrac{m\pi\xi}{a}\sin\dfrac{n\pi\eta}{b}}{abD\pi^4\left[\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2 - \dfrac{m^2 N_x}{\pi^2 a^2 D}\right]}$$

S.5.8

The guessed form of deflection is

$$w = A_{11}\left(1 - \frac{4x^2}{a^2}\right)\left(1 - \frac{4y^2}{a^2}\right) \tag{i}$$

Clearly when $x = \pm a/2$, $w = 0$ and when $y = \pm a/2$, $w = 0$. Therefore, the equation for the displaced shape satisfies the displacement boundary conditions.

From Eq. (i)

$$\frac{\partial^2 w}{\partial x^2} = -8\frac{A_{11}}{a^2}\left(1 - \frac{4y^2}{a^2}\right), \quad \frac{\partial^2 w}{\partial y^2} = -8\frac{A_{11}}{a^2}\left(1 - \frac{4x^2}{a^2}\right)$$

Substituting in Eq. (5.7)

$$M_x = -\frac{8A_{11}D}{a^2}\left[1 - \frac{4y^2}{a^2} + \nu\left(1 - \frac{4x^2}{a^2}\right)\right]$$

Clearly, when $x = \pm a/2$, $M_x \neq 0$ and when $y = \pm a/2$, $M_x \neq 0$. Similarly for M_y. Thus the assumed displaced shape does not satisfy the condition of zero moment at the simply supported edges.

From Eq. (i)

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{64A_{11}xy}{a^4}$$

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Substituting for $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, $\partial^2 w/\partial x \partial y$ and w in Eq. (5.46) and simplifying gives

$$U + V = \int_{-a/2}^{a/2} \int_{-a/2}^{a} \left\{ \frac{32A_{11}^2 D}{a^4} \left[4 - \frac{16}{a^2}(x^2 + y^2) + \frac{16}{a^4}(x^4 + 2x^2y^2 + y^4) - 1.4 \right.\right.$$
$$\left.\left. + \frac{5.6}{a^2}(x^2 + y^2) + \frac{67.2x^2y^2}{a^4} \right] - q_0 A_{11} \left(1 - \frac{4x^2}{a^2} - \frac{4y^2}{a^2} + \frac{16x^2y^2}{a^4} \right) \right\} \mathrm{d}x\,\mathrm{d}y$$

from which

$$U + V = \frac{62.4A_{11}^2 D}{a^2} - \frac{4q_0 A_{11} a^2}{9}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U+V)}{\partial A_{11}} = \frac{124.8A_{11}D}{a^2} - \frac{4q_0 a^2}{9} = 0$$

Hence, since $D = Et^3/12(1 - \nu^2)$

$$A_{11} = 0.0389 q_0 a^4/Et^3$$

S.5.9

From Eq. (5.36) the deflection of the plate from its initial curved position is

$$w_1 = B_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

in which

$$B_{11} = \frac{A_{11}N_x}{\dfrac{\pi^2 D}{a^2}\left(1 + \dfrac{a^2}{b^2}\right)^2 - N_x}$$

The total deflection, w, of the plate is given by

$$w = w_1 + w_0$$

i.e.

$$w = \left[\frac{A_{11}N_x}{\dfrac{\pi^2 D}{a^2}\left(1 + \dfrac{a^2}{b^2}\right)^2 - N_x} + A_{11} \right] \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

i.e.

$$w = \frac{A_{11}}{1 - \dfrac{N_x a^2}{\pi^2 D} \Big/ \left(1 + \dfrac{a^2}{b^2}\right)^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

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Solutions to Chapter 6 Problems

S.6.1

The forces on the bar AB are shown in Fig. S.6.1 where

$$M_B = K\left(\frac{dv}{dz}\right)_B \tag{i}$$

and P is the buckling load.

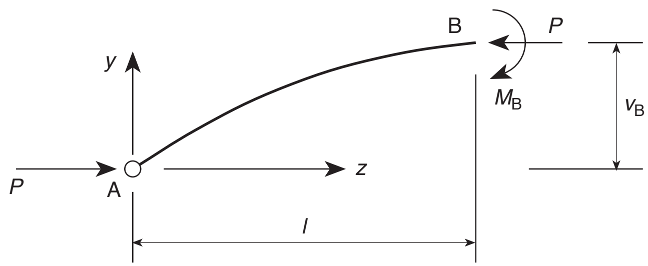

Fig. S.6.1

From Eq. (6.1)

$$EI\frac{d^2v}{dz^2} = -Pv \tag{ii}$$

The solution of Eq. (ii) is

$$v = A\cos\mu z + B\sin\mu z \tag{iii}$$

where $\mu^2 = P/EI$.

When $z = 0$, $v = 0$ so that, from Eq. (iii), $A = 0$. Hence

$$v = B\sin\mu z$$

Then

$$\frac{dv}{dz} = \mu B\cos\mu z$$

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and when $z = l$, $\mathrm{d}v/\mathrm{d}z = M_B/K$ from Eq. (i). Thus

$$B = M_B/\mu K \cos \mu l$$

and Eq. (iv) becomes

$$v = \frac{M_B}{\mu K \cos \mu l} \sin \mu z \tag{v}$$

Also, when $z = l$, $Pv_B = M_B$ from equilibrium. Hence, substituting in Eq. (v) for M_B

$$v_B = \frac{Pv_B}{\mu K \cos \mu l} \sin \mu l$$

from which

$$P = \mu K/\tan \mu l \tag{vi}$$

(a) When $K \to \infty$, $\tan \mu l \to \infty$ and $\mu l \to \pi/2$, i.e.

$$\sqrt{\frac{P}{EI}}\, l \to \frac{\pi}{2}$$

from which

$$P \to \frac{\pi^2 EI}{4l^2}$$

which is the Euler buckling load of a pin-ended column of length $2l$.
(b) When $EI \to \infty$, $\tan \mu l \to \mu l$ and Eq. (vi) becomes $P = K/l$ and the bars remain straight.

S.6.2

Suppose that the buckling load of the column is P. Then from Eq. (6.1) and referring to Fig. S.6.2, in AB

$$EI \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -Pv \tag{i}$$

and in BC

$$4EI \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -Pv \tag{ii}$$

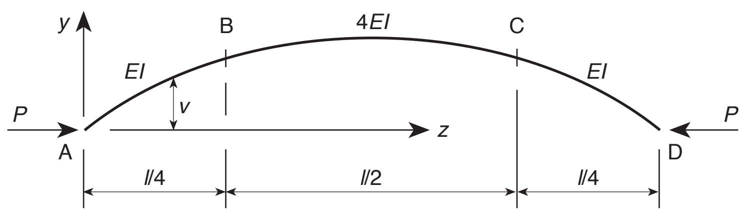

Fig. S.6.2

The solutions of Eqs (i) and (ii) are, respectively

$$v_{\mathrm{AB}} = A\cos\mu z + B\sin\mu z \tag{iii}$$

$$v_{\mathrm{BC}} = C\cos\frac{\mu}{2}z + D\sin\frac{\mu}{2}z \tag{iv}$$

in which

$$\mu^2 = \frac{P}{EI}$$

When $z = 0$, $v_{\mathrm{AB}} = 0$ so that, from Eq. (iii), $A = 0$. Thus

$$v_{\mathrm{AB}} = B\sin\mu z \tag{v}$$

Also, when $z = l/2$, $(\mathrm{d}v/\mathrm{d}z)_{\mathrm{BC}} = 0$. Hence, from Eq. (iv)

$$0 = -\frac{\mu}{2}C\sin\frac{\mu l}{4} + \frac{\mu}{2}D\cos\frac{\mu l}{4}$$

whence

$$D = C\tan\frac{\mu l}{4}$$

Then

$$v_{\mathrm{BC}} = C\left(\cos\frac{\mu}{2}z + \tan\frac{\mu l}{4}\sin\frac{\mu}{2}z\right) \tag{vi}$$

When $z = l/4$, $v_{\mathrm{AB}} = v_{\mathrm{BC}}$ so that, from Eqs (v) and (vi)

$$B\sin\frac{\mu l}{4} = C\left(\cos\frac{\mu l}{8} + \tan\frac{\mu l}{4}\sin\frac{\mu l}{8}\right)$$

which simplifies to

$$B\sin\frac{\mu l}{4} = C\sec\frac{\mu l}{4}\cos\frac{\mu l}{8} \tag{vii}$$

Further, when $z = l/4$, $(\mathrm{d}v/\mathrm{d}z)_{\mathrm{AB}} = (\mathrm{d}v/\mathrm{d}z)_{\mathrm{BC}}$. Again from Eqs (v) and (vi)

$$\mu B\cos\frac{\mu l}{4} = C\left(-\frac{\mu}{2}\sin\frac{\mu l}{8} + \frac{\mu}{2}\tan\frac{\mu l}{4}\cos\frac{\mu l}{8}\right)$$

from which

$$B\cos\frac{\mu l}{4} = \frac{C}{2}\sec\frac{\mu l}{4}\sin\frac{\mu l}{8} \tag{viii}$$

Dividing Eq. (vii) by Eq. (viii) gives

$$\tan\frac{\mu l}{4} = 2\bigg/\tan\frac{\mu l}{8}$$

or

$$\tan\frac{\mu l}{4}\tan\frac{\mu l}{8} = 2$$

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Hence

$$\frac{2\tan^2 \mu l/8}{1 - \tan^2 \mu l/8} = 2$$

from which

$$\tan\frac{\mu l}{8} = \frac{1}{\sqrt{2}}$$

and

$$\frac{\mu l}{8} = 35.26^\circ = 0.615\,\text{rad}$$

i.e.

$$\sqrt{\frac{P}{EI}}\frac{l}{8} = 0.615$$

so that

$$P = \frac{24.2EI}{l^2}$$

S.6.3

With the spring in position the forces acting on the column in its buckled state are shown in Fig. S.6.3. Thus, from Eq. (6.1)

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = 4P(\delta - v) - k\delta(l - z) \tag{i}$$

The solution of Eq. (i) is

$$v = A\cos\mu z + B\sin\mu z + \frac{\delta}{4P}[4P + k(z - l)] \tag{ii}$$

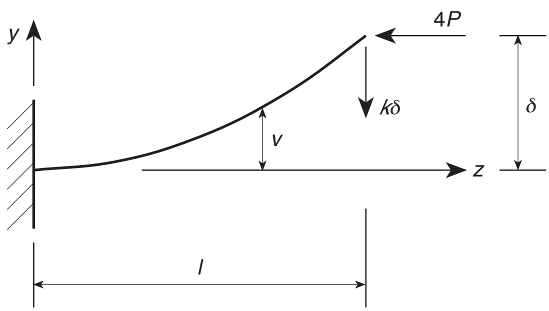

Fig. S.6.3

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where

$$\mu^2 = \frac{4P}{EI}$$

When $z = 0$, $v = 0$, hence, from Eq. (ii)

$$0 = A + \frac{\delta}{4P}(4P - kl)$$

from which

$$A = \delta(kl - 4P)/4P$$

Also when $z = 0$, $\mathrm{d}v/\mathrm{d}z = 0$ so that, from Eq. (ii)

$$0 = \mu B + \delta k/4P$$

and

$$B = -\delta k/4P\mu$$

Eq. (ii) then becomes

$$v = \frac{\delta}{4P}\left[(kl - 4P)\cos\mu z - \frac{k}{\mu}\sin\mu z + 4P + k(z - l)\right] \tag{iii}$$

When $z = l$, $v = \delta$. Substituting in Eq. (iii) gives

$$\delta = \frac{\delta}{4P}\left[(kl - 4P)\cos\mu l - \frac{k}{\mu}\sin\mu l + 4P\right]$$

from which

$$k = \frac{4P\mu}{\mu l - \tan\mu l}$$

S.6.4

The compressive load P will cause the column to be displaced from its initial curved position to that shown in Fig. S.6.4. Then, from Eq. (6.1) and noting that the bending moment at any point in the column is proportional to the change in curvature produced (see Eq. (6.22))

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - EI\frac{\mathrm{d}^2 v_0}{\mathrm{d}z^2} = -Pv \tag{i}$$

Now

$$v_0 = a\frac{4z}{l^2}(l - z)$$

so that

$$\frac{\mathrm{d}^2 v_0}{\mathrm{d}z^2} = -\frac{8a}{l^2}$$

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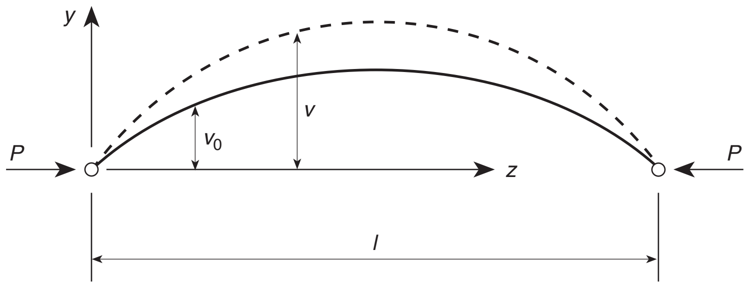

Fig. S.6.4

and Eq. (i) becomes

$$\frac{d^2v}{dz^2}+\frac{P}{EI}v=-\frac{8a}{l^2} \tag{ii}$$

The solution of Eq. (ii) is

$$v=A\cos\lambda z+B\sin\lambda z-8a/(\lambda l)^2 \tag{iii}$$

where $\lambda^2=P/EI$.

When $z=0$, $v=0$ so that $A=8a/(\lambda l)^2$. When $z=l/2$, $dv/dz=0$. Thus, from Eq. (iii)

$$0=-\lambda A\sin\frac{\lambda l}{2}+\lambda B\cos\frac{\lambda l}{2}$$

whence

$$B=\frac{8a}{(\lambda l)^2}\tan\frac{\lambda l}{2}$$

Eq. (iii) then becomes

$$v=\frac{8a}{(\lambda l)^2}\left(\cos\lambda z+\tan\frac{\lambda l}{2}\sin\lambda z-1\right) \tag{iv}$$

The maximum bending moment occurs when v is a maximum at $z=l/2$. Then, from Eq. (iv)

$$M(\max)=-Pv_{\max}=-\frac{8aP}{(\lambda l)^2}\left(\cos\frac{\lambda l}{2}+\tan\frac{\lambda l}{2}\sin\frac{\lambda l}{2}-1\right)$$

from which

$$M(\max)=-\frac{8aP}{(\lambda l)^2}\left(\sec\frac{\lambda l}{2}-1\right)$$

S.6.5

Under the action of the compressive load P the column will be displaced to the position shown in Fig. S.6.5. As in P.6.4 the bending moment at any point is

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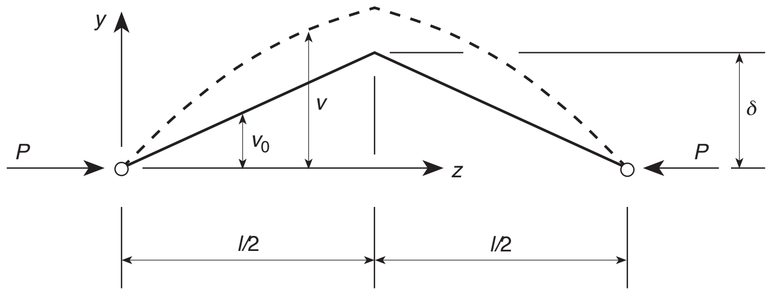

Fig. S.6.5

proportional to the change in curvature. Thus, from Eq. (6.1)

$$EI\frac{d^2v}{dz^2} - EI\frac{d^2v_0}{dz^2} = -Pv \tag{i}$$

In this case, since each half of the column is straight before the application of P, $d^2v_0/dz^2 = 0$ and Eq. (i) reduces to

$$EI\frac{d^2v}{dz^2} = -Pv \tag{ii}$$

The solution of Eq. (ii) is

$$v = A\cos\mu z + B\sin\mu z \tag{iii}$$

in which $\mu^2 = P/EI$. When $z = 0$, $v = 0$ so that $A = 0$ and Eq. (iii) becomes

$$v = B\sin\mu z \tag{iv}$$

The slope of the column at its mid-point in its unloaded position is $2\delta/l$. This must be the slope of the column at its mid-point in its loaded state since a change of slope over zero distance would require an infinite bending moment. Thus, from Eq. (iv)

$$\frac{dv}{dz} = \frac{2\delta}{l} = \mu B\cos\frac{\mu l}{2}$$

so that

$$B = \frac{2\delta}{\mu l\cos(\mu l/2)}$$

and

$$v = \frac{2\delta}{\mu l\cos(\mu l/2)}\sin\mu z \tag{v}$$

The maximum bending moment will occur when v is a maximum, i.e. at the mid-point of the column. Then

$$M(\max) = -Pv_{\max} = -\frac{2P\delta}{\mu l\cos(\mu l/2)}\sin\frac{\mu l}{2}$$

from which

$$M(\max) = -P\frac{2\delta}{l}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\frac{l}{2}$$

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S.6.6

Referring to Fig. S.6.6 the bending moment at any section z is given by

$$M = P(e + v) - \frac{wl}{2}z + w\frac{z^2}{2}$$

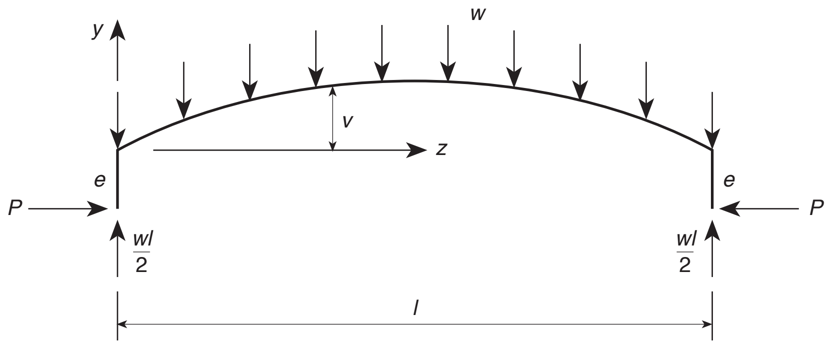

Fig. S.6.6

or

$$M = P(e + v) + \frac{w}{2}(z^2 - lz) \tag{i}$$

Substituting for M in Eq. (6.1)

$$EI\frac{d^2v}{dz^2} + Pv = -Pe - \frac{w}{2}(z^2 - lz)$$

or

$$\frac{d^2v}{dz^2} + \mu^2 v = -\mu^2 e - \frac{w\mu^2}{2P}(z^2 - lz) \tag{ii}$$

The solution of Eq. (ii) is

$$v = A\cos\mu z + B\sin\mu z - e + \frac{w}{2P}(lz - z^2) + \frac{w}{\mu^2 P} \tag{iii}$$

When $z = 0$, $v = 0$, hence $A = e - w/\mu^2 P$. When $z = l/2$, $dv/dz = 0$ which gives

$$B = A\tan\frac{\mu l}{2} = \left(e - \frac{w}{\mu^2 P}\right)\tan\frac{\mu l}{2}$$

Eq. (iii) then becomes

$$v = \left(e - \frac{w}{\mu^2 P}\right)\left[\frac{\cos\mu(z - l/2)}{\cos\mu l/2} - 1\right] + \frac{w}{2P}(lz - z^2) \tag{iv}$$

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The maximum bending moment will occur at mid-span where $z = l/2$ and $v = v_{\max}$. From Eq. (iv)

$$v_{\max} = \left(e - \frac{EIw}{P^2}\right)\left(\sec\frac{\mu l}{2} - 1\right) + \frac{wl^2}{8P}$$

and from Eq. (i)

$$M(\max) = Pe + Pv_{\max} - \frac{wl^2}{8}$$

whence

$$M(\max) = \left(Pe - \frac{w}{\mu^2}\right)\sec\frac{\mu l}{2} + \frac{w}{\mu^2} \tag{v}$$

For the maximum bending moment to be as small as possible the bending moment at the ends of the column must be numerically equal to the bending moment at mid-span. Thus

$$Pe + \left(Pe - \frac{w}{\mu^2}\right)\sec\frac{\mu l}{2} + \frac{w}{\mu^2} = 0$$

or

$$Pe\left(1 + \sec\frac{\mu l}{2}\right) = \frac{w}{\mu^2}\left(\sec\frac{\mu l}{2} - 1\right)$$

Then

$$e = \frac{w}{P\mu^2}\left(\frac{1 - \cos\mu l/2}{1 + \cos\mu l/2}\right)$$

i.e.

$$e = \left(\frac{w}{P\mu^2}\right)\tan^2\frac{\mu l}{4} \tag{vi}$$

From Eq. (vi) the end moment is

$$Pe = \frac{w}{\mu^2}\tan^2\frac{\mu l}{4} = \frac{wl^2}{16}\left(\frac{\tan\mu l/4}{\mu l/4}\right)\left(\frac{\tan\mu l/4}{\mu l/4}\right)$$

When $P \to 0$, $\tan\mu l/4 \to \mu l/4$ and the end moment becomes $wl^2/16$.

S.6.7

From Eq. (6.21) the buckling stress, σ_b, is given by

$$\sigma_b = \frac{\pi^2 E_t}{(l/r)^2} \tag{i}$$

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The stress–strain relationship is

$$10.5\times10^{6}\varepsilon=\sigma+21\,000\left(\frac{\sigma}{49\,000}\right)^{16} \tag{ii}$$

Hence

$$10.5\times10^{6}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\sigma}=1+\frac{16\times21\,000}{(49\,000)^{16}}\sigma^{15}$$

from which

$$E_{\mathrm{t}}=\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}=\frac{10.5\times10^{6}\times(49\,000)^{16}}{(49\,000)^{16}+16\times21\,000(\sigma)^{15}}$$

Then, from Eq. (i)

$$\left(\frac{l}{r}\right)^{2}=\frac{\pi^{2}E_{\mathrm{t}}}{\sigma_{\mathrm{b}}}=\frac{10.36\times10^{7}}{\sigma_{\mathrm{b}}+336\,000(\sigma_{\mathrm{b}}/49\,000)^{16}} \tag{iii}$$

From Eq. (iii) the following $\sigma_{\mathrm{b}}-l/r$ relationship is found

σ_{b}	4900	3×4900	6×4900	9×4900	49 000
l/r	145.4	84.0	59.3	31.2	16.4

For the given strut

$$r^{2}=\frac{I}{A}=\frac{\pi(D^{4}-d^{4})/64}{\pi(D^{2}-d^{2})/4}=\frac{1}{16}(D^{2}+d^{2})$$

i.e.

$$r^{2}=\frac{1}{16}(1.5^{2}+1.34^{2})=0.253\,\text{units}^{2}$$

Hence

$$r=0.503\,\text{units}$$

Thus

$$\frac{l}{r}=\frac{20}{0.503}=39.8$$

Then, from the $\sigma_{\mathrm{b}}-l/r$ relationship

$$\sigma_{\mathrm{b}}=40\,500\ \text{force units/units}^{2}$$

Hence the buckling load is

$$40\,500\times\frac{\pi}{4}(1.5^{2}-1.34^{2})$$

i.e.

$$\text{Buckling load}=14\,454\ \text{force units}$$

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S.6.8

The deflected shape of each of the members AB and BC is shown in Fig. S.6.8. For the member AB and from Eq. (6.1)

$$E\frac{d^2v_1}{dz_1^2} = -M_B$$

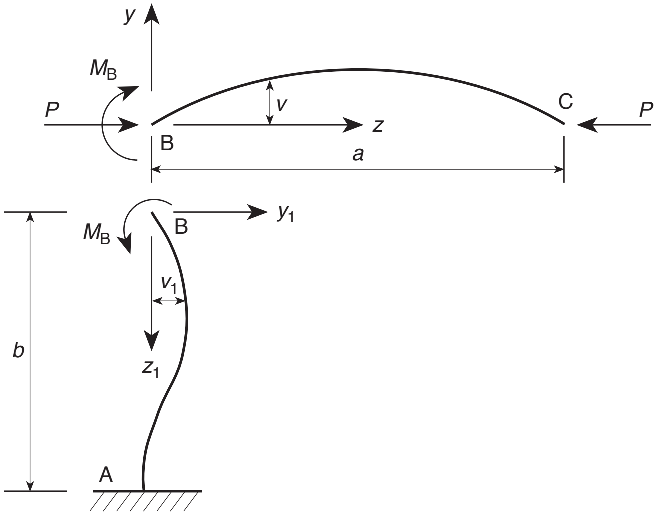

Fig. S.6.8

so that

$$EI\frac{dv_1}{dz_1} = -M_B z_1 + A$$

When $z_1 = b$, $dv_1/dz_1 = 0$. Thus $A = M_B b$ and

$$EI\frac{dv_1}{dz_1} = -M_B(z_1 - b) \qquad \text{(i)}$$

At B, when $z_1 = 0$, Eq. (i) gives

$$\frac{dv_1}{dz_1} = \frac{M_B b}{EI} \qquad \text{(ii)}$$

In BC Eq. (6.1) gives

$$EI\frac{d^2v}{dz^2} = -Pv + M_B$$

or

$$EI\frac{d^2v}{dz^2} + Pv = M_B \qquad \text{(iii)}$$

The solution of Eq. (iii) is

$$v = B\cos\lambda z + C\sin\lambda z + M_{\mathrm{B}}/P \tag{iv}$$

When $z = 0$, $v = 0$ so that $B = -M_{\mathrm{B}}/P$.

When $z = a/2$, $\mathrm{d}v/\mathrm{d}z = 0$ so that

$$C = B\tan\frac{\lambda a}{2} = -\frac{M_{\mathrm{B}}}{P}\tan\frac{\lambda a}{2}$$

Eq. (iv) then becomes

$$v = -\frac{M_{\mathrm{B}}}{P}\left(\cos\lambda z + \tan\frac{\lambda a}{2}\sin\lambda z - 1\right)$$

so that

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{M_{\mathrm{B}}}{P}\left(-\lambda\sin\lambda z + \lambda\tan\frac{\lambda a}{2}\cos\lambda z\right)$$

At B, when $z = 0$,

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{M_{\mathrm{B}}}{P}\lambda\tan\frac{\lambda a}{2} \tag{v}$$

Since $\mathrm{d}v_1/\mathrm{d}z_1 = \mathrm{d}v/\mathrm{d}z$ at B then, from Eqs (ii) and (v)

$$\frac{b}{EI} = -\frac{\lambda}{P}\tan\frac{\lambda a}{2}$$

whence

$$\frac{\lambda a}{2} = -\frac{1}{2}\left(\frac{a}{b}\right)\tan\frac{\lambda a}{2}$$

S.6.9

In an identical manner to S.6.4

$$EI\frac{\mathrm{d}^2 v'}{\mathrm{d}z^2} - EI\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -Pv'$$

where v' is the total displacement from the horizontal. Thus

$$\frac{\mathrm{d}^2 v'}{\mathrm{d}z^2} + \frac{P}{EI}v' = \frac{\mathrm{d}^2 v}{\mathrm{d}z^2}$$

or, since

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -\frac{\pi^2}{l^2}\delta\sin\frac{\pi}{l}z \quad \text{and} \quad \mu^2 = \frac{P}{EI}$$

$$\frac{\mathrm{d}^2 v'}{\mathrm{d}z^2} + \mu^2 v' = -\frac{\pi^2}{l^2}\delta\sin\frac{\pi z}{l} \tag{i}$$

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The solution of Eq. (i) is

$$v' = A\cos\mu z + B\sin\mu z + \frac{\pi^2\delta}{\pi^2 - \mu^2 l^2}\sin\frac{\pi z}{l} \tag{ii}$$

When $z = 0$ and l, $v' = 0$, hence $A = B = 0$ and Eq. (ii) becomes

$$v' = \frac{\pi^2\delta}{\pi^2 - \mu^2 l^2}\sin\frac{\pi z}{l}$$

The maximum bending moment occurs at the mid-point of the tube so that

$$M(\max) = Pv' = P\frac{\pi^2\delta}{\pi^2 - \mu^2 l^2} = \frac{P\delta}{1 - Pl^2/\pi^2 EI}$$

i.e.

$$M(\max) = \frac{P\delta}{1 - P/P_{\mathrm{e}}} = \frac{P\delta}{1 - \alpha}$$

The total maximum direct stress due to bending and axial load is then

$$\sigma(\max) = \frac{P}{\pi dt} + \left(\frac{P\delta}{1-\alpha}\right)\frac{d/2}{\pi d^3 t/8}$$

Hence

$$\sigma(\max) = \frac{P}{\pi dt}\left(1 + \frac{1}{1-\alpha}\frac{4\delta}{d}\right)$$

S.6.10

The forces acting on the members AB and BC are shown in Fig. S.6.10.

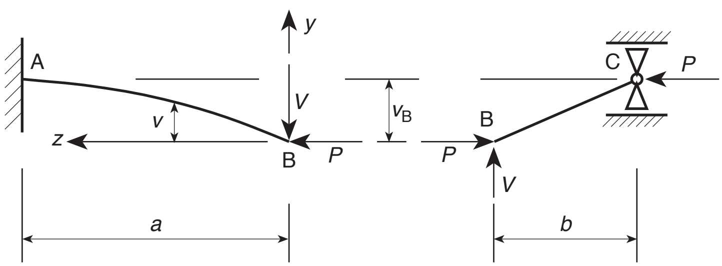

Fig. S.6.10

Considering first the moment equilibrium of BC about C

$$Pv_{\mathrm{B}} = Vb$$

from which

$$v_{\mathrm{B}} = \frac{Vb}{P} \tag{i}$$

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For the member AB and from Eq. (6.1)

$$EI\frac{d^2v}{dz^2} = -Pv - Vz$$

or

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = -\frac{Vz}{EI} \tag{ii}$$

The solution of Eq. (ii) is

$$v = A\cos\lambda z + B\sin\lambda z - \frac{Vz}{P} \tag{iii}$$

When $z = 0$, $v = 0$ so that $A = 0$. Also when $z = a$, $dv/dz = 0$, hence

$$0 = \lambda B\cos\lambda a - \frac{V}{P}$$

from which

$$B = \frac{V}{\lambda P\cos\lambda a}$$

and Eq. (iii) becomes

$$v = \frac{V}{P}\left(\frac{\sin\lambda z}{\lambda\cos\lambda a} - z\right)$$

When $z = a$, $v = v_B = Vb/P$ from Eq. (i). Thus

$$\frac{Vb}{P} = \frac{V}{P}\left(\frac{\sin\lambda a}{\lambda\cos\lambda a} - a\right)$$

from which

$$\lambda(a+b) = \tan\lambda a$$

S.6.11

The bending moment, M, at any section of the column is given by

$$M = P_{CR}v = P_{CR}k(lz - z^2) \tag{i}$$

Also

$$\frac{dv}{dz} = k(l - 2z) \tag{ii}$$

Substituting from Eqs (i) and (ii) in Eq. (6.47)

$$U + V = \frac{P_{CR}^2k^2}{2E}\left\{\frac{1}{I_1}\int_0^a(lz - z^2)^2\,dz + \frac{1}{I_2}\int_a^{l-a}(lz - z^2)^2\,dz + \frac{1}{I_1}\int_{l-a}^{l}(lz - z^2)^2\,dz\right\}$$
$$- \frac{P_{CR}k^2}{2}\int_0^l(l - 2z)^2\,dz$$

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i.e.

$$U + V = \frac{P_{\mathrm{CR}}^2 k^2}{2E}\left\{\frac{1}{I_1}\left[\frac{l^2 z^3}{3} - \frac{lz^4}{2} + \frac{z^5}{5}\right]_0^a + \frac{1}{I_2}\left[\frac{l^2 z^3}{3} - \frac{lz^4}{2} + \frac{z^5}{5}\right]_a^{l-a}\right.$$
$$\left. + \frac{1}{I_1}\left[\frac{l^2 z^3}{3} - \frac{lz^4}{2} + \frac{l^5}{5}\right]_{l-a}^{l}\right\} - \frac{P_{\mathrm{CR}} k^2}{2}\left[l^2 z - 2lz^2 + \frac{4z^3}{3}\right]_0^l$$

i.e.

$$U + V = \frac{P_{\mathrm{CR}}^2 k^2}{2EI_2}\left\{\left(\frac{I_2}{I_1} - 1\right)\left[\frac{l^2 a^3}{3} - \frac{la^4}{2} + \frac{a^5}{5} - \frac{l^2(l-a)^3}{3} + \frac{l(l-a)^4}{2} - \frac{(l-a)^5}{5}\right]\right.$$
$$\left. + \frac{I_2}{I_1}\frac{l^5}{30}\right\} - \frac{P_{\mathrm{CR}} k^2 l^3}{6}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U+V)}{\partial k} = \frac{P_{\mathrm{CR}}^2 k}{EI_2}\left\{\left(\frac{I_2}{I_1} - 1\right)\left[\frac{l^2 a^3}{3} - \frac{la^4}{2} + \frac{a^5}{5} - \frac{l^2(l-a)^3}{3} + \frac{l(l-a)^4}{2} - \frac{(l-a)^5}{5}\right]\right.$$
$$\left. + \frac{I_2}{I_1}\frac{l^5}{30}\right\} - \frac{P_{\mathrm{CR}} k l^3}{3} = 0$$

Hence

$$P_{\mathrm{CR}} = \frac{EI_2 l^3}{3\left\{\left(\frac{I_2}{I_1} - 1\right)\left[\frac{l^2 a^3}{3} - \frac{la^4}{2} + \frac{a^5}{5} - \frac{l^2(l-a)^3}{3} + \frac{l(l-a)^4}{2} - \frac{(l-a)^5}{5}\right] + \frac{I_2}{I_1}\frac{l^5}{30}\right\}} \tag{iii}$$

When $I_2 = 1.6I_1$ and $a = 0.2l$, Eq. (iii) becomes

$$P_{\mathrm{CR}} = \frac{14.96EI_1}{l^2} \tag{iv}$$

Without the reinforcement

$$P_{\mathrm{CR}} = \frac{\pi^2 EI_1}{l^2} \tag{v}$$

Therefore, from Eqs (iv) and (v) the increase in strength is

$$\frac{EI_1}{l^2}(14.96 - \pi^2)$$

Thus the percentage increase in strength is

$$\left[\frac{EI}{l^2}(14.96 - \pi^2)\bigg/\frac{l^2}{\pi^2 EI}\right] \times 100 = 52\%$$

Since the radius of gyration of the cross-section of the column remains unchanged

$$I_1 = A_1 r^2 \quad \text{and} \quad I_2 = A_2 r^2$$

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Hence

$$\frac{A_2}{A_1} = \frac{I_2}{I_1} = 1.6 \tag{vi}$$

The original weight of the column is $lA_1\rho$ where ρ is the density of the material of the column. Then, the increase in weight $= 0.4lA_1\rho + 0.6lA_2\rho - lA_1\rho = 0.6l\rho(A_2 - A_1)$. Substituting for A_2 from Eq. (vi)

$$\text{Increase in weight} = 0.6l\rho(1.6A_1 - A_1) = 0.36lA_1\rho$$

i.e. an increase of 36%.

S.6.12

The equation for the deflected centre line of the column is

$$v = \frac{4\delta}{l^2}z^2 \tag{i}$$

in which δ is the deflection at the ends of the column relative to its centre and the origin for z is at the centre of the column. Also, the second moment of area of its cross-section varies, from the centre to its ends, in accordance with the relationship

$$I = I_1\left(1 - 1.6\frac{z}{l}\right) \tag{ii}$$

At any section of the column the bending moment, M, is given by

$$M = P_{\text{CR}}(\delta - v) = P_{\text{CR}}\delta\left(1 - 4\frac{z^2}{l^2}\right) \tag{iii}$$

Also, from Eq. (i)

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{8\delta}{l^2}z \tag{iv}$$

Substituting in Eq. (6.47) for M, I and $\mathrm{d}v/\mathrm{d}z$

$$U + V = 2\int_0^{l/2} \frac{P_{\text{CR}}^2\delta^2(1 - 4z^2/l^2)^2}{2EI_1(1 - 1.6z/l)}\,\mathrm{d}z - \frac{P_{\text{CR}}}{2}2\int_0^{l/2}\frac{64\delta^2}{l^4}z^2\,\mathrm{d}z$$

or

$$U + V = \frac{P_{\text{CR}}^2\delta^2}{EI_1l^3}\int_0^{l/2}\frac{(l^2 - 4z^2)^2}{(l - 1.6z)}\,\mathrm{d}z - \frac{64P_{\text{CR}}\delta^2}{l^4}\int_0^{l/2} z^2\,\mathrm{d}z \tag{v}$$

Dividing the numerator by the denominator in the first integral in Eq. (v) gives

$$U + V = \frac{P_{\text{CR}}^2\delta^2}{EI_1l^3}\left[\int_0^{l/2}(-10z^3 - 6.25lz^2 + 1.09l^2z + 0.683l^3)\,\mathrm{d}z \right.$$
$$\left. + 0.317l^3\int_0^{l/2}\frac{\mathrm{d}z}{(1 - 1.6z/l)}\right] - \frac{64P_{\text{CR}}\delta^2}{l^4}\left[\frac{z^3}{3}\right]_0^{l/2}$$

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Hence

$$U + V = \frac{P_{\text{CR}}^2 \delta^2}{EIl^3} \left[-10\frac{z^4}{4} - 6.25l\frac{z^3}{3} + 1.09l^2\frac{z^2}{2} + 0.683l^3 z - \frac{0.317}{1.6} l^4 \log_e \left(1 - \frac{1.6z}{l}\right)\right]_0^{l/2} - \frac{8P_{\text{CR}}\delta^2}{3l}$$

i.e.

$$U + V = \frac{0.3803 P_{\text{CR}}^2 \delta^2 l}{EI_1} - \frac{8P_{\text{CR}}\delta^2}{3l}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U+V)}{\partial\delta} = \frac{0.7606 P_{\text{CR}}^2 \delta l}{EI_1} - \frac{16 P_{\text{CR}}\delta}{3l} = 0$$

Hence

$$P_{\text{CR}} = \frac{7.01 EI_1}{l^2}$$

For a column of constant thickness and second moment of area I_2,

$$P_{\text{CR}} = \frac{\pi^2 EI_2}{l^2} \quad \text{(see Eq. (6.5))}$$

For the columns to have the same buckling load

$$\frac{\pi^2 EI_2}{l^2} = \frac{7.01 EI_1}{l^2}$$

so that

$$I_2 = 0.7 I_1$$

Thus, since the radii of gyration are the same

$$A_2 = 0.7 A_1$$

Therefore, the weight of the constant thickness column is equal to $\rho A_2 l = 0.7\rho A_1 l$.
The weight of the tapered column $= \rho \times$ average thickness $\times\, l = \rho \times 0.6 A_1 l$.
Hence the saving in weight $= 0.7\rho A_1 l - 0.6\rho A_1 l = 0.1\rho A_1 l$.
Expressed as a percentage

$$\text{saving in weight} = \frac{0.1\rho A_1 l}{0.7\rho A_1 l} \times 100 = 14.3\%$$

S.6.13

There are four boundary conditions to be satisfied, namely, $v = 0$ at $z = 0$ and $z = l$, $\mathrm{d}v/\mathrm{d}z = 0$ at $z = 0$ and $\mathrm{d}^2 v/\mathrm{d}z^2$ (i.e. bending moment) $= 0$ at $z = l$. Thus, since only one arbitrary constant may be allowed for, there cannot be more than five terms in the

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polynomial. Suppose

$$v = a_0 + a_1\left(\frac{z}{l}\right) + a_2\left(\frac{z}{l}\right)^2 + a_3\left(\frac{z}{l}\right)^3 + a_4\left(\frac{z}{l}\right)^4 \tag{i}$$

Then, since $v = 0$ at $z = 0$, $a_0 = 0$. Also, since $\mathrm{d}v/\mathrm{d}z = 0$ at $z = 0$, $a_1 = 0$. Hence, Eq. (i) becomes

$$v = a_2\left(\frac{z}{l}\right)^2 + a_3\left(\frac{z}{l}\right)^3 + a_4\left(\frac{z}{l}\right)^4 \tag{ii}$$

When $z = l$, $v = 0$, thus

$$0 = a_2 + a_3 + a_4 \tag{iii}$$

When $z = l$, $\mathrm{d}^2v/\mathrm{d}z^2 = 0$, thus

$$0 = a_2 + 3a_3 + 6a_4 \tag{iv}$$

Subtracting Eq. (iv) from Eq. (ii)

$$0 = -2a_3 - 5a_4$$

from which $a_3 = -5a_4/2$.

Substituting for a_3 in Eq. (iii) gives $a_4 = 2a_2/3$ so that $a_3 = -5a_2/3$. Eq. (ii) then becomes

$$v = a_2\left(\frac{z}{l}\right)^2 - \frac{5a_2}{3}\left(\frac{z}{l}\right)^3 + \frac{2a_2}{3}\left(\frac{z}{l}\right)^4 \tag{v}$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}z} = 2a_2\frac{z}{l} - 5a_2\frac{z^2}{l^3} + \frac{8a_2}{3}\frac{z^3}{l^4} \tag{vi}$$

and

$$\frac{\mathrm{d}^2v}{\mathrm{d}z^2} = 2\frac{a_2}{l} - 10a_2\frac{z}{l^3} + 8a_2\frac{z^2}{l^4} \tag{vii}$$

The total strain energy of the column will be the sum of the strain energy due to bending and the strain energy due to the resistance of the elastic foundation. For the latter, consider an element, δz, of the column. The force on the element when subjected to a small displacement, v, is $k\delta z v$. Thus, the strain energy of the element is $\frac{1}{2}kv^2\delta z$ and the strain energy of the column due to the resistance of the elastic foundation is

$$\int_0^l \frac{1}{2}kv^2\,\mathrm{d}z$$

Substituting for v from Eq. (v)

$$U\text{ (elastic foundation)} = \frac{1}{2}k\frac{a_2^2}{l^4}\int_0^l\left(z^4 - \frac{10z^5}{3l} + \frac{37z^6}{9l^2} - \frac{20z^7}{9l^3} + \frac{4z^8}{9l^4}\right)\mathrm{d}z$$

i.e. U (elastic foundation) $= 0.0017ka_2^2 l$.

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Now substituting for d^2v/dz^2 and dv/dz in Eq. (6.48) and adding U (elastic foundation) gives

$$U + V = \frac{EI}{2}\int_0^l \frac{4a_2^2}{l^4}\left(1 - \frac{10z}{l} + \frac{33z^2}{l^2} - \frac{40z^3}{l^3} + \frac{16z^4}{l^4}\right)dz + 0.0017ka_2^2 l$$

$$- \frac{P_{CR}}{2}\int_0^l \frac{a_2^2}{l^4}\left(4z^2 - \frac{20z^3}{l} + \frac{107z^4}{3l^2} - \frac{80z^5}{3l^3} + \frac{64z^6}{9l^4}\right)dz \qquad \text{(viii)}$$

Eq. (viii) simplifies to

$$U + V = \frac{0.4EI}{l^3}a_2^2 + 0.0017ka_2^2 l - \frac{0.019a_2^2 P_{CR}}{l}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U + V)}{\partial a_2} = \frac{0.8EI}{l^3}a_2 + 0.0034ka_2 l - \frac{0.038a_2 P_{CR}}{l}$$

whence

$$P_{CR} = \frac{21.05EI}{l^2} + 0.09kl^2$$

S.6.14

Assuming that the elastic deflection, w, of the plate is of the same form as the initial curvature, then

$$w = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

Hence, from Eq. (5.36) in which $m = n = 1$, $a = b$ and $N_x = \sigma t$

$$w = \frac{\delta\sigma t}{(4\pi^2 D/a^2) - \sigma t}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a} \qquad \text{(i)}$$

The deflection, w_C, at the centre of the plate where $x = a/2$, $y = a/2$ is, from Eq. (i)

$$w_C = \frac{\delta\sigma t}{(4\pi^2 D/a^2) - \sigma t} \qquad \text{(ii)}$$

When $\sigma t \to 4\pi^2 D/a$, $w \to \infty$ and $\sigma t \to N_{x,CR}$, the buckling load of the plate. Eq. (ii) may then be written

$$w_C = \frac{\delta\sigma t}{N_{x,CR} - \sigma t} = \frac{\delta\sigma t/N_{x,CR}}{1 - \sigma t/N_{x,CR}}$$

from which

$$w_C = N_{x,CR}\frac{w_C}{\sigma t} - \delta \qquad \text{(iii)}$$

Therefore, from Eq. (iii), a graph of w_C against $w_C/\sigma t$ will be a straight line of slope $N_{x,CR}$ and intercept δ, i.e. a Southwell plot.

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S.6.15

The total potential energy of the plate is given by Eq. (6.52), i.e.

$$U + V = \frac{1}{2}\int_0^l \int_0^b \left[D\left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} - N_x \left(\frac{\partial w}{\partial x} \right)^2 \right] \mathrm{d}x\,\mathrm{d}y \qquad \text{(i)}$$

in which

$$w = a_{11} \sin\frac{m\pi x}{l} \sin^2\frac{\pi y}{b} \qquad \text{(ii)}$$

and

$$N_x = \sigma t$$

From Eq. (ii)

$$\frac{\partial w}{\partial x} = a_{11}\frac{m\pi}{l}\cos\frac{m\pi x}{l}\sin^2\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2} = -a_{11}\frac{m^2\pi^2}{l^2}\sin\frac{m\pi x}{l}\sin^2\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial y^2} = a_{11}\frac{2\pi^2}{b^2}\sin\frac{m\pi x}{l}\cos\frac{2\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x \partial y} = a_{11}\frac{m\pi^2}{bl}\cos\frac{m\pi x}{l}\sin\frac{2\pi y}{b}$$

Substituting these expressions in Eq. (i) and integrating gives

$$U + V = \frac{D}{2}a_{11}^2\pi^4\left(\frac{3m^4 b}{16 l^3} + \frac{m^2}{2lb} + \frac{l}{b^3} \right) - \frac{3\sigma t a_{11}^2 m^2 \pi^2 b}{32 l}$$

The total potential energy of the plate has a stationary value in the neutral equilibrium of its buckled state, i.e. when $\sigma = \sigma_{\mathrm{CR}}$. Thus

$$\frac{\partial(U+V)}{\partial a_{11}} = D a_{11}\pi^4\left(\frac{3m^4 b}{16 l^3} + \frac{m^2}{2lb} + \frac{l}{b^3} \right) - \frac{3\sigma_{\mathrm{CR}} t a_{11} m^2\pi^2 b}{16 l} = 0$$

whence

$$\sigma_{\mathrm{CR}} = \frac{16 l \pi^2 D}{3 t m^2 b}\left(\frac{3m^4 b}{16 l^3} + \frac{m^2}{2lb} + \frac{l}{b^3} \right) \qquad \text{(iii)}$$

When $l = 2b$, Eq. (iii) gives

$$\sigma_{\mathrm{CR}} = \frac{32\pi^2 D}{3tb^2}\left(\frac{3m^2}{128} + \frac{1}{4} + \frac{2}{m^2} \right) \qquad \text{(iv)}$$

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σ_{CR} will be a minimum when $\mathrm{d}\sigma_{\mathrm{CR}}/\mathrm{d}m = 0$, i.e. when

$$\frac{6m}{128} - \frac{4}{m^3} = 0$$

or

$$m^4 = \frac{4 \times 128}{6}$$

from which

$$m = 3.04$$

i.e.

$$m = 3$$

Substituting this value of m in Eq. (iv)

$$\sigma_{\mathrm{CR}} = \frac{71.9D}{tb^2}$$

whence

$$\sigma_{\mathrm{CR}} = \frac{6E}{(1-\nu^2)}\left(\frac{t}{b}\right)^2$$

S.6.16

(a) The length, l, of the panel is appreciably greater than the dimension b so that failure will occur due to buckling rather than yielding. The modes of buckling will then be those described on pages 176 and 177.

(1) *Buckling as a column of length l*
Consider a stiffener and an associated portion of sheet as shown in Fig. S.6.16. The critical stress, σ_{CR}, is given by Eq. (6.8), i.e.

$$\sigma_{\mathrm{CR}} = \frac{\pi^2 E}{(l/r)^2} \tag{i}$$

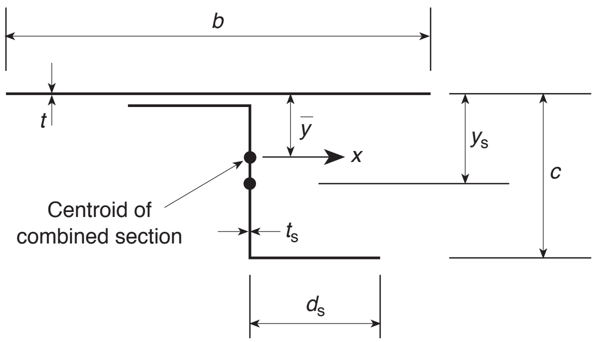

Fig. S.6.16

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In Eq. (i) r is the radius of gyration of the combined section. Thus, $r = \sqrt{I_x/A}$, where A and I_x are the cross-sectional area and the second moment of area of the combined section respectively. From Fig. S.6.16

$$A = bt + t_s(2d + c) = bt + A_s \tag{ii}$$

Also

$$(bt + A_s)\bar{y} = A_s y_s$$

so that

$$\bar{y} = \frac{A_s y_s}{bt + A_s}$$

Then

$$I_x = bt(\bar{y})^2 + 2dt_s\left(\frac{c}{2}\right)^2 + \frac{t_s c^3}{12} + A_s(\bar{y} - y_s)^2$$

or

$$I_x = bt(\bar{y})^2 + t_s\frac{c^2}{2}\left(d + \frac{c}{6}\right) + A_s(\bar{y} - y_s)^2 \tag{iii}$$

The radius of gyration follows from Eqs (ii) and (iii) and hence the critical stress from Eq. (i).

(2) *Buckling of the sheet between stiffeners*
The sheet may buckle as a long plate of length l and width, b, which is simply supported on all four edges. The buckling stress is then given by Eq. (6.58), i.e.

$$\sigma_{CR} = \frac{\eta k \pi^2 E}{12(1 - \nu^2)}\left(\frac{t}{b}\right)^2 \tag{iv}$$

Since l is very much greater than b, k is equal to 4 (from Fig. 6.15). Therefore, assuming that buckling takes place in the elastic range ($\eta = 1$), Eq. (iv) becomes

$$\sigma_{CR} = \frac{4\pi^2 E}{12(1 - \nu^2)}\left(\frac{t}{b}\right)^2 \tag{v}$$

(3) *Buckling of stiffener flange*
The stiffener flange may buckle as a long plate simply supported on three edges with one edge free. In this case $k = 0.43$ (see Fig. 6.16 (a)) and, assuming elastic buckling (i.e. $\eta = 1$)

$$\sigma_{CR} = \frac{0.43\pi^2 E}{12(1 - \nu^2)}\left(\frac{t_s}{d_s}\right)^2 \tag{vi}$$

(b) A suitable test would be a panel buckling test.

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S.6.17

(a) Consider, initially, the buckling of the panel as a pin-ended column. For a section comprising a width of sheet and associated stiffener as shown in Fig. S.6.17,

$$A = 120 \times 3 + 30 \times 3.5 = 465\,\text{mm}^2$$

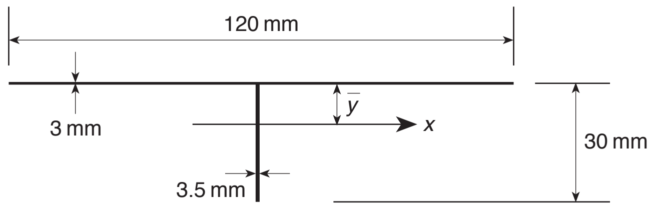

Fig. S.6.17

Then

$$465\bar{y} = 30 \times 3.5 \times 15 + 120 \times 3 \times 1.5$$

i.e.

$$\bar{y} = 4.5\,\text{mm}$$

Then

$$I_x = 120 \times 3 \times 4.5^2 + \frac{120 \times 3^3}{12} + \frac{3.5 \times 4.5^3}{3} + \frac{3.5 \times 25.5^3}{3}$$

i.e.

$$I_x = 27\,011\,\text{mm}^4$$

Hence

$$r = \sqrt{\frac{27\,011}{465}} = 7.62\,\text{mm}$$

From Eq. (6.8)

$$\sigma_{\text{CR}} = \frac{\pi^2 \times 70\,000}{(500/7.62)^2}$$

i.e.

$$\sigma_{\text{CR}} = 160.5\,\text{N/mm}^2$$

From Section 6.10 the equivalent skin thickness is

$$\bar{t} = \frac{30 \times 3.5}{120} + 3 = 3.875\,\text{mm}$$

Overall buckling of the panel will occur when

$$N_{x,\text{CR}} = \sigma_{\text{CR}}\bar{t} = 160.5 \times 3.875 = 621.9\,\text{N/mm} \qquad \text{(i)}$$

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Buckling of the sheet will occur when, from Eq. (6.57)

$$\sigma_{\mathrm{CR}} = 3.62E\left(\frac{t}{b}\right)^2 = 3.62 \times 70\,000\left(\frac{3}{120}\right)^2$$

i.e.

$$\sigma_{\mathrm{CR}} = 158.4\,\mathrm{N/mm}^2$$

Hence

$$N_{x,\mathrm{CR}} = 158.4 \times 3.875 = 613.8\,\mathrm{N/mm} \tag{ii}$$

Buckling of the stiffener will occur when, from Eq. (6.57)

$$\sigma_{\mathrm{CR}} = 0.385E\left(\frac{t}{b}\right)^2 = 0.385 \times 70\,000\left(\frac{3.5}{30}\right)^2$$

i.e.

$$\sigma_{\mathrm{CR}} = 366.8\,\mathrm{N/mm}^2$$

whence

$$N_{x,\mathrm{CR}} = 366.8 \times 3.875 = 1421.4\,\mathrm{N/mm} \tag{iii}$$

By comparison of Eqs (i), (ii) and (iii) the onset of buckling will occur when

$$N_{x,\mathrm{CR}} = 613.8\,\mathrm{N/mm}$$

(b) Since the stress in the sheet increases parabolically after reaching its critical value then

$$\sigma = CN_x^2 \tag{iv}$$

where C is some constant. From Eq. (iv)

$$\sigma_{\mathrm{CR}} = CN_{x,\mathrm{CR}}^2 \tag{v}$$

so that, combining Eqs (iv) and (v)

$$\frac{\sigma}{\sigma_{\mathrm{CR}}} = \left(\frac{N_x}{N_{x,\mathrm{CR}}}\right)^2 \tag{vi}$$

Suppose that $\sigma = \sigma_{\mathrm{F}}$, the failure stress, i.e. $\sigma_{\mathrm{F}} = 300\,\mathrm{N/mm}^2$. Then, from Eq. (vi)

$$N_{x,\mathrm{F}} = \sqrt{\frac{\sigma_{\mathrm{F}}}{\sigma_{\mathrm{CR}}}}N_{x,\mathrm{CR}}$$

or

$$N_{x,\mathrm{F}} = \sqrt{\frac{300}{158.4}} \times 613.8$$

i.e.

$$N_{x,\mathrm{F}} = 844.7\,\mathrm{N/mm}$$

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S.6.18

The purely flexural instability load is given by Eq. (6.7) in which, from Table 6.1, $l_e = 0.5l$ where l is the actual column length. Also it is clear that the least second moment of area of the column cross-section occurs about an axis coincident with the web. Thus

$$I = 2 \times \frac{2tb^3}{12} = \frac{tb^3}{3}$$

Then

$$P_{CR} = \frac{\pi^2 EI}{(0.5l)^2}$$

i.e.

$$P_{CR} = \frac{4\pi^2 Etb^3}{3l^2} \tag{i}$$

The purely torsional buckling load is given by the last of Eqs (6.90), i.e.

$$P_{CR(\theta)} = \frac{A}{I_0}\left(GJ + \frac{\pi^2 E\Gamma}{l^2}\right) \tag{ii}$$

In Eq. (ii) $A = 5bt$ and

$$I_0 = I_x + I_y = 2 \times 2tb\frac{b^2}{4} + \frac{tb^3}{12} + \frac{tb^3}{3}$$

i.e.

$$I_0 = \frac{17tb^3}{12}$$

Also, from Eq. (9.59)

$$J = \sum \frac{st^3}{3} = \frac{1}{3}(2b8t^3 + bt^3) = \frac{17bt^3}{3}$$

and, referring to S.11.13

$$\Gamma = \frac{tb^5}{12}$$

Then, from Eq. (ii)

$$P_{CR(\theta)} = \frac{20}{17b}\left(17Gt^3 + \frac{\pi^2 Etb^4}{l^2}\right) \tag{iii}$$

Now equating Eqs (i) and (iii)

$$\frac{4\pi^2 Etb^3}{3l^2} = \frac{20}{17b}\left(17Gt^3 + \frac{\pi^2 Etb^4}{l^2}\right)$$

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from which

$$l^2 = \frac{2\pi^2 Eb^4}{255Gt^2}$$

From Eq. (1.45), $E/G = 2(1 + \nu)$. Hence

$$l = \frac{2\pi b^2}{t}\sqrt{\frac{1+\nu}{255}}$$

Eqs (i) and (iii) may be written, respectively, as

$$P_{\mathrm{CR}} = \frac{1.33C_1}{l^2}$$

and

$$P_{\mathrm{CR}(\theta)} = C_2 + \frac{1.175C_1}{l^2}$$

where C_1 and C_2 are constants. Thus, if l were less than the value found, the increase in the last term in the expression for $P_{\mathrm{CR}(\theta)}$ would be less than the increase in the value of P_{CR}, i.e. $P_{\mathrm{CR}(\theta)} < P_{\mathrm{CR}}$ for a decrease in l and the column would fail in torsion.

S.6.19

In this case Eqs (6.90) do not apply since the ends of the column are not free to warp. From Eq. (6.83) and since, for the cross-section of the column, $x_{\mathrm{s}} = y_{\mathrm{s}} = 0$,

$$E\Gamma\frac{\mathrm{d}^4\theta}{\mathrm{d}z^4} + \left(I_0\frac{P}{A} - GJ\right)\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} = 0 \tag{i}$$

For buckling, $P = P_{\mathrm{CR}}$, the critical load and $P_{\mathrm{CR}}/A = \sigma_{\mathrm{CR}}$, the critical stress. Eq. (i) may then be written

$$\frac{\mathrm{d}^4\theta}{\mathrm{d}z^4} + \lambda^2\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} = 0 \tag{ii}$$

in which

$$\lambda^2 = \frac{(I_0\sigma_{\mathrm{CR}} - GJ)}{E\Gamma} \tag{iii}$$

The solution of Eq. (ii) is

$$\theta = A\cos\lambda z + B\sin\lambda z + Dz + F \tag{iv}$$

The boundary conditions are:

$$\theta = 0 \text{ at } z = 0 \text{ and } z = 2l$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = 0 \text{ at } z = 0 \text{ and } z = 2l \quad \text{(see Eq. (9.67))}$$

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Then $B = D = 0$, $F = -A$ and Eq. (iv) becomes

$$\theta = A(\cos \lambda z - 1) \qquad \text{(v)}$$

Since $\theta = 0$ when $z = 2l$

$$\cos \lambda 2l = 1$$

or

$$\lambda 2l = 2n\pi$$

Hence, for $n = 1$

$$\lambda^2 = \frac{\pi^2}{l^2}$$

i.e. from Eq. (iii)

$$\frac{I_0 \sigma_{\mathrm{CR}} - GJ}{E\Gamma} = \frac{\pi^2}{l^2}$$

so that

$$\sigma_{\mathrm{CR}} = \frac{1}{I_0}\left(GJ + \frac{\pi^2 E\Gamma}{l^2}\right) \qquad \text{(vi)}$$

For the cross-section of Fig. P.6.19

$$J = \sum \frac{st^3}{3} \quad \text{(see Eq. (9.59))}$$

i.e.

$$J = \frac{8bt^3}{3} = \frac{8 \times 25.0 \times 2.5^3}{3} = 1041.7\,\mathrm{mm}^4$$

and

$$I_{xx} = 4bt(b\cos 30^\circ)^2 + 2\frac{(2b)^3 t \sin^2 60^\circ}{12} \quad \text{(see Section 9.1)}$$

i.e.

$$I_{xx} = 4b^3 t = 4 \times 25.0^3 \times 2.5 = 156\,250.0\,\mathrm{mm}^4$$

Similarly

$$I_{yy} = 4\left(\frac{bt^3}{12} + btb^2\right) + 2\frac{(2b)^3 t \cos^2 60^\circ}{12} = \frac{14b^3 t}{3}$$

so that

$$I_{yy} = 14 \times 25.0^3 \times 2.5/3 = 182\,291.7\,\mathrm{mm}^4$$

Then

$$I_0 = I_{xx} + I_{yy} = 338\,541.7\,\mathrm{mm}^4$$

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The torsion-bending constant, Γ, is found by the method described in Section 11.5 and is given by

$$\Gamma = b^5 t = 25.0^5 \times 2.5 = 24.4 \times 10^6\,\text{mm}^4$$

Substituting these values in Eq. (vi) gives

$$\sigma_{\text{CR}} = 282.0\,\text{N/mm}^2$$

S.6.20

The three possible buckling modes of the column are given by Eqs (6.90), i.e.

$$P_{\text{CR}(xx)} = \frac{\pi^2 EI_{xx}}{L^2} \tag{i}$$

$$P_{\text{CR}(yy)} = \frac{\pi^2 EI_{yy}}{L^2} \tag{ii}$$

$$P_{\text{CR}(\theta)} = \frac{A}{I_0}\left(GJ + \frac{\pi^2 E\Gamma}{L^2}\right) \tag{iii}$$

From Fig P.6.20 and taking the x axis parallel to the flanges

$$A = (2 \times 20 + 40) \times 1.5 = 120\,\text{mm}^2$$

$$I_{xx} = 2 \times 20 \times 1.5 \times 20^2 + 1.5 \times 40^3/12 = 3.2 \times 10^4\,\text{mm}^4$$

$$I_{yy} = 1.5 \times 40^3/12 = 0.8 \times 10^4\,\text{mm}^4$$

$$I_0 = I_{xx} + I_{yy} = 4.0 \times 10^4\,\text{mm}^4$$

$$J = (20 + 40 + 20) \times 1.5^3/3 = 90.0\,\text{mm}^4 \quad \text{(see Eq. (9.59))}$$

$$\Gamma = \frac{1.5 \times 20^3 \times 40^2}{12}\left(\frac{2 \times 40 + 20}{40 + 2 \times 20}\right) = 2.0 \times 10^6\,\text{mm}^6 \quad \text{(see Eq. (ii) of Example 11.2)}$$

Substituting the appropriate values in Eqs (i), (ii) and (iii) gives

$$P_{\text{CR}(xx)} = 22\,107.9\,\text{N}$$

$$P_{\text{CR}(yy)} = 5527.0\,\text{N}$$

$$P_{\text{CR}(\theta)} = 10\,895.2\,\text{N}$$

Thus the column will buckle in bending about the y axis at a load of 5527.0 N.

S.6.21

The separate modes of buckling are obtained from Eqs (6.90), i.e.

$$P_{\text{CR}(xx)} = P_{\text{CR}(yy)} = \frac{\pi^2 EI}{L^2} \; (I_{xx} = I_{yy} = I, \text{ say}) \tag{i}$$

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and

$$P_{CR(\theta)} = \frac{A}{I_0}\left(GJ + \frac{\pi^2 E\Gamma}{L^2}\right) \tag{ii}$$

In this case

$$I_{xx} = I_{yy} = \pi r^3 t = \pi \times 40^3 \times 2.0 = 4.02 \times 10^5\,\text{mm}^4$$

$$A = 2\pi rt = 2\pi \times 40 \times 2.0 = 502.7\,\text{mm}^2$$

$$J = 2\pi rt^3/3 = 2\pi \times 40 \times 2.0^3/3 = 670.2\,\text{mm}^4$$

From Eq. (6.81)

$$I_0 = I_{xx} + I_{yy} + Ax_s^2 \quad (\text{note that } y_s = 0)$$

in which x_s is the distance of the shear centre of the section from its vertical diameter; it may be shown that $x_s = 80\,\text{mm}$ (see S.9.11). Then

$$I_0 = 2 \times 4.02 \times 10^5 + 502.7 \times 80^2 = 4.02 \times 10^6\,\text{mm}^4$$

The torsion-bending constant Γ is found in a similar manner to that for the section shown in Fig. P.11.2 and is given by

$$\Gamma = \pi r^5 t(\tfrac{2}{3}\pi^2 - 4)$$

i.e.

$$\Gamma = \pi \times 40^5 \times 2.0(\tfrac{2}{3}\pi^2 - 4) = 1.66 \times 10^9\,\text{mm}^6$$

(a) $$P_{CR(xx)} = P_{CR(yy)} = \frac{\pi^2 \times 70\,000 \times 4.02 \times 10^5}{(3.0 \times 10^3)^2} = 3.09 \times 10^4\,\text{N}$$

(b) $$P_{CR(\theta)} = \frac{502.7}{4.02 \times 10^6}\left(22\,000 \times 670.2 + \frac{\pi^2 \times 70\,000 \times 1.66 \times 10^9}{(3.0 \times 10^3)^2}\right) = 1.78 \times 10^4\,\text{N}$$

The flexural–torsional buckling load is obtained by expanding Eq. (6.92). Thus

$$(P - P_{CR(xx)})(P - P_{CR(\theta)})I_0/A - P^2 x_s^2 = 0$$

from which

$$P^2(1 - Ax_s^2/I_0) - P(P_{CR(xx)} + P_{CR(\theta)}) + P_{CR(xx)}P_{CR(\theta)} = 0 \tag{iii}$$

Substituting the appropriate values in Eq. (iii) gives

$$P^2 - 24.39 \times 10^4 P + 27.54 \times 10^8 = 0 \tag{iv}$$

The solutions of Eq. (iv) are

$$P = 1.19 \times 10^4\,\text{N} \quad \text{or} \quad 23.21 \times 10^4\,\text{N}$$

Therefore, the least flexural–torsional buckling load is 1.19×10^4 N.

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S.6.22

The beam may be regarded as two cantilevers each of length 1.2 m, built-in at the mid-span section and carrying loads at their free ends of 5 kN. The analysis of a complete tension field beam in Section 6.13 therefore applies directly. From Eq. (6.108)

$$\tan^4 \alpha = \frac{1 + 1.5 \times 350/2 \times 300}{1 + 1.5 \times 300/280} = 0.7192$$

hence

$$\alpha = 42.6^\circ$$

From Eq. (6.98)

$$F_\mathrm{T} = \frac{5 \times 1.2 \times 10^3}{350} + \frac{5}{2 \tan 42.6^\circ}$$

i.e.

$$F_\mathrm{T} = 19.9\,\mathrm{kN}$$

From Eq. (6.102)

$$P = \frac{5 \times 300 \tan 42.6^\circ}{350}$$

i.e.

$$P = 3.9\,\mathrm{kN}$$

S.6.23

(i) The shear stress buckling coefficient for the web is given as $K = 7.70[1 + 0.75(b/d)^2]$. Thus Eq. (6.112) may be rewritten as

$$\tau_\mathrm{CR} = KE\left(\frac{t}{b}\right)^2 = 7.70[1 + 0.75(b/d)^2]E\left(\frac{t}{b}\right)^2$$

Hence

$$\tau_\mathrm{CR} = 7.70[1 + 0.75(250/725)^2] \times 70\,000(t/250)^2$$

i.e.

$$\tau_\mathrm{CR} = 9.39t^2 \tag{i}$$

The actual shear stress in the web, τ, is

$$\tau = \frac{100\,000}{750t} = \frac{133.3}{t} \tag{ii}$$

Two conditions occur, firstly

$$\tau \leqslant 165\,\mathrm{N/mm^2}$$

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so that, from Eq. (ii) $t = 0.81$ mm and secondly

$$\tau \leqslant 15\tau_{CR}$$

so that, from Eqs (i) and (ii)

$$15 \times 9.39t^2 = 133.3/t$$

whence

$$t = 0.98\,\text{mm}$$

Therefore, from the range of standard thicknesses

$$t = 1.2\,\text{mm}$$

(ii) For $t = 1.2$ mm, τ_{CR} is obtained from Eq. (i) and is

$$\tau_{CR} = 13.5\,\text{N/mm}^2$$

and, from Eq. (ii), $\tau = 111.1\,\text{N/mm}^2$. Thus, $\tau/\tau_{CR} = 8.23$ and, from the table, the diagonal tension factor, k, is equal to 0.41.

The stiffener end load follows from Eq. (6.114) and is

$$Q_s = \sigma_s A_s = \frac{A_s k\tau \tan\alpha}{(A_s/tb) + 0.5(1-k)}$$

i.e.

$$Q_s = \frac{A_s \times 0.41 \times 111.1 \tan 40^\circ}{(A_s/1.2 \times 250) + 0.5(1 - 0.41)} = \frac{130A_s}{1 + 0.0113A_s}$$

The maximum secondary bending moment in the flanges is obtained from Eq. (6.104) multiplied by k, thus

$$\text{maximum secondary bending moment} = \frac{kWb^2 \tan\alpha}{12d}$$

i.e.

$$\text{maximum secondary bending moment} = \frac{0.41 \times 100\,000 \times 250^2 \times \tan 40^\circ}{12 \times 750}$$

$$= 238\,910\,\text{N mm}$$

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Solutions to Chapter 8 Problems

S.8.1

Suppose that the mass of the aircraft is m and its vertical deceleration is a. Then referring to Fig. S.8.1(a) and resolving forces in a vertical direction

$$ma + 135 - 2 \times 200 = 0$$

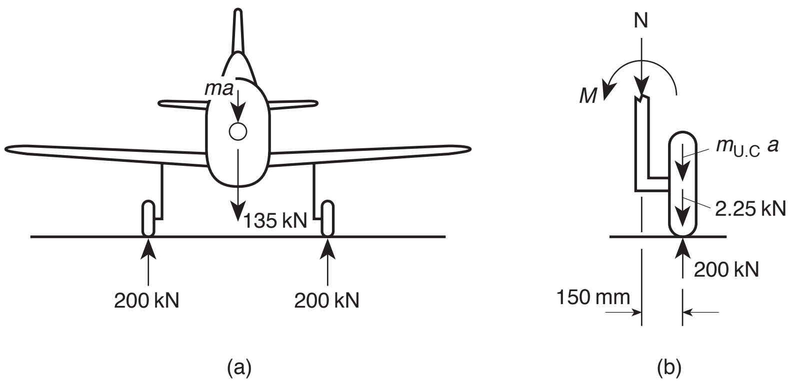

Fig. S.8.1

which gives

$$ma = 265\,\text{kN}$$

Therefore

$$a = \frac{265}{m} = \frac{265}{135/g}$$

i.e.

$$a = 1.96g$$

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Now consider the undercarriage shown in Fig. S.8.1(b) and suppose that its mass is $m_{\mathrm{U.C}}$. Then resolving forces vertically

$$N + m_{\mathrm{U.C}}a + 2.25 - 200 = 0 \tag{i}$$

in which

$$m_{\mathrm{U.C}}a = \frac{2.25}{g} \times 1.96g = 4.41\,\mathrm{kN}$$

Substituting in Eq. (i) gives

$$N = 193.3\,\mathrm{kN}$$

Now taking moments about the point of contact of the wheel and the ground

$$M + N \times 0.15 = 0$$

which gives

$$M = -29.0\,\mathrm{kNm} \quad \text{(i.e. clockwise)}$$

The vertical distance, s, through which the aircraft moves before its vertical velocity is zero, i.e. the shortening of the oleo strut, is obtained using elementary dynamics; the compression of the tyre is neglected here but in practice could be significant. Thus, assuming that the deceleration a remains constant

$$v^2 = v_0^2 + 2as$$

in which $v_0 = 3.5\,\mathrm{m/s}$ and $v = 0$. Then

$$s = -\frac{3.5^2}{2(-1.96g)} = \frac{3.5^2}{2 \times 1.96 \times 9.81}$$

i.e.

$$s = 0.32\,\mathrm{m}$$

Let the mass of the wing outboard of the section AA be m_{w}. Then, referring to Fig. S.8.1(c) and resolving forces vertically the shear force, S, at the section AA is given by

$$S - m_{\mathrm{w}}a - 6.6 = 0$$

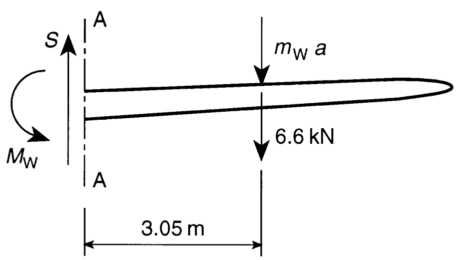

Fig. S.8.1(c)

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i.e.

$$S - \frac{6.6}{g} \times 1.96g - 6.6 = 0$$

which gives

$$S = 19.5\,\text{kN}$$

Now taking moments about the section AA

$$M_\text{w} - m_\text{w}a \times 3.05 - 6.6 \times 3.05 = 0$$

or

$$M_\text{w} = \frac{6.6}{g} \times 1.96g \times 3.05 + 6.6 \times 3.05$$

i.e.

$$M_\text{w} = 59.6\,\text{kNm}$$

S.8.2

From Example 8.2 the time taken for the vertical velocity of the aircraft to become zero is 0.099 s. During this time the aircraft moves through a vertical distance, s, which, from elementary dynamics, is given by

$$s = v_0 t + \tfrac{1}{2}at^2$$

where $v_0 = 3.7\,\text{m/s}$ and $a = -3.8g$ (see Example 8.2). Then

$$s = 3.7 \times 0.099 - \tfrac{1}{2} \times 3.8 \times 9.81 \times 0.099^2$$

i.e.

$$s = 0.184\,\text{m}$$

The angle of rotation, θ_1, during this time is given by

$$\theta_1 = \omega_0 t + \tfrac{1}{2}\alpha t^2$$

in which $\omega_0 = 0$ and $\alpha = 3.9\,\text{rad/s}^2$ (from Example 8.2). Then

$$\theta_1 = \tfrac{1}{2} \times 3.9 \times 0.099^2 = 0.019\,\text{rad}$$

The vertical distance, s_1, moved by the nose wheel during this rotation is, from Fig. 8.7

$$s_1 = 0.019 \times 5.0 = 0.095\,\text{m}$$

Therefore the distance, s_2, of the nose wheel from the ground after the vertical velocity at the CG of the aircraft has become zero is given by

$$s_2 = 1.0 - 0.184 - 0.095$$

i.e.

$$s_2 = 0.721\,\text{m}$$

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It follows that the aircraft must rotate through a further angle θ_2 for the nose wheel to hit the ground where

$$\theta_2 = \frac{0.721}{5.0} = 0.144\,\text{rad}$$

During the time taken for the vertical velocity of the aircraft to become zero the vertical ground reactions at the main undercarriage will decrease from 1200 kN to 250 kN and, assuming the same ratio, the horizontal ground reaction will decrease from 400 kN to $(250/1200) \times 400 = 83.3$ kN. Therefore, from Eqs (ii) and (iii) of Example 8.2, the angular acceleration of the aircraft when the vertical velocity of its CG becomes zero is

$$\alpha_1 = \frac{250}{1200} \times 3.9 = 0.81\,\text{rad/s}^2$$

Thus the angular velocity, ω_1, of the aircraft at the instant the nose wheel hits the ground is given by

$$\omega_1^2 = \omega_0^2 + 2\alpha_1\theta_2$$

where $\omega_0 = 0.39$ rad/s (see Example 8.2). Then

$$\omega_1^2 = 0.39^2 + 2 \times 0.81 \times 0.144$$

which gives

$$\omega_1 = 0.62\,\text{rad/s}$$

The vertical velocity, $v_{\text{N.W}}$, of the nose wheel is then

$$v_{\text{N.W}} = 0.62 \times 5.0$$

i.e.

$$v_{\text{N.W}} = 3.1\,\text{m/s}$$

S.8.3

With the usual notation the loads acting on the aircraft at the bottom of a symmetric manoeuvre are shown in Fig. S.8.3.

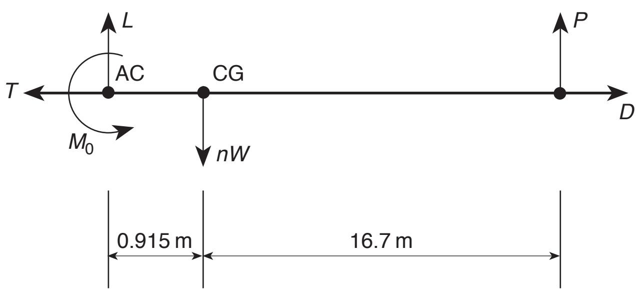

Fig. S.8.3

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Taking moments about the CG

$$0.915L - M_0 = 16.7P \quad \text{(i)}$$

and for vertical equilibrium

$$L + P = nW \quad \text{(ii)}$$

Further, the bending moment in the fuselage at the CG is given by

$$M_{\text{CG}} = nM_{\text{LEV.FLT}} - 16.7P \quad \text{(iii)}$$

Also

$$M_0 = \tfrac{1}{2}pV^2 S\bar{c}C_{\text{M},0} = \tfrac{1}{2} \times 1.223 \times 27.5 \times 3.05^2 \times 0.0638V^2$$

i.e.

$$M_0 = 9.98V^2 \quad \text{(iv)}$$

From Eqs (i) and (iii)

$$0.915(nW - P) - M_0 = 16.7P$$

Substituting for M_0 from Eq. (iv) and rearranging

$$P = 0.052nW - 0.567V^2 \quad \text{(v)}$$

In cruise conditions where, from Fig. P.8.3, $n = 1$ and $V = 152.5\,\text{m/s}$, P, from Eq. (v) is given by

$$P = -2994.3\,\text{N}$$

Then, from Eq. (iii) when $n = 1$

$$600\,000 = M_{\text{LEV.FLT}} + 16.7 \times 2994.3$$

which gives

$$M_{\text{LEV.FLT}} = 549\,995\,\text{Nm}$$

Now, from Eqs (iii) and (v)

$$M_{\text{CG}} = 549\,995n - 16.7(0.052nW - 0.567V^2)$$

or

$$M_{\text{CG}} = 379\,789n + 9.47V^2 \quad \text{(vi)}$$

From Eq. (vi) and Fig. P.8.3 it can be seen that the most critical cases are $n = 3.5$, $V = 152.5\,\text{m/s}$ and $n = 2.5$, $V = 183\,\text{m/s}$. For the former Eq. (vi) gives

$$M_{\text{CG}} = 1\,549\,500\,\text{Nm}$$

and for the latter

$$M_{\text{CG}} = 1\,266\,600\,\text{Nm}$$

Therefore the maximum bending moment is $1\,549\,500\,\text{Nm}$ at $n = 3.5$ and $V = 152.5\,\text{m/s}$.

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S.8.4

With the usual notation the loads acting on the aeroplane are shown in Fig. S.8.4; ΔP is the additional tail load required to check the angular velocity in pitch. Then

$$\Delta P \times 12.2 = 204\,000 \times 0.25$$

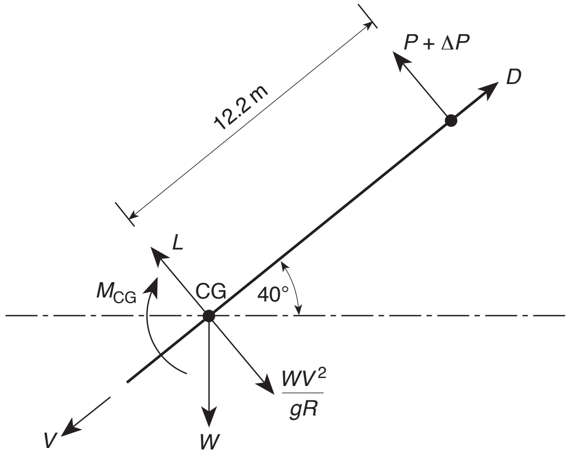

Fig. S.8.4

i.e.

$$\Delta P = 4180\,\text{N}$$

Now resolving perpendicularly to the flight path

$$L + (P + \Delta P) = WV^2/gR + W\cos 40^\circ \qquad \text{(i)}$$

Then resolving parallel to the flight path

$$fW + W\sin 40^\circ = D \qquad \text{(ii)}$$

where f is the forward inertia coefficient, and taking moments about the CG

$$(P + \Delta P) \times 12.2 = M_{\text{CG}} \qquad \text{(iii)}$$

Assume initially that

$$L = W\cos 40^\circ + WV^2/gR$$

i.e.

$$L = 238\,000\cos 40^\circ + 238\,000 \times 215^2/(9.81 \times 1525)$$

which gives

$$L = 917\,704\,\text{N}$$

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Then

$$C_{\mathrm{L}}=\frac{L}{\frac{1}{2}\rho V^2 S}=\frac{917\,704}{\frac{1}{2}\times 1.223\times(215)^2\times 88.5}=0.367$$

and

$$M_{\mathrm{CG}}=\tfrac{1}{2}\rho V^2 S(0.427C_{\mathrm{L}}-0.061)$$

i.e.

$$M_{\mathrm{CG}}=\tfrac{1}{2}\times 1.223\times 215^2\times 88.5(0.427\times 0.367-0.061)$$

from which

$$M_{\mathrm{CG}}=239\,425\,\mathrm{Nm}$$

Then, from Eq. (iii)

$$P+\Delta P=239\,425/12.2$$

i.e.

$$P+\Delta P=19\,625\,\mathrm{N}$$

Thus, a more accurate value for L is

$$L=917\,704-19\,625=898\,079\,\mathrm{N}$$

which then gives

$$C_{\mathrm{L}}=\frac{898\,079}{\frac{1}{2}\times 1.223\times 215^2\times 88.5}=0.359$$

Hence

$$M_{\mathrm{CG}}=\tfrac{1}{2}\times 1.223\times 215^2\times 88.5(0.427\times 0.359-0.061)$$

i.e.

$$M_{\mathrm{CG}}=230\,880\,\mathrm{Nm}$$

and, from Eq. (iii)

$$P+\Delta P=18\,925\,\mathrm{N}$$

Then

$$L=917\,704-18\,925=898\,779\,\mathrm{N}$$

so that

$$n=\frac{898\,779}{238\,000}=3.78$$

At the tail

$$\Delta n=\frac{\ddot{\theta}l}{g}=\frac{230\,880}{204\,000}\times\frac{12.2}{9.81}=1.41$$

Thus the total n at the tail $=3.78+1.41=5.19$.

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Now

$$C_D = 0.0075 + 0.045 \times \left(\frac{898\,779}{\frac{1}{2}\rho V^2 S} \right)^2 + 0.0128$$

i.e.

$$C_D = 0.026$$

so that

$$D = \tfrac{1}{2}\rho V^2 S \times 0.026 = 65\,041\,\text{N}$$

Thus, from Eq. (ii)

$$f = -0.370$$

S.8.5

From Eq. (8.21) ϕ, in Fig. 8.12, is given by

$$\tan\phi = \frac{V^2}{gR} = \frac{168^2}{9.81 \times 610} = 4.72$$

so that

$$\phi = 78.03^\circ$$

From Eq. (8.20)

$$n = \sec\phi = 4.82$$

Thus, the lift generated in the turn is given by

$$L = nW = 4.82 \times 133\,500 = 643\,470\,\text{N}$$

Then

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S} = \frac{643\,470}{\frac{1}{2} \times 1.223 \times 168^2 \times 46.5} = 0.80$$

Hence

$$C_D = 0.01 + 0.05 \times 0.80^2 = 0.042$$

and the drag

$$D = \tfrac{1}{2} \times 1.223 \times 168^2 \times 46.5 \times 0.042 = 33\,707\,\text{N}$$

The pitching moment M_0 is given by

$$M_0 = \tfrac{1}{2}\rho V^2 S\bar{c}C_{M,0} = -\tfrac{1}{2} \times 1.223 \times 168^2 \times 46.5 \times 3.0 \times 0.03$$

i.e.

$$M_0 = -72\,229\,\text{Nm} \quad \text{(i.e. nose down)}$$

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The wing incidence is given by

$$\alpha = \frac{C_L}{dC_L/d\alpha} = \frac{0.80}{4.5} \times \frac{180}{\pi} = 10.2^\circ$$

The loads acting on the aircraft are now as shown in Fig. S.8.5.

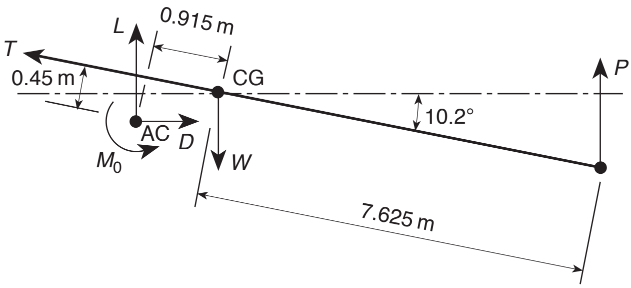

Fig. S.8.5

Taking moments about the CG

$$L(0.915\cos 10.2^\circ + 0.45\sin 10.2^\circ) - D(0.45\cos 10.2^\circ - 0.915\sin 10.2^\circ) - M_0$$
$$= P \times 7.625\cos 10.2 \qquad \text{(i)}$$

Substituting the values of L, D and M_0 in Eq. (i) gives

$$P = 73\,160\,\text{N}$$

S.8.6

(a) The forces acting on the aircraft in the pull-out are shown in Fig. S.8.6. Resolving forces perpendicularly to the flight path

$$L = \frac{WV^2}{gR} + W\cos\theta \qquad \text{(i)}$$

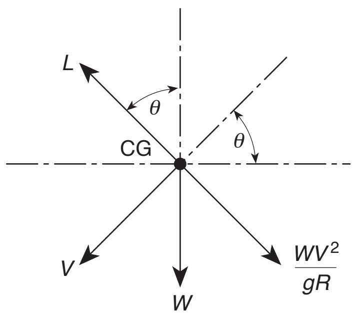

Fig. S.8.6

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The maximum allowable lift is $4.0W$ so that Eq. (i) becomes

$$\frac{V^2}{gR} = 4 - \cos\theta$$

or

$$\frac{V\omega}{g} = 4 - \cos\theta \tag{ii}$$

where $\omega\,(= V/R)$ is the angular velocity in pitch. In Eq. (ii) ω will be a maximum when $\cos\theta$ is a minimum, i.e. when θ reaches its maximum allowable value (60°). Then, from Eq. (ii)

$$\omega = \frac{g}{V}(4 - 0.5) = \frac{3.5g}{V} \tag{iii}$$

From Eq. (iii) ω will be a maximum when V is a minimum which occurs when $C_L = C_{L,MAX}$. Thus

$$\tfrac{1}{2}\rho V^2 S C_{L,MAX} = 4 \times \tfrac{1}{2}\rho V_s^2 S C_{L,MAX}$$

whence

$$V = 2V_s = 2 \times 46.5 = 93.0\,\text{m/s}$$

Therefore, from Eq. (iii)

$$\omega_{max} = \frac{3.5 \times 9.81}{93.0} = 0.37\,\text{rad/s}$$

(b) Referring to Fig. 8.12, Eq. (8.17) gives

$$nW\sin\phi = \frac{WV^2}{gR}$$

i.e.

$$4\sin\phi = \frac{V\omega}{g} \tag{iv}$$

Also, from Eq. (8.20) $\sec\phi = 4$ whence $\sin\phi = 0.9375$. Then Eq. (iv) becomes

$$\omega = 3.87\frac{g}{V} \tag{v}$$

Thus, ω is a maximum when V is a minimum, i.e. when $V = 2V_s$ as in (a). Therefore

$$\omega_{max} = \frac{3.87 \times 9.81}{2 \times 46.5} = 0.41\,\text{rad/s}$$

The maximum rate of yaw is $\omega_{max}\cos\phi$, i.e.

$$\text{maximum rate of yaw} = 0.103\,\text{rad/s}$$

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S.8.7

The forces acting on the airliner are shown in Fig. S.8.7 where α_{w} is the wing incidence. As a first approximation let $L = W$. Then

$$\frac{1}{2}\rho V^2 S\alpha_{\mathrm{w}}\frac{\partial C_{\mathrm{L}}}{\partial \alpha} = 1\,600\,000$$

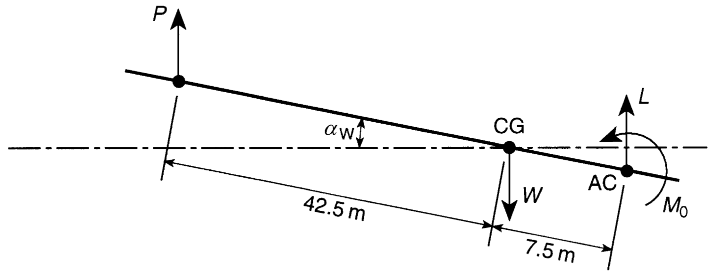

Fig. S.8.7

i.e.

$$\alpha_{\mathrm{w}} = \frac{1\,600\,000 \times 180}{\frac{1}{2} \times 0.116 \times 610^2 \times 280 \times 1.5 \times \pi}$$

so that

$$\alpha_{\mathrm{w}} = 10.1^\circ$$

From vertical equilibrium

$$L + P = W \tag{i}$$

and taking moments about the CG.

$$P \times 42.5\cos 10.1^\circ = L \times 7.5\cos 10.1^\circ + M_0 \tag{ii}$$

Substituting for L from Eq. (i) in Eq. (ii)

$$P \times 42.5\cos 10.1^\circ = (1\,600\,000 - P)7.5\cos 10.1^\circ + \tfrac{1}{2} \times 0.116 \times 610^2 \times 280 \times 22.8 \times 0.01$$

from which

$$P = 267\,963\,\mathrm{N}$$

Thus, from Eq. (i)

$$L = 1\,332\,037\,\mathrm{N}$$

giving

$$\alpha_{\mathrm{w}} = 8.4^\circ$$

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Then, taking moments about the CG

$$P \times 42.5 \cos 8.4^\circ = (1\,600\,000 - P)7.5 \cos 8.4^\circ + \tfrac{1}{2} \times 0.116 \times 610^2 \times 280 \times 22.8 \times 0.01$$

which gives

$$P = 267\,852\,\text{N}$$

This is sufficiently close to the previous value of tail load to make a second approximation unnecessary.

The change $\Delta\alpha$ in wing incidence due to the gust is given by

$$\Delta\alpha = \frac{18}{610} = 0.03\,\text{rad}$$

Thus the change ΔP in the tail load is

$$\Delta P = \frac{1}{2}\rho V^2 S_\text{T} \frac{\partial C_{\text{L,T}}}{\partial \alpha} \Delta\alpha$$

i.e.

$$\Delta P = \tfrac{1}{2} \times 0.116 \times 610^2 \times 28 \times 2.0 \times 0.03 = 36\,257\,\text{N}$$

Also, neglecting downwash effects, the change ΔL in wing lift is

$$\Delta L = \frac{1}{2}\rho V^2 S \frac{\partial C_\text{L}}{\partial \alpha} \Delta\alpha$$

i.e.

$$\Delta L = \tfrac{1}{2} \times 0.116 \times 610^2 \times 280 \times 1.5 \times 0.03 = 271\,931\,\text{N}$$

The resultant load factor, n, is then given by

$$n = 1 + \frac{36\,257 + 271\,931}{1\,600\,000}$$

i.e.

$$n = 1.19$$

S.8.8

As a first approximation let $L = W$. Then

$$\frac{1}{2}\rho V^2 S \frac{\text{d}C_\text{L}}{\text{d}\alpha} \alpha_\text{w} = 145\,000$$

Thus

$$\alpha_\text{w} = \frac{145\,000}{\tfrac{1}{2} \times 1.223 \times 250^2 \times 50 \times 4.8} = 0.0158\,\text{rad} = 0.91^\circ$$

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Also

$$C_{\text{D}} = 0.021 + 0.041 \times 0.08^2$$

i.e.

$$C_{\text{D}} = 0.0213$$

Referring to Fig. P.8.8 and taking moments about the CG and noting that $\cos 0.91^\circ \doteq 1$

$$L \times 0.5 - D \times 0.4 + M_0 = P \times 8.5$$

i.e.

$$0.5(145\,000 - P) - 0.4 \times \tfrac{1}{2}\rho V^2 S C_{\text{D}} + \tfrac{1}{2}\rho V^2 S\bar{c} C_{\text{M},0} = 8.5P$$

Thus

$$0.5(145\,000 - P) - 0.4 \times \tfrac{1}{2} \times 1.223 \times 250^2 \times 50 \times 0.0213 - \tfrac{1}{2}$$
$$\times 1.223 \times 250^2 \times 50 \times 2.5 \times 0.032 = 8$$

which gives

$$P = -10\,740\,\text{N}$$

Hence

$$L = W - P = 145\,000 + 10\,740 = 155\,740\,\text{N}$$

The change ΔP in the tail load due to the gust is given by

$$\Delta P = \frac{1}{2}\rho V^2 S_{\text{T}} \frac{\partial C_{\text{L,T}}}{\partial \alpha} \Delta\alpha$$

in which

$$\Delta\alpha = -\frac{6}{250} = -0.024\,\text{rad}$$

Thus

$$\Delta P = -\tfrac{1}{2} \times 1.223 \times 250^2 \times 9.0 \times 2.2 \times 0.024 = -18\,162\,\text{N}$$

Therefore the total tail load $= -10\,740 - 18\,162 = -28\,902\,\text{N}$.

The increase in wing lift ΔL due to the gust is given by

$$\Delta L = -\frac{1}{2}\rho V^2 S \frac{\partial C_{\text{L}}}{\partial \alpha} \Delta\alpha = -\frac{1}{2} \times 1.223 \times 250^2 \times 50 \times 4.8 \times 0.024$$

i.e.

$$\Delta L = -220\,140\,\text{N}$$

Hence

$$n = 1 - \frac{(220\,140 + 18\,162)}{145\,000} = -0.64$$

Finally the forward inertia force fW is given by

$$fW = D = \frac{1}{2}\rho V^2 S C_D = \frac{1}{2} \times 1.223 \times 250^2 \times 50 \times 0.0213$$

i.e.

$$fW = 40\,703\,\mathrm{N}$$

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Solutions to Chapter 9 Problems

S.9.1

From Section 9.1 the components of the bending moment about the x and y axes are, respectively

$$M_x = 3000 \times 10^3 \cos 30^\circ = 2.6 \times 10^6 \text{ Nmm}$$

$$M_y = 3000 \times 10^3 \sin 30^\circ = 1.5 \times 10^6 \text{ Nmm}$$

The direct stress distribution is given by Eq. (9.6) so that, initially, the position of the centroid of area, C, must be found. Referring to Fig. S.9.1 and taking moments of area about the edge BC

$$(100 \times 10 + 115 \times 10)\bar{x} = 100 \times 10 \times 50 + 115 \times 10 \times 5$$

i.e.

$$\bar{x} = 25.9 \text{ mm}$$

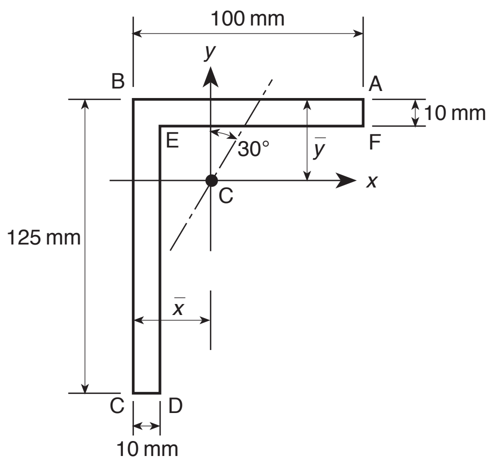

Fig. S.9.1

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Now taking moments of area about AB

$$(100 \times 10 + 115 \times 10)\bar{y} = 100 \times 10 \times 5 + 115 \times 10 \times 67.5$$

from which

$$\bar{y} = 38.4\,\text{mm}$$

The second moments of area are then

$$I_{xx} = \frac{100 \times 10^3}{12} + 100 \times 10 \times 33.4^2 + \frac{10 \times 115^3}{12} + 10 \times 115 \times 29.1^2$$

$$= 3.37 \times 10^6\,\text{mm}^4$$

$$I_{yy} = \frac{10 \times 100^3}{12} + 10 \times 100 \times 24.1^2 + \frac{115 \times 10^3}{12} + 115 \times 10 \times 20.9^2$$

$$= 1.93 \times 10^6\,\text{mm}^4$$

$$I_{xy} = 100 \times 10 \times 33.4 \times 24.1 + 115 \times 10(-20.9)(-29.1) = 1.50 \times 10^6\,\text{mm}^4$$

Substituting for M_x, M_y, I_{xx}, I_{yy} and I_{xy} in Eq. (9.6) gives

$$\sigma_z = 0.27x + 0.65y \tag{i}$$

Since the coefficients of x and y in Eq. (i) have the same sign the maximum value of direct stress will occur in either the first or third quadrants. Thus

$$\sigma_{z(\text{A})} = 0.27 \times 74.1 + 0.65 \times 38.4 = 45.0\,\text{N/mm}^2 \quad \text{(tension)}$$

$$\sigma_{z(\text{C})} = 0.27 \times (-25.9) + 0.65 \times (-86.6) = -63.3\,\text{N/mm}^2 \quad \text{(compression)}$$

The maximum direct stress therefore occurs at C and is $63.3\,\text{N/mm}^2$ compression.

S.9.2

From Eqs (9.17) the horizontal component of deflection, u, is given by

$$u'' = \frac{M_x I_{xy} - M_y I_{xx}}{E(I_{xx}I_{yy} - I_{xy}^2)} \tag{i}$$

in which, for the span BD, referring to Fig. P.9.2, $M_x = -R_\text{D}z$, $M_y = 0$, where R_D is the vertical reaction at the support at D. Taking moments about B

$$R_\text{D}2l + Wl = 0$$

so that

$$R_\text{D} = -W/2 \quad \text{(downward)}$$

Eq. (i) then becomes

$$u'' = \frac{WI_{xy}}{2E(I_{xx}I_{yy} - I_{xy}^2)}z \tag{ii}$$

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From Fig. P.9.2

$$I_{xx} = \frac{t(2a)^3}{12} + 2at(a)^2 + 2\left[\frac{t(a/2)^3}{12} + t\frac{a}{2}\left(\frac{3a}{4}\right)^2\right] = \frac{13a^3t}{4}$$

$$I_{yy} = \frac{t(2a)^3}{12} + 2\frac{a}{2}t(a)^2 = \frac{5a^3t}{3}$$

$$I_{xy} = \frac{a}{2}t(-a)\left(\frac{3a}{4}\right) + at\left(-\frac{a}{2}\right)(a) + \frac{a}{2}t(a)\left(-\frac{3a}{4}\right) + at\left(\frac{a}{2}\right)(-a) = -\frac{7a^3t}{4}$$

Eq. (ii) then becomes

$$u'' = -\frac{42W}{113Ea^3t} \tag{iii}$$

Integrating Eq. (iii) with respect to z

$$u' = -\frac{21W}{113Ea^3t}z^2 + A$$

and

$$u = -\frac{7W}{113Ea^3t}z^3 + Az + B \tag{iv}$$

When $z = 0$, $u = 0$ so that $B = 0$. Also $u = 0$ when $z = 2l$ which gives

$$A = -\frac{28Wl^2}{113Ea^3t}$$

Then

$$u = \frac{7W}{113Ea^3t}(-z^3 + 4l^2z) \tag{v}$$

At the mid-span point where $z = l$, Eq. (v) gives

$$u = \frac{0.186Wl^3}{Ea^3t}$$

Similarly

$$v = \frac{0.177Wl^3}{Ea^3t}$$

S.9.3

The beam is allowed to deflect in the horizontal direction at B so that the support reaction, R_B, at B is vertical. Then, from Eq. (4.18), the total complementary energy, C, of the beam is given by

$$C = \int_L \int_0^M d\theta\, dM - R_B\Delta_B - W\Delta_C \tag{i}$$

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From the principle of the stationary value of the total complementary energy of the beam and noting that $\Delta_B = 0$

$$\frac{\partial C}{\partial R_B} = \int_L d\theta \frac{\partial M}{\partial R_B} = 0$$

Thus

$$\frac{\partial C}{\partial R_B} = \int_L \frac{M}{EI}\frac{\partial M}{\partial R_B}\,dz = 0 \tag{ii}$$

In CB

$$M = W(2l - z) \quad \text{and} \quad \partial M/\partial R_B = 0$$

In BA

$$M = W(2l - z) - R_B(l - z) \quad \text{and} \quad \partial M/\partial R_B = -(l - z)$$

Substituting in Eq. (ii)

$$\int_0^l [W(2l - z) - R_B(l - z)](l - z)\,dz = 0$$

from which

$$R_B = \frac{5W}{2}$$

Then

$$M_C = 0, \quad M_B = Wl, \quad M_A = -Wl/2$$

and the bending moment diagram is as shown in Fig. S.9.3.

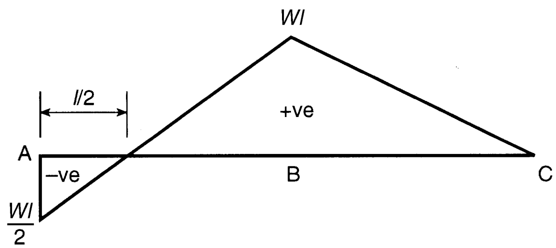

Fig. S.9.3

S.9.4

Initially, the section properties are determined. By inspection the centroid of area, C, is a horizontal distance $2a$ from the point 2. Now referring to Fig. S.9.4 and taking moments of area about the flange 23

$$(5a + 4a)t\bar{y} = 5at(3a/2)$$

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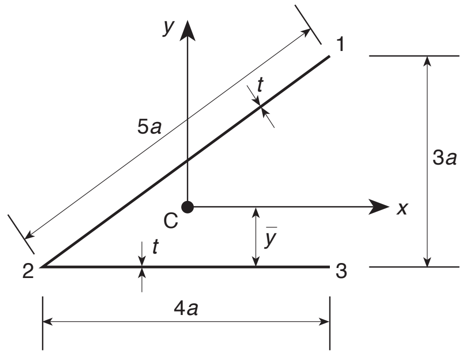

Fig. S.9.4

from which

$$\bar{y} = 5a/6$$

From Section 9.1

$$I_{xx} = 4at(5a/6)^2 + (5a)^3 t(3/5)^2/12 + 5at(2a/3)^2 = 105a^3t/12$$

$$I_{yy} = t(4a)^3/12 + (5a)^3 t(4/5)^2/12 = 12a^3 t$$

$$I_{xy} = (5a)^3(3/5)(4/5)/12 = 5a^3 t$$

From Fig. P.9.4 the maximum bending moment occurs at the mid-span section in a horizontal plane about the y axis. Thus

$$M_x = 0, \quad M_y(\max) = wl^2/8$$

Substituting these values and the values of I_{xx}, I_{yy} and I_{xy} in Eq. (9.6)

$$\sigma_z = \frac{wl^2}{8a^3t}\left(\frac{7}{64}x - \frac{1}{16}y\right) \tag{i}$$

From Eq. (i) it can be seen that σ_z varies linearly along each flange. Thus

At 1 where $x = 2a, y = 13a/6$, $\quad \sigma_{z,1} = wl^2/96a^2t$

At 2 where $x = -2a, y = -5a/6$, $\quad \sigma_{z,2} = -wl^2/48a^2t$

At 3 where $x = 2a, y = -5a/6$, $\quad \sigma_{z,3} = 13wl^2/384a^2t$

Therefore, the maximum stress occurs at 3 and is $13wl^2/384a^2t$.

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S.9.5

Referring to Fig. S.9.5:

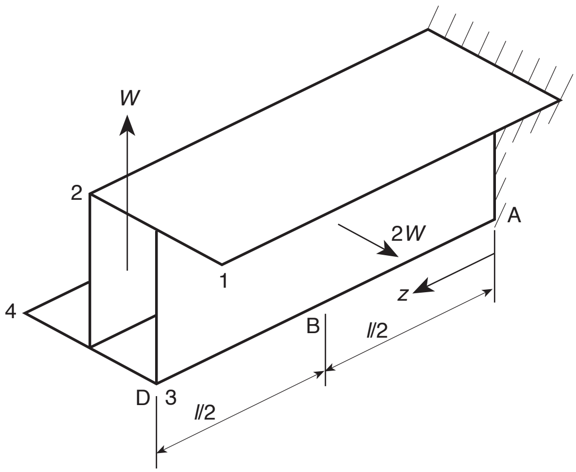

Fig. S.9.5

In DB

$$M_x = -W(l - z) \tag{i}$$

$$M_y = 0$$

In BA

$$M_x = -W(l - z) \tag{ii}$$

$$M_y = -2W\left(\frac{l}{2} - z\right) \tag{iii}$$

Now referring to Fig. P.9.5 the centroid of area, C, of the beam cross-section is at the centre of antisymmetry. Then

$$I_{xx} = 2\left[td\left(\frac{d}{2}\right)^2 + \frac{td^3}{12}\right] = \frac{2td^3}{3}$$

$$I_{yy} = 2\left[td\left(\frac{d}{4}\right)^2 + \frac{td^3}{12} + td\left(\frac{d}{4}\right)^2\right] = \frac{5td^3}{12}$$

$$I_{xy} = td\left(\frac{d}{4}\right)\left(\frac{d}{2}\right) + td\left(-\frac{d}{4}\right)\left(-\frac{d}{2}\right) = \frac{td^3}{4}$$

Substituting for I_{xx}, I_{yy} and I_{xy} in Eq. (9.6) gives

$$\sigma_z = \frac{1}{td^3}[(3.10M_y - 1.16M_x)x + (1.94M_x - 1.16M_y)y] \tag{iv}$$

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Along the edge 1, $x = 3d/4$, $y = d/2$. Eq. (iv) then becomes

$$\sigma_{z,1} = \frac{1}{td^2}(1.75M_y + 0.1M_x) \tag{v}$$

Along the edge 2, $x = -d/4$, $y = d/2$. Eq. (iv) then becomes

$$\sigma_{z,2} = \frac{1}{td^2}(-1.36M_y + 1.26M_x) \tag{vi}$$

From Eqs (i), (ii), (iii), (v) and (vi)

In DB

$$\sigma_{z,1} = -\frac{0.1W}{td^2}(l - z) \text{ whence } \sigma_{z,1}(\mathrm{B}) = -\frac{0.05Wl}{td^2}$$

$$\sigma_{z,2} = -\frac{1.26W}{td^2}(l - z) \text{ whence } \sigma_{z,2}(\mathrm{B}) = -\frac{0.63Wl}{td^2}$$

In BA

$$\sigma_{z,1} = \frac{W}{td^2}(3.6z - 1.85l) \text{ whence } \sigma_{z,1}(\mathrm{A}) = -\frac{1.85Wl}{td^2}$$

$$\sigma_{z,2} = \frac{W}{td^2}(-1.46z + 0.1l) \text{ whence } \sigma_{z,2}(\mathrm{A}) = \frac{0.1Wl}{td^2}$$

S.9.6

(a) From Eqs (9.17)

$$u'' = \frac{M_x I_{xy} - M_y I_{xx}}{E(I_{xx}I_{yy} - I_{xy}^2)} \tag{i}$$

Referring to Fig. P.9.6

$$M_x = -\frac{w}{2}(l - z)^2 \tag{ii}$$

and

$$M_y = -T(l - z) \tag{iii}$$

in which T is the tension in the link. Substituting for M_x and M_y from Eqs (ii) and (iii) in Eq. (i)

$$u'' = -\frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{xy}}{2}(l - z)^2 - TI_{xx}(l - z)\right]$$

Then

$$u' = -\frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{xy}}{2}\left(l^2 z - lz^2 + \frac{z^3}{3}\right) - TI_{xx}\left(lz - \frac{z^2}{2}\right) + A\right]$$

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When $z = 0$, $u' = 0$ so that $A = 0$. Hence

$$u = -\frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{xy}}{2}\left(l^2\frac{z^2}{2} - l\frac{z^3}{3} + \frac{z^4}{12}\right) - TI_{xx}\left(l\frac{z^2}{2} - \frac{z^3}{6}\right) + B\right]$$

When $z = 0$, $u = 0$ so that $B = 0$. Hence

$$u = -\frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{xy}}{2}\left(l^2\frac{z^2}{2} - l\frac{z^3}{3} + \frac{z^4}{12}\right) - TI_{xx}\left(l\frac{z^2}{2} - \frac{z^3}{6}\right)\right] \quad \text{(iv)}$$

Since the link prevents horizontal movement of the free end of the beam, $u = 0$ when $z = l$. Hence, from Eq. (iv)

$$w\frac{I_{xy}}{2}\left(\frac{l^4}{2} - \frac{l^4}{3} + \frac{l^4}{12}\right) - TI_{xx}\left(\frac{l^3}{2} - \frac{l^3}{6}\right) = 0$$

whence

$$T = \frac{3wlI_{xy}}{8I_{xx}}$$

(b) From Eqs (9.17)

$$v'' = \frac{M_xI_{yy} - M_yI_{xy}}{E(I_{xx}I_{yy} - I_{xy}^2)} \quad \text{(v)}$$

The equation for v may be deduced from Eq. (iv) by comparing Eqs (v) and (i). Thus

$$v = \frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{yy}}{2}\left(l^2\frac{z^2}{2} - l\frac{z^3}{3} + \frac{z^4}{12}\right) - TI_{xy}\left(l\frac{z^2}{2} - \frac{z^3}{6}\right)\right] \quad \text{(vi)}$$

At the free end of the beam where $z = l$

$$v_{\mathrm{FE}} = \frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left(\frac{wI_{yy}l^4}{8} - TI_{xy}\frac{l^3}{3}\right)$$

which becomes, since $T = 3wlI_{xy}/8I_{xx}$

$$v_{\mathrm{FE}} = \frac{wl^4}{8EI_{xx}}$$

S.9.7

Referring to Fig. P.9.7, at the built-in end of the beam

$$M_x = 50 \times 100 - 50 \times 200 = -5000\,\mathrm{Nmm}$$

$$M_y = 80 \times 200 = 16\,000\,\mathrm{Nmm}$$

and at the half-way section

$$M_x = -50 \times 100 = -5000\,\mathrm{Nmm}$$

$$M_y = 80 \times 100 = 8000\,\mathrm{Nmm}$$

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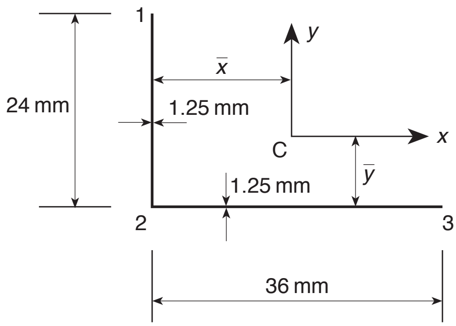

Fig. S.9.7

Now referring to Fig. S.9.7 and taking moments of areas about 12

$$(24 \times 1.25 + 36 \times 1.25)\bar{x} = 36 \times 1.25 \times 18$$

which gives

$$\bar{x} = 10.8\,\text{mm}$$

Taking moments of areas about 23

$$(24 \times 1.25 + 36 \times 1.25)\bar{y} = 24 \times 1.25 \times 12$$

which gives

$$\bar{y} = 4.8\,\text{mm}$$

Then

$$I_{xx} = \frac{1.25 \times 24^3}{12} + 1.25 \times 24 \times 7.2^2 + 1.25 \times 36 \times 4.8^2 = 4032\,\text{mm}^4$$

$$I_{yy} = 1.25 \times 24 \times 10.8^2 + \frac{1.25 \times 36^3}{12} + 1.25 \times 36 \times 7.2^2 = 10\,692\,\text{mm}^4$$

$$I_{xy} = 1.25 \times 24 \times (-10.8)(7.2) + 1.25 \times 36 \times (7.2)(-4.8) = -3888\,\text{mm}^4$$

Substituting for I_{xx}, I_{yy} and I_{xy} in Eq. (9.6) gives

$$\sigma_z = (1.44M_y + 1.39M_x) \times 10^{-4}x + (3.82M_x + 1.39M_y) \times 10^{-4}y \qquad \text{(i)}$$

Thus, at the built-in end Eq. (i) becomes

$$\sigma_z = 1.61x + 0.31y \qquad \text{(ii)}$$

whence $\sigma_{z,1} = -11.4\,\text{N/mm}^2$, $\sigma_{z,2} = -18.9\,\text{N/mm}^2$, $\sigma_{z,3} = 39.1\,\text{N/mm}^2$. At the half-way section Eq. (i) becomes

$$\sigma_z = 0.46x - 0.80y \qquad \text{(iii)}$$

whence $\sigma_{z,1} = -20.3\,\text{N/mm}^2$, $\sigma_{z,2} = -1.1\,\text{N/mm}^2$, $\sigma_{z,3} = 15.4\,\text{N/mm}^2$.

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S.9.8

The section properties are, from Fig. S.9.8

$$I_{xx} = 2\int_0^{\pi} t(r - r\cos\theta)^2 r\,\mathrm{d}\theta = 3\pi t r^3$$

$$I_{yy} = 2\int_0^{\pi} t(r\sin\theta)^2 r\,\mathrm{d}\theta = \pi t r^3$$

$$I_{xy} = 2\int_0^{\pi} t(-r\sin\theta)(r - r\cos\theta) r\,\mathrm{d}\theta = -4tr^3$$

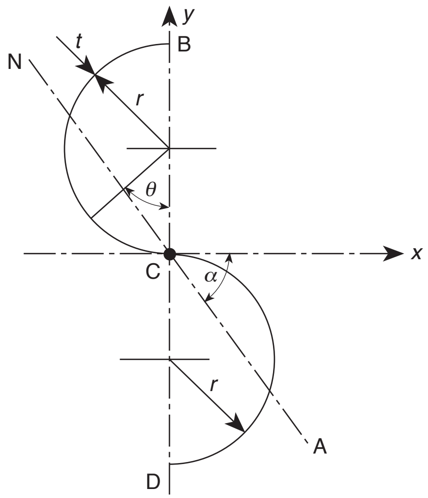

Fig. S.9.8

Since $M_y = 0$, Eq. (9.10) reduces to

$$\tan\alpha = -\frac{I_{xy}}{I_{yy}} = \frac{4tr^3}{\pi t r^3}$$

i.e.

$$\alpha = 51.9^\circ$$

Substituting for $M_x = 3.5 \times 10^3$ Nmm and $M_y = 0$, Eq. (9.6) becomes

$$\sigma_z = \frac{10^3}{tr^3}(1.029x + 0.808y) \tag{i}$$

The maximum value of direct stress will occur at a point a perpendicular distance furthest from the neutral axis, i.e. by inspection at B or D. Thus

$$\sigma_z(\max) = \frac{10^3}{0.64 \times 5^3}(0.808 \times 2 \times 5)$$

i.e.

$$\sigma_z(\max) = 101.0\,\mathrm{N/mm^2}$$

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Alternatively Eq. (i) may be written

$$\sigma_z = \frac{10^3}{tr^3}[1.029(-r\sin\theta) + 0.808(r - r\cos\theta)]$$

or

$$\sigma_z = \frac{808}{tr^2}(1 - \cos\theta - 1.27\sin\theta) \qquad \text{(ii)}$$

The expression in brackets has its greatest value when $\theta = \pi$, i.e. at B (or D).

S.9.9

In Fig. S.9.9 the x axis is an axis of symmetry (i.e. $I_{xy} = 0$) and the shear centre, S, lies on this axis. Suppose S is a distance ξ_S from the web 24. To find ξ_S an arbitrary shear load S_y is applied through S and the internal shear flow distribution determined. Since $I_{xy} = 0$ and $S_x = 0$, Eq. (9.34) reduces to

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s \qquad \text{(i)}$$

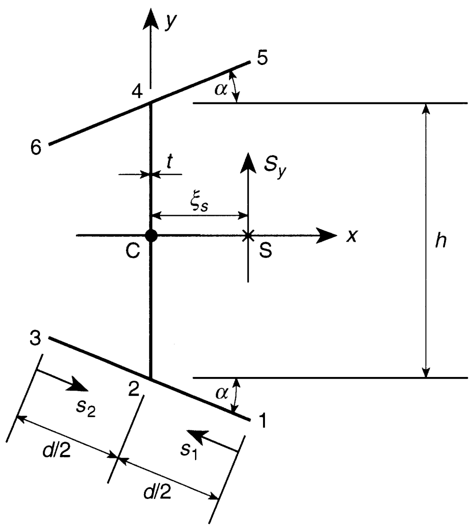

Fig. S.9.9

in which

$$I_{xx} = \frac{th^3}{12} + 2\left[\frac{td^3\sin^2\alpha}{12} + td\left(\frac{h}{2}\right)^2\right]$$

i.e.

$$I_{xx} = \frac{th^3}{12}(1 + 6\rho + 2\rho^3\sin^2\alpha) \qquad \text{(ii)}$$

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Then

$$q_{12} = -\frac{S_y}{I_{xx}} \int_0^{s_1} ty \, \mathrm{d}s_1$$

i.e.

$$q_{12} = \frac{S_y}{I_{xx}} \int_0^{s_1} t \left[\frac{h}{2} + \left(\frac{d}{2} - s_1 \right) \sin\alpha \right] \mathrm{d}s_1$$

so that

$$q_{12} = \frac{S_y t}{2I_{xx}} (hs_1 + ds_1 \sin\alpha - s_1^2 \sin\alpha) \tag{iii}$$

Also

$$q_{32} = -\frac{S_y}{I_{xx}} \int_0^{s_2} ty \, \mathrm{d}s_2 = \frac{S_y t}{I_{xx}} \int_0^{s_2} \left[\frac{h}{2} - \left(\frac{d}{2} - s_2 \right) \sin\alpha \right] \mathrm{d}s_2$$

whence

$$q_{32} = \frac{S_y t}{2I_{xx}} (hs_2 - ds_2 \sin\alpha + s_2^2 \sin\alpha) \tag{iv}$$

Taking moments about C in Fig. S.9.9

$$S_y \xi_S = -2 \int_0^{d/2} q_{12} \frac{h}{2} \cos\alpha \, \mathrm{d}s_1 + 2 \int_0^{d/2} q_{32} \frac{h}{2} \cos\alpha \, \mathrm{d}s_2 \tag{v}$$

Substituting in Eq. (v) for q_{12} and q_{32} from Eqs (iii) and (iv)

$$S_y \xi_S = \frac{S_y th \cos\alpha}{I_{xx}} \left[\int_0^{d/2} -(hs_1 + ds_1 \sin\alpha - s_1^2 \sin\alpha) \, \mathrm{d}s_1 + \int_0^{d/2} (hs_2 - ds_2 \sin\alpha + s_2^2 \sin\alpha) \, \mathrm{d}s_2 \right]$$

from which

$$\xi_S = -\frac{thd^3 \sin\alpha \cos\alpha}{12 I_{xx}} \tag{vi}$$

Now substituting for I_{xx} from Eq. (ii) in Eq. (vi)

$$\xi_S = -d \frac{\rho^2 \sin\alpha \cos\alpha}{1 + 6\rho + 2\rho^3 \sin^2\alpha}$$

S.9.10

The x axis is an axis of symmetry so that $I_{xy} = 0$ and the shear centre, S, lies on this axis (see Fig. S.9.10). Therefore, an arbitrary shear force, S_y, is applied through S and the internal shear flow distribution determined.

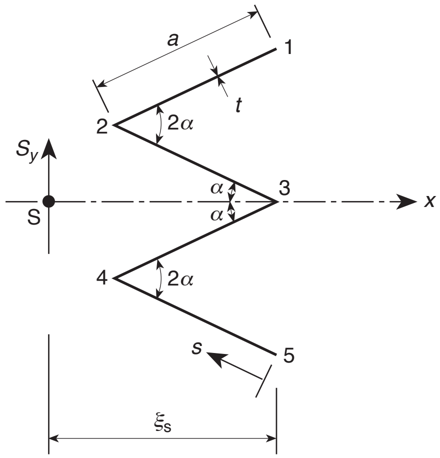

Fig. S.9.10

Since $S_x = 0$ and $I_{xy} = 0$, Eq. (9.34) reduces to

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s \tag{i}$$

in which, from Fig. S.9.10,

$$I_{xx} = 2\left[\frac{a^3 t \sin^2\alpha}{12} + at\left(a\sin\alpha + \frac{a}{2}\sin\alpha\right)^2 + \frac{a^3 t \sin^2\alpha}{12} + at\left(\frac{a}{2}\sin\alpha\right)^2\right]$$

which gives

$$I_{xx} = \frac{16a^3 t \sin^2\alpha}{3} \tag{ii}$$

For the flange 54, from Eq. (i)

$$q_{54} = -\frac{S_y}{I_{xx}}\int_0^s t(s-2a)\sin\alpha\,\mathrm{d}s$$

from which

$$q_{54} = -\frac{S_y t\sin\alpha}{I_{xx}}\left(\frac{s^2}{2} - 2as\right) \tag{iii}$$

Taking moments about the point 3

$$S_y\xi_S = 2\int_0^a q_{54} a\sin 2\alpha\,\mathrm{d}s \tag{iv}$$

Substituting in Eq. (iv) for q_{54} from Eq. (iii)

$$S_y\xi_S = -\frac{2a\sin 2\alpha S_y t\sin\alpha}{I_{xx}}\int_0^a\left(\frac{s^2}{2} - 2as\right)\mathrm{d}s$$

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which gives

$$\xi_S = \frac{2at \sin 2\alpha \sin\alpha}{I_{xx}}\left(\frac{5a^3}{6}\right) \tag{v}$$

Substituting for I_{xx} from Eq. (ii) in Eq. (v) gives

$$\xi_S = \frac{5a\cos\alpha}{8}$$

S.9.11

The shear centre, S, lies on the axis of symmetry a distance ξ_S from the point 2 as shown in Fig. S.9.11. Thus, an arbitrary shear load, S_y, is applied through S and since $I_{xy} = 0$, $S_x = 0$, Eq. (9.34) simplifies to

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s \tag{i}$$

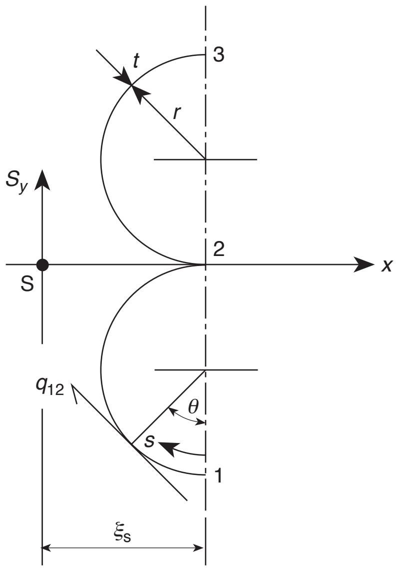

Fig. S.9.11

in which I_{xx} has the same value as the section in S.9.8, i.e. $3\pi r^3 t$. Then Eq. (i) becomes

$$q_{12} = \frac{S_y}{I_{xx}}\int_0^\theta t(r + r\cos\theta)r\,\mathrm{d}\theta$$

or

$$q_{12} = \frac{S_y}{3\pi r}\left[\theta + \sin\theta\right]_0^\theta$$

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i.e.

$$q_{12} = \frac{S_y}{3\pi r}(\theta + \sin\theta) \tag{ii}$$

Taking moments about the point 2

$$S_y \xi_S = 2\int_0^{\pi} q_{12}(r + r\cos\theta) r \, d\theta \tag{iii}$$

Substituting in Eq. (iii) for q_{12} from Eq. (ii)

$$S_y \xi_S = \frac{2S_y r}{3\pi}\int_0^{\pi}(\theta + \sin\theta)(1 + \cos\theta)\, d\theta$$

Thus

$$\xi_S = \frac{2r}{3\pi}\int_0^{\pi}(\theta + \theta\cos\theta + \sin\theta + \sin\theta\cos\theta)\, d\theta$$

i.e.

$$\xi_S = \frac{2r}{3\pi}\left[\frac{\theta^2}{2} + \theta\sin\theta - \frac{\cos 2\theta}{4}\right]_0^{\pi}$$

from which

$$\xi_S = \frac{\pi r}{3}$$

S.9.12

The x axis is an axis of symmetry so that $I_{xy} = 0$ and the shear centre, S, lies on this axis (see Fig. S.9.12). Further $S_x = 0$ so that Eq. (9.34) reduces to

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\, ds \tag{i}$$

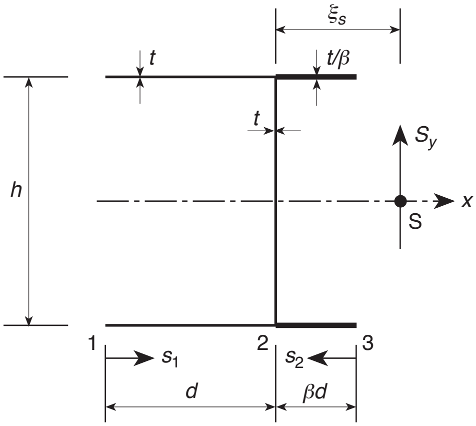

Fig. S.9.12

Referring to Fig. S.9.12

$$I_{xx} = \frac{th^3}{12} + 2\left[td\left(\frac{h}{2}\right)^2 + \frac{t}{\beta}\beta d\left(\frac{h}{2}\right)^2\right] = th^2\left(\frac{h}{12} + d\right)$$

From Eq. (i)

$$q_{12} = -\frac{S_y}{I_{xx}} \int_0^{s_1} t\left(-\frac{h}{2}\right) \mathrm{d}s_1$$

i.e.

$$q_{12} = \frac{S_y th}{2I_{xx}} s_1 \tag{ii}$$

Also

$$q_{32} = -\frac{S_y}{I_{xx}} \int_0^{s_2} \frac{t}{\beta}\left(-\frac{h}{2}\right) \mathrm{d}s_2$$

so that

$$q_{32} = \frac{S_y th}{2\beta I_{xx}} s_2 \tag{iii}$$

Taking moments about the mid-point of the web

$$S_y \xi_S = 2\int_0^d q_{12} \frac{h}{2}\, \mathrm{d}s_1 - 2\int_0^{\beta d} q_{32} \frac{h}{2}\, \mathrm{d}s_2 \tag{iv}$$

Substituting from Eqs (ii) and (iii) in Eq. (iv) for q_{12} and q_{32}

$$S_y \xi_S = \frac{S_y th^2}{2I_{xx}} \int_0^d s_1\, \mathrm{d}s_1 - \frac{S_y th^2}{2\beta I_{xx}} \int_0^{\beta d} s_2\, \mathrm{d}s_2$$

i.e.

$$\xi_S = \frac{th^2}{2I_{xx}}\left(\frac{d^2}{2} - \beta\frac{d^2}{2}\right)$$

i.e.

$$\xi_S = \frac{th^2 d^2(1-\beta)}{4th^3(1+12d/h)/12}$$

so that

$$\frac{\xi_S}{d} = \frac{3\rho(1-\beta)}{(1+12\rho)}$$

S.9.13

Referring to Fig. S.9.13 the shear centre, S, lies on the axis of symmetry, the x axis, so that $I_{xy} = 0$. Therefore, apply an arbitrary shear load, S_y, through the shear centre

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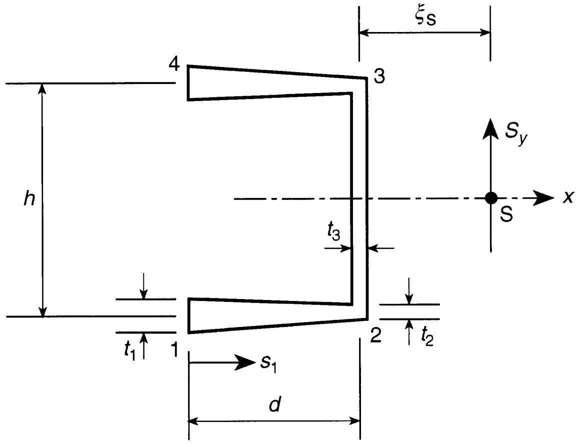

Fig. S.9.13

and determine the internal shear flow distribution. Thus, since $S_x = 0$, Eq. (9.34) becomes

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s ty \, ds \tag{i}$$

in which

$$I_{xx} = \frac{t_3 h^3}{12} + 2\frac{(t_1 + t_2)}{2} d \left(\frac{h}{2}\right)^2$$

i.e.

$$I_{xx} = \frac{h^2}{12}[t_3 h + 3(t_1 + t_2)d] \tag{ii}$$

The thickness t in the flange 12 at any point s_1 is given by

$$t = t_1 - \frac{(t_1 - t_2)}{d} s_1 \tag{iii}$$

Substituting for t from Eq. (iii) in Eq. (i)

$$q_{12} = -\frac{S_y}{I_{xx}} \int_0^{s_1} \left[t_1 - \frac{(t_1 - t_2)}{d} s_1\right]\left(-\frac{h}{2}\right) ds_1$$

Hence

$$q_{12} = \frac{S_y h}{2I_{xx}}\left[t_1 s_1 - \frac{(t_1 - t_2)}{d} \frac{s_1^2}{2}\right] \tag{iv}$$

Taking moments about the mid-point of the web

$$S_y \xi_S = 2\int_0^d q_{12}\left(\frac{h}{2}\right) ds_1$$

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i.e.

$$S_y\xi_S = \frac{S_y h^2}{2I_{xx}}\left[t_1\frac{s_1^2}{2} - \frac{(t_1 - t_2)}{d}\frac{s_1^3}{6}\right]_0^d$$

from which

$$\xi_S = \frac{h^2 d^2}{12I_{xx}}(2t_1 + t_2)$$

Substituting for I_{xx} from Eq. (ii)

$$\xi_S = \frac{d^2(2t_1 + t_2)}{3d(t_1 + t_2) + ht_3}$$

S.9.14

The beam section is shown in Fig. S.9.14(a). Clearly the x axis is an axis of symmetry so that $I_{xy} = 0$ and the shear centre, S, lies on this axis. Thus, apply an arbitrary shear load, S_y, through S and determine the internal shear flow distribution. Since $S_x = 0$, Eq. (9.34) simplifies to

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s \tag{i}$$

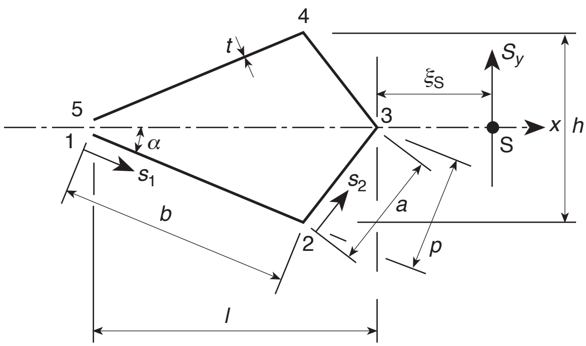

Fig. S.9.14(a)

in which, from Fig. S.9.14(a)

$$I_{xx} = 2\left[\int_0^b t\left(\frac{h}{2b}s\right)^2 \mathrm{d}s + \int_0^a t\left(\frac{h}{2a}s\right)^2 \mathrm{d}s\right] \tag{ii}$$

where the origin of s in the first integral is the point 1 and the origin of s in the second integral is the point 3. Eq. (ii) then gives

$$I_{xx} = th^2(b + a)/6 \tag{iii}$$

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From Eq. (i)

$$q_{12} = -\frac{S_y}{I_{xx}}\int_0^{s_1} t\left(-\frac{h}{2b}s_1\right)\mathrm{d}s_1$$

from which

$$q_{12} = \frac{S_y th}{2bI_{xx}}\frac{s_1^2}{2}$$

or, substituting for I_{xx} from Eq. (iii)

$$q_{12} = \frac{3S_y}{2bh(b+a)}s_1^2 \qquad \text{(iv)}$$

and

$$q_2 = \frac{3S_y b}{2h(b+a)} \qquad \text{(v)}$$

Also

$$q_{23} = -\frac{S_y}{I_{xx}}\int_0^{s_2} t\left[-\frac{h}{2a}(a-s_2)\right]\mathrm{d}s_2 + q_2$$

Substituting for I_{xx} from Eq. (iii) and q_2 from Eq. (v)

$$q_{23} = \frac{3S_y}{h(b+a)}\left(s_2 - \frac{s_2^2}{2a} + \frac{b}{2}\right) \qquad \text{(vi)}$$

and

$$q_3 = \frac{3S_y}{2h} \qquad \text{(vii)}$$

Eq. (iv) shows that q_{12} varies parabolically but does not change sign between 1 and 2; also $\mathrm{d}q_{12}/\mathrm{d}s_1 = 0$ when $s_1 = 0$. From Eq. (vi) $q_{23} = 0$ when $s_2 - s_2^2/2a + b/2 = 0$, i.e. when

$$s_2^2 - 2as_2 - ba = 0 \qquad \text{(viii)}$$

Solving Eq. (viii)

$$s_2 = a \pm \sqrt{a^2 + ba}$$

Thus, q_{23} does not change sign between 2 and 3. Further

$$\frac{\mathrm{d}q_{23}}{\mathrm{d}s_2} = \frac{3S_y}{h(b+a)}\left(1 - \frac{s_2}{a}\right) = 0 \text{ when } s_2 = a$$

Therefore q_{23} has a turning value at 3. The shear flow distributions in the walls 34 and 45 follow from antisymmetry; the complete distribution is shown in Fig. S.9.14(b).

Referring to Fig. S.9.14(a) and taking moments about the point 3

$$S_y\xi_S = 2\int_0^b q_{12}p\,\mathrm{d}s_1 \qquad \text{(ix)}$$

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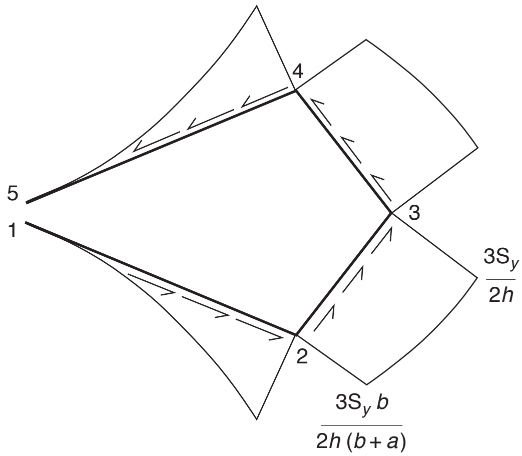

Fig. S.9.14(b)

where p is given by

$$\frac{p}{l} = \sin\alpha = \frac{h}{2b}, \quad \text{i.e. } p = \frac{hl}{2b}$$

Substituting for p and q_{12} from Eq. (iv) in Eq. (ix) gives

$$S_y \xi_S = \frac{3S_y}{bh(b+a)} \int_0^b \frac{hl}{2b} s_1^2 \, ds_1$$

from which

$$\xi_S = \frac{l}{2(1 + a/b)}$$

S.9.15

Initially the position of the centroid, C, must be found. From Fig. S.9.15, by inspection $\bar{y} = a$. Also taking moments about the web 23

$$(2at + 2a2t + a2t)\bar{x} = a2t\frac{a}{2} + 2ata$$

from which $\bar{x} = 3a/8$.

To find the horizontal position of the shear centre, S, apply an arbitrary shear load, S_y, through S. Since $S_x = 0$ Eq. (9.34) simplifies to

$$q_s = \frac{S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2} \int_0^s tx \, ds - \frac{S_y I_{yy}}{I_{xx}I_{yy} - I_{xy}^2} \int_0^s ty \, ds$$

i.e.

$$q_s = \frac{S_y}{I_{xx}I_{yy} - I_{xy}^2} \left(I_{xy} \int_0^s tx \, ds - I_{yy} \int_0^s ty \, ds \right) \qquad \text{(i)}$$

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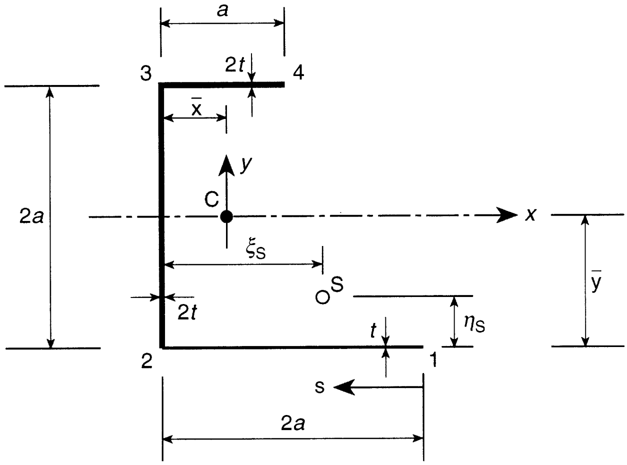

Fig. S.9.15

in which, referring to Fig. S.9.15

$$I_{xx} = a2t(a)^2 + 2at(a)^2 + t(2a)^3/12 = 16a^3t/3$$

$$I_{yy} = 2ta^3/12 + 2ta(a/8)^2 + t(2a)^3/12 + 2at(5a/8)^2 + 4at(3a/8)^2 = 53a^3t/24$$

$$I_{xy} = a2t(a/8)(a) + 2at(5a/8)(-a) = -a^3t$$

Substituting for I_{xx}, I_{yy} and I_{xy} in Eq. (i) gives

$$q_s = \frac{9S_y}{97a^3t}\left(-\int_0^s tx\,\mathrm{d}s - \frac{53}{24}\int_0^s ty\,\mathrm{d}s\right) \tag{ii}$$

from which

$$q_{12} = \frac{9S_y}{97a^3}\left[-\int_0^s t\left(\frac{13a}{8} - s\right)\mathrm{d}s - \frac{53}{24}\int_0^s t(-a)\,\mathrm{d}s\right] \tag{iii}$$

i.e.

$$q_{12} = \frac{9S_y}{97a^3}\left(\frac{7as}{12} + \frac{s^2}{2}\right) \tag{iv}$$

Taking moments about the corner 3 of the section

$$S_y\xi_S = -\int_0^{2a} q_{12}(2a)\,\mathrm{d}s \tag{v}$$

Substituting for q_{12} from Eq. (iv) in Eq. (v)

$$S_y\xi_S = -\frac{18S_y}{97a^2}\int_0^{2a}\left(\frac{7as}{12} + \frac{s^2}{2}\right)\mathrm{d}s$$

from which

$$\xi_S = -\frac{45a}{97}$$

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Now apply an arbitrary shear load S_x through the shear centre, S. Since $S_y = 0$ Eq. (9.34) simplifies to

$$q_s = -\frac{S_x}{I_{xx}I_{yy} - I_{xy}^2}\left(I_{xx}\int_0^s tx\,ds - I_{xy}\int_0^s ty\,ds\right)$$

from which, by comparison with Eq. (iii)

$$q_{12} = -\frac{9S_x}{97a^3 t}\left[\frac{16}{3}\int_0^s t\left(\frac{13a}{8} - s\right) ds + \int_0^s t(-a)\,ds\right]$$

i.e.

$$q_{12} = -\frac{3S_x}{97a^3}(23as - 8s^2) \tag{vi}$$

Taking moments about the corner 3

$$S_x(2a - \eta_S) = -\int_0^{2a} q_{12}(2a)\,ds$$

Substituting for q_{12} from Eq. (vi)

$$S_x(2a - \eta_S) = \frac{6S_x}{97a^2}\int_0^{2a}(23as - 8s^2)\,ds$$

which gives

$$\eta_S = \frac{46a}{97}$$

S.9.16

Since the section is doubly symmetrical the centroid of area, C, and the shear centre, S, coincide. The applied shear load, S, may be replaced by a shear load, S, acting through the shear centre together with a torque, T, as shown in Fig. S.9.16. Then

$$T = Sa\cos 30^\circ = 0.866Sa \tag{i}$$

The shear flow distribution produced by this torque is given by Eq. (9.49), i.e.

$$q_T = \frac{T}{2A} = \frac{0.866Sa}{2A} \quad \text{(from Eq. (i))}$$

where

$$A = a2a\cos 30^\circ + 2\times a\cos 30^\circ \times a\sin 30^\circ = 2.6a^2$$

Then

$$q_T = \frac{0.17S}{a} \quad \text{(clockwise)} \tag{ii}$$

The rate of twist is obtained from Eq. (9.52) and is

$$\frac{d\theta}{dz} = \frac{0.866Sa}{4(2.6a^2)^2 G}\left(\frac{6a}{t}\right)$$

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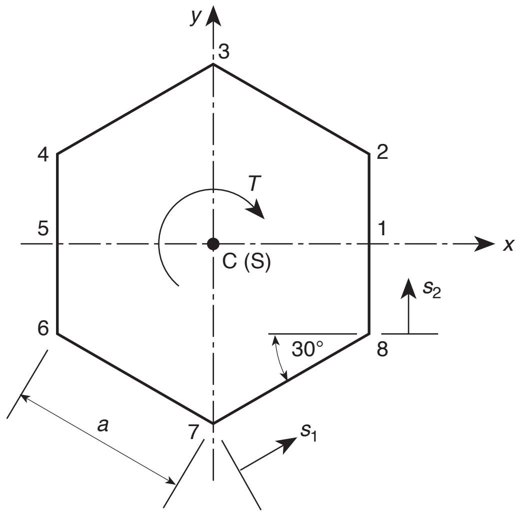

Fig. S.9.16

i.e.

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{0.192S}{Gta^2} \tag{iii}$$

The shear load, S, through the shear centre produces a shear flow distribution given by Eq. (9.35) in which $S_y = -S$, $S_x = 0$ and $I_{xy} = 0$. Hence

$$q_s = \frac{S}{I_{xx}} \int_0^s ty\,\mathrm{d}s + q_{s,0} \tag{iv}$$

in which

$$I_{xx} = 2\frac{ta^3}{12} + 4\int_0^a t(-a + s_1 \sin 30^\circ)^2\,\mathrm{d}s_1 = \frac{5a^3t}{2}$$

Also on the vertical axis of symmetry the shear flow is zero, i.e. at points 7 and 3. Therefore choose 7 as the origin of s in which case $q_{s,0}$ in Eq. (iv) is zero and

$$q_s = \frac{S}{I_{xx}} \int_0^s ty\,\mathrm{d}s \tag{v}$$

From Eq. (v) and referring to Fig. S.9.16

$$q_{78} = \frac{S}{I_{xx}} \int_0^{s_1} t(-a + s_1 \sin 30^\circ)\,\mathrm{d}s_1$$

i.e.

$$q_{78} = \frac{2S}{5a^3} \int_0^s \left(-a + \frac{s_1}{2}\right)\mathrm{d}s$$

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so that

$$q_{78} = -\frac{S}{5a^3}\left(2as_1 - \frac{s_1^2}{2}\right) \tag{vi}$$

and

$$q_8 = -\frac{3S}{10a} \tag{vii}$$

Also

$$q_{81} = \frac{S}{I_{xx}}\int_0^{s_2} t\left(-\frac{a}{2} + s_2\right) \mathrm{d}s_2 + q_8$$

i.e.

$$q_{81} = \frac{2S}{5a^3}\int_0^{s_2} \left(-\frac{a}{2} + s_2\right) \mathrm{d}s_2 - \frac{3S}{10a}$$

from which

$$q_{81} = \frac{S}{10a^3}(-2as_2 + 2s_2^2 - 3a^2) \tag{viii}$$

Thus

$$q_1 = -\frac{7S}{20a}$$

The remaining distribution follows from symmetry.

The complete shear flow distribution is now found by superimposing the shear flow produced by the torque, T, (Eq. (ii)) and the shear flows produced by the shear load acting through the shear centre. Thus, taking anticlockwise shear flows as negative

$$q_1 = -\frac{0.17S}{a} - \frac{0.35S}{a} = -\frac{0.52S}{a}$$

$$q_2 = q_8 = -\frac{0.17S}{a} - \frac{0.3S}{a} = -\frac{0.47S}{a} \quad \text{(from Eq. (vii))}$$

$$q_3 = q_7 = -\frac{0.17S}{a}$$

$$q_4 = q_6 = -\frac{0.17S}{a} + \frac{0.3S}{a} = \frac{0.13S}{a}$$

$$q_5 = -\frac{0.17S}{a} + \frac{0.35S}{a} = \frac{0.18S}{a}$$

The distribution in all walls is parabolic.

S.9.17

The shear centre, S, lies on the horizontal axis of symmetry, the x axis. Therefore apply an arbitrary shear load, S_y, through S (Fig. S.9.17(a)). The internal shear

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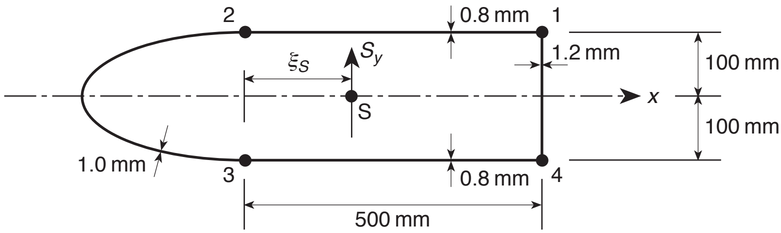

Fig. S.9.17(a)

flow distribution is given by Eq. (9.80) which, since $I_{xy} = 0$, $S_x = 0$ and $t_D = 0$, simplifies to

$$q_s = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r + q_{s,0} \tag{i}$$

in which

$$I_{xx} = 2 \times 450 \times 100^2 + 2 \times 550 \times 100^2 = 20 \times 10^6 \text{ mm}^4$$

Eq. (i) then becomes

$$q_s = -0.5 \times 10^{-7} S_y \sum_{r=1}^{n} B_r y_r + q_{s,0} \tag{ii}$$

The first term on the right-hand side of Eq. (ii) is the q_b distribution (see Eq. (9.36)). To determine q_b 'cut' the section in the wall 23. Then

$$q_{b,23} = 0$$

$$q_{b,34} = -0.5 \times 10^{-7} S_y \times 550 \times (-100) = 2.75 \times 10^{-3} S_y = q_{b,12}$$

$$q_{b,41} = 2.75 \times 10^{-3} S_y - 0.5 \times 10^{-7} S_y \times 450 \times (-100) = 5.0 \times 10^{-3} S_y$$

The value of shear flow at the 'cut' is obtained using Eq. (9.47) which, since $G =$ constant becomes

$$q_{s,0} = -\frac{\oint (q_b/t)\, ds}{\oint ds/t} \tag{iii}$$

In Eq. (iii)

$$\oint \frac{ds}{t} = \frac{580}{1.0} + 2 \times \frac{500}{0.8} + \frac{200}{1.2} = 1996.7$$

Then, from Eq. (iii) and the above q_b distribution

$$q_{s,0} = -\frac{S_y}{1996.7}\left(2 \times \frac{2.75 \times 10^{-3} \times 500}{0.8} + \frac{5.0 \times 10^{-3} \times 200}{1.2}\right)$$

i.e.

$$q_{s,0} = -2.14 \times 10^{-3} S_y$$

The complete shear flow distribution is shown in Fig. S.9.17(b).

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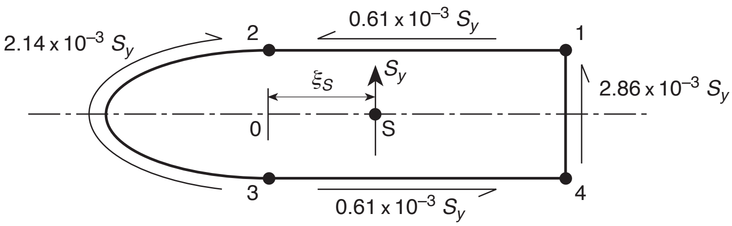

Fig. S.9.17(b)

Now taking moments about O in Fig. S.9.17(b) and using the result of Eq. (9.79)

$$S_y\xi_S = 2 \times 0.61 \times 10^{-3} S_y \times 500 \times 100 + 2.86 \times 10^{-3} S_y \times 200 \times 500$$
$$- 2.14 \times 10^{-3} S_y \times 2(135\,000 - 500 \times 200)$$

which gives

$$\xi_S = 197.2\,\text{mm}$$

S.9.18

Referring to Fig. S.9.18 the x axis is an axis of symmetry so that $I_{xy} = 0$ and since $S_x = 0$ Eq. (9.35) reduces to

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s ty\,\mathrm{d}s + q_{s,0} \tag{i}$$

in which

$$I_{xx} = \frac{(2r)^3 t \sin^2 45^\circ}{12} + 2\int_0^{\pi/2} t(r\sin\theta)^2 r\,\mathrm{d}\theta$$

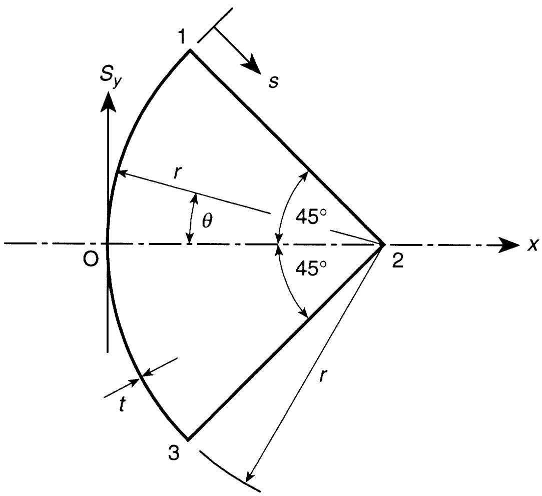

Fig. S.9.18

i.e.

$$I_{xx} = 0.62tr^3$$

'Cut' the section at O. Then, from the first term on the right-hand side of Eq. (i)

$$q_{\text{b,O1}} = -\frac{S_y}{0.62tr^3}\int_0^\theta tr\sin\theta r\,\mathrm{d}\theta$$

i.e.

$$q_{\text{b,O1}} = -\frac{S_y}{0.62r}\left[-\cos\theta\right]_0^\theta$$

so that

$$q_{\text{b,O1}} = \frac{S_y}{0.62r}(\cos\theta - 1) = 1.61\frac{S_y}{r}(\cos\theta - 1) \qquad \text{(ii)}$$

and

$$q_{\text{b},1} = -0.47S_y/r$$

Also

$$q_{\text{b},12} = -\frac{S_y}{0.62tr^3}\int_0^s t(r-s)\sin 45^\circ\,\mathrm{d}s - \frac{0.47S_y}{r}$$

which gives

$$q_{\text{b},12} = \frac{S_y}{r}(-1.14rs + 0.57s^2 - 0.47) \qquad \text{(iii)}$$

Now take moments about the point 2

$$S_y r = 2\int_0^{\pi/4} q_{\text{b,O1}} rr\,\mathrm{d}\theta + 2\times\frac{\pi r^2}{4}q_{s,0}$$

Substituting in Eq. (iv) for $q_{\text{b,O1}}$ from Eq. (ii)

$$S_y r = 2\int_0^{\pi/4} 1.61\frac{S_y}{r}(\cos\theta - 1)r^2\,\mathrm{d}\theta + \frac{\pi r^2}{2}q_{s,0}$$

i.e.

$$S_y r = 3.22S_y r\left[\sin\theta - \theta\right]_0^{\pi/4} + \frac{\pi r^2}{2}q_{s,0}$$

so that

$$q_{s,0} = \frac{0.80S_y}{r}$$

Then, from Eq. (ii)

$$q_{\text{O1}} = \frac{S_y}{r}(1.61\cos\theta - 0.80)$$

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and from Eq. (iii)

$$q_{12} = \frac{S_y}{r}(0.57s^2 - 1.14rs - 0.33)$$

The remaining distribution follows from symmetry.

S.9.19

The x axis is an axis of symmetry so that $I_{xy} = 0$ and, since $S_x = 0$, Eq. (9.35) simplifies to

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s + q_{s,0} \tag{i}$$

in which, from Fig. S.9.19(a)

$$I_{xx} = \frac{th^3}{12} + \frac{(2d)^3 t\sin^2\alpha}{12} = \frac{th^2}{12}(h + 2d) \tag{ii}$$

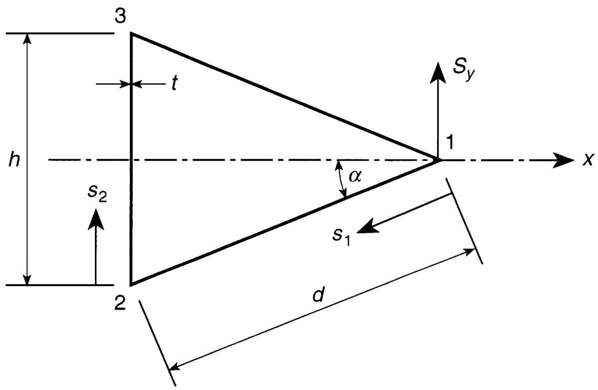

Fig. S.9.19(a)

'Cut' the section at 1. Then, from the first term on the right-hand side of Eq. (i)

$$q_{\mathrm{b},12} = -\frac{S_y}{I_{xx}}\int_0^{s_1} t(-s_1\sin\alpha)\,\mathrm{d}s_1 = \frac{S_y t\sin\alpha}{2I_{xx}}s_1^2$$

Substituting for I_{xx} and $\sin\alpha$

$$q_{\mathrm{b},12} = \frac{3S_y}{hd(h + 2d)}s_1^2 \tag{iii}$$

and

$$q_{\mathrm{b},2} = \frac{3S_y d}{h(h + 2d)}$$

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Also

$$q_{\mathrm{b},23} = -\frac{S_y}{I_{xx}}\int_0^{s_2} t\left(-\frac{h}{2}+s_2\right)\mathrm{d}s_2 + q_{\mathrm{b},2}$$

so that

$$q_{\mathrm{b},23} = \frac{6S_y}{h(h+2d)}\left(s_2 - \frac{s_2^2}{h} + \frac{d}{2}\right) \qquad \text{(iv)}$$

Now taking moments about the point 1 (see Eq. (9.38))

$$0 = \int_0^h q_{\mathrm{b},23} d\cos\alpha\,\mathrm{d}s_2 + 2\frac{h}{2}d\cos\alpha\,q_{s,0}$$

i.e.

$$0 = \int_0^h q_{\mathrm{b},23}\,\mathrm{d}s_2 + hq_{s,0} \qquad \text{(v)}$$

Substituting in Eq. (v) for $q_{\mathrm{b},23}$ from Eq. (iv)

$$0 = \frac{6S_y}{h(h+2d)}\int_0^h\left(s_2 - \frac{s_2^2}{h} + \frac{d}{2}\right)\mathrm{d}s_2 + hq_{s,0}$$

which gives

$$q_{s,0} = -\frac{S_y(h+3d)}{h(h+2d)} \qquad \text{(vi)}$$

Then, from Eqs (iii) and (i)

$$q_{12} = \frac{3S_y}{hd(h+2d)}s_1^2 - \frac{S_y(h+3d)}{h(h+2d)}$$

i.e.

$$q_{12} = \frac{S_y}{h(h+2d)}\left(\frac{3s_1^2}{d} - h - 3d\right) \qquad \text{(vii)}$$

and from Eqs (iv) and (vi)

$$q_{23} = \frac{S_y}{h(h+2d)}\left(6s_2 - \frac{6s_2^2}{h} - h\right) \qquad \text{(viii)}$$

The remaining distribution follows from symmetry.

From Eq. (vii), q_{12} is zero when $s_1^2 = (hd/3) + d^2$, i.e. when $s_1 > d$. Thus there is no change of sign of q_{12} between 1 and 2. Further

$$\frac{\mathrm{d}q_{12}}{\mathrm{d}s_1} = \frac{6s_1}{d} = 0 \text{ when } s_1 = 0$$

and

$$q_1 = -\frac{S_y(h+3d)}{h(h+2d)}$$

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Also, when $s_1 = d$

$$q_2 = -\frac{S_y}{(h+2d)}$$

From Eq. (viii) q_{23} is zero when $6s_2 - (6s_2^2/h) - h = 0$, i.e. when $s_2^2 - s_2 h + (h^2/6) = 0$. Then

$$s_2 = \frac{h}{2} \pm \frac{h}{\sqrt{12}}$$

Thus q_{23} is zero at points a distance $h/\sqrt{12}$ either side of the x axis. Further, from Eq. (viii), q_{23} will be a maximum when $s_2 = h/2$ and $q_{23}(\max) = S_y/2(h+2d)$. The complete distribution is shown in Fig. S.9.19(b).

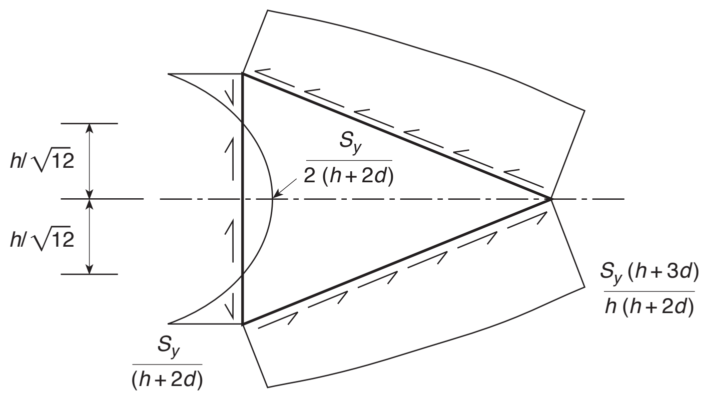

Fig. S.9.19(b)

S.9.20

The x axis is an axis of symmetry so that $I_{xy} = 0$, also the shear centre, S, lies on this axis. Apply an arbitrary shear load, S_y, through S. The internal shear flow distribution is then given by Eq. (9.80) in which $S_x = 0$ and $I_{xy} = 0$. Thus

$$q_s = -\frac{S_y}{I_{xx}}\left(\int_0^s t_D y\,\mathrm{d}s + \sum_{r=1}^{n} B_r y_r\right) + q_{s,0} \tag{i}$$

in which from Fig. S.9.20

$$I_{xx} = 4 \times 100 \times 40^2 + 2 \times 0.64 \times 240 \times 40^2 + \frac{0.36 \times 80^3}{12} + \frac{0.64 \times 80^3}{12}$$

i.e.

$$I_{xx} = 1.17 \times 10^6\,\mathrm{mm}^4$$

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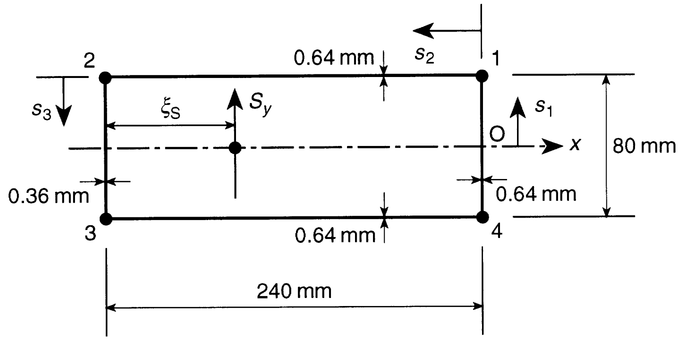

Fig. S.9.20

'Cut' the section at O. Then, from the first two terms on the right-hand side of Eq. (i)

$$q_{\mathrm{b,O1}} = -\frac{S_y}{I_{xx}} \int_0^{s_1} 0.64 s_1 \, \mathrm{d}s_1$$

i.e.

$$q_{\mathrm{b,O1}} = -0.27 \times 10^{-6} S_y s_1^2 \tag{ii}$$

and

$$q_{\mathrm{b,1}} = -4.32 \times 10^{-4} S_y$$

Also

$$q_{\mathrm{b,12}} = -\frac{S_y}{I_{xx}} \left(\int_0^{s_2} 0.64 \times 40 \, \mathrm{d}s_2 + 100 \times 40 \right) - 4.32 \times 10^{-4} S_y$$

i.e.

$$q_{\mathrm{b,12}} = -10^{-4} S_y (0.22 s_2 + 38.52) \tag{iii}$$

whence

$$q_{\mathrm{b,2}} = -91.32 \times 10^{-4} S_y$$

Finally

$$q_{\mathrm{b,23}} = -\frac{S_y}{I_{xx}} \left[\int_0^{s_3} 0.36(40 - s_3) \, \mathrm{d}s_3 + 100 \times 40 \right] - 91.32 \times 10^{-4} S_y$$

i.e.

$$q_{\mathrm{b,23}} = -10^{-4} S_y (0.12 s_3 - 0.15 \times 10^{-2} s_3^2 + 125.52) \tag{iv}$$

The remaining q_{b} distribution follows from symmetry. From Eq. (9.47)

$$q_{s,0} = -\frac{\oint (q_{\mathrm{b}}/t) \, \mathrm{d}s}{\oint \mathrm{d}s/t} \tag{v}$$

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in which

$$\oint \frac{\mathrm{d}s}{t} = \frac{80}{0.64} + \frac{2 \times 240}{0.64} + \frac{80}{0.36} = 1097.2$$

Now substituting in Eq. (v) for $q_{\mathrm{b,O1}}$, $q_{\mathrm{b,12}}$ and $q_{\mathrm{b,23}}$ from Eqs (ii), (iii) and (iv) respectively

$$q_{s,0} = \frac{2 \times 10^{-4} S_y}{1097.2}\left[\int_0^{40} \frac{0.27 \times 10^{-2}}{0.64} s_1^2 \, \mathrm{d}s_1 + \int_0^{240} \frac{1}{0.64}(0.22 s_2 + 38.52)\, \mathrm{d}s_2 + \int_0^{40} \frac{1}{0.64}(0.12 s_3 - 0.15 \times 10^{-2} s_3^2 + 125.52)\, \mathrm{d}s_3\right]$$

from which

$$q_{s,0} = 70.3 \times 10^{-4} S_y$$

The complete shear flow distribution is then

$$q_{\mathrm{O1}} = -10^{-4} S_y (0.27 \times 10^{-2} s_1^2 - 70.3) \qquad \text{(vi)}$$

$$q_{12} = q_{34} = -10^{-4} S_y (0.22 s_2 - 31.78) \qquad \text{(vii)}$$

$$q_{23} = -10^{-4} S_y (0.12 s_3 - 0.15 \times 10^{-2} s_3^2 - 55.22) \qquad \text{(viii)}$$

Taking moments about the mid-point of the wall 23

$$S_y \xi_{\mathrm{S}} = 2\left[\int_0^{40} q_{\mathrm{O1}} \times 240 \, \mathrm{d}s_1 + \int_0^{240} q_{12} \times 40 \, \mathrm{d}s_2\right] \qquad \text{(ix)}$$

Substituting for q_{O1} and q_{12} from Eqs (vi) and (vii) in Eq. (ix)

$$S_y \xi_{\mathrm{S}} = -2 \times 10^{-4} S_y \left[\int_0^{40} (0.27 \times 10^{-2} s_1^2 - 70.3) \times 240 \, \mathrm{d}s_1 + \int_0^{240} (0.22 s_2 - 31.78) \times 40 \, \mathrm{d}s_2\right]$$

from which

$$\xi_{\mathrm{S}} = 142.5 \, \mathrm{mm}$$

S.9.21

Referring to Fig. P.9.21 the maximum torque occurs at the built-in end of the beam and is given by

$$T_{\max} = 20 \times 2.5 \times 10^3 = 50\,000 \, \mathrm{Nm}$$

From Eq. (9.49)

$$\tau_{\max} = \frac{q_{\max}}{t_{\min}} = \frac{T_{\max}}{2 A t_{\min}}$$

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i.e.

$$\tau_{\max} = \frac{50\,000 \times 10^3}{2 \times 250 \times 1000 \times 1.2}$$

so that

$$\tau_{\max} = 83.3\,\text{N/mm}^2$$

From Eq. (9.52)

$$\frac{d\theta}{dz} = \frac{T}{4A^2} \oint \frac{ds}{Gt}$$

i.e.

$$\frac{d\theta}{dz} = \frac{20(2500 - z) \times 10^3 \times 2}{4 \times (250 \times 1000)^2} \left(\frac{1000}{18\,000 \times 1.2} + \frac{250}{26\,000 \times 2.1} \right)$$

which gives

$$\frac{d\theta}{dz} = 8.14 \times 10^{-9}(2500 - z)$$

Then

$$\theta = 8.14 \times 10^{-9}\left(2500z - \frac{z^2}{2}\right) + C_1$$

When $z = 0$, $\theta = 0$ so that $C_1 = 0$, hence

$$\theta = 8.14 \times 10^{-9}\left(2500z - \frac{z^2}{2}\right)$$

Thus θ varies parabolically along the length of the beam and when $z = 2500\,\text{mm}$

$$\theta = 0.0254\,\text{rad} \quad \text{or} \quad 1.46^\circ$$

S.9.22

The shear modulus of the walls of the beam is constant so that Eq. (9.53) may be written

$$w_s - w_0 = \frac{T\delta}{2AG}\left(\frac{\delta_{Os}}{\delta} - \frac{A_{Os}}{A}\right) \tag{i}$$

in which

$$\delta = \oint \frac{ds}{t} \quad \text{and} \quad \delta_{Os} = \int_0^s \frac{ds}{t}$$

Also, the warping displacement will be zero on the axis of symmetry, i.e. at the mid-points of the walls 61 and 34. Therefore take the origin for s at the mid-point of the

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wall 61, then Eq. (i) becomes

$$w_s = \frac{T\delta}{2AG}\left(\frac{\delta_{Os}}{\delta} - \frac{A_{Os}}{A}\right) \tag{ii}$$

in which

$$l_{23} = \sqrt{500^2 + 100^2} = 509.9\,\text{mm} \quad \text{and} \quad l_{12} = \sqrt{890^2 + 150^2} = 902.6\,\text{mm}$$

Then

$$\delta = \frac{200}{2.0} + \frac{300}{2.5} + \frac{2 \times 509.9}{1.25} + \frac{2 \times 902.6}{1.25} = 2479.9$$

and

$$A = \tfrac{1}{2}(500 + 200) \times 890 + \tfrac{1}{2}(500 + 300) \times 500 = 511\,500\,\text{mm}^2$$

Eq. (ii) then becomes

$$w_s = \frac{90\,500 \times 10^3 \times 2479.9}{2 \times 511\,500 \times 27\,500}\left(\frac{\delta_{Os}}{2479.9} - \frac{A_{Os}}{511\,500}\right)$$

i.e.

$$w_s = 7.98 \times 10^{-4}(4.03\delta_{Os} - 0.0196A_{Os}) \tag{iii}$$

The walls of the section are straight so that δ_{Os} and A_{Os} vary linearly within each wall. It follows from Eq. (iii) that w_s varies linearly within each wall so that it is only necessary to calculate the warping displacement at the corners of the section. Thus, referring to Fig. P.9.22

$$w_1 = 7.98 \times 10^{-4}\left(4.03 \times \frac{100}{2.0} - 0.0196 \times \frac{1}{2} \times 890 \times 100\right)$$

i.e.

$$w_1 = -0.53\,\text{mm} = -w_6 \text{ from antisymmetry}$$

Also

$$w_2 = 7.98 \times 10^{-4}\left(4.03 \times \frac{902.6}{1.25} - 0.0196 \times \frac{1}{2} \times 250 \times 890\right) - 0.53$$

i.e.

$$w_2 = 0.05\,\text{mm} = -w_5$$

Finally

$$w_3 = 7.98 \times 10^{-4}\left(4.03 \times \frac{509.9}{1.25} - 0.0196 \times \frac{1}{2} \times 250 \times 500\right) + 0.05$$

i.e.

$$w_3 = 0.38\,\text{mm} = -w_4$$

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S.9.23

Referring to Fig. P.9.23 and considering the rotational equilibrium of the beam

$$2R = 2 \times 450 + 1.0 \times 2000$$

so that

$$R = 1450\,\text{Nm}$$

In the central portion of the beam

$$T = 450 + 1.0(1000 - z) - 1450 = -z\,\text{Nm} \quad (z \text{ in mm}) \tag{i}$$

and in the outer portions

$$T = 450 + 1.0(1000 - z) = 1450 - z\,\text{Nm} \quad (z \text{ in mm}) \tag{ii}$$

From Eq. (i) it can be seen that T varies linearly from zero at the mid-span of the beam to -500 Nm at the supports. Further, from Eq. (ii) the torque in the outer portions of the beam varies linearly from 950 Nm at the support to 450 Nm at the end. Therefore $T_{\text{max}} = 950$ Nm and from Eq. (9.49)

$$\tau_{\text{max}} = \frac{q_{\text{max}}}{t_{\text{min}}} = \frac{T_{\text{max}}}{2At_{\text{min}}}$$

i.e.

$$\tau_{\text{max}} = \frac{950 \times 10^3}{2 \times \pi \times 50^2 \times 2.5} = 24.2\,\text{N/mm}^2$$

For convenience the datum for the angle of twist may be taken at the mid-span section and angles of twist measured relative to this point. Thus, from Eq. (9.52) and Eq. (i), in the central portion of the beam

$$\frac{d\theta}{dz} = -\frac{z \times 10^3 \times \pi \times 100}{4(\pi \times 50^2)^2 \times 30\,000 \times 2.5}$$

i.e.

$$\frac{d\theta}{dz} = -1.70 \times 10^{-8} z$$

Then

$$\theta = -1.70 \times 10^{-8} \frac{z^2}{2} + B$$

When $z = 0$, $\theta = 0$ (datum point) so that $B = 0$. Then

$$\theta = -0.85 \times 10^{-8} z^2 \tag{iii}$$

In the outer portions of the beam, from Eq. (9.52) and Eq. (ii)

$$\frac{d\theta}{dz} = \frac{(1450 - z) \times 10^3 \times \pi \times 100}{4(\pi \times 50^2)^2 \times 30\,000 \times 2.5}$$

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i.e.

$$\frac{d\theta}{dz} = 1.70 \times 10^{-8}(1450 - z)$$

Hence

$$\theta = 1.70 \times 10^{-8}\left(1450z - \frac{z^2}{2}\right) + C \qquad \text{(iv)}$$

When $z = 500\,\text{mm}$, $\theta = -2.13 \times 10^{-3}$ rad from Eq. (iii). Thus, substituting this value in Eq. (iv) gives $C = -12.33 \times 10^{-3}$ and Eq. (iv) becomes

$$\theta = 1.70 \times 10^{-8}\left(1450z - \frac{z^2}{2}\right) - 12.33 \times 10^{-3}\,\text{rad} \qquad \text{(v)}$$

The distribution of twist along the beam is then obtained from Eqs (iii) and (v) and is shown in Fig. S.9.23. Note that the distribution would be displaced upwards by 2.13×10^{-3} rad if it were assumed that the angle of twist was zero at the supports.

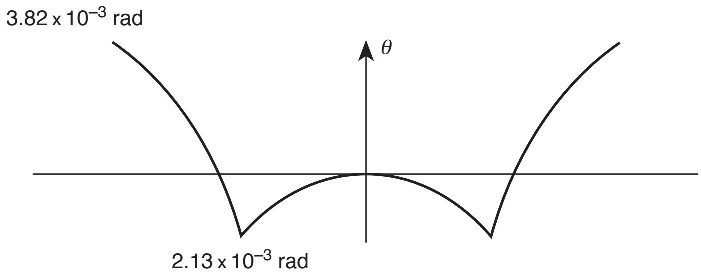

Fig. S.9.23

S.9.24

In Eq. (9.52), i.e.

$$\frac{d\theta}{dz} = \frac{T}{4A^2}\oint \frac{ds}{Gt}$$

$Gt = \text{constant} = 44\,000\,\text{N/mm}$. Thus, referring to Fig. S.9.24

$$\frac{d\theta}{dz} = \frac{4500 \times 10^3}{4(100 \times 200 + \pi \times 50^2/2)^2}\left(\frac{2 \times 200 + 100 + \pi \times 50}{44\,000}\right)$$

i.e.

$$\frac{d\theta}{dz} = 29.3 \times 10^{-6}\,\text{rad/mm}$$

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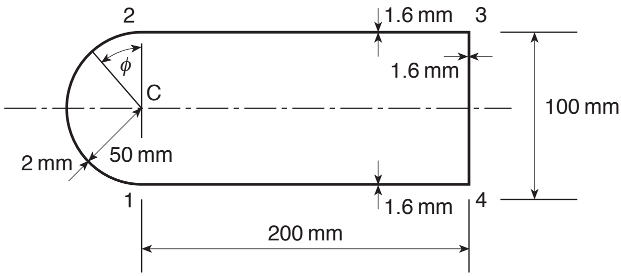

Fig. S.9.24

The warping displacement is zero on the axis of symmetry so that Eq. (9.53) becomes

$$w_s = \frac{T\delta}{2A}\left(\frac{\delta_{Os}}{\delta} - \frac{A_{Os}}{A}\right) \tag{i}$$

where

$$\delta = \oint \frac{ds}{Gt} \quad \text{and} \quad \delta_{Os} = \int_0^s \frac{ds}{Gt}$$

Since Gt = constant, Eq. (i) may be written

$$w_s = \frac{T}{2AGt}\oint ds\left(\frac{\int_0^s ds}{\oint ds} - \frac{A_{Os}}{A}\right) \tag{ii}$$

in which

$$\oint ds = 2 \times 200 + 100 + \pi \times 50 = 657.1\,\text{mm}$$

and

$$A = 100 \times 200 + \pi \times 50^2/2 = 23\,927.0\,\text{mm}^2$$

Eq. (ii) then becomes

$$w_s = \frac{4500 \times 10^3 \times 657.1}{2 \times 23\,927.0 \times 44\,000}\left(\frac{\int_0^s ds}{657.1} - \frac{A_{Os}}{23\,927.0}\right)$$

i.e.

$$w_s = 1.40 \times 10^{-3}\left(1.52\int_0^s ds - 4.18 \times 10^{-2} A_{Os}\right) \tag{iii}$$

In the straight walls $\int_0^s ds$ and A_{Os} are linear so that it is only necessary to calculate the warping displacement at the corners. Thus

$$w_3 = -w_4 = 1.40 \times 10^{-3}(1.52 \times 50 - 4.18 \times 10^{-2} \times \tfrac{1}{2} \times 200 \times 50) = -0.19\,\text{mm}$$

$$w_2 = -w_1 = 1.40 \times 10^{-3}(1.52 \times 200 - 4.18 \times 10^{-2} \times \tfrac{1}{2} \times 200 \times 50) - 0.19$$

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i.e.

$$w_2 = -w_1 = -0.056\,\text{mm}$$

In the wall 21

$$\int_0^s \mathrm{d}s = 50\phi \quad \text{and} \quad A_{\mathrm{O}s} = \tfrac{1}{2} \times 50^2\phi$$

Then Eq. (iii) becomes

$$w_{21} = 1.40 \times 10^{-3}(1.52 \times 50\phi - 4.18 \times 10^{-2} \times \tfrac{1}{2} \times 50^2\phi) - 0.056$$

i.e.

$$w_{21} = 0.033\phi - 0.056 \qquad \text{(iv)}$$

Thus w_{21} varies linearly with ϕ and when $\phi = \pi/2$ the warping displacement should be zero. From Eq. (iv), when $\phi = \pi/2$, $w_{21} = -0.004\,\text{mm}$; the discrepancy is due to rounding off errors.

S.9.25

Suppose the mass density of the covers is ρ_a and of the webs ρ_b. Then

$$\rho_a = k_1 G_a, \quad \rho_b = k_1 G_b$$

Let W be the weight/unit span. Then

$$W = 2at_a\rho_a g + 2bt_b\rho_b g$$

so that, substituting for ρ_a and ρ_b

$$W = 2k_1 g(at_a G_a + bt_b G_b) \qquad \text{(i)}$$

The torsional stiffness may be defined as $T/(\mathrm{d}\theta/\mathrm{d}z)$ and from Eq. (9.52)

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{4a^2b^2}\left(\frac{2a}{G_a t_a} + \frac{2b}{G_b t_b}\right) \qquad \text{(ii)}$$

Thus, for a given torsional stiffness, $\mathrm{d}\theta/\mathrm{d}z = \text{constant}$, i.e.

$$\frac{a}{G_a t_a} + \frac{b}{G_b t_b} = \text{constant} = k_2 \qquad \text{(iii)}$$

Let $t_b/t_a = \lambda$. Eq. (iii) then becomes

$$t_a = \frac{1}{k_2}\left(\frac{a}{G_a} + \frac{b}{\lambda G_b}\right)$$

and substituting for t_a in Eq. (i)

$$W = 2k_1 g t_a(aG_a + \lambda bG_b) = 2\frac{k_1}{k_2}g\left(a^2 + b^2 + \frac{abG_a}{\lambda G_b} + \frac{\lambda abG_b}{G_a}\right)$$

For a maximum

$$\frac{\mathrm{d}W}{\mathrm{d}\lambda} = 0$$

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i.e.

$$\lambda^2 = \left(\frac{G_a}{G_b}\right)^2$$

from which

$$\lambda = \frac{G_a}{G_b} = \frac{t_b}{t_a}$$

For the condition $G_a t_a = G_b t_b$ knowing that a and b can vary, Eq. (i) becomes

$$W = 2k_1 G_a t_a g(a + b) \tag{iv}$$

From Eq. (ii), for constant torsional stiffness

$$\frac{a+b}{a^2 b^2} = \text{constant} = k_3 \tag{v}$$

Let $b/a = x$. Eq. (iv) may then be written

$$W = 2k_1 G_a t_a g a(1 + x) \tag{vi}$$

and Eq. (v) becomes

$$k_3 = \frac{1+x}{a^3 x^2}$$

which gives

$$a^3 = \frac{1+x}{k_3 x^2}$$

Substituting for a in Eq. (vi)

$$W = \frac{2k_1 G_a t_a g}{k_3^{1/3}} \left(\frac{1+x}{x^2}\right)^{1/3} (1 + x)$$

i.e.

$$W = \frac{2k_1 G_a t_a g}{k_3^{1/3}} \frac{(1+x)^{4/3}}{x^{2/3}}$$

Hence for $(\mathrm{d}W/\mathrm{d}x) = 0$

$$0 = \frac{4}{3}\frac{(1+x)^{1/3}}{x^{2/3}} - \frac{2}{3}x^{-5/3}(1+x)^{4/3}$$

i.e.

$$4x - 2(1 + x) = 0$$

so that

$$x = 1 = b/a$$

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S.9.26

From the second of Eqs (9.61) the maximum shear stress is given by

$$\tau_{\max} = \pm\frac{tT}{J} \tag{i}$$

in which J, from Eqs (9.59), is given by (see Fig. P.9.26)

$$J = \frac{100 \times 2.54^3}{3} + 2 \times \frac{38 \times 1.27^3}{3} + \frac{2}{3}\int_0^{50}\left(1.27 + 1.27\frac{s}{50}\right) \mathrm{d}s$$

where the origin for s is at the corner 2 (or 5). Thus

$$J = 854.2\,\mathrm{mm}^4$$

Substituting in Eq. (i)

$$\tau_{\max} = \pm\frac{2.54 \times 100 \times 10^3}{854.2} = \pm 297.4\,\mathrm{N/mm}^2$$

The warping distribution is given by Eq. (9.68) and is a function of the swept area, A_R (see Fig. 9.37). Since the walls of the section are straight, A_R varies linearly around the cross-section. Also, the warping is zero at the mid-point of the web so that it is only necessary to calculate the warping at the extremity of each wall. Thus

$$\begin{aligned} w_1 &= -2A_\mathrm{R}\frac{T}{GJ} = -2 \times \frac{1}{2} \times 25 \times 50 \times \frac{100 \times 10^3}{26\,700 \times 854.2} \\ &= -5.48\,\mathrm{mm} = -w_6 \text{ from antisymmetry} \end{aligned}$$

Note that p_R, and therefore A_R, is positive in the wall 61.

$$w_2 = -5.48 + 2 \times \frac{1}{2} \times 50 \times 50 \times \frac{100 \times 10^3}{26\,700 \times 854.2} = 5.48\,\mathrm{mm} = -w_5$$

(p_R is negative in the wall 12)

$$w_3 = 5.48 + 2 \times \frac{1}{2} \times 38 \times 75 \times \frac{100 \times 10^3}{26\,700 \times 854.2} = 17.98\,\mathrm{mm} = -w_4$$

(p_R is negative in the wall 23)

S.9.27

The maximum shear stress in the section is given by the second of Eqs (9.61), i.e.

$$\tau_{\max} = \pm\frac{t_{\max}T_{\max}}{J} \tag{i}$$

in which $t_{\max} = t_0$ and the torsion constant J is obtained using the second of Eqs (9.59). Thus

$$J = 2\left[\frac{1}{3}\int_0^a \left(\frac{s}{a}t_0\right)^3 \mathrm{d}s + \frac{1}{3}\int_0^{3a}\left(\frac{s}{3a}t_0\right)^3 \mathrm{d}s + \frac{at_0^3}{3}\right]$$

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In the first integral s is measured from the point 7 while in the second s is measured from the point 1. Thus

$$J = \frac{4at_0^3}{3}$$

Substituting in Eq. (i)

$$\tau_{\max} = \pm\frac{t_0 T}{4at_0^3/3} = \pm\frac{3T}{4at_0^2}$$

The warping distribution is given by Eq. (9.67). Thus, for unit rate of twist

$$w_s = -2A_R \tag{ii}$$

Since the walls are straight A_R varies linearly in each wall so that it is only necessary to calculate the warping displacement at the extremities of the walls. Further, the section is constrained to twist about O so that $w_0 = w_3 = w_4 = 0$. Then

$$w_7 = -2\times\tfrac{1}{2}aa = -a^2 = -w_8 \quad (p_R \text{ is positive in 37})$$

$$w_2 = 2\times\tfrac{1}{2}a2a\cos 45^\circ = \sqrt{2}a^2 = -w_5 \quad (p_R \text{ is negative in 32})$$

$$w_1 = \sqrt{2}a^2 + 2\times\tfrac{1}{2}a(2a\sin 45^\circ + a) = a^2(1+2\sqrt{2}) = -w_6 \quad (p_R \text{ is negative in 21})$$

S.9.28

Referring to Fig. S.9.28(a) the x axis of the beam cross-section is an axis of symmetry so that $I_{xy} = 0$. Further, S_y at the end A is equal to -4450 N and $S_x = 0$. The total deflection, Δ, at one end of the beam is then, from Eqs (9.86) and (9.88)

$$\Delta = \int_L \frac{M_{x,1}M_{x,0}}{EI_{xx}}\,\mathrm{d}z + \int_L\left(\int_{\text{section}}\frac{q_0 q_1}{Gt}\,\mathrm{d}s\right)\mathrm{d}z \tag{i}$$

in which q_0, from Eqs (9.89) and (9.80) is given by

$$q_0 = -\frac{S_{y,0}}{I_{xx}}\sum_{r=1}^{n} B_r y_r + q_{s,0} \tag{ii}$$

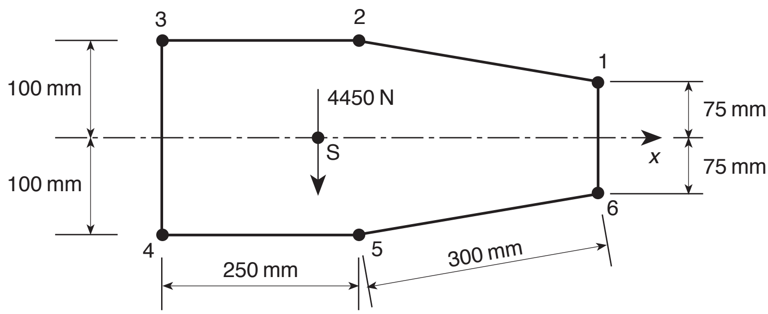

Fig. S.9.28(a)

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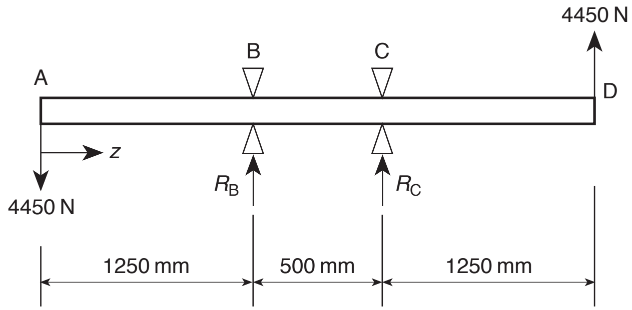

Fig. S.9.28(b)

and

$$q_1 = q_0/4450$$

Since the booms carrying all the direct stresses, I_{xx} in Eq. (i) is, from Fig. S.9.28(a)

$$I_{xx} = 2 \times 650 \times 100^2 + 2 \times 650 \times 75^2 + 2 \times 1300 \times 100^2 = 46.3 \times 10^6\,\text{mm}^4$$

Also, from Fig. S.9.28(b) and taking moments about C

$$R_B \times 500 - 4450 \times 1750 - 4450 \times 1250 = 0$$

from which

$$R_B = 26\,700\,\text{N}$$

Therefore in AB

$$M_{x,0} = 4450z, \quad M_{x,1} = z$$

and in BC

$$M_{x,0} = 33.4 \times 10^6 - 22\,250z, \quad M_{x,1} = 7500 - 5z$$

Thus the deflection, Δ_M, due to bending at the end A of the beam is, from the first term on the right-hand side of Eq. (i)

$$\Delta_M = \frac{1}{EI_{xx}}\left\{\int_0^{1250} 4450z^2\,dz + \int_{1250}^{1500} 4450(7500 - 5z)^2\,dz\right\}$$

i.e.

$$\Delta_M = \frac{4450}{69\,000 \times 46.3 \times 10^6}\left\{\left[\frac{z^3}{3}\right]_0^{1250} - \frac{1}{15}\left[(7500 - 5z)^3\right]_{1250}^{1500}\right\}$$

from which

$$\Delta_M = 1.09\,\text{mm}$$

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Now 'cut' the beam section in the wall 12. From Eq. (9.80), i.e.

$$q_s = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r + q_{s,0} \tag{iii}$$

$$q_{b,12} = 0$$

$$q_{b,23} = -\frac{S_y}{I_{xx}} \times 1300 \times 100 = -130\,000\frac{S_y}{I_{xx}}$$

$$q_{b,34} = -130\,000\frac{S_y}{I_{xx}} - \frac{S_y}{I_{xx}} \times 650 \times 100 = -195\,000\frac{S_y}{I_{xx}}$$

$$q_{b,16} = -\frac{S_y}{I_{xx}} \times 650 \times 75 = -48\,750\frac{S_y}{I_{xx}}$$

The remaining distribution follows from symmetry. The shear load is applied through the shear centre of the cross-section so that $\mathrm{d}\theta/\mathrm{d}z = 0$ and $q_{s,0}$ is given by Eq. (9.48), i.e.

$$q_{s,0} = -\frac{\oint q_b \,\mathrm{d}s}{\oint \mathrm{d}s} \quad (t = \text{constant})$$

in which

$$\oint \mathrm{d}s = 2 \times 300 + 2 \times 250 + 2 \times 100 + 2 \times 75 = 1450\,\text{mm}$$

i.e.

$$q_{s,0} = -\frac{2S_y}{1450 I_{xx}}(-130\,000 \times 250 - 195\,000 \times 100 + 48\,750 \times 75)$$

from which

$$q_{s,0} = 66\,681 S_y/I_{xx}$$

Then

$$q_{12} = 66\,681 S_y/I_{xx}$$
$$q_{23} = -63\,319 S_y/I_{xx}$$
$$q_{34} = -128\,319 S_y/I_{xx}$$
$$q_{16} = -115\,431 S_y/I_{xx}$$

Therefore the deflection, Δ_S, due to shear is, from the second term in Eq. (i)

$$\Delta_S = \int_L \left(\int_{\text{section}} \frac{q_0 q_1}{Gt} \,\mathrm{d}s \right) \mathrm{d}z$$

i.e.

$$\Delta_S = \int_L \left\{ 2\frac{S_{y,0} S_{y,1}}{Gt I_{xx}^2}(115\,431^2 \times 75 + 66\,681^2 \times 300 + 63\,319^2 \times 250 + 128\,319^2 \times 100) \right\} \mathrm{d}z$$

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Thus

$$\Delta_S = \int_L 2\frac{S_{y,0}S_{y,1} \times 4.98 \times 10^{12}}{26\,700 \times 2.5 \times (46.3 \times 10^6)^2}\,dz = \int_L 6.96 \times 10^{-8} S_{y,0}S_{y,1}\,dz$$

Then

$$\Delta_S = \int_0^{1250} 6.96 \times 10^{-8} \times 4450 \times 1\,dz + \int_{1250}^{1500} 6.96 \times 10^{-8} \times 22\,250 \times 5\,dz$$

from which

$$\Delta_S = 2.32\,\text{mm}$$

The total deflection, Δ, is then

$$\Delta = \Delta_M + \Delta_S = 1.09 + 2.32 = 3.41\,\text{mm}$$

S.9.29

At any section of the beam the applied loading is equivalent to bending moments in vertical and horizontal planes, to vertical and horizontal shear forces through the shear centre (the centre of symmetry C) plus a torque. However, only the vertical deflection of A is required so that the bending moments and shear forces in the horizontal plane do not contribute directly to this deflection. The total deflection is, from Eqs (9.83), (9.86) and (9.88)

$$\Delta = \int_L \frac{T_0 T_1}{GJ}\,dz + \int_L \frac{M_{x,1}M_{x,0}}{EI_{xx}}\,dz + \int_L \left(\int_{\text{section}} \frac{q_0 q_1}{Gt}\,ds\right)dz \qquad \text{(i)}$$

Referring to Fig. S.9.29 the vertical force/unit length on the beam is

$$1.2p_0\frac{c}{2} + p_0\frac{c}{2} + 0.8p_0\frac{c}{2} - p_0\frac{c}{2} = p_0 c \quad \text{(upwards)}$$

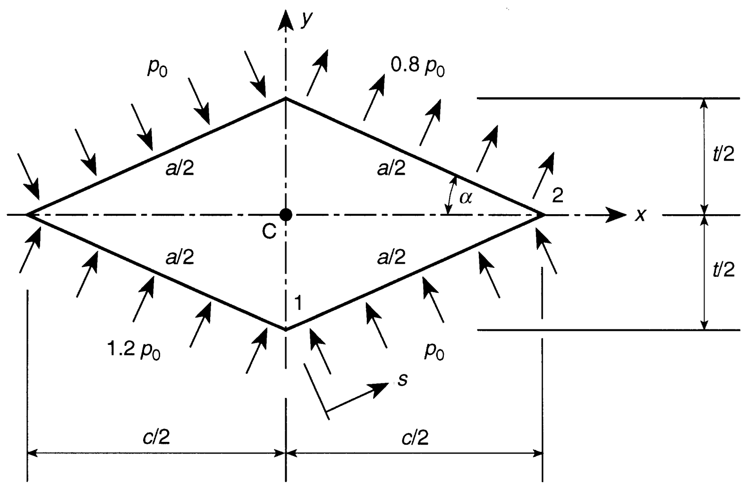

Fig. S.9.29

acting at a distance of $0.2c$ to the right of the vertical axis of symmetry. Also the horizontal force/unit length on the beam is

$$1.2p_0\frac{t}{2} + p_0\frac{t}{2} + 0.8p_0\frac{t}{2} - p_0\frac{t}{2} = p_0 t$$

acting to the right and at a distance $0.2t$ above the horizontal axis of symmetry. Thus, the torque/unit length on the beam is

$$p_0 c \times 0.2c - p_0 t \times 0.2t = 0.2p_0(c^2 - t^2)$$

acting in an anticlockwise sense. Then, at any section, a distance z from the built-in end of the beam

$$T_0 = 0.2p_0(c^2 - t^2)(L - z), \quad T_1 = -1\frac{c}{2} \quad \text{(unit load acting upwards at } A\text{)}$$

Comparing Eqs (3.12) and (9.52)

$$J = \frac{4A^2}{\oint \dfrac{ds}{t}}$$

i.e.

$$J = 4\left(2\frac{t}{2}\frac{c}{2}\right)^2 \bigg/ \frac{2a}{t_0} = \frac{t^2c^2t_0}{2a}$$

Then

$$\int_0^L \frac{T_0T_1}{GJ}\,dz = -\int_0^L \frac{0.1p_0(c^2 - t^2)c}{Gt^2c^2t_0/2a}(L - z)\,dz = \frac{0.1p_0aL^2(t^2 - c^2)}{Gt^2t_0c} \qquad \text{(ii)}$$

The bending moment due to the applied loading at any section a distance z from the built-in end is given by

$$M_{x,0} = -\frac{p_0c}{2}(L - z)^2, \quad \text{also } M_{x,1} = -1(L - z)$$

Thus

$$\int_0^L \frac{M_{x,1}M_{x,0}}{EI_{xx}}\,dz = \frac{p_0c}{2EI_{xx}}\int_0^L (L - z)^3\,dz$$

in which

$$I_{xx} = 2\frac{(a)^3t_0\sin^2\alpha}{12} = \frac{a^3t_0}{6}\left(\frac{t/2}{a/2}\right)^2 = \frac{at^2t_0}{6}$$

Then

$$\int_0^L \frac{M_{x,1}M_{x,0}}{EI_{xx}}\,dz = \frac{3p_0c}{Eat^2t_0}\left[-\frac{1}{4}(L - z)^4\right]_0^L = \frac{3p_0cL^4}{4Eat^2t_0} \qquad \text{(iii)}$$

The shear load at any section a distance z from the built-in end produced by the actual loading system is given by

$$S_{y,0} = p_0c(L - z), \quad \text{also } S_{y,1} = 1$$

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From Eq. (9.35), in which $I_{xy} = 0$ and $S_x = 0$

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s + q_{s,0} \qquad \text{(iv)}$$

If the origin of s is taken at the point 1, $q_{s,0} = 0$ since the shear load is applied on the vertical axis of symmetry, Eq. (iv) then becomes

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s$$

and

$$q_{12} = -\frac{6S_y}{at^2 t_0}\int_0^s t_0\left(-\frac{t}{2} + s\sin\alpha\right)\mathrm{d}s$$

i.e.

$$q_{12} = \frac{6S_y}{at^2}\left(\frac{t}{2}s - \frac{t}{a}\frac{s^2}{2}\right)$$

Thus

$$q_{12} = \frac{3S_y}{at}\left(s - \frac{s^2}{a}\right)$$

The remaining distribution follows from symmetry. Then

$$\int_{\text{section}}\frac{q_0 q_1}{Gt}\,\mathrm{d}s = 4 \times \frac{9p_0 c(L-z)}{Ga^2t^2t_0}\int_0^{a/2}\left(s - \frac{s^2}{a}\right)^2\mathrm{d}s$$

i.e.

$$\int_{\text{section}}\frac{q_0 q_1}{Gt}\,\mathrm{d}s = \frac{3p_0 ca(L-z)}{5Gt^2 t_0}$$

Then

$$\int_0^L\left(\int_{\text{section}}\frac{q_0 q_1}{Gt}\,\mathrm{d}s\right)\mathrm{d}z = \frac{3p_0 ca}{5Gt^2t_0}\int_0^L (L-z)\,\mathrm{d}z = \frac{3p_0 caL^2}{10Gt^2t_0} \qquad \text{(v)}$$

Now substituting in Eq. (i) from Eqs (ii), (iii) and (v)

$$\Delta = \frac{0.1p_0 aL^2(t^2 - c^2)}{Gt^2t_0c} + \frac{3p_0cL^4}{4Eat^2t_0} + \frac{3p_0caL^2}{10Gt^2t_0}$$

i.e.

$$\Delta = \frac{p_0L^2}{t^2t_0}\left[\frac{a(t^2-c^2)}{10Gc} + \frac{3cL^2}{4Ea} + \frac{3ca}{10G}\right]$$

Substituting the given values and taking $a \simeq c$

$$\Delta = \frac{p_0(2c)^2}{(0.05c)^2t_0}\left[\frac{c[(0.05c)^2 - c^2]}{4E} + \frac{3c(2c)^2}{4Ec} + \frac{3c^2}{4E}\right]$$

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Neglecting the term $(0.05c)^2$ in $[(0.05c)^2 - c^2]$ gives

$$\Delta = \frac{5600 p_0 c^2}{E t_0}$$

S.9.30

The pressure loading is equivalent to a shear force/unit length of $3bp_0/2$ acting in the vertical plane of symmetry together with a torque $= 3bp_0(3b/2 - b)/2 = 3b^2p_0/4$ as shown in Fig. S.9.30. The deflection of the beam is then, from Eqs (9.83), (9.86) and (9.88)

$$\Delta = \int_L \frac{T_0 T_1}{GJ}\,\mathrm{d}z + \int_L \frac{M_{x,1} M_{x,0}}{EI_{xx}}\,\mathrm{d}z + \int_L \left(\int_{\text{section}} \frac{q_0 q_1}{Gt}\,\mathrm{d}s \right) \mathrm{d}z \qquad \text{(i)}$$

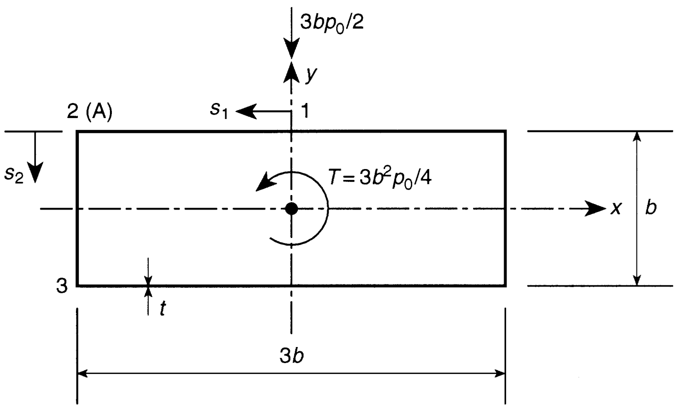

Fig. S.9.30

Now

$$T_0 = 3b^2 p_0 (L - z)/4, \quad T_1 = 3b/2$$

Also, from Eqs (3.12) and (9.52)

$$J = \frac{4A^2}{\oint \mathrm{d}s/t} = \frac{4(3b^2)^2}{8b/t} = \frac{9b^3 t}{2}$$

Thus

$$\int_L \frac{T_0 T_1}{GJ}\,\mathrm{d}z = \int_0^L \frac{p_0}{4Gt}(L - z)\,\mathrm{d}z = \frac{p_0 L^2}{8Gt} \qquad \text{(ii)}$$

Also

$$M_{x,0} = 3bp_0(L - z)^2/4, \quad M_{x,1} = 1(L - z)$$

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Then

$$\int_L \frac{M_{x,1}M_{x,0}}{EI_{xx}}\,\mathrm{d}z = \int_0^L \frac{3bp_0}{4EI_{xx}}(L-z)^3\,\mathrm{d}z$$

in which

$$I_{xx} = 2 \times 3bt \times (b/2)^2 + 2tb^3/12 = 5b^3t/3$$

Thus

$$\int_L \frac{M_{x,1}M_{x,0}}{EI_{xx}}\,\mathrm{d}z = \frac{9p_0}{20Eb^2t}\int_0^L (L-z)^3\,\mathrm{d}z = \frac{9p_0L^4}{80Eb^2t} \tag{iii}$$

Further

$$S_{y,0} = -\frac{3bp_0}{2}(L-z), \quad S_{y,1} = -1$$

Taking the origin for s at 1 in the plane of symmetry where $q_{s,0} = 0$ and since $I_{xy} = 0$ and $S_x = 0$, Eq. (9.35) simplifies to

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s$$

Then

$$q_{12} = -\frac{3S_y}{5b^3t}\int_0^{s_1} t\left(\frac{b}{2}\right)\mathrm{d}s_1$$

i.e.

$$q_{12} = -\frac{3S_y}{10b^2}s_1$$

from which

$$q_2 = -\frac{9S_y}{20b}$$

Also

$$q_{23} = -\frac{S_y}{I_{xx}}\int_0^{s_2} t\left(\frac{b}{2} - s_2\right)\mathrm{d}s_2 - \frac{9S_y}{20b}$$

i.e.

$$q_{23} = -\frac{3S_y}{5b^3}\left(\frac{b}{2}s_2 - \frac{s_2^2}{2}\right) - \frac{9S_y}{20b}$$

Hence

$$q_{23} = -\frac{3S_y}{20b}\left(2\frac{s_2}{b} - 2\frac{s_2^2}{b^2} + 3\right)$$

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Then

$$\int_{\text{section}} \frac{q_0 q_1}{Gt}\,\mathrm{d}s = 4\int_0^{3b/2} \frac{3bp_0(L-z)}{2Gt}\left(\frac{3}{10b^2}\right)^2 s_1^2\,\mathrm{d}s_1$$

$$+\,2\int_{3b/2}^{b} \frac{3bp_0(L-z)}{2Gt}\left(\frac{3}{20}\right)^2\left(2\frac{s_2}{b}-2\frac{s_2^2}{b^2}+3\right)^2\mathrm{d}s_2$$

which gives

$$\int_{\text{section}} \frac{q_0 q_1}{Gt}\,\mathrm{d}s = \frac{1359p_0}{1000Gt}(L-z)$$

Hence

$$\int_0^L\left(\int_{\text{section}} \frac{q_0 q_1}{Gt}\,\mathrm{d}s\right)\mathrm{d}z = \frac{1359p_0}{1000Gt}\int_0^L (L-z)\,\mathrm{d}z = \frac{1359p_0L^2}{2000Gt} \qquad \text{(iv)}$$

Substituting in Eq. (i) from Eqs (ii), (iii) and (iv) gives

$$\Delta = \frac{p_0L^2}{8Gt} + \frac{9p_0L^4}{80Eb^2t} + \frac{1359p_0L^2}{2000Gt}$$

Thus

$$\Delta = \frac{p_0L^2}{t}\left(\frac{9L^2}{80Eb^2} + \frac{1609}{2000G}\right)$$

Solutions to Chapter 10 Problems

S.10.1

Referring to Fig. P.10.1 the bending moment at section 1 is given by

$$M_1 = \frac{15 \times 1^2}{2} = 7.5\,\text{kNm}$$

Thus

$$P_{z,\text{U}} = -P_{z,\text{L}} = \frac{7.5}{300 \times 10^{-3}} = 25\,\text{kN}$$

Also

$$P_{y,\text{U}} = 0 \quad \text{and} \quad P_{y,\text{L}} = -25 \times \frac{100}{1 \times 10^3} = -2.5\,\text{kN} \quad \text{(see Eqs (10.1))}$$

Then

$$P_\text{U} = \sqrt{P_{z,\text{U}}^2 + P_{y,\text{U}}^2} = 25\,\text{kN} \quad \text{(tension)}$$

$$P_\text{L} = -\sqrt{25^2 + 2.5^2} = -25.1\,\text{kN} \quad \text{(compression)}$$

The shear force at section 1 is $15 \times 1 = 15\,\text{kN}$. This is resisted by $P_{y,\text{L}}$, the shear force in the web. Thus

$$\text{shear in web} = 15 - 2.5 = 12.5\,\text{kN}$$

Hence

$$q = \frac{12.5 \times 10^3}{300} = 41.7\,\text{N/mm}$$

At section 2 the bending moment is

$$M_2 = \frac{15 \times 2^2}{2} = 30\,\text{kNm}$$

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Hence

$$P_{z,\mathrm{U}} = -P_{z,\mathrm{L}} = \frac{30}{400 \times 10^{-3}} = 75\,\mathrm{kN}$$

Also

$$P_{y,\mathrm{U}} = 0 \quad \text{and} \quad P_{y,\mathrm{L}} = -75 \times \frac{200}{2 \times 10^3} = -7.5\,\mathrm{kN}$$

Then

$$P_\mathrm{U} = 75\,\mathrm{kN} \quad \text{(tension)}$$

and

$$P_\mathrm{L} = -\sqrt{75^2 + 7.5^2} = -75.4\,\mathrm{kN} \quad \text{(compression)}$$

The shear force at section 2 is $= 15 \times 2 = 30\,\mathrm{kN}$. Hence the shear force in the web $= 30 - 7.5 = 22.5\,\mathrm{kN}$ which gives

$$q = \frac{22.5 \times 10^3}{400} = 56.3\,\mathrm{N/mm}$$

S.10.2

The bending moment at section 1 is given by

$$M = \frac{15 \times 1^2}{2} = 7.5\,\mathrm{kNm}$$

The second moment of area of the beam cross-section at section 1 is

$$I_{xx} = 2 \times 500 \times 150^2 + \frac{2 \times 300^3}{12} = 2.7 \times 10^7\,\mathrm{mm}^4$$

The direct stresses in the flanges in the z direction are, from Eq. (9.6)

$$\sigma_{z,\mathrm{U}} = -\sigma_{z,\mathrm{L}} = \frac{7.5 \times 10^6 \times 150}{2.7 \times 10^7} = 41.7\,\mathrm{N/mm}^2$$

Then

$$P_{z,\mathrm{U}} = 41.7 \times 500 = 20\,850\,\mathrm{N} = P_\mathrm{U} \quad \text{(tension)}$$

Also

$$P_{z,\mathrm{L}} = -20\,850\,\mathrm{N} \quad \text{(compression)}$$

Hence

$$P_{y,\mathrm{L}} = -20\,850 \times \frac{100}{1 \times 10^3} = -2085\,\mathrm{N} \quad \text{(compression)}$$

Therefore, the shear force in the web at section 1 is given by

$$S_y = -15 \times 1 \times 10^3 + 2085 = -12\,915\,\mathrm{N}$$

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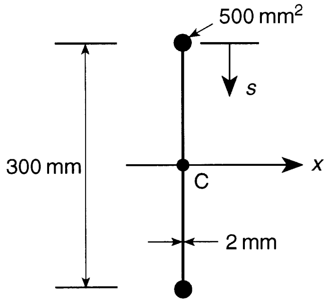

Fig. S.10.2

The shear flow distribution is obtained using Eq. (10.6). Thus, referring to Fig. S.10.2

$$q = \frac{12\,915}{2.7 \times 10^7}\left[\int_0^s 2(150 - s)\,\mathrm{d}s + 500 \times 150\right]$$

Hence

$$q = 4.8 \times 10^{-4}(300s - s^2 + 75\,000)$$

The maximum value of q occurs when $s = 150$ mm, i.e.

$$q_{\max} = 46.8\,\text{N/mm}$$

S.10.3

The beam section at a distance of 1.5 m from the built-in end is shown in Fig. S.10.3. The bending moment, M, at this section is given by

$$M = -40 \times 1.5 = -60\,\text{kNm}$$

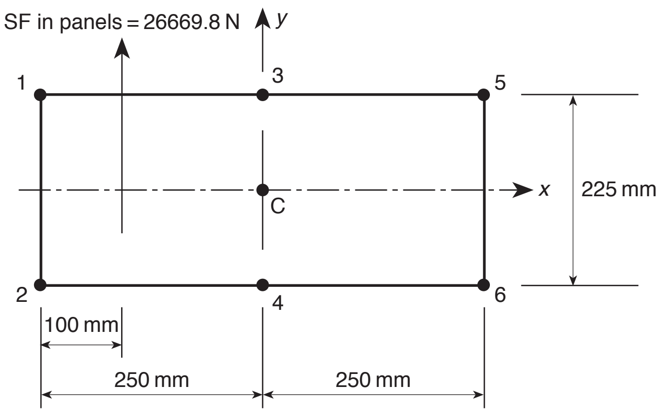

Fig. S.10.3

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Since the x axis is an axis of symmetry $I_{xy} = 0$; also $M_y = 0$. The direct stress distribution is then, from Eq. (9.6)

$$\sigma_z = \frac{M_x}{I_{xx}} y \qquad \text{(i)}$$

in which $I_{xx} = 2 \times 1000 \times 112.5^2 + 4 \times 500 \times 112.5^2 = 50.63 \times 10^6\,\text{mm}^4$. Then, from Eq. (i), the direct stresses in the flanges and stringers are

$$\sigma_z = \pm \frac{60 \times 10^6 \times 112.5}{50.63 \times 10^6} = \pm 133.3\,\text{N/mm}^2$$

Therefore

$$P_{z,1} = -P_{z,2} = -133.3 \times 1000 = -133\,300\,\text{N}$$

and

$$P_{z,3} = P_{z,5} = -P_{z,4} = -P_{z,6} = -133.3 \times 500 = -66\,650\,\text{N}$$

From Eq. (10.9)

$$P_{y,1} = P_{y,2} = 133\,300 \times \frac{75}{3 \times 10^3} = 3332.5\,\text{N}$$

and

$$P_{y,3} = P_{y,4} = P_{y,5} = P_{y,6} = 66\,650 \times \frac{75}{3 \times 10^3} = 1666.3\,\text{N}$$

Thus the total vertical load in the flanges and stringers is

$$2 \times 3332.5 + 4 \times 1666.3 = 13\,330.2\,\text{N}$$

Hence the total shear force carried by the panels is

$$40 \times 10^3 - 13\,330.2 = 26\,669.8\,\text{N}$$

The shear flow distribution is given by Eq. (9.80) which, since $I_{xy} = 0$, $S_x = 0$ and $t_D = 0$ reduces to

$$q_s = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r + q_{s,0}$$

i.e.

$$q_s = -\frac{26\,669.8}{50.63 \times 10^6} \sum_{r=1}^{n} B_r y_r + q_{s,0}$$

or

$$q_s = -5.27 \times 10^{-4} \sum_{r=1}^{n} B_r y_r + q_{s,0} \qquad \text{(ii)}$$

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From Eq. (ii)

$$q_{b,13} = 0$$

$$q_{b,35} = -5.27 \times 10^{-4} \times 500 \times 112.5 = -29.6\,\text{N/mm}$$

$$q_{b,56} = -29.6 - 5.27 \times 10^{-4} \times 500 \times 112.5 = -59.2\,\text{N/mm}$$

$$q_{b,12} = -5.27 \times 10^{-4} \times 1000 \times 112.5 = -59.3\,\text{N/mm}$$

The remaining distribution follows from symmetry. Now taking moments about the point 2 (see Eq. (9.37))

$$26\,669.8 \times 100 = 59.2 \times 225 \times 500 + 29.6 \times 250 \times 225 + 2 \times 500 \times 225 q_{s,0}$$

from which

$$q_{s,0} = -36.9\,\text{N/mm} \quad \text{(i.e. clockwise)}$$

Then

$$q_{13} = 36.9\,\text{N/mm} = q_{42}$$

$$q_{35} = 36.9 - 29.6 = 7.3\,\text{N/mm} = q_{64}$$

$$q_{65} = 59.2 - 36.9 = 22.3\,\text{N/mm}$$

$$q_{21} = 36.9 + 59.3 = 96.2\,\text{N/mm}$$

Finally

$$P_1 = -\sqrt{P_{z,1}^2 + P_{y,1}^2} = -\left(\sqrt{133\,300^2 + 3332.5^2}\right) \times 10^{-3} = -133.3\,\text{kN} = -P_2$$

$$P_3 = -\sqrt{P_{z,3}^2 + P_{y,3}^2} = -\left(\sqrt{66\,650^2 + 1666.3^2}\right) \times 10^{-3}$$
$$= -66.7\,\text{kN} = P_5 = -P_4 = -P_6$$

S.10.4

The direct stresses in the booms are obtained from Eq. (9.6) in which $I_{xy} = 0$ and $M_y = 0$. Thus

$$\sigma_z = \frac{M_x}{I_{xx}} y \qquad \text{(i)}$$

From Fig. P.10.4 the y coordinates of the booms are:

$$y_1 = -y_6 = 750\,\text{mm}$$

$$y_2 = y_{10} = -y_5 = -y_7 = 250 + 500 \sin 45^\circ = 603.6\,\text{mm}$$

$$y_3 = y_9 = -y_4 = -y_8 = 250\,\text{mm}$$

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Then $I_{xx} = 2 \times 150(750^2 + 2 \times 603.6^2 + 2 \times 250^2) = 4.25 \times 10^8\,\text{mm}^4$. Hence, from Eq. (i)

$$\sigma_z = \frac{100 \times 10^6}{4.25 \times 10^8} y$$

i.e.

$$\sigma_z = 0.24y$$

Thus

Boom	1	2, 10	3, 9	4, 8	5, 7	6
$\sigma_z(\text{N/mm}^2)$	180.0	144.9	60.0	−60.0	−144.9	−180.0

From Eq. (9.80)

$$q_s = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r + q_{s,0}$$

i.e.

$$q_s = -\frac{50 \times 10^3 \times 150}{4.25 \times 10^8} \sum y_r + q_{s,0}$$

so that

$$q_s = -0.018 y_r + q_{s,0} \tag{ii}$$

'Cut' the wall 89. Then, from the first term on the right-hand side of Eq. (ii)

$$q_{b,89} = 0$$

$$q_{b,9\,10} = -0.018 \times 250 = -4.5\,\text{N/mm}$$

$$q_{b,10\,1} = -4.5 - 0.018 \times 603.6 = -15.4\,\text{N/mm}$$

$$q_{b,12} = -15.4 - 0.018 \times 750 = -28.9\,\text{N/mm}$$

$$q_{b,23} = -28.9 - 0.018 \times 603.6 = -39.8\,\text{N/mm}$$

$$q_{b,34} = -39.8 - 0.018 \times 250 = -44.3\,\text{N/mm}$$

The remaining q_b distribution follows from symmetry and the complete distribution is shown in Fig. S.10.4. The moment of a constant shear flow in a panel about a specific point is given by Eq. (9.79). Thus, taking moments about C (see Eq. (9.37))

$$50 \times 10^3 \times 250 = 2(-2 \times 4.5 A_{9\,10} - 2 \times 15.4 A_{10\,1} - 2 \times 28.9 A_{12} - 2 \times 39.8 A_{23}$$
$$- 2 \times 44.3 A_{34}) - 2Aq_{s,0} \tag{iii}$$

in which

$$A_{34} = \tfrac{1}{2} \times 500 \times 250 = 62\,500\,\text{mm}^2$$

$$A_{23} = A_{9\,10} = 62\,500 + \tfrac{45}{360} \times \pi \times 500^2 - \tfrac{1}{2} \times 250 \times 353.6 = 116\,474.8\,\text{mm}^2$$

$$A_{12} = A_{10\,1} = \tfrac{1}{2} \times 250 \times 353.6 + \tfrac{45}{360} \times \pi \times 500^2 = 142\,374.8\,\text{mm}^2$$

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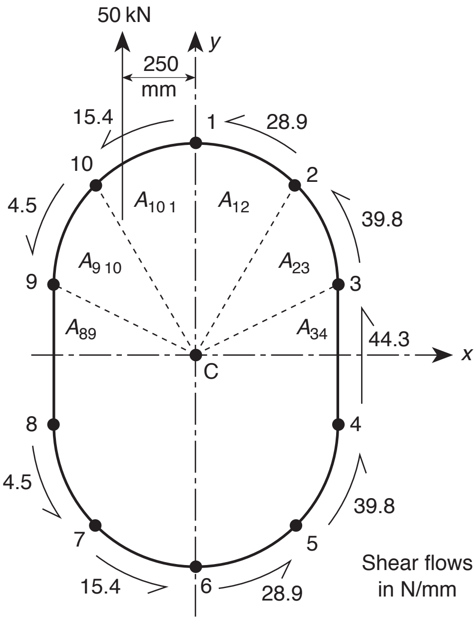

Fig. S.10.4

Also the total area, A, of the cross-section is

$$A = 500 \times 1000 + \pi \times 500^2 = 1\,285\,398.2\,\text{mm}^2$$

Eq. (iii) then becomes

$$50 \times 10^3 \times 250 = -2 \times 2(4.5 \times 116\,474.8 + 15.4 \times 142\,374.8 + 28.9 \times 142\,374.8 + 39.8 \times 116\,474.8 + 44.3 \times 62\,500) - 2 \times 1\,285\,398.2 q_{s,0}$$

from which

$$q_{s,0} = -27.0\,\text{N/mm} \quad \text{(clockwise)}$$

Then

$$q_{89} = 27.0\,\text{N/mm}, \quad q_{9\,10} = q_{78} = 22.5\,\text{N/mm}, \quad q_{10\,1} = q_{67} = 11.6\,\text{N/mm},$$
$$q_{21} = q_{65} = 1.9\,\text{N/mm}, \quad q_{32} = q_{54} = 12.8\,\text{N/mm}, \quad q_{43} = 17.3\,\text{N/mm}$$

S.10.6

The beam section is unsymmetrical and $M_x = -120\,000\,\text{Nm}$, $M_y = -30\,000\,\text{Nm}$. Therefore, the direct stresses in the booms are given by Eq. (9.6), i.e.

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) y \tag{i}$$

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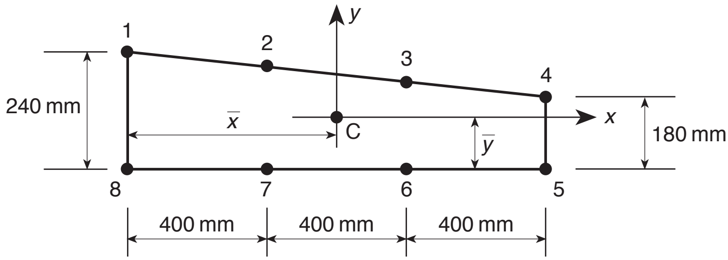

Fig. S.10.6

In Fig. S.10.6 $\bar{x} = 600\,\text{mm}$ by inspection. Also, taking moments of area about the line of the bottom booms

$$(4 \times 1000 + 4 \times 600)\bar{y} = 1000 \times 240 + 1000 \times 180 + 600 \times 220 + 600 \times 200$$

from which

$$\bar{y} = 105\,\text{mm}$$

Then

$$I_{xx} = 2 \times 1000 \times 105^2 + 2 \times 600 \times 105^2 + 1000 \times 135^2 + 1000 \times 75^2 + 600 \times 115^2 + 600 \times 95^2 = 72.5 \times 10^6\,\text{mm}^4$$

$$I_{yy} = 4 \times 1000 \times 600^2 + 4 \times 600 \times 200^2 = 1536.0 \times 10^6\,\text{mm}^4$$

$$I_{xy} = 1000[(-600)(135) + (600)(75)] + 600[(-200)(115) + (200)(95)]$$

$$= -38.4 \times 10^6\,\text{mm}^4$$

Note that the sum of the contributions of booms 5, 6, 7 and 8 to I_{xy} is zero. Substituting for M_x, M_y, I_{xx} etc in Eq. (i) gives

$$\sigma_z = -0.062x - 1.688y \tag{ii}$$

The solution is completed in Table S.10.6.

Table S.10.6

Boom	1	2	3	4	5	6	7	8
x (mm)	−600	−200	200	600	600	200	−200	−600
y (mm)	135	115	95	75	−105	−105	−105	−105
σ_z (N/mm^2)	−190.7	−181.7	−172.8	−163.8	140.0	164.8	189.6	214.4

S.10.7

From Eq. (10.24) for Cell I

$$\frac{d\theta}{dz} = \frac{1}{2A_{\text{I}}G}[q_{\text{I}}(\delta_{21} + \delta_{16} + \delta_{65} + \delta_{52}) - q_{\text{II}}\delta_{52}] \tag{i}$$

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and for Cell II

$$\frac{d\theta}{dz} = \frac{1}{2A_{II}G}[-q_I\delta_{52} + q_{II}(\delta_{32} + \delta_{25} + \delta_{54} + \delta_{43})] \qquad \text{(ii)}$$

In Eqs (i) and (ii)

$$A_I = 7750 + (250 + 600) \times 500/2 = 220\,250\,\text{mm}^2$$

$$A_{II} = 6450 + (150 + 600) \times 920/2 = 351\,450\,\text{mm}^2$$

$$\delta_{21} = \left(\sqrt{250^2 + 500^2}\right)/1.63 = 343.0, \quad \delta_{16} = 300/2.03 = 147.8$$

$$\delta_{65} = \left(\sqrt{100^2 + 500^2}\right)/0.92 = 554.2, \quad \delta_{52} = 600/2.54 = 236.2$$

$$\delta_{54} = \left(\sqrt{250^2 + 920^2}\right)/0.92 = 1036.3, \quad \delta_{43} = 250/0.56 = 446.4$$

$$\delta_{32} = \left(\sqrt{200^2 + 920^2}\right)/0.92 = 1023.4$$

Substituting these values in Eqs (i) and (ii) gives, for Cell I

$$\frac{d\theta}{dz} = \frac{1}{2 \times 220\,250G}(1281.2q_I - 236.2q_{II}) \qquad \text{(iii)}$$

and for Cell II

$$\frac{d\theta}{dz} = \frac{1}{2 \times 351\,450G}(-236.2q_I + 2742.3q_{II}) \qquad \text{(iv)}$$

Equating Eqs (iii) and (iv) gives

$$q_{II} = 0.73q_I \qquad \text{(v)}$$

Then, in Cell I

$$\tau_{max} = \tau_{65} = \frac{q_I}{0.92} = 1.087q_I$$

and in Cell II

$$\tau_{max} = \frac{q_{II}}{0.56} = 1.304q_I$$

In the wall 52

$$\tau_{52} = \frac{q_I - q_{II}}{2.54} = 0.106q_I$$

Therefore

$$\tau_{max} = 1.304q_I = 140\,\text{N/mm}^2$$

which gives

$$q_I = 107.4\,\text{N/mm}$$

and, from Eq. (v)

$$q_{II} = 78.4\,\text{N/mm}$$

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Substituting for q_{I} and q_{II} in Eq. (10.22)

$$T = (2 \times 220\,250 \times 107.4 + 2 \times 351\,450 \times 78.4) \times 10^{-3}$$

i.e.

$$T = 102\,417\,\mathrm{Nm}$$

From Eq. (iii) (or Eq. (iv))

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2 \times 220\,250 \times 26\,600}(1281.2 \times 107.4 - 236.2 \times 78.4)$$

i.e.

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = 1.02 \times 10^{-5}\,\mathrm{rad/mm}$$

Hence

$$\theta = 1.02 \times 10^{-5} \times 2500 \times (180/\pi) = 1.46^\circ$$

The torsional stiffness is obtained from Eq. (3.12), thus

$$GJ = T/(\mathrm{d}\theta/\mathrm{d}z) = 102\,417 \times 10^3/(1.02 \times 10^{-5}) = 10 \times 10^{12}\,\mathrm{Nmm^2/rad}$$

S.10.8

From Eq. (10.24) for Cell I

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{I}}G}[q_{\mathrm{I}}(\delta_{45^{\mathrm{o}}} + \delta_{45^{\mathrm{i}}}) - q_{\mathrm{II}}\delta_{45^{\mathrm{i}}}] \qquad \text{(i)}$$

For Cell II

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{II}}G}[-q_{\mathrm{I}}\delta_{45^{\mathrm{i}}} + q_{\mathrm{II}}(\delta_{34} + \delta_{45^{\mathrm{i}}} + \delta_{56} + \delta_{63}) - q_{\mathrm{III}}\delta_{63}] \qquad \text{(ii)}$$

For Cell III

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{III}}G}[-q_{\mathrm{II}}\delta_{63} + q_{\mathrm{III}}(\delta_{23} + \delta_{36} + \delta_{67} + \delta_{72}) - q_{\mathrm{IV}}\delta_{72}] \qquad \text{(iii)}$$

For Cell IV

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{IV}}G}[-q_{\mathrm{III}}\delta_{72} + q_{\mathrm{IV}}(\delta_{27} + \delta_{78} + \delta_{81} + \delta_{12})] \qquad \text{(iv)}$$

where

$$\delta_{12} = \delta_{78} = 762/0.915 = 832.8, \quad \delta_{23} = \delta_{67} = \delta_{34} = \delta_{56} = 812/0.915 = 887.4$$

$$\delta_{45^{\mathrm{i}}} = 356/1.220 = 291.8, \quad \delta_{45^{\mathrm{o}}} = 1525/0.711 = 2144.9$$

$$\delta_{36} = 406/1.625 = 249.8, \quad \delta_{72} = 356/1.22 = 291.8, \quad \delta_{81} = 254/0.915 = 277.6$$

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Substituting these values in Eqs (i), (ii), (iii) and (iv)

$$\frac{d\theta}{dz} = \frac{1}{2 \times 161\,500G}(2436.7q_{\mathrm{I}} - 291.8q_{\mathrm{II}}) \tag{v}$$

$$\frac{d\theta}{dz} = \frac{1}{2 \times 291\,000G}(-291.8q_{\mathrm{I}} + 2316.4q_{\mathrm{II}} - 249.8q_{\mathrm{III}}) \tag{vi}$$

$$\frac{d\theta}{dz} = \frac{1}{2 \times 291\,000G}(-249.8q_{\mathrm{II}} + 2316.4q_{\mathrm{III}} - 291.8q_{\mathrm{IV}}) \tag{vii}$$

$$\frac{d\theta}{dz} = \frac{1}{2 \times 226\,000G}(-291.8q_{\mathrm{III}} + 2235.0q_{\mathrm{IV}}) \tag{viii}$$

Also, from Eq. (10.22)

$$T = 2(161\,500q_{\mathrm{I}} + 291\,000q_{\mathrm{II}} + 291\,000q_{\mathrm{III}} + 226\,000q_{\mathrm{IV}}) \tag{ix}$$

Equating Eqs (v) and (vi)

$$q_{\mathrm{I}} - 0.607q_{\mathrm{II}} + 0.053q_{\mathrm{III}} = 0 \tag{x}$$

Now equating Eqs (v) and (vii)

$$q_{\mathrm{I}} - 0.063q_{\mathrm{II}} - 0.528q_{\mathrm{III}} + 0.066q_{\mathrm{IV}} = 0 \tag{xi}$$

Equating Eqs (v) and (viii)

$$q_{\mathrm{I}} - 0.120q_{\mathrm{II}} + 0.089q_{\mathrm{III}} - 0.655q_{\mathrm{IV}} = 0 \tag{xii}$$

From Eq. (ix)

$$q_{\mathrm{I}} + 1.802q_{\mathrm{II}} + 1.802q_{\mathrm{III}} + 1.399q_{\mathrm{IV}} = 3.096 \times 10^{-6}T \tag{xiii}$$

Subtracting Eq. (xi) from Eq. (x)

$$q_{\mathrm{II}} - 1.068q_{\mathrm{III}} + 0.121q_{\mathrm{IV}} = 0 \tag{xiv}$$

Subtracting Eq. (xii) from Eq. (x)

$$q_{\mathrm{II}} + 0.074q_{\mathrm{III}} - 1.345q_{\mathrm{IV}} = 0 \tag{xv}$$

Subtracting Eq. (xiii) from Eq. (x)

$$q_{\mathrm{II}} + 0.726q_{\mathrm{III}} + 0.581q_{\mathrm{IV}} = 0 \tag{xvi}$$

Now subtracting Eq. (xv) from Eq. (xiv)

$$q_{\mathrm{III}} - 1.284q_{\mathrm{IV}} = 0 \tag{xvii}$$

Subtracting Eq. (xvi) from Eq. (xiv)

$$q_{\mathrm{III}} + 0.256q_{\mathrm{IV}} = 0.716 \times 10^{-6}T \tag{xviii}$$

Finally, subtracting Eq. (xviii) from Eq. (xvii)

$$q_{\mathrm{IV}} = 0.465 \times 10^{-6}T$$

and from Eq. (xvii)

$$q_{\mathrm{III}} = 0.597 \times 10^{-6}T$$

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Substituting for q_{III} and q_{IV} in Eq. (viii)

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1.914 \times 10^{-9} T}{G}$$

so that

$$T/(\mathrm{d}\theta/\mathrm{d}z) = 522.5 \times 10^6 G \, \mathrm{Nmm}^2/\mathrm{rad}$$

S.10.9

In this problem the cells are not connected consecutively so that Eq. (10.24) does not apply. Therefore, from Eq. (10.23) for Cell I

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{I}}G}[q_{\mathrm{I}}(\delta_{12^{\mathrm{U}}} + \delta_{23} + \delta_{34^{\mathrm{U}}} + \delta_{41}) - q_{\mathrm{II}}\delta_{34^{\mathrm{U}}} - q_{\mathrm{III}}(\delta_{23} + \delta_{41})] \tag{i}$$

For Cell II

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{II}}G}[-q_{\mathrm{I}}\delta_{34^{\mathrm{U}}} + q_{\mathrm{II}}(\delta_{34^{\mathrm{U}}} + \delta_{34^{\mathrm{L}}}) - q_{\mathrm{III}}\delta_{34^{\mathrm{L}}}] \tag{ii}$$

For Cell III

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{III}}G}[-q_{\mathrm{I}}(\delta_{23} + \delta_{41}) - q_{\mathrm{II}}\delta_{34^{\mathrm{L}}} + q_{\mathrm{III}}(\delta_{14} + \delta_{43^{\mathrm{L}}} + \delta_{32} + \delta_{21^{\mathrm{L}}})] \tag{iii}$$

In Eqs (i)–(iii)

$$\delta_{12^{\mathrm{U}}} = 1084/1.220 = 888.5, \quad \delta_{12^{\mathrm{L}}} = 2160/1.625 = 1329.2$$
$$\delta_{14} = \delta_{23} = 127/0.915 = 138.8, \quad \delta_{34^{\mathrm{U}}} = \delta_{34^{\mathrm{L}}} = 797/0.915 = 871.0$$

Substituting these values in Eqs (i)–(iii)

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2 \times 108\,400G}(2037.1q_{\mathrm{I}} - 871.0q_{\mathrm{II}} - 277.6q_{\mathrm{III}}) \tag{iv}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2 \times 202\,500G}(-871.0q_{\mathrm{I}} + 1742.0q_{\mathrm{II}} - 871.0q_{\mathrm{III}}) \tag{v}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2 \times 528\,000G}(-277.6q_{\mathrm{I}} - 871.0q_{\mathrm{II}} + 2477.8q_{\mathrm{III}}) \tag{vi}$$

Also, from Eq. (10.22)

$$565\,000 \times 10^3 = 2(108\,400q_{\mathrm{I}} + 202\,500q_{\mathrm{II}} + 528\,000q_{\mathrm{III}}) \tag{vii}$$

Equating Eqs (iv) and (v)

$$q_{\mathrm{I}} - 0.720q_{\mathrm{II}} + 0.075q_{\mathrm{III}} = 0 \tag{viii}$$

Equating Eqs (iv) and (vi)

$$q_{\mathrm{I}} - 0.331q_{\mathrm{II}} - 0.375q_{\mathrm{III}} = 0 \tag{ix}$$

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From Eq. (vii)

$$q_{\mathrm{I}} + 1.868q_{\mathrm{II}} + 4.871q_{\mathrm{III}} = 260.61 \tag{x}$$

Now subtracting Eq. (ix) from Eq. (viii)

$$q_{\mathrm{II}} - 1.157q_{\mathrm{III}} = 0 \tag{xi}$$

Subtracting Eq. (x) from Eq. (viii)

$$q_{\mathrm{II}} + 1.853q_{\mathrm{III}} = 100.70 \tag{xii}$$

Finally, subtracting Eq. (xii) from Eq. (xi)

$$q_{\mathrm{III}} = 33.5\,\mathrm{N/mm}$$

Then, from Eq. (xi)

$$q_{\mathrm{II}} = 38.8\,\mathrm{N/mm}$$

and from Eq. (ix)

$$q_{\mathrm{I}} = 25.4\,\mathrm{N/mm}$$

Thus

$$q_{12^{\mathrm{U}}} = 25.4\,\mathrm{N/mm}, \quad q_{21^{\mathrm{L}}} = 33.5\,\mathrm{N/mm}, \quad q_{14} = q_{32} = 33.5 - 25.4 = 8.1\,\mathrm{N/mm}$$

$$q_{43^{\mathrm{U}}} = 38.8 - 25.4 = 13.4\,\mathrm{N/mm}, \quad q_{34^{\mathrm{L}}} = 38.8 - 33.5 = 5.3\,\mathrm{N/mm}$$

S.10.10

In Eq. (10.28) the q_{b} shear flow distribution is given by Eq. (9.75) in which, since the x axis is an axis of symmetry (Fig. S.10.10), $I_{xy} = 0$; also $S_x = 0$. Thus

$$q_{\mathrm{b}} = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r \tag{i}$$

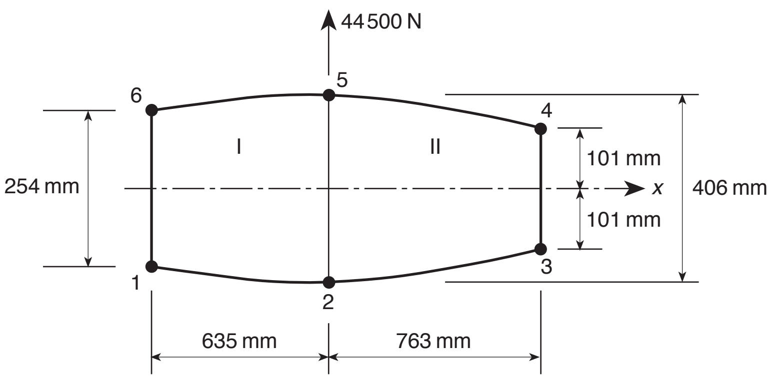

Fig. S.10.10

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in which

$$I_{xx} = 2 \times 1290 \times 127^2 + 2 \times 1936 \times 203^2 + 2 \times 645 \times 101^2 = 214.3 \times 10^6\,\text{mm}^4$$

Then Eq. (i) becomes

$$q_b = -\frac{44\,500}{214.3 \times 10^6}\sum_{r=1}^{n} B_r y_r = -2.08 \times 10^{-4}\sum_{r=1}^{n} B_r y_r$$

'Cut' the walls 65 and 54. Then

$$q_{b,65} = q_{b,54} = 0$$

$$q_{b,61} = -2.08 \times 10^{-4} \times 1290 \times 127 = -32.8\,\text{N/mm}$$

$$q_{b,12} = q_{b,23} = 0 \quad \text{(from symmetry)}$$

$$q_{b,25} = -2.08 \times 10^{-4} \times 1936(-203) = 81.7\,\text{N/mm}$$

$$q_{b,34} = -2.08 \times 10^{-4} \times 645(-101) = 13.6\,\text{N/mm}$$

From Eq. (10.28) for Cell I

$$\frac{d\theta}{dz} = \frac{1}{2A_I G}[q_{s,0,I}(\delta_{56} + \delta_{61} + \delta_{12} + \delta_{24}) - q_{s,0,II}\delta_{25} + q_{b,25}\delta_{25} + q_{b,61}\delta_{61}] \qquad \text{(ii)}$$

For Cell II

$$\frac{d\theta}{dz} = \frac{1}{2A_{II} G}[-q_{s,0,I}\delta_{25} + q_{s,0,II}(\delta_{45} + \delta_{52} + \delta_{23} + \delta_{34}) + q_{b,34}\delta_{34} + q_{b,52}\delta_{25}] \qquad \text{(iii)}$$

in which

$$\delta_{56} = \delta_{12} = 647/0.915 = 707.1, \quad \delta_{45} = \delta_{23} = 775/0.559 = 1386.4$$

$$\delta_{61} = 254/1.625 = 156.3, \quad \delta_{52} = 406/2.032 = 199.8, \quad \delta_{34} = 202/1.220 = 165.6$$

Substituting these values in Eqs (ii) and (iii)

$$\frac{d\theta}{dz} = \frac{1}{2 \times 232\,000G}(1770.3q_{s,0,I} - 199.8q_{s,0,II} + 11\,197.0) \qquad \text{(iv)}$$

$$\frac{d\theta}{dz} = \frac{1}{2 \times 258\,000G}(-199.8q_{s,0,I} + 3138.2q_{s,0,II} - 14\,071.5) \qquad \text{(v)}$$

Also, taking moments about the mid-point of the web 25 and from Eq. (10.39) (or Eq. (10.30))

$$0 = 13.6 \times 202 \times 763 - 32.8 \times 254 \times 635 + 2A_I q_{s,0,I} + 2A_{II} q_{s,0,II} \qquad \text{(vi)}$$

Equating Eqs (iv) and (v)

$$q_{s,0,I} - 1.55q_{s,0,II} + 12.23 = 0 \qquad \text{(vii)}$$

From Eq. (vi)

$$q_{s,0,I} + 1.11q_{s,0,II} - 6.88 = 0 \qquad \text{(viii)}$$

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Subtracting Eq. (viii) from Eq. (vii) gives

$$q_{s,0,\mathrm{II}} = 7.2\,\mathrm{N/mm}$$

Then, from Eq. (vii)

$$q_{s,0,\mathrm{I}} = -1.1\,\mathrm{N/mm}$$

Thus

$$q_{16} = 32.8 + 1.1 = 33.9\,\mathrm{N/mm}, \quad q_{65} = q_{21} = 1.1\,\mathrm{N/mm},$$
$$q_{45} = q_{23} = 7.2\,\mathrm{N/mm}, \quad q_{34} = 13.6 + 7.2 = 20.8\,\mathrm{N/mm},$$
$$q_{25} = 81.7 - 1.1 - 7.2 = 73.4\,\mathrm{N/mm}$$

S.10.11

Referring to Fig. P.10.11, the horizontal x axis is an axis of symmetry so that $I_{xy} = 0$. Further, $S_x = 0$ so that, from Eq. (9.75)

$$q_{\mathrm{b}} = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r \tag{i}$$

in which

$$I_{xx} = 4 \times 1290 \times 153^2 + 4 \times 645 \times 153^2 = 181.2 \times 10^6\,\mathrm{mm}^4$$

Eq. (i) then becomes

$$q_{\mathrm{b}} = -\frac{66\,750}{181.2 \times 10^6} \sum_{r=1}^{n} B_r y_r = -3.68 \times 10^{-4} \sum_{r=1}^{n} B_r y_r$$

Now, 'cutting' Cell I in the wall 45 and Cell II in the wall 12

$$q_{\mathrm{b},45} = 0 = q_{\mathrm{b},12}$$
$$q_{\mathrm{b},43} = -3.68 \times 10^{-4} \times 645 \times 153 = -36.3\,\mathrm{N/mm} = q_{\mathrm{b},65} \quad \text{(from symmetry)}$$
$$q_{\mathrm{b},18} = -3.68 \times 10^{-4} \times 1290 \times 153 = -72.6\,\mathrm{N/mm}$$
$$q_{\mathrm{b},78} = 0 \quad \text{(from symmetry)}$$
$$q_{\mathrm{b},76} = -3.68 \times 10^{-4} \times 645 \times (-153) = 36.3\,\mathrm{N/mm} = q_{\mathrm{b},32} \quad \text{(from symmetry)}$$
$$q_{\mathrm{b},63} = 36.3 + 36.3 - 3.68 \times 10^{-4} \times 1290 \times (-153) = 145.2\,\mathrm{N/mm}$$

The shear load is applied through the shear centre of the section so that the rate of twist of the section, $\mathrm{d}\theta/\mathrm{d}z$, is zero and Eq. (10.28) for Cell I simplifies to

$$0 = \frac{1}{2A_{\mathrm{I}}G_{\mathrm{REF}}}[q_{s,0,\mathrm{I}}(\delta_{34} + \delta_{45} + \delta_{56} + \delta_{63}) - q_{s,0,\mathrm{II}}\delta_{63} + q_{\mathrm{b},63}\delta_{63} + q_{\mathrm{b},34}\delta_{34} + q_{\mathrm{b},56}\delta_{56}] \tag{ii}$$

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and for Cell II

$$0 = \frac{1}{2A_{\mathrm{II}}G_{\mathrm{REF}}}[-q_{s,0,\mathrm{I}}\delta_{63} + q_{s,0,\mathrm{II}}(\delta_{12} + \delta_{23} + \delta_{36} + \delta_{67} + \delta_{78} + \delta_{81}) + q_{\mathrm{b},81}\delta_{81}$$
$$+ q_{\mathrm{b},23}\delta_{23} + q_{\mathrm{b},36}\delta_{36} + q_{\mathrm{b},67}\delta_{67}] \qquad \text{(iii)}$$

in which $G_{\mathrm{REF}} = 24\,200\,\mathrm{N/mm^2}$. Then, from Eq. (10.27)

$$t^*_{34} = t^*_{56} = \frac{20\,700}{24\,200} \times 0.915 = 0.783\,\mathrm{mm}$$

$$t^*_{36} = t^*_{81} = t^*_{45} = \frac{24\,800}{24\,200} \times 1.220 = 1.250\,\mathrm{mm}$$

Thus

$$\delta_{34} = \delta_{56} = 380/0.783 = 485.3$$
$$\delta_{12} = \delta_{23} = \delta_{67} = \delta_{78} = 356/0.915 = 389.1$$
$$\delta_{36} = \delta_{81} = 306/1.250 = 244.8$$
$$\delta_{45} = 610/1.250 = 488.0$$

Eq. (ii) then becomes

$$1703.4q_{s,0,\mathrm{I}} - 244.8q_{s,0,\mathrm{II}} + 70\,777.7 = 0$$

or

$$q_{s,0,\mathrm{I}} - 0.144q_{s,0,\mathrm{II}} + 41.55 = 0 \qquad \text{(iv)}$$

and Eq. (iii) becomes

$$-244.8q_{s,0,\mathrm{I}} + 2046q_{s,0,\mathrm{II}} - 46\,021.1 = 0$$

or

$$q_{s,0,\mathrm{I}} - 8.358q_{s,0,\mathrm{II}} + 188.0 = 0 \qquad \text{(v)}$$

Subtracting Eq. (v) from Eq. (iv) gives

$$q_{s,0,\mathrm{II}} = 17.8\,\mathrm{N/mm}$$

Then, from Eq. (v)

$$q_{s,0,\mathrm{I}} = -39.2\,\mathrm{N/mm}$$

The resulting shear flows are then

$$q_{12} = q_{78} = 17.8\,\mathrm{N/mm}, \quad q_{32} = q_{76} = 36.3 - 17.8 = 18.5\,\mathrm{N/mm}$$
$$q_{63} = 145.2 - 17.8 - 39.2 = 88.2\,\mathrm{N/mm}$$
$$q_{43} = q_{65} = 39.2 - 36.3 = 2.9\,\mathrm{N/mm}, \quad q_{54} = 39.2\,\mathrm{N/mm}$$
$$q_{81} = 72.6 + 17.8 = 90.4\,\mathrm{N/mm}$$

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Now taking moments about the mid-point of the web 63

$$66\,750x_{\mathrm{S}} = -2 \times q_{76} \times 356 \times 153 + 2 \times q_{78} \times 356 \times 153 + q_{81} \times 306 \times 712$$
$$-2 \times q_{43} \times 380 \times 153 - q_{54} \times 2(51\,500 + 153 \times 380) \quad \text{(see Eq. (9.79))}$$

from which

$$x_{\mathrm{S}} = 160.1\,\mathrm{mm}$$

S.10.12

Referring to Fig. P.10.12 the horizontal x axis is an axis of symmetry so that $I_{xy} = 0$ and the shear centre lies on this axis. Further, applying an arbitrary shear load, S_y, through the shear centre then $S_x = 0$ and Eq. (9.75) simplifies to

$$q_{\mathrm{b}} = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r \tag{i}$$

in which

$$I_{xx} = 2 \times 645 \times 102^2 + 2 \times 1290 \times 152^2 + 2 \times 1935 \times 152^2 = 162.4 \times 10^6\,\mathrm{mm}^4$$

Eq. (i) then becomes

$$q_{\mathrm{b}} = -6.16 \times 10^{-9} S_y \sum_{r=1}^{n} B_r y_r \tag{ii}$$

'Cut' the walls 34^{o} and 23. Then, from Eq. (ii)

$$q_{\mathrm{b},34^{\mathrm{o}}} = q_{\mathrm{b},23} = 0 = q_{\mathrm{b},45} \quad \text{(from symmetry)}$$

$$q_{\mathrm{b},43^{\mathrm{i}}} = -6.16 \times 10^{-9} S_y \times 1935 \times (-152) = 1.81 \times 10^{-3} S_y\,\mathrm{N/mm}$$

$$q_{\mathrm{b},65} = -6.16 \times 10^{-9} S_y \times 645 \times (-102) = 0.41 \times 10^{-3} S_y\,\mathrm{N/mm} = q_{\mathrm{b},21}$$
(from symmetry)

$$q_{\mathrm{b},52} = 0.41 \times 10^{-3} S_y - 6.16 \times 10^{-9} S_y \times 1290 \times (-152) = 1.62 \times 10^{-3} S_y\,\mathrm{N/mm}$$

Since the shear load, S_y, is applied through the shear centre of the section the rate of twist, $\mathrm{d}\theta/\mathrm{d}z$, is zero. Thus, for Cell I, Eq. (10.28) reduces to

$$0 = q_{s,0,\mathrm{I}}(\delta_{34^{\mathrm{o}}} + \delta_{34^{\mathrm{i}}}) - q_{s,0,\mathrm{II}}\delta_{34^{\mathrm{i}}} + q_{\mathrm{b},43^{\mathrm{i}}}\delta_{34^{\mathrm{i}}} \tag{iii}$$

and for Cell II

$$0 = -q_{s,0,\mathrm{I}}\delta_{43^{\mathrm{i}}} + q_{s,0,\mathrm{II}}(\delta_{23} + \delta_{34^{\mathrm{i}}} + \delta_{45} + \delta_{52}) + q_{\mathrm{b},52}\delta_{52} - q_{\mathrm{b},43^{\mathrm{i}}}\delta_{43^{\mathrm{i}}} \tag{iv}$$

in which

$$\delta_{34^{\mathrm{o}}} = 1015/0.559 = 1815.7, \quad \delta_{34^{\mathrm{i}}} = 304/2.030 = 149.8$$
$$\delta_{23} = \delta_{45} = 765/0.915 = 836.1$$
$$\delta_{25} = 304/1.625 = 187.1$$

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Thus Eq. (iii) becomes

$$1965.5q_{s,0,\mathrm{I}} - 149.8q_{s,0,\mathrm{II}} + 0.271S_y = 0$$

or

$$q_{s,0,\mathrm{I}} - 0.076q_{s,0,\mathrm{II}} + 0.138 \times 10^{-3}S_y = 0 \tag{v}$$

and Eq. (iv) becomes

$$-149.8q_{s,0,\mathrm{I}} + 2009.1q_{s,0,\mathrm{II}} + 319.64 \times 10^{-4}S_y = 0$$

or

$$q_{s,0,\mathrm{I}} - 13.411q_{s,0,\mathrm{II}} - 0.213 \times 10^{-3}S_y = 0 \tag{vi}$$

Subtracting Eq. (vi) from Eq. (v)

$$13.335q_{s,0,\mathrm{II}} + 0.351 \times 10^{-3}S_y = 0$$

whence

$$q_{s,0,\mathrm{II}} = -0.026 \times 10^{-3}S_y$$

Then from Eq. (vi)

$$q_{s,0,\mathrm{I}} = -0.139 \times 10^{-3}S_y$$

Now taking moments about the mid-point of the web 43

$$\begin{aligned} S_y x_s = &-2q_{\mathrm{b},21}(508 \times 152 + 50 \times 762) + q_{\mathrm{b},52} \times 304 \times 762 + 2 \times 258\,000q_{s,0,\mathrm{II}} \\ &+2 \times 93\,000q_{s,0,\mathrm{I}} \end{aligned}$$

from which

$$x_s = 241.4\,\mathrm{mm}$$

S.10.13

The direct stresses in the booms are given by the first of Eqs (9.9) in which, referring to Fig. P.10.13, at the larger cross-section

$$I_{xx} = 2 \times 600 \times 105^2 + 4 \times 800 \times 160^2 = 95.2 \times 10^6\,\mathrm{mm}^4$$

Then, from Eq. (10.8)

$$P_{z,r} = \sigma_{z,r}B_r = \frac{M_x B_r}{I_{xx}} y_r$$

or

$$P_{z,r} = \frac{1800 \times 10^3}{95.2 \times 10^6} B_r y_r = 1.89 \times 10^{-2} B_r y_r \tag{i}$$

The components of boom load in the y and x directions (see Fig. 10.4(a) for the axis system) are found using Eqs (10.9) and (10.10). Then, choosing the intersection of the

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Table S.10.13

Boom	$P_{z,r}$ (N)	$\delta y_r/\delta z$	$\delta x_r/\delta z$	$P_{y,r}$ (N)	$P_{x,r}$ (N)	P_r (N)	η_r (mm)	ξ_r (mm)	$P_{y,r}\xi_r$ (Nmm)	$P_{x,r}\eta_r$ (Nm)
1	1190.7	0.045	−0.12	53.6	−142.9	1200.4	590	105	31 624	−15 004.5
2	2419.2	0.060	0	145.2	0	2423.6	0	160	0	0
3	2419.2	0.060	0.18	145.2	435.5	2462.4	790	160	−114 708	69 680
4	−2419.2	−0.060	0.18	145.2	−435.5	−2462.4	790	160	−114 708	69 680
5	−2419.2	−0.060	0	145.2	0	−2423.6	0	160	0	0
6	−1190.7	−0.045	−0.12	53.6	142.9	−1200.4	590	105	31 624	−15 004.5

web 52 and the horizontal axis of symmetry (the x axis) as the moment centre and defining the boom positions in relation to the moment centre as in Fig. 10.5 the moments corresponding to the boom loads are calculated in Table S.10.13. In Table S.10.13 anticlockwise moments about the moment centre are positive, clockwise negative. Also

$$\sum_{r=1}^{n} P_{x,r} = 0$$

$$\sum_{r=1}^{n} P_{y,r} = 688.0\,\text{N}$$

$$\sum_{r=1}^{n} P_{y,r}\xi_r = -166\,168\,\text{Nmm}$$

$$\sum_{r=1}^{n} P_{x,r}\eta_r = 109\,351\,\text{Nmm}$$

The shear load resisted by the shear stresses in the webs and panels is then

$$S_y = 12\,000 - 688 = 11\,312\,\text{N}$$

'Cut' the walls 12, 23 and 34$^{\text{o}}$ in the larger cross-section. Then, from Eq. (9.75) and noting that $I_{xy} = 0$

$$q_{\text{b}} = -\frac{S_y}{I_{xx}}\sum_{r=1}^{n} B_r y_r$$

i.e.

$$q_{\text{b}} = -\frac{11\,312}{95.2 \times 10^6}\sum_{r=1}^{n} B_r y_r = -1.188 \times 10^{-4}\sum_{r=1}^{n} B_r y_r$$

Thus

$$q_{\text{b},12} = q_{\text{b},23} = q_{\text{b},34^{\text{o}}} = q_{\text{b},45} = q_{\text{b},56} = 0$$

$$q_{\text{b},61} = -1.188 \times 10^{-4} \times 600 \times (-105) = 7.48\,\text{N/mm}$$

$$q_{\text{b},52} = -1.188 \times 10^{-4} \times 800 \times (-160) = 15.21\,\text{N/mm}$$

$$q_{\text{b},43^{\text{i}}} = -1.188 \times 10^{-4} \times 800 \times (-160) = 15.21\,\text{N/mm}$$

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From Eq. (10.28) for Cell I

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{I}}G}[q_{s,0,\mathrm{I}}(\delta_{34^{\mathrm{o}}} + \delta_{34^{\mathrm{i}}}) - q_{s,0,\mathrm{II}}\delta_{34^{\mathrm{i}}} + q_{\mathrm{b},43^{\mathrm{i}}}\delta_{43^{\mathrm{i}}}] \tag{ii}$$

For Cell II

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{II}}G}[-q_{s,0,\mathrm{I}}\delta_{34^{\mathrm{i}}} + q_{s,0,\mathrm{II}}(\delta_{23} + \delta_{34^{\mathrm{i}}} + \delta_{45} + \delta_{52}) - q_{s,0,\mathrm{III}}\delta_{52} + q_{\mathrm{b},52}\delta_{52} - q_{\mathrm{b},43^{\mathrm{i}}}\delta_{43^{\mathrm{i}}}] \tag{iii}$$

For Cell III

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{III}}G}[-q_{s,0,\mathrm{II}}\delta_{52} + q_{s,0,\mathrm{III}}(\delta_{12} + \delta_{25} + \delta_{56} + \delta_{61}) + q_{\mathrm{b},61}\delta_{61} - q_{\mathrm{b},52}\delta_{52}] \tag{iv}$$

in which

$$\delta_{12} = \delta_{56} = 600/1.0 = 600, \quad \delta_{23} = \delta_{45} = 800/1.0 = 800$$
$$\delta_{34^{\mathrm{o}}} = 1200/0.6 = 2000, \quad \delta_{34^{\mathrm{i}}} = 320/2.0 = 160, \quad \delta_{52} = 320/2.0 = 160$$
$$\delta_{61} = 210/1.5 = 140$$

Substituting these values in Eqs (ii), (iii) and (iv)

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2 \times 100\,000G}(2160q_{s,0,\mathrm{I}} - 160q_{s,0,\mathrm{II}} + 2433.6) \tag{v}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2 \times 260\,000G}(-160q_{s,0,\mathrm{I}} + 1920q_{s,0,\mathrm{II}} - 160q_{s,0,\mathrm{III}}) \tag{vi}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2 \times 180\,000G}(-160q_{s,0,\mathrm{II}} + 1500q_{\mathrm{s},0,\mathrm{III}} - 1384.8) \tag{vii}$$

Also, taking moments about the mid-point of web 52, i.e. the moment centre (see Eq. (10.31))

$$0 = q_{\mathrm{b},61} \times 210 \times 590 - q_{\mathrm{b},43^{\mathrm{i}}} \times 320 \times 790 + 2A_{\mathrm{I}}q_{s,0,\mathrm{I}} + 2A_{\mathrm{II}}q_{s,0,\mathrm{II}} + 2A_{\mathrm{III}}q_{s,0,\mathrm{III}} + \sum_{r=1}^{n} P_{x,r}\eta_r + \sum_{r=1}^{n} P_{y,r}\xi_r \tag{viii}$$

Substituting the appropriate values in Eq. (viii) and simplifying gives

$$q_{s,0,\mathrm{I}} + 2.6q_{s,0,\mathrm{II}} + 1.8q_{s,0,\mathrm{III}} - 14.88 = 0 \tag{ix}$$

Equating Eqs (v) and (vi)

$$q_{s,0,\mathrm{I}} - 0.404q_{s,0,\mathrm{II}} + 0.028q_{s,0,\mathrm{III}} + 1.095 = 0 \tag{x}$$

Equating Eqs (v) and (vii)

$$q_{s,0,\mathrm{I}} - 0.033q_{s,0,\mathrm{II}} - 0.386q_{s,0,\mathrm{III}} + 1.483 = 0 \tag{xi}$$

Now subtracting Eq. (x) from Eq. (ix)

$$q_{s,0,\mathrm{II}} + 0.590q_{s,0,\mathrm{III}} - 5.318 = 0 \tag{xii}$$

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and subtracting Eq. (xi) from Eq. (ix)

$$q_{s,0,\mathrm{II}} + 0.830q_{s,0,\mathrm{III}} - 6.215 = 0 \tag{xiii}$$

Finally, subtracting Eq. (xiii) from Eq. (xii) gives

$$q_{s,0,\mathrm{III}} = 3.74\,\mathrm{N/mm}$$

Then, from Eq. (xiii)

$$q_{s,0,\mathrm{II}} = 3.11\,\mathrm{N/mm}$$

and from Eq. (ix)

$$q_{s,0,\mathrm{I}} = 0.06\,\mathrm{N/mm}$$

The complete shear flow distribution is then

$$q_{12} = q_{56} = 3.74\,\mathrm{N/mm}, \qquad q_{32} = q_{45} = 3.11\,\mathrm{N/mm}$$

$$q_{34^{\mathrm{o}}} = 0.06\,\mathrm{N/mm}, \qquad q_{43^{\mathrm{i}}} = 12.16\,\mathrm{N/mm}$$

$$q_{52} = 14.58\,\mathrm{N/mm}, \qquad q_{61} = 11.22\,\mathrm{N/mm}$$

S.10.14

From S.10.8

$$\delta_{\mathrm{I}} = 2437 \text{ (see Eq. (v))}, \quad \delta_{\mathrm{II}} = 2316 \text{ (see Eq. (vi))}$$

$$\delta_{\mathrm{III}} = 2316 \text{ (see Eq. (vii))}, \quad \delta_{\mathrm{IV}} = 2235 \text{ (see Eq. (viii))}$$

$$\delta_{\mathrm{I,II}} = 292, \quad \delta_{\mathrm{II,III}} = 250, \quad \delta_{\mathrm{III,IV}} = 292$$

Then, from Eqs (10.34)

$$C_{\mathrm{I,II}} = \frac{292}{2316} = 0.126, \qquad C_{\mathrm{II,I}} = \frac{292}{2437} = 0.120$$

$$C_{\mathrm{II,III}} = \frac{250}{2316} = 0.108, \qquad C_{\mathrm{III,II}} = \frac{250}{2316} = 0.108$$

$$C_{\mathrm{III,IV}} = \frac{292}{2235} = 0.131, \qquad C_{\mathrm{IV,III}} = \frac{292}{2316} = 0.126$$

Also

$$q_{\mathrm{I}} = \frac{2A_{\mathrm{I}}}{\delta_{\mathrm{I}}} = \frac{2 \times 161\,500}{2437} = 132.5\,\mathrm{N/mm}$$

$$q_{\mathrm{II}} = \frac{2A_{\mathrm{II}}}{\delta_{\mathrm{II}}} = \frac{2 \times 291\,000}{2316} = 251.3\,\mathrm{N/mm}$$

$$q_{\mathrm{III}} = \frac{2A_{\mathrm{III}}}{\delta_{\mathrm{III}}} = \frac{2 \times 291\,000}{2316} = 251.3\,\mathrm{N/mm}$$

$$q_{\mathrm{IV}} = \frac{2A_{\mathrm{IV}}}{\delta_{\mathrm{IV}}} = \frac{2 \times 226\,000}{2235} = 202.2\,\mathrm{N/mm}$$

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Table S.10.14

	Cell I	Cell II		Cell III		Cell IV
Cs	0.126	0.120	0.108	0.108	0.131	0.126
Assumed q (N/mm)	133	251		251		202
COq	30.12	16.76	27.11	27.11	25.45	32.88
COq	2.01	3.80	2.93	2.93	4.14	3.33
COq	0.46	0.25	0.32	0.32	0.42	0.54
COq	0.03	0.06	0.03	0.03	0.07	0.06
Final q (N/mm)	165.6	302.3		311.5		238.8
$2Aq$ (Nmm)	5.35×10^7	17.59×10^7		18.13×10^7		10.79×10^7
Total T (Nmm)			51.86×10^7			
Actual q (N/mm) (Torque $= T$)	$3.19 \times 10^{-7}T$	$5.83 \times 10^{-7}T$		$6.01 \times 10^{-7}T$		$4.60 \times 10^{-7}T$

The solution is continued in Table S.10.14. Substituting for q_{I} and q_{II} in Eq. (v) of S.10.8 gives

$$\frac{d\theta}{dz} = \frac{1.880 \times 10^{-9} T}{G}$$

whence

$$T/(d\theta/dz) = 532.0 \times 10^6 G \,\mathrm{N\,mm^2/rad}$$

which differs only slightly from the exact solution in S.10.8.

S.10.15

From Fig. P.10.15 and the data given in P.10.15

$$\delta_{\mathrm{I}} = \frac{1500}{2.5} + \frac{300}{3.0} = 700, \quad \delta_{\mathrm{II}} = \frac{2 \times 605}{3.0} + \frac{300}{3.0} + \frac{400}{3.0} = 636.7$$

$$\delta_{\mathrm{III}} = \frac{2 \times 603}{3.0} + \frac{2 \times 400}{3.0} = 668.7, \quad \delta_{\mathrm{IV}} = \delta_{\mathrm{II}} = 636.7$$

$$\delta_{\mathrm{V}} = \frac{300}{3.0} + \frac{2 \times 605}{2.5} + \frac{200}{3.0} = 650.7, \quad \delta_{\mathrm{I,II}} = \frac{300}{3.0} = 100$$

$$\delta_{\mathrm{II,III}} = \frac{400}{3.0} = 133.3, \quad \delta_{\mathrm{III,IV}} = \frac{400}{3.0} = 133.3, \quad \delta_{\mathrm{IV,V}} = \frac{300}{3.0} = 100$$

From Eqs (10.34)

$$C_{\mathrm{I,II}} = \frac{100}{636.7} = 0.157, \quad C_{\mathrm{II,I}} = \frac{100}{700} = 0.143$$

$$C_{\mathrm{II,III}} = \frac{133.3}{668.7} = 0.199, \quad C_{\mathrm{III,II}} = \frac{133.3}{636.7} = 0.209$$

$$C_{\mathrm{III,IV}} = \frac{133.3}{636.7} = 0.209, \quad C_{\mathrm{IV,III}} = \frac{133.3}{668.7} = 0.199$$

$$C_{\mathrm{IV,V}} = \frac{100}{650.7} = 0.154, \quad C_{\mathrm{V,IV}} = \frac{100}{636.7} = 0.157$$

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From Eq. (10.35) in which $I_{xy} = 0$ and $S_x = 0$

$$q_b = -\frac{S_y}{I_{xx}}\sum_{r=1}^{n} B_r y_r \tag{i}$$

in which

$$I_{xx} = 2000(4 \times 150^2 + 4 \times 200^2 + 2 \times 100^2) = 5.4 \times 10^8\,\text{mm}^4$$

Eq. (i) then becomes

$$q_b = -\frac{100 \times 10^3 \times 2000}{5.4 \times 10^8}\sum_{r=1}^{n} y_r = -0.37\sum_{r=1}^{n} y_r \tag{ii}$$

'Cut' each cell in the top panel. Then, from Eq. (ii)

$$q_{b,65^i} = -0.37 \times (-150) = 55.5\,\text{N/mm}$$

$$q_{b,74} = -0.37 \times (-200) = 74.0\,\text{N/mm}$$

$$q_{b,83} = -0.37 \times (-200) = 74.0\,\text{N/mm}$$

$$q_{b,92} = -0.37 \times (-150) = 55.5\,\text{N/mm}$$

$$q_{b,10\,1} = -0.37 \times (-100) = 37.0\,\text{N/mm}$$

Thus

$$\oint_{\text{I}} \frac{q_b\,\text{d}s}{t} = 55.5 \times 100 = 5550\,\text{N/mm}$$

$$\oint_{\text{II}} q_b\frac{\text{d}s}{t} = 74.0 \times 133.3 - 55.5 \times 100 = 4314.2\,\text{N/mm}$$

$$\oint_{\text{III}} q_b\frac{\text{d}s}{t} = 74.0 \times 133.3 - 74.0 \times 133.3 = 0$$

$$\oint_{\text{IV}} q_b\frac{\text{d}s}{t} = 55.5 \times 100 - 74.0 \times 133.3 = -4314.2\,\text{N/mm}$$

$$\oint_{\text{V}} \frac{q_b\,\text{d}s}{t} = 37.0 \times \frac{200}{3} - 55.5 \times 100 = -3083.3\,\text{N/mm}$$

The solution is continued in Table S.10.15. The final shear flows are then

$$q_{65^o} = 9.1\,\text{N/mm}, \quad q_{65^i} = 55.5 + 8.2 - 9.1 = 54.6\,\text{N/mm}$$

$$q_{54} = q_{76} = 8.2\,\text{N/mm}, \quad q_{74} = 74.0 + 0.1 - 8.2 = 65.9\,\text{N/mm}$$

$$q_{43} = q_{87} = 0.1\,\text{N/mm}, \quad q_{83} = 74.0 - 0.1 - 7.7 = 66.2\,\text{N/mm}$$

$$q_{23} = q_{89} = 7.7\,\text{N/mm}, \quad q_{92} = 55.5 + 7.7 - 5.9 = 57.3\,\text{N/mm}$$

$$q_{9\,10} = q_{12} = 5.9\,\text{N/mm}, \quad q_{10\,1} = 37.0 + 5.9 = 42.9\,\text{N/mm}$$

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Table S.10.15

	Cell I		Cell II			Cell III			Cell IV		Cell V
$\oint q_b \, ds/t$	5550		4314.2			0			−4314.2		−3083.3
δ	700		636.7			668.7			636.7		650.7
Cs	0.157	0.143		0.199	0.209		0.209	0.199		0.154	0.157
$q'[= (-\oint q_b \, ds/t)/\delta]$	−7.93		−6.78			0			6.78		4.74
COq	−0.97	−1.25		0	−1.35		1.35	0		0.74	1.04
COq	−0.18	−0.15		0	−0.25		0.15	0		0.16	0.11
COq	−0.02	−0.03		−0.02	−0.03		0.03	−0.02		0.02	0.02
Corrective qs	−9.1		−8.2			−0.1			7.7		5.9

Taking moments about the mid-point of web 74

$$100 \times 10^3 x_s = -q_{\mathrm{b},65^{\mathrm{i}}} \times 300 \times 600 + q_{\mathrm{b},83} \times 400 \times 600 + q_{\mathrm{b},92} \times 300 \times 1200$$
$$+q_{\mathrm{b},101} \times 200 \times 1800 + 2A_{\mathrm{I}} q_{s,0,\mathrm{I}} + 2A_{\mathrm{II}} q_{s,0,\mathrm{II}} + 2A_{\mathrm{III}} q_{s,0,\mathrm{III}}$$
$$+2A_{\mathrm{IV}} q_{s,0,\mathrm{IV}} + 2A_{\mathrm{V}} q_{s,0,\mathrm{V}}$$

Thus

$$100 \times 10^3 x_s = -55.5 \times 300 \times 600 + 74.0 \times 400 \times 600 + 55.5 \times 300 \times 1200$$
$$+37.0 \times 200 \times 1800 - 2 \times 120\,000 \times 9.1 - 2 \times 215\,000 \times 8.2$$
$$-2 \times 250\,000 \times 0.1 + 2 \times 215\,000 \times 7.7 + 2 \times 155\,000 \times 5.9$$

which gives

$$x_s = 404.5\,\mathrm{mm}$$

S.10.16

From the overall equilibrium of the beam in Fig. S.10.16(a)

$$R_{\mathrm{F}} = 4\,\mathrm{kN}, \quad R_{\mathrm{D}} = 2\,\mathrm{kN}$$

The shear load in the panel ABEF is therefore 4 kN and the shear flow q is given by

$$q_1 = \frac{4 \times 10^3}{1000} = 4\,\mathrm{N/mm}$$

Similarly

$$q_2 = \frac{2 \times 10^3}{1000} = 2\,\mathrm{N/mm}$$

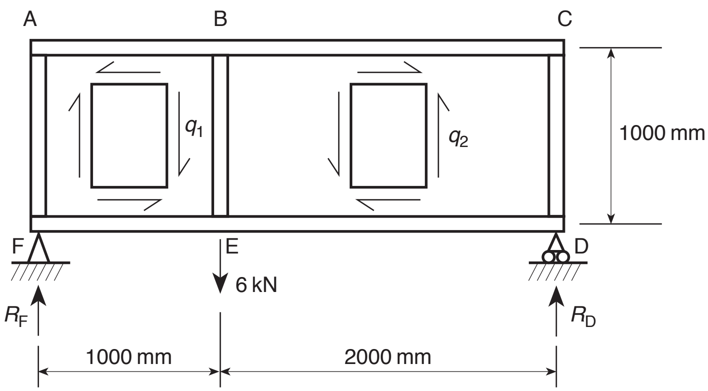

Fig. S.10.16(a)

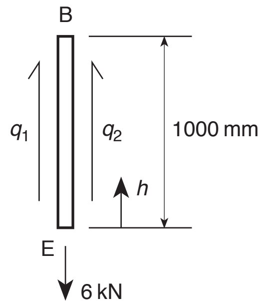

Fig. S.10.16(b)

Considering the vertical equilibrium of the length h of the stiffener BE in Fig. S.10.16(b)

$$P_{\mathrm{EB}} + (q_1 + q_2)h = 6 \times 10^3$$

where P_{EB} is the tensile load in the stiffener at the height h, i.e.

$$P_{\mathrm{EB}} = 6 \times 10^3 - 6h \tag{i}$$

Then, from Eq. (i), when $h = 0$, $P_{\mathrm{EB}} = 6000\,\mathrm{N}$ and when $h = 1000\,\mathrm{mm}$, $P_{\mathrm{EB}} = 0$. Therefore the stiffener load varies linearly from zero at B to 6000 N at E.

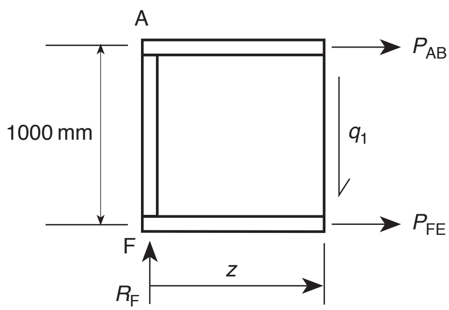

Fig. S.10.16(c)

Consider now the length z of the beam in Fig. S.10.16(c). Taking moments about the bottom flange at the section z,

$$P_{\mathrm{AB}} \times 1000 + R_{\mathrm{F}}z = 0$$

whence

$$P_{\mathrm{AB}} = -4z\,\mathrm{N}$$

Thus P_{AB} varies linearly from zero at A to 4000 N (compression) at B. Similarly P_{CB} varies linearly from zero at C to 4000 N (compression) at B.

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S.10.17

Referring to Fig. P.10.17 and considering the vertical equilibrium of the stiffener CDF

$$8000 \sin 30^\circ - q_1 \times 200 - q_2 \times 200 = 0$$

from which

$$q_1 + q_2 = 20 \qquad \text{(i)}$$

Now considering the horizontal equilibrium of the stiffener ED

$$8000 \cos 30^\circ - q_1 \times 300 + q_2 \times 300 = 0$$

whence

$$q_1 - q_2 = 23.1 \qquad \text{(ii)}$$

Adding Eqs (i) and (ii)

$$2q_1 = 43.1$$

i.e.

$$q_1 = 21.6\,\text{N/mm}$$

so that, from Eq. (i)

$$q_2 = -1.6\,\text{N/mm}$$

The vertical shear load at any section in the panel ABEGH is $8000 \sin 30^\circ = 4000$ N. Hence

$$400q_3 = 4000$$

i.e.

$$q_3 = 10\,\text{N/mm}$$

Now consider the equilibrium of the flange ABC in Fig. S.10.17(a). At any section z between C and B

$$P_{\text{CB}} = 21.6z \qquad \text{(iii)}$$

so that P_{CB} varies linearly from zero at C to 6480 N (tension) at B. Also at any section z between B and A

$$P_{\text{BA}} = 21.6 \times 300 + 10(z - 300)$$

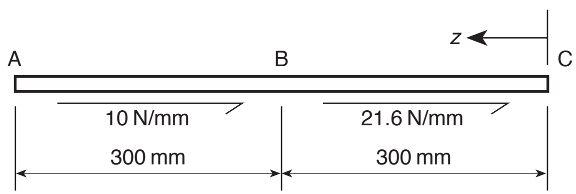

Fig. S.10.17(a)

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i.e.

$$P_{\mathrm{BA}} = 3480 + 10z \qquad \text{(iv)}$$

Thus P_{BA} varies linearly from 6480 N (tension) at B to 9480 N (tension) at A.

Referring to Fig. S.10.17(b) for the bottom flange HGF, the flange load P_{FG} at any section z is given by

$$P_{\mathrm{FG}} = 1.6z \qquad \text{(v)}$$

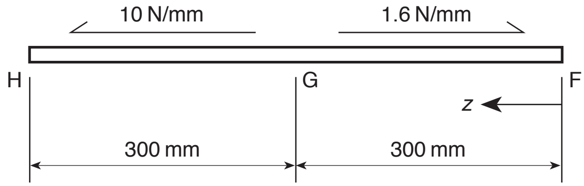

Fig. S.10.17(b)

Thus P_{FG} varies linearly from zero at F to 480 N (tension) at G. Also at any section z between G and H

$$P_{\mathrm{GH}} + 10(z - 300) - 1.6 \times 300 = 0$$

i.e.

$$P_{\mathrm{GH}} = 3480 - 10z \qquad \text{(vi)}$$

Hence P_{GH} varies linearly from 480 N (tension) at G to −2520 N (compression) at H.

The forces acting on the stiffener DE are shown in Fig. S.10.17(c). At any section a distance z from D

$$P_{\mathrm{DE}} + 21.6z + 1.6z - 8000\cos 30^\circ = 0$$

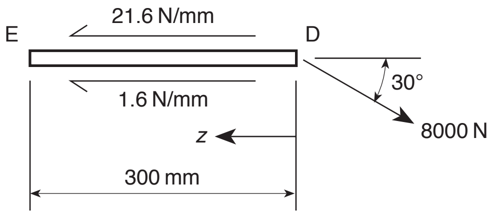

Fig. S.10.17(c)

i.e.

$$P_{\mathrm{DE}} = -23.2z + 6928.2 \qquad \text{(vii)}$$

Therefore, P_{DE} varies linearly from 6928 N (tension) at D to zero at E. (The small value of P_{DE} at E given by Eq. (vii) is due to rounding off errors in the values of the shear flows.)

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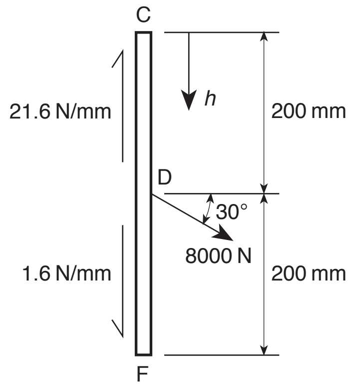

Fig. S.10.17(d)

The forces in the stiffener CDF are shown in Fig. S.10.17(d). At any section in CD a distance h from C the stiffener load, P_{CD}, is given by

$$P_{CD} = 21.6h \tag{viii}$$

so that P_{CD} varies linearly from zero at C to 4320 N (tension) at D. In DF

$$P_{DF} + 8000 \sin 30^\circ + 1.6(h - 200) - 21.6 \times 200 = 0$$

from which

$$P_{DF} = 640 - 1.6h \tag{ix}$$

Hence, P_{DF} varies linearly from 320 N (tension) at D to zero at F.

The stiffener BEG is shown in Fig. S.10.17(e). In BE at any section a distance h from B

$$P_{BE} + 21.6h - 10h = 0$$

i.e.

$$P_{BE} = -11.6h \tag{x}$$

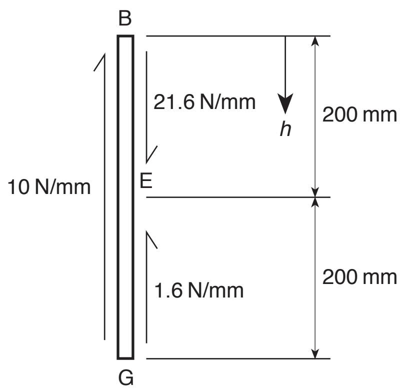

Fig. S.10.17(e)

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P_{BE} therefore varies linearly from zero at B to −2320 N (compression) at E. In EG

$$P_{EG} - 1.6(h - 200) + 21.6 \times 200 - 10h = 0$$

i.e.

$$P_{EG} = 11.6h - 4640 \qquad \text{(xi)}$$

Thus P_{EG} varies linearly from −2320 N (compression) at E to zero at G.

S.10.18

A three flange wing section is statically determinate (see Section 10.3) so that the shear flows applied to the wing rib may be found by considering the equilibrium of the wing rib. From Fig. S.10.18(a) and resolving forces horizontally

$$600q_{12} - 600q_{34} - 1200 = 0$$

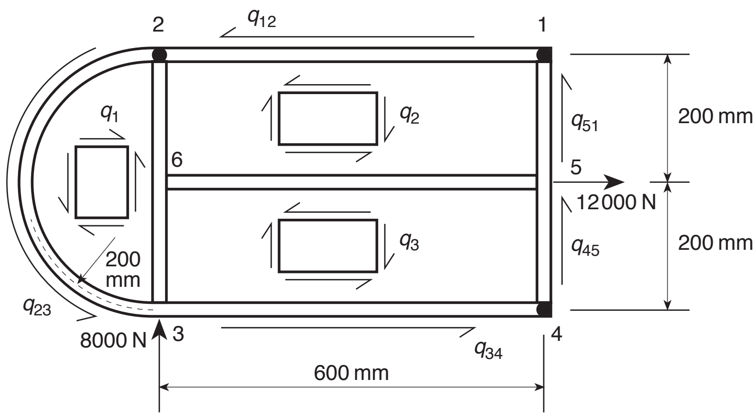

Fig. S.10.18(a)

whence

$$q_{12} - q_{34} = 20 \qquad \text{(i)}$$

Now resolving vertically and noting that $q_{51} = q_{45}$

$$400q_{45} - 400q_{23} + 8000 = 0$$

i.e.

$$q_{45} - q_{23} = -20 \qquad \text{(ii)}$$

Taking moments about 4

$$q_{12} \times 600 \times 400 + 2\left(\frac{\pi \times 200^2}{2} + \frac{1}{2} \times 400 \times 600\right)q_{23} - 12\,000 \times 200 - 8000 \times 600 = 0$$

so that

$$q_{12} + 1.52q_{23} = 30 \qquad \text{(iii)}$$

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Subtracting Eq. (iii) from Eq. (i) and noting that $q_{34} = q_{23}$

$$-2.52q_{23} = -10$$

or

$$q_{23} = 4.0\,\text{N/mm} = q_{34}$$

Then from Eq. (i)

$$q_{12} = 24.0\,\text{N/mm}$$

and from Eq. (ii)

$$q_{45} = -16.0\,\text{N/mm} = q_{51}$$

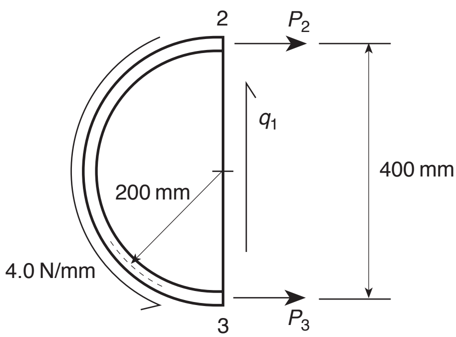

Fig. S.10.18(b)

Consider the nose portion of the wing rib in Fig. S.10.18(b). Taking moments about 3

$$P_2 \times 400 - 2 \times \frac{\pi \times 200^2}{2} \times 4.0 = 0$$

from which

$$P_2 = 1256.6\,\text{N} \quad \text{(tension)}$$

From horizontal equilibrium

$$P_3 + P_2 = 0$$

whence

$$P_3 = -1256.6\,\text{N} \quad \text{(compression)}$$

and from vertical equilibrium

$$q_1 = 4.0\,\text{N/mm}$$

From the vertical equilibrium of the stiffener 154 in Fig. S.10.18(c)

$$q_2 \times 200 + q_3 \times 200 - 16 \times 400 = 0$$

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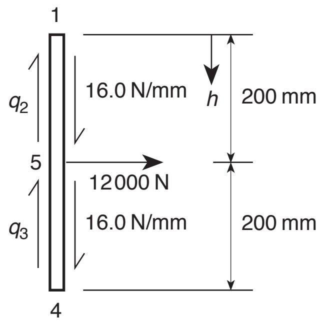

Fig. S.10.18(c)

i.e.

$$q_2 + q_3 = 32 \tag{iv}$$

Also, in 15 at any distance h from 1

$$P_{15} + 16h - q_2 h = 0$$

i.e.

$$P_{15} = (q_2 - 16)h \tag{v}$$

and in 54

$$P_{54} + 16h - q_2 \times 200 - q_3(h - 200) = 0$$

whence

$$P_{54} = 200(q_2 - q_3) + (q_3 - 16)h \tag{vi}$$

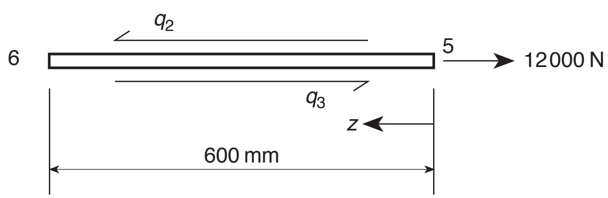

Fig. S.10.18(d)

Fig. S.10.18(d) shows the stiffener 56. From horizontal equilibrium

$$600q_2 - 600q_3 - 12\,000 = 0$$

or

$$q_2 - q_3 = 20 \tag{vii}$$

Adding Eqs (iv) and (vii)

$$2q_2 = 52$$

i.e.

$$q_2 = 26\,\text{N/mm}$$

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and from Eq. (iv)

$$q_3 = 6\,\text{N/mm}$$

Then, from Eq. (v)

$$P_{15} = 10h \tag{viii}$$

and P_{15} varies linearly from zero at 1 to 2000 N (tension) at 5. From Eq. (vi)

$$P_{54} = 200(26 - 6) + (6 - 16)h$$

i.e.

$$P_{54} = 4000 - 10h \tag{ix}$$

so that P_{54} varies linearly from 2000 N (tension) at 5 to zero at 4. Now from Fig. S.10.18(d) at any section z

$$P_{56} + q_2 z - q_3 z - 12\,000 = 0$$

i.e.

$$P_{56} = -20z + 12\,000 \tag{x}$$

Thus P_{56} varies linearly from 12 000 N (tension) at 5 to zero at 6.

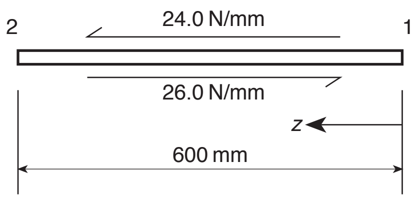

Fig. S.10.18(e)

Consider the flange 12 in Fig. S.10.18(e). At any section a distance z from 1

$$P_{12} + 24.0z - 26z = 0$$

i.e.

$$P_{12} = 2z \tag{xi}$$

Hence P_{12} varies linearly from zero at 1 to 1200 N (tension) at 2.

Now consider the bottom flange in Fig. S.10.18(f). At any section a distance z from 4

$$P_{43} + 6z - 4z = 0$$

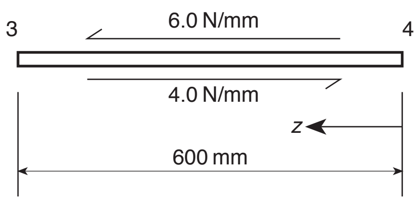

Fig. S.10.18(f)

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i.e.

$$P_{43} = -2z \tag{xii}$$

Thus P_{43} varies linearly from zero at 4 to $-1200\,\text{N}$ (compression) at 3. (The discrepancy between P_2 in 12 and P_2 in 23 and between P_3 in 43 and P_3 in 23 is due to the rounding off error in the shear flow q_1.)

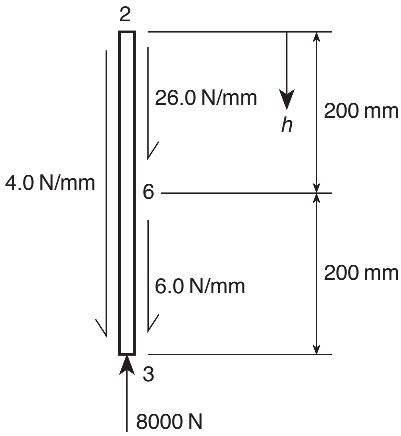

Fig. S.10.18(g)

In Fig. S.10.18(g) the load in the stiffener at any section a distance h from 2 is given by

$$P_{26} + 26h + 4h = 0$$

i.e.

$$P_{26} = -30h \tag{xiii}$$

Therefore P_{26} varies linearly from zero at 2 to $-6000\,\text{N}$ (compression) at 6. In 63

$$P_{63} + 26 \times 200 + 4.0h + 6(h - 200) = 0$$

i.e.

$$P_{63} = -4000 - 10h \tag{xiv}$$

Thus P_{63} varies linearly from $-6000\,\text{N}$ (compression) at 6 to $-8000\,\text{N}$ (compression) at 3.

S.10.19

Consider first the flange loads and shear flows produced by the shear load acting through the shear centre of the wing box. Referring to Fig. S.10.19(a), in bay ① the shear load is resisted by the shear flows q_1 in the spar webs. Thus

$$q_1 = \frac{2000}{2 \times 200} = 5\,\text{N/mm}$$

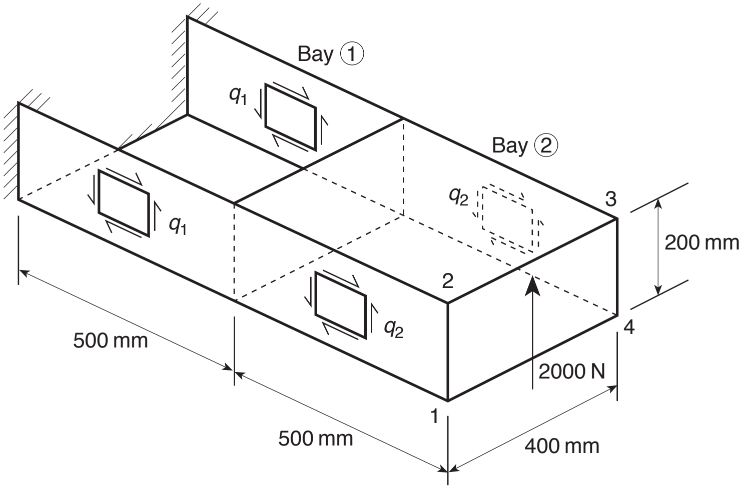

Fig. S.10.19(a)

Similarly in bay ②

$$q_2 = \frac{2000}{2 \times 200} = 5\,\mathrm{N/mm}$$

From symmetry the bending moment produced by the shear load will produce equal but opposite loads in the top and bottom flanges. These flange loads will increase with bending moment, i.e. linearly, from zero at the free end to

$$\pm\frac{2000 \times 1000}{2 \times 200} = \pm 5000\,\mathrm{N}$$

at the built-in end. Thus, at the built-in end

$$P_1 = P_4 = -P_2 = -P_3 = 5000\,\mathrm{N}$$

Alternatively, the flange loads may be determined by considering the equilibrium of a single flange subjected to the flange load and the shear flows in the adjacent spar webs.

Now consider the action of the applied torque in Fig. S.10.19(b). In bay ① the torque is resisted by differential bending of the spar webs. Thus

$$q_1 \times 200 \times 400 = 1000 \times 10^3$$

which gives

$$q_1 = 12.5\,\mathrm{N/mm}$$

The differential bending of the spar webs in bay ① induces flange loads as shown in Fig. S.10.19(c). For equilibrium of flange 1

$$2P_1 = 500q_1 = 500 \times 12.5$$

so that

$$P_1 = 3125\,\mathrm{N}$$

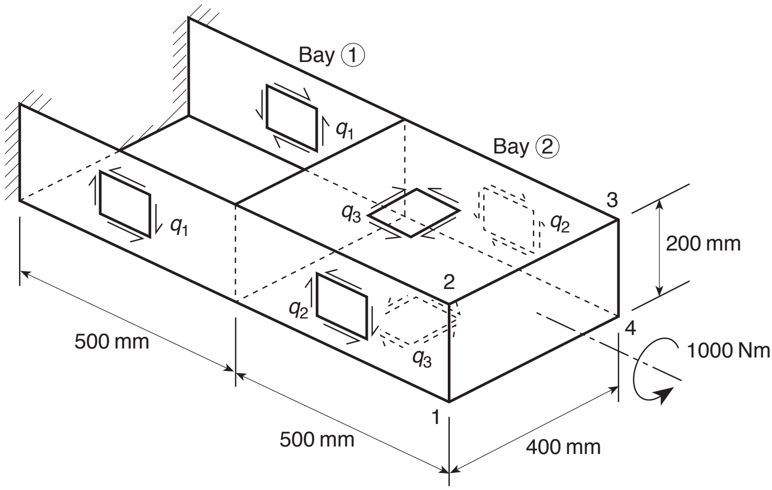

Fig. S.10.19(b)

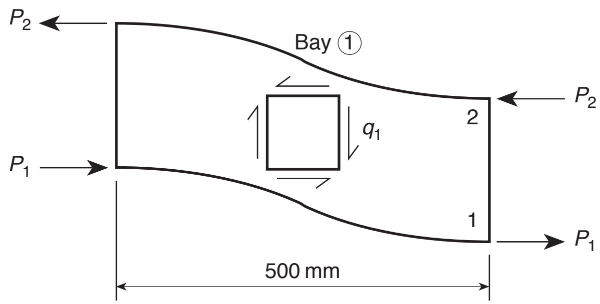

Fig. S.10.19(c)

Now considering the equilibrium of flange 1 in bay ②

$$P_1 + q_2 \times 500 - q_3 \times 500 = 0$$

whence

$$q_2 - q_3 = -6.25 \tag{i}$$

Also, the resultant of the shear flows in the spar webs and skin panels in bay ② is equivalent to the applied torque. Thus

$$2 \times 2 \times \tfrac{1}{2} \times 200 \times 200 q_2 + 2 \times 2 \times \tfrac{1}{2} \times 400 \times 100 q_3 = 1000 \times 10^3$$

i.e.

$$q_2 + q_3 = 12.5 \tag{ii}$$

Adding Eqs (i) and (ii) gives

$$q_2 = 3.125\,\text{N/mm}$$

whence

$$q_3 = 9.375\,\mathrm{N/mm}$$

The shear flows due to the combined action of the shear and torsional loads are then as follows:

Bay ①

Spar webs: $q = 12.5 - 5 = 7.5\,\mathrm{N/mm}$

Bay ②

Spar webs: $q = 5 - 3.125 = 1.875\,\mathrm{N/mm}$

Skin panels: $q = 9.375\,\mathrm{N/mm}$

The flange loads are:

Bay ①

At the built-in end: $P_1 = 5000 - 3125 = 1875\,\mathrm{N}$ (tension)

At the central rib: $P_1 = 2500 + 3125 = 5625\,\mathrm{N}$ (tension)

Bay ②

At the central rib: $P_1 = 3625\,\mathrm{N}$ (tension)

At the free end: $P_1 = 0$

Finally the shear flows on the central rib are:

On the horizontal edges: $q = 9.375\,\mathrm{N/mm}$

On the vertical edges: $q = 7.5 + 1.875 = 9.375\,\mathrm{N/mm}$

S.10.20

From Eq. (10.48) the modulus of the bar is given by

$$E_1 = 140\,000 \times \frac{100 \times 10}{100 \times 55} + 3000 \times \frac{100 \times 45}{100 \times 55}$$

i.e.

$$E_1 = 27\,909.1\,\mathrm{N/mm^2}$$

The overall direct stress in the longitudinal direction is given by

$$\sigma_1 = \frac{500 \times 10^3}{100 \times 55} = 90.9\,\mathrm{N/mm^2}$$

Therefore, from Eq. (10.45), the longitudinal strain in the bar is

$$\varepsilon_1 = \frac{90.9}{27\,909.1} = 3.26 \times 10^{-3}$$

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The shortening, Δ_l, of the bar is then

$$\Delta_l = 3.26 \times 10^{-3} \times 1 \times 10^3 = 3.26\,\text{mm}$$

The major Poisson's ratio for the bar is obtained using Eq. (10.50). Thus

$$\nu_{lt} = \frac{100 \times 45}{100 \times 55} \times 0.16 + \frac{100 \times 10}{100 \times 55} \times 0.28 = 0.18$$

Hence the strain across the thickness of the bar is

$$\varepsilon_t = 0.18 \times 3.26 \times 10^{-3} = 5.87 \times 10^{-4}$$

so that the increase in thickness of the bar is

$$\Delta_t = 5.87 \times 10^{-4} \times 55$$

i.e.

$$\Delta_t = 0.032\,\text{mm}$$

The stresses in the polyester and Kevlar are found from Eqs (10.46). Hence

$$\sigma_m(\text{polyester}) = 3000 \times 3.26 \times 10^{-3} = 9.78\,\text{N/mm}^2$$

$$\sigma_f(\text{Kevlar}) = 140\,000 \times 3.26 \times 10^{-3} = 456.4\,\text{N/mm}^2$$

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Solutions to Chapter 11 Problems

S.11.1

In Fig. S.11.1 $\alpha = \tan^{-1} 127/305 = 22.6^\circ$. Choose O as the origin of axes then, from Eq. (11.1), since all the walls of the section are straight, the shear flow in each wall is constant. Then

$$q_{12} = 1.625G(254\theta' - u') \tag{i}$$

$$q_{23} = 1.625G(254\theta' \cos 22.6^\circ - u' \cos 22.6^\circ - v' \sin 22.6^\circ)$$

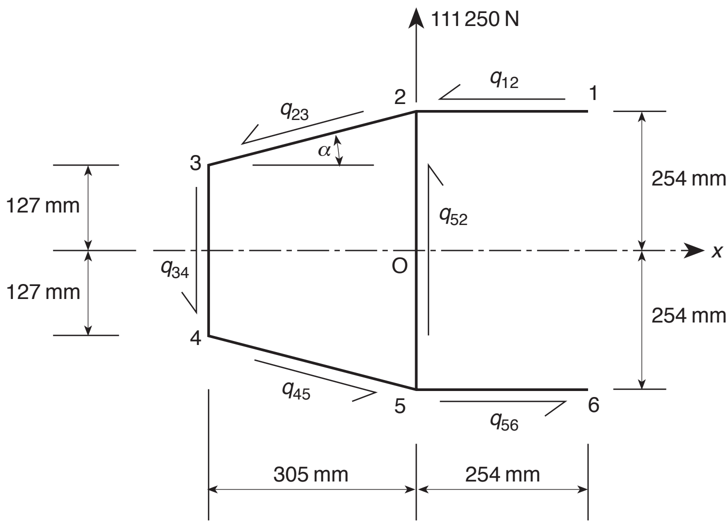

Fig. S.11.1

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i.e.

$$q_{23} = 1.625G(234.5\theta' - 0.923u' - 0.384v') \tag{ii}$$

$$q_{34} = 2.03G(305\theta' - v') \tag{iii}$$

$$q_{52} = 2.54Gv' \tag{iv}$$

$$q_{45} = 1.625G(234.5\theta' + 0.923u' - 0.384v') \tag{v}$$

$$q_{56} = 1.625G(254\theta' + u') \tag{vi}$$

From symmetry $q_{12} = q_{56}$ and $q_{23} = q_{45}$ so that, from Eqs (i) and (vi) (or Eqs (ii) and (v)) $u' = 0$. Now resolving forces vertically

$$q_{52} \times 508 - q_{23} \times 127 - q_{34} \times 254 - q_{45} \times 127 = 111\,250$$

i.e.

$$508q_{52} - 2 \times 127q_{23} - 254q_{34} = 111\,250$$

Substituting for q_{52}, q_{23} and q_{34} from Eqs (iv), (ii) and (iii) respectively gives

$$v' - 129.3\theta' = 56.63/G \tag{vii}$$

Now taking moments about O

$$2q_{12} \times 254 \times 254 + 2q_{23} \times 305 \times 254 + q_{34} \times 254 \times 305 = 0$$

Substituting for q_{12}, q_{23} and q_{34} from Eqs (i), (ii) and (iii) respectively gives

$$v' - 631.1\theta' = 0 \tag{viii}$$

Subtracting Eq. (viii) from Eq. (vii) gives

$$\theta' = 0.113/G \tag{ix}$$

Hence, from Eq. (viii)

$$v' = 71.2/G \tag{x}$$

Now substituting for θ' and v' from Eqs (ix) and (x) in Eqs (i)–(vi) gives

$$q_{12} = q_{56} = 46.6\,\text{N/mm}, \quad q_{32} = q_{54} = 1.4\,\text{N/mm}$$

$$q_{43} = 74.6\,\text{N/mm}, \quad q_{52} = 180.8\,\text{N/mm}$$

Finally, from Eq. (9.31)

$$x_R = -\frac{v'}{\theta'} = -\frac{71.2}{0.113} = -630.1\,\text{mm}, \quad y_R = \frac{u'}{\theta'} = 0$$

S.11.2

In Fig. S.11.2, $\alpha = \tan^{-1} 125/300 = 22.6°$. Also, since the walls of the beam section are straight the shear flow in each wall, from Eq. (11.1), is constant. Choosing O, the mid-point of the wall 42, as the origin, then, from Eq. (11.1) and referring to

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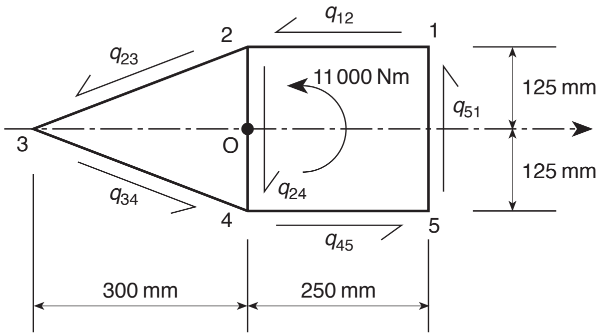

Fig. S.11.2

Fig. S.11.2

$$q_{51} = 1.6G(250\theta' + v') \tag{i}$$

$$q_{12} = 1.2G(125\theta' - u') \tag{ii}$$

$$q_{23} = 1.0G(125\theta' \cos 22.6^\circ - u' \cos 22.6^\circ - v' \sin 22.6^\circ)$$

i.e.

$$q_{23} = 1.0G(115.4\theta' - 0.923u' - 0.384v') \tag{iii}$$

$$q_{34} = 1.0G(115.4\theta' + 0.923u' - 0.384v') \tag{iv}$$

$$q_{45} = 1.2G(125\theta' + u') \tag{v}$$

$$q_{24} = 1.6G(-v') \tag{vi}$$

From antisymmetry $q_{12} = q_{45}$ and $q_{23} = q_{34}$. Thus, from Eqs (ii) and (v) (or Eqs (iii) and (iv)), $u' = 0$. Resolving forces vertically

$$q_{51} \times 250 - q_{24} \times 250 - q_{23} \times 125 - q_{34} \times 125 = 0$$

i.e.

$$q_{51} - q_{24} - q_{23} = 0 \tag{vii}$$

Substituting in Eq. (vii) for q_{51}, q_{24} and q_{23} from Eqs (i), (vi) and (iii) respectively gives

$$v' + 79.41\theta' = 0 \tag{viii}$$

Now taking moments about O

$$2q_{12} \times 250 \times 125 + 2q_{23} \times 300 \times 125 + q_{51} \times 250 \times 250 = 11\,000 \times 10^3$$

i.e.

$$q_{12} + 1.2q_{23} + q_{51} = 176 \tag{ix}$$

Substituting in Eq. (ix) for q_{12}, q_{23} and q_{51} from Eqs (ii), (iii) and (i) respectively gives

$$v' + 604.4\theta' = 154.5/G \tag{x}$$

Subtracting Eq. (x) from Eq. (viii) gives

$$\theta' = 0.294/G \tag{xi}$$

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whence, from Eq. (viii)

$$v' = -23.37/G \tag{xii}$$

Substituting for θ' and v' from Eqs (xi) and (xii) in Eqs (i)–(vi) gives

$$q_{51} = 80.2\,\text{N/mm}, \quad q_{12} = q_{45} = 44.1\,\text{N/mm}$$

$$q_{23} = q_{34} = 42.9\,\text{N/mm}, \quad q_{24} = 37.4\,\text{N/mm}$$

The centre of twist referred to O has coordinates, from Eq. (9.31)

$$x_R = -\frac{v'}{\theta'} = \frac{23.37}{0.294} = 79.5\,\text{mm}, \quad y_R = \frac{u'}{\theta'} = 0$$

S.11.3

Referring to Fig. S.11.3 the shear flows in the walls 12 and 23 are constant since the walls are straight (see Eq. (11.1)). Choosing O as the origin of axes, from Eq. (11.1)

$$q_{12} = Gt(\theta' R \cos 30^\circ + u' \cos 30^\circ + v' \sin 30^\circ)$$

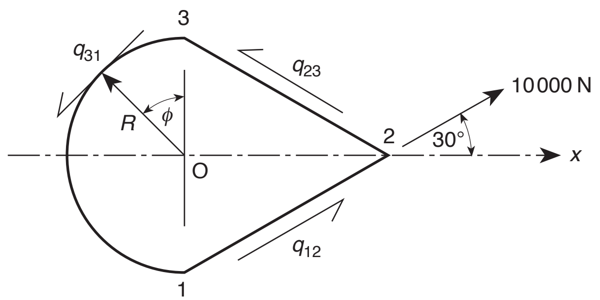

Fig. S.11.3

i.e.

$$q_{12} = Gt(0.866R\theta' + 0.866u' + 0.5v') \tag{i}$$

$$q_{23} = Gt(0.866R\theta' - 0.866u' + 0.5v') \tag{ii}$$

$$q_{31} = Gt(R\theta' - u' \cos\phi - v' \sin\phi) \tag{iii}$$

Resolving forces vertically

$$q_{12}R + q_{23}R - \int_0^\pi q_{31} \sin\phi\, R\, \mathrm{d}\phi = 10\,000 \sin 30^\circ$$

i.e.

$$q_{12} + q_{23} - \int_0^\pi q_{31} \sin\phi\, \mathrm{d}\phi = 5000/R \tag{iv}$$

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Substituting in Eq. (iv) for q_{12}, q_{23} and q_{31} from Eqs (i), (ii) and (iii) respectively gives

$$R\theta' - 9.59v' = -18\,656.7/GtR \tag{v}$$

Resolving forces horizontally

$$q_{12}(R/\tan 30^\circ) - q_{23}(R/\tan 30^\circ) - \int_0^\pi q_{31}\cos\phi\, R\,\mathrm{d}\phi = 10\,000\cos 30^\circ$$

i.e.

$$1.732q_{12} - 1.732q_{23} - \int_0^\pi q_{31}\cos\phi\,\mathrm{d}\phi = 8660.3/R \tag{vi}$$

Substituting in Eq. (vi) for q_{12}, q_{23} and q_{31} from Eqs (i), (ii) and (iii) respectively gives

$$u' = 1894.7/GtR \tag{vii}$$

Taking moments about O

$$q_{12}(R/\tan 30^\circ)R + q_{23}(R/\tan 30^\circ)R + \int_0^\pi q_{31}R^2\,\mathrm{d}\phi = 10\,000R\cos 30^\circ$$

i.e.

$$1.732q_{12} + 1.732q_{23} + \int_0^\pi q_{31}\,\mathrm{d}\phi = 8660.3/R \tag{viii}$$

Substituting in Eq. (viii) for q_{12}, q_{23} and q_{31} from Eqs (i), (ii) and (iii) respectively gives

$$R\theta' - 0.044v' = 1410.0/GtR \tag{ix}$$

Now subtracting Eq. (ix) from Eq. (v)

$$-9.546v' = -20\,066.7/GtR$$

whence

$$v' = 2102.1/GtR \tag{x}$$

Then, from Eq. (v)

$$R\theta' = 1502.4/GtR \tag{xi}$$

Substituting for u', v' and θ' from Eqs (vii), (x) and (xi) respectively in Eqs (i)–(iii) gives

$$q_{12} = 3992.9/R\,\mathrm{N/mm}, \quad q_{23} = 711.3/R\,\mathrm{N/mm}$$

$$q_{31} = (1502.4 - 1894.7\cos\phi - 2102.1\sin\phi)/R\,\mathrm{N/mm}$$

S.11.4

From Fig. P.11.4 the torque at any section of the beam is given by

$$T = 20\times 10^3(2500 - z)\,\mathrm{Nmm} \tag{i}$$

Eq. (11.16) for the warping distribution along boom 4 then becomes

$$w = C\cosh\mu z + D\sinh\mu z + \frac{20\times 10^3(2500 - z)}{8abG}\left(\frac{b}{t_b} - \frac{a}{t_a}\right) \tag{ii}$$

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where

$$\mu^2 = \frac{8Gt_b t_a}{BE(bt_a + at_b)}$$

Comparing Figs 11.5 and P.11.4, $t_b = t_a = 1.0\,\text{mm}$, $a = 500\,\text{mm}$, $b = 200\,\text{mm}$ and $B = 800\,\text{mm}$. Then

$$\mu^2 = \frac{8 \times 0.36 \times 1.0 \times 1.0}{800(200 \times 1.0 + 500 \times 1.0)}$$

from which

$$\mu = 2.27 \times 10^{-3}$$

Eq. (ii) then becomes

$$w = C \cosh 2.27 \times 10^{-3} z + D \sinh 2.27 \times 10^{-3} z - 3.75 \times 10^{-4}(2500 - z) \qquad \text{(iii)}$$

When $z = 0$, $w = 0$, hence, from Eq. (iii), $C = 0.9375$. At the free end the direct stress in boom 4 is zero so that the direct strain $\partial w/\partial z = 0$ at the free end. Hence, from Eq. (iii), $D = -0.9386$ and the warping distribution along boom 4 is given by

$$w = 0.9375 \cosh 2.27 \times 10^{-3} z - 0.9386 \sinh 2.27 \times 10^{-3} z$$
$$- 3.75 \times 10^{-4}(2500 - z) \qquad \text{(iv)}$$

Substituting for w from Eq. (iv) and T from Eq. (i) in Eq. (11.11)

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = -10^{-5}[1.6069 \cosh 2.27 \times 10^{-3} z - 1.6088 \sinh 2.27 \times 10^{-3} z$$
$$- 3.4998 \times 10^{-3}(2500 - z)] \qquad \text{(v)}$$

Then

$$\theta = -10^{-5}\left[\frac{1.6069}{2.27 \times 10^{-3}} \sinh 2.27 \times 10^{-3} z - \frac{1.6088}{2.27 \times 10^{-3}} \cosh 2.27 \times 10^{-3} z\right.$$
$$\left. - 3.4998 \times 10^{-3}\left(2500z - \frac{z^2}{2}\right)\right] + F \qquad \text{(vi)}$$

When $z = 0$, $\theta = 0$ so that, from Eq. (vi)

$$F = -10^{-5} \times 1.6088/2.27 \times 10^{-3}$$

and

$$\theta = -10^{-5}\left[707.9 \sinh 2.27 \times 10^{-3} z - 708.7 \cosh 2.27 \times 10^{-3} z\right.$$
$$\left. - 3.4998 \times 10^{-3}\left(2500z - \frac{z^2}{2}\right) + 708.7\right] \qquad \text{(vii)}$$

At the free end where $z = 2500\,\text{mm}$ Eq. (vii) gives

$$\theta = 0.1036\,\text{rad} = 5.9^\circ$$

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S.11.5

The warping distribution along the top right-hand corner boom is given by Eq. (11.16), i.e.

$$w = C \cosh \mu z + D \sinh \mu z + w_0 \tag{i}$$

where

$$\mu^2 = \frac{8Gt_2t_1}{BE(bt_1 + at_2)} \quad \text{and} \quad w_0 = \frac{T}{8abG}\left(\frac{b}{t_2} - \frac{a}{t_1}\right)$$

At each end of the beam the warping is completely suppressed, i.e. $w = 0$ at $z = 0$ and $z = l$. Thus, from Eq. (i)

$$0 = C + w_0$$

i.e.

$$C = -w_0$$

and

$$0 = C \cosh \mu l + D \sinh \mu l + w_0$$

which gives

$$D = \frac{w_0}{\sinh \mu l}(\cosh \mu l - 1)$$

Hence, Eq. (i) becomes

$$w = w_0\left[1 - \cosh \mu z + \frac{(\cosh \mu l - 1)}{\sinh \mu l}\sinh \mu z\right] \tag{ii}$$

The direct load, P, in the boom is then given by

$$P = \sigma_z B = BE\frac{\partial w}{\partial z}$$

Thus, from Eq. (ii)

$$P = \mu BEw_0\left[-\sinh \mu z + \frac{(\cosh \mu l - 1)}{\sinh \mu l}\cosh \mu z\right] \tag{iii}$$

or, substituting for w_0 from above

$$P = \frac{\mu BET}{8abGt_1t_2}(bt_1 - at_2)\left[-\sinh \mu z + \frac{(\cosh \mu l - 1)}{\sinh \mu l}\cosh \mu z\right] \tag{iv}$$

For a positive torque, i.e. T is anticlockwise when viewed along the z axis to the origin of z, the term in square brackets in Eq. (iv) becomes, when $z = 0$

$$\frac{\cosh \mu l - 1}{\sinh \mu l}$$

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which is positive. Thus at $z = 0$ the load in the boom is tensile. At $z = l$ the term in square brackets in Eq. (iv) becomes

$$\frac{1 - \cosh \mu l}{\sinh \mu l}$$

which is negative. Thus at $z = l$ the load in the boom is compressive. Also, from Eq. (iv) $\partial P/\partial z = 0$ at $z = l/2$ and the distribution of boom load is that shown in Fig. S.11.5. The reverse situation occurs for a negative, i.e. a clockwise, torque.

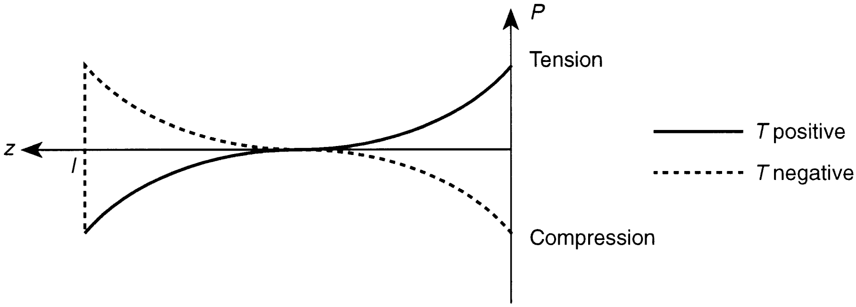

Fig. S.11.5

S.11.6

Initially directions for the shear flows, q, are chosen as shown in Fig. S.11.6(a). The panel is symmetrical about its horizontal centre line so that only half need be considered.

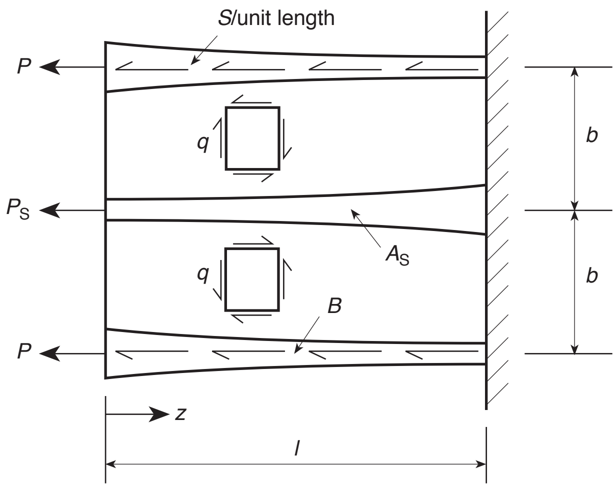

Fig. S.11.6(a)

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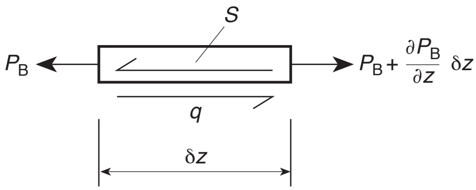

Fig. S.11.6(b)

For equilibrium of the element of the top boom shown in Fig. S.11.6(b)

$$P_{\mathrm{B}} + \frac{\partial P_{\mathrm{B}}}{\partial z}\delta z - P_{\mathrm{B}} - S\delta z + q\delta z = 0$$

i.e.

$$\frac{\partial P_{\mathrm{B}}}{\partial z} = S - q \tag{i}$$

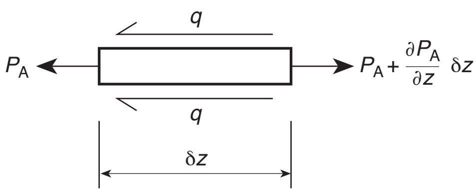

Fig. S.11.6(c)

Also, for equilibrium of the element of the central stringer shown in Fig. S.11.6(c)

$$P_{\mathrm{A}} + \frac{\partial P_{\mathrm{A}}}{\partial z}\delta z - P_{\mathrm{A}} - 2q\delta z = 0$$

i.e.

$$\frac{\partial P_{\mathrm{A}}}{\partial z} = 2q \tag{ii}$$

For equilibrium of the length, z, of the panel shown in Fig. S.11.6(d)

$$2P_{\mathrm{B}} + P_{\mathrm{A}} - 2Sz - 2P - P_{\mathrm{S}} = 0$$

i.e.

$$P_{\mathrm{A}} = 2P + P_{\mathrm{S}} + 2Sz - 2P_{\mathrm{B}} \tag{iii}$$

The compatibility of displacement condition for the top boom and central stringer is shown in Fig. S.11.6(e). Thus

$$(1 + \varepsilon_{\mathrm{A}})\delta z = (1 + \varepsilon_{\mathrm{B}})\delta z + b\frac{\mathrm{d}\gamma}{\mathrm{d}z}\delta z$$

i.e.

$$\frac{\mathrm{d}\gamma}{\mathrm{d}z} = \frac{1}{b}(\varepsilon_{\mathrm{A}} - \varepsilon_{\mathrm{B}}) \tag{iv}$$

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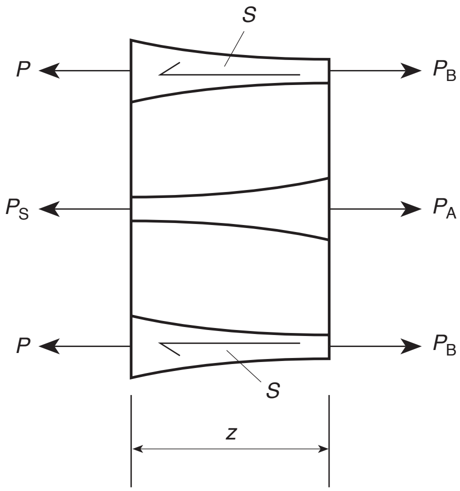

Fig. S.11.6(d)

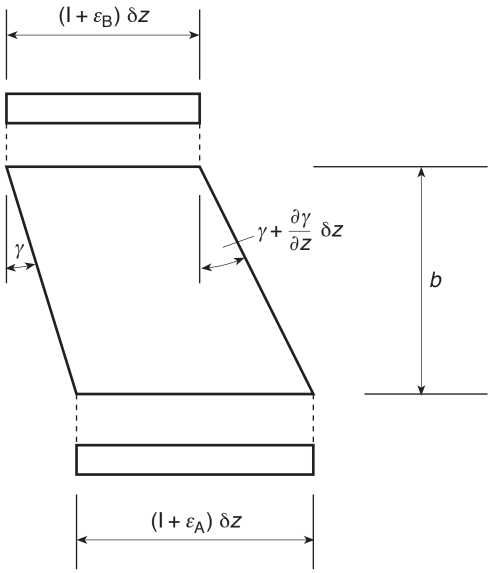

Fig. S.11.6(e)

Now

$$\varepsilon_A = \frac{\sigma_A}{E} \quad \text{and} \quad \varepsilon_B = \frac{\sigma_B}{E} = \frac{\sigma_e}{E} = \text{constant}$$

Also $\sigma_A = 0.8\sigma_e$ so that Eq. (iv) becomes

$$\frac{d\gamma}{dz} = -\frac{0.2\sigma_e}{bE} \tag{v}$$

In Eq. (v) $\gamma = q/Gt$, hence

$$\frac{dq}{dz} = -\frac{0.2Gt}{bE}\sigma_e \tag{vi}$$

Substituting for q in Eq. (vi) from Eq. (i) gives

$$\frac{\partial^2 P_B}{\partial z^2} = \frac{0.2Gt}{bE}\sigma_e$$

so that

$$P_B = \frac{0.1Gt\sigma_e}{bE}z^2 + Cz + D \tag{vii}$$

When $z = 0$, $P_B = P$ so that, from Eq. (vii), $D = P$. Also, when $z = l$, $q = 0$ so that, from Eq. (i), $\partial P_B/\partial z = S$ at $z = l$. Hence, from Eq. (vii)

$$C = -\frac{0.2Gt\sigma_e}{bE}l + S$$

and

$$P_B = \frac{0.1Gt\sigma_e}{bE}z^2 + \left(S - \frac{0.2Gt\sigma_e l}{bE}\right)z + P \tag{viii}$$

Now $P_B = \sigma_e B$ so that, from Eq. (viii)

$$B = \frac{0.1Gt}{bE}z^2 + \frac{1}{\sigma_e}\left[\left(S - \frac{0.2Gt\sigma_e l}{bE}\right)z + P\right] \tag{ix}$$

Substituting for P_B from Eq. (viii) in Eq. (iii) gives

$$P_A = \frac{0.4Gt\sigma_e}{bE}\left(lz - \frac{z^2}{2}\right) + P_S \tag{x}$$

But $P_A = A_S 0.8\sigma_e$ so that, from Eq. (x)

$$A_S = \frac{Gt}{2bE}\left(lz - \frac{z^2}{2}\right) + \frac{1.25P_S}{\sigma_e} \tag{xi}$$

Substituting the given values in Eqs (ix) and (xi) gives

$$B = 3.8 \times 10^{-4}z^2 + 0.3227z + 1636.4 \tag{xii}$$

and

$$A_S = 2.375z - 9.5 \times 10^{-4}z^2 + 659.1 \tag{xiii}$$

From Eq. (xiii) when $z = 1250$ mm, $A_S = 2143.5$ mm. Then

$$P_A = 0.8\sigma_e A_S = 0.8 \times 275 \times 2143.5 = 471\,570\,\text{N}$$

The total load, P_T, carried by the panel at the built-in end is

$$P_T = 2 \times 450\,000 + 145\,000 + 2 \times 350 \times 1250 = 1\,920\,000\,\text{N}$$

Therefore, the fraction of the load carried by the stringer is $471\,570/1\,920\,000 = 0.25$.

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S.11.7

The panel is symmetrical about its vertical centre line and therefore each half may be regarded as a panel with a built-in end as shown in Fig. S.11.7(a). Further, the panel is symmetrical about its horizontal centre line so that only the top half need be considered; the assumed directions of the shear flows are shown.

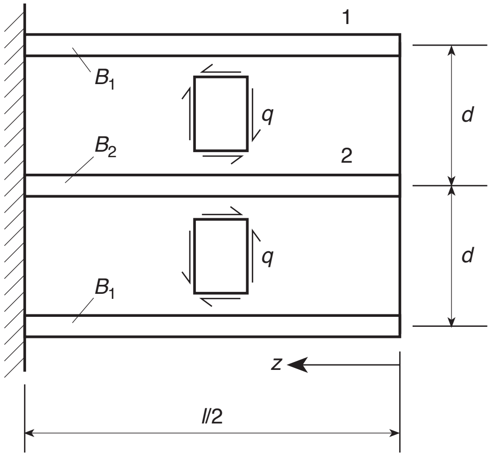

Fig. S.11.7(a)

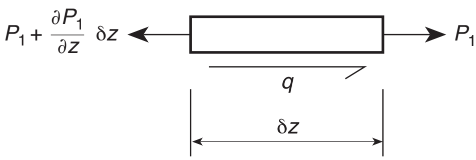

Fig. S.11.7(b)

Consider the equilibrium of the element of longeron 1 shown in Fig. S.11.7(b).

$$P_1 + \frac{\partial P_1}{\partial z}\delta z - P_1 - q\delta z = 0$$

Hence

$$\frac{\partial P_1}{\partial z} = q \tag{i}$$

Now consider the equilibrium of the element of longeron 2 shown in Fig. S.11.7(c).

$$P_2 + \frac{\partial P_2}{\partial z}\delta z - P_2 + 2q\delta z = 0$$

whence

$$\frac{\partial P_2}{\partial z} = -2q \tag{ii}$$

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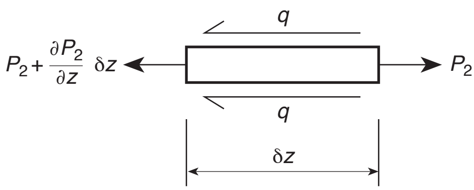

Fig. S.11.7(c)

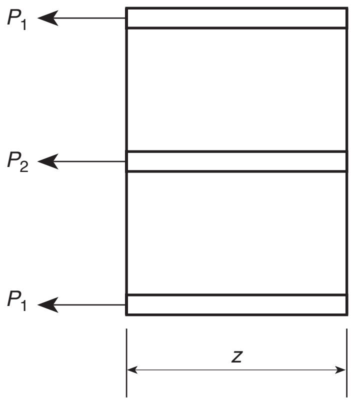

Fig. S11.7(d)

From the overall equilibrium of the length z of the panel shown in Fig. S.11.7(d)

$$2P_1 + P_2 = 0 \tag{iii}$$

The compatibility condition for an element of the top half of the panel is shown in Fig. S.11.7(e). Thus

$$(1 + \varepsilon_1)\delta z = (1 + \varepsilon_2)\delta z + d\frac{\mathrm{d}\gamma}{\mathrm{d}z}\delta z$$

i.e.

$$\frac{\mathrm{d}\gamma}{\mathrm{d}z} = \frac{1}{d}(\varepsilon_1 - \varepsilon_2) \tag{iv}$$

In Eq. (iv)

$$\varepsilon_1 = \frac{P_1}{B_1 E}$$

Also, an element, δz, of the central longeron would, without restraint, increase in length by an amount $\alpha T \delta z$. The element therefore suffers an effective strain equal to $(\varepsilon_2 - \alpha T)\delta z/\delta z$. Thus

$$\frac{P_2}{B_2 E} = \varepsilon_2 - \alpha T$$

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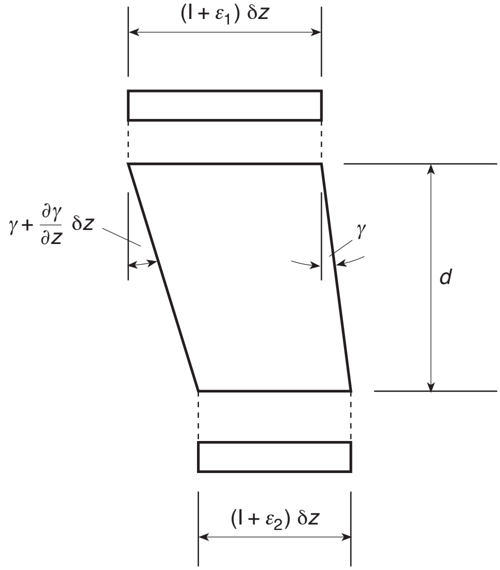

Fig. S.11.7(e)

so that Eq. (iv) becomes

$$\frac{\mathrm{d}\gamma}{\mathrm{d}z} = \frac{1}{dE}\left(\frac{P_1}{B_1} - \frac{P_2}{B_2} - \alpha TE\right) \tag{v}$$

Also $\gamma = q/Gt$ and from Eq. (ii) $q = -(\partial P_2/\partial z)/2$. Therefore, substituting for γ and then q in Eq. (v) and for P_1 from Eq. (iii) in Eq. (v)

$$-\frac{1}{2}\frac{\partial^2 P_2}{\partial z^2} = \frac{Gt}{dE}\left(-\frac{P_2}{2B_1} - \frac{P_2}{B_2} - \alpha TE\right)$$

or

$$\frac{\partial^2 P_2}{\partial z^2} - \frac{2GT}{dE}\left(\frac{1}{2B_1} + \frac{1}{B_2}\right) = \frac{2Gt\alpha T}{d} \tag{vi}$$

The solution of Eq. (vi) is

$$P_2 = C\cosh \mu z + D\sinh \mu z - \frac{2Gt\alpha T}{\mu^2 d} \tag{vii}$$

where

$$\mu^2 = \frac{2Gt}{dE}\left(\frac{1}{2B_1} + \frac{1}{B_2}\right)$$

When $z = 0$, $P_2 = 0$ so that, from Eq. (vii)

$$C = \frac{2Gt\alpha T}{\mu^2 d}$$

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Also when $z = l/2$, $q = 0$ and, from Eq. (ii), $\partial P_2/\partial z = 0$. Hence, from Eq. (vii)

$$0 = \mu C \sinh \mu \frac{l}{2} + \mu D \cosh \mu \frac{l}{2}$$

from which

$$D = -C \tanh \frac{\mu l}{2} = -\frac{2Gt\alpha T}{\mu^2 d} \tanh \frac{\mu l}{2}$$

Thus,

$$P_2 = \frac{2Gt\alpha T}{\mu^2 d} \left(\cosh \mu z - \tanh \frac{\mu l}{2} \sinh \mu z - 1 \right) \qquad \text{(viii)}$$

or, substituting for μ^2

$$P_2 = E\alpha T \left(\cosh \mu z - \tanh \frac{\mu l}{2} \sinh \mu z - 1 \right) \Big/ \left(\frac{1}{2B_1} + \frac{1}{B_2} \right) \qquad \text{(ix)}$$

From Fig. S.11.7(e) the relative displacement of the central longeron at one end of the panel is $d(\gamma)_{z=0}$. Now

$$\gamma_{z=0} = \left(\frac{q}{Gt} \right)_{z=0} = -\frac{1}{2Gt} \left(\frac{\partial P_2}{\partial z} \right)_{z=0} \qquad \text{(from Eq. (ii))}$$

Hence, from Eq. (ix)

$$\text{relative displacement} = \frac{\alpha T}{\mu} \tanh \frac{\mu l}{2}$$

S.11.8

The panel is unsymmetrical so that the shear flows in the top and bottom halves will have different values as shown in Fig. S.11.8(a).

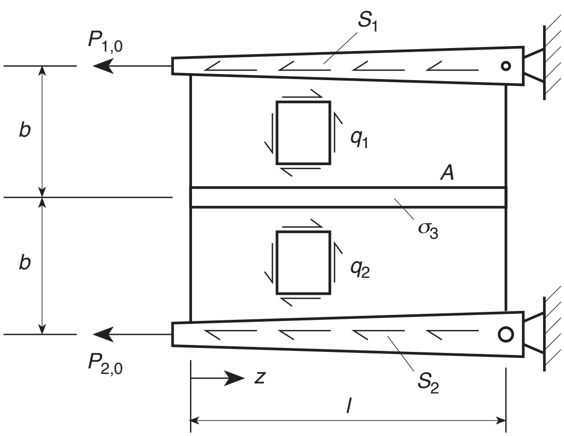

Fig. S.11.8(a)

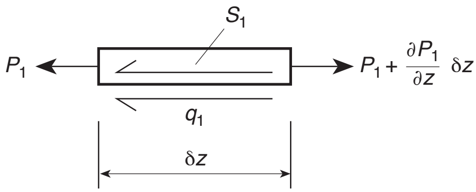

Fig. S.11.8(b)

For equilibrium of the element of the top member shown in Fig. S.11.8(b)

$$P_1 + \frac{\partial P_1}{\partial z}\delta z - P_1 - S_1\delta z - q_1\delta z = 0$$

i.e.

$$\frac{\partial P_1}{\partial z} = S_1 + q_1 \tag{i}$$

Similarly, for the equilibrium of the element of the central stringer shown in Fig. S.11.8(c)

$$P_3 + \frac{\partial P_3}{\partial z}\delta z - P_3 - q_2\delta z + q_1\delta z = 0$$

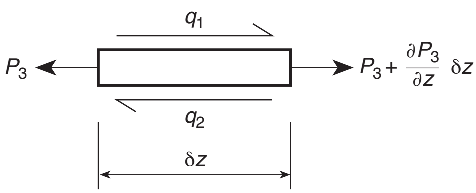

Fig. S.11.8(c)

i.e.

$$\frac{\partial P_3}{\partial z} = q_2 - q_1 \tag{ii}$$

Also, from Fig. S.11.8(d)

$$P_2 + \frac{\partial P_2}{\partial z}\delta z - P_2 - S_2\delta z + q_2\delta z = 0$$

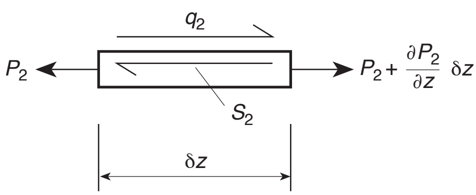

Fig. S.11.8(d)

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whence

$$\frac{\partial P_2}{\partial z} = S_2 - q_2 \tag{iii}$$

Now, from the longitudinal equilibrium of a length z of the panel (Fig. S.11.8(e))

$$P_1 + P_3 + P_2 - P_{1,0} - P_{2,0} - S_1 z - S_2 z = 0$$

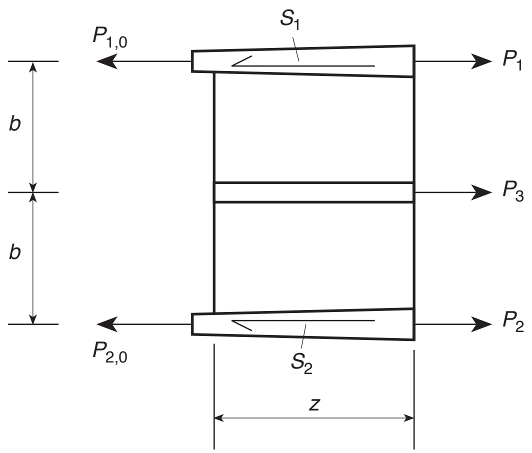

Fig. S.11.8(e)

i.e.

$$P_1 + P_3 + P_2 = P_{1,0} + P_{2,0} + (S_1 + S_2)z \tag{iv}$$

and from its moment equilibrium about the bottom edge member

$$P_1 2b + P_3 b - P_{1,0} 2b - S_1 z 2b = 0$$

i.e.

$$2P_1 + P_3 = 2P_{1,0} + 2S_1 z \tag{v}$$

From the compatibility condition between elements of the top edge member and the central stringer in Fig. S.11.8(f)

$$(1 + \varepsilon_1)\delta z = (1 + \varepsilon_3)\delta z + b\left(\frac{d\gamma_1}{dz} + \frac{\partial^2 v}{\partial z^2}\right)\delta z$$

or

$$\frac{d\gamma_1}{dz} = \frac{1}{b}(\varepsilon_1 - \varepsilon_3) - \frac{\partial^2 v}{\partial z^2} \tag{vi}$$

Similarly for elements of the central stringer and the bottom edge member

$$\frac{d\gamma_2}{dz} = \frac{1}{b}(\varepsilon_3 - \varepsilon_2) - \frac{\partial^2 v}{\partial z^2} \tag{vii}$$

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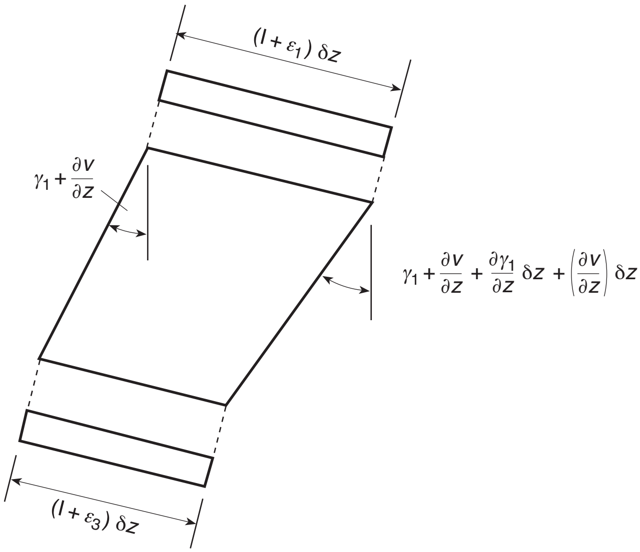

Fig. S.11.8(f)

Subtracting Eq. (vii) from Eq. (vi)

$$\frac{d\gamma_1}{dz} - \frac{d\gamma_2}{dz} = \frac{1}{b}(\varepsilon_1 - 2\varepsilon_3 + \varepsilon_2) \qquad \text{(viii)}$$

Now $\gamma = q/Gt$, $\varepsilon_1 = \sigma_1/E$, $\varepsilon_3 = \sigma_3/E$ and $\varepsilon_2 = \sigma_2/E$. Eq. (viii) may then be written

$$\frac{dq_1}{dz} - \frac{dq_2}{dz} = \frac{Gt}{bE}(\sigma_1 - 2\sigma_3 + \sigma_2)$$

or, from Eq. (i)

$$-\frac{\partial^2 P_3}{\partial z^2} = \frac{Gt}{bE}(\sigma_1 - 2\sigma_3 + \sigma_2)$$

Then, since $\sigma_3 = P_3/A$

$$\frac{\partial^2 \sigma_3}{\partial z^2} = \frac{Gt}{bEA}(2\sigma_3 - \sigma_1 - \sigma_2)$$

or

$$\frac{\partial^2 \sigma_3}{\partial z^2} - \frac{2Gt}{bEA}\sigma_3 = -\frac{Gt}{bEA}(\sigma_1 + \sigma_2) \qquad \text{(ix)}$$

The solution of Eq. (ix) is

$$\sigma_3 = C\cosh\mu z + D\sinh\mu z + (\sigma_1 + \sigma_2)/2$$

where $\mu^2 = 2Gt/bEA$.

When $z = 0$, $\sigma_3 = 0$ so that $C = -(\sigma_1 + \sigma_2)/2$. When $z = l$, $\sigma_3 = 0$ which gives

$$D = \frac{\sigma_1 + \sigma_2}{2\sinh\mu l}(\cosh\mu l - 1)$$

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Thus

$$\sigma_3 = \left(\frac{\sigma_1 + \sigma_2}{2}\right)\left[1 - \cosh \mu z - \frac{(1 - \cosh \mu l)}{\sinh \mu l} \sinh \mu z\right] \qquad \text{(x)}$$

From Eq. (i)

$$q_1 = \frac{\partial P_1}{\partial z} - S_1 \qquad \text{(xi)}$$

Substituting for P_1 from Eq. (v) in Eq. (xi)

$$q_1 = -\frac{1}{2}\frac{\partial P_3}{\partial z} = -\frac{1}{2}A\frac{\partial \sigma_3}{\partial z}$$

Therefore, from Eq. (x)

$$q_1 = A\left(\frac{\sigma_1 + \sigma_2}{4}\right)\mu\left[\sinh \mu z + \frac{(1 - \cosh \mu l)}{\sinh \mu l} \cosh \mu z\right]$$

S.11.9

This problem is similar to that of the six-boom beam analysed in Section 11.4 (Fig. 11.10) and thus the top cover of the beam is subjected to the loads shown in Fig. S.11.9(a). From symmetry the shear flow in the central panel of the cover is zero.

Considering the equilibrium of the element δz of the corner longeron (1) in Fig. S.11.9(b)

$$P_1 + \frac{\partial P_1}{\partial z}\delta z - P_1 + \frac{S}{2h}\delta z - q\delta z = 0$$

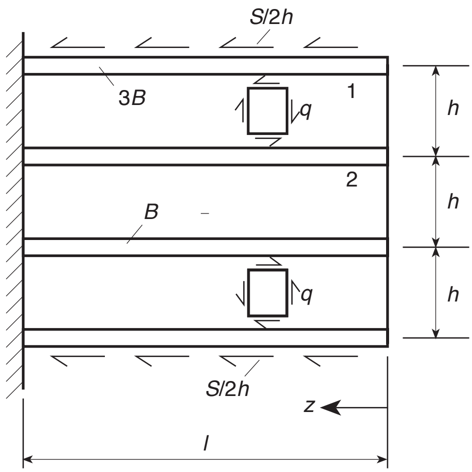

Fig. S.11.9(a)

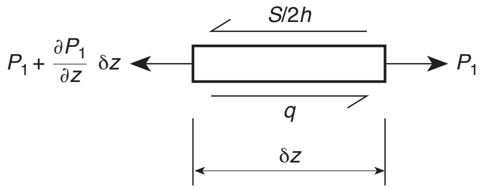

Fig. S.11.9(b)

i.e.

$$\frac{\partial P_1}{\partial z} = q - \frac{S}{2h} \tag{i}$$

Now considering the equilibrium of the element δz of longeron 2 in Fig. S.11.9(c)

$$P_2 + \frac{\partial P_2}{\partial z}\delta z - P_2 + q\delta z = 0$$

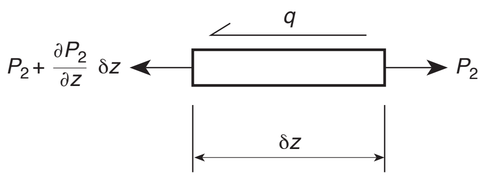

Fig. S.11.9(c)

which gives

$$\frac{\partial P_2}{\partial z} = -q \tag{ii}$$

From the equilibrium of the length z of the panel shown in Fig. S.11.9(d)

$$2P_1 + 2P_2 + 2\frac{S}{2h}z = 0$$

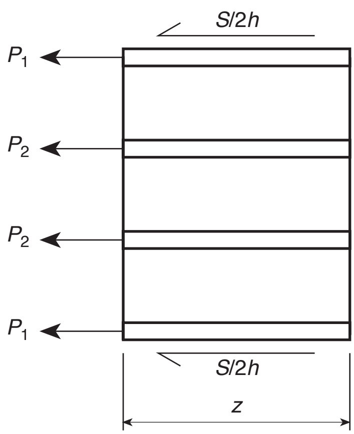

Fig. S.11.9(d)

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or

$$P_1 + P_2 = -\frac{Sz}{2h} \tag{iii}$$

The compatibility of the displacement condition between longerons 1 and 2 is shown in Fig. S.11.9(e). Thus

$$(1+\varepsilon_1)\delta z = (1+\varepsilon_2)\delta z + h\frac{d\gamma}{dz}\delta z$$

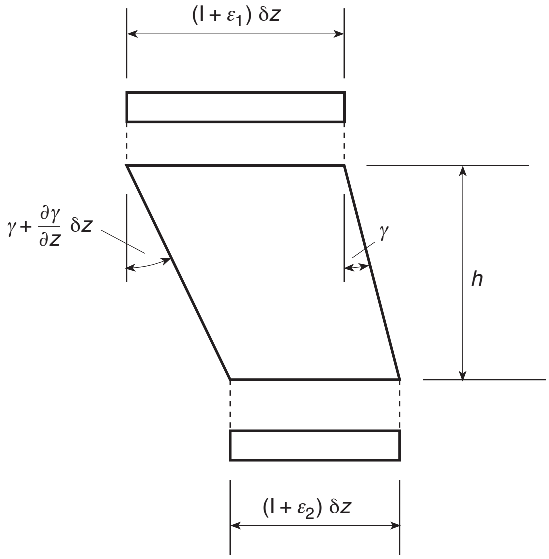

Fig. S.11.9(e)

from which

$$\frac{d\gamma}{dz} = \frac{1}{h}(\varepsilon_1 - \varepsilon_2) \tag{iv}$$

In Eq. (iv) $\gamma = q/Gt$, $\varepsilon_1 = P_1/3BE$, and $\varepsilon_2 = P_2/BE$. Eq. (iv) then becomes

$$\frac{dq}{dz} = \frac{Gt}{hBE}\left(\frac{P_1}{3} - P_2\right) \tag{v}$$

From Eq. (i)

$$\frac{dq}{dz} = \frac{\partial^2 P_1}{\partial z^2}$$

and from Eq. (iii)

$$P_2 = -P_1 - \frac{Sz}{2h}$$

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Substituting in Eq. (v)

$$\frac{\partial^2 P_1}{\partial z^2} = \frac{Gt}{hBE}\left(\frac{4P_1}{3} + \frac{Sz}{2h}\right)$$

or

$$\frac{\partial^2 P_1}{\partial z^2} - \frac{4Gt}{3hBE}P_1 = \frac{GtSz}{2h^2 BE} \tag{vi}$$

The solution of Eq. (vi) is

$$P_1 = C\cosh\mu z + D\sinh\mu z - \frac{3Sz}{8h} \tag{vii}$$

where $\mu^2 = 4Gt/3hBE$.

When $z = 0$, $P_1 = 0$ so that, from Eq. (vii), $C = 0$. When $z = l$, $q = 0$ so that, from Eq. (i), $\partial P_1/\partial z = -S/2h$. Hence from Eq. (vii)

$$D = -\frac{S}{8h\mu\cosh\mu l}$$

and Eq. (vii) becomes

$$P_1 = -\frac{S}{8h}\left(\frac{\sinh\mu z}{\mu\cosh\mu l} + 3z\right) \tag{viii}$$

Substituting for P_1 in Eq. (i) gives

$$q = -\frac{S}{8h}\left(\frac{\cosh\mu z}{\cosh\mu l} - 1\right) \tag{ix}$$

If the effect of shear lag is neglected then Eq. (ix) reduces to

$$q = \frac{S}{8h}$$

and the shear flow distribution is that shown in Fig. S.11.9(f) in which $q_{12} = q_{43} = q_{65} = q_{78} = S/8h$ and $q_{81} = q_{54} = S/2h$. The deflection Δ due to bending and shear is given by Eqs (9.86) and (9.88) in which

$$M_0 = -Sz \quad \text{and} \quad M_1 = -1 \times z$$

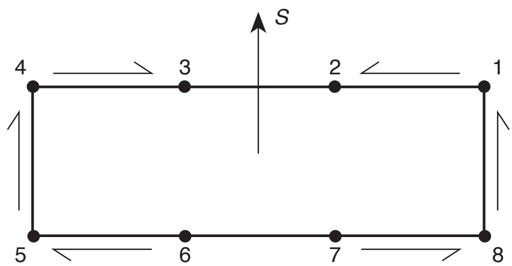

Fig. S.11.9(f)

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Also $I_{xx} = 4 \times 3B \times (h/2)^2 + 4 \times B \times (h/2)^2 = 4Bh^2$ and $q_1 = q_0/S$. Thus

$$\Delta = \int_0^L \frac{Sz^2}{4Bh^2E}\,\mathrm{d}z + \int_0^l \left(\oint \frac{q_0q_1}{Gt}\,\mathrm{d}s\right)\mathrm{d}z \tag{x}$$

In Eq. (x)

$$\oint \frac{q_0q_1}{Gt}\,\mathrm{d}s = \frac{S}{G}\left(\frac{4h}{64h^2t} + \frac{2h}{4h^2 3t}\right) = \frac{11S}{48Ght}$$

Hence, substituting in Eq. (x)

$$\Delta = \frac{Sl}{12h}\left(\frac{l^2}{BhE} + \frac{11}{4Gt}\right)$$

S.11.10

The position of the shear centre, S, is given and is also obvious by inspection (see Fig. S.11.10(a)). Initially, then, the swept area, $2A_{\mathrm{R},0}$ (see Section 11.5) is determined as a function of s. In 12, $2A_{\mathrm{R},0} = 2sd/2 = sd$. Hence, at 2, $2A_{\mathrm{R},0} = d^2$. In 23, $2A_{\mathrm{R},0} = 2(s/2)(d/2) + d^2 = sd/2 + d^2$. Therefore at 3, $2A_{\mathrm{R},0} = 3d^2/2$. In 34, $2A_{\mathrm{R},0}$ remains constant since $p = 0$. The remaining distribution follows from anti-symmetry and the complete distribution is shown in Fig. S.11.10(b). The centre of gravity of the 'wire' $1'2'3'4'5'6'$ (i.e. $2A'_{\mathrm{R}}$) is found by taking moments about the s axis. Thus

$$2A'_{\mathrm{R}}5dt = dt\left(\frac{d^2}{2} + \frac{5d^2}{4} + \frac{3d^2}{2} + \frac{5d^2}{4} + \frac{d^2}{2}\right)$$

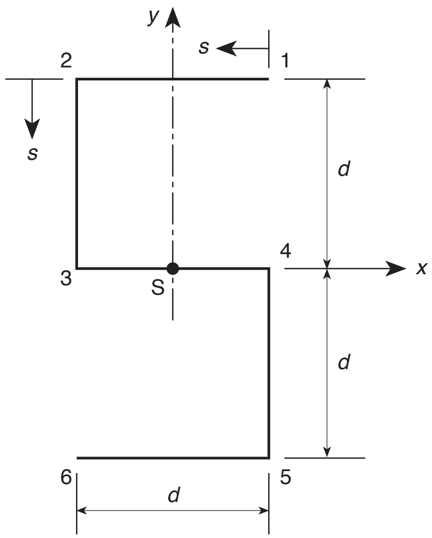

Fig. S.11.10(a)

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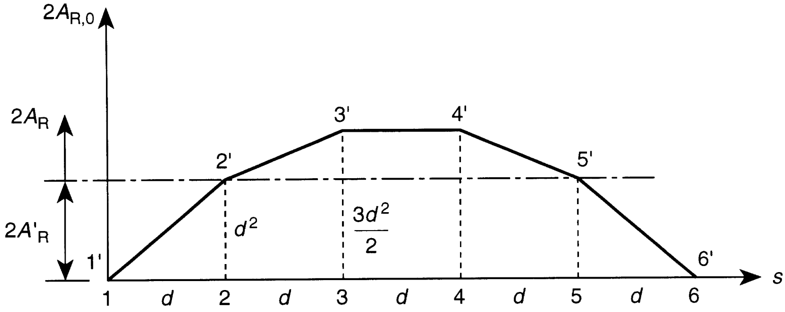

Fig. S.11.10(b)

which gives $2A'_R = d^2$. Therefore, instead of using Eq. (11.62), the moment of inertia of the wire (i.e. Γ_R) may be found directly, i.e.

$$\Gamma_R = 2dt\frac{(d^2)^2}{3} + 2dt\frac{(d^2/2)^2}{3} + dt\left(\frac{d^2}{2}\right)^2$$

which gives

$$\Gamma_R = \frac{13d^5 t}{12}$$

S.11.11

By inspection the shear centre, S, lies at the mid-point of the wall 34 (Fig. S.11.11(a)). The swept area, $2A_{R,0}$, is then determined as follows. In 12, $2A_{R,0} = (2sa\sin 2\alpha)/2$, i.e. $2A_{R,0} = a^2 \sin 2\alpha$. In 23, $2A_{R,0} = 2 \times \frac{1}{2}sa\sin 2\alpha + a^2 \sin 2\alpha = (sa + a^2)\sin 2\alpha$ and at 3, $2A_{R,0} = 2a^2 \sin 2\alpha$.

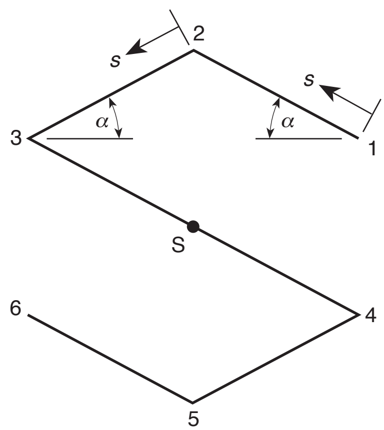

Fig. S.11.11(a)

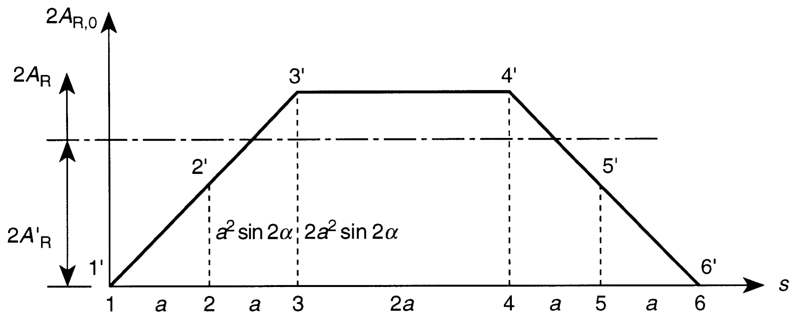

Fig. S.11.11(b)

In 34 there is no contribution to $2A_{\mathrm{R},0}$ since $p = 0$. The remaining distribution follows from antisymmetry and the complete distribution is shown in Fig. S.11.11(b).

The centre of gravity of the 'wire' $1'2'3'4'5'6'$ (i.e. $2A'_{\mathrm{R}}$) is found by taking moments about the s axis. Thus

$$2A'_{\mathrm{R}}6at = at(2 \times 2a^2 \sin 2\alpha + 2 \times 2a^2 \sin 2\alpha)$$

i.e.

$$2A'_{\mathrm{R}} = \tfrac{4}{3}a^2 \sin 2\alpha$$

Then, from Eq. (11.62)

$$\Gamma_{\mathrm{R}} = 2 \times 2at\frac{(2a^2 \sin 2\alpha)^2}{3} + 2at(2a^2 \sin 2\alpha)^2 - \left(\frac{4}{3}a^2 \sin 2\alpha\right)^2 6at$$

which gives

$$\Gamma_{\mathrm{R}} = \tfrac{8}{3}a^5 t \sin^2 2\alpha$$

S.11.12

The shear centre, S, of the section is at a distance $\pi r/3$ above the horizontal through the centres of the semicircular arcs (see P.9.11). Consider the left-hand portion of the section in Fig. S.11.12(a).

$$\begin{aligned} 2A_{\mathrm{R},0} &= -2(\text{Area BCS} - \text{Area BSO}) \\ &= -2(\text{Area CSF} + \text{Area CFOD} + \text{Area BCD} - \text{Area BSO}) \end{aligned}$$

i.e.

$$2A_{\mathrm{R},0} = -2\left[\frac{1}{2}(r\cos\theta_1 + r)\left(\frac{\pi r}{3} - r\sin\theta_1\right) + \frac{1}{2}(2r + r\cos\theta_1)r\sin\theta_1 + \frac{1}{2}r^2\theta_1 - \frac{1}{2}2r\frac{\pi r}{3}\right]$$

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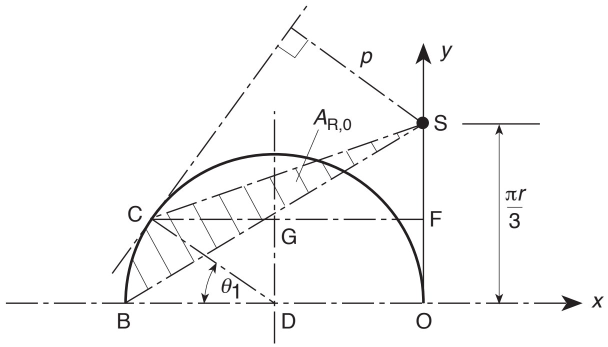

Fig. S.11.12(a)

i.e.

$$2A_{R,0} = r^2\left(\frac{\pi}{3} - \theta_1 - \sin\theta_1 - \frac{\pi}{3}\cos\theta_1\right) \qquad \text{(i)}$$

When $\theta_1 = \pi$, $2A_{R,0} = -\pi r^2/3$.

Note that in Eq. (i) $A_{R,0}$ is negative for the tangent in the position shown.

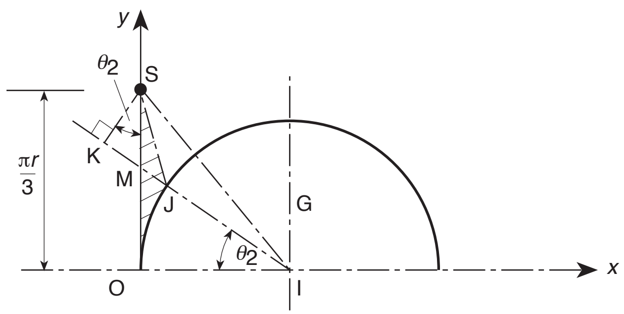

Fig. S.11.12(b)

Consider now the right-hand portion of the section shown in Fig. S.11.12(b). The swept area $2A_{R,0}$ is given by

$$2A_{R,0} = 2 \text{ Area OSJ} - \pi r^2/3$$

i.e.

$$2A_{R,0} = 2(\text{Area OSI} - \text{Area OJI} - \text{Area SJI}) - \pi r^2/3$$

which gives

$$2A_{R,0} = 2\left[\frac{1}{2}r\frac{\pi r}{3} - \frac{1}{2}r^2\theta_2 - \frac{1}{2}r\text{KS}\right] - \frac{\pi r^2}{3} \qquad \text{(ii)}$$

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In Eq. (ii)

$$\mathrm{KS} = \mathrm{MS}\cos\theta_2 = \left(\frac{\pi r}{3} - r\tan\theta_2\right)\cos\theta_2$$

i.e.

$$\mathrm{KS} = \frac{\pi r}{3}\cos\theta_2 - r\sin\theta_2$$

Substituting in Eq. (ii) gives

$$2A_{\mathrm{R},0} = r^2\left(\sin\theta_2 - \theta_2 - \frac{\pi}{3}\cos\theta_2\right) \tag{iii}$$

In Eq. (11.56)

$$\frac{\int_C 2A_{\mathrm{R},0}t\,\mathrm{d}s}{\int_C t\,\mathrm{d}s} = \frac{1}{2\pi r}\left[\int_0^\pi r^3\left(\frac{\pi}{3} - \theta_1 - \sin\theta_1 - \frac{\pi}{3}\cos\theta_1\right)\mathrm{d}\theta_1 + \int_0^\pi r^3\left(\sin\theta_2 - \theta_2 - \frac{\pi}{3}\cos\theta_2\right)\mathrm{d}\theta_2\right]$$

i.e.

$$\frac{\int_C 2A_{\mathrm{R},0}t\,\mathrm{d}s}{\int_C t\,\mathrm{d}s} = -\frac{\pi r^2}{3}$$

Hence, Eq. (11.56) becomes

$$2A_{\mathrm{R}} = 2A_{\mathrm{R},0} + \frac{\pi r^2}{3}$$

Then

$$\Gamma_{\mathrm{R}} = \int_C (2A_{\mathrm{R}})^2 t\,\mathrm{d}s = \int_0^\pi r^4\left(\frac{\pi}{3} - \theta_1 - \sin\theta_1 - \frac{\pi}{3}\cos\theta_1 + \frac{\pi}{3}\right)^2\mathrm{d}\theta_1 + \int_0^\pi r^4\left(\sin\theta_2 - \theta_2 - \frac{\pi}{3}\cos\theta_2 + \frac{\pi}{3}\right)^2\mathrm{d}\theta_2$$

which gives

$$\Gamma_{\mathrm{R}} = \pi^2 r^5 t\left(\frac{\pi}{3} - \frac{3}{\pi}\right)$$

S.11.13

The applied loading is equivalent to a shear load, P, through the shear centre (the centre of symmetry) of the beam section together with a torque $T = -Ph/2$. The

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direct stress distribution at the built-in end of the beam is then, from Eqs (9.9) and (11.4)

$$\sigma = \frac{M_x}{I_{xx}} y - 2A_R E \frac{d^2\theta}{dz^2} \tag{i}$$

In Eq. (i)

$$M_x = Pl \tag{ii}$$

and

$$I_{xx} = 2td^3/12 = td^3/6 \tag{iii}$$

Also $d^2\theta/dz^2$ is obtained from Eq. (11.59), i.e.

$$T = GJ \frac{d\theta}{dz} - E\Gamma_R \frac{d^3\theta}{dz^3}$$

or, rearranging

$$\frac{d^3\theta}{dz^3} - \mu^2 \frac{d\theta}{dz} = -\mu^2 \frac{T}{GJ} \tag{iv}$$

in which $\mu^2 = GJ/E\Gamma_R$. The solution of Eq. (iv) is

$$\frac{d\theta}{dz} = C \cosh \mu z + D \sinh \mu z + \frac{T}{GJ} \tag{v}$$

At the built-in end the warping is zero so that, from Eq. (9.67), $d\theta/dz = 0$ at the built-in end. Thus, from Eq. (v), $C = -T/GJ$. At the free end the direct stress, σ_Γ, is zero so that, from Eq. (11.54), $d^2\theta/dz^2 = 0$ at the free end. Then, from Eq. (v)

$$D = (T/GJ) \tanh \mu l$$

and Eq. (iii) becomes

$$\frac{d\theta}{dz} = \frac{T}{GJ}\left[1 - \frac{\cosh \mu(l - z)}{\cosh \mu l}\right] \tag{vi}$$

Differentiating Eq. (vi) with respect to z gives

$$\frac{d^2\theta}{dz^2} = \frac{T}{GJ} \mu \frac{\sinh \mu(l - z)}{\cosh \mu l} \tag{vii}$$

Hence, from Eq. (11.54)

$$\sigma_\Gamma = -2A_R E \frac{T}{GJ} \mu \frac{\sinh \mu(l - z)}{\cosh \mu l}$$

which, at the built-in end becomes

$$\sigma_\Gamma = -\sqrt{\frac{E}{GJ\Gamma_R}} T 2A_R \tanh \mu l \tag{viii}$$

In Eq. (viii)

$$J = (h + 2d)t^3/3 \quad \text{(see Eqs (9.59))} \tag{ix}$$

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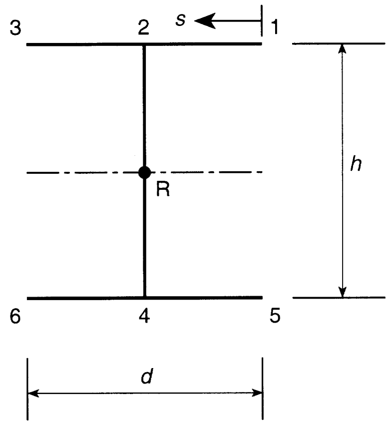

Fig. S.11.13(a)

The torsion bending constant, Γ_R, is found using the method described in Section 11.5. Thus, referring to Fig. S.11.13(a), in 12, $2A_{R,0} = sh/2$ and at 2, $2A_{R,0} = hd/4$. Also, at 3, $2A_{R,0} = hd/2$. Between 2 and 4, $2A_{R,0}$ remains constant and equal to $hd/4$. At 5, $2A_{R,0} = hd/4 + hd/4 = hd/2$ and at 6, $2A_{R,0} = hd/4 - hd/4 = 0$. The complete distribution is shown in Fig. S.11.13(b). By inspection $2A'_R = hd/4$. Then

$$\Gamma_R = 4t\frac{d}{2}\frac{1}{3}\left(\frac{hd}{4}\right)^2$$

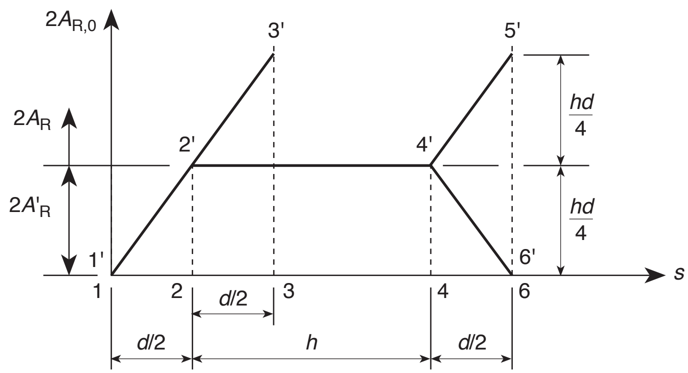

Fig. S.11.13(b)

i.e.

$$\Gamma_R = \frac{td^3h^2}{24} \tag{x}$$

Substituting the given values in Eqs (ii), (iii), (ix) and (x) gives

$$M_x = 200 \times 375 = 75\,000\,\text{Nmm}$$

$$I_{xx} = 2.5 \times 37.5^3/6 = 21\,973.0\,\text{mm}^4$$

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$$J = (75 + 2 \times 37.5)2.5^3/3 = 781.3\,\text{mm}^4$$

$$\Gamma_R = 2.5 \times 37.5^3 \times 75^2/24 = 3.09 \times 10^7\,\text{mm}^6$$

Then

$$\mu^2 = 781.3/(2.6 \times 3.09 \times 10^7) = 9.72 \times 10^{-6}$$

and

$$\mu = 3.12 \times 10^{-3}$$

Thus from Eqs (i) and (viii)

$$\sigma = 3.41y + 0.064(2A_R) \tag{xi}$$

Then, at 1 where $y = -d/2 = -18.75$ mm and $2A_R = -hd/4 = -703.1\,\text{mm}^2$,

$$\sigma_1 = -108.9\,\text{N/mm}^2 = -\sigma_3$$

Similarly

$$\sigma_5 = -18.9\,\text{N/mm}^2 = -\sigma_6$$

and

$$\sigma_2 = \sigma_4 = \sigma_{24} = 0$$

S.11.14

The rate of twist in each half of the beam is obtained from the solution of Eq. (11.59). Thus, referring to Fig. S.11.14, for BC

$$\frac{d\theta}{dz_1} = \frac{T}{8GJ} + A\cosh 2\mu z_1 + B\sinh 2\mu z_1 \tag{i}$$

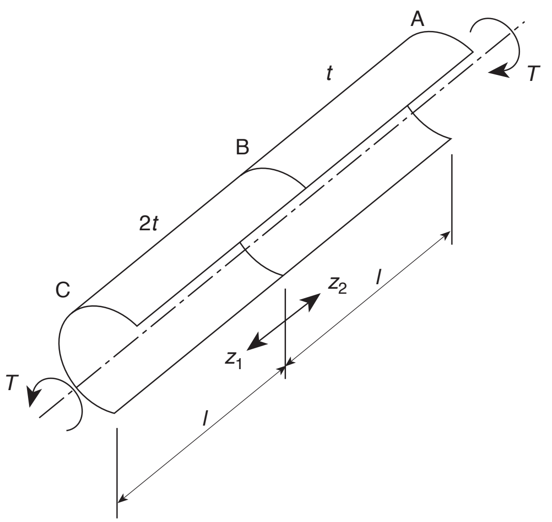

Fig. S.11.14

where $\mu^2 = GJ/E\Gamma$ and for BA

$$\frac{\mathrm{d}\theta}{\mathrm{d}z_2} = \frac{T}{GJ} + C\cosh\mu z_2 + D\sinh\mu z_2 \qquad \text{(ii)}$$

The boundary conditions are as follows:
When

$$z_1 = z_2 = 0, \quad \mathrm{d}\theta/\mathrm{d}z_1 = \mathrm{d}\theta/\mathrm{d}z_2 \qquad \text{(iii)}$$

When

$$z_1 = z_2 = l, \quad \mathrm{d}^2\theta/\mathrm{d}z_1^2 = \mathrm{d}^2\theta/\mathrm{d}z_2^2 = 0 \quad \text{(see Eq. (11.54))} \qquad \text{(iv)}$$

When

$$z_1 = z_2 = 0, \quad 2\,\mathrm{d}^2\theta/\mathrm{d}z_1^2 = -\mathrm{d}^2\theta/\mathrm{d}z_2^2 \qquad \text{(v)}$$

(since the loads at B in each half of the section are equal and opposite). From Eqs (i), (ii) and (iv)

$$B = -A\tanh 2\mu l \qquad \text{(vi)}$$

$$D = -C\tanh\mu l \qquad \text{(vii)}$$

From Eqs (i), (ii) and (iii)

$$\frac{T}{8GJ} + A = \frac{T}{GJ} + C$$

i.e.

$$A - C = \frac{7T}{8GJ} \qquad \text{(viii)}$$

From Eqs (i), (ii) and (v)

$$D = -4B \qquad \text{(ix)}$$

Solving Eqs (vi)–(ix) gives

$$B = -\frac{7T\tanh\mu l\tanh 2\mu l}{8GJ(4\tanh 2\mu l + \tanh\mu l)}$$

$$D = \frac{7T(4\tanh\mu l\tanh 2\mu l)}{8GJ(4\tanh 2\mu l + \tanh\mu l)}$$

$$A = \frac{7T\tanh\mu l}{8GJ(4\tanh 2\mu l + \tanh\mu l)}$$

$$C = -\frac{7T(4\tanh 2\mu l)}{8GJ(4\tanh 2\mu l + \tanh\mu l)}$$

Integrating Eq. (i)

$$\theta_1 = \frac{T}{8GJ}z_1 + \frac{A}{2\mu}\sinh 2\mu z_1 + \frac{B}{2\mu}\cosh 2\mu z_1 + F$$

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When $z_1 = 0$, $\theta_1 = 0$ so that $F = -B/2\mu$. Integrating Eq. (ii)

$$\theta_2 = \frac{T}{GJ} z_2 + \frac{C}{\mu} \sinh \mu z_2 + \frac{D}{\mu} \cosh \mu z_2 + H$$

When $z_2 = 0$, $\theta_2 = 0$ so that $H = -D/\mu$. Hence, when $z_1 = l$ and $z_2 = l$ the angle of twist of one end of the beam relative to the other is

$$\begin{aligned}\theta_1 + \theta_2 = {} & \frac{T}{8GJ}(l + 8l) + \frac{7T}{8GJ\mu(4 \tanh 2\mu l + \tanh \mu l)} \\ & \times [\tfrac{1}{2}(\tanh \mu l \sinh 2\mu l - \tanh \mu l \tanh 2\mu l \cosh 2\mu l - 4 \tanh 2\mu l \sinh \mu l \\ & + 4 \tanh \mu l \tanh 2\mu l \cosh \mu l - \tfrac{7}{2}(\tanh \mu l \tanh 2\mu l)]\end{aligned}$$

which simplifies to

$$\theta_1 + \theta_2 = \frac{Tl}{8GJ}\left[9 - \frac{49 \sinh 2\mu l}{2\mu l(10 \cosh^2 \mu l - 1)}\right]$$

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Solutions to Chapter 12 Problems

S.12.1

Referring to Fig. P.12.1 and Fig. 12.3

Member	12	23	34	41	13
Length	L	L	L	L	$\sqrt{2}L$
$\lambda(\cos\theta)$	$1/\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
$\mu(\sin\theta)$	$1/\sqrt{2}$	$1/\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1

The stiffness matrix for each member is obtained using Eq. (12.30). Thus

$$[K_{12}] = \frac{AE}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad [K_{23}] = \frac{AE}{2L}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$[K_{34}] = \frac{AE}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad [K_{41}] = \frac{AE}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$[K_{13}] = \frac{AE}{\sqrt{2}L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The stiffness matrix for the complete framework is now assembled using the method

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described in Example 12.1. Eq. (12.29) then becomes

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \\ F_{x,4} \\ F_{y,4} \end{Bmatrix} = \frac{AE}{2L} \begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 2+\sqrt{2} & -1 & -1 & 0 & -\sqrt{2} & 1 & -1 \\ -1 & -1 & 2 & 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 & -1 & -1 \\ 0 & -\sqrt{2} & 1 & -1 & 0 & 2+\sqrt{2} & -1 & -1 \\ -1 & 1 & 0 & 0 & -1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 0 & -1 & -1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ v_1 \\ u_2 = 0 \\ v_2 = 0 \\ u_3 = 0 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} \qquad \text{(i)}$$

In Eq. (i)

$$F_{y,1} = -P, \quad F_{x,1} = F_{x,3} = F_{y,3} = 0$$

Then

$$F_{y,1} = -P = \frac{AE}{2L}[(2+\sqrt{2})v_1 - \sqrt{2}v_3] \qquad \text{(ii)}$$

$$F_{y,3} = 0 = \frac{AE}{2L}[-\sqrt{2}v_1 + (2+\sqrt{2})v_3] \qquad \text{(iii)}$$

From Eq. (iii)

$$v_1 = (1+\sqrt{2})v_3 \qquad \text{(iv)}$$

Substituting for v_1 in Eq. (ii) gives

$$v_3 = -\frac{0.293PL}{AE}$$

Hence, from Eq. (iv)

$$v_1 = -\frac{0.707PL}{AE}$$

The forces in the members are obtained using Eq. (12.32), thus

$$S_{12} = \frac{AE}{\sqrt{2}L}[1 \quad 1]\begin{Bmatrix} 0-0 \\ 0+0.707PL/AE \end{Bmatrix} = \frac{P}{2} = S_{14} \text{ from symmetry}$$

$$S_{13} = \frac{AE}{\sqrt{2}L}[0 \quad 1]\begin{Bmatrix} 0-0 \\ -0.293PL/AE + 0.707PL/AE \end{Bmatrix} = 0.293P$$

$$S_{23} = \frac{AE}{\sqrt{2}L}[-1 \quad 1]\begin{Bmatrix} 0-0 \\ -0.293PL/AE - 0 \end{Bmatrix} = -0.207P = S_{43} \text{ from symmetry}$$

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The support reactions are $F_{x,2}$, $F_{y,2}$, $F_{x,4}$ and $F_{y,4}$. From Eq. (i)

$$F_{x,2} = \frac{AE}{2L}(-v_1 + v_3) = 0.207P$$

$$F_{y,2} = \frac{AE}{2L}(-v_1 - v_3) = 0.5P$$

$$F_{x,4} = \frac{AE}{2L}(v_1 - v_3) = -0.207P$$

$$F_{y,4} = \frac{AE}{2L}(-v_1 - v_3) = 0.5P$$

S.12.2

Referring to Fig. P.12.2 and Fig. 12.3

Member	12	23	34	31	24
Length	$l/\sqrt{3}$	$l/\sqrt{3}$	l	l	$l/\sqrt{3}$
$\lambda(\cos\theta)$	$\sqrt{3}/2$	0	1/2	−1/2	$\sqrt{3}/2$
$\mu(\sin\theta)$	1/2	1	$-\sqrt{3}/2$	$-\sqrt{3}/2$	−1/2

From Eq. (12.30) the member stiffness matrices are

$$[K_{12}] = \frac{AE}{l}\begin{bmatrix} 3\sqrt{3}/4 & 3/4 & -3\sqrt{3}/4 & -3/4 \\ 3/4 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 \\ -3\sqrt{3}/4 & -3/4 & 3\sqrt{3}/4 & 3/4 \\ -3/4 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 \end{bmatrix}$$

$$[K_{23}] = \frac{AE}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & \sqrt{3} \end{bmatrix}$$

$$[K_{34}] = \frac{AE}{l}\begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{31}] = \frac{AE}{l}\begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix}$$

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$$[K_{24}] = \frac{AE}{l}\begin{bmatrix} 3\sqrt{3}/4 & -3/4 & -3\sqrt{3}/4 & 3/4 \\ -3/4 & \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 \\ -3\sqrt{3}/4 & 3/4 & 3\sqrt{3}/4 & -3/4 \\ 3/4 & -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 \end{bmatrix}$$

The stiffness matrix for the complete framework is now assembled using the method described in Example 12.1. Eq. (12.29) then becomes

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \\ F_{x,4} \\ F_{y,4} \end{Bmatrix} = \frac{AE}{l}\left|\begin{matrix} \frac{1+3\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} & -\frac{3\sqrt{3}}{4} & -\frac{3}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 & 0 \\ \frac{3+\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & 0 \\ -\frac{3\sqrt{3}}{4} & -\frac{3}{4} & \frac{3\sqrt{3}}{2} & 0 & 0 & 0 & -\frac{3\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & 0 & \frac{3\sqrt{3}}{2} & 0 & -\sqrt{3} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & -\sqrt{3} & 0 & \frac{3}{2}+\sqrt{3} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ 0 & 0 & -\frac{3\sqrt{3}}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1+3\sqrt{3}}{4} & -\frac{3+\sqrt{3}}{4} \\ 0 & 0 & \frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{3+\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} \end{matrix}\right|\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = 0 \\ v_2 \\ u_3 = 0 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} \tag{i}$$

In Eq. (i) $F_{x,2} = F_{y,2} = 0$, $F_{x,3} = 0$, $F_{y,3} = -P$, $F_{x,4} = -H$. Then

$$F_{y,2} = 0 = \frac{AE}{l}\left(\frac{3\sqrt{3}}{2}v_2 - \sqrt{3}v_3\right) \tag{ii}$$

and

$$F_{y,3} = -P = \frac{AE}{l}\left[-\sqrt{3}v_2 + \left(\frac{3}{2} + \sqrt{3}\right)v_3\right] \tag{iii}$$

From Eq. (ii)

$$v_2 = \frac{2}{3}v_3 \tag{iv}$$

Now substituting for v_2 in Eq. (iii)

$$-\frac{Pl}{AE} = -\frac{2\sqrt{3}}{3}v_3 + \frac{3}{2}v_3 + \sqrt{3}v_3$$

Hence

$$v_3 = -\frac{6Pl}{(9 + 2\sqrt{3})AE}$$

and, from Eq. (iv)

$$v_2 = -\frac{4Pl}{(9 + 2\sqrt{3})AE}$$

Also from Eq. (i)

$$F_{x,4} = -H = \frac{AE}{l}\left(\frac{3}{4}v_2 + \frac{\sqrt{3}}{4}v_3\right)$$

Substituting for v_2 and v_3 gives

$$H = 0.449P$$

S.12.3

Referring to Fig. P.12.3 and Fig. 12.3

Member	12	23	34	45	24
Length	l	l	l	l	l
$\lambda(\cos\theta)$	$-1/2$	$1/2$	$-1/2$	$1/2$	1
$\mu(\sin\theta)$	$\sqrt{3}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	0

From Eq. (12.30) the member stiffness matrices are

$$[K_{12}] = \frac{AE}{l}\begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{23}] = \frac{AE}{l}\begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{34}] = \frac{AE}{l}\begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{45}] = \frac{AE}{l}\begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix}$$

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$$[K_{24}] = \frac{AE}{l}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The stiffness matrix for the complete truss is now assembled using the method described in Example 12.1. Eq. (12.29) then becomes

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \\ F_{x,4} \\ F_{y,4} \\ F_{x,5} \\ F_{y,5} \end{Bmatrix} = \frac{AE}{4l}\left|\begin{matrix} 1 & -\sqrt{3} & -1 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 3 & \sqrt{3} & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & \sqrt{3} & 6 & 0 & -1 & -\sqrt{3} & -4 & 0 & 0 & 0 \\ \sqrt{3} & -3 & 0 & 6 & -\sqrt{3} & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} & 2 & 0 & -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & -\sqrt{3} & -3 & 0 & 6 & \sqrt{3} & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 & -1 & \sqrt{3} & 6 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{3} & -3 & 0 & 6 & -\sqrt{3} & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\sqrt{3} & 1 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & -3 & \sqrt{3} & 3 \end{matrix}\right| \begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 \\ u_3 = 0 \\ v_3 = 0 \\ u_4 \\ v_4 \\ u_5 = 0 \\ v_5 = 0 \end{Bmatrix} \tag{i}$$

In Eq. (i) $F_{x,2} = F_{y,2} = 0$, $F_{x,4} = 0$, $F_{y,4} = -P$. Thus from Eq. (i)

$$F_{x,2} = 0 = \frac{AE}{4l}(6u_2 - 4u_4) \tag{ii}$$

$$F_{y,2} = 0 = \frac{AE}{4l}(6v_2) \tag{iii}$$

$$F_{x,4} = 0 = \frac{AE}{4l}(-4u_2 + 6u_4) \tag{iv}$$

$$F_{y,4} = -P = \frac{AE}{4l}(6v_4) \tag{v}$$

From Eq. (v)

$$v_4 = -\frac{2Pl}{3AE}$$

From Eq. (iii)

$$v_2 = 0$$

and from Eqs (ii) and (iv)

$$u_2 = u_4 = 0$$

Hence, from Eq. (12.32)

$$S_{24} = \frac{AE}{l}[1 \quad 0]\begin{Bmatrix} 0 - 0 \\ -2Pl/3AE - 0 \end{Bmatrix}$$

which gives

$$S_{24} = 0$$

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S.12.4

The uniformly distributed load on the member 26 is equivalent to concentrated loads of $wl/4$ at nodes 2 and 6 together with a concentrated load of $wl/2$ at node 4. Thus, referring to Fig. P.12.4 and Fig. 12.3

Member	12	23	24	46	56	67
Length	l	l	$l/2$	$l/2$	l	l
$\lambda(\cos\theta)$	0	$-1/\sqrt{2}$	1	1	0	$1/\sqrt{2}$
$\mu(\sin\theta)$	1	$1/\sqrt{2}$	0	0	1	$1/\sqrt{2}$

From Eq. (12.47) and using the alternative form of Eq. (12.44)

$$[K_{12}] = \frac{EI}{l^3}\begin{bmatrix} 12 & & & & & \text{SYM} \\ 0 & 0 & & & & \\ 6 & 0 & 4 & & & \\ -12 & 0 & -6 & 12 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 6 & 0 & 2 & 6 & 0 & 0 \end{bmatrix}$$

$$[K_{23}] = \frac{EI}{l^3}\begin{bmatrix} 6 & & & & & \text{SYM} \\ 6 & 6 & & & & \\ 6/\sqrt{2} & 6/\sqrt{2} & 4 & & & \\ 6 & 6 & -6/\sqrt{2} & 6 & & \\ -6 & -6 & -6/\sqrt{2} & 6 & 6 & \\ 6/\sqrt{2} & 6/\sqrt{2} & 2 & 6/\sqrt{2} & -6/\sqrt{2} & -4/\sqrt{2} \end{bmatrix}$$

$$[K_{24}] = [K_{46}] = \frac{EI}{l^3}\begin{bmatrix} 0 & & & & & \text{SYM} \\ 0 & 96 & & & & \\ 0 & -24 & 8 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -96 & 24 & 0 & 96 & \\ 0 & -24 & 4 & 0 & 24 & 8 \end{bmatrix}$$

$$[K_{56}] = \frac{EI}{l^3}\begin{bmatrix} 12 & & & & & \text{SYM} \\ 0 & 0 & & & & \\ 6 & 0 & 4 & & & \\ -12 & 0 & -6 & 12 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 6 & 0 & 2 & 6 & 0 & 0 \end{bmatrix}$$

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$$[K_{67}] = \frac{EI}{l^3}\begin{bmatrix} 6 & & & & & \text{SYM} \\ -6 & 6 & & & & \\ 6/\sqrt{2} & -6/\sqrt{2} & 4 & & & \\ -6 & 6 & -6/\sqrt{2} & 6 & & \\ 6 & -6 & 6/\sqrt{2} & -6 & 6 & \\ 6/\sqrt{2} & -6/\sqrt{2} & 2 & 6/\sqrt{2} & 6/\sqrt{2} & 4/\sqrt{2} \end{bmatrix}$$

The member stiffness matrices are then assembled into a 21×21 symmetrical matrix using the method described in Example 12.1. The known nodal displacements are $u_1 = v_1 = \theta_1 = u_5 = v_5 = \theta_5 = u_2 = u_4 = u_6 = \theta_3 = \theta_7 = 0$ and the support reactions are obtained from $\{F\} = [K]\{\delta\}$. Having obtained the support reactions the internal shear force and bending moment distributions in each member follow (see Example 12.2).

S.12.5

Referring to Fig. P.12.5, $u_2 = 0$ from symmetry. Consider the members 23 and 29. The forces acting on the member 23 are shown in Fig. S.12.5(a) in which F_{29} is the force applied at 2 in the member 23 due to the axial force in the member 29. Suppose that the node 2 suffers a vertical displacement v_2. The shortening in the member 29 is then $v_2 \cos\theta$ and the corresponding strain is $-(v_2 \cos\theta)/l$. Thus the compressive stress in 29 is $-(Ev_2 \cos\theta)/l$ and the corresponding compressive force is $-(AEv_2 \cos\theta)/l$. Thus

$$F_{29} = -(AEv_2 \cos^2\theta)/l$$

Now $AE = 6\sqrt{2}EI/L^2$, $\theta = 45^\circ$ and $l = \sqrt{2}L$. Hence

$$F_{29} = -\frac{3EI}{L^3}v_2$$

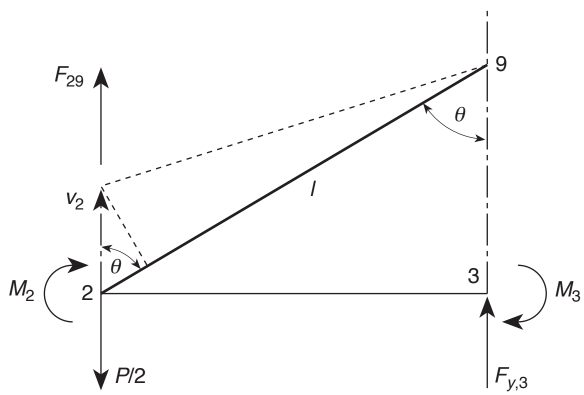

Fig. S.12.5(a)

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and

$$F_{y,2} = -\frac{P}{2} - \frac{3EI}{L^3} v_2 \tag{i}$$

Further, from Eq. (3.12)

$$M_3 = GJ\frac{d\theta}{dz} = -2 \times 0.8EI\frac{\theta_3}{0.8L} = -\frac{2EI}{L}\theta_3 \tag{ii}$$

From the alternative form of Eq. (12.44), for the member 23

$$\begin{Bmatrix} F_{y,2} \\ M_2/L \\ F_{y,3} \\ M_3/L \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 L = 0 \\ v_3 = 0 \\ \theta_3 L \end{Bmatrix} \tag{iii}$$

Then, from Eqs (i) and (iii)

$$F_{y,2} = -\frac{P}{2} - \frac{3EI}{L^3}v_2 = \frac{12EI}{L^3}v_2 - \frac{6EI}{L^2}\theta_3$$

Hence

$$15v_2 - 6\theta_3 L = -\frac{PL^3}{2EI} \tag{iv}$$

From Eqs (ii) and (iii)

$$\frac{M_3}{L} = -\frac{2EI}{L^2}\theta_3 = -\frac{6EI}{L^3}v_2 + \frac{4EI}{L^2}\theta_3$$

which gives $\theta_3 = v_2/L$.

Substituting for θ_3 in Eq. (iv) gives

$$v_2 = -\frac{PL^3}{18EI}$$

Then

$$\theta_3 = -\frac{PL^2}{18EI}$$

From Eq. (i)

$$F_{y,2} = -\frac{P}{2} + \frac{3EI}{L^3}\frac{PL^3}{18EI} = -\frac{P}{3}$$

and from Eq. (ii)

$$M_3 = \frac{2EI}{L}\frac{PL^2}{18EI} = \frac{PL}{9} = -M_1$$

Now, from Eq. (iii)

$$\frac{M_2}{L} = -\frac{EI}{L^3}6v_2 + \frac{2EI}{L^3}\theta_3 L = \frac{2PL}{9}$$

$$F_{y,3} = -\frac{12EI}{L^3}v_2 + \frac{6EI}{L^3}\theta_3 L = \frac{P}{3}$$

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The force in the member 29 is $F_{29}/\cos\theta = \sqrt{2}F_{29}$. Thus

$$S_{29} = S_{28} = \sqrt{2}\frac{3EI}{L^3}\frac{PL^3}{18EI} = \frac{\sqrt{2}P}{6} \quad \text{(tension)}$$

The torques in the members 36 and 37 are given by $M_3/2$, i.e.

$$M_{36} = M_{37} = PL/18$$

The shear force and bending moment diagrams for the member 123 follow and are shown in Figs S.12.5(b) and (c) respectively.

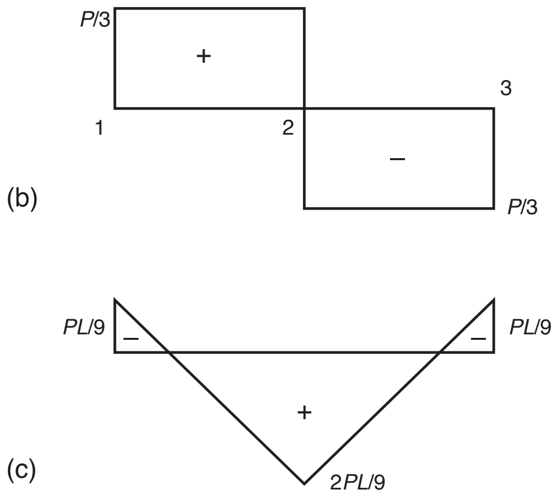

Fig. S.12.5(b) and (c)

S.12.6

The stiffness matrix for each element of the beam is obtained using the given force–displacement relationship, the complete stiffness matrix for the beam is then obtained using the method described in Example 12.1. This gives

$$\begin{Bmatrix} F_{y,1} \\ M_1/L \\ F_{y,2} \\ M_2/L \\ F_{y,3} \\ M_3/L \\ F_{y,4} \\ M_4/L \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & -12 & -24 & -12 & & & & \\ -12 & 8 & 12 & 4 & & & & \\ -24 & 12 & 36 & 6 & -12 & -6 & & \\ -12 & 4 & 6 & 12 & 6 & 2 & & \\ & & -12 & 6 & 36 & -24 & -24 & -12 \\ & & -6 & 2 & -6 & 12 & 12 & 4 \\ & & & & -24 & 12 & 24 & 12 \\ & & & & -12 & 4 & 12 & 8 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 L \\ v_2 \\ \theta_2 L \\ v_3 \\ \theta_3 L \\ v_4 \\ \theta_4 L \end{Bmatrix} \quad \text{(i)}$$

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The ties FB, CH, EB and CG produce vertically upward forces F_2 and F_3 at B and C respectively. These may be found using the method described in S.12.5. Thus

$$F_2 = -\left(\frac{a_1 E \cos^2 60^\circ}{2L/\sqrt{3}} + \frac{a_2 E \cos^2 45^\circ}{\sqrt{2}L}\right) v_2$$

But $a_1 = 384I/5\sqrt{3}L^2$ and $a_2 = 192I/5\sqrt{2}L^2$ so that

$$F_2 = -\frac{96EI}{5L^3} v_2$$

Similarly

$$F_3 = -\frac{96EI}{5L^3} v_3$$

Then

$$F_{y,2} = -P - \frac{96EI}{5L^3} v_2 \quad \text{and} \quad F_{y,3} = -P - \frac{96EI}{5L^3} v_3$$

In Eq. (i), $v_1 = \theta_1 = v_4 = \theta_4 = 0$ and $M_2 = M_3 = 0$. Also, from symmetry, $v_2 = v_3$, and $\theta_2 = -\theta_3$. Then, from Eq. (i)

$$M_2 = 0 = 6v_2 + 12\theta_2 L + 6v_3 + 2\theta_3 L$$

i.e.

$$12v_2 + 10\theta_2 L = 0$$

which gives

$$\theta_2 = -\frac{6}{5L} v_2$$

Also from Eq. (i)

$$F_{y,2} = -P - \frac{96EI}{5L^3} v_2 = \frac{EI}{L^3}(36v_2 + 6\theta_2 L - 12v_3 - 6\theta_3 L)$$

i.e.

$$-P - \frac{96EI}{5L^3} v_2 = \frac{48EI}{5L^3} v_2$$

whence

$$v_2 = -\frac{5PL^3}{144EI} = v_3$$

and

$$\theta_2 = \frac{PL^2}{24EI} = -\theta_3$$

The reactions at the ends of the beam now follow from the above values and Eq. (i). Thus

$$F_{y,1} = \frac{EI}{L^3}(-24v_2 - 12\theta_2 L) = \frac{P}{3} = F_{y,4}$$

$$M_1 = \frac{EI}{L^2}(12v_2 + 4\theta_2 L) = -\frac{PL}{4} = -M_4$$

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Also

$$F_2 = F_3 = \frac{96EI}{5L^3}\frac{5PL^3}{144EI} = \frac{2P}{3}$$

The forces on the beam are then as shown in Fig. S.12.6(a). The shear force and bending moment diagrams for the beam follow and are shown in Figs S.12.6(b) and (c) respectively.

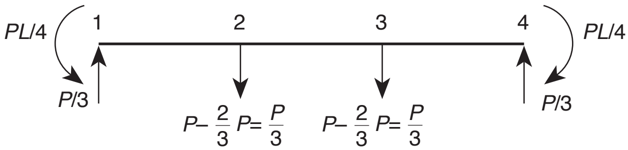

Fig. S.12.6(a)

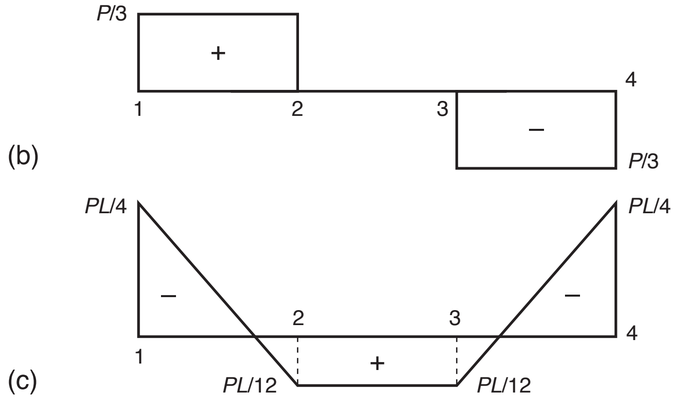

Fig. S.12.6 (b) and (c)

The forces in the ties are obtained using Eq. (12.32). Thus

$$S_{\text{BF}} = S_{\text{CH}} = \frac{a_1E}{2L/\sqrt{3}}\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{Bmatrix} 0-0 \\ v_2-0 \end{Bmatrix}$$

i.e.

$$S_{\text{BF}} = S_{\text{CH}} = \frac{384EI\sqrt{3}}{5\sqrt{3}\times 2L^3}\frac{1}{2}\frac{5PL^3}{144EI} = \frac{2}{3}P$$

and

$$S_{\text{BE}} = S_{\text{CG}} = \frac{a_2E}{\sqrt{2}L}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{Bmatrix} 0-0 \\ v_2-0 \end{Bmatrix}$$

i.e.

$$S_{\text{BE}} = S_{\text{CG}} = \frac{192EI}{5\sqrt{2}\times\sqrt{2}L^3}\frac{1}{\sqrt{2}}\frac{5PL^3}{144EI} = \frac{\sqrt{2}P}{3}$$

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S.12.7

The forces acting on the member 123 are shown in Fig. S.12.7(a). The moment M_2 arises from the torsion of the members 26 and 28 and, from Eq. (3.12), is given by

$$M_2 = -2GJ\frac{\theta_2}{1.6l} = -EI\frac{\theta_2}{l} \tag{i}$$

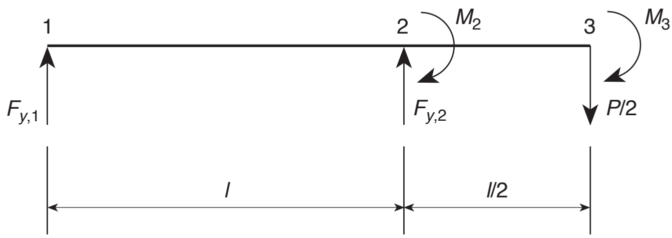

Fig. S.12.7(a)

Now using the alternative form of Eq. (12.44) for the member 12

$$\begin{Bmatrix} F_{y,1} \\ M_1/l \\ F_{y,2} \\ M_2/l \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 L \\ v_2 \\ \theta_2 L \end{Bmatrix} \tag{ii}$$

and for the member 23

$$\begin{Bmatrix} F_{y,2} \\ M_2/l \\ F_{y,3} \\ M_3/l \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 96 & -24 & -96 & -24 \\ -24 & 8 & 24 & 4 \\ -96 & 24 & 96 & 24 \\ -24 & 4 & 24 & 8 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 L \\ v_3 \\ \theta_3 L \end{Bmatrix} \tag{iii}$$

Combining Eqs (ii) and (iii) using the method described in Example 12.1

$$\begin{Bmatrix} F_{y,1} \\ M_1/l \\ F_{y,2} \\ M_2/l \\ F_{y,3} \\ M_3/l \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & -6 & -12 & -6 & 0 & 0 \\ -6 & 4 & 6 & 2 & 0 & 0 \\ -12 & 6 & 108 & -18 & -96 & -24 \\ -6 & 2 & -18 & 12 & 24 & 4 \\ 0 & 0 & -96 & 24 & 96 & 24 \\ 0 & 0 & -24 & 4 & 24 & 8 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 l \\ v_2 \\ \theta_2 l \\ v_3 \\ \theta_3 l \end{Bmatrix} \tag{iv}$$

In Eq. (iv) $v_1 = v_2 = 0$ and $\theta_3 = 0$. Also $M_1 = 0$ and $F_{y,3} = -P/2$. Then from Eq. (iv)

$$\frac{M_1}{l} = 0 = \frac{EI}{l^3}(4\theta_1 l + 2\theta_2 l)$$

from which

$$\theta_1 = -\frac{\theta_2}{2} \tag{v}$$

Also, from Eqs (i) and (iv)

$$\frac{M_2}{l} = -\frac{EI}{l^2}\theta_2 = \frac{EI}{l^3}(2\theta_1 l + 12\theta_2 l + 24v_3)$$

so that

$$13\theta_2 l + 2\theta_1 l + 24v_3 = 0 \tag{vi}$$

Finally from Eq. (iv)

$$F_{y,3} = -\frac{P}{2} = \frac{EI}{l^3}(24\theta_2 l + 96v_3)$$

which gives

$$v_3 = -\frac{Pl^3}{192EI} - \frac{\theta_2 l}{4} \tag{vii}$$

Substituting in Eq. (vi) for θ_1 from Eq. (v) and v_3 from Eq. (vii) gives

$$\theta_2 = \frac{Pl^2}{48EI}$$

Then, from Eq. (v)

$$\theta_1 = -\frac{Pl^2}{96EI}$$

and from Eq. (vii)

$$v_3 = -\frac{Pl^3}{96EI}$$

Now substituting for θ_1, θ_2 and v_3 in Eq. (iv) gives $F_{y,1} = -P/16$, $F_{y,2} = 9P/16$, $M_2 = -Pl/48$ (from Eq. (i)) and $M_3 = -Pl/6$. Then the bending moment at 2 in 12 is $F_{y,1}l = -Pl/12$ and the bending moment at 2 in 32 is $-(P/2)(l/2) + M_3 = -Pl/12$. Also $M_3 = -Pl/6$ so that the bending moment diagram for the member 123 is that shown in Fig. S.12.7(b).

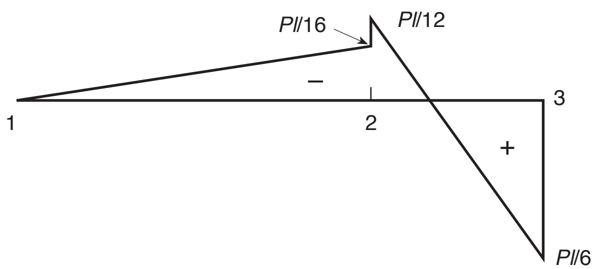

Fig. S.12.7(b)

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S.12.8

(a) The element is shown in Fig. S.12.8. The displacement functions for a triangular element are given by Eqs (12.82). Thus

$$\left.\begin{aligned} u_1 &= \alpha_1, \quad v_1 = \alpha_4 \\ u_2 &= \alpha_1 + a\alpha_2, \quad v_2 = \alpha_4 + a\alpha_5 \\ u_3 &= \alpha_1 + a\alpha_3, \quad v_3 = \alpha_4 + a\alpha_6 \end{aligned}\right\} \tag{i}$$

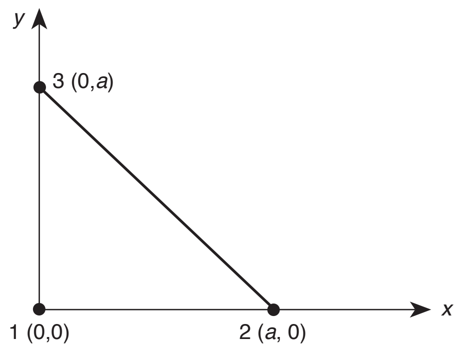

Fig. S.12.8

From Eqs (i)

$$\alpha_1 = u_1, \quad \alpha_2 = (u_2 - u_1)/a, \quad \alpha_3 = (u_3 - u_1)/a$$

$$\alpha_4 = v_1, \quad \alpha_5 = (v_2 - v_1)/a, \quad \alpha_6 = (v_3 - v_1)/a$$

Hence in matrix form

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1/a & 0 & 1/a & 0 & 0 & 0 \\ -1/a & 0 & 0 & 0 & 1/a & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1/a & 0 & 1/a & 0 & 0 \\ 0 & -1/a & 0 & 0 & 0 & 1/a \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

which is of the form

$$\{x\} = [A^{-1}]\{\delta^{\mathrm{e}}\}$$

Also, from Eq. (12.89)

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

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Hence

$$[B] = [C][A^{-1}] = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & -1/a & 0 & 0 & 0 & 1/a \\ -1/a & -1/a & 0 & 1/a & 1/a & 0 \end{bmatrix}$$

(b) From Eq. (12.94)

$$[K^{\mathrm{e}}] = \begin{bmatrix} -1/a & 0 & -1/a \\ 0 & -1/a & -1/a \\ 1/a & 0 & 0 \\ 0 & 0 & 1/a \\ 0 & 0 & 1/a \\ 0 & 1/a & 0 \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

$$\times \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & -1/a & 0 & 0 & 0 & 1/a \\ -1/a & -1/a & 0 & 1/a & 1/a & 0 \end{bmatrix} \frac{1}{2}a^2 t$$

which gives

$$[K^{\mathrm{e}}] = \frac{Et}{4(1-\nu^2)} \begin{bmatrix} 3-\nu & 1+\nu & -2 & -(1-\nu) & -(1-\nu) & -2\nu \\ 1+\nu & 3-\nu & -2\nu & -(1-\nu) & -(1-\nu) & -2 \\ -2 & -2\nu & 2 & 0 & 0 & 2\nu \\ -(1-\nu) & -(1-\nu) & 0 & 1-\nu & 1-\nu & 0 \\ -(1-\nu) & -(1-\nu) & 0 & 1-\nu & 1-\nu & 0 \\ -2\nu & -2 & -2\nu & 0 & 0 & 2 \end{bmatrix}$$

Continuity of displacement is only ensured at nodes, not along their edges.

S.12.9

(a) There are six degrees of freedom so that the displacement field must include six coefficients. Thus

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{i}$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y \tag{ii}$$

(b) From Eqs (i) and (ii) and referring to Fig. S.12.9

$$u_1 = \alpha_1 + \alpha_2 + \alpha_3, \quad v_1 = \alpha_4 + \alpha_5 + \alpha_6$$

$$u_2 = \alpha_1 + 2\alpha_2 + \alpha_3, \quad v_2 = \alpha_4 + 2\alpha_5 + \alpha_6$$

$$u_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3, \quad v_3 = \alpha_4 + 2\alpha_5 + 2\alpha_6$$

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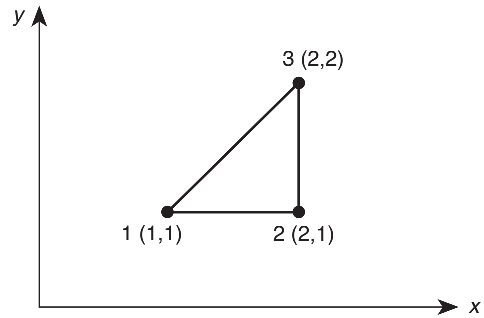

Fig. S.12.9

Thus

$$\alpha_2 = u_2 - u_1, \quad \alpha_3 = u_3 - u_2, \quad \alpha_1 = 2u_1 - u_3$$

$$\alpha_5 = v_2 - v_1, \quad \alpha_6 = v_3 - v_2, \quad \alpha_4 = 2v_1 - v_3$$

Therefore

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{iii}$$

which is of the form

$$\{\alpha\} = [A^{-1}]\{\delta^{\mathrm{e}}\}$$

From Eq. (12.89)

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Hence

$$[B] = [C][A^{-1}] = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

(c) From Eq. (12.69)

$$\{\sigma\} = [D][B]\{\delta^{\mathrm{e}}\}$$

Thus, for plane stress problems (see Eq. (12.92))

$$[D][B] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

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i.e.

$$[D][B] = \frac{E}{1-\nu^2}\begin{bmatrix} -1 & 2\nu & 1 & 0 & 0 & -\nu \\ -\nu & 2 & \nu & 0 & 0 & -1 \\ 0 & -\frac{1}{2}(1-\nu) & -\frac{1}{2}(1-\nu) & \frac{1}{2}(1-\nu) & \frac{1}{2}(1-\nu) & 0 \end{bmatrix}$$

For plain strain problems (see Eq. (12.93))

$$[D][B] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix}$$

$$\times\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

$$[D][B] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} -1 & \frac{2\nu}{1-\nu} & 1 & 0 & 0 & -\frac{\nu}{1-\nu} \\ -\frac{\nu}{1-\nu} & 2 & \frac{\nu}{1-\nu} & 0 & 0 & -1 \\ 0 & -\frac{1-2\nu}{2(1-\nu)} & -\frac{1-2\nu}{2(1-\nu)} & \frac{1-2\nu}{2(1-\nu)} & \frac{1-2\nu}{2(1-\nu)} & 0 \end{bmatrix}$$

S.12.10

(a) The element is shown in Fig. S.12.10. There are eight degrees of freedom so that a displacement field must include eight coefficients. Therefore assume

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \tag{i}$$

$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy \tag{ii}$$

(b) From Eqs (12.88) and Eqs (i) and (ii)

$$\varepsilon_x = \frac{\partial u}{\partial x} = \alpha_2 + \alpha_4 y$$

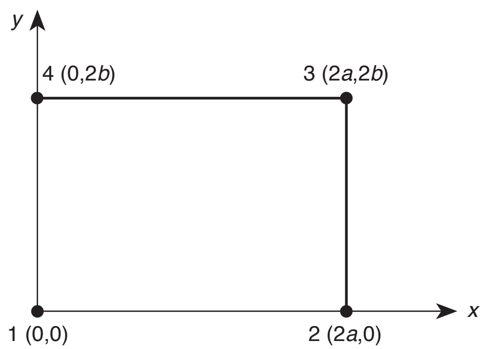

Fig. S.12.10

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$$\varepsilon_y = \frac{\partial v}{\partial y} = \alpha_7 + \alpha_8 x$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \alpha_3 + \alpha_4 x + \alpha_6 + \alpha_8 y$$

Thus since $\{\varepsilon\} = [C]\{\alpha\}$

$$[C] = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \tag{iii}$$

(c) From Eq. (iii)

$$[C]^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ y & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & x \\ 0 & x & y \end{bmatrix}$$

and from Eq. (12.92)

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

Thus

$$\int_{\mathrm{vol}} [C]^{\mathrm{T}}[D][C]\,\mathrm{d}V = \int_0^{2a}\int_0^{2b} [C]^{\mathrm{T}}[D][C]t\,\mathrm{d}x\,\mathrm{d}y \tag{iv}$$

Substituting in Eq. (iv) for $[C]^{\mathrm{T}}$, $[D]$ and $[C]$ and multiplying out gives

$$\int_0^{2a}\int_0^{2b} [C]^{\mathrm{T}}[D][C]t\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{Et}{1-\nu^2}\int_0^{2a}\int_0^{2b}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y & 0 & 0 & \nu & \nu x \\ 0 & 0 & \frac{1}{2}(1-\nu) & \frac{x}{2}(1-\nu) & 0 & \frac{1}{2}(1-\nu) & 0 & \frac{y}{2}(1-\nu) \\ 0 & y & \frac{x}{2}(1-\nu) & y^2 + \frac{x^2(1-\nu)}{2} & 0 & \frac{x}{2}(1-\nu) & \nu y & \nu xy + \frac{xy}{2}(1-\nu) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) & \frac{x}{2}(1-\nu) & 0 & \frac{1}{2}(1-\nu) & 0 & \frac{y}{2}(1-\nu) \\ 0 & \nu & 0 & \nu y & 0 & 0 & 1 & x \\ 0 & \nu x & \frac{y}{2}(1-\nu) & \nu xy + \frac{xy}{2}(1-\nu) & 0 & \frac{y}{2}(1-\nu) & x & x^2 + \frac{y^2}{2}(1-\nu) \end{bmatrix}\mathrm{d}x\,\mathrm{d}y$$

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$$
= \frac{Et}{1-\nu^2}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4ab & 0 & 4ab^2 & 0 & 0 & 4ab\nu & 4ba^2\nu \\
0 & 0 & 2ab(1-\nu) & 2a^2b(1-\nu) & 0 & 2ab(1-\nu) & 0 & 2ab^2(1-\nu) \\
0 & 4ab^2 & 2a^2b(1-\nu) & \frac{8}{3}\{2ab^3 + a^3b(1-\nu)\} & 0 & 2a^2b(1-\nu) & 4ab^2\nu & 2a^2b^2(1+\nu) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2ab(1-\nu) & 2a^2b(1-\nu) & 0 & 2ab(1-\nu) & 0 & 2ab^2(1-\nu) \\
0 & 4ab\nu & 0 & 4ab^2\nu & 0 & 0 & 4ab & 4a^2b \\
0 & 4a^2b\nu & 2ab^2(1-\nu) & 2a^2b^2(1+\nu) & 0 & 2ab^2(1-\nu) & 4a^2b & \frac{8}{3}\{2a^3b + ab^3(1-\nu)\}
\end{bmatrix}
$$

S.12.11

From the first of Eqs (12.96)

$$u_1 = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 0.1/10^3 \tag{i}$$

$$u_2 = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0.3/10^3 \tag{ii}$$

$$u_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0.6/10^3 \tag{iii}$$

$$u_4 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = 0.1/10^3 \tag{iv}$$

Adding Eqs (i) and (ii)

$$u_1 + u_2 = 2\alpha_1 - 2\alpha_3 = 0.4/10^3$$

i.e.

$$\alpha_1 - \alpha_3 = 0.2/10^3 \tag{v}$$

Adding Eqs (iii) and (iv)

$$u_3 + u_4 = 2\alpha_1 + 2\alpha_3 = 0.7/10^3$$

i.e.

$$\alpha_1 + \alpha_3 = 0.35/10^3 \tag{vi}$$

Adding Eqs (v) and (vi)

$$\alpha_1 = 0.275/10^3$$

Then from Eq. (v)

$$\alpha_3 = 0.075/10^3$$

Now subtracting Eq. (ii) from Eq. (i)

$$u_1 - u_2 = -2\alpha_2 + 2\alpha_4 = -0.2/10^3$$

i.e.

$$\alpha_2 - \alpha_4 = 0.1/10^3 \tag{vii}$$

Subtracting Eq. (iv) from Eq. (iii)

$$u_3 - u_4 = 2\alpha_2 + 2\alpha_4 = 0.5/10^3$$

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i.e.

$$\alpha_2 + \alpha_4 = 0.25/10^3 \tag{viii}$$

Now adding Eqs (vii) and (viii)

$$2\alpha_2 = 0.35/10^3$$

whence

$$\alpha_2 = 0.175/10^3$$

Then from Eq. (vii)

$$\alpha_4 = 0.075/10^3$$

From the second of Eqs (12.96)

$$v_1 = \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = 0.1/10^3 \tag{ix}$$

$$v_2 = \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0.3/10^3 \tag{x}$$

$$v_3 = \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0.7/10^3 \tag{xi}$$

$$v_4 = \alpha_5 - \alpha_6 + \alpha_7 - \alpha_8 = 0.5/10^3 \tag{xii}$$

Then, in a similar manner to the above

$$\alpha_5 = 0.4/10^3$$

$$\alpha_7 = 0.2/10^3$$

$$\alpha_6 = 0.1/10^3$$

$$\alpha_8 = 0$$

Eqs (12.96) are now written

$$u_i = (0.275 + 0.175x + 0.075y + 0.075xy) \times 10^{-3}$$

$$v_i = (0.4 + 0.1x + 0.2y) \times 10^{-3}$$

Then, from Eqs (12.88)

$$\varepsilon_x = (0.175 + 0.075y) \times 10^{-3}$$

$$\varepsilon_y = 0.2 \times 10^{-3}$$

$$\gamma_{xy} = (0.075 + 0.075x + 0.1) \times 10^{-3} = (0.175 + 0.075x) \times 10^{-3}$$

At the centre of the element $x = y = 0$. Then

$$\varepsilon_x = 0.175 \times 10^{-3}$$

$$\varepsilon_y = 0.2 \times 10^{-3}$$

$$\gamma_{xy} = 0.175 \times 10^{-3}$$

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so that, from Eqs (12.92)

$$\sigma_x = \frac{200\,000}{1-0.3^2}(0.175 + 0.3 \times 0.2) \times 10^{-3} = 51.65\,\mathrm{N/mm^2}$$

$$\sigma_y = \frac{200\,000}{1-0.3^2}(0.2 + 0.3 \times 0.175) \times 10^{-3} = 55.49\,\mathrm{N/mm^2}$$

$$\tau_{xy} = \frac{200\,000}{2(1+0.3)} \times 0.175 \times 10^{-3} = 13.46\,\mathrm{N/mm^2}$$

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Solutions to Chapter 13 Problems

S.13.1

The solution is obtained directly from Eq. (13.9) in which $\partial c_l/\partial\alpha = a$, $\alpha = \alpha_0$ and $c_{m,0} = C_{M,0}$. Thus

$$\theta = \left(\frac{C_{M,0}}{ea} + \alpha_0\right)\left[\frac{\cos\lambda(s-y)}{\cos\lambda s} - 1\right]$$

which gives

$$\theta = \left(\frac{C_{M,0}}{ea} + \alpha_0\right)\frac{\cos\lambda(s-y)}{\cos\lambda s} - \frac{C_{M,0}}{ea} - \alpha_0$$

Thus

$$\theta + \alpha_0 = \left(\frac{C_{M,0}}{ea} + \alpha_0\right)\frac{\cos\lambda(s-y)}{\cos\lambda s} - \frac{C_{M,0}}{ea}$$

where

$$\lambda^2 = \frac{ea\frac{1}{2}\rho V^2 c^2}{GJ}$$

Also, from Eq. (13.11) the divergence speed V_d is given by

$$V_d = \sqrt{\frac{\pi^2 GJ}{2\rho ec^2 s^2 a}}$$

S.13.2

Since the additional lift due to operation of the aileron is at a distance hc aft of the flexural axis the moment equilibrium equation (13.25) for an elemental strip becomes

$$\frac{dT}{dy}\delta y - \Delta Lec - \Delta L_\xi hc = 0 \qquad \text{(i)}$$

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in which, from Eq. (13.23)

$$\Delta L = \frac{1}{2}\rho V^2 c\delta y\left[a_1\left(\theta - \frac{py}{V}\right) + a_2 f_{\mathrm{a}}(y)\xi\right]$$

where $f_{\mathrm{a}}(y) = 0$ for $0 \leqslant y \leqslant ks$ and $f_{\mathrm{a}}(y) = 1$ for $ks \leqslant y \leqslant s$. Also

$$\Delta L_\xi = \tfrac{1}{2}\rho V^2 c\delta y a_2 f_{\mathrm{a}}(y)\xi$$

Then, substituting for $T\,(= GJ\,\mathrm{d}\theta/\mathrm{d}y)$, ΔL and ΔL_ξ in Eq. (i) and writing $\lambda^2 = \rho V^2 ec^2 a_1/2GJ$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \lambda^2\theta = \lambda^2\frac{py}{V} + \lambda^2\frac{h}{e}\frac{a_2}{a_1}f_{\mathrm{a}}(y)\xi \tag{ii}$$

The solution of Eq. (ii) is obtained by comparison with Eq. (13.29). Thus

$$\theta_1(0 - ks) = \frac{p}{V}\left(y - \frac{\sin\lambda y}{\lambda\cos\lambda s}\right) - \frac{ha_2\xi}{ea_1}(\tan\lambda s\cos\lambda ks - \sin\lambda ks)\sin\lambda y \tag{iii}$$

and

$$\theta_2(ks - s) = \frac{p}{V}\left(y - \frac{\sin\lambda y}{\lambda\cos\lambda s}\right) + \frac{ha_2\xi}{ea_1}(1 - \cos\lambda y\cos\lambda ks - \tan\lambda s\cos\lambda ks\sin\lambda y) \tag{iv}$$

Then, from Eq. (13.32)

$$\int_0^{ks} a_1\left(\theta_1 - \frac{py}{V}\right)y\,\mathrm{d}y + \int_{ks}^{s} a_1\left(\theta_2 - \frac{py}{V}\right)y\,\mathrm{d}y = -a_2\xi\int_{ks}^{s} y\,\mathrm{d}y \tag{v}$$

Substituting for θ_1 and θ_2 in Eq. (v) from Eqs (iii) and (iv) gives

$$-\tan\lambda s\int_0^s y\sin\lambda y\,\mathrm{d}y + \tan\lambda ks\int_0^{ks} y\sin\lambda y\,\mathrm{d}y - \int_{ks}^{s} y\cos\lambda y\,\mathrm{d}y + \frac{(e+h)}{h\cos\lambda ks}\int_{ks}^{s} y\,\mathrm{d}y$$
$$= \frac{pea_1}{ha_2\xi\lambda V\cos\lambda s\cos\lambda ks}\int_0^s y\sin\lambda y\,\mathrm{d}y$$

Hence the aileron effectiveness is given by

$$\frac{(ps/V)}{\xi} = \frac{-\tan\lambda s\displaystyle\int_0^s y\sin\lambda y\,\mathrm{d}y + \tan\lambda ks\int_0^{ks} y\sin\lambda y\,\mathrm{d}y - \int_{ks}^{s} y\cos\lambda y\,\mathrm{d}y + \frac{(e+h)}{2h\cos\lambda ks}[s^2 - (ks)^2]}{\dfrac{ea}{ha_2\lambda s\cos\lambda s\cos\lambda ks}\displaystyle\int_0^s y\sin\lambda y\,\mathrm{d}y} \tag{vi}$$

The aileron effectiveness is zero, i.e. aileron reversal takes place, when the numerator on the right-hand side of Eq. (vi) is zero, i.e. when

$$\tan\lambda ks\int_0^{ks} y\sin\lambda y\,\mathrm{d}y - \tan\lambda s\int_0^s y\sin\lambda y\,\mathrm{d}y - \int_{ks}^{s} y\cos\lambda y\,\mathrm{d}y = \frac{(e+h)}{2h\cos\lambda ks}[(ks)^2 - s^2]$$

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S.13.3

Referring to Fig. S.13.3(a), with unit load at D (1), $R_C = 2$. Then

$$M_1 = 1z \quad (0 \leqslant z \leqslant l)$$

$$M_1 = 1z - R_C(z - l) = 2l - z \quad (l \leqslant z \leqslant 2l)$$

$$M_1 = -1(z - 2l) \quad (2l \leqslant z \leqslant 3l)$$

$$M_2 = 0 \quad (0 \leqslant z \leqslant 2l)$$

$$M_2 = 1(z - 2l) \quad (2l \leqslant z \leqslant 3l)$$

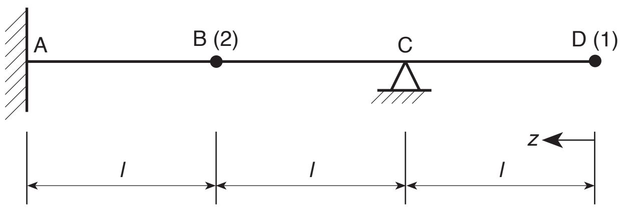

Fig. S.13.3(a)

Hence, from the first of Eqs (4.27)

$$\delta_{11} = \frac{1}{EI}\int_0^l M_1^2\,dz + \frac{1}{EI}\int_l^{2l} M_1^2\,dz + \frac{1}{EI}\int_{2l}^{3l} M_1^2\,dz$$

Substituting for M_1 from the above

$$\delta_{11} = \frac{1}{EI}\left[\int_0^l z^2\,dz + \int_l^{2l}(2l - z)^2\,dz + \int_{2l}^{3l}(z - 2l)^2\,dz\right]$$

which gives

$$\delta_{11} = \frac{l^3}{EI}$$

Also

$$\delta_{22} = \frac{1}{EI}\int_{2l}^{3l}(z - 2l)^2\,dz$$

from which

$$\delta_{22} = \frac{l^3}{3EI}$$

and

$$\delta_{12} = \delta_{21} = \frac{1}{EI}\int_{2l}^{3l} -(z - 2l)^2\,dz$$

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i.e.

$$\delta_{12} = \delta_{21} = -\frac{l^3}{3EI}$$

From Eqs (13.40) the equations of motion are

$$m\ddot{v}_1\delta_{11} + 2m\ddot{v}_2\delta_{12} + v_1 = 0 \tag{i}$$

$$m\ddot{v}_1\delta_{21} + 2m\ddot{v}_2\delta_{22} + v_2 = 0 \tag{ii}$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin \omega t$ and substituting for δ_{11}, δ_{12} and δ_{22}, Eqs (i) and (ii) become

$$-3\lambda\omega^2 v_1 + 2\lambda\omega^2 v_2 + v_1 = 0$$

$$\lambda\omega^2 v_1 - 2\lambda\omega^2 v_2 + v_2 = 0$$

in which $\lambda = ml^3/3EI$ or, rearranging

$$(1 - 3\lambda\omega^2)v_1 + 2\lambda\omega^2 v_2 = 0 \tag{iii}$$

$$\lambda\omega^2 v_1 + (1 - 2\lambda\omega^2)v_2 = 0 \tag{iv}$$

From Eq. (13.42) and Eqs (iii) and (iv)

$$\begin{vmatrix} (1 - 3\lambda\omega^2) & 2\lambda\omega^2 \\ \lambda\omega^2 & (1 - 2\lambda\omega^2) \end{vmatrix} = 0$$

from which

$$(1 - 3\lambda\omega^2)(1 - 2\lambda\omega^2) - 2(\lambda\omega^2)^2 = 0$$

or

$$4(\lambda\omega^2)^2 - 5\lambda\omega^2 + 1 = 0$$

i.e.

$$(4\lambda\omega^2 - 1)(\lambda\omega^2 - 1) = 0 \tag{v}$$

Hence

$$\lambda\omega^2 = \tfrac{1}{4} \quad \text{or} \quad 1$$

so that

$$\omega^2 = \frac{3EI}{4ml^3} \quad \text{or} \quad \omega^2 = \frac{3EI}{ml^3}$$

Hence

$$\omega_1 = \sqrt{\frac{3EI}{4ml^3}}, \quad \omega_2 = \sqrt{\frac{3EI}{ml^3}}$$

The frequencies of vibration are then

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3EI}{4ml^3}}, \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{3EI}{ml^3}}$$

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From Eq. (iii)

$$\frac{v_1}{v_2} = -\frac{2\lambda\omega^2}{1 - 3\lambda\omega^2} \tag{vi}$$

When $\omega = \omega_1$, v_1/v_2 is negative and when $\omega = \omega_2$, v_1/v_2 is positive. The modes of vibration are therefore as shown in Fig. S.13.3(b) and (c).

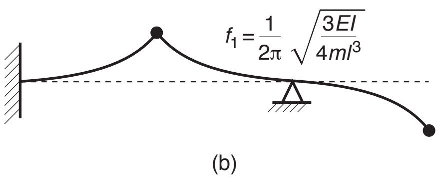

Fig. S.13.3 (b) and (c)

S.13.4

Referring to Fig. S.13.4

$$M_2 = -\tfrac{1}{2}z \quad (0 \leqslant z \leqslant l)$$
$$M_2 = -\tfrac{1}{2}(2l - z) \quad (l \leqslant z \leqslant 2l)$$
$$M_2 = 0 \quad (0 \leqslant x \leqslant l)$$
$$M_4 = 1x \quad (0 \leqslant x \leqslant l)$$
$$M_4 = \tfrac{1}{2}z \quad (0 \leqslant z \leqslant l)$$
$$M_4 = -\tfrac{1}{2}(2l - z) \quad (l \leqslant z \leqslant 2l)$$

Then from the first of Eqs (4.27)

$$\delta_{22} = \frac{1}{3EI}\int_0^l \frac{z^2}{4}\,\mathrm{d}z + \frac{1}{EI}\int_l^{2l} \frac{(2l - z)^2}{4}\,\mathrm{d}z$$

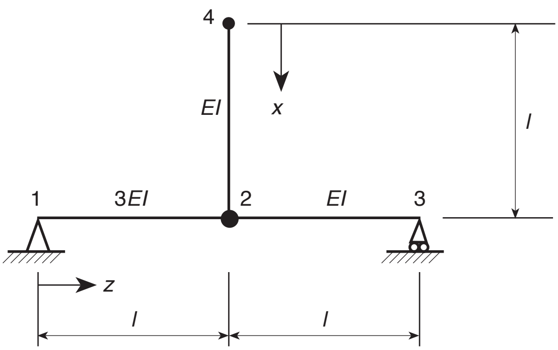

Fig. S.13.4

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which gives

$$\delta_{22} = \frac{l^3}{9EI}$$

Also

$$\delta_{44} = \frac{1}{EI}\int_0^l x^2\,\mathrm{d}x + \frac{1}{3EI}\int_0^l \frac{z^2}{4}\,\mathrm{d}z + \frac{1}{EI}\int_l^{2l} \frac{(2l-z)^2}{4}\,\mathrm{d}z$$

from which

$$\delta_{44} = \frac{4l^3}{9EI}$$

and

$$\delta_{42} = \delta_{24} = -\frac{1}{3EI}\int_0^l \frac{z^2}{4}\,\mathrm{d}z + \frac{1}{EI}\int_l^{2l} \frac{(2l-z)^2}{4}\,\mathrm{d}z$$

Thus

$$\delta_{42} = \delta_{24} = \frac{l^3}{18EI}$$

From Eqs (13.40) the equations of motion are

$$m\ddot{v}_4\delta_{44} + 2m\ddot{v}_2\delta_{42} + v_4 = 0 \qquad \text{(i)}$$

$$m\ddot{v}_4\delta_{24} + 2m\ddot{v}_2\delta_{22} + v_2 = 0 \qquad \text{(ii)}$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin\omega t$ and substituting for δ_{44}, δ_{42} and δ_{22}, Eqs (i) and (ii) become

$$-8\lambda\omega^2 v_4 - 2\lambda\omega^2 v_2 + v_4 = 0 \qquad \text{(iii)}$$

$$-\lambda\omega^2 v_4 - 4\lambda\omega^2 v_2 + v_2 = 0 \qquad \text{(iv)}$$

in which $\lambda = ml^3/18EI$. Then, from Eq. (13.42)

$$\begin{vmatrix} (1-8\lambda\omega^2) & -2\lambda\omega^2 \\ -\lambda\omega^2 & (1-4\lambda\omega^2) \end{vmatrix} = 0$$

which gives

$$(1-8\lambda\omega^2)(1-4\lambda\omega^2) - 2(\lambda\omega^2)^2 = 0$$

i.e.

$$30(\lambda\omega^2)^2 - 12\lambda\omega^2 + 1 = 0 \qquad \text{(v)}$$

Solving Eq. (v)

$$\lambda\omega^2 = 0.118 \quad \text{or} \quad \lambda\omega^2 = 0.282$$

Hence

$$\omega^2 = 0.118 \times \frac{18EI}{ml^3} \quad \text{or} \quad \omega^2 = 0.282 \times \frac{18EI}{ml^3}$$

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Then, since $f = \omega/2\pi$, the natural frequencies of vibration are

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{2.13EI}{ml^3}}, \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{5.08EI}{ml^3}}$$

S.13.5

The second moment of area, I, of the tube cross-section is given by

$$I = \frac{\pi}{64}(D^4 - d^4)$$

in which D and d are the outer and inner diameters respectively. Now,

$$D = 25 + 1.25 = 26.25\,\text{mm}, \quad d = 25 - 1.25 = 23.75\,\text{mm}$$

Thus

$$I = \frac{\pi}{64}(26.25^4 - 23.75^4) = 7689.1\,\text{mm}^4$$

The polar second moment of area, J, for a circular section is $2I$, i.e. $J = 15\,378.2\,\text{mm}^4$. From Eqs (4.27)

$$\delta_{ij} = \int_L \frac{M_i M_j}{EI}\,\text{d}s + \int_L \frac{T_i T_j}{GJ}\,\text{d}s \tag{i}$$

Then, referring to Fig. S.13.5(a)

$$M_1 = 1y \quad (0 \leqslant y \leqslant a)$$
$$M_1 = 1z \quad (0 \leqslant z \leqslant 2a)$$
$$T_1 = 0 \quad (0 \leqslant y \leqslant a)$$
$$T_1 = 1a \quad (0 \leqslant z \leqslant 2a)$$
$$M_2 = 1 \quad (0 \leqslant y \leqslant a)$$
$$T_2 = 1 \quad (0 \leqslant z \leqslant 2a)$$

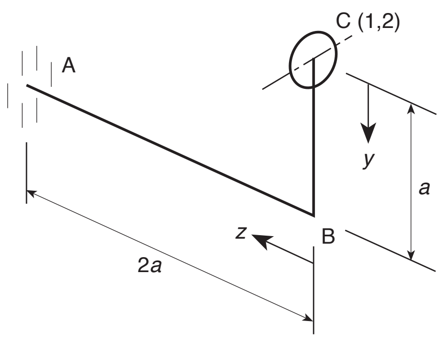

Fig. S.13.5(a)

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Thus, from Eq. (i)

$$\delta_{11} = \int_0^a \frac{y^2}{EI}\,dy + \int_0^{2a} \frac{z^2}{EI}\,dz + \int_0^{2a} \frac{a^2}{GJ}\,dz$$

which gives

$$\delta_{11} = a^3\left(\frac{3}{EI} + \frac{2}{GJ}\right) = 250^3\left(\frac{3}{70\,000 \times 7689.1} + \frac{2}{28\,000 \times 15\,378.2}\right)$$

i.e.

$$\delta_{11} = 0.16$$

Also

$$\delta_{22} = \int_0^a \frac{1^2}{EI}\,dy + \int_0^{2a} \frac{1^2}{GJ}\,dz$$

i.e.

$$\delta_{22} = a\left(\frac{1}{EI} + \frac{2}{GJ}\right) = 250\left(\frac{1}{70\,000 \times 7689.1} + \frac{2}{28\,000 \times 15\,378.2}\right)$$

which gives

$$\delta_{22} = 1.63 \times 10^{-6}$$

Finally

$$\delta_{12} = \delta_{21} = \int_0^a \frac{y}{EI}\,dy + \int_0^{2a} \frac{a}{GJ}\,dz$$

so that

$$\delta_{12} = \delta_{21} = a^2\left(\frac{1}{2EI} + \frac{2}{GJ}\right) = 250^2\left(\frac{1}{2 \times 70\,000 \times 7689.1} + \frac{2}{28\,000 \times 15\,378.2}\right)$$

Thus

$$\delta_{12} = \delta_{21} = 3.48 \times 10^{-4}$$

The equations of motion are then, from Eqs (13.40)

$$m\ddot{v}\delta_{11} + mr^2\ddot{\theta}\delta_{12} + v = 0 \qquad \text{(ii)}$$

$$m\ddot{v}\delta_{21} + mr^2\ddot{\theta}\delta_{22} + \theta = 0 \qquad \text{(iii)}$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin\omega t$ and $\theta = \theta_0 \sin\omega t$, Eqs (i) and (ii) may be written

$$-m\delta_{11}\omega^2 v - mr^2\delta_{12}\omega^2\theta + v = 0$$

$$-m\delta_{21}\omega^2 v - mr^2\delta_{22}\omega^2\theta + \theta = 0$$

Substituting for m, r and δ_{11} etc,

$$-20 \times 0.16\omega^2 v - 20 \times 62.5^2 \times 3.48 \times 10^{-4}\omega^2\theta + v = 0$$

$$-20 \times 3.48 \times 10^{-4}\omega^2 v - 20 \times 62.5^2 \times 1.63 \times 10^{-6}\omega^2\theta + \theta = 0$$

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which simplify to

$$v(1 - 3.2\omega^2) - 27.2\omega^2\theta = 0 \qquad \text{(iv)}$$

$$-0.007\omega^2 v + \theta(1 - 0.127\omega^2) = 0 \qquad \text{(v)}$$

Hence, from Eqs (13.42)

$$\begin{vmatrix} (1 - 3.2\omega^2) & -27.2\omega^2 \\ -0.007\omega^2 & (1 - 0.127\omega^2) \end{vmatrix} = 0$$

which gives

$$(1 - 3.2\omega^2)(1 - 0.127\omega^2) - 0.19\omega^4 = 0$$

or

$$\omega^4 - 15.4\omega^2 + 4.63 = 0 \qquad \text{(vi)}$$

Solving Eq. (vi) gives

$$\omega^2 = 15.1 \quad \text{or} \quad 0.31$$

Hence the natural frequencies are

$$f = 0.62\,\text{Hz} \quad \text{and} \quad 0.09\,\text{Hz}$$

From Eq. (iv)

$$\frac{v}{\theta} = \frac{27.2\omega^2}{1 - 3.2\omega^2}$$

Thus, when $\omega^2 = 15.1$, v/θ is negative and when $\omega^2 = 0.31$, v/θ is positive. The modes of vibration are then as shown in Figs S.13.5(b) and (c).

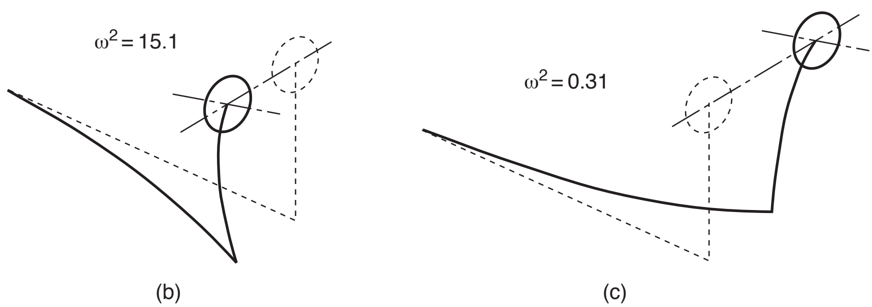

Fig. S.13.5 (b) and (c)

S.13.6

Choosing the origin for z at the free end of the tube

$$M_1 = z, \quad S_1 = 1 \quad \text{and} \quad T_1 = 0$$

$$M_2 = z, \quad S_2 = 1 \quad \text{and} \quad T_2 = 2a$$

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in which the point 1 is at the axis of the tube and point 2 at the free end of the rigid bar.

From Eqs (4.27) and Eq. (9.88)

$$\delta_{ij} = \int_0^L \frac{M_i M_j}{EI}\,dz + \int_0^L \frac{T_i T_j}{GJ}\,dz + \int_0^L \left(\oint \frac{q_i q_j}{Gt}\,ds\right) dz \qquad \text{(i)}$$

in which q_i and q_j are obtained from Eq. (9.35) in which $S_{y,i} = S_{y,j} = 1$, $S_x = 0$ and $I_{xy} = 0$. Thus

$$q_i = q_j = -\frac{1}{I_{xx}} \int_0^s ty\,ds + q_{s,0}$$

'Cutting' the tube at its lowest point in its vertical plane of symmetry gives $q_{s,0} = 0$. Then, referring to Fig. S.13.6

$$q_i = q_j = \frac{1}{I_{xx}} \int_0^\theta ta\cos\theta a\,d\theta$$

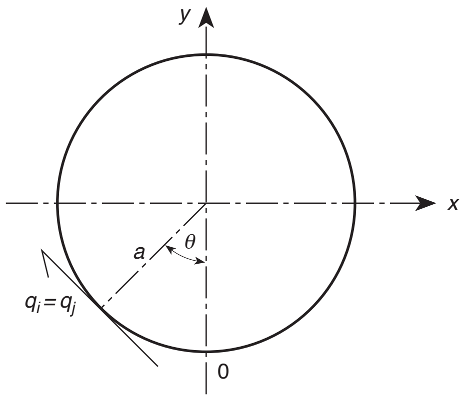

Fig. S.13.6

i.e.

$$q_i = q_j = \frac{a^2 t \sin\theta}{I_{xx}}$$

From Fig. 9.11, $I_{xx} = \pi a^3 t$. Hence $q_i = q_j = \sin\theta/\pi a$ and

$$\oint \frac{q_i q_j}{Gt}\,ds = 2\int_0^\pi \frac{\sin^2\theta}{G\pi^2 a^2 t} a\,d\theta = \frac{1}{G\pi a t}$$

Also in Eq. (i) the torsion J is obtained from Eq. (9.52), i.e.

$$J = \frac{4A^2}{\oint ds/t} = \frac{4(\pi a^2)^2}{2\pi a/t} = 2\pi a^3 t$$

Therefore from Eq. (i)

$$\delta_{11} = \int_0^L \frac{z^2}{EI}\,dz + \int_0^L \frac{1}{G\pi a t}\,dz = \frac{L^3}{3EI} + \frac{L}{G\pi a t} \qquad \text{(ii)}$$

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Putting $\lambda = 3Ea^2/GL^2$, Eq. (ii) becomes

$$\delta_{11} = \frac{L^3}{3EI}(1 + \lambda)$$

Also

$$\delta_{22} = \int_0^L \frac{z^2}{EI}\,dz + \int_0^L \frac{4a^2}{G2\pi a^3 t}\,dz + \int_0^L \frac{1}{G\pi at}\,dz$$

which gives

$$\delta_{22} = \frac{L^3}{3EI}(1 + 3\lambda)$$

Finally

$$\delta_{12} = \delta_{21} = \int_0^L \frac{z^2}{EI}\,dz + \int_0^L \frac{1}{G\pi at}\,dz$$

i.e.

$$\delta_{12} = \delta_{21} = \frac{L^3}{3EI}(1 + \lambda)$$

From Eqs (13.40) the equations of motion are

$$m\ddot{v}_1\delta_{11} + m\ddot{v}_2\delta_{12} + v_1 = 0 \qquad \text{(iii)}$$

$$m\ddot{v}_1\delta_{21} + m\ddot{v}_2\delta_{22} + v_2 = 0 \qquad \text{(iv)}$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin \omega t$, Eqs (iii) and (iv) become

$$-m\delta_{11}\omega^2 v_1 - m\delta_{12}\omega^2 v_2 + v_1 = 0$$

$$-m\delta_{21}\omega^2 v_1 - m\delta_{22}\omega^2 v_2 + v_2 = 0$$

Substituting for δ_{11}, δ_{22} and δ_{12} and writing $\mu = L^3/3EI$ gives

$$v_1[1 - m\omega^2\mu(1 + \lambda)] - m\omega^2\mu(1 + \lambda)v_2 = 0$$

$$-m\omega^2\mu(1 + \lambda)v_1 + v_2[1 - m\omega^2\mu(1 + 3\lambda)] = 0$$

Hence, from Eqs (13.42)

$$\begin{vmatrix} [1 - m\omega^2\mu(1 + \lambda)] & -m\omega^2\mu(1 + \lambda) \\ -m\omega^2\mu(1 + \lambda) & [1 - m\omega^2\mu(1 + 3\lambda)] \end{vmatrix} = 0$$

Then

$$[1 - m\omega^2\mu(1 + \lambda)][1 - m\omega^2\mu(1 + 3\lambda)] - m^2\omega^4\mu^2(1 + \lambda)^2 = 0$$

which simplifies to

$$\frac{1}{\omega^4} - \frac{1}{\omega^2}2m\mu(1 + 2\lambda) + 2m^2\mu^2\lambda(1 + \lambda) = 0$$

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Solving gives

$$\frac{1}{\omega^2} = m\mu(1 + 2\lambda) \pm m\mu(1 + 2\lambda + 2\lambda^2)^{1/2}$$

i.e.

$$\frac{1}{\omega^2} = \frac{mL^3}{3E\pi a^3 t}[1 + 2\lambda \pm (1 + 2\lambda + 2\lambda^2)^{1/2}]$$

S.13.7

Choosing the origin for z at the free end of the beam

$$M_1 = z, \quad S_1 = 1$$

Also, from Eqs (4.27) and Eq. (9.88)

$$\delta_{ij} = \int_0^L \frac{M_i M_j}{EI}\,\mathrm{d}z + \int_0^L \left(\oint \frac{q_i q_j}{Gt}\,\mathrm{d}s\right)\mathrm{d}z \tag{i}$$

in which q_i and q_j are obtained from Eq. (9.80) and in which $S_{y,i} = S_{y,j} = 1$, $S_x = 0$, $I_{xy} = 0$ and $t_\mathrm{D} = 0$. Thus

$$q_i = q_j = -\frac{1}{I_{xx}}\sum_{r=1}^{n} B_r y_r + q_{s,0}$$

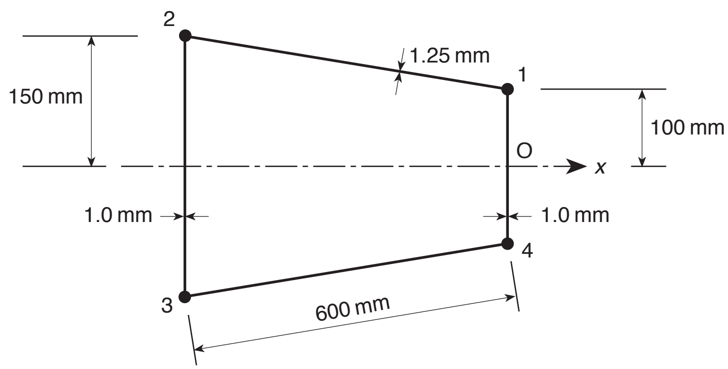

Fig. S.13.7

where I_{xx} is given by (see Fig. S.13.7)

$$I_{xx} = 2 \times 970 \times 100^2 + 2 \times 970 \times 150^2 = 6.305 \times 10^7\,\mathrm{mm}^4$$

Thus

$$q_{\mathrm{b},i} = q_{\mathrm{b},j} = -\frac{1}{6.305 \times 10^7}\sum_{r=1}^{n} B_r y_r$$

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Hence, cutting the tube at O,

$$q_{\text{b,O1}} = 0$$

$$q_{\text{b,12}} = -\frac{970 \times 100}{6.305 \times 10^7} = -0.0015\,\text{N/mm}$$

$$q_{\text{b,23}} = -0.0015 - \frac{970 \times 150}{6.305 \times 10^7} = -0.0038\,\text{N/mm}$$

Then, from Eq. (9.47)

$$q_{s,0} = -\frac{2}{2(100/1.0 + 600/1.25 + 150/1.0)}\left(-\frac{0.0015 \times 600}{1.25} - \frac{0.0038 \times 150}{1.0}\right)$$

i.e.

$$q_{s,0} = 0.0018\,\text{N/mm}$$

Therefore

$$q_{i,\text{O1}} = q_{j,\text{O1}} = 0.0018\,\text{N/mm}$$

$$q_{i,12} = q_{j,12} = -0.0015 + 0.0018 = 0.0003\,\text{N/mm}$$

$$q_{i,23} = q_{j,23} = -0.0038 + 0.0018 = 0.002\,\text{N/mm}$$

Then

$$\oint \frac{q_i q_j}{Gt}\,\mathrm{d}s = \frac{2}{26\,500}\left(\frac{0.0018^2 \times 100}{1.0} + \frac{0.0003^2 \times 600}{1.25} + \frac{0.002^2 \times 150}{1.0}\right) = 7.3 \times 10^{-8}$$

Hence

$$\delta_{11} = \int_0^{1525} \frac{z^2}{EI}\,\mathrm{d}z + \int_0^{1525} 7.3 \times 10^{-8}\,\mathrm{d}z$$

i.e.

$$\delta_{11} = \frac{1525^3}{3 \times 70\,000 \times 6.305 \times 10^7} + 7.3 \times 10^{-8} \times 1525 = 3.79 \times 10^{-4}$$

For flexural vibrations in a vertical plane the equation of motion is, from Eqs (13.40)

$$m\ddot{v}_1\delta_{11} + v_1 = 0$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin \omega t$ Eq. (ii) becomes

$$-m\delta_{11}\omega^2 v_1 + v_1 = 0$$

i.e.

$$\omega^2 = \frac{1}{m\delta_{11}} = \frac{9.81 \times 10^3}{4450 \times 3.79 \times 10^{-4}} = 5816.6$$

Hence

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{5816.6} = 12.1\,\text{Hz}$$

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S.13.8

Assume a deflected shape given by

$$V = \cos\frac{2\pi z}{l} - 1 \tag{i}$$

where z is measured from the left-hand end of the beam. Eq. (i) satisfies the boundary conditions of $V = 0$ at $z = 0$ and $z = l$ and also $\mathrm{d}V/\mathrm{d}z = 0$ at $z = 0$ and $z = l$. From Eq. (i)

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -\frac{2\pi}{l}\sin\frac{2\pi z}{l}$$

and

$$\frac{\mathrm{d}^2 V}{\mathrm{d}z^2} = -\frac{4\pi^2}{l^2}\cos\frac{2\pi z}{l}$$

Substituting these expressions in Eq. (13.57)

$$\omega^2 = \frac{2\left[\int_0^{l/4} 4EI(4\pi^2/l^2)^2\cos^2(2\pi z/l)\,\mathrm{d}z + \int_{l/4}^{l/2} EI(4\pi^2/l^2)^2\cos^2(2\pi z/l)\,\mathrm{d}z\right]}{2\left[\int_0^{l/4} 2m\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z + \int_{l/4}^{l/2} m\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z\right] + 2\frac{1}{2}ml(-1)^2 + \frac{1}{4}ml(2)^2}$$

which simplifies to

$$\omega^2 = \frac{EI(4\pi^2/l^2)^2\left[\int_0^{l/4} 4\cos^2(2\pi z/l)\,\mathrm{d}s + \int_{l/4}^{l/2}\cos^2(2\pi z/l)\,\mathrm{d}z\right]}{m\left[\int_0^{l/4} 2\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z + \int_{l/4}^{l/2}\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z + l\right]} \tag{ii}$$

Now

$$\int_0^{l/4}\cos^2\frac{2\pi z}{l}\,\mathrm{d}z = \frac{1}{2}\left[z + \frac{l}{4\pi}\sin\frac{4\pi z}{l}\right]_0^{l/4} = \frac{l}{8}$$

$$\int_{l/4}^{l/2}\cos^2\frac{2\pi z}{l}\,\mathrm{d}z = \frac{1}{2}\left[z + \frac{l}{4\pi}\sin\frac{4\pi z}{l}\right]_{l/4}^{l/2} = \frac{l}{8}$$

$$\int_0^{l/4}\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z = \int_0^{l/4}\left[\frac{1}{2}\left(1 + \cos\frac{4\pi z}{l}\right) - 2\cos\frac{2\pi z}{l} + 1\right]\mathrm{d}z$$

$$= \left[\frac{1}{2}\left(z + \frac{l}{4\pi}\sin\frac{4\pi z}{l}\right) - \frac{l}{\pi}\sin\frac{2\pi z}{l} + z\right]_0^{l/4} = \frac{3l}{8} - \frac{l}{\pi}$$

Similarly

$$\int_{l/4}^{l/2}\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z = \frac{3l}{8} + \frac{l}{\pi}$$

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Substituting these values in Eq. (ii)

$$\omega^2 = \frac{EI(4\pi^2/l^2)^2(4l/8 + l/8)}{m\left[2\left(\frac{3l}{8} - \frac{l}{\pi}\right) + \frac{3l}{8} + \frac{l}{\pi} + l\right]}$$

i.e.

$$\omega^2 = 539.2\frac{EI}{ml^4}$$

Then

$$f = \frac{\omega}{2\pi} = 3.7\sqrt{\frac{EI}{ml^4}}$$

The accuracy of the solution may be improved by assuming a series for the deflected shape, i.e.

$$V(z) = \sum_{s=1}^{n} B_s V_s(z) \quad \text{(Eq. (13.58))}$$

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