JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 18, No. 3, May–June 1995

Transfer Orbits in Restricted Problem

Antonio Fernando Bertachini de Almeida Prado*
Instituto Naçional de Pesquisas Espaciasis, Brazil, São dos Campos, 1222-7-010 Brazil
and
Roger Broucke†
University of Texas at Austin, Austin, Texas 78712

This paper studies transfer orbits in the planar restricted three-body problem. In particular, we are searching for orbits that can be used in two situations: 1) to transfer a spacecraft from one body back to the same body (known in the literature as Hénon's problem) and 2) to transfer a spacecraft from one body to the respective Lagrangian points L_4 and L_5. To avoid numerical problems during close approaches, the global Lemaître regularization is used. Under this model, Hénon's problem becomes a Lambert three-body problem. After the simulations, we found orbits to transfer a spacecraft between any two points in the group formed by Earth and the Lagrangian points L_3, L_4, L_5 (in the Earth–sun system) with near-zero ΔV (near 10^{-2} in canonical units). We also found several orbits that can be used to make a tour to the Lagrangian points for reconnaissance purposes with near-zero ΔV for the entire tour. The method employed was to solve the two-point boundary value problem for each transfer using the results available from the two-body version of this problem as a first guess.

Introduction

To solve the problem defined here, we study each situation individually. In the first situation, attention is given to the family of transfer orbits involving no more than one revolution of the spacecraft around the primary body (multirevolution transfers are not allowed). The systems under study are the ones with more important practical applications: the Earth–sun and the Earth–moon systems. Five families of transfer orbits are found in the region studied and the results are plotted in terms of the true anomaly. The same plots also show the evolution of the Jacobian constant. A special effort is made to reproduce some of the previously found[1,2] transfer orbits with small ΔV under this improved model.

In the second situation, the problem of sending a spacecraft from Earth to the Lagrangian points L_4 and L_5 (in the sun–Earth system) is treated as a natural extension of the problem of sending a spacecraft from one body back to the same body. Two transfer orbits from Earth to L_4 and two transfer orbits from Earth to L_5 are found. Next, the numerical integration is extended beyond the desired Lagrangian point and it is found that, for all four orbits, the spacecraft passes near the Lagrangian points L_3, L_4, and L_5 and comes back to the neighborhood of Earth. In general, the orbits found here can be applied as follows:

1) A spacecraft can be transferred between any two points in the group formed by Earth and the Lagrangian points L_3, L_4, and L_5 (in the Earth–sun system) with near-zero ΔV (near 10^{-2} in canonical units).

2) A tour to the Lagrangian points can be made for reconnaissance purposes[3] with near-zero ΔV for the entire tour. The small relative velocities during the close approaches are ideal for the data acquisition phase or for a rendezvous with another spacecraft. There is also a possibility to recover the spacecraft after the tour, since it returns to Earth's neighborhood.

3) A cycler transportation system can be built linking all the points involved or only two of them. In a system like that, a heavy spacecraft can stay in one of the orbits shown here and a small spacecraft can make a "taxi service" and rendezvous with the heavy vehicle to transport persons and/or materials to/from it, similar to what happens in the systems proposed for Earth and the moon[4,5] or for Earth and Mars.[5]

Mathematical Model and Some Properties

The model used in all phases of this paper is the well-known planar circular restricted three-body problem. This model assumes that two main bodies (M_1 and M_2) are orbiting their common center of mass in circular Keplerian orbits and a third body (M_3), with negligible mass, is orbiting these two primaries. The motion of M_3 is supposed to stay in the plane of the motion of M_1 and M_2 and it is affected by both primaries, but it does not affect their motion.[6] The standard canonical system of units associated with this model is used (the unit of distance is the distance M_1–M_2 and the unit of time is chosen such that the period of the motion of M_2 around M_1 is 2π). Under this model, the equations of motion are

$$\ddot{x} - 2\dot{y} = x - \frac{\partial V}{\partial x} = \frac{\partial \Omega}{\partial x} \tag{1a}$$

$$\ddot{y} + 2\dot{x} = y - \frac{\partial V}{\partial y} = \frac{\partial \Omega}{\partial y} \tag{1b}$$

where Ω is the pseudopotential function given by

$$\Omega = \tfrac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{2}$$

and x and y are two perpendicular axes with the origin in the center of mass of the system, with x pointing from M_1 (that has coordinates $x = -\mu$, $y = 0$) to M_2 (that has coordinates $x = 1 - \mu$, $y = 0$).

One of the most important reasons why the rotating frame is more suitable to describe the motion of M_3 in the three-body problem is the existence of an invariant, called a Jacobi integral (or energy integral). There are many ways to define the Jacobi integral and the reference system used to describe this problem (see Ref. 6, p. 449). In this paper the definitions used by Broucke[7] are followed. Under this version, the Jacobi integral is given by

$$J = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega(x, y) = \text{const} \tag{3}$$

Another important property needed in this paper is the mirror image theorem.[8] It is an important and useful property of the planar circular restricted three-body problem. It says: In the rotating coordinate system, for each trajectory defined by $x(t)$, $y(t)$, $\dot{x}(t)$, $\dot{y}(t)$ that is found, there is a symmetric (in relation to the x axis) trajectory defined by $x(-t)$, $-y(-t)$, $-\dot{x}(-t)$, $\dot{y}(-t)$.

