Lab 3 : Convex Optimisation

Gradient Descent

Write the code following the instructions to obtain the desired results

Import all the required libraries

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl
from mpl_toolkits.mplot3d import Axes3D
%matplotlib inline
```

Find the value of x at which $f(x)$ is minimum :

1. Find x analytically
2. Write the update equation of gradient descent
3. Find x using gradient descent method

Example 1 : $f(x) = x^2 + x + 2$

Analytical :

$$\frac{d}{dx}f(x) = 2x + 1 = 0$$

$$\frac{d^2}{dx^2}f(x) = 2 \ (Minima)$$

$$x = -\frac{1}{2} \ (analytical \ solution)$$

Gradient Descent Update equation :

$$x_{init} = 4$$

$$x_{updt} = x_{old} - \lambda(\frac{d}{dx}f(x)|x = x_{old})$$

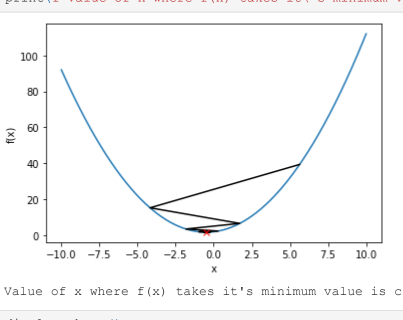
$$x_{updt} = x_{old} - \lambda(2x_{old} + 1)$$

Gradient Descent Method :

Follow the below steps and write your code in the block below

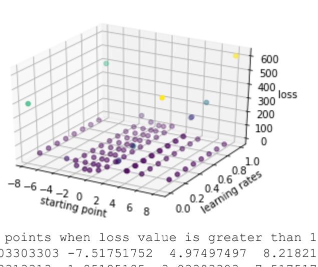
1. Generate x, 1000 data points from -10 to 10
2. Generate and Plot the function $f(x) = x^2 + x + 2$
3. Initialize the starting point (x_{init}) and learning rate (λ)
4. Use Gradient descent algorithm to compute value of x at which the function $f(x)$ is minimum
5. Also vary the learning rate and initialisation point and plot your observations

In [127...
```
# Generating 1000 points from -10 to 10
x = np.linspace(-10,10,1000)
# Genrating the plot for f(x)
y = x**2 + x + 2
plt.plot(x,y)
plt.xlabel('x')
plt.ylabel('f(x)')
# Initializing the starting point and learning rate
x_init = np.random.choice(x)
lr = 0.8
# Gradient descent algorithm
iter = 1800
def loss(x):
    return (grad(x)**2)

def grad(x):
    return 2*x + 1

def double_diff(x):
    return 2        # stationary point is minima

def grad_desc(x_init,lr,display = False,iter = 1800):
    x = x_init
    num = 0
    traversed_data = {}
    if display:
        while num < iter:
            x_updt =  x - (lr * grad(x))
            x = x_updt
            num += 1
        return loss(x)
    else:
        while num < iter:
            traversed_data[x] = x**2 + x + 2
            x_updt =  x - (lr * grad(x))
            x = x_updt
            num += 1
        return traversed_data

def display_obser():
    ls_lr = np.linspace(0,1,10)
    ls_x = np.random.choice(x,size = ls_lr.shape)
    ls_x = np.array(ls_x)
    ls_x,ls_lr = np.meshgrid(ls_x,ls_lr)
    loss_arr = grad_desc(ls_x,ls_lr,True,100)
    fig = plt.figure()
    ax = plt.axes(projection = '3d')
    ax.scatter3D(ls_x,ls_lr,loss_arr,cmap = 'viridis',c = loss_arr)
    ax.set_xlabel('starting point',linespacing = 3.3)
    ax.set_ylabel('learning rates',linespacing = 3.2)
    ax.set_zlabel('loss', linespacing = 3.0)
    ax.dist = 11
    plt.show()
    print(f'Start points when loss value is greater than 10\n {ls_x[loss_arr > 10]}\n')
    print(f'learning rates when loss value is greater than 10\n {ls_lr[loss_arr > 10]}')
    return

traversed_data = grad_desc(x_init,lr)
key,value = list(traversed_data.keys()),list(traversed_data.values())
plt.plot(key,value,'black')
plt.plot(key[-1],value[-1],'rx')
plt.show()
print(f'Value of x where f(x) takes it\'s minimum value is close to {key[-1]}')
```




