LAMBERT PROBLEM SOLUTION IN THE HILL MODEL OF MOTION

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Abstract. The goal of this paper is obtaining a solution of the Lambert problem in the restricted three-body problem described by the Hill equations. This solution is based on the use of pre determinate reference orbits of different types giving the first guess and defining the sought-for transfer type. A mathematical procedure giving the Lambert problem solution is described. This procedure provides step-by-step transformation of the reference orbit to the sought-for transfer orbit. Numerical examples of the procedure application to the transfers in the Sun–Earth system are considered. These examples include transfer between two specified positions in a given time, a periodic orbit design, a halo orbit design, halo-to-halo transfers, LEO-to-halo transfer, analysis of a family of the halo-to-halo transfer orbits. The proposed method of the Lambert problem solution can be used for the two-point boundary value problem solution in any model of motion if a set of typical reference orbits can be found.

Key words: Lambert problem, three-body problem, Hill equations, Lagrange points, halo orbit

1. Introduction

A high interest in the Sun–Earth L_1 and L_2 Lagrange (or libration) points and their vicinities for the spacecraft positioning was demonstrated in the last three decades. This is because these points are very convenient for solving many scientific and applied problems related to the Solar-Terrestrial physics (Farquhar, 1969, 1991). The advantage of the Lagrange points is that the spacecraft can stay in the so-called halo orbit around one of them as long as necessary with a very small propellant consumption. The transfer from the Lagrange point vicinity back to Earth or to another Lagrange point also can be performed with a small ΔV. Some kind of the Lambert problem solution (i.e. determination of a transfer orbit between two given positions in a given time) would be very helpful for the design of the transfers in the extended Earth vicinity including the Lagrange points. However, this solution for the restricted three-body problem is complicated, comparing with the two-body problem, by the following difficulties:

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– an exact analytical solution of the Lambert problem having been obtained for the two-body problem does not exist for the three-body problem;
– the Lambert problem usually has many different solutions in the three-body problem for any specified initial data set (such as the terminal transfer positions, transfer time, and number of complete orbits around the attracting center).

There are many papers in the literature that refers to the transfers in the extended planet vicinity including the L_1 and L_2 Lagrange points. In a general form, it is possible to divide them in three groups:

(1) Papers that concentrate on the halo orbit analysis, like: Farquhar (1972, 1991), Farquhar and Kamel (1973), Farquhar et al. (1977), Breakwell and Brown (1979), Richardson (1980a, b, c), Howell and Breakwell (1984), Popescu (1986), Popescu and Cardos (1995) and Felipe et al. (2000). These papers describe the halo orbits and show approximate analytical solutions, that can later be used as a starting point to find accurate numerical orbits.
(2) Papers that consider the problem of transferring the spacecraft between a parking orbit around the Earth and a halo orbit. Among these, we can mention, D'Amario and Edelbaum (1974), Stalos et al. (1993), Howell et al. (1994), Starchville and Melton (1997) and Kechichian (2001).
(3) A third line of research studies maneuvers between halo orbits. Some good samples are Farquhar et al. (1980), Farquhar (1980), Popescu (1985), Simó et al. (1987), Hiday and Howell (1992), Howell and Gordon (1992), Gordon and Howell (1992) and Howell and Pernicka (1993). In this category it is also possible to include some papers that consider the Rendezvous between two spacecraft, like: Jones and Bishop (1993a, b, 1994). An excellent compilation of the results combining all the topics are available in four volumes in Gómez et al. (2001a, b, c, d).

The present paper uses a different approach to the considered transfer problem solution similar to one suggested by Prado and Broucke (1995, 1996). Namely, the paper suggests a solution based on use of several reference orbits of different types. For every specific transfer, the reference orbit of a desired type should be selected; this reference orbit is used then as a first guess providing uniqueness of the solution. The suggested approach and the mathematical procedure based on it have proven their effectiveness for solving the following problems using a unique mathematical procedure:

– design of a transfer of a specified type between two given positions in the extended Earth vicinity in a given time;
– design of a transfer of a specified type both between the L_1 and L_2 halo orbits and between two halo orbits around the same Lagrange point;

– design of a periodic orbit around the Earth in the extended Earth vicinity;
– a halo orbit design;
– analysis of a family of transfers of a given type between two specified positions with the transfer time as the parameter of the family and determination of the interval of the transfer times for which the transfer is possible.

However, in some cases the suggested approach does not allow direct determination of a specified transfer and one or more intermediate steps are needed. For example, if a three-dimensional transfer is sought for, sometimes it cannot be found directly; first a planar transfer should be found and then it is used as a first guess for the sought-for 3D-transfer calculation. This is illustrated by examples in the paper.

The Hill equations are used as the mathematical model of motion. These equations describe the motion in the extended Earth vicinity including the Sun–Earth L_1 and L_2 Lagrange points with accuracy sufficient for a preliminary analysis. However, solution of the considered transfer problem can be easily adapted to the general equations of the restricted three-body problem. This generalized solution can be applied, for instance, to the Moon vicinity including the Earth–Moon L_1 and L_2 Lagrange points.

2. Equations of Motion and Formulation of the Transfer Problem

The model used in this paper is the well-known Hill approximation of the circular restricted three-body problem. The Sun–Earth system is considered below for simplicity, although of course the model can be applied to other systems.

The rotating frame with its origin in the Earth mass center, x-axis along the Sun–Earth direction and y-axis in the ecliptic plane is considered. The angular velocity ω of the frame rotation assumed to be constant and equal to $\omega = \sqrt{\mu_0}/a^{3/2}$ where μ_0 is the Sun gravity constant and a is the Sun to Earth mean distance. Then the Hill equations for the position $\mathbf{r} = \{x, y, z\}$ and velocity $\mathbf{v} = \{v_x, v_y, v_z\}$ can be written in the form

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \omega^2 \mathbf{N}\mathbf{r} + 2\omega \mathbf{M}\mathbf{v} - \frac{\mu}{r^3}\mathbf{r} \tag{1}$$

where μ is the Earth gravity constant, $r = |\mathbf{r}|$,

$$\mathbf{N} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There are two stationary solutions of Equation (1) (i.e. solutions with $\mathbf{r} = \text{const}$, $\dot{\mathbf{r}} = \ddot{\mathbf{r}} = \cdots = \mathbf{0}$) in the Earth vicinity (collinear L_1 and L_2 Lagrange points) given by $\mathbf{r}_L = \{x_L, 0, 0\}$ where

$$x_L = \pm a\left(\frac{\mu}{3\mu_0}\right)^{1/3} = \pm 1.49656 \times 10^6 \text{ km} \tag{2}$$

with '$-$' for L_1 and '$+$' for L_2 (Note that value (2) slightly differs from which the accurate three-body model gives). There exist a special class of unstable three-dimensional periodic orbits (halo orbits) around the L_1 and L_2 points.[1]

The state transition matrix of the system is

$$\mathbf{\Phi} = \mathbf{\Phi}(t, t_0) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t_0)} \tag{3}$$

where $\mathbf{x} = \{\mathbf{r}, \mathbf{v}\}$ is the state vector. Matrix $\mathbf{\Phi}$ can be represented in the form

$$\mathbf{\Phi} = \mathbf{\Phi}(t, t_0) = \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{\Phi}_{12} \\ \mathbf{\Phi}_{21} & \mathbf{\Phi}_{22} \end{bmatrix} \tag{4}$$

with

$$\mathbf{\Phi}_{11} = \frac{\partial \mathbf{r}(t)}{\partial \mathbf{r}(t_0)}, \quad \mathbf{\Phi}_{12} = \frac{\partial \mathbf{r}(t)}{\partial \mathbf{v}(t_0)}, \quad \mathbf{\Phi}_{21} = \frac{\partial \mathbf{v}(t)}{\partial \mathbf{r}(t_0)}, \quad \mathbf{\Phi}_{22} = \frac{\partial \mathbf{v}(t)}{\partial \mathbf{v}(t_0)} \tag{5}$$

Matrix (3) satisfies the variational equation

$$\dot{\mathbf{\Phi}} = \mathbf{F\Phi}, \quad \mathbf{\Phi}(t_0, t_0) = \mathbf{I}_6 \tag{6}$$

where

$$\mathbf{F} = \begin{bmatrix} 0 & \mathbf{I}_3 \\ \omega^2\mathbf{N} + \mathbf{G} & 2\omega\mathbf{M} \end{bmatrix}, \quad \mathbf{G} = \frac{\mu}{r^3}\left(3\frac{\mathbf{r}\mathbf{r}^{\mathrm{T}}}{r^2} - \mathbf{I}_3\right)$$

$\mathbf{I}_6$ and $\mathbf{I}_3$ are unit matrices of sixth and third order respectively, superscript 'T' denotes transition.

