UNIT-2 CALCULUS

30/01/22

$$u(x,y) = x^2y + y^2x - 2x + 3y.$$

$$\frac{\partial u}{\partial x} = 2xy + y^2 \cdot (1) - 2 + 0.$$

$$\frac{\partial u}{\partial y} = x^2 \cdot (1) + 2y \cdot x - 0 + 3.$$

Double integrals:

$$\int_{x=0}^{2} \int_{y=1}^{4} x^2 y \, dy \, dx.$$

outer, inner: $\iint f(x,y) \, dx \, dy.$

① the limits of the order integral must be constant.

② The order of the integration is decided based on the limits.

$$\int_{x=0}^{2} \int_{y=3}^{5} xy \, dy \, dx \qquad \int_{0}^{2} \int_{3}^{5} xy \, dy \, dx$$

$$\int_{0}^{1} \int_{-y}^{2+3y} x^2 y \, dx \, dy$$

③ If the limit is a function of y then the integral is w.r.t. x. If the limit is a function of x then the integral is w.r.t. y.

$$\int_{-1}^{1} \int_{x}^{3x} xy \, dx \, dy \qquad \int_{-1}^{1} \int_{1}^{3} f(x,y) \, dx \, dy$$

④ If all the limits are constants then the region of integration is a rectangular region.

⑤ If we want to integrate first w.r.t. x and then w.r.t. y then we have to take

HORIZONTAL strip.

(6) If we want to integrate first w.r.t. y and w.r.t. x. then we have to take VERTICAL strip

(7) only one strip should be taken.

(8)

strip	lower	upper
Hor	Left	Right
vert	Bottom	Top

(9) when Horizontal strip is taken. the bottom most point is the lower limit for y and the top most point is the upper limit for y w.r.t x axis

(10) when vertical strip is taken the left most point is the lower limit for x and the right most point is the upper limit for x. y axis w.r.t

Type: I

Evaluation of Double integrals when the limits are given explicitly

(1) Evaluate $\int_{x=0}^{2}\int_{y=1}^{3} xy\, dy\, dx$

Soln:

$$\text{Let } I = \int_{x=0}^{2}\int_{y=1}^{3} xy\, dy\, dx$$

$$= \int_{x=0}^{2}\left[\int_{y=1}^{3} x y\, dy\right] dx$$

$$= \int_{x=0}^{2} \left(\frac{xy^2}{2}\right)_1^3 dx$$

$$= \int_{x=0}^{2} \left[\frac{x \cdot 3^2}{2} - \frac{x \cdot 1^2}{2}\right] dx$$

$$= \int_{x=0}^{2} \left[\frac{9x}{2} - \frac{x}{2}\right] dx$$

$$= \int_{x=0}^{2} \frac{\cancel{8}^{4} x}{\cancel{2}} dx$$

$$= \int_{x=0}^{2} 4x \, dx.$$

$$= 4\left[\frac{x^2}{2}\right]_0^2$$

$$= 4\left[\frac{4}{2} - 0\right]$$

$$= \frac{16}{2}$$

$$\boxed{I = 8}$$

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2. Evaluate $\int_0^1 \int_0^x xy \, dy \, dx$.

Soln.

$$\text{let } I = \int_0^1 \int_0^x xy \, dy \, dx$$

$$= \int_0^1 \left[\int_0^x xy \, dy\right] dx$$

$$= \int_0^1 \left[\frac{x \cdot y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left[x \cdot \frac{x^2}{2} - 0\right] dx$$

$$= \int_0^1 \left[\frac{x^3}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 x^3\,dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{2}\left(\frac{1}{4} - 0\right)$$

$$\boxed{I = \frac{1}{8}}$$

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3. Evaluate $\int_{-1}^{1}\int_{2}^{3} x^2y^2\,dx\,dy$.

Soln: Let $I = \int_{-1}^{1}\int_{2}^{3} x^2y^2\,dx\,dy$.

$$\int_{-1}^{1}\left[\int_2^3 x^2y^2\,dx\right]dy$$

$$= \int_{-1}^{1} \frac{x^3}{3}y^2\Big|_2^3\,dy$$

$$= \int_{-1}^{1}\left(\frac{3^3}{3}y^2 - \frac{2^3}{3}y^1\right)dy$$

$$= \int_{-1}^{1}\left[9y^2 - \frac{8}{3}y^1\right]dy$$

$$= \int_{-1}^{1}\frac{19}{3}y^2\,dy$$

$$= \frac{19}{3}\left(\frac{y^3}{3}\right)_{-1}^{1}$$

$$= \frac{19}{3}\left(\frac{1}{3} - \frac{(-1)^3}{3}\right)$$

$$= \frac{19}{3} \times \frac{2}{3} = \frac{38}{9}$$

4. Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$.

Soln:

Let $I = \int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$.

$= \int_0^{\pi/2}\left[\int_0^{\sin\theta} r\,dr\right]d\theta$

$= \int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{\sin\theta}$

$= \int_0^{\pi/2}\left[\frac{\sin^2\theta}{2} - 0\right]d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,d\theta$ $\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{\pi}{8}$

Reduction formula:

$$\int_0^{\pi/2} \sin^n\theta\,d\theta = \int_0^{\pi/2} \cos^n\theta\,d\theta = \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \frac{1}{2} \times \frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \frac{2}{3} \times 1 & \text{if } n \text{ is odd} \end{cases}$$

Type: II

Evaluation of double integrals when the limits are not given explicity.

1. Evaluate $\iint_R (x+y)\,dx\,dy$

where R is the region bounded by $x=0$, $y=0$, and $x+y=1$.

