CIRCULAR RESTRICTED PROBLEM OF THREE BODIES (CR3BP)

Motion of a 3^{rd} body that has negligible mass with respect to two massive bodies, termed the PRIMARIES, i.e.

$$m := m_3 \ll m_2 < m_1$$

In other words, the third body does not affect the remaining two bodies 1 and 2, which are assumed to describe circular orbits around their mass center

The angular velocity of the two primaries is

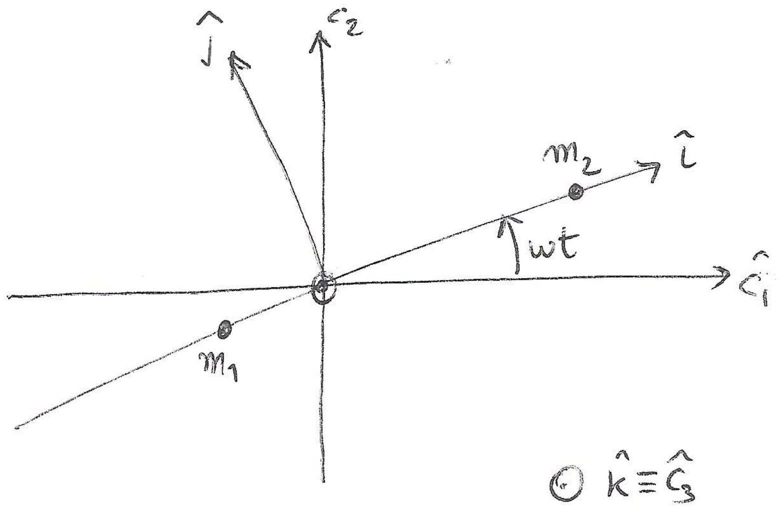
$$\omega = \sqrt{\frac{GM}{R^3}} \quad \text{where} \quad R = \text{their constant distance}$$

- Frames

Synodic REFERENCE FRAME

$\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ rotates together with the two primaries with $\hat{k} \uparrow\uparrow \underline{H}$

(angular momentum $\underline{H}$)



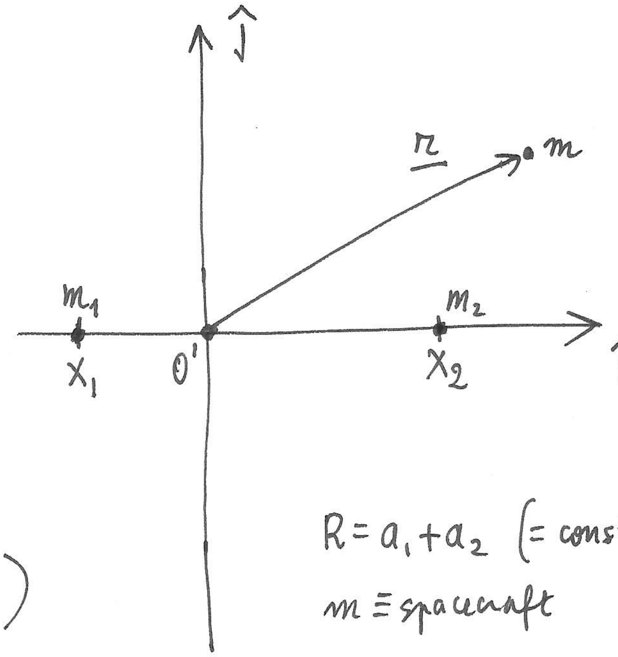
$\{\hat{c}_1, \hat{c}_2, \hat{c}_3\}$ inertial axes

$\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ synodic axes

At $t=0$ they coincide

$$\begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}$$

The position of m_1 and m_2 with respect to O' (center of mass) is given by



$R = a_1 + a_2$ (= const)

$m \equiv$ spacecraft

$$\begin{cases} x_2 - x_1 = R \\ m x_1 + m_2 x_2 = 0 \end{cases} \longrightarrow$$

$$\rightarrow \begin{cases} x_1 = -\dfrac{m_2}{M} R \\ x_2 = \dfrac{m_1}{M} R \end{cases}$$

Letting $\mu = \dfrac{m_2}{m_1 + m_2}$ (mass parameter)

and $\begin{cases} DU = R & \text{distance unit} \\ TU = \omega^{-1} & \text{time unit} \end{cases} \Rightarrow G(m_1 + m_2) = 1 \dfrac{DU^3}{TU^2}$

one obtains

$$\begin{cases} x_1 = -\mu R = -\mu\, DU \\ x_2 = (1-\mu) R = (1-\mu)\, DU \end{cases} \qquad \begin{cases} G m_2 = \mu \dfrac{DU^3}{TU^2} \\ G m_1 = (1-\mu) \dfrac{DU^3}{TU^2} \end{cases}$$

One can choose 1 and 2 such that $m_1 > m_2 \Rightarrow \mu < \frac{1}{2}$

- <u>Equations of motion</u>

$$\frac{d^2 \underline{r}}{dt^2} = -\frac{(1-\mu)(\underline{r} - \underline{R}_1)}{|\underline{r} - \underline{R}_1|^3} - \frac{\mu(\underline{r} - \underline{R}_2)}{|\underline{r} - \underline{R}_2|^3}$$

omitting DU and TU henceforth

The position vector can be written in terms of its components in the rotating frame $(\hat{\imath}, \hat{\jmath}, \hat{k})$,

$$\underline{r} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Because $\quad \underline{\omega}\times\hat{i}=\hat{j}\omega, \quad \underline{\omega}\times\hat{j}=-\hat{i}\omega, \quad \underline{\omega}\times\hat{k}=\underline{0}$

$(\underline{\omega}=\omega\hat{k})$ the left hand side of the previous vector equation becomes

$$\frac{d^2\underline{r}}{dt^2}=\frac{d}{dt}\left[\dot{x}\hat{i}+\dot{y}\hat{j}+\dot{z}\hat{k}+x(\underline{\omega}\times\hat{i})+y(\underline{\omega}\times\hat{j})+z(\underline{\omega}\times\hat{k})\right]=$$

$$=\frac{d}{dt}\left[(\dot{x}-\omega y)\hat{i}+(\dot{y}+\omega x)\hat{j}+\dot{z}\right]=$$

$$=(\ddot{x}-\omega\dot{y})\hat{i}+(\ddot{y}+\omega\dot{x})\hat{j}+\ddot{z}+\omega(\dot{x}-\omega y)\hat{j}-\omega(\dot{y}+\omega x)\hat{i}=$$

$$=(\ddot{x}-2\omega\dot{y}-\omega^2x)\hat{i}+(\ddot{y}+2\omega\dot{x}-\omega^2y)\hat{j}+\ddot{z}$$

Therefore, along the three rotating axes

$$\hat{i})\quad \ddot{x}-2\omega\dot{y}-\omega^2x=-\frac{(1-\mu)(x+\mu)}{\left[(x+\mu)^2+y^2+z^2\right]^{3/2}}-\frac{\mu(x+\mu-1)}{\left[(x+\mu-1)^2+y^2+z^2\right]^{3/2}}$$

$$\hat{j})\quad \ddot{y}+2\omega\dot{x}-\omega^2y=-\frac{(1-\mu)y}{\left[(x+\mu)^2+y^2+z^2\right]^{3/2}}-\frac{\mu y}{\left[(x+\mu-1)^2+y^2+z^2\right]^{3/2}}$$

$$\hat{k})\quad \ddot{z}=-\frac{(1-\mu)z}{\left[(x+\mu)^2+y^2+z^2\right]^{3/2}}-\frac{\mu z}{\left[(x+\mu-1)^2+y^2+z^2\right]^{3/2}}$$

In the previous expressions the denominators contain the instantaneous distance from mass 1 and mass 2.

The physical unit of $(1-\mu)$ and μ in numerators is $\frac{DU^3}{TU^2}$

The physical unit of $(x+\mu)$ and $(x+\mu-1)$ in denominators is DU as well as in numerators

- Jacobi integral

Letting
$$\Omega = \frac{\omega^2}{2}(x^2+y^2) + \frac{1-\mu}{[(x+\mu)^2+y^2+z^2]^{1/2}} + \frac{\mu}{[(x+\mu-1)^2+y^2+z^2]^{1/2}}$$

(Ω is also termed "potential function")

the equations of motion can be rewritten as

$$\begin{cases} \ddot{x} - 2\omega\dot{y} = \dfrac{\partial\Omega}{\partial x} & (1) \\ \ddot{y} + 2\omega\dot{x} = \dfrac{\partial\Omega}{\partial y} & (2) \\ \ddot{z} = \dfrac{\partial\Omega}{\partial z} & (3) \end{cases}$$

(1) is multiplied by $\dot{x}$, (2) by $\dot{y}$, (3) by $\dot{z}$, then one adds and obtains

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \dot{x}\frac{\partial\Omega}{\partial x} + \dot{y}\frac{\partial\Omega}{\partial y} + \dot{z}\frac{\partial\Omega}{\partial z}$$

$$\rightarrow \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = \frac{d\Omega}{dt} \longrightarrow \frac{d}{dt}\left[\Omega - \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)\right] = 0$$

This means that the quantity

$$C := 2\Omega - \left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) \quad \text{is CONSTANT}$$

This is referred to as the JACOBI INTEGRAL.

As $C \propto -\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$ it is intuitive that C is related to energy. In fact, C decreases as the energy increases; of course, for specified initial conditions, the value of C does not change in time, and, due to this, C is an INTEGRAL of motion in the CR3BP

- Zero velocity surfaces and curves

Zero velocity surfaces (in 3-d) and curves (in 2-d) are the loci where $\dot{x} = \dot{y} = \dot{z} = 0$.

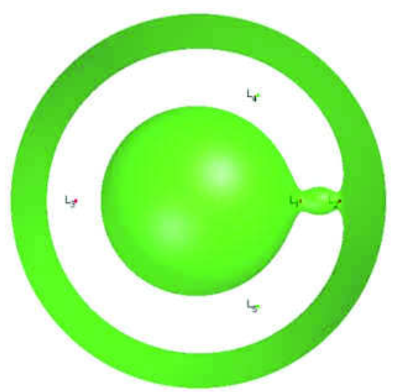
These surfaces (and curves) constrain the region where the spacecraft motion can take place. In fact

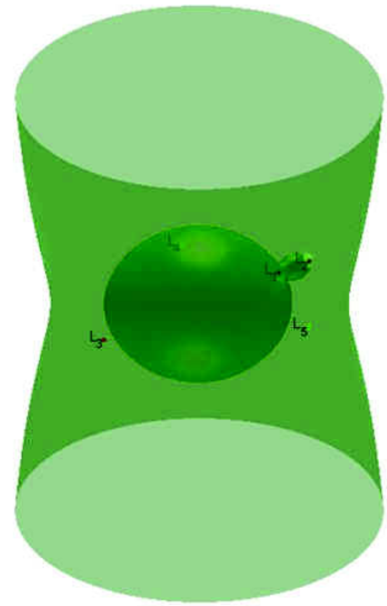
$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega(x,y,z) - C \geqslant 0$$

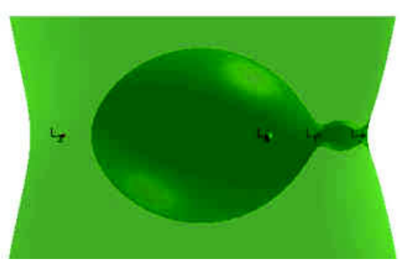
Because Ω is a function of the space coordinates only (x,y,z), the inequality at the right-hand-side defines the region of allowed motion, which is termed also HILL'S REGION.

Looking at
$$2\Omega = \omega^2(x^2+y^2) + \frac{2(1-\mu)}{\left[(x+\mu)^2+y^2+z^2\right]^{1/2}} + \frac{2\mu}{\left[(x+\mu-1)^2+y^2+z^2\right]^{1/2}}$$

(i) if x, y are large
→ first term prevails, and is associated with a cylinder with axis z







(ii) if $(x+\mu)^2+y^2+z^2$ is small

OR

if $(x+\mu-1)^2+y^2+z^2$ is small

→ either 2^{nd} OR 3^{rd} term prevails, with two associated surfaces:

⇒ 2^{nd} term is a near-sphere about primary 1
3^{rd} term is a near-sphere about primary 2

In the previous figure, the zero velocity surfaces are illustrated for a particular value of C. Motion is allowed

(i) In the proximity of primary 1, i.e. inside the near-sphere that surrounds primary 1;

(ii) In the proximity of primary 2, i.e. inside the near-sphere that surrounds primary 2;

(iii) Outside the near-cylindrical surface with axis z.

Zero velocity curves are the sections of zero velocity surfaces with the (x,y)-plane, and will be described in greater detail in the following.

- <u>Libration points</u>

Libration (or Lagrange) points are equilibrium points in the synodic frame, where the 3rd body (i.e. the spacecraft) remains indefinitely, provided that it is located at these points with $\dot{x} = \dot{y} = \dot{z} = 0$ (zero velocity in (x,y,z))

These points are sought in the (x,y)-plane, i.e.

$z = 0$ and $\dot{z} = 0$ hold in the following.

Equilibrium means that $\dot{x} = 0$ and $\dot{y} = 0$

and also $\ddot{x} = 0$ and $\ddot{y} = 0$

at libration points

These conditions yield

$$(A) \quad \ddot{x} = \omega^2 x - \frac{(1-\mu)(x+\mu)}{\left[(x+\mu)^2+y^2\right]^{3/2}} - \frac{\mu\,(x+\mu-1)}{\left[(x+\mu-1)^2+y^2\right]^{3/2}} = 0$$

$$(B) \quad \ddot{y} = \omega^2 y - \frac{(1-\mu)\,y}{\left[(x+\mu)^2+y^2\right]^{3/2}} - \frac{\mu\, y}{\left[(x+\mu-1)^2+y^2\right]^{3/2}} = 0$$

(1) COLLINEAR LIBRATION POINTS

Equilibrium points are sought along the x-axis (i.e. $y=0$). Only (A) is needed, because (B) is satisfied if $y=0$; (A) becomes

$$\omega^2 x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu\,(x+\mu-1)}{|x+\mu-1|^3} = 0$$