```
Value of x where f(x) takes it's minimum value is close to -0.5
```

In [128...
```
display_obser()
```




```
Start points when loss value is greater than 10
 [-3.03303303 -7.51751752  4.97497497  8.21821822  3.25325325  0.81081081
  3.13313313 -1.85185185 -3.03303303 -7.51751752  4.97497497  8.21821822
  3.25325325  0.81081081  3.13313313 -1.85185185]

learning rates when loss value is greater than 10
 [0. 0. 0. 0. 0. 0. 0. 0. 1. 1. 1. 1. 1. 1. 1. 1.]
```

Example 2 : $f(x) = xsinx$

Analytical : Find solution analytically

Gradient Descent Update equation : Write Gradient descent update equations

Gradient Descent Method :

Follow the below steps and write your code in the block below

1. Generate x, 1000 data points from -10 to 10
2. Generate and Plot the function $f(x) = x^2 + x + 2$
3. Initialize the starting point (x_{init}) and learning rate (λ)
4. Use Gradient descent algorithm to compute value of x at which the function $f(x)$ is minimum
5. Also vary the learning rate and initialisation point and plot your observations

Analytical Solution:

$$f(x) = xsinx$$

$$\frac{d}{dx}f(x) = xcosx + sinx = 0$$

$$\frac{d^2}{dx^2}f(x) = cosx - xsinx + cosx > 0 \ (Minima)$$

$$f(x) = xsinx = f(-x) \tag{1}$$

$$f(x) \ is \ symmetric \ and \ has \ infinite \ minima \ and \ For \ x \ in \ the \ range(-10,10) \ we \ have \ 3 \ minimas \ at,$$
$$x = -4.91318 \ , 0 \ , 4.91318$$

Gradient Descent Update Equations:

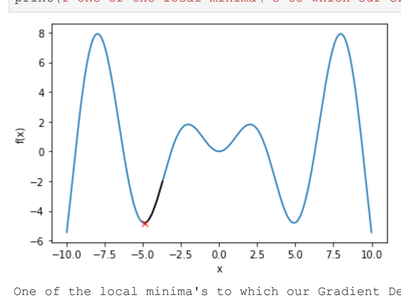
$$x_{init} = 2$$

$$x_{updt} = x_{old} - \lambda\left(\frac{d}{dx}f(x)|x = x_{old}\right)$$

$$x_{updt} = x_{old} - \lambda\left(xcosx + sinx\right)$$

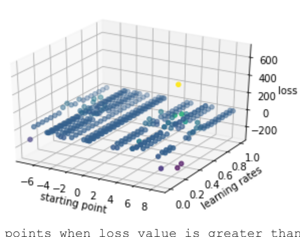
Gradient Descent Method :

In [151...
```
# Generating 1000 points from -10 to 10
x = np.linspace(-10,10,1000)
# Genrating the plot for f(x)
y = x * np.sin(x)
plt.plot(x,y)
plt.xlabel('x')
plt.ylabel('f(x)')
# Initializing the starting point and learning rate
x_init = np.random.choice(x)
lr = 0.1
# Gradient descent algorithm
iter = 1800
def loss(x):
    return (grad(x) ** 2) * double_diff(x) # double_diff(x) is used to know whether the stationary point in min

def grad(x):
    return x * np.cos(x) + np.sin(x)

def double_diff(x):
    return 2 * np.cos(x) - x * np.sin(x) # stationary point can be positive or negative