Received June 2, 1994; presented as Paper 94-3745 at the AIAA/AAS Astrodynamics Conference, Scottsdale, AZ, Aug. 1–3, 1994; revision received Oct. 6, 1994; accepted for publication Nov. 19, 1994.

*Research Engineer, Department of Space Mechanics and Control. Student Member AIAA.

†Professor, Department of Aerospace Engineering and Engineering Mechanics. Associate Fellow AIAA.

Lemaître Regularization

The equations of motion (1) are right, but they are not suitable for numerical integration in trajectories passing near one of the primaries. The reason is that the positions of both primaries are singularities in the potential V (since r_1 or r_2 goes to zero or near zero) and the precision of the numerical integration is affected every time this situation occurs.

The solution for this problem is to use regularization, which consists of a substitution of the variables for position $(x - y)$ and time (t) by another set of variables $(\omega_1, \omega_2, \tau)$ such that the singularities are eliminated in these new variables. Several transformations with this goal are available in the literature (see Ref. 6, Chapter 3), like Thiele–Burrau, Lemaître, and Birkhoff. They are called global regularization to emphasize that both singularities are eliminated at the same time. The case where only one singularity is eliminated at a time is called local regularization. For the present research the Lemaître's regularization is used. To perform the required transformation, it is necessary first to define a new complex variable $q = q_1 + iq_2$ (i is the imaginary unit), with q_1 and q_2 given by

$$q_1 = x + \tfrac{1}{2} - \mu \tag{4}$$

$$q_2 = y \tag{5}$$

Now, in terms of q, the transformation involved in Lemaître regularization is given by

$$q = f(\omega) = \frac{1}{4}\left(\omega^2 + \frac{1}{\omega^2}\right) \tag{6}$$

for the old variables for position $(x–y)$ and

$$\frac{\partial t}{\partial \tau} = |f'(\omega)|^2 = \frac{|\omega^4 - 1|^2}{4|\omega|^6} \tag{7}$$

where $f'(\omega)$ denotes $\partial f / \partial \omega$ for the time.

In these new variables the equation of motion of the system is

$$\omega'' + 2i|f'(\omega)|^2 \omega' = \mathrm{grad}_\omega \Omega^* \tag{8}$$

where $\omega = \omega_1 + i\omega_2$ is the new complex variable for position, ω' and ω'' denote first and second derivatives of ω with respect to the regularized time τ, $\mathrm{grad}_\omega \Omega^*$ represents $(\partial\Omega^*/\partial\omega_1) + i(\partial\Omega^*/\partial\omega_2)$, and Ω^* is the transformed pseudopotential given by

$$\Omega^* = \left(\Omega - \tfrac{1}{2}C\right)|f'(\omega)|^2 \tag{9}$$

where $C = \mu(1 - \mu) - 2J$.

Equation (8) in complex variable can be separated in two second-order equations in the real variables ω_1 and ω_2 and organized in the standard first-order form, which is more suitable for numerical integration. The final form, after defining the regularized velocity components ω_3 and ω_4 as $\omega_1^1 = \omega_3$ and $\omega_2^1 = \omega_4$, is

$$\omega_1^1 = \omega_3 \tag{10a}$$

$$\omega_2^1 = \omega_4 \tag{10b}$$

$$\omega_3^1 = 2\omega_4 |f'(\omega)|^2 + \frac{\partial \Omega^*}{\partial \omega_1} \tag{10c}$$

$$\omega_4^1 = -2\omega_3 |f'(\omega)|^2 + \frac{\partial \Omega^*}{\partial \omega_2} \tag{10d}$$

Another set of equations necessary for this research is the one that relates velocity components from one set of variables to another:

$$\dot{q}_1 = \frac{f'(\omega)}{|f'(\omega)|^2}\omega_3 \tag{11a}$$

$$\dot{q}_2 = \frac{f'(\omega)}{|f'(\omega)|^2}\omega_4 \tag{11b}$$

Results to Transfer Spacecraft from One Body Back to Same Body

The theory developed in the last few sections to solve the problem of transferring a spacecraft from one body back to the same body (called the three-body Lambert problem) can be used here to solve Hénon's problem in the case $\mu \neq 0$. The approach used here is to solve the three-body Lambert problem with the following input data: 1) the initial position for M_3, that is, the position of M_2 at the time that M_3 departs from M_2; 2) the final position of M_3, that is, the position of M_2 at the time that M_3 arrives at M_2; and 3) the time of flight, that is, $2\pi(\tau/\pi) = 2\tau$, where τ is half of the transfer time in canonical units.

