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UNIT-I

Probability of an event:- (Probability) I[st] subdivision

* samplespace → All possible outcomes of an experiment
* Event → Subset of a sample space.
* Probability of the event → $\frac{\text{No. of event in the space}}{\text{No. of samplespace}}$.

$$P(E) = \frac{n(E)}{n(S)}$$

It lies b/w $0 \le P(E) \le 1$.

* Random variables. →

define a function

$f : S$ to a Real no.

S → Sample space, $\mathbb{R}$ → representation

this is called random variable. $(x, y, \ldots)$

Event	probability	
E_1	a_1	real no.
E_2	a_2	b/w 0 & 1.
⋮	⋮	
E_n	a_n	

$f : S \to \mathbb{R}$

- Random variable
- single dimension Random variable (RV) (only one rv)

Probability density function:/Probability mass fnc..

$$P(X=x) = \begin{cases} f(x) & x \in S. \\ 0 & \text{otherwise} \end{cases}$$

Discrete type

$$\sum_{x \in S} P(X=x) = 1. \qquad \int_a^b f(x)\,dx = 1.$$

Distribution function:-

$$F(x) = P(\underset{(RV)}{X} \le x)$$

$\sum_{x \le a} P(X=x)$ → Discrete type

$\int_{-\infty}^{a} f(x)\,dx$ → continuous type

Two dimensional random variable:-

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(x, y).

Probability density function:-

$P(x, y)$

$$\sum_y \sum_x P(x, y) = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1$$

$$f(x, y) = \begin{cases} \frac{xy}{2} & 0 \le x \le 1 \\ & 0 \le y \le 1 \end{cases}$$

(Eg: In the random placement of three variables in three cells, describe the possible outcomes of the experiment. Let X_i denote the no. of marbles in cell $i = 1, 2, 3$ & N, the no. of cells occupied obtain the joint distribution of (X_1, N) & (X_1, X_2).

Soln:-

Let a, b, c be the marbles.

$\checkmark \to P(2)$
$0 \to P(3)$
$\Delta \to P(1)$

Possibles:-

1) a - b - c
2) a - c - b
3) b - a - c
4) b - c - a
5) c - a - b
6) c - b - a
7) ab - c - ☆
8) ab - ☆ - c
9) ☆ - ab - c
10) ac - b - ☆
11) ac - ☆ - b
12) ☆ - ac - b
13) bc - a - ☆
14) bc - ☆ - a
15) ☆ - bc - a
16) a - bc - ☆
17) a - ☆ - bc
18) ☆ - a - bc
19) b - ca - ☆
20) b - ☆ - ca
21) ☆ - b - ca
22) c - ab - ☆
23) c - ☆ - ab
24) ☆ - c - ab
25) abc - ☆ - ☆
26) ☆ - abc - ☆
27) ☆ - ☆ - abc.

$X_1 \to$ one marble in a cell

$X_2 \to$ Two marbles in a cell

$N \to$ No. of cells occupied.

$P(N=1) = \frac{3}{27}$, $P(N=2) = \frac{18}{27}$, $P(N=3) = \frac{6}{27}$

$X_1 \to$ no. of marbles placed in first cell.

$P(X_1=0) = \frac{8}{27}$, $P(X_1=1) = \frac{12}{27}$, $P(X_1=2) = \frac{6}{27}$.

$\downarrow$ No marbles placed in the 1st cell

$\downarrow$ one marbles placed in 1st cell in

$\downarrow$ Two marble placed in 1st cell

Joint distribution of N & Xi.

9, 12, 15 18, 21, 24, 26 27

N / Xi	1 ($X_1=0$, $N=1$)	2 ($X_1=1$, $N=2$)	3 ($X_1=2$, $N=3$)	Distribution of X_1
0	2/27	6/27	0	8/27
1	0	6/27	6/27	12/27
2	0	6/27	0	6/27
3	1/27	0	0	1/27
Distribution of N	3/27	18/27	6/27	1

$P(X_1=0, N=1)$

= Prob {No. of marbles in 1st cell & No. of cells occupied is 1}. 2/27

$P(X_1=1, N=2)$

No. of cells in 1M

1M in 1st cell and no. of cells occupied is 2

$X_1=0, N=1 \Rightarrow 2/27$

$X_1=0, N=2 \Rightarrow 6/27$

$X_1=0, N=3 \Rightarrow 0$.

$X_1=1, N=1 \Rightarrow 0$

$X_1=1, N=2 \Rightarrow 6/27$

$X_1=1, N=3 \Rightarrow 6/27$

$X_1=2, N=1 \Rightarrow 0$

$X_1=2, N=2 \Rightarrow 6/27$

$X_1=2, N=3 \Rightarrow 0$

$X_1=3, N=1 \Rightarrow 1/27$.

$X_1=3, N=2 \Rightarrow 0$

$X_1=3, N=3 \Rightarrow 0$

Joint distribution of X_1 & X_2.

$X_2 \backslash X_1$	0	1	2	3	Distribution of X_2
0	1/27	3/27	3/27	1/27	8/27
1	3/27	6/27	3/27	0	12/27
2	3/27	3/27	0	0	6/27
3	1/27	0	0	0	1/27
Distribution of X_1	8/27	12/27	6/27	1/27	1

$X_1 \rightarrow$ No. of marbles in $cell_1$

$X_2 \rightarrow$ No. of marbles in $cell_2$

$X_{\cancel{0}}=0, X_2=0 \Rightarrow 1/27$

$X_1=0, X_2=1 \Rightarrow 3/27$

$X_1=0, X_2=2 \Rightarrow 3/27$

$X_1=0, X_2=3 \Rightarrow 1/27$

$X_1=1 \rightarrow X_2=0 \Rightarrow 3/27$

$X_1=1, X_2=1 \Rightarrow 6/27$

$X_1=1, X_2=2 \Rightarrow 3/27$

$X_1=1, X_2=3 \Rightarrow 0$

$X_1=2, X_2=0 \Rightarrow 2/27$

$X_1=2, X_2=1 \Rightarrow 3/27$

$X_1=2, X_2=2 \Rightarrow 0$

$X_1=2, X_2=3 \Rightarrow 0$

$X_1=3, X_2=0 \Rightarrow 1/27$

$X_1=3, X_2=1 \Rightarrow 0$

$X_1=3, X_2=2 \Rightarrow 0$

$X_1=3, X_2=3 \Rightarrow 0$

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1. Given prob distribution of x, y.
Find;
(i) $P(X \le 1, Y = 2)$ (ii) $P(X \le 1)$ (iii) $P(Y \le 3)$ (iv) $P(X < 3, Y \le 4)$

x \ Y	1	2	3	4	5	6	Distribution of x
0	0	0	1/32	2/32	2/32	3/32	$8/32 = \frac{16}{64}$
1	1/16	1/16	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	1/8.	$10/16 = \frac{40}{64}$
2	1/32	1/32	$\frac{1}{64}$	$\frac{1}{64}$	0	2/64	8/640
Distribution of Y	$\frac{6}{64}$	$\frac{6}{64}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{12}{64}$	$\frac{16}{64}$	

(1)

(i) $P(X \le 1, Y = 2)$

$= P(X=0, Y=2) + P(X=1, Y=2)$

$= 0 + 1/16 = 1/16.$

(ii) $P(X \le 1)$.

$= P(X=0) + P(X=1)$

$= \frac{16}{64} + \frac{40}{64} = \frac{7}{8}.$

(iii) $P(Y \le 3)$

$= P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3)$

$= \frac{6}{64} + \frac{6}{64} + \frac{11}{64} \cancel{+12} = \frac{23}{64}.$

(iv) $P(X < 3, Y \le 4)$

$= P(X=0, Y=1) + P(X=1, Y=2) + P(X=3, Y=3)$

$+ P(X=0, Y=4) + P(X=1, Y=1) +$

$+ P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4) +$

$+ P(X=2, Y=1) + P(X=2, Y=2) + P(X=2, Y=3) +$

$+ P(X=2, Y=4).$

$$= \frac{3}{32} + \frac{6}{16} + \frac{6}{64} = \frac{6+24+6}{24}$$

$$= \frac{36}{64}$$

Marginal Prob function:

Let (X, Y) be a discrete two-dimensional r.v with values (x_i, y_i). The prob distribution of X

$$P_X(x_i) = \sum_{i=1}^{m} P(x_i, y_i)$$

$$P_Y(y_j) = \sum_{i=1}^{n} P(x_j, y_j)$$

$x \backslash y$	y_1	y_2	...	y_m
x_1	P_{11}	P_{12}	...	P_{1m}
x_2	P_{21}	P_{22}		P_{2m}
⋮	⋮			
x_n	P_{n1}	P_{n2}		P_{nm}

conditional prob for

$$P_{X/Y}(x, y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

Joint point distribution:

$X \backslash Y$	1	2	3	4	Total
1	4/36	3/36	2/36	1/36	10/36
2	1/36	3/36	3/36	2/36	9/36
3	5/36	1/36	1/36	1/36	8/36
4	1/36	2/36	1/36	5/36	9/36
Total	11/36	9/36	7/36	9/36	1

(i) Find marginal distribution of x & Y.