def grad_desc(x_init,lr,display = False,iter = 100):
    x = x_init
    num = 0
    traversed_data = {}
    if display:
        while num < iter:
            x_updt =  x - (lr * grad(x))
            x = x_updt
            num += 1
        return loss(x)
    else:
        while num < iter:
            traversed_data[x] = x * np.sin(x)
            x_updt =  x - (lr * grad(x))
            x = x_updt
            num += 1
        return traversed_data

def display_obser():
    ls_lr = np.linspace(0,1,20)
    ls_x = np.random.choice(x,size = ls_lr.shape)
    ls_x = np.array(ls_x)
    ls_x,ls_lr = np.meshgrid(ls_x,ls_lr)
    loss_arr = grad_desc(ls_x,ls_lr,True,200)
    fig = plt.figure()
    ax = plt.axes(projection = '3d')
    ax.scatter3D(ls_x,ls_lr,loss_arr,cmap = 'viridis',c = loss_arr)
    ax.set_xlabel('starting point',linespacing = 3.3)
    ax.set_ylabel('learning rates',linespacing = 3.2)
    ax.set_zlabel('loss', linespacing = 3.0)
    ax.dist = 11
    plt.show()
    print(f'Start points when loss value is greater than 200\n {ls_x[loss_arr > 200]}\n')
    print(f'learning rates when loss value is greater than 200\n {ls_lr[loss_arr > 200]}')
    return

traversed_data = grad_desc(x_init,lr)
key,value = list(traversed_data.keys()),list(traversed_data.values())
plt.plot(key,value,'black')
plt.plot(key[-1],value[-1],'rx')
plt.show()
print(f'One of the local minima\'s to which our Gradient Descent converges is close to {key[-1]}')
```




```
One of the local minima's to which our Gradient Descent converges is close to -4.913180439434884
```

In [152...
```
display_obser()
```




```
Start points when loss value is greater than 200
 [8.15815816 8.4984985  8.4984985  8.4984985 ]

learning rates when loss value is greater than 200
 [0.21052632 0.21052632 0.26315789 0.31578947]
```

Find the value of x and y at which $f(x, y)$ is minimum :

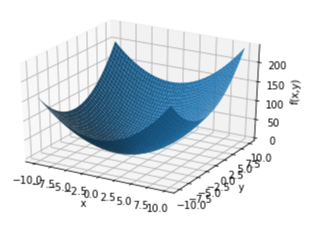
Example 1 : $f(x, y) = x^2 + y^2 + 2x + 2y$

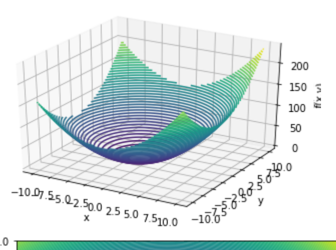
Gradient Descent Method :

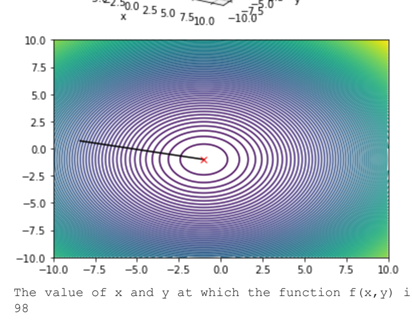
Follow the below steps and write your code in the block below

1. Generate x and y, 1000 data points from -10 to 10
2. Generate and Plot the function $f(x, y) = x^2 + y^2 + 2x + 2y$
3. Initialize the starting point (x_{init},y_{init}) and learning rate (λ)
4. Use Gradient descent algorithm to compute value of x and y at which the function $f(x, y)$ is minimum
5. Also vary the learning rate and initialisation point and plot your observations

In [182...
```
#1. Generate x,y 1000 data points from -10 to 10
x = np.linspace(-10,10,1000)
y = np.linspace(-10,10,1000)
#2. Plot the function f(x,y)
def f(x,y):
    return x**2 + y**2 + 2*x + 2*y
X,Y = np.meshgrid(x,y)
fig = plt.figure()
ax = plt.axes(projection='3d')
ax.plot_surface(X,Y,f(X,Y))
ax.set_xlabel('x',linespacing = 3.3)
ax.set_ylabel('y', linespacing = 3.3)
ax.set_zlabel('f(x,y)', linespacing = 3.0)
ax.dist = 11
plt.show()
#3. Initialize the starting point and learning rate
x_init = np.random.choice(x)
y_init = np.random.choice(y)
lr = 0.1
iter = 1800
#4. Using Gradient Descent
def loss(x,y):
    return (grad_x(x) ** 2) * (grad_y(y) ** 2)