The solution of the problem (output of the three-body Lambert problem) is the desired transfer orbit (in the restricted three-body context), ready to be plotted as a point in the equivalent of the Hénon diagram.[9] This diagram is a graphic that has τ/π in the horizontal axes and ν/π, ν defined by Eqs. (12) in the vertical axes and where we plot one point for each solution that we can find. Figures 1 and 2 are two examples of those diagrams and are used here to show the solutions obtained. The scheme looks very simple, but it is not so easy to implement. The difficulty arises from the fact that, to get convergence in the solution of the two-point boundary value

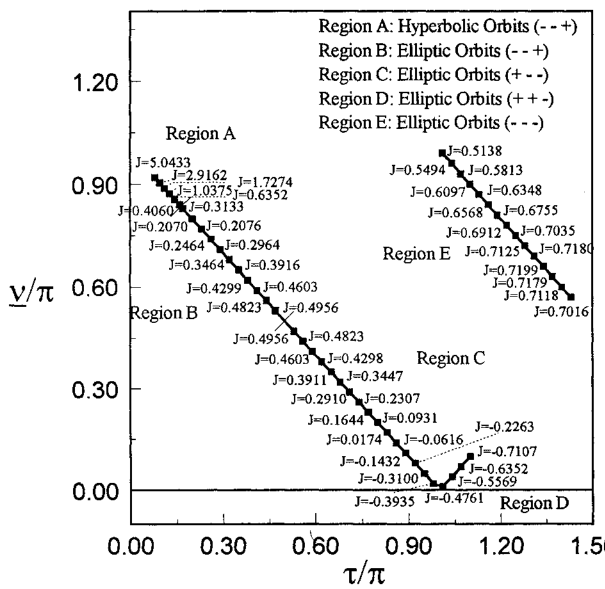

Fig. 1 Equivalent of Hénon's diagram for sun–Earth system.

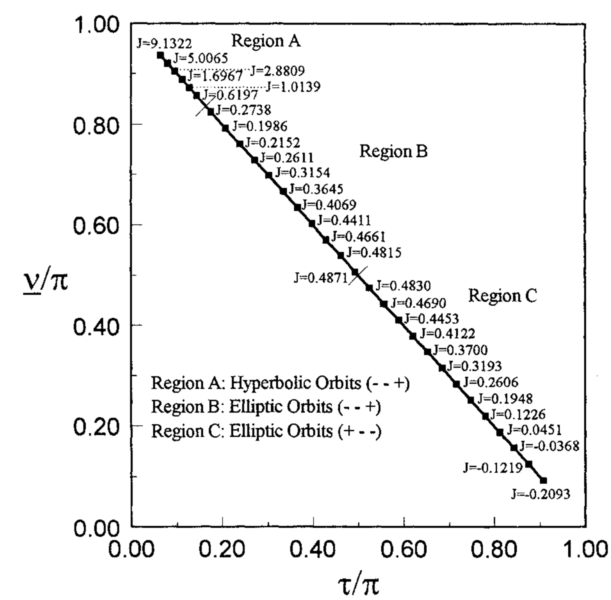

Fig. 2 Equivalent of Hénon's diagram for Earth-moon system.

problem (TPBVP) involved, an accurate first guess is required for each transfer orbit considered. The first "good first guess" available is the solution of the related two-body Lambert problem (same initial and final position and time of flight, but with $\mu = 0$, using two-body celestial mechanics equations), as solved in Refs. 1 and 2. If μ is small (such as the Earth–sun system, where $\mu = 0.000003$), this first guess is good enough to get convergence in the TPBVP for transfers with transfer time between 0.4 and about 9.0 canonical units of time (0.06–1.43 years in the sun–Earth system). For transfer orbits with transfer time greater than that and/or involving more than one revolution for M_3, a method to find a more accurate first guess has to be developed. Figure 1 shows the results obtained by the numerical simulations of the sun–Earth system for the range $0 \leq \tau/\pi \leq 1.43$. It is the "equivalent" of the Hénon diagram, which means that the differences from the original diagram are the following:

1) The orbit of M_3 is no longer a conic (because this is a three-body problem and not a two-body problem), so the concepts of eccentric anomaly and true anomaly do not exist anymore. For the graphs presented in this research, we plot τ/π against $\boldsymbol{\nu}/\pi$, where $\boldsymbol{\nu}$ is defined by

$$\boldsymbol{\nu} = \nu \qquad \text{if} \qquad M_3 \text{ passes periapse at } \tau = 0 \tag{12a}$$

$$\boldsymbol{\nu} = \pi - \nu \qquad \text{if} \qquad M_3 \text{ passes apoapse at } \tau = 0 \tag{12b}$$

and ν is half of the angle traveled by M_2 during the transfer. This definition has the goal of making ν a generalized true anomaly (which becomes the true anomaly in the case $\mu = 0$) and $\boldsymbol{\nu}$ is the generalization of the $\boldsymbol{\nu}$ used in Ref. 2 (they also become the same quantity when $\mu = 0$). Then, ν and τ (and so $\boldsymbol{\nu}$ and τ) are linearly related, since the motion of M_2 around M_1 is circular (remember that M_3 has negligible mass, which means that M_1 and M_2 form a two-body system).

2) The value of the Jacobi constant J [Eq. (3)] is given for several points in the diagram, since this is an important invariant in the restricted three-body problem.

To study these results in further detail, it is necessary to make an analogy between the two-body and the three-body problem. It means that the name hyperbolic orbit (in the three-body context) is given to an orbit that comes from a two-body hyperbolic orbit with the inclusion of the perturbation of the third body. The same analogy applies to an elliptic orbit and the important parameters (ε, ε', ε''), defined by Hénon,[9] are used again here: $\varepsilon = +1(-1)$ if the periapse is in an positive (negative) abscissa; $\varepsilon' = +1(-1)$ if the sense of the orbit is posigrade (retrograde); $\varepsilon'' = +1(-1)$ if the passage at $\tau = 0$ is at pariapse (apoapse).