Three cases occur

(a) $x+\mu<0 \longrightarrow x<-\mu$

(b) $x+\mu>0$ and $x+\mu-1<0 \longrightarrow -\mu<x<1-\mu$

(c) $x+\mu-1>0 \longrightarrow x>1-\mu$

In each case a quintic equation can be found (not reported for the sake of brevity): the only real admissible solution in the respective range (a, b, or c) provides the x-coordinate of the equilibrium point, in the previous 3 cases:

(a) Left exterior collinear libration point, denoted with L_3

(b) Interior collinear libration point, denoted with L_1

(c) Right exterior collinear libration point, denoted with L_2

(2) TRIANGULAR (OR EQUILATERAL) LIBRATION POINTS

If $y \neq 0$, then both (A) and (B) must vanish.

Letting
$$\begin{cases} r_1 := \sqrt{(x+\mu)^2 + y^2} & \text{distance from primary 1} \\ r_2 := \sqrt{(x+\mu-1)^2 + y^2} & \text{distance from primary 2} \end{cases}$$

$$\text{(A)} \quad \omega^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} = 0$$

where $\omega = 1\ TU^{-1}$
(due to definition of TU)

$$\text{(B)} \quad \omega^2 y - \frac{\mu y}{r_2^3} - \frac{(1-\mu) y}{r_1^3} = 0$$

Using $\omega = 1\ TU^{-1}$ (B) becomes

$$y\left[r_1^3 r_2^3 - \mu r_1^3 - r_2^3 + \mu r_2^3 \right] = 0$$

the term in parentheses vanishes if $r_1 = r_2 = 1$ (DU) regardless of μ. Using $r_1 = r_2 = 1$ (DU) in (A), one gets

$$x - (1-\mu)(x+\mu) - \mu(x+\mu-1) = 0$$

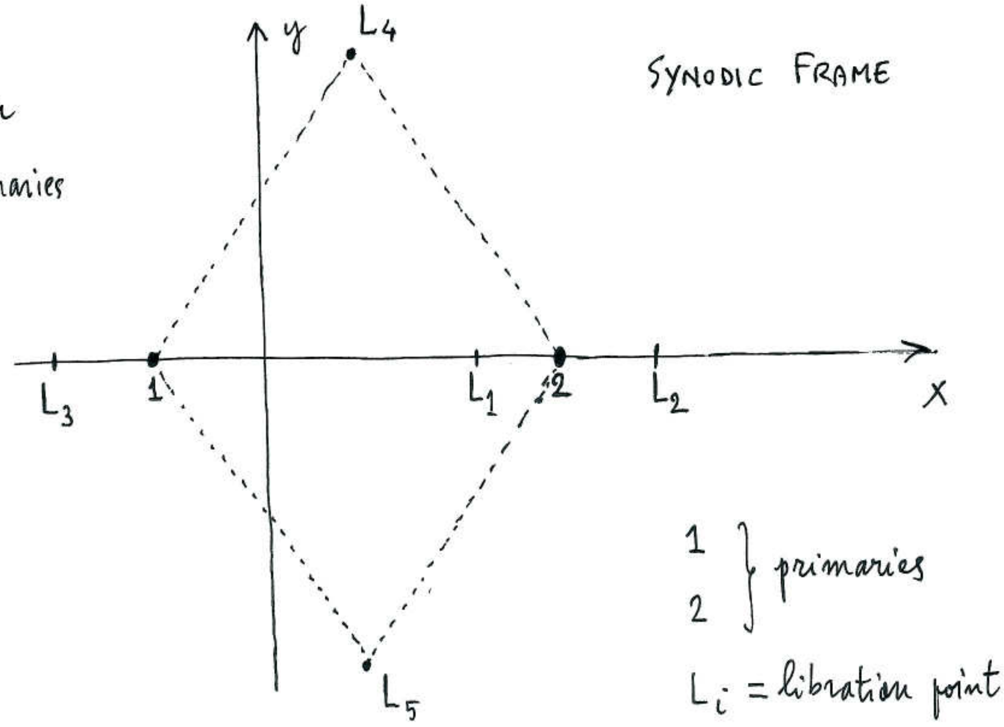
Therefore also (A) is fulfilled, and this means that in the (x, y) the points $r_1 = r_2 = 1$ DU are equilibrium points. Two such points exist, located above and below the x-axis, and termed equilateral or triangular points because each triangular libration point forms an equilateral triangle with the two primaries. It is common to denote with

L_4 the triangular point above the x-axis

L_5 the triangular point under the x-axis

Location of libration point for two primaries (when $\mu < 0.5$)

L_4 and L_5 form two equilateral triangles



- Function Ω at L_i

The function Ω is stationary at L_i; in fact

$$\frac{\partial \Omega}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial y} = 0 \quad \text{at } L_i$$

However, Ω can have a minimum or maximum at L_i (or can be simply stationary). In order to find out if Ω has min or max at L_i, these are the steps:

(a) Calculate symbolically $\Omega_{xx}, \Omega_{yy}, \Omega_{xy}$ and evaluate these at L_i

(b) Calculate

$$\det \begin{bmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{yx} & \Omega_{yy} \end{bmatrix} =: H$$

(c) Four cases can occur:

(i) $\Omega_{xx}, \Omega_{yy} > 0$ and $H > 0$ $\rightarrow$ Ω has min value at L_i

(ii) $\Omega_{xx}, \Omega_{yy} < 0$ and $H > 0$ $\rightarrow$ Ω has max value at L_i

(iii) $H = 0$ $\rightarrow$ further derivatives needed

(iv) $H < 0$ $\rightarrow$ not max nor min

The results of the study of Ω at L_i are

(a) At L_1, L_2, L_3 (collinear points) Ω has not a max or min value, i.e. it is simply stationary

(b) At L_4, L_5 Ω has the minimum value Ω_{min}

$$\Omega_{min} = \Omega(L_4, L_5) = \frac{3}{2} - \frac{\mu}{2}(1-\mu)$$

As Ω has the minimum value at L_4 and L_5, the inequality $2\Omega - C \geqslant 0$ (HILL'S REGION of allowed motion) is satisfied in the entire space if

$$C < 2\Omega_{min} = 3 - \mu(1-\mu)$$

In other words, if the initial conditions for the spacecraft are such that $C < 2\Omega_{min}$, then it can travel in the entire space, because no zero velocity surface exists.

- Special values of C

If the spacecraft is placed at L_i with zero velocity, then

$$C_i = 2\Omega(L_i)$$

Because the velocity is zero at L_i, the libration point belongs to the zero velocity surface (and curve, in the (x, y)-plane)

From the previous discussion

$$C_{5,4} = 2\Omega(L_4, L_5) \leqslant 2\Omega \quad \text{at all points}$$

Hence, the motion can take place in the entire space if

$$C < C_{4,5}$$

This means also that the zero velocity surfaces (and curves) disappear at $C = C_{4,5}$.

The geometry of the zero velocity curves vary as C varies, i.e. when the spacecraft initial conditions change

Let C_i = value of C when the zero velocity curve contains L_i.

several cases can occur:

(1) $C < C_{4,5}$: motion allowed in the entire space

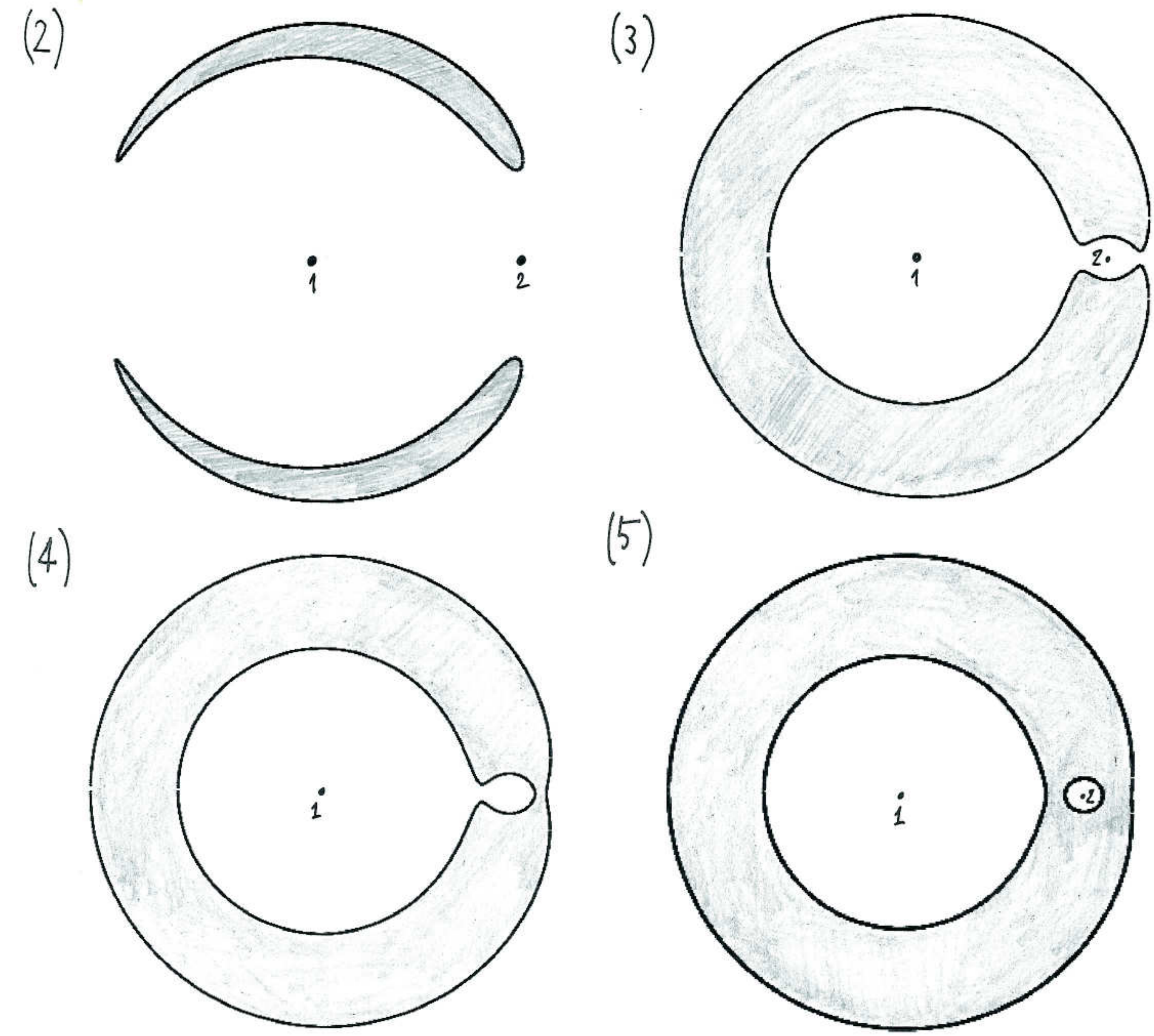
(2) $C_{4,5} < C < C_3$: motion forbidden only in the proximity of L_4 and L_5

(3) $C_3 < C < C_2$: interior and exterior transfers from primary 1 to 2 are feasible (interior transfers pass close to L_1, exterior transfers pass close to L_2)

(4) $C_2 < C < C_1$: interior transfers from 1 to 2 are feasible; motion outside the greater curve is feasible no exterior transfer at L_2 is feasible

(5) $C_1 < C$: no interior or exterior transfer is feasible; the spacecraft remains confined either (i) in the proximity of primary 1, or (ii) in the proximity of primary 2, or (iii) outside the greater zero velocity curve

It is apparent that $C_1, C_2, C_3, C_{4,5}$ represent very meaningful values for understanding feasibility of a trajectory.



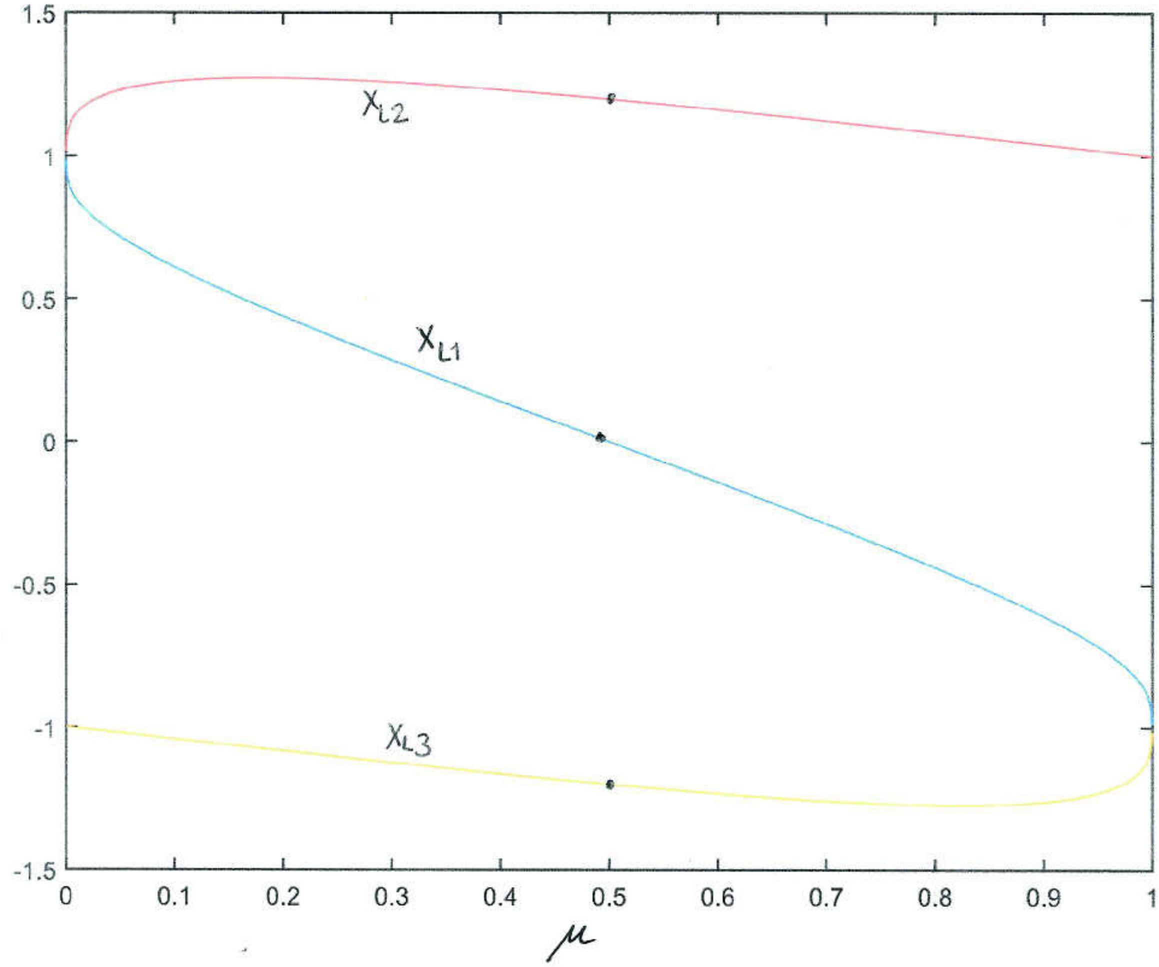
In these figures the forbidden region is shaded (grey).

From (2) to (5) the energy decreases, and this means that C increases

The next figure portrays the coordinates of L_i as μ varies.

Moreover, for the Earth-Moon system $\left(\mu = \frac{1}{82.27}\right)$ the characteristic values of C_1 through C_4 are reported.

Position of the libration points (collinear) as a function of μ.



$\mu = \frac{1}{82.27}$ for the Moon

$C_1 = 3.18838273477815 \ \frac{DU^2}{TU^2}$ when $C_1 = 2\Omega(L_1)$

$C_2 = 3.17219608074121 \ \frac{DU^2}{TU^2}$ $C_2 = 2\Omega(L_2)$

$C_3 = 3.01215166144792 \ \frac{DU^2}{TU^2}$ $C_3 = 2\Omega(L_3)$

$C_4 = 2.98799264736 92 \ \frac{DU^2}{TU^2}$ $C_4 = 2\Omega(L_4)$

- Stability of libration points

Let (x_0, y_0) be the coordinates of a libration point;

(ξ, η) are small displacements, i.e.

$$\begin{cases} x = x_0 + \xi \\ y = y_0 + \eta \end{cases} \qquad \underline{r_0} \leftrightarrow (x_0, y_0)$$

Using $\omega = 1\ TU^{-1}$, the equations of motion are expanded to first order, to yield

$$\begin{cases} \ddot{x} - 2\dot{y} = \Omega_x \longrightarrow \ddot{x}_0 + \ddot{\xi} - 2(\dot{y}_0 + \dot{\eta}) = \Omega_x|_{\underline{r_0}} + \Omega_{xx}|_{\underline{r_0}}\xi + \Omega_{xy}|_{\underline{r_0}}\eta \\ \ddot{y} + 2\dot{x} = \Omega_y \longrightarrow \ddot{y}_0 + \ddot{\eta} + 2(\dot{x}_0 + \dot{\xi}) = \Omega_y|_{\underline{r_0}} + \Omega_{yx}|_{\underline{r_0}}\xi + \Omega_{yy}|_{\underline{r_0}}\eta \end{cases}$$

The partial derivatives of Ω are evaluated at $\underline{r_0}$, here and henceforth, but $|_{\underline{r_0}}$ is omitted, for the sake of brevity.

Because $\ddot{x}_0 = \ddot{y}_0 = 0$ and $\Omega_x = \Omega_y = 0$ (at $\underline{r_0}$), one obtains the linear, time-independent differential system

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{xx}\xi + \Omega_{xy}\eta$$

$$\ddot{\eta} + 2\dot{\xi} = \Omega_{xy}\xi + \Omega_{yy}\eta$$

Letting $\underline{z} = [\xi \ \ \dot{\xi} \ \ \eta \ \ \dot{\eta}]^T$, the previous system can be rewritten as

$$\underline{\dot{z}} = A\underline{z} \quad , \text{ where } \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \Omega_{xx} & 0 & \Omega_{xy} & 2 \\ 0 & 0 & 0 & 1 \\ \Omega_{yx} & -2 & \Omega_{yy} & 0 \end{bmatrix}$$

Stability of this linear differential system depends on the eigenvalues of A, given by solving

$$\det(A - \lambda I) = 0$$

$$\rightarrow \quad \lambda^4 - \lambda^2(\Omega_{xx} + \Omega_{yy} - 4) + \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = 0$$

where $\{\Omega_{xx}, \Omega_{yy}, \Omega_{xy}\}$ are evaluated at L_i

(1) COLLINEAR LIBRATION POINTS

After several calculations one can obtain

$$\begin{cases} \Omega_{xx} + \Omega_{yy} - 4 = k_1 - 2 \\ \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = (1+2k_1)(1-k_1) \end{cases}$$

where k_1 is a constant, which can be proven to be $k_1 > 1$ for all collinear points.