def grad_x(x):
    return 2*x + 2

def grad_y(y):
    return 2*y + 2

def double_diff(x,y):
    return 2,2  # double derivatives w.r.t both x,y are positive. Stationary point can be either saddle point o

def grad_desc_3d(x_init,y_init,lr,display = False,iter = 1800):
    x = x_init
    y = y_init
    num = 0
    traversed_data_3d = {}
    if display:
        while num < iter:
            x_updt =  x - (lr * grad_x(x))
            y_updt =  y - (lr * grad_y(y))
            x = x_updt
            y = y_updt
            num += 1
        return loss(x,y)
    else:
        while num < iter:
            traversed_data_3d[x] = y
            x_updt =  x - (lr * grad_x(x))
            y_updt = y - (lr * grad_y(y))
            x = x_updt
            y = y_updt
            num += 1
        return traversed_data_3d

def display_obser():
    lr_arr = np.linspace(0,1,20)
    x_arr = np.random.choice(x,size = lr_arr.shape)
    y_arr = np.random.choice(y,size = lr_arr.shape)
    x_arr,y_arr,lr_arr = np.meshgrid(x_arr,y_arr,lr_arr)
    loss_arr = grad_desc_3d(x_arr,y_arr,lr_arr,True,50)
    fig = plt.figure()
    ax = plt.axes(projection = '3d')
    ax.scatter3D(x_arr,y_arr,loss_arr,cmap = 'viridis',c = lr_arr)
    ax.set_xlabel('x_coordinate',linespacing = 3.3)
    ax.set_ylabel('y_coordinate',linespacing = 3.2)
    ax.set_zlabel('loss', linespacing = 3.0)
    ax.dist = 11
    plt.show()
    points = np.column_stack((x_arr[loss_arr > 10],y_arr[loss_arr > 10]))
    print(f'Start points when loss value is greater than 10\n {points}\n')
    print(f'learning rates when loss value is greater than 10\n {lr_arr[loss_arr > 10]}')
    return

traversed_data_3d = grad_desc_3d(x_init,y_init,lr)
fig2 = plt.figure()
ax = plt.axes(projection='3d')
ax.contour3D(X,Y,f(X,Y),50,cmap = 'viridis')
ax.set_xlabel('x',linespacing = 3.3)
ax.set_ylabel('y', linespacing = 3.3)
ax.set_zlabel('f(x,y)', linespacing = 3.0)
ax.dist = 10
plt.show()

fig3 = plt.figure()
ax = plt.axes()
ax.contour(X,Y,f(X,Y),100,cmap = 'viridis')
x,y = zip(*traversed_data_3d.items())
ax.plot(x,y,'black')
ax.plot(x[-1],y[-1],'rx')
plt.show()

print(f'\nThe value of x and y at which the function f(x,y) is minimum is close to {x[-1]}, {y[-1]}')
```

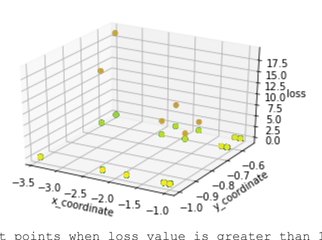






```
The value of x and y at which the function f(x,y) is minimum is close to -1.0000000000000004, -0.99999999999999
98
```

In [183...
```
display_obser()
```