It is important to have always in mind that these parameters refer to the two-body elliptic orbits and are applied here as a valid approximation since μ is small.

It is possible to see in Fig. 1 the appearance of five distinct regions when the analogy with the two-body problem is considered. They are called regions A to E and they are identified as follows:

1) Region A is composed of hyperbolic orbits and it goes from $\tau/\pi = 0.0796$ (the first point that gives convergence to the TPBVP) to $\tau/\pi = 0.16393$ (the frontier with region B). They have the highest values for J (it implies that the initial impulses also have the highest values) and the shortest transfer times, as expected. The notation $--+$ is a short form of $\varepsilon = -1$, $\varepsilon' = -1$, and $\varepsilon'' = +1$.

2) Region B is composed of elliptic orbits that have their periapse with a negative abscissa ($\varepsilon = -1$) and travel in a retrograde (opposite to the motion of M_2) direction ($\varepsilon' = -1$) and M_3 passes periapse at $\tau = 0$, the middle of the transfer ($\varepsilon'' = +1$). This region starts at $\tau/\pi = 0.16393$, the boundary with region A, and extends to $\tau/\pi = 0.5$, the boundary with region C.

3) Region C is composed of elliptic orbits that have their periapse with a positive abscissa ($\varepsilon = +1$) and travel in a retrograde (opposite to the motion of M_2) direction ($\varepsilon' = -1$) and M_3 passes apoapse at $\tau = 0$, the middle of the transfer ($\varepsilon'' = -1$). This region starts at $\tau/\pi = 0.5$, the boundary with region B, and extends to $\tau/\pi = 1.0$, the boundary with region D.

4) Region D is composed of elliptic orbits that have their periapse with a positive abscissa ($\varepsilon = +1$) and travel in a posigrade (the same of the motion of M_2) direction ($\varepsilon' = +1$) and M_3 passes apoapse at $\tau = 0$, the middle of the transfer ($\varepsilon'' = -1$). This region starts at $\tau/\pi = 1.0$, the boundary with region C, and extends to $\tau/\pi = 1.1$, the last point that gives convergence for the TPBVP in this region.

5) Region E is composed of elliptic orbits that have their periapse with a negative abscissa ($\varepsilon = -1$) and travel in a retrograde (opposite to the motion of M_2) direction ($\varepsilon' = -1$) and M_3 passes apoapse at $\tau = 0$, the middle of the transfer ($\varepsilon'' = -1$). This region starts at $\tau/\pi = 1.0$ and extends to $\tau/\pi = 1.43$, the last point that gives convergence for the TPBVP in this region.

Results for Earth–Moon System

After that the attention is turned to the Earth–moon system. This is a case with more practical interest and near-term applications, but it is also a more difficult case due to the high value of the mass parameter ($\mu = 0.0121505$). Figure 2 shows the results obtained using the same definitions and conventions used for the sun–Earth system. The main difference is that the single conic approximation for the first guess works only in the range $0.0637 \leq \tau/\pi \leq 0.9072$. As a consequence, only the regions A, B, and C in the diagram can be found.

The results shown in this research do not give the whole picture of the equivalent of Hénon's diagram.[9] It shows only the portion that can be constructed by using the conic trajectory given by the case $\mu = 0$ as the first guess for the Lambert three-body problem routine. However, the method outlined here can be used to generate the whole picture if a procedure to find a more accurate first guess can be developed.

Transfer Orbits with Minimum ΔV

An important characteristic of this problem[1,2] is the family of transfer orbits with near-zero ΔV to transfer a spacecraft from M_2 back to M_2 again. These orbits, which exist in the two-body problem model (case $\mu = 0$ of the three-body problem), are important enough to deserve a study in the more realistic case $\mu \neq 0$. In the present section this research is performed in the sun–Earth system. The Earth–moon system is also very important and it is under study now (the high value of μ makes this study much more difficult). The two-body solution is used to give us the initial positions and a first guess for the initial velocity. Then, a trial-and-error technique in the initial velocity is used to find the solution. The equations of motion are integrated several times for slight changes (on the order of 0.01 or less in canonical units) in the initial velocity until a satisfactory trajectory is found. The gradient method (where the difference between the position of the spacecraft after a specified transfer time and the position of the Lagrangian point is the performance index to be minimized) did not work with those first guesses, due to the high nonlinear characteristic of the system in this particular case. Figure 3 shows the trajectory as seen in the rotating frame. The initial conditions to reproduce this trajectory are $x = 1.000000$, $y = -0.000043$, $\dot{x} = 0.096957$, and $\dot{y} = -.371500$. It is important to note that the ΔV for escape velocity from Earth is 0.3735 canonical units (the absolute minimal for any transfer from the surface of a celestial body), which means that the ΔV found in this transfer orbit