Let us put $t_0 = 0$ and assume that two positions $\mathbf{r}_0$ and $\mathbf{r}_1$ and time T are given. The problem is to find the transfer orbit between these two positions such that $\mathbf{r}_0 = \mathbf{r}(0), \mathbf{r}_1 = \mathbf{r}(T)$ in the orbit. This is the Lambert problem for the Hill model of motion.

3. Reference Orbits

The problem solution is not unique, i.e. there may exist a variety of transfer orbits of different types for any specified positions and transfer time. This is illustrated by Figure 1, showing some of the possible transfers between two positions with coordinates $\{-1.3 \times 10^6, -0.5 \times 10^6\,\text{km}, 0\}$ and $\{1.72\times$

[1]Some class of these orbits is called Lissajous orbits in literature. Although here all 3-dimensional orbits about Lagrange points will be called halo orbits for simplicity.

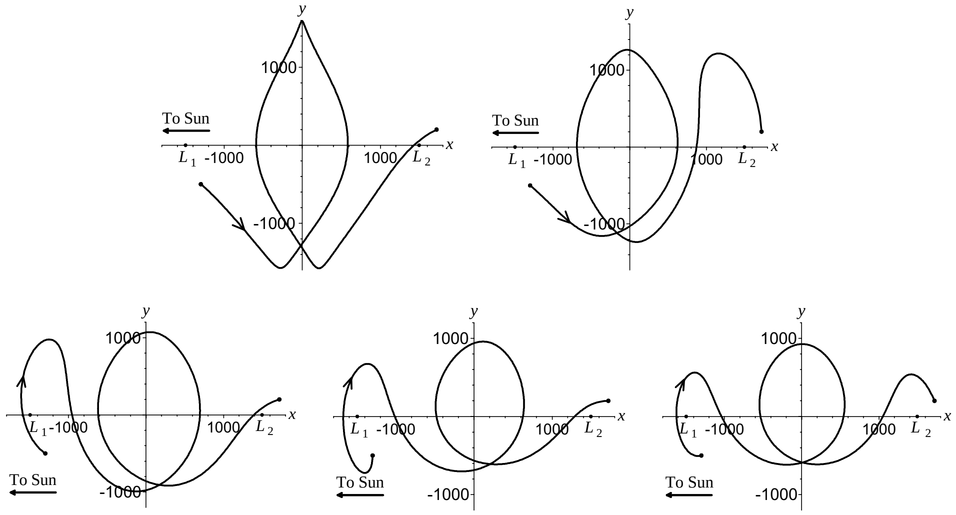


Figure.1. Different transfer orbits between two given positions in a given time.

$10^6, 0.2 \times 10^6\,\text{km}, 0\}$, with transfer time 320 days and one complete orbit around the Earth. Axis labels in Figure 1 and in other Figures below are given in 10^3 km.

The reference orbits play two roles in the solution of the considered transfer problem: they specify the orbit type and they are used as a first guess in the solving procedure.

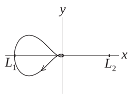
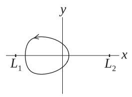
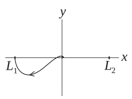
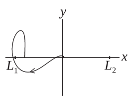
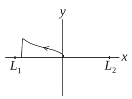
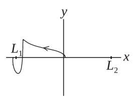
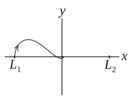
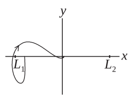
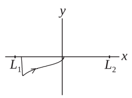
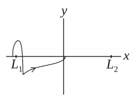
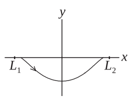
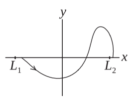
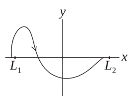
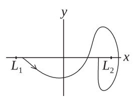
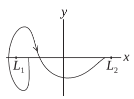
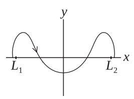
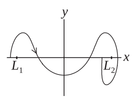
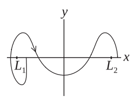
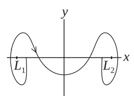
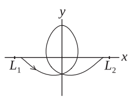
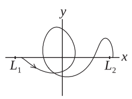
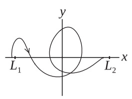
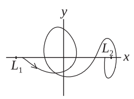
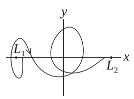
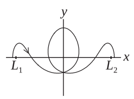
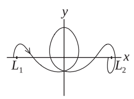
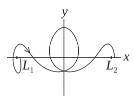
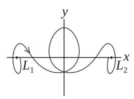
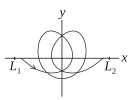
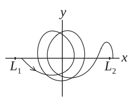
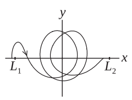
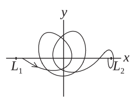
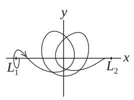
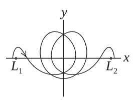
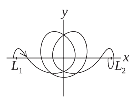
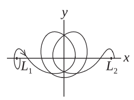
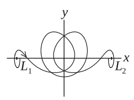
The following transfer classes are considered: Earth to Earth via a Lagrange point vicinity (EE class), Earth to a Lagrange point vicinity (EL class), a Lagrange point vicinity to Earth (LE class), and the transfer between two Lagrange points (LL class). The corresponding reference orbits are given in Table I. The reference orbits of the EE class begin and end at the minimum distance from the Earth. As is seen in Table I, transfers with one and two complete orbits around the Earth are included for the LL class.

The reference orbits given in Table I were obtained using the method described below in the following way. First, for each reference orbit a transfer of the same type was approximately calculated by means of varying the initial velocity vector. Then this approximation was used as a first guess for specified terminal positions and transfer time of the reference orbit.

Notice that in some cases different reference orbits can give the same transfer. For example, the last transfer orbit shown in Figure 1 corresponds to the LL-class reference orbits of sixth and eighth types with one complete revolution (see Table I).

Table I gives reference orbits only for the Earth-to-L_1, L_1-to-Earth, and L_1-to-L_2 transfers. However, since the Hill equations are symmetrical with

TABLE I
Reference orbits

Transfer class	No. of complete orbits	Orbit type								
		1	2	3	4	5	6	7	8	9
EE	0			—	—	—	—	—	—	—
EL	0					—	—	—	—	—
LE	0					—	—	—	—	—
	0									
LL	1									
	2									

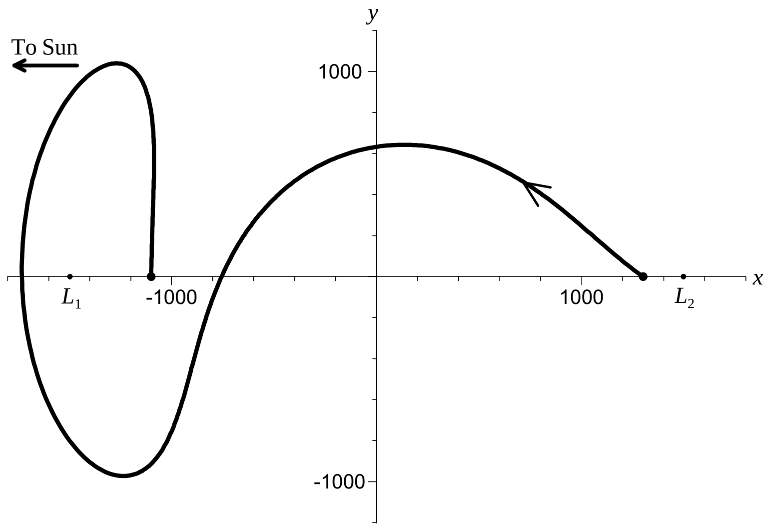


Figure 2. Transfer orbit of the fourth type with zero complete orbits from L_2 to L_1.

respect to the frame origin (i.e. if $\mathbf{r}(t), \mathbf{v}(t)$ is a solution of (1) then $-\mathbf{r}(t)$, $-\mathbf{v}(t)$ also is a solution), these orbits also can be used for the Earth-to-L_2, L_2-to-Earth, and L_2-to-L_1 transfers. For that the initial state vector of the respective reference orbit from Table I should be taken with sign '$-$'. For example, if $\mathbf{x}_{0\text{ref}}$ is the initial state vector of an L_1-to-L_2 reference orbit (LL class) then $-\mathbf{x}_{0\text{ref}}$ is the state vector of the respective L_2-to-L_1 reference orbit of the same type. Figure 2 shows the L_2-to-L_1 reference orbit of fourth type with zero complete orbits around the Earth.

Note that the reference orbits given in Table I cover not all possible transfer types. For example, retrograde orbits (i.e. orbits flying by the Earth in the clockwise direction) are not represented. Table I gives reference orbits for most practically important cases. However, more reference orbits can be added if necessary.

All suggested reference orbits are planar. Nevertheless, they can be used also for three-dimensional transfers. Parameters of the reference orbits (such as their terminal positions and transfer time) are not given here because they are not very important and can be varied in certain limits.

4. Mathematical Procedure of the Transfer Problem Solution

Let us assume that the reference orbit is given by its initial state vector

$$\mathbf{x}_{0\text{ref}} = \mathbf{x}_{\text{ref}}(0) = \{\mathbf{r}_{0\text{ref}}, \mathbf{v}_0\} \tag{7}$$

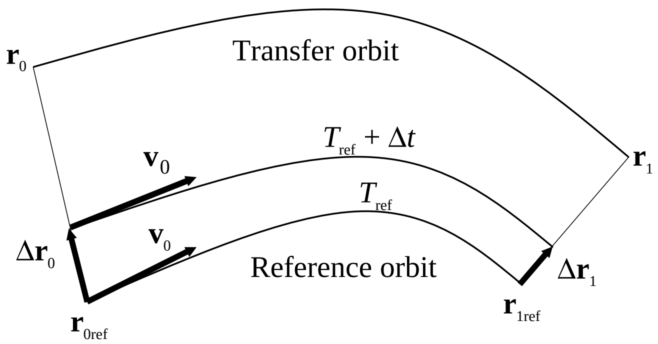


Figure 3. First step of the transfer problem solution.

and the time T_{ref}. Propagation of the vector (7) to the time T_{ref} by means of numerical integration of Equations (1) gives the final state vector of the reference orbit

$$\mathbf{x}_{1\text{ref}} = \mathbf{x}_{\text{ref}}(T_{\text{ref}}) = \{\mathbf{r}_{1\text{ref}}, \mathbf{v}_1\} \tag{8}$$

Let us introduce variations of the reference parameters given by

$$\Delta\mathbf{r}_0 = \varepsilon(\mathbf{r}_0 - \mathbf{r}_{0\text{ref}}), \quad \Delta\mathbf{r}_1 = \varepsilon(\mathbf{r}_1 - \mathbf{r}_{1\text{ref}}), \quad \Delta t = \varepsilon(T - T_{\text{ref}}) \tag{9}$$

where $0 < \varepsilon \leq 1$ is a parameter. Now we will find the transfer orbit between the positions $\mathbf{r}'_{0\text{ref}} = \mathbf{r}_{0\text{ref}} + \Delta\mathbf{r}_0, \mathbf{r}'_{1\text{ref}} = \mathbf{r}_{1\text{ref}} + \Delta\mathbf{r}_1$ with the transfer time $T_{\text{ref}} + \Delta t$. Let $\mathbf{v}'_0$, $\mathbf{v}'_1$ be the velocities at the beginning and end of this orbit, $\Delta\mathbf{v}_0 = \mathbf{v}'_0 - \mathbf{v}_0, \Delta\mathbf{v}_1 = \mathbf{v}'_1 - \mathbf{v}_1$. In order to find the transfer orbit between $\mathbf{r}'_{0\text{ref}}$, $\mathbf{r}'_{1\text{ref}}$ it is sufficient to find the velocity $\mathbf{v}'_0$ (see Figure 3). If the ε value is small enough then the following equation is approximately fulfilled:

$$\Delta\mathbf{x}_1 = \mathbf{\Phi}\,\Delta\mathbf{x}_0 + \dot{\mathbf{x}}_{1\text{ref}}\,\Delta t \tag{10}$$

where state transition matrix $\mathbf{\Phi} = \mathbf{\Phi}(0, T_{\text{ref}})$ is calculated in the reference orbit,

$$\Delta\mathbf{x}_0 = \{\Delta\mathbf{r}_0, \Delta\mathbf{v}_0\}, \quad \Delta\mathbf{x}_1 = \{\Delta\mathbf{r}_1, \Delta\mathbf{v}_1\} \tag{11}$$

Equation (10) using (4, 5, 11) gives

$$\Delta\mathbf{r}_1 = \mathbf{\Phi}_{11}\,\Delta\mathbf{r}_0 + \mathbf{\Phi}_{12}\,\Delta\mathbf{v}_0 + \mathbf{v}_1\,\Delta t \tag{12}$$

Now an approximate value of the velocity $\mathbf{v}'_0$ can be found from (12) as follows:

$$\mathbf{v}_0'^{(0)} = \mathbf{v}_0 + \mathbf{\Phi}_{12}^{-1}(\Delta\mathbf{r}_1 - \mathbf{\Phi}_{11}\,\Delta\mathbf{r}_0 - \mathbf{v}_1\,\Delta t) \tag{13}$$

Since the value of $\mathbf{v}'_0$ given by (13) is approximate, the propagation of the state vector

$$\mathbf{x}'_{\text{ref}} = \{\mathbf{r}'_{0\text{ref}}, \mathbf{v}'_0\} \tag{14}$$

in time $T_{\text{ref}} + \Delta t$ gives the final position $\mathbf{r}'_{1\text{ref}} + \Delta\mathbf{r}^{(0)}$ (see Figure 4). In order to find the accurate value of the velocity $\mathbf{v}'_0$ the Newton–Raphson method can be used at nth iteration of which the correction is given by

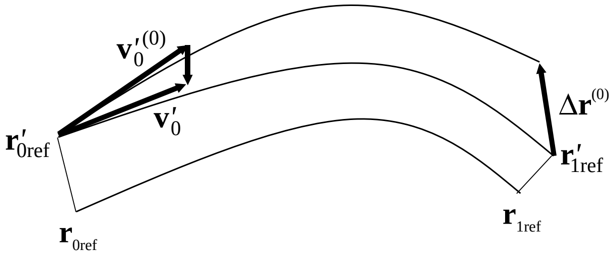


Figure 4. Determination of initial velocity of the intermediate transfer at the first step of the problem solution.

$$\mathbf{v}_0'^{(n+1)} = \mathbf{v}_0'^{(n)} - \left(\mathbf{\Phi}_{12}^{(n)}\right)^{-1} \Delta\mathbf{r}^{(n)}, \quad n = 0, 1, \ldots \tag{15}$$

Matrix $\mathbf{\Phi}^{(n)}$ for (15) is calculated in the orbit given by $\mathbf{x}_{0\text{ref}}'^{(n)} = \{\mathbf{r}'_{0\text{ref}}, \mathbf{v}_0'^{(n)}\}$, vector $\Delta\mathbf{r}^{(n)}$ in (15) is a result of the propagation of $\mathbf{x}_{0\text{ref}}'^{(n)}$. The value of ε in (9) should be selected so that to provide convergence of the procedure (15).

The mathematical procedure given by (9–13, 15) and illustrated by Figures 3 and 4 is a first step of the suggested Lambert problem solution. After the value $\mathbf{v}'_0$ is found the intermediate transfer orbit with the initial state (14) is taken as a new reference orbit and the procedure is repeated (second step) etc.

Notice that the values of parameter ϵ can be different at each step of the suggested procedure of the Lambert problem solution. This procedure converges to the sought-for transfer orbit in N steps if

$$\sum_{i=1}^{N} \varepsilon_i = 1 \tag{16}$$

where ε_i is the ε value at ith step. The procedure may not converge to the sought-for transfer orbit if

$$\sum_{i=1}^{\infty} \varepsilon_i < 1 \tag{17}$$

for any values ε_i providing convergence of the procedure at each step. The reason of (17) may be one of the following:

(1) Transfer of the specified type between the given positions is impossible.
(2) Transfer in the specified time T is impossible.
(3) The transfer orbit of the specified type between the given positions in the given time exists, but the suggested procedure does not provide convergence to it.

In first two cases it is necessary to change the transfer type or transfer time respectively. Examples of the third case are considered below and it is shown how the problem can be solved in this case.

5. Examples of the Transfers

In this section a few examples of the suggested method application are given. The transfer as well as the corresponding reference orbit used for the solution will be denoted as 'Transfer class-orbit type-(only for the LL class) number of complete orbits' according to Table I. For example, the LL-class reference orbit of seventh type with zero complete orbits will be designated as LL-7-0, the LE-class orbit of third type – as LE-3 (see Table I). All coordinates and velocities below are given in kilometers and seconds.

5.1. TRANSFER BETWEEN TWO SPECIFIED POSITIONS IN A GIVEN TIME

Let us consider LL-6-1 transfer between the positions

$$\mathbf{r}_0 = \{-1.4 \times 10^6, -0.8 \times 10^6, 0.3 \times 10^6\}, \quad \mathbf{r}_1 = \{1.8 \times 10^6, -0.5 \times 10^6, -0.2 \times 10^6\} \tag{18}$$

in 300 days. The mathematical procedure described in Section 4 does not provide the solution directly: it stops when the terminal points of the transfer orbit reach coordinates $\{-1.47 \times 10^6, -0.53 \times 10^6, 0.2 \times 10^6\}$ and $\{1.81 \times 10^6, -0.35 \times 10^6, -0.15 \times 10^6\}$ and the transfer time is 287 days. At this intermediate transfer orbit the procedure (15) does not converges for any positive value of the parameter ε (i.e. the inequality (17) takes place). However, the problem can be solved in the following way. First, an intermediate planar LL-6-1 transfer between the positions $\mathbf{r}_0 = \{-1.4 \times 10^6, -0.8 \times 10^6, 0\}$ and $\mathbf{r}_1 = \{1.8 \times 10^6, -0.5 \times 10^6, 0\}$ in 260 days is sought-for. This transfer shown in Figure 5 (as well as the reference LL-6-1 orbit) is used then as a first guess for the sought-for 300-day transfer between the positions (18). This final transfer also is shown in two projections in Figure 5.

Note that the LL-8-1 reference orbit (see Table I) also can be used for obtaining the intermediate 260-day transfer shown in Figure 5.

5.2. PERIODIC ORBIT

The following problem will be considered here as an example: find a planar periodic orbit simply rounding the Earth with period 5–6 months and passing through the point

$$\mathbf{r}_0 = \mathbf{r}_1 = \{-0.2 \times 10^6, 1.2 \times 10^6, 0\} \tag{19}$$

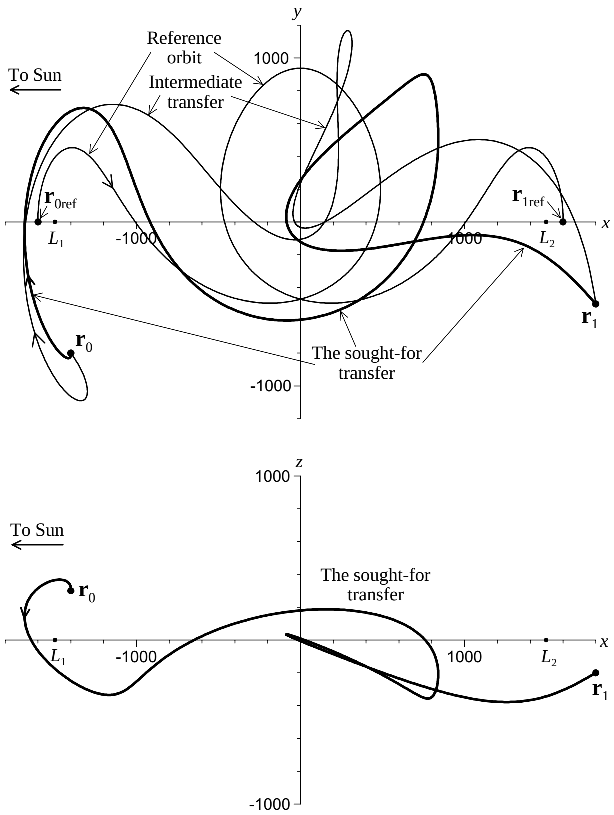


Figure 5. 3D transfer of the LL-6-1 type between two given positions.

Let us take the orbital period equal to 165 days as a first guess and use LL-1-1 reference orbit (see Table I) for the problem solution; this reference orbit is to give two revolutions around the Earth. However, the problem cannot be solved directly, in one step: the transfer shown in Figure 6 with the transfer time 330 days is the direct problem solution using LL-1-1 reference orbit.

In order to solve the problem a 300-day transfer can be found first between $\mathbf{r}_0$ given by (19) and $\mathbf{r}_1 = \{-0.2 \times 10^6, 1.2 \times 10^6, 0\}$ using the LL-1-1 reference orbit. This transfer is shown in Figure 7(a) Now the end of this orbit can be propagated in 30 days more and the respective orbit shown in Figure 7(b) can be used as the reference one for the 330-day transfer between $\mathbf{r}_0, \mathbf{r}_1$ given by (19). Although this is not a periodic orbit yet. In order to find the periodic orbit, the transfer time between the positions $\mathbf{r}_0, \mathbf{r}_1$ given by (19) should be varied so that the beginning and end velocities of the transfer were equal to each other. This periodic orbit is shown in Figure 8, the orbital period is 167.36 days. Now this orbit can be used as a reference one for a

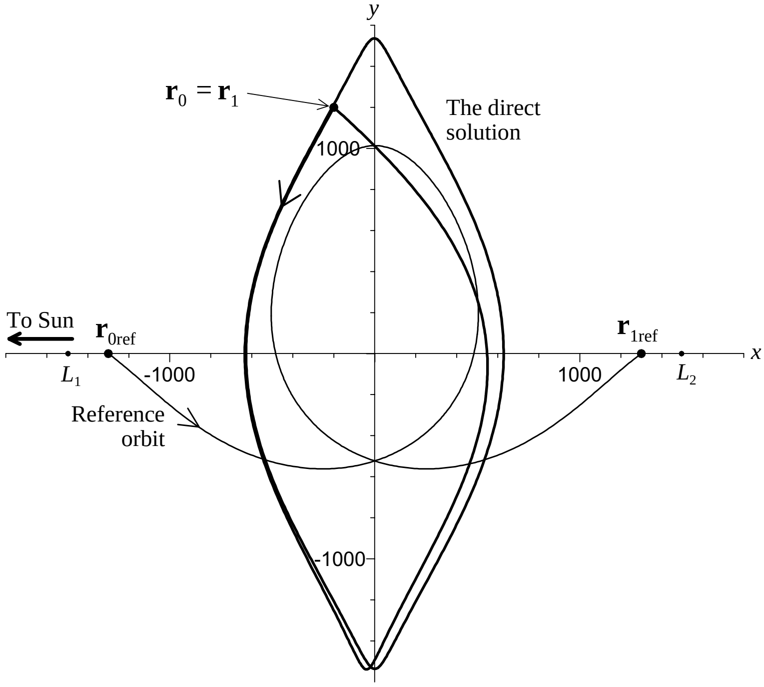


Figure 6. Failed attempt to design the periodic orbit directly.

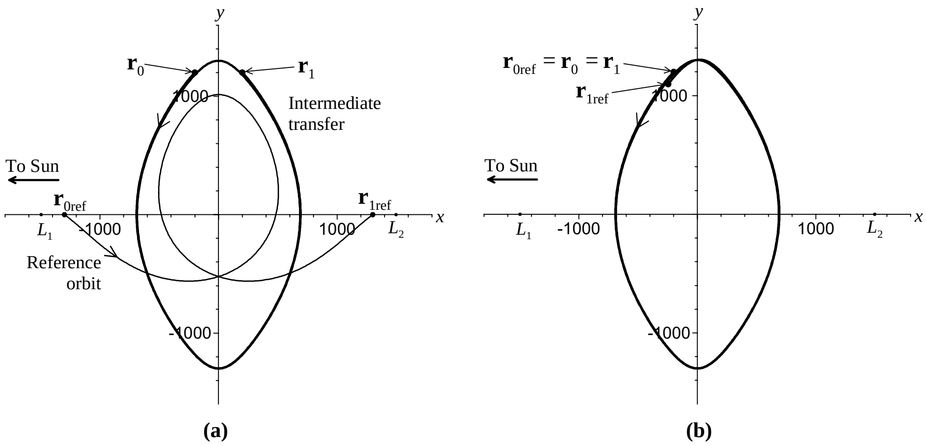


Figure 7. Intermediate transfer orbits in the periodic orbit design.

three-dimensional periodic or quasi-periodic orbit. An example of such quasi-periodic orbit beginning and ending in the position

$$\mathbf{r}_0 = \mathbf{r}_1 = \{-0.2 \times 10^6, 1.2 \times 10^6, 0.2 \times 10^6\}$$

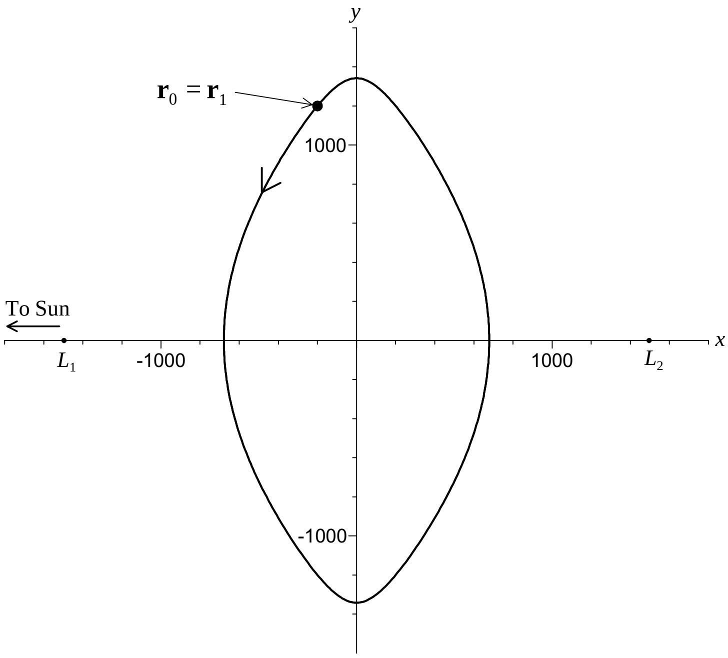


Figure 8. Planar periodic orbit.

is shown in two projections in Figure 9. Period of the orbit is ≈ 170 days for x and y coordinates and ≈ 340 days for z coordinate.

5.3. HALO ORBIT DESIGN

Here an application of the suggested method of the Lambert problem solution to the halo orbit design is given. Let us consider an example of the planar L_1 halo orbit starting and ending in the point

$$\mathbf{r}_0 = \mathbf{r}_1 = \{-1.29656 \times 10^6, 0, 0\}$$

(i.e. 200,000 km apart from the L_1 point, see (2)). Knowing that the halo orbit period is close to 180 days let us take this figure as a first guess for the period. A 180-day part of the LL-5-0 reference orbit (see Table I) can be taken as the reference orbit in this case. This part gives the 180-day transfer between $\mathbf{r}_0$ and $\mathbf{r}_1$ (see Figure 10). Now this transfer can be used as the reference orbit for obtaining the halo orbit in the following way: varying the transfer time in order to get the initial and end velocities in the transfer orbit equal to each other. This halo orbit also is shown in Figure 10, its period is equal to 178.295 day, the initial and end velocity vectors are equal to $\{0, -241.45\,\text{m/s}, 0\}$.

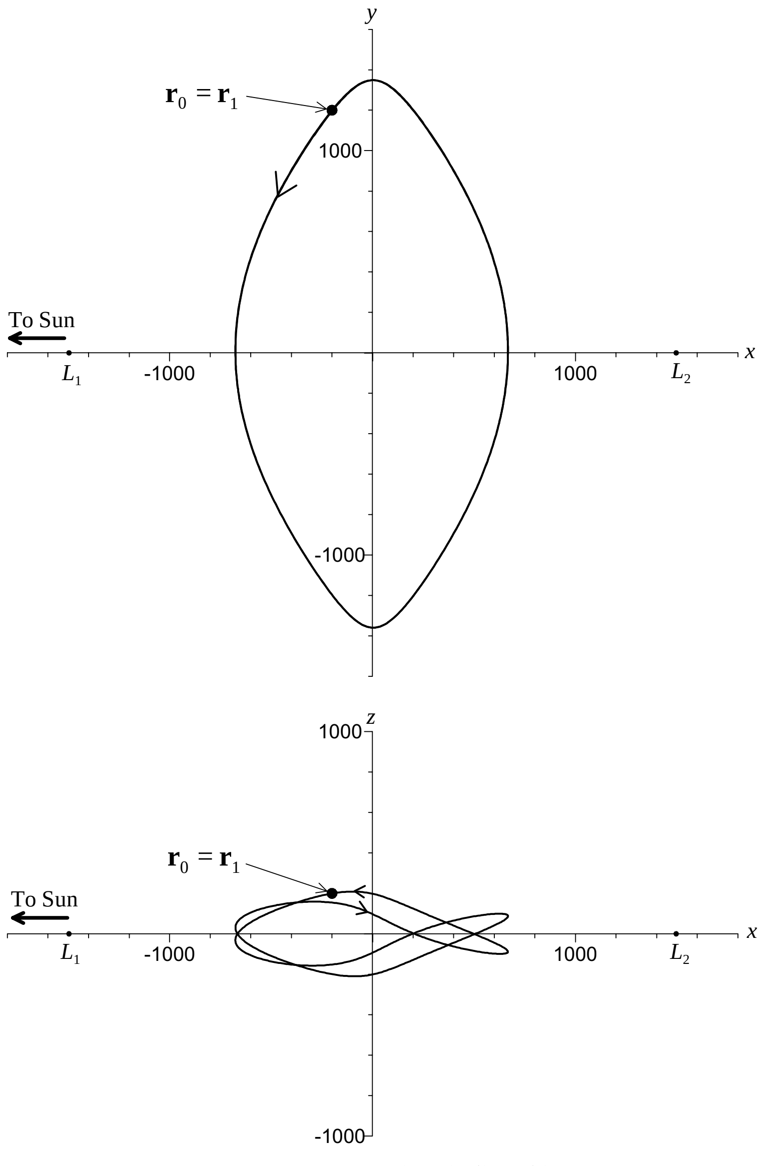


Figure 9. 3D quasi-periodic orbit.

This planar halo orbit can be used as a reference orbit for three-dimensional halo orbit design. Let us consider the following example of this design: the planar halo orbit shown in Figure 10 is propagated in one more orbital period and this 2-revolution orbit is used as a reference one for the 358-day transfer between the positions

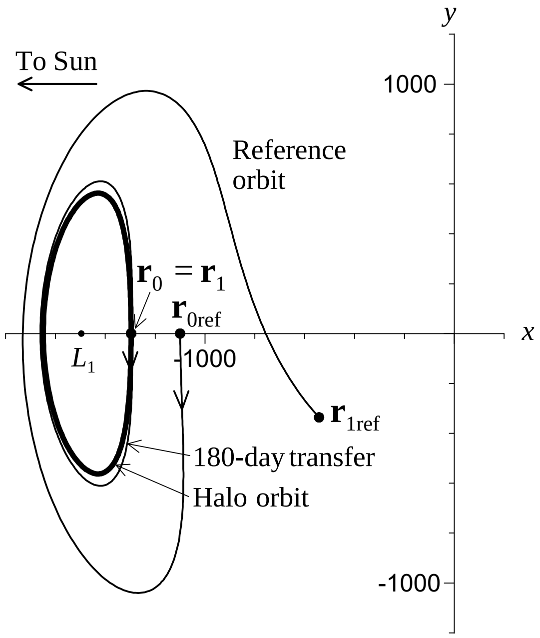


Figure 10. Planar halo orbit design.

$$\mathbf{r}_0 = \{-1.29656 \times 10^6, 0, 0.1 \times 10^6\}, \quad \mathbf{r}_1 = \{-1.29656 \times 10^6, 0, -0.1 \times 10^6\}$$

This transfer is shown in three projections in Figure 11.

As was demonstrated in this subsection, one of the reference orbits given in Table I can be used for the halo orbit design. Although a reference halo orbit can be added to ones given in Table I for this purpose.

5.4. HALO TO HALO TRANSFER

The transfers between two given halo orbits principally does not differ from the transfer between two given positions (see Section 5.1). Below the halo orbit will be specified by parameters $\Delta x_0, z_0, v_0, \varphi_0$ giving the initial halo state vector

$$\{x_L - \Delta x_0, 0, z_0, 0, v_0 \cos \varphi_0, v_0 \sin \varphi_0\} \tag{20}$$

where x_L is given by (2). The position of the first halo exit and second halo entry (i.e. the terminal transfer positions) will be given by the time of flight t_* along the halo starting at the position (20).

The L_1 halo to L_2 halo transfer of the LL-6-1 type is shown in three projections in Figure 12. The transfer time is 220 days. The halo orbit

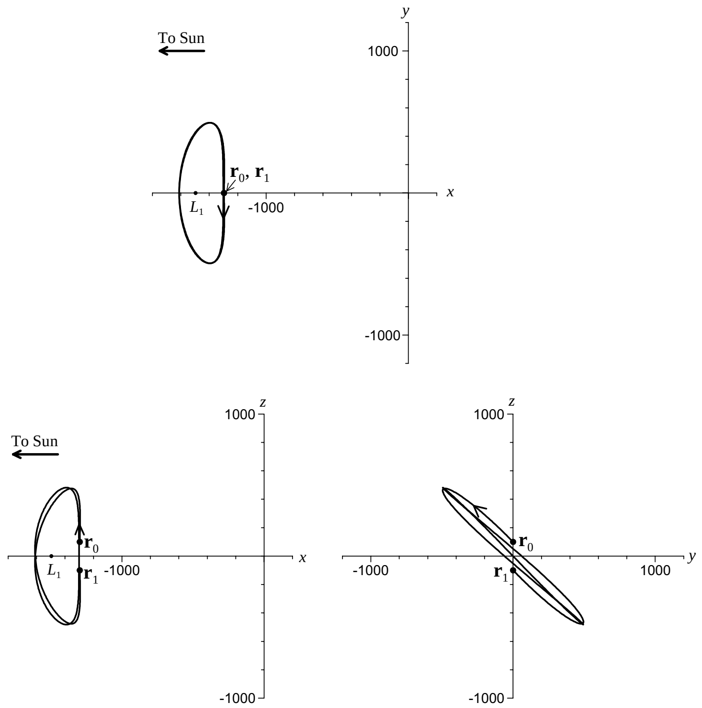


Figure 11. 3D halo orbit design.

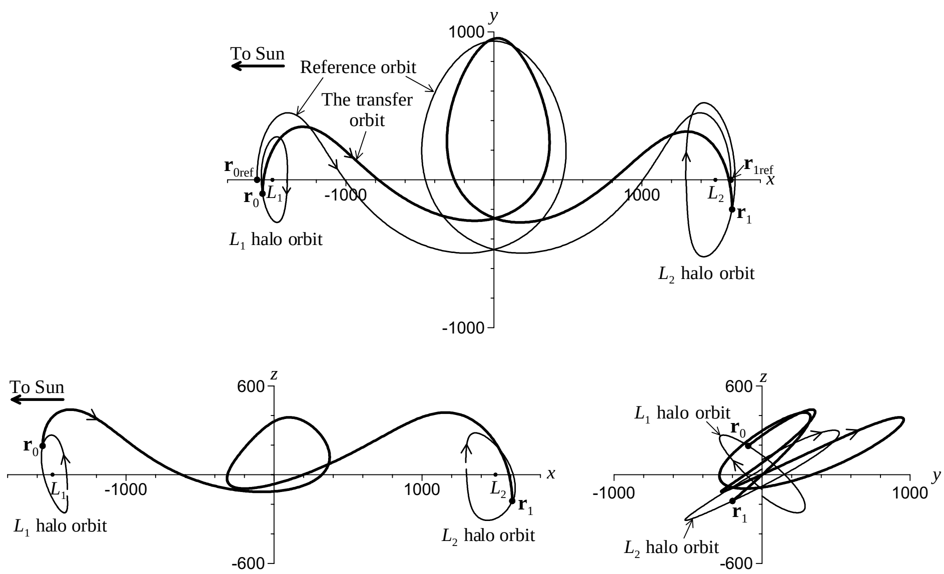


Figure 12. 3D L_1 halo to L_2 halo transfer.

parameters are the following: $\Delta x_0 = -0.1 \times 10^6$ km, $z_0 = -0.1 \times 10^6$ km, $\varphi_0 = 140°$ for the L_1 halo and $\Delta x_0 = 0.2 \times 10^6$ km, $z_0 = 0.1 \times 10^6$ km, $\varphi_0 = 30°$ for the L_2 halo. The velocity v_0 is selected to provide periodicity of the halo and is equal to 155.1 m/s for the L_1 halo and 254.3 m/s for the L_2 halo. The positions of the L_1 halo exit and L_2 halo entry are given by the times $t_* = 80$ day and $t_* = 100$ day respectively.

Three projections of the 70-day transfer between two L_1 halo orbits are shown in Figure 13. The 70-day part of the LL-3-0 reference orbit (see Table I) also shown in Figure 13 was used to find the transfer. The following parameter values specify the halo orbits: $\Delta x_0 = -0.1 \times 10^6$ km, $z_0 = -0.1 \times 10^6$ km, $\varphi_0 = 140°$ for the first halo and $\Delta x_0 = 0.25 \times 10^6$ km, $z_0 = 0.1 \times 10^6$ km, $\varphi_0 = 150°$ for the second halo. The velocities v_0 providing the halo orbit periodicity are equal to 155.1 m/s for the first halo and 310.7 m/s for the second halo. The time t_* is equal to 80 and 170 day for the first and second halo respectively. Note that in this case, as well as for the halo orbit design (see Section 5.3), there was no need in a special reference

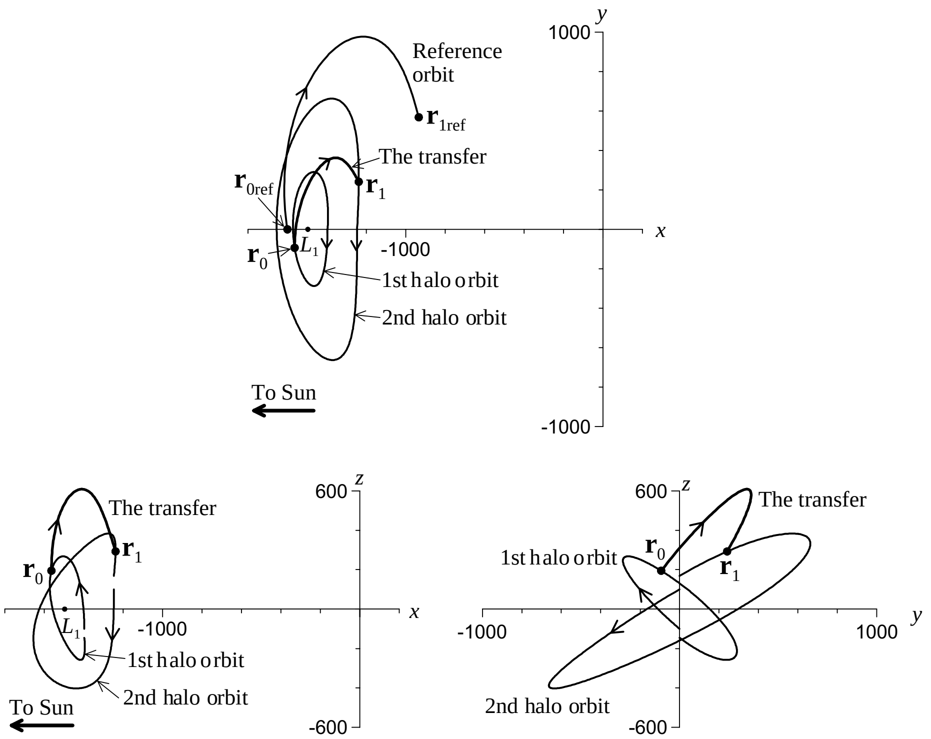


Figure 13. 3D L_1 halo to another L_1 halo transfer.

orbit. Although a reference halo orbit can be added to Table I and used to find transfer between two halo orbits near the same Lagrange point.

5.5. LEO TO HALO TRANSFER

The solution of the problem of the low Earth orbit (LEO) to halo transfer is very similar to the solution for the halo-to-halo transfer and does not need a special consideration. However, the suggested method of the Lambert problem solution can be used also to find a LEO-to-halo transfer with free insertion into the halo orbit. The example below illustrates this possibility. A planar transfer from LEO of 7000-km radius to a halo orbit is considered in this example. First, a 235-day transfer between the positions

$$\mathbf{r}_0 = \{7000, 0, 0\} \tag{21}$$

and

$$\mathbf{r}_1 = \{-1.35 \times 10^6, 0.8 \times 10^6, 0\} \tag{22}$$

is found (first transfer orbit in Figure 14) using the EL-4 reference orbit. Note that position (22) and the transfer time are selected quite arbitrarily. Then

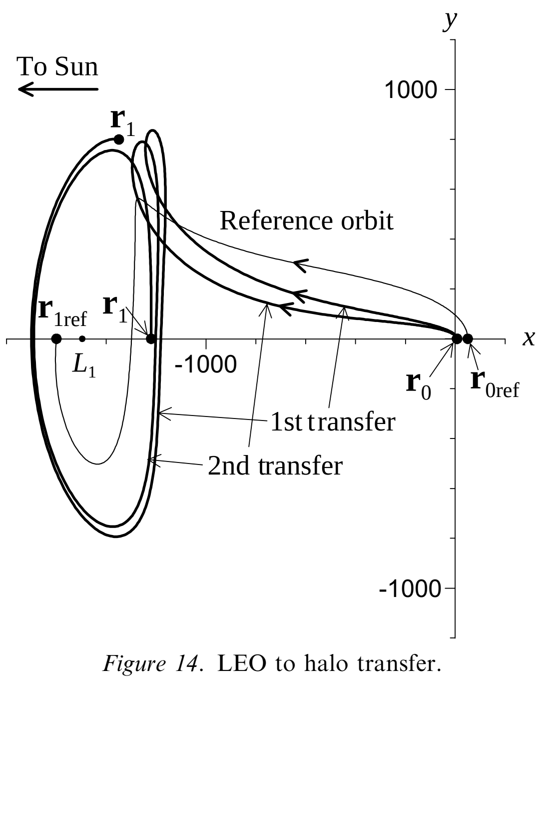


Figure 14. LEO to halo transfer.

this transfer is used as a reference orbit to find a 280-day transfer between positions (21) and

$$\mathbf{r}_1 = \{-1.22 \times 10^6, 0, 0\} \tag{23}$$

(second transfer orbit in Figure 14). The transfer time and position (23) also are taken quite arbitrarily and depend on the desired halo orbit parameters. This second transfer orbit can be used as a reference orbit for obtaining transfer between position (21) and, for instance, $\mathbf{r}_1 = \{-1.35\times 10^6, -0.75 \times 10^6, 0\}$ etc. This procedure can provide a transfer with free insertion into a halo orbit with any desired revolutions around the L_1 point and also can be used for three-dimensional transfers and for free halo-to-halo transfers. Although it must be mentioned that perhaps it is not the most effective solution of the transfer problem with free exit of and insertion into the halo orbit.

5.6. FAMILY OF THE TRANSFER ORBITS

The suggested method also can be applied to analysis of the family of transfers between two given positions $\mathbf{r}_0$ and $\mathbf{r}_1$ in time $T \in [T_0, T_1]$ with given T_0, T_1 and step ΔT. In order to find this family the transfer in time $T = T_0$ is determined first and then the transfer orbit for each time $T = T_0, T_0 + \Delta T, T_0 + 2\Delta T, \ldots$ is used as a reference orbit for finding next transfer in time $T + \Delta T$. Figure 15 shows the family of the halo-to-halo transfer for $T_0 = 180\,\text{day}$, $T_1 = 230\,\text{day}$ and step $\Delta t = 5\,\text{day}$. Parameters of the halo orbits are the same as in the example of the L_1 halo to L_2 halo transfer in Section 5.4. The transfer parameters, for instance the ΔVs necessary for the transfer, can be analyzed using this family. Figure 16 shows ΔV_0 of launch from the L_1 halo, ΔV_1 of insertion into the L_2 halo, and the total $\Delta V = \Delta V_0 + \Delta V_1$ versus the transfer time. As is seen in Figure 16, the total ΔV reaches its minimum in the 212-day transfer.

6. Recommendations

As was seen in Section 5, the suggested method of the transfer problem solution did not give the solution directly in most of the considered examples and intermediate transfer orbits were needed in these cases. In each case the intermediate transfer orbit was used as a new reference orbit for the sought-for transfer or for the next intermediate transfer. The following ways of forming the intermediate transfer can be recommended to the potential users of the method:

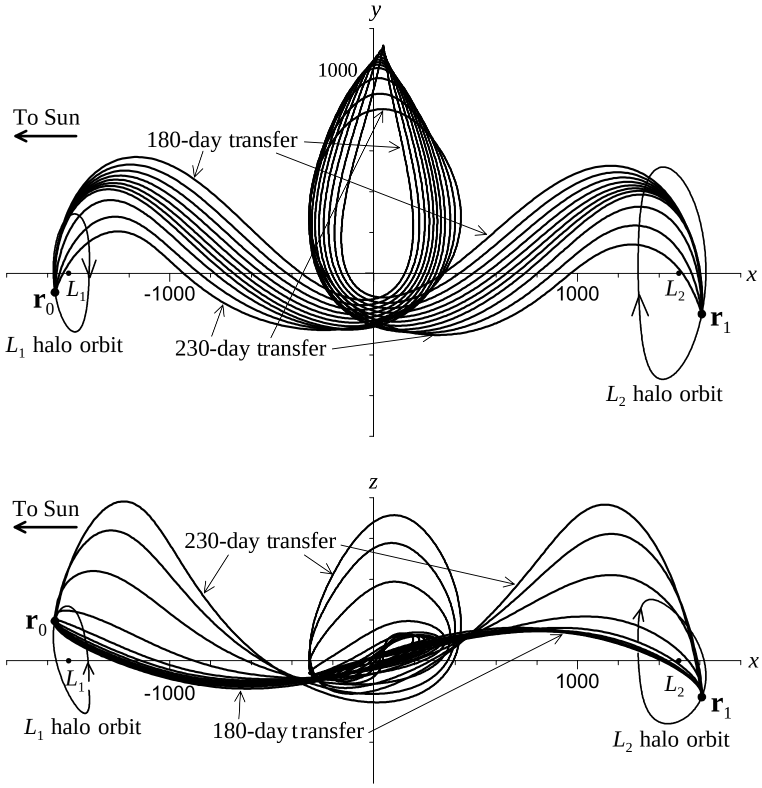


Figure 15. Family of the L_1 halo to L_2 halo transfers.

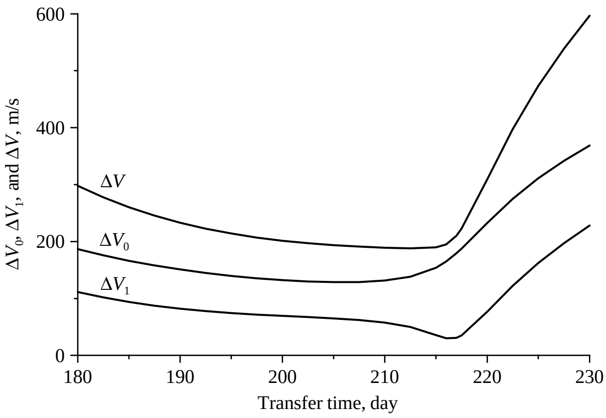


Figure 16. ΔV of launch from the L_1 halo (ΔV_0), of insertion into the L_2 halo (ΔV_1) and total ΔV for the family of the L_1 halo to L_2 halo transfers.

1. Planar intermediate transfer with zero z coordinates in $\mathbf{r}_0$ and $\mathbf{r}_1$ (see example in Section 5.1).
2. Propagation of the reference orbit or an intermediate transfer to the given time T of the sought-for transfer like it was done in Sections 5.2, 5.3, 5.4 (see Figures 7(b), 10 and 13).
3. Subsequent shift of the end position of the intermediate transfers along the sought-for transfer like it was suggested in Section 5.5 (see Figure 14).
4. Changing time of the reference orbit to T first, i.e. intermediate transfer with the terminal positions $\mathbf{r}_0 = \mathbf{r}_{0\text{ref}}$, $\mathbf{r}_1 = \mathbf{r}_{1\text{ref}}$ and transfer time T.
5. Changing time of the sought-for transfer to T_{ref} first, i.e. intermediate transfer with the terminal positions $\mathbf{r}_0$, $\mathbf{r}_1$ and transfer time T_{ref}.

7. Number of Steps of the Calculation Procedure and the Calculation Time

It should be reminded that the subsequent transformation of the reference orbit to the sought-for transfer orbit described in Section 4 includes N steps. Each step changes the terminal positions of the intermediate transfer orbit in $\Delta\mathbf{r}_0$ and $\Delta\mathbf{r}_1$ respectively and the transfer time in Δt (see (9–15) and Figures 3 and 4). The number N as well as the calculation time were not given in Section 5 because they depend on the used integrator, on the practical realization of the mathematical procedure described in Section 4, and of course on the computer power. Here typical values of the number N and calculation time will be given.

The Runge–Kutta integrator of fourth order with variable step was used for the calculations. Not more than four iterations were allowed for the Newton–Raphson procedure (15). The Lambert problem solution needed just a few steps for the planar halo orbit design (see Section 5.3) and halo-to-halo transfers (Section 5.4), less than 100 steps for the transfer between two specified positions (Section 5.1) and for the periodic orbit design (Section 5.2), and up to 2500 for the three-dimensional halo orbit design (Section 5.3) and LEO-to-halo transfer (Section 5.5). The calculations can take up to 3 min in the 2.5 MHz Pentium 4 in the last case (i.e. for the 2500-step procedure). Although both the number of steps and the calculation time can be reduced by means of the computer program optimization.

8. Comparison with Other Methods

A comparison of the proposed method with two other methods of the two-point boundary value problem solution was done. Those methods are the finite difference method with deferred corrections and multiple shooting method which are realized in the standard Fortran routines DBVPFD and DBVPMS available in the Microsoft IMSL Library. The comparison gave the following results:

(1) Both of the standard routines need a first guess with the same transfer time as the sought-for orbit. The first guess for the DBVPMS routine (multiple shooting method) should have the same terminal positions as the sought-for transfer. It is not necessary for the suggested method, i.e. it operates with first guesses (the reference orbits) which have terminal positions and transfer time substantially different from the given ones for the sought-for transfer.
(2) The standard routines do not provide a reliable convergence to the desired transfer type even with a good first step. For example, reference orbits from Table I were used as first guesses for the DBVPFD routine, but this routine converged to a completely wrong transfer type. Contrary to that, the suggested method provides convergence to a desired transfer type.
(3) Both standard routines work much faster than the proposed method.

Thus, the advantages of the method proposed in this paper are:

– the user does not need to prepare a special first guess for each transfer orbit, it is sufficient just to point out the transfer type;
– this method always converges to the orbit of a specified type if this orbit exists.
– at last, the proposed method is simpler than the other ones.

Disadvantages of this method are its slowness comparing with the other methods and necessity of an intermediate transfer orbit in some cases.

9. Conclusions

The method suggested in the paper gives the Lambert problem solution in the extended Earth vicinity including the collinear L_1 and L_2 Lagrange points. The Hill model of motion was used; however, this method can be easily adapted to the general equations of the restricted three-body problem an applied to transfers in any body vicinity. Moreover, this method can be applied to any model of motion where a set of typical reference orbits can be formed. The suggested method has proven its effectiveness in many practically important cases, such as planar halo orbit design, halo-to-halo transfer etc. Although in some other cases (three-dimensional halo orbit design etc.) an obtaining of an intermediate transfer orbit and/or a few thousand steps are needed for the sought-for transfer determination. This problem can be solved by means of use of more effective mathematical procedure and computer programs.

The proposed method is slower than other ones, but contrary to them does not need a special first guess for each sought-for transfer and always provide convergence to the transfer of a specified type.

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