```
Start points when loss value is greater than 10
 [[-3.42430719 -0.64123796]
 [-3.42430719 -0.64123796]
 [-3.42430719 -0.64123796]
 [-3.42430719 -0.64123796]
 [-3.42430719 -0.55154745]
 [-3.42430719 -0.55154745]
 [-3.42430719 -0.55154745]
 [-3.42430719 -0.55154745]]

learning rates when loss value is greater than 10
 [0. 1. 0. 1. 0. 1. 0. 1.]
```

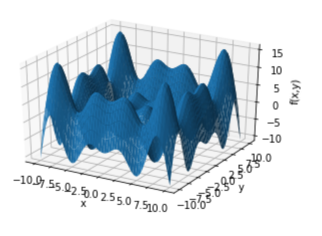
Example 2 : $f(x, y) = xsin(x) + ysin(y)$

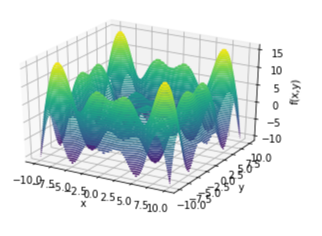
Gradient Descent Method :

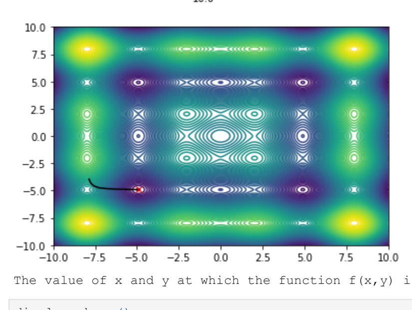
Follow the below steps and write your code in the block below

1. Generate x and y, 1000 data points from -10 to 10
2. Generate and Plot the function $f(x, y) = xsin(x) + ysin(y)$
3. Initialize the starting point (x_{init},y_{init}) and learning rate (λ)
4. Use Gradient descent algorithm to compute value of x and y at which the function $f(x, y)$ is minimum
5. Also vary the learning rate and initialisation point and plot your observations

In [218...
```
#1. Generate x,y 1000 data points from -10 to 10
x = np.linspace(-10,10,1000)
y = np.linspace(-10,10,1000)
#2. Plot the function f(x,y)
def f(x,y):
    return x * np.sin(x) + y * np.sin(y)
X,Y = np.meshgrid(x,y)
fig = plt.figure()
ax = plt.axes(projection='3d')
ax.plot_surface(X,Y,f(X,Y))
ax.set_xlabel('x',linespacing = 3.3)
ax.set_ylabel('y', linespacing = 3.3)
ax.set_zlabel('f(x,y)', linespacing = 3.0)
ax.dist = 11
plt.show()
#3. Initialize the starting point and learning rate
x_init = np.random.choice(x)
y_init = np.random.choice(y)
lr = 0.1
iter = 1800
#4. Using Gradient Descent
def loss(x,y):
    return grad_x(x) * grad_y(y) * is_max(x) * check_max(y)

def grad_x(x):
    return x * np.cos(x) + np.sin(x)

def grad_y(y):
    return y * np.cos(y) + np.sin(y)

def is_max(x):
    temp = 2 * np.cos(x) - x * np.sin(x)
    return 1 if np.all(temp > 0)  else -1

def check_max(y):
    temp = 2 * np.cos(y) - y * np.sin(y)
    return 1 if np.all(temp > 0) else -1

def grad_desc_3d(x_init,y_init,lr,display = False,iter = 1800):
    x = x_init
    y = y_init
    num = 0
    traversed_data_3d = {}
    if display == True:
        while num < iter:
            x_updt =  x - (lr * grad_x(x))
            y_updt = y - (lr * grad_y(y))
            x = x_updt
            y = y_updt
            num += 1
        return loss(x,y)
    else:
        while num < iter:
            traversed_data_3d[x] = y
            x_updt =  x - (lr * grad_x(x))
            y_updt = y - (lr * grad_y(y))
            x = x_updt
            y = y_updt
            num += 1
        return traversed_data_3d