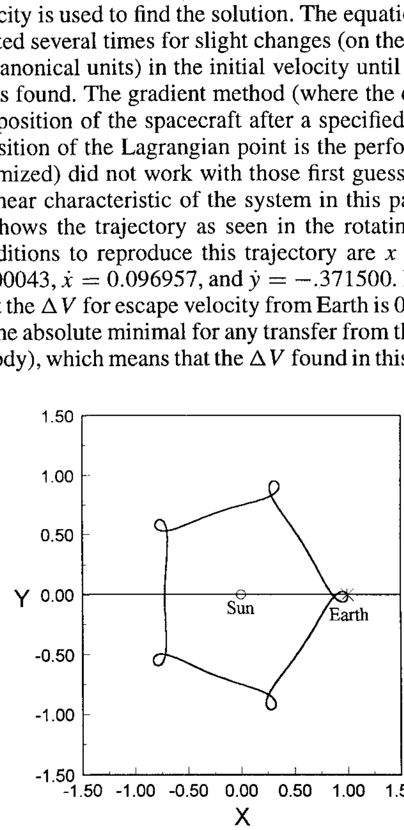

Fig. 3 Transfer orbit (Earth–Earth) with minimum ΔV (rotating frame).

(0.3839 canonical units) is only slightly above it (0.0104 canonical units), and there is not much improvement left to be done, as far as fuel savings are concerned.

Results for Transfers Between Earth and Lagrangian Points

In this section, the theory developed in the first sections of this paper is used to find transfer orbit between Earth and the Lagrangian points with minimum ΔV. Since the results are different for each of the trajectories studied, it is necessary to study them in detail one by one. For identification purposes the definition of the following nomenclature is made: LONG-4-5 is the orbit that goes to L_4 first, just after leaving Earth, and then goes to L_3 and L_5 and has a long period (about 25 years); SHORT-4-5 is the orbit that visits the Lagrangian points in the same order but with a shorter period (about 13 years); LONG-5-4 is the orbit that visits the Lagrangian points in an opposite order (L_5 first and then L_3 and L_4) with a long period (about 28 years); and SHORT-5-4 is the orbit similar to LONG-5-4 (same order of points visited) but with a shorter period (about 11 years). This is the orbit with the shortest period of all the orbits studied. Each one of these orbits is described in detail in the following sections.

SHORT-5-4 Orbit

In this orbit the spacecraft M_3 leaves Earth and visits the Lagrangian points in the order L_5 (in 2.12 years), L_3 (in 6.46 years), and L_4 (in 10.88 years) and then returns to Earth's neighborhood (in 13.05 years). Table 1 shows the coordinates x and y, the distance R from the Lagrangian point, the velocity components V_x and V_y, the magnitude V of the velocity vector, and the time (t) lapsed from departure for the passage by all the important points, referred to the rotating frame for all orbits studied in this paper.

It is important to remember again that the ΔV required for Earth's escape is 11,180 m/s (the absolute minimal for any transfer from the surface of Earth), which means that all the ΔV involved in Earth's escape that are shown in this research are of this order of magnitude. Then, there is not much room left for improvements, as far as fuel savings are concerned, in those maneuvers too. All the ΔV of this order of magnitude, when the spacecraft is leaving Earth, or near zero, when the spacecraft is far from Earth, are called *near-zero* ΔV in this research. Figure 4 shows the first two revolutions of this trajectory. The particularly important points of this orbit are as follows:

1) A shorter time is required for all transfers involved, when compared with the two LONG transfers. A period for the total tour (from Earth back to Earth) is about 13 years. The legs connecting L_4 and L_5 to Earth have a little more than 2.1 years each.

2) It also has closer approaches to the Lagrangian points visited when compared to the two LONG transfers.

3) After the close approach with Earth, at the end of the first revolution, this orbit continues for a second revolution in the same direction of motion. The trajectory followed in the second revolution is not much different from the trajectory followed in the first one, and there are 12 "crossing points." Those are points that belong to the trajectory followed by the spacecraft in the first and in the second revolution. Those 12 crossing points are candidates for a one-burn maneuver that transfers the spacecraft from the trajectory it follows in the second revolution to the trajectory it follows in the first revolution. After this maneuver the spacecraft starts again its journey to L_5, L_3, L_4, and Earth. A cycler transportation that links all the points involved in about 13 years is achieved.

The final result is a periodic trajectory linking Earth and the Lagrangian points L_3, L_4, and L_5 that has a period of about 13 years

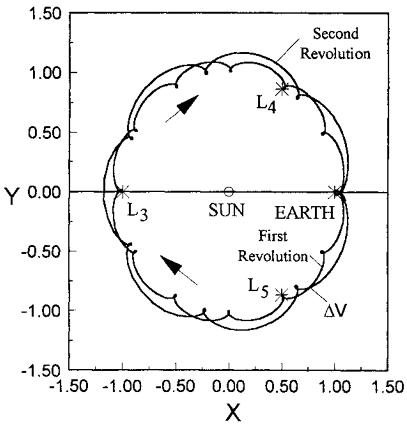

Fig. 4 Orbit SHORT-5-4.

Table 1 Position, velocity, and time for passages by Lagrangian points

Point	x	y	R, $\times 10^6$ km	V_x, m/s	V_y, m/s	V, m/s	t, years[a]
Orbit SHORT-5-4							
Earth	—	—	—	0.0	11,130	11,130	0.00
L_5	74.9	−130.1	0.55	306.8	590	664	2.12
L_3	−150.0	1.32	1.38	253.2	−610	661	6.46
L_4	76.1	129.7	1.30	−128.1	685	697	10.88
Earth	150.4	0.00	0.81	479.5	1,111	1,209	13.05
Orbit LONG-5-4							
Earth	—	—	—	0.0	11,107	11,107	0.00
L_5	78.1	−129.6	3.34	−51	−497	500	4.24
L_3	−153.7	0.0	4.07	−197	1,311	1,337	12.74
L_4	74.9	130.6	1.09	27	48	57	20.81
Earth	149.6	0.7	0.75	−938	−253	971	24.88
Orbit SHORT-4-5							
Earth	—	—	—	0	−11,140	11,140	0.00
L_4	74.9	129.2	0.37	715	−334	786	1.81
L_3	−149.3	−0.4	0.46	18	789	792	5.49
L_5	74.6	−128.9	0.69	−807	119	816	9.20
Earth	149.6	−0.1	0.12	2302	−1,346	2,666	11.00
Orbit LONG-4-5							
Earth	—	—	—	0.0	−11,101	11,101	0.00
L_4	73.7	127.9	1.99	−294.9	378	479	4.69
L_3	−144.4	−0.1	5.21	−53.6	−1,748	1,751	13.96
L_5	72.8	−127.4	2.86	512	673	846	23.29
Earth	149.6	0.0	0.00	24,085	−10,381	10,657	27.84

[a]Referred to rotating frame.

and that requires $\Delta V = 0.0667$ (1986.7 m/s) per revolution for nominal operation.

LONG-4-5 Orbit

In this orbit the spacecraft M_3 leaves Earth and visits the Lagrangian points in the order L_4 (in 4.69 years), L_3 (in 13.96 years), and L_5 (in 23.29 years) and then returns to Earth's neighborhood (in 27.84 years). Figure 5 shows the first two revolutions of this trajectory. The particularly important points of this orbit are as follows:

1) It has the closest approach with Earth at the end of the first revolution. This is an important characteristic if a capture of the spacecraft is planned after the tour.

2) After this close approach, the orbit is slightly deviated by Earth, but very close approaches to the Lagrangian points and Earth again exist in at least two more revolutions, with no nominal corrections required. It makes this orbit the best one for a continuous cycler without nominal corrections.

3) This orbit has the characteristic of reversing the direction of its motion after some of the "swing-bys"[1,2] with Earth. It means that some of the swing-bys with Earth have the effect of changing the direction of the motion in the rotating frame. During the first five revolutions of this trajectory, it occurs twice: The first one reverses the counterclockwise motion to a clockwise motion at the end of the third revolution (the first three revolutions are in the counterclockwise direction), and the second one reverses the motion of the spacecraft to a counterclockwise motion, again at the end of the fourth revolution.

SHORT-4-5 Orbit

In this orbit the spacecraft M_3 leaves the Earth and visits the Lagrangian points in the order L_4 (in 1.81 years), L_3 (in 5.49 years), and L_5 (in 9.20 years) and then returns to Earth's neighborhood (in 11.00 years). Figure 6 shows the first two revolutions of this trajectory. The particularly important points of this orbit are as follows:

1) After the close approach with Earth (in the end of the first revolution) the spacecraft starts a new tour to the Lagrangian points, in the reverse orbit. It means that the swing-by with Earth has the effect of changing the direction of its motion in the rotating frame. Even more curious, integrating this trajectory for a longer time, it is possible to see that the first five revolutions have alternating directions of motion. It means that there are four consecutive swing-bys that have the property of reversing the direction of the motion of the spacecraft. It is also noted that the second revolution has very close approaches to the Lagrangian points visited. It makes this orbit very suitable for a double tour to the Lagrangian points, with no impulses required for nominal operation.

2) It has the shortest transfer time (in the first revolution) of all orbits described. The period for an Earth-to-Earth trip is about 11 years and the legs connecting Earth and the Lagrangian points L_4 and L_5 last about 1.8 years each way.

3) It has the closest approaches to the Lagrangian points visited (during the first and second revolutions).

Of course, maneuvers can be made to get any desirable result, like repetition of the first revolution only, repetition of the first two revolutions, and so on. Several crossing points are available for a one-burn impulsive maneuver, if desirable. However, the most interesting application for the curious swing-by found in this trajectory is to build a cycler transportation system between Earth and the Lagrangian points L_4 and L_5, as explained in the next section.

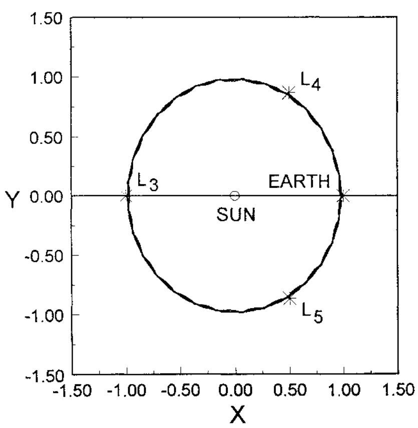

Fig. 5 Orbit LONG-4-5.

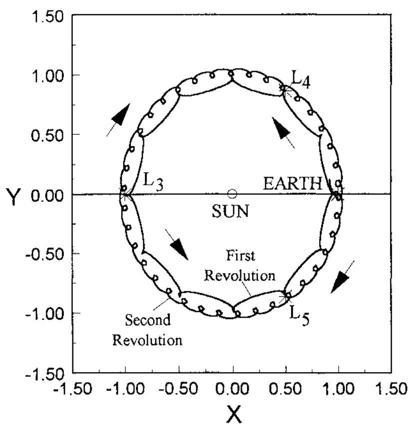

Fig. 6 Orbit SHORT-4-5.

Cycler Transportation System Between Earth and Lagrangian Points L_4 and L_5

The swing-by discovered in the previous section can be used to build a cycler transportation system between Earth and the Lagrangian point L_5. Suppose that the spacecraft starts at L_5 with zero velocity. It is possible to apply an impulse of 0.0274 (816 m/s) such that its velocity goes to $V_x = -0.0271$ and $V_y = 0.0040$. With this velocity, the spacecraft follows one trajectory that is part of the SHORT-4-5 trajectory, as shown in Fig. 7. Then, it goes to Earth, makes the swing-by, and returns to L_5, arriving there with velocity $V_x = -0.0018$, $V_y = 0.0263$. At this point, it is possible to apply an impulse $\Delta V = 0.0337$ (1003.8 m/s) such that its velocity goes to $V_x = -0.0271$, $V_y = 0.0040$; again and it starts the cycler one more time. The timeline for a complete cycler is shown in Table 2.

Table 2 Timeline for a complete cycler between the Earth and L_5

$t = 0$	Spacecraft leaves L_5 from rest (as seen in rotating frame) with impulse of $\Delta V = 0.0274$ (816 m/s)
$t = 1.80$ years	Spacecraft arrives at Earth, makes swing-by to reverse sense of motion and starts going back to L_5
$t = 7.62$ years	Spacecraft arrives at L_5; new impulse of $\Delta V = 0.0377$ (1003.8 m/s) is applied to send it back to Earth and to start cycle again

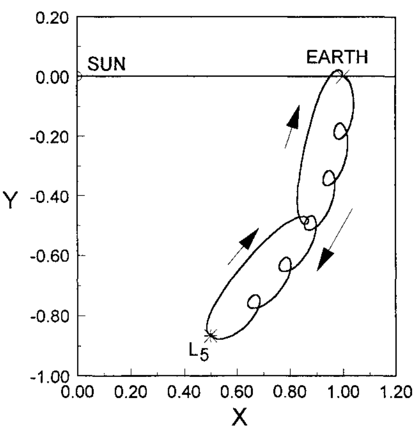

Fig. 7 Cycler system between Earth and L_5.

Another possibility to start the cycler again is to divide this last impulse in two parts. The first one has a magnitude of $\Delta V = 0.0264$ (786.1 m/s) and brings the spacecraft to a complete stop at L_5. The second one is similar to the impulse applied in the beginning of the first revolution ($\Delta V = 0.0274 = 816$ m/s) and it starts the next revolution of the cycler in the same way that it started the previous one. The advantage of this double-impulse maneuver is that it is possible to keep the spacecraft parked at L_5 for as long as needed, allowing time to refuel, transfer of cargo, repairs, etc. The disadvantage is obviously the larger ΔV required. The one-impulse maneuver requires 1003.8 m/s and the two-impulse maneuver requires a total of 1602 m/s, that, is 598.2 m/s more expensive. To reproduce this cycler system for the Lagrangian point L_4, there is no need for further calculations. By using the mirror image theorem, it is only necessary to find the mirror image of the previous trajectory linking Earth and the Lagrangian point L_5. Note that the mirror image of the legs for an Earth-bound trip in now an L_4-bound trip and the mirror image of the L_5-bound leg is now the Earth-bound leg.

LONG-5-4 Orbit

In this orbit the spacecraft M_3 leaves Earth at $t = 0$ and goes to L_5 (in 4.24 years), L_3 (in 12.74) years), L_4 (in 20.81 years), and back to Earth's neighborhood again (in 24.88 years). Figure 8 shows the first two revolutions of this trajectory. The special properties of this orbit are as follows:

1) This is the orbit with smaller residual velocity during the close approaches with the Lagrangian points among all the four orbits studied. This is important to facilitate the data acquisition and/or a rendezvous with another spacecraft and to reduce the magnitude of the impulse required to stop the spacecraft at the Lagrangian point.

2) After completing the first revolution, the spacecraft makes a swing-by with Earth, changes its direction of motion (as seen in the rotating frame) from "clockwise" to "counterclockwise," and goes back to pass near L_4, L_3, L_5, and Earth, in a second revolution. The closest distance between the Lagrangian points and the spacecraft is a little bigger in the second revolution than in the first one, but a maneuver with a small ΔV can provide closer approaches, if desirable. This orbit has the disadvantage of longer transfer times than the ones found in the SHORT versions, but it has the advantage of requiring smaller ΔV. The characteristic of changing its direction of motion can be used in the whole orbit to make a complete tour to the points and start it again in the reverse order or in part of it to build a cycler transportation system between Earth and the Lagrangian points L_4 and L_5, as done before. The details of this new version of this cycler system are explained better in the next section.

Option for Faster Cycler Transportation System Between Earth and L_5 or L_4

The characteristic of reversing the direction of motion of the LONG-5-4 orbit can be used to build a new version of a system for permanent transportation between Earth and the Lagrangian point L_4. In this version, the spacecraft leaves L_4 (by applying an impulse such that $V_x = 26.8$ m/s and $V_y = 47.7$ m/s), goes to Earth, and returns to L_4 with the impulse given by Earth's swing-by (with no necessity of fuel expenditure). Next, an extra impulse is applied, to make a fine adjustment that allows M_3 to arrive at the Lagrangian point L_4. Optimization techniques are not applied (although there is freedom to choose the position for the maneuver and the time of flight from this point to the destination point L_4) to find the maneuver with minimum ΔV for this case. A simple trial case (guessing a position for the impulse and a subsequent time of flight to L_4) shows that an impulse of less than 0.02 canonical units (about 560 m/s) can satisfy all the requirements. Then, after M_3 arrives at L_4, it is necessary to apply another impulse to reverse its motion and send it back to Earth, following the same trajectory it did in the first revolution. Again, a trial case that satisfies all the requirements but without any optimization technique shows that a maneuver with ΔV less than 0.05 in canonical units (about 1500 m/s) is sufficient.

The final result is a trajectory that requires 4.0728 years for the Earth-bound trip, 1.7825 years for the L_4-bound trip, and about 2060 m/s per revolution in maneuvers. It is a little more expensive than the previous cycler transportation system shown before (2060 vs 1812 m/s), but it is faster (5.86 vs 7.62 years). The decision for which trajectory to use depends on the specific requirements of the mission considered.

Again, a similar system can be built between Earth and the Lagrangian point L_5 using the mirror image theorem in the same way it was used before. The mirror image of the legs for an Earth-bound trip in now an L_5-bound trip and the mirror image of the L_4-bound leg is now the Earth-bound leg.

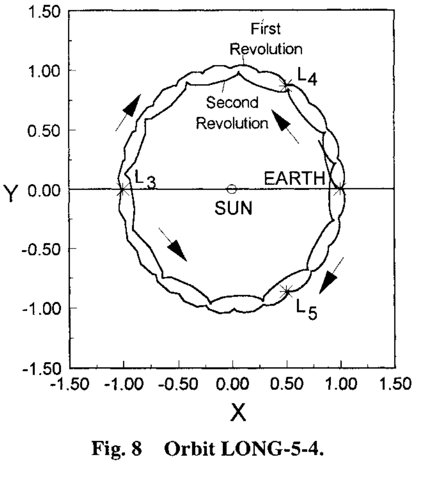

Fig. 8 Orbit LONG-5-4.

Conclusions

In this paper, the problem of transferring a specraft from one body back to the same body in the planar restricted three-body problem is considered. Solutions are found for the Earth–sun and Earth–moon systems. Trajectories under this model with near-zero ΔV to move a spacecraft between any two points on the group formed by Earth and the Lagrangian points L_3, L_4, and L_5 in the Earth-sun system are found. It is shown how to apply these results to build a cycler transportation system to link all the points in this group. It is also shown how to use one or more swing-by with Earth to build a cycler transportation system between Earth and the Lagrangian points L_4 and L_5, with small ΔV required for maneuvers in nominal operation. Those trajectories are only representative examples, and many others can be found by changing slightly the initial velocity. Those trajectories are generated using the restricted three-body problem, and it needs small corrective maneuvers to be used in the "real world." A suggestion for future research is to simulate those trajectories in improved models (closer to the real world) to evaluate the magnitude of those corrections.

Acknowledgments

The authors express their thanks to CAPES (Federal Agency for Post-Graduate Education, Brazil) and INPE (National Institute for Space Research, Brazil) for supporting this research.

References

[1]Prado, A. F. B. A., "Optimal Transfer and Swing-By Orbits in the Two- and Three-Body Problems," Ph.D. Dissertation, Univ. of Texas at Austin, TX, Dec. 1993.

[2]Broucke, R. A., and Prado, A. F. B. A., "Jupiter Swing-By Trajectories Passing Near the Earth," AIAA Paper 93-0177, Feb. 1993; also *Advances in the Astronautical Sciences*, Vol. 82, 1993, pp. 1159–1176.

[3]Bender, D. F., "A Suggested Trajectory for a Venus–Sun, Earth–Sun Lagrange Points Mission, VELA," AAS Paper 79-112, June 1979.

[4]Uphoff, C., and Crouch, M. A., "Lunar Cycler Orbits with Alternating Semi-Monthly Transfer Windows," AAS Paper 91-105, Feb. 1991.

[5]Aldrin, B., "Cyclic Trajectory Concepts," SAIC Presentation to the Interplanetary Rapid Transit meeting, Jet Propulsion Laboratory, 1985.

[6]Szebehely, V. G., *Theory of Orbits*, Academic, New York, 1967.

[7]Broucke, R. A., "Traveling Between the Lagrange Points and the Moon," *Journal of Guidance, Control, and Dynamics*, Vol. 2, No. 4, 1979, pp. 257–263.

[8]Miele, A., "Theorem of Image Trajectories in the Earth-Moon Space," *Astronautica Acta*, Vol. 6, 1960, pp. 225–232.

[9]Hénon, M., "Sur les Orbites Interplanétaires qui Rencontrent Deux Fois la Terre," *Bull. Astron.*, Vol. 3, 1968, pp. 377–402.