(ii) conditional distribution of x given y = 1 & that of Y given x = 2.

Soln:-

(i) Marginal distribution of x.

x	1	2	3	4
P_i	10/36	9/36	8/36	9/36

Marginal distribution of Y.

Y	1	2	3	4
P_i	11/36	9/36	7/36	9/36

(ii) $P(X=1/Y=1)$, $P(X=2/Y=1)$, $P(X=3/Y=1)$, $P(X=4/Y=1)$

$$\frac{P(X=1, Y=1)}{P(Y=1)} = \frac{4/36}{11/36} = \frac{4}{11}$$

$$\frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

$$\frac{P(X=3, Y=1)}{P(Y=1)} = \frac{5/36}{11/36} = \frac{5}{11}$$

$$\frac{P(X=4, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

conditional distribution of x gn. y=1 is.

x:	1	2	3	4
P_i:	4/11	1/11	5/11	1/11.

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1. A two-dimensional r.v (X, Y) have the bivariate distribution given by: $P(X=x, Y=y) = \frac{x^2+y}{32}$ for $x = 0, 1, 2, 3$ & $y = 0, 1$. Find the marginal distribution of X & Y.

$y \backslash x$	0	1	2	3	Distribution of Y
0	0	1/32	4/32	9/32	14/32
1	1/32	2/32	5/32	10/32	18/32
Distribution of x	$\frac{1}{32}$	$\frac{3}{32}$	9/32	19/32	1

circled values are calculated by $\frac{x^2+y}{32}$.

y:	0	1
fy:	14/32	18/32

x	0	1	2	3
fx:	1/32	3/32	9/32	19/32

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2. Two discrete r.v $(X$ & $Y)$ have the joint prob-density function

$$P_{XY}(x, y) = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!} \qquad y = 0, 1, 2, \ldots x \quad x = 0, 1, 2, \ldots$$

where λ, p are constants with $\lambda > 0$ & $0 < p < 1$.

Find the (i) marginal prob-density func. of X & Y

(ii) conditional distribution of Y for a given X.

(iii) conditional distribution of x for a given y.

soln:

(1) Marginal density for of X.

$$P_X(x) = \sum_{y=0}^{x} P_{XY}(x,y).$$

$$= \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{\lfloor y \lfloor x-y}$$

$$P_X(x) = \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} (1-p)^{x-y} p^y}{y!\,(x-y)!}$$

$$= \lambda^x e^{-\lambda} \sum_{y=0}^{x} \frac{(1-p)^{x-y}}{\lfloor y \lfloor x-y} p^y$$

$$= \frac{\lambda^x e^{-\lambda}}{\lfloor x} \sum_{y=0}^{x} \frac{\lfloor x \, p^y (1-p)^{x-y}}{\lfloor y \lfloor x-y}$$

$$= \frac{\lambda^x e^{-\lambda}}{\lfloor x} \sum_{y=0}^{x} {}^{x}c_y \cdot p^y (1-p)^{x-y}.$$

$$= \frac{\lambda^x e^{-\lambda}}{\lfloor x} \{p + 1 - p\}^x = \frac{\lambda^x e^{-\lambda}}{\lfloor x}$$

Marginal density for of Y.

$$P_Y(y) = \sum_{x=0}^{\infty} P_{xy}(x,y)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x p^y (1-p)^{x-y}}{\lfloor y \lfloor x-y}$$

$$= \frac{e^{-\lambda} p^y}{\lfloor y} \sum_{x=0}^{\infty} \frac{\lambda^x (1-p)^{x-y}}{\lfloor x-y}$$

$$\sum_{y=0}^{x} {}^{x}c_y p^y (1-p)^{x-y}$$

$${}^{x}c_0 p^0 (1-p)^{x-0}$$

$${}^{x}c_0 p^0 (1-p)^x$$

$${}^{x}c_1 p (1-p)^{x-1}$$

$${}^{x}c_2 p (1-p)^{x-2}$$

$$+ \cdots +$$

$${}^{x}c_x p^x (1-p)^0$$

$$= p^x + {}^{x}c_{x-1} \, {}^{x}c_{x-2} (1-p)^2 + \cdots + (1-p)^x +$$

$$\{p + 1 - p\}^x$$

$$= \frac{\cancel{\lambda}^y e^{-\lambda} p^y}{\cancel{\lambda^y} \lfloor y} \sum_{x=0}^{\infty} \frac{\lambda^{x-y} (1-p)^{x-y}}{\lfloor x-y} \rightarrow e^{\lambda(1-p)}$$

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \dots$$

$\times$ and $\div$ by λ^y in this step

$$\cancel{P_y(y) = \frac{e^{-\lambda} p^y}{\lambda^y \lfloor y} e^{\lambda} e^{-\lambda p}}$$

$$\cancel{= \frac{e^{-\lambda p} p^y}{\lambda^y \lfloor y}}$$

$$P_y(y) = \frac{\lambda^y \cancel{e^{-\lambda}} p^y}{\lfloor y} \cancel{e^{\lambda}} e^{-\lambda p}$$

$$= \frac{\lambda^y p^y e^{-\lambda p}}{\lfloor y}$$

$$= \frac{e^{-\lambda p} (\lambda p)^y}{\lfloor y}$$

(ii)

$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}$$

$$= \frac{\cancel{\lambda^x e^{-\lambda}} p^y (1-p)^{x-y} \lfloor x}{\lfloor y \lfloor x-y \; \cancel{\lambda^x e^{-\lambda}}}$$

$$= \frac{p^y (1-p)^{x-y} \lfloor x}{\lfloor y \lfloor x-y}$$

$$= \cancel{p^y} \; {}^{x}C_y \, p^y (1-p)^{x-y}$$

$$x \geq y, \quad y = 0, 1, 2, \dots x$$

(iii) $P_{X/Y}(x/y) = \dfrac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y} \cdot \cancel{\lfloor y}}{\lfloor y \lfloor x-y \quad e^{-\lambda p} (\lambda p)^y}$.

$x \geq y$ & $x = y, y+1, y+2, \ldots$

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3. suppose that 2 D r.v (continous) has joint PDF $f(x,y) = \begin{cases} 6x^2 y & 0 < x < 1, 0 < y < 1. \\ 0 & \text{elsewhere}. \end{cases}$

(i) verify integral $\int_0^1 \int_0^1 f(x,y)\, dx\, dy = 1$.

(ii) find $P(0 < x < 3/4, \; 1/3 < y < 2)$.

(iii) $P(X + Y < 1)$.

(iv) $P(X > Y)$

(v) $P(X < 1 / Y < 2)$.

Soln:

(i) $\int_0^1 \int_0^1 f(x,y)\, dx\, dy = \int_0^1 \int_0^1 6x^2 y\, dx\, dy$

$= \int_0^1 \left(\frac{\overset{2}{\cancel{6}} x^3}{\cancel{3}}\right)_0^1 \cdot y\, dy$

$= \int_0^1 (2x^3)_0^1 \, y\, dy$

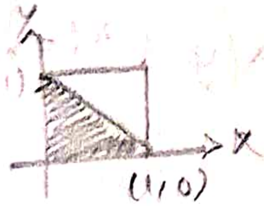
$= \int_0^1 2y\, dy = \left(\frac{2y^2}{2}\right)_0^1 = 1$.

(ii) $P(0 < x < 3/4, \; 1/3 < y < 2)$

$= \int_0^{3/4} \int_{1/3}^1 6x^2 y \cdot dy\, dx \quad \{0 < y < 1\}$.