def display_obser():
    lr_arr = np.linspace(0,1,10)
    x_arr = np.random.choice(x,size = lr_arr.shape)
    y_arr = np.random.choice(y,size = lr_arr.shape)
    x_arr,y_arr,lr_arr = np.meshgrid(x_arr,y_arr,lr_arr)
    loss_arr = grad_desc_3d(x_arr,y_arr,lr_arr,True,50)
    fig = plt.figure()
    ax = plt.axes(projection = '3d')
    ax.scatter3D(x_arr,y_arr,loss_arr,cmap = 'viridis',c = lr_arr)
    ax.set_xlabel('x_coordinate',linespacing = 3.3)
    ax.set_ylabel('y_coordinate',linespacing = 3.2)
    ax.set_zlabel('loss', linespacing = 3.0)
    ax.dist = 11
    plt.show()
    points = np.column_stack((x_arr[loss_arr > 10],y_arr[loss_arr > 10]))
    print(f'Start points when loss value is greater than 10\n {points}\n')
    print(f'learning rates when loss value is greater than 10\n {lr_arr[loss_arr > 10]}')
    return

traversed_data_3d = grad_desc_3d(x_init,y_init,lr)

fig2 = plt.figure()
ax = plt.axes(projection='3d')
ax.contour3D(X,Y,f(X,Y),50,cmap = 'viridis')
ax.set_xlabel('x',linespacing = 3.3)
ax.set_ylabel('y', linespacing = 3.3)
ax.set_zlabel('f(x,y)', linespacing = 3.0)
ax.dist = 11
plt.show()

fig3 = plt.figure()
ax = plt.axes()
ax.contour(X,Y,f(X,Y),100,cmap = 'viridis')
x,y = zip(*traversed_data_3d.items())
ax.plot(x,y,'black')
ax.plot(x[-1],y[-1],'rx')
plt.show()

print(f'\nThe value of x and y at which the function f(x,y) is minimum is close to {x[-1]}, {y[-1]}')
```

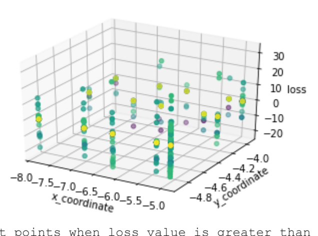






```
The value of x and y at which the function f(x,y) is minimum is close to -4.913180439434884, -4.913180439434884
```

In [219...
```
display_obser()
```




```
Start points when loss value is greater than 10
 [[-4.91318966 -4.91318044]
 [-6.18510529 -4.91318044]
 [-7.85785786 -4.91318044]
 [-4.91318046 -4.91318044]
 [-4.91322068 -4.91318044]
 [-6.82196723 -4.91318044]
 [-5.21871875 -4.91318044]
 [-4.91318046 -3.95395395]
 [-4.91322068 -3.95395395]
 [-6.82196723 -3.95395395]
 [-5.21871875 -3.95395395]
 [-4.91318046 -4.91291803]
 [-4.9131997  -4.91291803]
 [-6.18510529 -4.91291803]
 [-4.91318046 -4.2984928 ]
 [-4.9131997  -4.2984928 ]
 [-6.18510529 -4.2984928 ]
 [-4.91318966 -4.91318044]
 [-4.91318046 -4.91318044]
 [-4.91318044 -4.91318044]
 [-4.91318966 -4.91318044]
 [-4.91322068 -4.91318044]
 [-6.82196723 -4.91318044]
 [-5.21871875 -4.91318044]
 [-4.91318046 -4.90819099]
 [-4.91322068 -4.90819099]
 [-6.82196723 -4.90819099]
 [-5.21871875 -4.90819099]
 [-4.91318966 -4.91318044]]

learning rates when loss value is greater than 10
 [0.66666667 0.55555556 0.55555556 0.55555556 0.55555556 0.55555556
 0.55555556 0.55555556 0.55555556 0.55555556 0.55555556 0.66666667
 0.66666667 0.66666667 0.66666667 0.66666667 0.66666667 0.66666667
 0.55555556 0.55555556 0.66666667 0.55555556 0.55555556 0.55555556
 0.55555556 0.55555556 0.55555556 0.55555556 0.66666667]
```

In [ ]: