Estimation and Tracking
Bar-Shalom & Li, 1993, Artech House

5.3 EXAMPLE OF A FILTER

5.3.1 The Model

Given the system with state

$$x = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} \tag{5.3.1-1}$$

which evolves according to

$$x(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} v(k) \qquad k = 0, 1, \ldots, 99 \tag{5.3.1-2}$$

with initial condition

$$x(0) = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \tag{5.3.1-3}$$

This represents a one-dimensional motion with position ξ and velocity $\dot{\xi}$ sampled at intervals T, which will be assumed as unity in the sequel.

Note that (5.3.1-2) is of the form

$$x(k+1) = Fx(k) + \Gamma v(k) \tag{5.3.1-4}$$

The process noise, a scalar, which models the acceleration, is a zero-mean white sequence with variance

$$E[v(k)^2] = q \tag{5.3.1-5}$$

The measurements consist of position corrupted by additive noise

$$z(k) = [1 \quad 0]\, x(k) + w(k) \qquad k = 1, \ldots, 100 \tag{5.3.1-6}$$

where the measurement noise is a zero-mean white sequence with variance

$$E[w(k)^2] = r = 1 \tag{5.3.1-7}$$

The two noise sequences are mutually independent.

The filter was initialized according to the "two-point differencing" procedure discussed later in Section 5.5 in (5.5.3-3) to (5.5.3-5).

5.5.3 A Practical Implementation in Tracking

The practical implementation of the initialization (5.5.2-1) can be done as follows. Consider two state components, say, position ξ and velocity $\dot{\xi}$ in a given coordinate. If only position measurements

$$z(k) = \xi(k) + w(k) \tag{5.5.3-1}$$

are available, then for the true values $\xi(k)$, $k = -1, 0$, one generates the corresponding measurement noises, say

$$w(k) \sim \mathcal{N}[0, R] \tag{5.5.3-2}$$

Then, denoting by T the sampling interval, one has

$$\hat{\xi}(0|0) = z(0) \tag{5.5.3-3}$$

$$\hat{\dot{\xi}}(0|0) = \frac{z(0) - z(-1)}{T} \tag{5.5.3-4}$$

and the corresponding 2×2 block of the initial covariance matrix is then

$$P(0|0) = \begin{bmatrix} R & R/T \\ R/T & 2R/T^2 \end{bmatrix} \tag{5.5.3-5}$$

This method, called ***two-point differencing***, guarantees consistency of the initialization of the filter, which starts updating the state at $k = 1$.

If several (Monte Carlo) runs are made, then the same initialization procedure has to be followed with new (*independent*) noises generated in every run according to (5.5.3-2). "Reuse" of the same initial conditions in Monte Carlo runs will lead to biased estimates (see problem 5-6).

Remark

The above amounts to a ***first order polynomial fitting*** of the first two measurements. If (and only if) there are significant higher derivatives, then one should use more than two points and a ***higher order polynomial fitting*** via LS is to be carried out.

5.6 NOTES AND PROBLEMS

5.6.1 Bibliographical Notes

The Kalman filter originated with the work of Kalman and Bucy [Kal60, KB61]. The idea of recursive estimation appeared earlier in the work of Swerling [Swe59]. The topic of the Kalman filtering is covered in many texts, for instance, [Bal84, Med69, SM71, Gel74, AM79, Kai81, May79, May82, Lew86]. The proof of the sequential estimation for dynamic systems as a direct extension of the static case presented in Section 5.2 is simpler than the many different proofs in the literature. Issues of observability and controllability and their implications on the Kalman filter are discussed in detail in [AM79].

The concept of consistency of a state estimator (Section 5.4) has been mentioned in the literature under the name of "covariance matching." The discussion of Section 5.4 is based on [BB83]. The lack of consistency of a Kalman filter has been called "divergence" (that is how severe it was in some cases) and has been the subject of extensive investigations [Sor85]. Sensitivity analysis of mismatched filters and the use of reduced order models are discussed in detail in [Gel74, Lew86].

The technique of initialization of filters presented in Section 5.5 has been known for many years, but overlooked in many instances.

The models considered here had all known "parameters" — the matrices F, G, H, Q, and R. Techniques for the "identification" of these system parameters can be found in [Lju87].

Treatment of a constant bias in recursive filtering using state augmentation is discussed in [Fri69].

5.6.2 Problems

5-1 **Filter implementation and consistency checking.** Consider the scalar system

$$x(k+1) = fx(k) + u(k) + v(k)$$

where $u(k) = 1$ and $v(k) \sim \mathcal{N}(0, q)$ and white, with measurement

$$z(k) = hx(k) + w(k)$$

where $w(k) \sim \mathcal{N}(0, r)$ and white. The initial condition for the system is $x(0)$.

1. Find the expression of the steady-state variance

$$P_\infty = \lim_{k\to\infty} P(k|k)$$

of the estimated state $\hat{x}(k|k)$.

2. With the parameters of the problem $f = h = 1$, $q = 0.01$, $r = 1$, simulate a trajectory using a random number generator starting from $x(0) = 0$ for $k = 1, \ldots, 50$.

3. Let the initial estimate of the state be $\hat{x}(0|0) = z(0)/h$. Determine the corresponding $P(0|0)$ as a function of the parameters of the problem (h, r).

4. For the values given in (2), estimate the state up to $k = 50$, starting as in (3). List the following:

$$k, x(k), v(k), w(k), z(k), \hat{x}(k|k-1), P(k|k-1), \hat{z}(k|k-1), S(k),$$
$$\nu(k), \nu(k)/\sqrt{S(k)}, W(k), \hat{x}(k|k), P(k|k), \tilde{x}(k|k)/\sqrt{P(k|k)}$$

5. Compare the values of $P(k|k)$ from (4) to the result of (1).

6. List $P(k|k)$ for $k = 0, 1, \ldots, 50$ for the following values of the initial variance: $P(0|0) =$ 0, 1, 10.

7. One desires to run the filter as in (4) with the various values of $P(0|0)$ as in (6). What should be changed in the filter simulation?

5-2 **Random number generator testing.** Describe the tests for a "correct" $\mathcal{N}(0,1)$ random number generator from which we have n numbers. Indicate the distributions and ~~confidence~~ probability regions for the

1. Sample mean.

2. Sample variance.

3. Sample correlation.

5-3 **Covariance of the state versus covariance of the estimation error.** Prove that, with $\tilde{x} \triangleq x - E[x|z]$, one has $\text{cov}[x|z] = \text{cov}[\tilde{x}|z]$.

5-4 **Asymptotic distribution of the sample correlation.** Show that the sample correlation (5.4.2-9) tends to $\mathcal{N}(0, 1/N)$. Hint: use the law of large numbers and the central limit theorem.

5-5 **MSE of the state for an update with an arbitrary gain.**

1. Prove that the Joseph form covariance update (5.2.3-18) holds for *arbitrary* gain at time $k+1$. Hint: write the propagation equation of the error from $\tilde{x}(k|k)$ to $\tilde{x}(k+1|k+1)$.

2. State the condition for its stability. k+1

3. Check this stability condition on the results of problem 5-1.

5-6 **Initialization of a filter (how NOT to do it) — a real story.** A tracking filter was initialized in a set of Monte Carlo runs with the target's initial range of 80kft, initial range estimate of 100kft and initial range variance of 10^6ft^2.

1. Characterize the assumed quality of the initial estimate.

2. How will the average estimation error over the Monte Carlo runs behave?

3. You are a "young Turk" engineer who wants to prove mastery of estimation: suggest a simple fix to the above initialization procedure that involves changing only the initial variance.

5-7 Orthogonality of the innovations to the state prediction. Show that

$$\nu(k) \perp \hat{x}(j|k-1) \qquad \forall j > k-1$$

5-8 Orthogonality of estimation error to previous estimates. Show that

$$\tilde{x}(i|k) \perp \hat{x}(i|j) \qquad \forall j \leq k$$

5-9 Alternative derivation of the Kalman filter gain. Show that the minimization of the trace of (5.2.3-18) with respect to the filter gain yields (5.2.3-11). Hint: use the formulas from problem 1-10.

5-10 State estimation errors' autocorrelation. Prove that the state estimation errors are not white:

$$E[\tilde{x}(k+1|k+1)\tilde{x}(k|k)'] = [I - W(k+1)H(k+1)]F(k)P(k|k)$$

5-11 Kalman filter with nonzero noise means. Derive the Kalman filter equations for the formulation from Subsection 5.2.1 with the following modifications:

$$E[v(k)] = \bar{v}(k) \qquad\qquad E[w(k)] = \bar{w}(k)$$

where the nonzero noise means are known. Their known covariances are Q and R. All the remaining assumptions are the same. Provide the derivations only for the equations that will be different than those in Subsection 5.2.4. Indicate which equations are not modified and why.

5-12 Bias in the measurements. Consider the problem from Section 5.3 with the modification that the measurement noise has an *unknown mean* $\bar{w}$ (the sensor bias). Append this to the state as an extra component assuming it to be constant in time.

1. Indicate the new state space representation with the augmented state and *zero-mean noises* (specify the matrices F, Γ, and H).

2. What happens if we run a Kalman filter on this representation? Can one estimate this sensor bias? Justify mathematically your answer.