$f(x,y) = \begin{cases} 6x^2 y & 0 < x < 1, \; y < 1 \\ 0 & \text{elsewhere} \end{cases}$

(iii) $P(x+y<1)$.



$$= \int_0^1 \int_0^{1-x} 6x^2 y \, dy \, dx.$$

$$= \int_0^1 \cancel{6}^{3} x^2 \left[\frac{y^2}{\cancel{2}}\right]_0^{1-x} dx$$

$$= \int_0^1 3x^2 (1-x)^2 \, dx$$

$$= \int_0^1 3x^2 (1-2x+x^2) \, dx$$

$$= 3 \left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right)\Big|_0^1$$

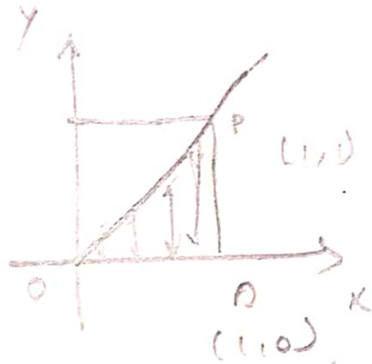
$$= 3\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$= 3\left(\frac{10-15+6}{30}\right)$$

$$= \frac{\cancel{3}(1)}{\cancel{30}\,10}$$

$$= \frac{1}{10}$$

(iv) $P(x>y)$.



$$P(x>y) = \int_0^1 \int_0^x 6x^2 y \, dy \, dx$$

$$= \int_0^1 \cancel{6}^{3} x^2 \left(\frac{y^2}{\cancel{2}}\right)_0^x dx$$

$$= \int_0^1 3x^4 \, dx = 3\left(\frac{x^5}{5}\right)_0^1$$

$$= \frac{3}{5}$$

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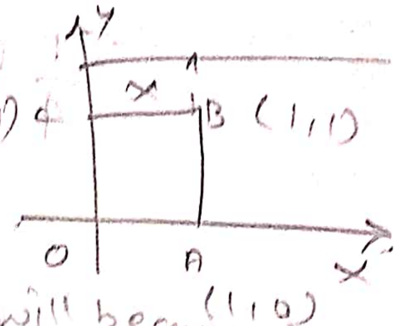
(v) $P(X<1/Y<2) = \dfrac{P(X<1, Y<2)}{P(Y<2)}$

$$\left\{ P_{X/Y}(x<a/y<b) = \frac{P(X<a, Y<b)}{P(Y<b)} \right\}$$

$$P(X<1, Y<2) = \int_0^1\int_0^1 f(x,y)\,dy\,dx + \cancel{\int_0^1\int_1^2 f(x,y)\,dy\,dx}$$



this part will become 0 -> b<2 f(x,y) is 0.

$$= \int_0^1\int_0^1 6x^2y\,dy\,dx = 1.$$

$$P(Y<2) = \int_0^1\int_0^2 6x^2y\,dy\,dx$$

$$= \int_0^1\int_0^1 6x^2y\,dy\,dx + \int_0^1\int_1^2 \cancel{f(x,y)}\,dy\,dx$$

$$= 1 \qquad\qquad 0.$$

$$P(X<1/Y<2) = \frac{1}{1} = 1$$

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1. The joint Prob. density fnc (PDF) $f(x,y)$=

$$f(x,y) = \begin{cases} \dfrac{1+xy}{4} & |x|\leq 1, |y|\leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) calculate marginal density func. of X & Y.

(ii) Are X and Y independent.

Soln

(ii) $f_X(x) = \int_{-1}^{1} \left\{ \frac{1+xy}{4} \right\} dy$

$$= \left(\frac{y + x\left(\frac{y^2}{2}\right)}{4} \right)_{-1}^{1} dy$$

$$= \frac{1}{4} \left\{ 2 + \frac{x}{2} \underbrace{\cancel{(1-1)}}_{0} \right\}$$

$$= \frac{1}{2}$$

$|x| \leq 1$

$\Rightarrow -1 \leq x \leq 1$

$|y| \leq 1$

$\Rightarrow -1 \leq y \leq 1$

$f_Y(y) = \int_{-1}^{1} \left\{ \frac{1+xy}{4} \right\} dx$

$$= \frac{1}{2}.$$

(iii) $f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1+xy}{4}}{\frac{1}{2}} = \frac{1+xy}{2}$

$\Rightarrow f_{XY}(x,y) = \frac{1+xy}{2} \neq \frac{1}{2} = f_X(x)$

$\therefore$ x and y are not independent.

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2. Joint PDF $f(x,y) = c(x^2-y^2)e^{-x}$ $0 \leq x < \infty$, $-x \leq y \leq x$

$|y| \leq x$

Find (i) const c

(ii) Marginal density for of X & Y

Soln:

(i) $\int_0^\infty \int_{-x}^{x} c(x^2-y^2)e^{-x} \, dy \, dx = 1$

$\Rightarrow \frac{1}{c} = \int_0^\infty \int_{-x}^{x} (x^2-y^2)e^{-x} \, dy \, dx$

$$= \int_0^\infty e^{-x} \left\{ x^2 (y)_{-x}^{x} - \left(\frac{y^3}{3}\right)_{-x}^{x} \right\} dx$$

$$= \int_0^\infty e^{-x} \left\{ x^2(2x) - \frac{1}{3}(2x^3) \right\} dx$$

$$= \int_0^\infty 2x^3 e^{-x} \left\{ 1 - \frac{1}{3} \right\} dx.$$

$$= \frac{4}{3} \int_0^\infty e^{-x} x^3 \, dx.$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$\downarrow$

Gamma integral

We can prove

$\lfloor 0 = 1$

$\Gamma(n) = \lfloor n-1$

$$= \frac{4}{3} \lfloor 3$$

$$= \frac{4}{\cancel{3}} \times \cancel{3} \times 2 \times 1$$

$$= 8$$

$$\frac{1}{c} = 8 \Rightarrow \boxed{c = \frac{1}{8}}$$

(ii) Marginal density func. of x.

$$f_X(x) = \int_{-x}^{x} c(x^2 - y^2) e^{-x} dy$$

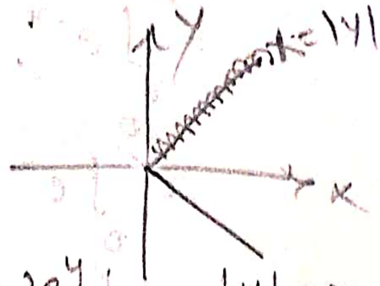
$$= \frac{1}{8} e^{-x} \int_{-x}^{x} \{x^2 - y^2\} dy.$$

$$= \frac{e^{-x}}{8} \left\{ x^2 (y)_{-x}^{x} - \left(\frac{y^3}{3}\right)_{-x}^{x} \right\}$$

$$= \frac{e^{-x}}{8} \left\{ 2x^3 - \frac{2x^3}{3} \right\}$$

$$= \frac{\cancel{4} e^{-x} x^3}{\cancel{24}\, 6} = \frac{e^{-x} x^3}{6}$$

$$f_Y(y) = \begin{cases} \int_y^\infty f(x,y)\,dx & y \geq 0 \\ \int_{-y}^\infty f(x,y)\,dx & y < 0 \end{cases}$$



$\hookrightarrow -c\{y^2e^y + 2(-y)e^y + 2e^y\} = -cy^2e^y$

$\int c(x^2-y^2)e^{-x}\,dx$

$= c\int x^2e^{-x}\,dx - cy^2\int e^{-x}\,dx$

$= c\{x^2e^{-x} - 2x\frac{e^{-x}}{-1} + 2e^{-x}\} - cy^2\left(\frac{e^{-x}}{-1}\right)$

$\int_y^\infty f(x,y)\,dx = 0 - \{c(y^2e^{-y} + 2ye^{-y} + 2e^{-y}\}$

$- cy^2e^{-y}$.

$\int c(x^2-y^2)e^{-x}\,dx$

$= \frac{1}{8}\int x^2e^{-x}\,dx - \frac{y^2}{8}\int e^{-x}\,dx$

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$= \frac{1}{8}\left\{x^2e^{-x} - 2x\left(\frac{e^{-x}}{-1}\right) + 2\left(\frac{e^{-x}}{1}\right)\right\} - \frac{y^2}{8}\left(\frac{e^{-x}}{-1}\right)$

$= \int_y^\infty$

22/11/21

One-dimensional r.v.:

1. Expection of a r.v $\to$ Average mean. $\to$ central moments
2. Moments of a r.v $\to$ μ'_r, $\mu_r = \sum_{x \in S} (x-\mu)^r$

22/11/21

UNIT

Subdivision - (III) in unit I.

One-dimensional r.v:-

1. Expection of a r.v → Average mean
2. Moments of a r.v → μ_r', $\mu_r = \sum_{x \in S} (x-\mu)^r$

μ_r'

↓

$\sum_{x \in S} (x-A)^r$

↳ raw moments.

Relation b/w raw moments and central moments.

Define random variable (r.v):

R.V is a fnc. X with domain S and range $(-\infty, \infty)$ such that for every real no. a the event $\{w \cdot X(w) \leq a\}$.

Eg: Tossing a coin 100 times.

$f: X \to [(1,100)$

$f(50) \to$ getting head or tail in the 50th throw

NOTE:-

X_1 & X_2 are r.v's & c is a constant, then $X_1 + X_2$ is a r.v; cX_1 is a r.v & $X_1 X_2$ is a r.v, ~~$X_1 - X_2$~~

$X_1 - X_2$ is a r.v.

Distribution fnc & (prob. density fnc (or) mass fnc.)

Distribution fnc:- Prob. density fnc

$$F(x) = P(X \le x)$$

(cummulative frequency)

$$P(X=x) = \begin{cases} P(x_i) & x = x_i \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\sum P(X=x_i) = 1}$$

$$F'(x) = f(x)$$

---

1. The diameter of an electric cable, say x, is assumed to be a continous R.V with prob. density fnc. $f(x) = 6x(1-x) \quad 0 \le x \le 1$

(I) check $f(x)$ is a prob. density fnc.

(II) Determine b such that $P(X<b) = P(X>b)$.

Solun.:-

(I) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ {Pdf}

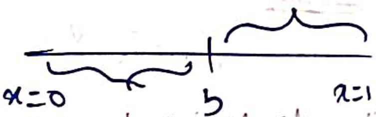
consider $\int_0^1 f(x)\,dx = \int_0^1 6x(1-x)\,dx = 6\{\int_0^1 (x-x^2)$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 6\left(\frac{1}{2} - \frac{1}{3}\right) = \cancel{6}\left(\frac{1}{\cancel{6}}\right)$$

$$= 1$$

$\therefore f(x)$ is a pdf.

(II) $\int_0^b f(x)\,dx = \int_b^1 f(x)\,dx.$



+ middle value median same as upside.

$$\int_0^b \cancel{6}(x-x^2)\,dx = \int_b^1 \cancel{6}(x-x^2)\,dx$$

$$\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_b^1$$

$$\Rightarrow \frac{b^2}{2} - \frac{b^3}{3} = \frac{1}{2} - \frac{1}{3} - \frac{b^2}{2} + \frac{b^3}{3}$$

$$\Rightarrow 2\left(\frac{b^2}{2} - \frac{b^3}{3}\right) = \frac{1}{2} - \frac{1}{3}.$$

$$\Rightarrow 2\frac{(3b^2 - 2b^3)}{\cancel{6}} = \frac{1}{\cancel{6}}$$

$$\Rightarrow 3b^2 - 2b^3 = \frac{1}{2}$$

$$\Rightarrow \cancel{b}^2(3 \cancel{-} 2\cancel{b}) = \frac{\cancel{1}}{2}.$$

$$\Rightarrow 6b^2 - 4b^3 = 1 \quad \text{(or)}$$

$$\Rightarrow 4b^3 - 6b^2 + 1 = 0.$$

$4b^3 - 6b^2$

$$(2b-1)(2b^2 - 2b - 1) = 0$$

$$\downarrow$$

$$b = \frac{2 \pm \sqrt{4+8}}{4}$$

$$= \frac{\cancel{2}^{1} \pm 2\sqrt{3}}{\cancel{4}_{2}}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$0 \leq x \leq 1$ & $0 \leq b \leq 1$.

$$\boxed{b = \frac{1}{2},\ b = \frac{1+\sqrt{3}}{2},\ b = \frac{1-\sqrt{3}}{2}}.$$

$\to$ -ve (skip this)

$\hookrightarrow b > 1.$

Expection of a r.v, denoted as $E(X)$ {Average}.

$E(x) = \sum_i x P_i$ (Discrete case)

$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$ (continous case)

$E(X^2) - (E(X))^2 = \text{Variance of } X \to (SD)^2$

$\hookrightarrow E(X - E(X))^2$

Property:

$$E(cX) = cE(X).$$

$$E(cX) = \int_{-\infty}^{\infty} cx\, f(x)\,dx.$$

$$= c\int_{-\infty}^{\infty} x f(x)\,dx.$$

$$= cE(x).$$

$var(cx) = ?$ {variance of x = $var(x)$}

c - constant

$$var(cx) = E(c^2x^2) - (E(cx))^2$$

(by expection)

$$= c^2E(x^2) - c^2(E(x))^2$$

$$= c^2\{E(x^2) - (E(x))^2\}$$

$$= c^2\, var(x)$$

$$E(X+Y) = E(x) + E(y)$$

$$\cancel{E(x+y) = \int_{-\infty}^{\infty}(f(x) + g(y))\,dx.}$$

---

1. A r.v x is distributed at random b/w 0 & 1 so that its pdf in $f(x) = kx^2(1-x^3)$ where k is a constant. Find

(i) k

(ii) mean

(iii) Variance.

Soln:

$$(1) \int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 kx^2 \overbrace{(1-x^3)}^{(x^2-x^5)} dx = 1.$$

$$k\left\{\frac{x^3}{3} - \frac{x^6}{6}\right\}_0^1$$

$$k\left[\frac{1}{3} - \frac{1}{6}\right] = 1$$

$$k\left(\frac{3}{18}\right) = 1$$

$$\boxed{k = 6}$$

$$(11)\ E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= 6\int_0^1 x\{x^2 - x^5\}\,dx$$

$$= 6\left\{\frac{x^4}{4} - \frac{x^7}{7}\right\}_0^1$$

$$= 6\left(\frac{1}{4} - \frac{1}{7}\right)$$

$$= 6\left(\frac{7-4}{28}\right)$$

$$= \cancel{6}^{3}\left(\frac{3}{\cancel{28}_{14}}\right)$$

$$\Rightarrow \frac{9}{14}$$

$$\text{Mean} = \frac{9}{14}.$$

$$(111)\ E(x^2) = (E(x))^2$$

$$E(x^2) = 6\int_0^1 x^2\{x^2 - x^5\}\,dx$$

$$= 6\left\{\frac{x^5}{5} - \frac{x^8}{8}\right\}_0^1$$

$$= 6\left(\frac{1}{5} - \frac{1}{8}\right)$$

$= \frac{18}{40}$

$= \frac{9}{20}$

Variance $= \frac{9}{20}$.

$var(x) = \frac{9}{20} - \left(\frac{9}{14}\right)^2$

$Var(x) = E(x)^2 - (E(x))^2$

$= \frac{9}{20} - \frac{81}{196}$.

$= \frac{196(9) - 81(20)}{20 \times 196}$

$= \frac{1764 - 1620}{3920}$

$= \frac{144}{3920}$

$= 0.036$ //

$\begin{array}{r} 196 \\ \times 9 \\ \hline 1764 \end{array}$ $\begin{array}{r} 5 \\ 8 \end{array}$

$\begin{array}{r} 196 \\ \times 21 \\ \hline 3920 \end{array}$ $\begin{array}{r} 81 \\ \times 2 \\ \hline 162 \end{array}$

0.036.

$\begin{array}{r} 1764 \\ 1620 \\ \hline 144 \\ \hline \end{array}$

23/11/21

1. Find the pdf of a r.v. X whose distribution function $F(x) = \begin{cases} 0 & x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & -a \le x \le a \\ 1 & x > a \end{cases}$

Soln:

W.K.t $f(x) = \frac{d}{dx} F(x)$

$\frac{d}{dx}\left\{\frac{1}{2}\left(\frac{x}{a}+1\right)\right\} = \frac{1}{2a}$

$$f(x) = \begin{cases} \frac{1}{2a} & -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

To verify $f(x)$ is a pdf

i.e; $\int_{-\infty}^{\infty} f(x)\,dx = 1$

consider $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-a}^{a} \frac{1}{2a}\,dx = \frac{1}{2a}\,(x)_{-a}^{a}$

$= \frac{1}{2a}(2a) = 1.$

---

2. let x be a continous random variable with pdf given by

$$f(x) = \begin{cases} kx & 0 \le x < 1 \\ k & 1 \le x < 2 \\ -kx+3k & 2 \le x < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Determine (i) k.

(ii) distribution for of x.

soln. wkt $\int_{-\infty}^{\infty} f(x)\,\cancel{a}\,dx = 1.$

$F(x) = P(X \le x)$

$F(x)$ such that $2 \le x < 3$

$P(x) = \int_{-\infty}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$

(i)

$$\int_0^1 kx\,dx + \int_1^2 kx\,dx + \int_2^3 (3k - kx)\,dx = 1$$

$$k\left\{ \left(\frac{x^2}{2}\right)_0^1 + (x)_1^2 + \left(3x - \frac{x^2}{2}\right)_2^3 \right\} = 1$$

$$k\left\{ \cancel{\frac{1}{2}} + 2 - \cancel{1} + \cancel{9} - \frac{9}{2} - \cancel{6} + 2 \right\} = 1$$

$k(7-5) = 1$

$k\{2\} = 1$

$2k = 1$

$$\boxed{k = \frac{1}{2}}$$

(ii) Distribution for $F(x) = P(X \le x)$.

(i) For any x, such that $-\infty < x < 0$, $F(x) = 0$.

(ii) For any x, such that $0 \le x < 1$.

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} \frac{1}{2} x\,dx = \frac{1}{2}\frac{x^2}{2} = \frac{x^2}{4}.$$

(iii) For any x, such that $1 \le x < 2$

$$F(x) = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= \frac{x^2}{4} + \int_{1}^{x} \frac{1}{2}\,dx = \frac{x^2}{4} + \frac{(x-1)}{2}.$$

(iv) For any x, such that $2 \le x < 3$.

$$F(x) = \int_{-\infty}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx = \frac{x^2}{4} + \frac{x-1}{2} +$$

$$\int_{2}^{x} \left\{ -\frac{x}{2} + \frac{3}{2} \right\} dx$$

$$= \frac{x^2}{4} + \frac{x-1}{2} + \left(-\frac{x^2}{4} + \frac{3x}{2} \right)_2^x$$

$$= \cancel{\frac{x^2}{4}} + \frac{x-1}{2} + \left(\cancel{\frac{-x^2}{4}} + \frac{3x}{2} \right) + \cancel{\frac{4}{4}} - 3$$

$$= 2x - \frac{5}{2}$$

$$F(x) = \begin{cases} 0 & -\infty \le x < 0. \\ \frac{x^2}{4} & 0 \le x < 1 \\ \frac{x^2+2x-2}{4} & 1 \le x < 2 \\ 2x - 5/2 & 2 \le x < 3 \\ 1 & x > 3 \end{cases}$$

NOTE: Two r.v are independent if

$f(x,y) = f_X(x)\, f_Y(y)$ → marginal density function of Y.

↓ Joint pdf   ↳ mdf of X

---

3. ⇒ Joint prob. density fnc. of X and Y is given by $f(x,y) = 4xy\,e^{-(x^2+y^2)}$. $x, y \geq 0$.

So Test wheter X & Y are independent. Also find the conditional density function of X given Y = y.

Soln:

$$f_X(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty 4xy\,e^{-(x^2+y^2)}\,dy$$

$$= 4x\,e^{-x^2} \int_0^\infty y\,e^{-y^2}\,dy$$

$y^2 = u \Rightarrow 2y\,dy = du.$

$$= 4x e^{-x^2} \int_0^\infty e^{-u} \frac{du}{2} = \frac{4xe^{-x^2}}{2} \left\{\frac{e^{-u}}{-1}\right\}_0^\infty$$

$$= 2x e^{-x^2}.$$

$$f_Y(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty 4xy\,e^{-(x^2+y^2)}\,dx$$

$$= 4y\,e^{-y^2} \int_0^\infty x e^{-x^2}\,dx$$

$$= \cancel{4ye^{-y^2}\int_0^\infty} \qquad = 2ye^{-y^2}$$

$$f_X(x)\, f_Y(y) = 2xe^{-x^2} \cdot 2ye^{-y^2}$$

$$= 4xy\,e^{-(x^2+y^2)} = f(x,y).$$

∴ X & Y are independent.

$$f_{X|Y} = \frac{f(x,y)}{f_Y(y)} = \frac{4xy\,e^{-(x^2+y^2)}}{2ye^{-y^2}} = 2xe^{-x^2}$$

$x \geq 0.$

---

4. Let X be a r.v with following prob-distribution

x :	-3	6	9
$P(X=x)$:	$1/6$	$1/2$	$1/3$.

Find $E(X)$ & $E(2X+1)^2$.

Soln.

$$E(X) = \sum_x x\,P(X=x)$$

$$E(X) = (-3)\frac{1}{6} + 6\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right)$$

$$= -\frac{1}{2} + 3 + 3 = 6 - \frac{1}{2} = \frac{11}{2}.$$

$$\boxed{E(X) = \frac{11}{2}}$$

$4 \times \frac{11}{2}$

$$E(2X+1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + E(1)$$

$$E(X^2) = \sum x^2 P(X=x)$$

$$= (-3)^2 \times \frac{1}{6} + 6^2 \times \frac{1}{2} + 9^2 \times \frac{1}{3}.$$

$$= \frac{9}{6} + \frac{36}{2} + \frac{81}{3} = \frac{3}{2} + 18 + 27 = \frac{93}{2}.$$

$$\therefore E(2X+1)^2 = 4\left(\frac{93}{2}\right) + 4\left(\frac{11}{2}\right) + 1$$

$$= 186 + 22 + 1 = 209$$

---

5. Let X be a r.v with $P(X=0) = P(X=2) = p$.

$P(X=1) = 1-2p$; $0 \leq p \leq \frac{1}{2}$.

For what p is var (X) a maximum.

Soln.

$Var(x) = E(x^2) - (E(x))^2.$

$E(x) = \sum_1 x \; P(X=x).$

$= 1(1-2p) + 0(p) + 2p$

$= 1 - \cancel{2p} + \cancel{2p} = 1$

$E(x^2) = \sum_1 x^2 \, P(X=x)$

$= 1(1-2p) + 0(p) + 2^2 p$

$= 1 - 2p + 4p.$

$= 1 - 2p$

$= 2p + 1$

$Var(x) = E(x^2) - (E(x)^2)$

$= \cancel{1} + 2p - \cancel{1}$

$= 2p$

Max value of $Var(x)$ is max value of $2p$.

$\Rightarrow$ Max value of $Var(x)$ is obtained $p = \frac{1}{2}$.

$2p = 1$

$\boxed{p = \frac{1}{2}}$

24/11/21

1. If t is very positive real no. S.T the function defined by $P(x) = e^{-t}(1-e^{-t})^{x-1}$ can represent a probability func. of a r.v X assume the values $X = 1, 2, 3, \ldots$ Find $E(x)$ & $Var(x)$ of the distribution.

Soln:

$P(x)$ is a pdf is $\sum_{x} P(x) = 1$

consider $\sum_{x=1}^{\infty} P(x) = \sum_{x=1}^{\infty} e^{-t}(1-e^{-t})^{x-1}$

$1+\alpha+\alpha^2+\cdots$

$= e^{-t}\{1+(1-e^{-t})+(1-e^{-t})^2+\cdots\}$

$= e^{-t}\left\{\frac{1}{1-\alpha}\right\} = \frac{e^{-t}}{1-1+e^{-t}} = 1.$

$\therefore$ $P(x)$ is a pdf of r.v. X..

$E(x) = \sum_{x=1}^{\infty} x\, P(x)$

$= \sum_{x=1}^{\infty} x \cdot e^{-t}(1-e^{-t})^{x-1}$ $\boxed{1-e^{-t} = \alpha}$ $\boxed{|\alpha|<1}$

$= e^{-t}\sum_{x=1}^{\infty} x \cdot \alpha^{x-1}$

$= e^{-t}\{1+2\alpha+3\alpha^2+\cdots\}$

$= \frac{e^{-t}}{(1-\alpha)^2} = \frac{e^{-t}}{(e^{-t})^2}$

$= e^{t}$ //

$Var(x) = E(x^2) - (E(x))^2$

$E(x^2) = \sum_{x=1}^{\infty} x^2 P(x)$

$= \sum_{x=1}^{\infty} x^2 e^{-t} \alpha^{x-1}$ $\{\alpha = 1-e^{-t}\}$

$$= e^{-t}\{1+4\alpha+9\alpha^2+16\alpha^3+\dots\}.$$

$$\begin{cases} S = 1+4a+9a^2+16a^3+\dots \\ (1-a)^3 S = (1-a)^3\{1+4a+9a^2+16a^3+\}\end{cases}$$

$$= \{1-3a+3a^2-a^3\}\{1+4a+9a^2+\dots\}$$

$$(1-a)^3 S = (1-3a+3a^2-a^3)(1+4a+9a^2+16a^3+\dots$$

$$= 1+4a+9a^2+16a^3+\dots$$

$$= -3a-12a^2-48a^3-\dots$$

$$+3a^2+12a^3+27a^4+\dots$$

$$-a^3-4a^4-9a^5-\dots$$

$$\boxed{1+a \Rightarrow S = \frac{(1+a)}{(1-a)^3}}$$

$$\therefore E(x^2) = e^{-t}\frac{(1+\alpha)}{(1-\alpha)^3} = \frac{e^{-t}\{1+1-e^{-t}\}}{(1-1+e^{-t})^3}$$

$$= e^{t}(2-e^{-t})e^{-2t}$$

$$=$$

$$\mathrm{var}(x) = E(x^2)-(E(x))^2$$

$$= e^{2t}\,\cancel{(e^{t}+2)}\,(2-e^{-t}) - e^{2t}.$$

$$= \cancel{e^{t}+2e^{2t}-e^{-2t}}$$

$$= \cancel{e^{t}+e^{2t}}$$

$$= e^{t}(e^{t}+1).$$

$$= 2e^{2t}-e^{t}-e^{+2t}.$$

$$= e^{2t}-e^{t}$$

$$= e^{t}(e^{t}-1)$$

Moments: $\mu_r = E(X-\mu)^r$

$$r=1$$

$$\mu_1 = E(X-\mu)$$

$$= E(X)-\mu$$

$$= \mu-\mu = 0$$

$$\mu_2 = E(x-\mu)^2$$
$$= E(x^2 - 2x\mu + \mu^2)$$
$$= E(x^2) - 2\mu E(x) + \mu^2$$
$$= E(x^2) - (E(x))^2.$$

Expection of fnc. of $g(x,y)$ of 2D-r.v (x,y) with pdf $f(x,y)$ is given by

$$E(g(x,y)) = \begin{cases} \sum_i \sum_j x_i y_j P_{ij} & \text{(Discrete)} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy & \text{(joint pdf)} \end{cases}$$

Q. Let x and y be two r.v's each takes the values -1, 0, 1, & having pdf as

Marginal density of Y

Y \ X	-1	0	1	f(y)
-1	0	0.1	0.1	0.2
0	0.2	0.2	0.2	0.6
1	0.	0.1	0.1	0.2
f(x)	0.2	0.4	0.4	1

Marginal density fnc. of X.

find $E(x)$, $E(y)$, $E(xy)$. Is x & y are Independent

$$E(x) = \sum x f_x(x) = (-1)(0.2) + 0(0.4) + 1(0.4)$$
$$= -0.2 + 0.4$$
$$= 0.2$$

$$E(y) = \sum y f_y(y) = (-1)(0.2) + 0(0.6) + 1(0.2)$$
$$= 0.$$

$$E(xy) = \sum_{-1}^{1}\sum_{-1}^{1} xy\, P(x_i, y_j) = 1(0)^{(0\cdot 0)} + (-1)(0)(0\cdot 1)$$
$$+ (-1)(1)(0\cdot 1) + 0(\cdots) +$$
$$1(-1)\cdot(0) + 1(0)(0\cdot 1) + 1\cdot(1)(0\cdot 1)$$

$$E(xy) = -0\cdot 1 + 0\cdot 1 = 0$$

$$E(xy) = 0 = (0\cdot 2)(0) = E(x)\,E(y)$$

$\Rightarrow$ x & y are independent.

25/11/21

Expection and ~~random~~ variance in 2D random variable ~~in 2D~~

Marginal density fn.

$E(x)$ & $E(y)$

$E(xy) = \sum_{i,j} ij\, P_{ij}$

conditional distribution fnc.

$E(Y/X=x)$,

$var(Y/X=x)$

---

1. Two r.v X & Y as the following joint pdf

$$f(x,y) = \begin{cases} 2-x-y & 0 \le x \le 1;\ 0 \le y \le 1. \\ 0 & \text{otherwise}. \end{cases}$$

Find (i) Marginal prob density fnc. (pdf) of x & y.

(ii) conditional density fnc. of x & y

(iii) var(x) & var(y)

(iv) Find covariance b/w x & y.

Formula $\rightarrow$ $\boxed{cov(x,y) = E(xy) - E(x)E(y)}$

Soln:

(i) $f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_0^1 (2-x-y)\,dy = (2-x)(y)_0^1 - \left(\frac{y^2}{2}\right)_0^1$$

$$= 2-x-\frac{1}{2} = \frac{3}{2}-x$$

$$f_X(x) = \begin{cases} \frac{3}{2}-x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (2-x-y)\,dx$$

$$= \frac{3}{2}-y.$$

$$f_Y(y) = \begin{cases} \frac{3}{2}-y & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(Note:

$$f_X(x)\,f_Y(y) = \left(\frac{3}{2}-x\right)\left(\frac{3}{2}-y\right) \neq 2-x-y = f(x,y).$$

$\therefore$ x & y are not independent)

(ii) $$f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)} = \frac{2-x-y}{\frac{3}{2}-y} \quad 0<x,y<1$$

$$f_{Y|X}(x,y) = \frac{f(x,y)}{f_X(x)} = \frac{2-x-y}{\frac{3}{2}-x} \quad 0\leq x,y<1$$

(iv) $$E(XY) = \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy$$ Expecting XY

$$= \int_0^1 \left\{ \frac{2x^2}{2}y - \frac{x^3}{3}y - \frac{x^2y^2}{2} \right\}_0^1 dy.$$

$$= \int_0^1 \left(y-\frac{y}{3}-\frac{y^2}{2}\right)dy = \int_0^1 \left(\frac{2y}{3}-\frac{y^2}{2}\right)dy$$

$$= \left\{ \frac{2y^2}{3(2)} - \frac{y^3}{2(3)} \right\}$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$ Expection of x

$$= \int_0^1 x\left(\frac{3}{2} - x\right) dx$$

$$= \left\{ \frac{3}{2}\left(\frac{x^2}{2}\right) - \frac{x^3}{3} \right\}_0^1$$

$$= \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12}$$

$$= \frac{5}{12}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$ Expection of y

$$= \int_0^1 y\left(\frac{3}{2} - y\right) dy$$

$$= \frac{5}{12}.$$

$$\boxed{\operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y)}$$ formula

$$= \frac{1}{6} - \frac{5}{12} \times \frac{5}{12}$$

$$= \frac{24-25}{144} = \frac{-1}{144}$$

(iii) $\boxed{\operatorname{var}(X) = E(X^2) - (E(X))^2}$ Formula

$$E(X^2) = \int_0^1 x^2\left(\frac{3}{2} - x\right) dx$$

$$= \int_0^1 \left\{ \frac{3x^2}{2} - x^3 \right\} dx$$

$$= \left(\frac{3}{2}\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right)_0^1$$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

Formulas:

- $r = \frac{cov(x,y)}{\sqrt{var(x)}\sqrt{var(y)}}$
- $Var(x) = E(x^2) - (E(x))^2$
- $Var(y) = E(y^2) - (E(y))^2$
- $cov(x,y) = E(xy) - E(x)E(y)$

$$Var(x) = \frac{1}{4} - \left(\frac{5}{12}\right)^2$$

$$= \frac{1}{4} - \frac{25}{144}$$

$$= \frac{36 - 25}{144}$$

$$= \frac{11}{144}$$

$$\boxed{Var(y) = E(y^2) - (E(y))^2}$$ Formula

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2$$

$$= \frac{11}{144}$$

Formul
$$\boxed{r = \frac{cov(x,y)}{\sqrt{var(x)}\sqrt{var(y)}}}$$

$$r = \frac{N\sum xy - \sum x \sum y}{\sqrt{N\sum x^2 - (\sum x)^2}\sqrt{N\sum y^2 - (\sum y)^2}}$$

cov(x,y), SD(x), SD(y)

$$= \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = \frac{-\frac{1}{144}}{\frac{\sqrt{11}}{12}\frac{\sqrt{11}}{12}}$$

$$= \frac{-1}{11}$$

---

2. Let $f(x,y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$

Find (i) $E(y)$ gn. x. ie; $E(y|x=x)$

(ii) $E(xy|x=x)$ (iii) $var(y)$ gn. $x=x$ ie; $var(y|x=x)$

Soln:

(i) $E(Y|X=x) = \int_{-\infty}^{+\infty} y \, f_{Y|X} \, dy$

But $f_{Y|X} = \frac{f(x,y)}{f_X(x)}$, $f_X(x) = \int_x^1 8xy \, dy$

$= 8x \left(\frac{y^2}{2}\right)_x^1$

$= 4x(1-x^2) \quad 0<x<1. \quad = \frac{2y}{1-x^2}$

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^2} & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(i) $E(Y|X=x) = \int_x^1 y \cdot \frac{2y}{1-x^2} \, dy$

$= \frac{2}{1-x^2} \left\{\frac{y^3}{3}\right\}_x^1$

$= \frac{2(1-x^3)}{3(1-x^2)}$

$= \frac{2(\cancel{1-x})(1+x+x^2)}{3(\cancel{1-x})(1+x)}$

$= \frac{2(1+x+x^2)}{3(1+x)}$

(ii) $E(XY/X=x) = x \cdot E(Y|X=x)$

$= x \cdot \frac{2}{3} \frac{(1+x+x^2)}{(1+x)}$

$= \frac{2x(1+x+x^2)}{3(1+x)}$

(iii) $\text{var}(Y/X=x) = E(Y^2/X=x) - (E(Y|X=x))^2$

$E(Y^2/X=x) = \int_x^1 y^2 \frac{2y}{1-x^2} \, dy$

$$= \frac{2}{1-x^2}\left\{\frac{y^4}{4_2}\right\}_x^1$$

$$= \frac{1-x^4}{2(1-x^2)}$$

$$= \frac{(1+x^2)\cancel{(1-x^2)}}{2\cancel{(1-x^2)}}$$

$$= \frac{1+x^2}{2}$$

$$var(Y/X=x) = \frac{1+x^2}{2} - \left(\frac{2}{3}\left(\frac{1+x+x^2}{1+x}\right)\right)^2.$$

Moment generating fnc. (mgf) :-

The mgf of a r.v. X about origin is given by $E(e^{tx})$, and is denoted by ~~Mx~~ $M_X(t) = E(e^{tX})$..

$$= \begin{cases} \sum e^{tx} \; P(X=x) & \text{discrete case} \\ \int_{-\infty}^{\infty} e^{tx} \; f(x)\,dx & \text{continuous case} \end{cases}$$

$$M_X(t) = E(e^{tX})$$

$$= E\left\{1 + \frac{tx}{\lfloor 1} + \frac{t^2x^2}{\lfloor 2} + \frac{t^3x^3}{\lfloor 3} + \cdots\right\}$$

$$= E(1) + \frac{t}{\lfloor 1} E(X) + \frac{t^2}{\lfloor 2} E(X^2) + \cdots$$

$$= 1 + \frac{t}{\lfloor 1}\mu_1' + \frac{t^2}{\lfloor 2}\mu_2' + \frac{t^3}{\lfloor 3}\mu_3' + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{\lfloor r} \mu_r'$$

raw moment $\mu_r' = E(x^r)$

$\mu_r = E(x-\mu)^r$ central moment

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{\lfloor r} \mu_r'$$

$$\left\{ \frac{d^r}{dt^r} M_X(t) \right\}_{t=0} = \mu_r'$$

$$\Downarrow$$

$$\boxed{\mu_r' = \frac{d^r}{dt^r} M_X(0)}$$

---

1. Find mgf of the r.v whose $\mu_r' = (r+1)! \cdot 2^r$

Soln:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{\lfloor r} (r+1)!\, 2^r$$

$$= \sum \frac{(r+1)!}{r!} (2t)^r$$

$$= \sum (r+1)(2t)^r$$

$$= \{ 1 + 2(2t)^1 + 3(2t)^2 + \cdots \}$$

$$= (1-2t)^{-2}$$

CIA - I Revision.

1. The mean and variance of a group of 100 items are respectively 15 & 9. The mean and variance of another group of 150 items are each equal to 16. If the two groups are merged, find the mean & variance of combined group.

Soln :

$n_1 = 100,\ n_2 = 150,\ \bar{x}_1 = 15,\ \sigma_1^2 = 9,\ \bar{x}_2 = 16 = \sigma_2^2$

$$\boxed{\text{Combined mean } \bar{X} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}}$$ Formula

$$= \frac{100(15) + 150(16)}{100+150}$$

$$= \frac{1500 + 2400}{250} = \frac{3900}{250} = \frac{390\cancel{0}}{25\cancel{0}}$$

$$= 15.6$$

Combined variance

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}$$

$d_1 = \bar{x}_1 - \bar{x}$ & $d_2 = \bar{x}_2 - \bar{x}$

$= 15 - 15.6 = 0.6$ & $= 16 - 15.6 = 0.4$.

$$\sigma^2 = \frac{100(9) + 150(16) + 100(-0.6)^2 + 150(0.4)^2}{100+150}$$

$$\sigma^2 = \frac{900 + 2400 + 36 + 24}{250} = \frac{336\cancel{0}}{25\cancel{0}} = 13.44$$

$\bar{X}_i$ combined mean $= \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2}$

combined variance $= \dfrac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}$

where $d_1 = \bar{x}_1 - \bar{x}$
$d_2 = \bar{x}_2 - \bar{x}$
$d_i = \bar{x}_i - \bar{x}$

$\bar{x}$ → combined mean.

2. Two players scored the following runs in the last few matches Determine the consistent player & better player → averages.

consistent player → Co-efficient of variation

A :- 14 18 23 32 17

B :- 34 9 17 11 12.

Soln:

$$\bar{X}_A = \frac{14+18+23+32+17}{5} = 20.8$$

$$\bar{X}_B = \frac{34+9+17+11+12}{5} = 16.6.$$

$\bar{X}_A > \bar{X}_B$ → A is the better player.

$$\left.\begin{matrix}\text{Coefficient of the} \\ \text{variation}\end{matrix}\right\} = \frac{\sigma}{\bar{x}} \times 100$$

						Total
X_A =	14	18	23	32	17	104
X_A^2 =	196	324	529	1024	289	2362.

$N_A = 5$, $X_A = 104$, $X_A^2 = 2362$.

$$\sigma_A^2 = \frac{N_A \Sigma x_A^2 - (\Sigma X_A)^2}{\cancel{N_A^2}}$$

$$= \frac{5(2362) - (104)^2}{25}$$

$$= 994/25 = 39.76$$

$$\sigma_A = \sqrt{\frac{994}{25}} = 31.72 \quad \sigma_A = 6.3$$

$$COV(A) = \frac{\sigma_A}{\bar{X}_A} \times 100$$

$$= \frac{\cancel{31.52}\ 6.30}{20.8} \times 100.$$

$COV(A) = 30.28.$

						total
$X_B =$	34	9	17	11	12	83
$X_B^2 =$	1156	81	289	121	144	1791

$N_B = 5$, $\cancel{\sigma_B^2}$ $X_B = 83$, $X_B^2 = 1791$, $N_B^2 = 25$.

$$\sigma_B^2 = \frac{N_B \sum X_B^2 - (\sum X_B)^2}{N_B^2}$$

$$= 5(1791) - (83)^2 / 25$$

$$= 45.45.$$

$$CV(B) = \frac{\sigma_B}{\bar{X}_B} \times 100$$

$$= \frac{\cancel{45.45}\ 9.69}{16.6} \times 100$$

$$= \cancel{273.21}\ 54.76.$$

$CV(A) < CV(B)$

A is conistent player //

---

continous case:

3.

CI	f	m	$d = \frac{m-A}{i}$	fd	fd^2
0-10	7	5	-3	-21	63
10-20	13	15	-2	-26	52
20-30	18	25	-1	-18	18
30-40	23	35 (A)	0	0	0
40-50	16	45	1	16	16
50-60	9	55	2	18	36
	86			-31	185

$$\bar{X} = A + \frac{\sum fd}{\sum f} \times i$$

$= 35 + \left(\frac{-31}{86}\right) \times 100$

$= 35 - \frac{310}{86}$

$= 31.39.$

SD $\rightarrow$ $\sigma^2 = \frac{\sum_i f d^2}{\sum_i f} - \left(\frac{\sum_i f d}{\sum_i f}\right)^2$

$= \frac{185}{86} - \cancel{(31.39)^2} \left(\frac{-31}{86}\right)^2$

$\sigma^2 = 2.15 - 0.1299$

$\sigma^2 = 2.0201.$

30/11/21

1. Is $f(x) = \begin{cases} 2x & 0 < x < 1 \\ 4-2x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$ a pdf. Justify your answer?

Soln:

If $f(x)$ is a pdf then $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Consider, $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 2x\,dx + \int_1^2 (4-2x)\,dx$

$= \cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_0^1 + \left(4x - \frac{\cancel{2}x^2}{\cancel{2}}\right)_1^2$

$= 1 + 8 - 4 - 4 + 1 = 2 \neq 1$

$\therefore f(x)$ is not a pdf.

---

2. Let x and y be two r.v's, then

$f(x,y) = \begin{cases} c(2x+y) & 0 < x < 1;\ 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$

Find (i) c (ii) Marginal pdf of x & Y

(iii) find conditional pdf.

Soln:

(i) wkt $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

$$\int_0^2\int_0^1 c\,(2x+y)\,dx\,dy = 1$$

$$c\int_0^2 \left\{\frac{2x^2}{2} + y(x)\right\}_0^1 dy = 1$$

$$\frac{1}{c} = \int_0^2 (1+y)\,dy$$

$$\frac{1}{c} = \{2+2\} \Rightarrow c = \frac{1}{4}. \Rightarrow \frac{1}{c} = \left(y + \frac{y^2}{2}\right)_0^2$$

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & 0<x<1,\ 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Marginal density of X: $f_X(x)$

$$f_X(x) = \int_0^2 f(x,y)\,dy = \int_0^2 \frac{1}{4}(2x+y)\,dy$$

$$= \frac{1}{4}\left\{2xy + \frac{y^2}{2}\right\}_0^2$$

$$= \frac{1}{4}\{8x+2\} = \frac{1}{2}(4x+1)$$

Marginal density of Y: $f_Y(y)$

$$f_Y(y) = \int_0^1 \frac{1}{4}(2x+y)\,dx$$

$$= \frac{1}{4}\left\{\frac{\cancel{2}x^2}{\cancel{2}} + y(x)\right\}_0^1 = \frac{1}{4}(1+y)$$

$$f_{X/Y} = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{4}(2x+y)}{\frac{1}{4}(1+y)} \quad 0<x<1,\ 0<y<2$$

$$f_{y/x} = \frac{f(x,y)}{f_x(x)} = \frac{\frac{1}{4}(2x+y)}{\frac{1}{2}(1+4x)} = \frac{2x+y}{2(1+4x)}, \quad 0<y<2, \; 0<x<1$$

3. verify whether the gn-distribution is a Joint-PDF.

Y \ X	1	2	3
-1	1/9	1/18	1/18
0	1/18	2/9	3/9
1	1/18	1/18	1/18

soln:-

Y \ X	1	2	3	Total f_Y
-1	1/9	1/18	1/18	4/18
0	1/18	2/9	3/9	11/18
1	1/18	1/18	1/18	3/18
Total f_X	4/18	6/18	8/18	1

Mdf (Y)

Y :	-1	0	1
f_Y :	$\frac{4}{18}$	$\frac{11}{18}$	$\frac{3}{18}$

Mdf (X)

X :	1	2	3
f_X :	$\frac{4}{18}$	$\frac{6}{18}$	$\frac{8}{18}$

$P(X/Y=0)$:	1	2	3
$f_{X/Y}$:	$\frac{1}{18}$	$\frac{2}{9}$	$\frac{3}{9}$

$P(X \le 2, Y \le -1)$

$= P(X=1, Y=-1) + P(X=2, Y=-1)$

$= \frac{1}{9} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6}$

If $b_{xy} = 0.5$, $b_{yx} = -0.5$ find correlation co-eff b/w x & y. $r^2 = b_{xy} b_{yx} = -0.5$

$P(X \le 2, Y \le 0) = P(X=1, Y=-1) + P(X=2, Y=-1) + P(X=1, Y=0) + P(X=2, Y=0)$

$= \frac{1}{6} + \frac{1}{18} + \frac{4}{18} = \frac{8}{18} = \frac{4}{9}.$

⇒ Find the regression lines for the foll. regression.

x: 23 27 24 32 36.

y: 18 19 28 22 10

x	y	x^2	y^2	xy
23	18	529	324	414
27	19	729	361	513
24	28	576	784	612
32	22	1024	484	704
36	10	1296	100	360.
142	97	4154	2053	2663

$$b_{xy} = \frac{N\sum xy - \sum x \sum y}{N\sum y^2 - (\sum y)^2}$$

$$b_{yx} = \frac{N\sum xy - \sum x \sum y}{N\sum x^2 - (\sum x)^2}$$

$$b_{xy} = \frac{5(2663) - (142)(97)}{5(2053) - 97^2} \; ; \; b_{yx} = \frac{\cancel{5(2663)} - 459}{5(4154) -}$$

$$= \frac{-459}{856} = -0.5362 \qquad = -0.7574$$

$b_{xy} = -0.5362$ $\quad \bar{x} = \frac{\sum x}{N} = \frac{142}{5} = 28.4$

$b_{yx} = -0.7574$ $\quad \bar{y} = \frac{\sum y}{N} = \frac{97}{5} = 19.4$

x on y $= (x - \bar{x}) = b_{xy}(y - \bar{y})$

$x - 28.4 = -0.5362(y - 19.4)$

y on x $\quad (y-\bar{y}) = b_{yx}(x-\bar{x})$

$y - 19.4 = -0.7574(x - 28.4)$

Moments: Revision

1. Find the 1st three moments for the gn. data

x: 1 2 3 4

f: 23 27 18 14

Soln.

x	f	fx	d = x − 2.28	fd	fd²	fd³
1	23	23	−1.28	−29.44	37.68	−48.23
2	27	54	−0.28	−7.56	2.11	−0.592
3	18	54	0.72	12.96	9.33	6.71
4	14	56	1.72	24.08	41.41	71.23
	82	187		0.04	90.53	29.11

$$\bar{X} = \frac{\sum fx}{\sum f} = \frac{187}{82} = 2.28$$

$$\mu_1 = \frac{\sum fd}{\sum f} = \frac{0.04}{82}$$

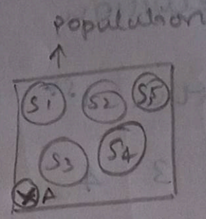
$$\mu_2 = \frac{\sum fd^2}{\sum f} = \frac{90.53}{82}$$

$$\mu_3 = \frac{\sum fd^3}{\sum f} = \frac{29.11}{82}$$

06/12/21

UNIT-IV

SAMPLING TECHNIQUES:-



sampling distributions:

(x) A → conclude whether A has come from this population.

Random sampling:

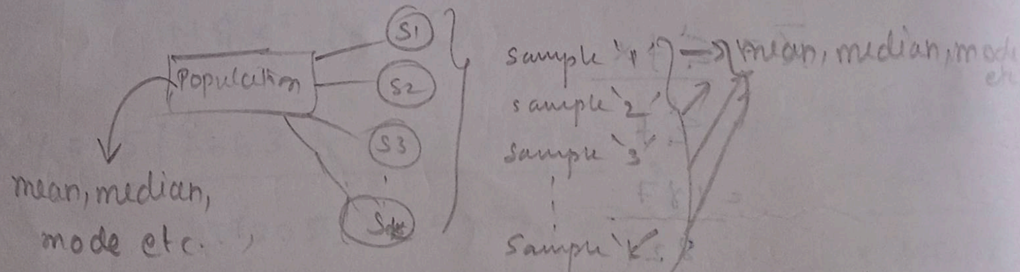
stratified sampling.

Population:

A statistical population is the set of all possible measurements on data corresponding to the entire collection of units for which an inference is to be made

Sample:

A sample is a part of the statistical population



Note:

* Measure of the population is parameter
* " " " samples is statistic.

Conclusion.

- Mean of the population = Mean of the samplespace

07/12/21

Parameter & statistic:-

Mean, mode, median, etc...., are some characterizations of a statistical distribution. These