Because $k_1 > 1$, $\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 < 0$ and therefore a solution for λ^2 exists such that $\lambda^2 > 0$

But $\lambda^2 > 0 \Rightarrow$ a positive and a negative real eigenvalue

$\Rightarrow$ λ real and positive implies INSTABILITY

i.e. collinear libration points are UNSTABLE

This means that if the spacecraft is placed in the proximity of L_i $(i = 1,2,3)$, with small ξ and η, then its dynamics is divergent (UNSTABLE equilibrium)

Linear periodic solutions are found only if the spacecraft is placed "along" the stable eigenvector.

(2) TRIANGULAR LIBRATION POINTS

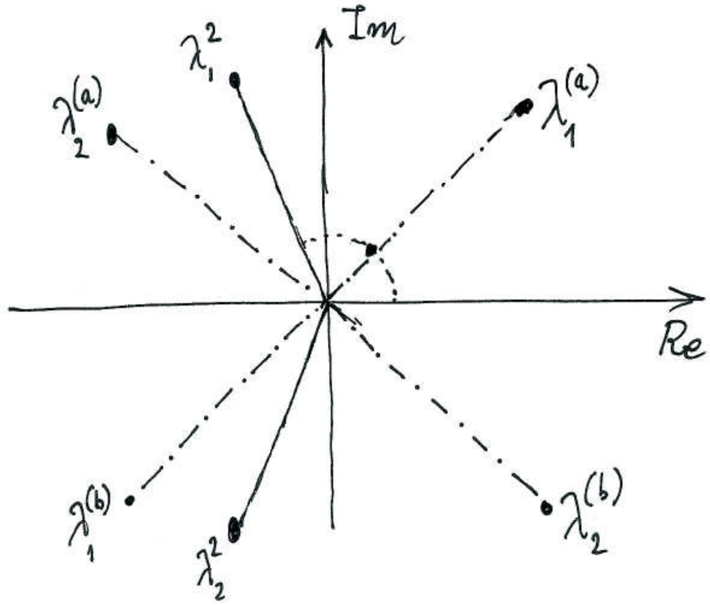
After several calculations one obtains

$$\Omega_{xx} = \frac{3}{4} \qquad \Omega_{yy} = \frac{9}{4} \qquad \Omega_{xy} = \pm \frac{3\sqrt{3}}{4}(1-2\mu)$$

$$\rightarrow \begin{cases} \Omega_{xx} + \Omega_{yy} - 4 = -1 < 0 & (a) \\ \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = \frac{27}{4}\mu(1-\mu) > 0 & (b) \end{cases}$$

Condition (a) implies that λ^2 has negative real part

However any pair of complex conjugate $\lambda^2_{1,2}$ with negative real part admits $\{\lambda_1^{(a)}, \lambda_1^{(b)}, \lambda_2^{(a)}, \lambda_2^{(b)}\}$ with positive real part (2 of them, see figure)



The only way for $\{\lambda_1^{(a)}, \lambda_2^{(b)}, \lambda_2^{(a)}, \lambda_2^{(b)}\}$ with real part ≤ 0 is having $\lambda^2_{1,2}$ real and negative. In addition to (a) and (b) the following condition must hold

$$\left(\Omega_{xx} + \Omega_{yy} - 4\right)^2 - 4\left(\Omega_{xx}\Omega_{yy} - \Omega_{xy}\right) \geq 0$$

$$\rightarrow \quad 27\mu^2 - 27\mu + 1 \geq 0$$

This inequality has the following solution

$$0 \leq \mu \leq \underbrace{\frac{1}{2} - \frac{\sqrt{69}}{18}}_{\sim 0.0385} \quad \text{OR} \quad \underbrace{\frac{1}{2} + \frac{\sqrt{69}}{18}}_{\sim 0.9615} \leq \mu \leq 1$$

(a) If the two primaries have μ that satisfies one of these two inequalities, the equilibrium around L_4, L_5 is NEUTRALLY STABLE (according to the linear analysis)

(b) Instead, if $\frac{1}{2} - \frac{\sqrt{69}}{18} < \mu < \frac{1}{2} + \frac{\sqrt{69}}{18}$

then the equilibrium about L_4, L_5 is UNSTABLE

It is easy to check that for the Earth-Moon system condition (a) is satisfied

In summary

(a) Around L_1, L_2, L_3 : UNSTABLE equilibrium

(b) Around L_4, L_5: equilibrium is

> NEUTRALLY STABLE if $0 \leq \mu \leq \frac{1}{2} - \frac{\sqrt{69}}{18}$ or $\frac{1}{2} + \frac{\sqrt{69}}{18} \leq \mu \leq 1$

> UNSTABLE if $\frac{1}{2} - \frac{\sqrt{69}}{18} \leq \mu \leq \frac{1}{2} + \frac{\sqrt{69}}{18}$