Soln:

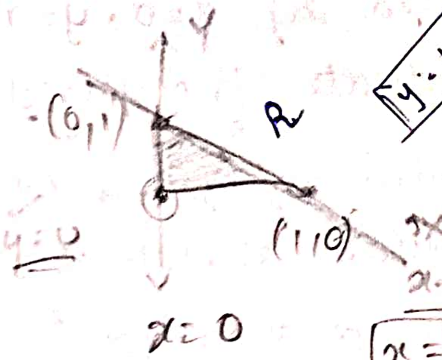
Let $I = \iint_R (x+y)\,dx\,dy$

$y_0, x=1$ $x=0$ $0+y=1$. $(0,1)$ $(1,0)$

where R is the region bounded by $x=0, y=0$ and $x+y=1$

The region of integration is shown in the foll. figure.

$\boxed{y = 1 - x}$



$x+y=1$

$\boxed{x = 1-y}$

x varies from 0 to $1-y$

y varies from 0 to 1.

$$I = \int_{y=0}^{1} \int_{x=0}^{1-y} (x+y)\,dx\,dy$$

$x\,dx + x\,dy$ $\frac{x^2}{2} + \frac{x^2}{2}y$ $\int dx = x$

$$= \int_{y=0}^{1} \left[\frac{x^2}{2} + xy\right]_0^{1-y} dy$$

$$= \int_{y=0}^{1} \left[\frac{(1-y)^2}{2} + y(1-y)\right] dy.$$

$$= \int_{y=0}^{1} \left[\frac{1-2y+y^2}{2} + y - y^2\right] dy$$

$$= \frac{1}{2}\int_{y=0}^{1} \left[1 - \cancel{2y} + y^2 + \cancel{2y} - 2y^2\right] dy$$

$$= \frac{1}{2}\int_{y=0}^{1} (1-y^2)\,dy = \frac{1}{2}\left[y - \frac{y^3}{3}\right]_0^1$$

$$I = \frac{1}{2}\left[1 - \frac{1}{3}\right] = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

H.W:-

1. Evaluate $\int_{x=1}^{2}\int_{y=2}^{4} xy\,dy\,dx$.

2. Evaluate $\int_0^1 \int_0^2 (x+y)\, dx\, dy$

3. Evaluate $\int_0^3 \int_x^{1-x} xy\, dy\, dx$

4. Evaluate $\iint_R xy\, dx\, dy$ where R is the region bounded by $y=0$ and $y=3$ & $x=y$, $x=0$.

5. Evaulate $\iint_R xy\, dx\, dy$ where R is the region bounded by $x=0$, $y=x^2$ and $x+y=1$.

6. Evaluate $\iint_R xy\, dx\, dy$ where R is the region bounded by $y=x$, $x+y=2$ and $y=0$.

7. ~~Evaluate~~ sketch the region of integration for integration for $\int_{x=0}^{4} \int_{y=0}^{9} xy\, dx\, dy$ and $\int_{x=1}^{3} \int_{y=1}^{3} xy\, dx\, dy$, Also make your comments

8. Evaluate $\iint_R \frac{e^{-y}}{y}\, dx\, dy$ where R is the region bounded by $x=0$, $x=y$ and $y=\infty$.

9. Evaluate $\iint_R xy\, dx\, dy$ where R is bounded by $y=x^2$ and $x=y^2$.

Answers:-

1. $\int_{x=1}^{2} \int_{y=2}^{4} xy\, dy\, dx$

$= \int_1^2 \left\{ x \int_2^4 y\, dy \right\} dx$

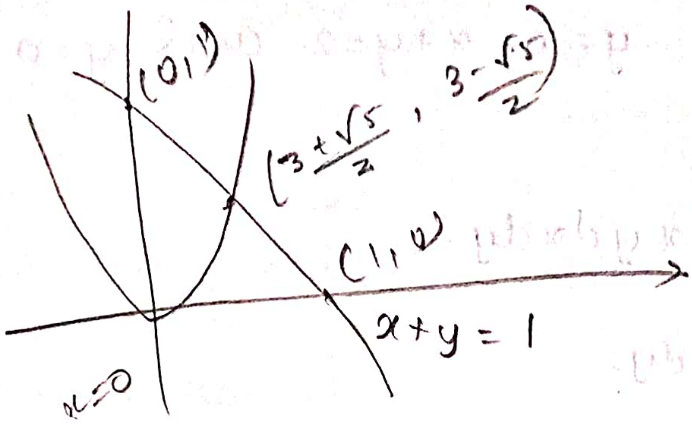
$= \int_1^2 \left\{ \frac{x}{2} (y^2)_2^4 \right\} dx$

$= \int_1^2 (6x)\, dx$

$= \frac{6}{2} (x^2)_1^2$

$\boxed{I = 9}$

5.



$y = x^2$

$x + y = 1$

$x + x^2 = 1$

$x^2 + x - 1 = 0$

$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt5}{2}, \frac{-1-\sqrt5}{2}$

+ve, -ve

$\frac{-1+\sqrt5}{2} + y = 1$

$y = 1 - \left(\frac{-1+\sqrt5}{2}\right)$

$= \frac{2+1-\sqrt5}{2}$

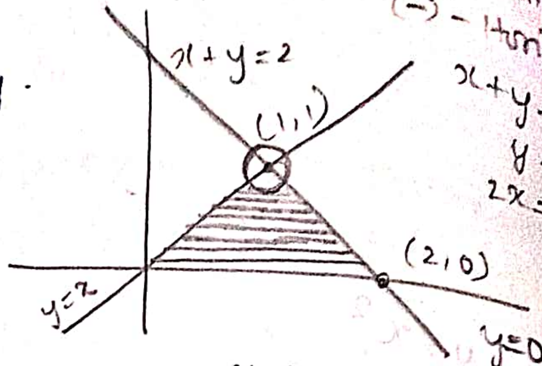
$= \frac{3-\sqrt5}{2}$

6.

6. $\iint_R xy\,dx\,dy$ $\quad y=x,\ x+y=2$ and $y=0$

(|) vertical
(-) - horizontal

Soln :-



$x+y=2$
$y=x$
$2x=2$
$x=1$
$y=1$

Let $I = \iint_R xy\,dx\,dy$.

$$I = \int_0^1 \int_y^{2-y} xy\,dx\,dy.$$

$x \to y$ to $2-y$

$y \to 0$ to 1

$$= \int_0^1 \left[\frac{x^2 y}{2}\right]_y^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 \left[y(2-y)^2 - y^3\right] dy$$

$$= \frac{1}{2}\int_0^1 \left[y\cancel{(4-y)}^2(4-4y+y^2) - y^3\right] dy$$

$$= \frac{1}{2}\int_0^1 \left[4y-4y^2+\cancel{y^3}-\cancel{y^3}\right] dy$$

$$= \frac{1}{2}\int_0^1 \left[4y-4y^2\right] dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2} - \frac{4}{3}y^3\right]_0^1$$

$$= \frac{1}{2}\left[\left(\frac{4}{2}-\frac{4}{3}\right)-(0)\right]$$

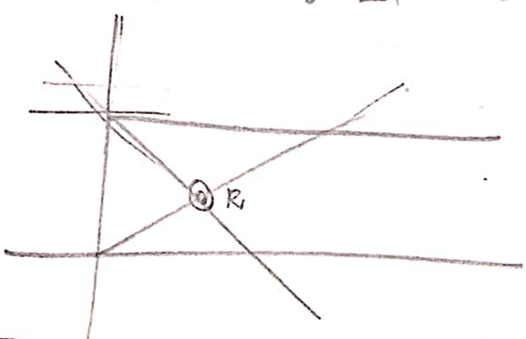
$$= \frac{1}{2}\left[2-\frac{4}{3}\right]$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right)$$

$$\boxed{I = \frac{1}{3}}$$

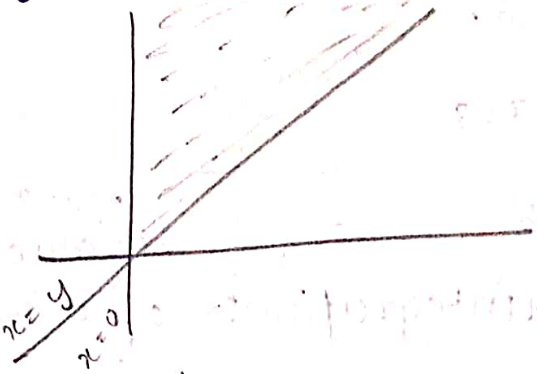
$x+y=2$, $y=x$, $y=2$, $x=0$

Sample



HORIZONTAL STRIP

8. $\iint_R \frac{e^{-y}}{y}\, dx\, dy$ $x=0,\ x=y$ and $y=\infty$.



x varies from 0 to y

y varies from 0 to ∞

$$I = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\, dx\, dy.$$

$$= \int_0^{\infty} \frac{e^{-y}}{y}\, x\Big|_0^{y} dy$$

$$= \int_0^{\infty}\left[\frac{e^{-y}}{y}\, y - (0)\right] dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{\cancel{y}}\, \cancel{y}\, dy$$

$$= \int_0^{\infty} e^{-y}\, dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= \left[-e^{-y}\right]_0^{\infty} = \left[-e^{-\infty} - (-e^{0})\right]$$

$$= [0+1] = 1.$$

$$= 1$$

$\boxed{e^{-\infty} = 0}$

$= \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$\boxed{e^{0} = 1.}$

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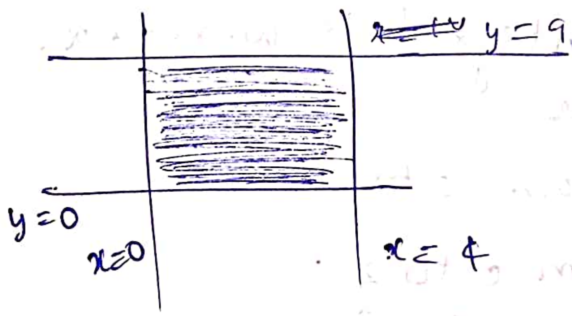
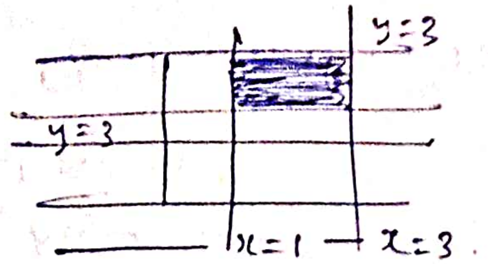
7. $\int_{x=0}^{4}\int_{y=0}^{9} xy\, dx\, dy$ $\int_{x=1}^{3}\int_{y=1}^{3} xy\, dx\, dy$

(iv) point

(i)



(iii)



Type III :- (X) Part a, b

Type I
Evaluation of double integrals when the limits are gn. ~~explicitly~~ limits

Type II :-
Evaluation of double Integrals when the ~~limits~~ gn region is gn.

change of order of integration:-

* Like in part A they will say change of order of integration in
* In part B change the order of integration in and hence evaluate
* If V want $\int$ w.r.t 'x' take "HORIZONTAL STRIP" $\int$ w.r.t 'y' take VERTICAL STRIP

Step 1:- Write down the gn. order

Step 2:- change the order (How to change means if the gn. order is xy change to yx vice versa)

Step 3:- Find the region of $\int$

Step 4:- choose the strip according to new order

(1) change the order of integration in $\int_{x=0}^{2}\int_{y=0}^{x}(x+y)\,dy\,dx$. and hence evaluate it.

Soln:

Let $I = \int_{x=0}^{2}\int_{y=0}^{x}(x+y)\,dy\,dx$

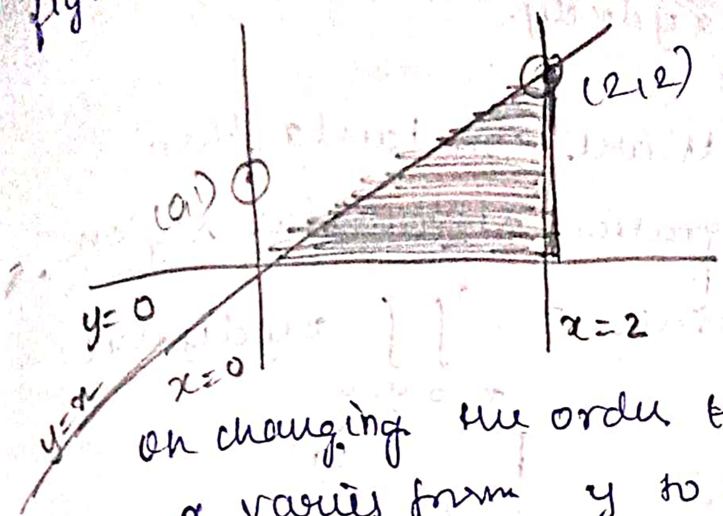
Given order is inner integral is w.r.t. y and ~~order~~ outer integral is w.r.t. x.

Given limits are

y varies from 0 to x

x varies from 0 to 2.

The region of integration is show in the follo figure.



$x=0, y=1$

$0 < y < x$

$\boxed{\cancel{0 < 1 < 0}}$

HORIZONTAL STRIP

On changing the order the new limits are

x varies from y to 2

y varies from 0 to 2.

$x=2, y=x$

$$I = \int_{y=0}^{2} \int_{x=y}^{2} (x+y)\, dx\, dy.$$

$\int x\,dx + y\,dx$

$\frac{x^2}{2} \quad xy$

$$I = \int_{y=0}^{2} \left(\frac{x^2}{2} + xy\right)_{x=y}^{2} dy.$$

$$I = \int_{y=0}^{2} \left[[2 + 2y] - \left(\frac{y^2}{2} + y^2\right)\right] dy$$

$$I = \int_{y=0}^{2} \left(2+2y - \frac{3}{2}y^2\right) dy$$

$$I = \int_{y=0}^{2} \left(2y + y^2 - \frac{y^3}{2}\right) dy$$

$$I = \left[y^2 + \frac{y^3}{3} - \frac{y^4}{8}\right]_0^2$$

$$I = \left[\left(4 + \frac{8}{3} - \frac{16}{2}\right) - (0)\right]$$

$$\boxed{I = 2}$$

2. Change the order of integration in $\int_{y=0}^{2} \int_{x=0}^{3} xy\, dx\, dy$ and hence evaluate it.

Soln: Let $I = \int_{y=0}^{2}\int_{x=0}^{3} x y \, dx \, dy.$

If all the elements are constants then the region of integration rectangular region.

On changing the order $I = \int_{x=0}^{3}\int_{y=0}^{2} xy \, dy \, dx.$

$$= \int_{x=0}^{3} \frac{x y^2}{2}\Big|_0^3 \, dx$$

$$= \frac{1}{2}\int_{x=0}^{3} [(2x) - (0)] \, dx$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \int_0^3 x \, dx.$$

$$\Rightarrow 3 \; x\Big|_0^3$$

$$= 3 - 0$$

$$\boxed{\therefore I = 3}$$

(or)

$\int_{y=0}^{2}\int_{x=0}^{3} x y \, dx \, dy$

$\int_{y=0}^{2}\int [x.$

x varies from 0 to 3

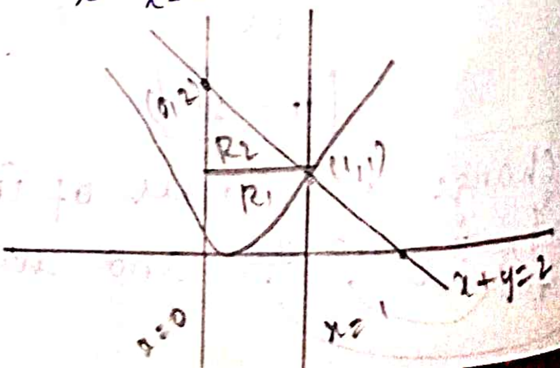
y varies from 0 to 2

3. Change the order of integration in $\int_{x=0}^{2}\int_{x^2}^{2-x} x y \, dy dx$ and hence evaluate.

Soln: Let $I = \int_{x=0}^{2}\int_{x^2}^{2-x} x y \, dy \, dx$ — y varies from x^2 to $2-x$; x " " " 0 to 2.

~~On changing the order~~ $I = \int_{x^2}^{2-x}\int_{x=0}^{2} x y \, dx \, dy.$

$$= \int_{x^2}^{2-x} \frac{x y^2}{2}\Big|_0^2 \, dx$$



On changing the order, the new limits are

<u>On R1</u> x v from 0 to $\sqrt{y}$ $x = \pm\sqrt{y}$

y v from 0 to 1 VERTICAL

<u>On R2</u> x v from 0 to $2-y$ HORIZONTAL

y v from 1 to 2.

$I = I_1 + I_2$

$$I_1 = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy$$

$$= \int_{y=0}^{1}\left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy$$

$$= \frac{1}{2}\int_{y=0}^{1}(y^2 - 0)\,dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y^2\,dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3} - 0\right) = \frac{1}{6}.$$

$$\boxed{I_1 = \frac{1}{6}}$$

$$I_2 = \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy.$$

$$= \int_{y=1}^{2}\left[\frac{x^2}{2}y\right]_0^{2-y} dy.$$

$$= \frac{1}{2}\int_{y=1}^{2}\left[y(2-y)^2 - (0)\right] dy$$

$$= \frac{1}{2}\int_{y=1}^{2} y\,(4-4y+y^2)\,dy$$

$$= \frac{1}{2}\int_{y=1}^{2}(4y - 4y^2 + y^3)\,dy$$

$$= \frac{1}{2}\left(\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right)_1^2$$

$$= \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{2}\left[12 - \frac{32}{3} - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{120 - 112 - 3}{12}\right]$$

$$= \frac{1}{2}\left(\frac{5}{12}\right) = \frac{5}{24}$$

$$\boxed{I_2 = 5/24}$$

01/02/22

Type IV: Evaluation of $\iiint$ integrals when the limits are gn.

1. Evaluate $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} xyz \, dz\, dy\, dx$

Soln:

Let $I = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} xyz \, dz\, dy\, dx.$

$$= \int_{x=0}^{1}\int_{y=0}^{2} \frac{xyz^2}{2}\Big|_0^3 \, dy\, dx.$$

$$= \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{2} (9xy - 0)\, dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{2} 9xy \, dy\, dx$$

$$= \frac{9}{2}\int_{x=0}^{1} \left\{\frac{xy^2}{2}\right|_0^2 dx$$

$$= \frac{9}{4}\int_{x=0}^{1} \frac{xy^2}{2}\Big|_0^2 dx$$

$$= \frac{9}{4} \int_{x=0}^{1} (4x - 0)\, dx$$

$$= \frac{9}{4} \int_{x=0}^{1} (4x)\, dx$$

$$= \frac{9}{4} \times 4 \int_{x=0}^{1} x\, dx$$

$$= \frac{9}{\cancel{4}} \times \cancel{4} \int_{x=0}^{1} \left(\frac{x^2}{2}\right)$$

$$= 9 \left(\frac{x^2}{2}\right)_0^1$$

$$= 9 \left(\frac{1}{2} - 0\right)$$

$$\boxed{I = \frac{9}{2}}$$

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2. Evaluate $\int_0^2 \int_0^1 \int_0^4 (x+y+z)\, dx\, dy\, dz$.

<u>soln</u>: Let $I = \int_0^2 \int_0^1 \int_0^4 (x+y+z)\, dx\, dy\, dz$

$$= \int_0^2 \int_0^1 \left(\frac{x^2}{2} + xy + xz\right)_0^4 dy\, dz$$

$$= \int_0^2 \int_0^1 \left[(8 + 4y + 4z) - 0\right] dy\, dz$$

$$= \int_0^2 \int_0^1 (8 + 4y + 4z)\, dy\, dz$$

$$= \int_0^2 \left(8y + \frac{\overset{2}{\cancel{4}}y^2}{\cancel{2}} + \underline{4yz}\right)_0^1 dz$$

$$= \int_0^2 \left((8 + 2(1) + 4(1)z) - (0)\right) dz$$

$$= \int_0^2 (8 + 2 + 4z)\, dz$$

$$= \int_0^2 (10+4z)\, dz$$

$$= \left(10z + \frac{4}{2} z^2\right)_0^2$$

$$= \left[20 + \frac{\cancel{4}}{\cancel{2}} (2)^2 - (0)\right]$$

$$= \left[20 + 2(4) - (0)\right]$$

$$= 20 + 8$$

$$\boxed{I = 28}.$$

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3. Evaluate $\int_0^1 \int_0^y \int_0^{x+y} (x^2 y z)\, dx\, dy\, dz$.

Soln:

Let $I = \int_0^1 \int_0^y \int_0^{x+y} (x^2 y z)\, dx\, dy\, dz$

$$= \int_0^1 \int_0^y \frac{x^2 y z^2}{2} \Big|_0^{x+y} dx\, dy.$$

$$= \frac{1}{2} \int_0^1 \int_0^y \left[\left(x^2 y (x+y)^2\right) - 0\right] dx\, dy.$$

$$= \frac{1}{2} \int_0^1 \int_0^y \left[x^4 y + 2x^3 y^2 + x^2 y^3\right] dx\, dy$$

$$= \frac{1}{2} \int_0^1 \left[\frac{x^5 y}{5} + \frac{2x^4 y^2}{4} + \frac{x^3 y^3}{3}\right]_0^y dy.$$

$$= \frac{1}{2} \int_0^1 \left[\left(\frac{y^6}{5} + \frac{y^6}{2} + \frac{y^6}{3}\right) - (0)\right] dy$$

$$= \frac{1}{2} \int_0^1 y^6 \left(\frac{6+15+10}{30}\right) dy$$

$$= \frac{31}{60} \int_0^1 y^6 \, dy = \frac{31}{60} \left(\frac{y^7}{7} \right)_0^1$$

$$= \frac{31}{60} \left(\frac{1}{7} - 0 \right)$$

$$\boxed{I = \frac{31}{420}}$$

4. Evaluate $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dx \, dy \, dz$

Soln:

$$\text{Let } I = \int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{x+y} e^{z} \right]_0^{x+y} dy \, dx.$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{x+y} e^{x+y} - e^{x+y} e^{0} \right] dy \, dx.$$

$$= \int_0^{\log 2} \int_0^{x} \left[e^{2x+2y} - e^{x+y} \right] dy \, dx.$$

$$= \int_0^{\log 2} \left[\frac{e^{2x+2y}}{2} - e^{x+y} \right]_0^{x} dx$$

$$= \int_0^{\log 2} \left[\left(\frac{e^{2x+2x}}{2} - e^{x+x} \right) - \left(\frac{e^{2x}}{2} - e^{x} \right) \right] dx$$

$$= \int_0^{\log 2} \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x} \right] dx$$

$$= \int_0^{\log 2} \frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \Big] dx$$

$$= \frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x} \Big|_0^{\log 2}$$

$$= \left(\frac{e^{4\log 2}}{8} - \frac{3}{4} e^{2\log 2} + e^{\log 2}\right)$$

$$- \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$= \left(\frac{e^{\log 2^4}}{8} - \frac{3}{4} e^{\log 2^2} + e^{\log 2}\right) - \left(\frac{3}{8}\right)$$

$$= \frac{16}{8} - \frac{3}{4} \times 4 + 2 - \frac{3}{8}$$

$$= 2 - 3 + 2 - \frac{3}{8}$$

$$= 1 - \frac{3}{8}$$

$$\boxed{I = 5/8}$$

---

5. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz \, dz \, dy \, dz$

Soln:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz \, dz \, dy \, d\cancel{z}x$$

$$= \int_0^1 \int_0^{1-x} \left. \frac{xyz^2}{2} \right|_0^{1-x-y} dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[xy(1-x-y)^2 - 0\right] dy \, dx.$$

$$= \cancel{\frac{1}{2} \int_0^1 \int_0^{1-x} \left[xy - x^2y - xy^2\right] dy \, dx.}$$

$$= \cancel{\frac{1}{2} \int_0^1 \left. \frac{xy^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3} \right|_0^{1-x} dx}$$

$$= \cancel{\frac{1}{2}\int_0^1 \frac{x}{2}(1-x)^2 - \frac{x^2}{2}(1-x)^2 - \frac{x}{3}(1-x)^3} - 0 \Big] dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x} xy\left[1+x^2+y^2-2x-2y+2xy\right] dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x} \left[xy + x^3y + xy^3 - 2x^2y - 2xy^2 + 2x^2y^2\right] dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{xy^2}{2} + \frac{x^3y^2}{2} + \frac{xy^4}{4} - \frac{2x^2y^2}{2} - \frac{2xy^3}{3} + \frac{2x^2y^3}{3}\right]_0^{1-x}$$

$$= \frac{1}{2}\int_0^1 \frac{x}{2}(1-x)^2 + \frac{x^3}{3}(1-x)^2 + \frac{x}{4}(1-x)^4 - x^2(1-x)^2 - \frac{2}{3}x(1-x)^3 + \frac{2}{3}x^2(1-x)^3\Big] dx$$

$$= \frac{1}{2}\int_0^1 \left\{(1-x)^2\left[\frac{x}{2} + \frac{x^3}{2} - x^2\right] + (1-x)^3\left[\frac{2}{3}x^2 - \frac{2}{3}x\right] + \frac{x}{4}(1-x)^4\right\} dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\left[\frac{x}{2} - x^2 + \frac{x^3}{2} + \frac{2}{3}(1-x)(x^2-x) + \frac{(1-x)^2 x}{4}\right] dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\left[\frac{x}{2} - x^2 + \frac{x^3}{2} + \frac{2}{3}\left(x^2 - x - x^3 + x^2\right) + \frac{x}{4}(1-2x+x^2)\right] dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\left[\frac{x}{2} - x^2 + \frac{x^3}{2} + \frac{4x^2}{3} - \frac{2}{3}x - \frac{2}{3}x^3 + \frac{x}{4}\right.$$

$$= \frac{x^2}{2} + \frac{x^3}{4} \Big] dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2 \left[\frac{x}{12} - \frac{x^2}{6} + \frac{x^3}{12}\right] dx$$

$$= \frac{1}{24}\int_0^1 \cancel{(1-x)^2}\,(1-2x+x^2)(x-2x^2+x^3)\,dx$$

$$= \frac{1}{24}\int_0^1 (x-2x^2+x^3-2x^2+4x^3-2x^4+x^3-2x^4+x^5)\,dx$$

$$= \frac{1}{24}\int_0^1 (x^5-4x^4+6x^3-4x^2+x)\,dx.$$

$$= \frac{1}{24}\left[\frac{x^6}{6} - \frac{4}{5}x^5 + \frac{6}{4}x^4 - \frac{4}{3}x^3 + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{24}\left[\frac{1}{6} - \frac{4}{5} + \frac{3}{2} - \frac{4}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{24}\left[\frac{5-24+45-40+15}{30}\right]$$

$$= \frac{1}{24}\left[\frac{1}{30}\right]$$

$$\boxed{I = \frac{1}{720}}$$

Type V: Evaluation of $\iiint$ integrals when the limits are not given explicitly.

1. Evaluate $\iiint_V dx\,dy\,dz$ where V is a finite region of space bounded by a planes ~~planes~~ $x=0$, $y=0$, $z=0$ and $2x+3y+4z=12$.

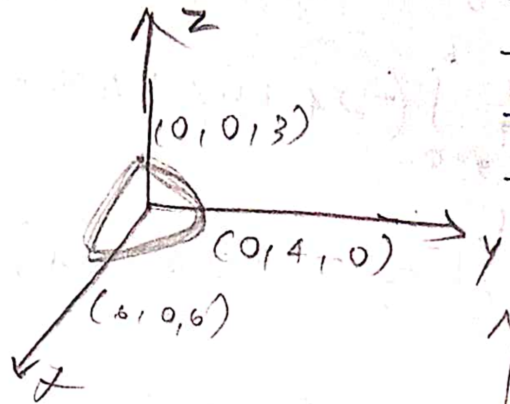
plane.

Soln.

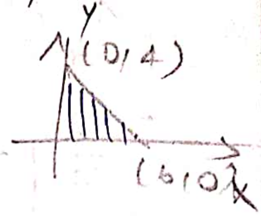
Let $I = \iiint_V dx\,dy\,dz$, where V is the region of space

$x=0$, $y=0$, $z=0$ and $2x+3y+4z=12$

x	y	z	
+	+	+	(+)ve octant
+	+	−	
+	−	+	
+	−	−	
−	+	+	
−	+	−	
−	−	+	
−	−	−	(−)ve octant



x	y	
+	+	(I)
−	−	(III)
+	−	(IV)
−	+	(II)



The limits are z varies from 0 to $\frac{1}{4}(12-2x-3y)$

y varies from 0 to $\frac{1}{3}(12-2x)$

x varies from 0 to 6.

$$I = \int_{x=0}^{6}\int_{y=0}^{1/3(12-2x)}\int_{z=0}^{1/4(12-2x-3y)} dz\,dy\,dx$$

$$I = \int_{x=0}^{6}\int_{y=0}^{1/3(12-2x)} [z]_0^{1/4(12-2x-3y)}$$

$$= \int_{x=0}^{6}\int_{y=0}^{1/3(12-2x)} (\tfrac{1}{4}(12-2x-3y)]\,dy\,dx$$

$$= \frac{1}{4}\int_{x=0}^{6} [12y-2xy-\frac{3}{2}y^2]_0^{1/3(12-2x)} dx$$

$$= \frac{1}{4}\int_{x=0}^{6} (4(12-2x)-\frac{2}{3}x(12-2x)-\frac{1}{2}(12-2x)]\,dx$$

$$= \frac{1}{4}\int_{0}^{6}(12-2x)[4-\frac{2}{3}x-\frac{1}{2}(12-2x)]\,dx$$

$$= \frac{1}{4}\int_{x=0}^{6}(12-2x)[4-\frac{2}{3}x-6+x]\,dx$$

$$= \frac{1}{4}\int_{x=0}^{6} (12-2x)\left[-2+\frac{x}{3}\right] dx$$

$$= \frac{1}{4}\int_{x=0}^{6}\left(-24+4x+4x-\frac{2}{3}x^2\right] dx$$

$$= \frac{1}{4}\int_{x=0}^{6}\left(-\frac{2}{3}x^2+8x-24\right) dx$$

$$= \frac{1}{4}\left[-\frac{2}{9}x^3+4x^2-24x\right]_0^6$$

$$= \frac{1}{4}\left[\left(-\frac{2}{9}\times 6^3+4\times 6^2-24\times 6\right)-(0)\right]$$

$$= \frac{1}{4}\left[-48+\cancel{144}-\cancel{144}\right]$$

$$= \frac{1}{4}\left[-48\right]$$

$$= \frac{1}{\cancel{4}}\left[-\overset{12}{\cancel{48}}\right]$$

$$\boxed{I = -12}$$

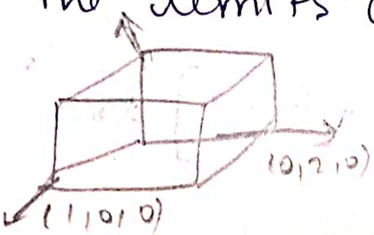
---

2. Evaluate $\iiint_V xyz\,dx\,dy\,dz$ where V is the region of space bounded by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$.

Soln:

Let $I = \iiint_V xyz\,dx\,dy\,dz$ where V is the region of space bounded by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$.

The limits are z varies from 0 to 3



y varies from 0 to 2

x varies from 0 to 1

$$I = \int_0^1 \int_0^2 \int_0^3 xyz \, dz \, dy \, dx.$$

$$= \int_{x=0}^{1} \int_{y=0}^{2} \frac{xyz^2}{2} \Big|_0^3 \, dy \, dx.$$

$$= \frac{9}{2} \int_{x=0} \int_{y=0}^{2} xy \, dy \, dx = \frac{9}{2} \int_{x=0}^{1} \frac{xy^2}{2} \Big|_0^2 dx$$

$$= \frac{9}{4} \int_{x=0}^{1} \cancel{4}\, 4x \, dx = 9 \int_{x=0}^{1} dx$$

$$= 9 \left[\frac{x^2}{2} \right]_0^1 = \frac{9}{2}$$

$$\boxed{I = \frac{9}{2}}$$

04/02/21

Limits:- $f(x)$ at $x = a$

$$\lim_{x \to a^-} f(x) = l_1 \text{ (say)} \quad \text{LHS limit}$$

$$\lim_{x \to a^+} f(x) = l_2 \text{ (say)} \quad \text{RHS limit}$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

$$l_1 = l_2 = l \text{ (say)}.$$

$$\boxed{\lim_{x \to a} f(x) = l}$$

Note:

1) If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$

then

(i) $\lim\limits_{x \to a} [f(x) \pm g(x)] = l \pm m$

(ii) $\lim\limits_{x \to a} [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)] =$

$$c_1 \lim_{x \to a} f_1(x) + c_2 \lim_{x \to a} f_2(x) + \dots$$

$$c_n \lim_{x \to a} f(x).$$

where $c_1, c_2, \dots c_n$ are constants.

(iii) $\lim_{x \to a} [f(x) \cdot g(x)] = lm$

(iv) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}$, provided $m \neq 0$.

Results:-

① $\lim_{x \to a} \left[\frac{x^n - a^n}{x - a}\right] = na^{n-1}$, for all rational values of n and $a \neq 0$.

② $\lim_{\theta \to 0} \left[\frac{\sin\theta}{\theta}\right] = 1$

③ $\lim_{x \to \infty} \left[1 + \frac{1}{x}\right]^x = e$ $(2 < e < 3)$.

④ $\lim_{x \to 0} (1+x)^{1/x} = e$

⑤ $\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = 1$

⑥ $\lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log_e a$

⑦ $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a.$

---

1. Find $\lim_{x \to 2} \left[\frac{x^4 - 2^4}{x - 2}\right]$

W.K.T; $\lim_{x \to a} \left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}.$

$a = 2, n = 4$

$\lim_{x \to 2} \left[\frac{2^4 - 2^4}{2 - 2}\right] = 4 \times 2^{4-1}$

$\frac{16 - 16}{4} = 0$

$= 4 \times 2^3$

$= 32$ //

2. Find $\lim_{n \to \infty} \left[\frac{2^{n+1} + 3^{n+1}}{2^n - 3^n}\right]$.

Soln:- $\lim_{n \to \infty} \left[\frac{2^{n+1} + 3^{n+1}}{2^n - 3^n}\right] = \lim_{n \to \infty} \left(\frac{2 \cdot 2^n + 3 \cdot 3^n}{2^n - 3^n}\right)$

$= \lim_{n \to \infty} \left[\frac{\cancel{3^n}\left[2\left(\frac{2}{3}\right)^n + 3\right]}{\cancel{3^n}\left[(2/3)^n - 1\right]}\right] = \lim_{n \to \infty} \left[\frac{\left[2(2/3)^n + 3\right]}{(2/3)^n - 1}\right]$

$\therefore \lim_{n \to \infty} \left[\frac{2^{n+1} + 3^{n+1}}{2^n - 3^n}\right] = -3$

3. Evaluate $\lim_{x \to 0} \left[\frac{\tan x - \sin x}{x^3}\right]$.

Soln:- $\lim_{x \to 0} \left(\frac{\tan x - \sin x}{x^3}\right) = \lim_{x \to 0} \left(\frac{\frac{\sin x}{\cos x} - \sin x}{x^3}\right)$

$= \lim_{x \to 0} \left[\frac{\sin x - \sin x \cos x}{x^3 \cos x}\right]$

$= \lim_{x \to 0} \left[\frac{\sin x (1 - \cos x)}{x^3 \cos x}\right]$

$= \lim_{x \to 0} \left[\frac{\sin x}{x} \cdot \frac{(1 - \cos x)}{x^2 \cos x}\right]$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1 - \cos x}{x^2 \cos x}$

② result $\to = 1 \cdot \lim_{x \to 0} \frac{2 \sin^2 x/2}{x^2 \cos x} \to (1 - \cos x)$.

$= \frac{1}{2} \lim_{x \to 0} \left[\frac{\sin^2 x/2}{\frac{x^2}{2} \cos x}\right]$

$$= \lim_{x\to 0}\left[\frac{\sin^2 x/2}{x^2/2}\right]\lim_{x\to 0}\left(\frac{1}{\cos x}\right).$$

$$= \lim_{x\to 0}\frac{\sin^2 x/2}{x^2/2}\cdot 1$$

$$= \lim_{x\to 0}\frac{\sin^2 x/2}{x^2/2} = \lim_{x\to 0}\frac{\sin^2 x/2}{\left(\frac{x}{2}\right)\cdot 2}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin x/2}{x/2}\right)^2$$

$$= \frac{1}{2}\cdot 1 = \frac{1}{2}//$$

---

4. Evaluate $\lim_{x\to 0}\left[x\left(\operatorname{cosec} x + 2\operatorname{cosec} 2x\right)\right]$

$$= \lim_{x\to 0}\left[x\left(\frac{1}{\sin x}+2\cdot\frac{1}{\sin 2x}\right)\right]$$

$$= \lim_{x\to 0}\left[x\left(\frac{1}{\sin x}+\frac{2}{2\sin x\cos x}\right)\right]$$

$$= \lim_{x\to 0}\left[x\left[\frac{\cos x+1}{\sin x\cos x}\right]\right]$$

$$= \lim_{x\to 0}\left[\frac{\cancel{2}x\cos^2 x/2}{\sin x\cos x}\right].$$

$$= \lim_{x\to 0}\left[\frac{2x\cos^2 x/2}{\sin x\cos x}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x\cos^2 x/2}{2\sin\frac{x}{2}\quad\cos x}\right]$$

$$= \lim_{x\to 0}$$

$$= \lim_{x\to 0}\left[\frac{x/2\cdot 2}{\sin x/2}\right]\lim_{x\to 0}\left(\frac{\cos x}{\cos x/2}\right)$$

$$= 2\lim_{x\to 0}\left[\frac{1}{\frac{\sin x/2}{x/2}}\right]\cdot 1$$

$$= 2\left[\frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\frac{\sin x/2}{x/2}}\right]$$

$= 2\left[\frac{1}{1}\right] = 2$

5. Find $\lim_{x \to 0} \left[1+\sin x\right]^{\cot x}$

Unit - 2, 3.

Soln: $\lim_{x \to 0} \left[1+\sin x\right]^{\frac{\cos x}{\sin x}}$

$= \lim_{x \to 0} \left[(1+\sin x)^{\frac{1}{\sin x}}\right]^{\cos x}$

$= \lim_{x \to 0} e^{\cos x}$

$= e$

6. Find $\lim_{x \to y} \left(\frac{a^x - a^y}{x-y}\right)$

Soln: $\lim_{x \to y} \left(\frac{a^x - a^y}{x-y}\right) = \lim_{x \to y} a^y \left(\frac{\frac{a^x}{a^y} - 1}{x-y}\right)$

$= \lim_{x \to y} a^y \left(\frac{a^{x-y} - 1}{x-y}\right)$

$= a^y \lim_{x \to y} \left(\frac{a^{x-y} - 1}{x-y}\right)$

Put $x - y = u$, $\Rightarrow x \to y$ becomes $u \to 0$.

$\lim_{x \to y} \frac{a^x - a^y}{x-y} = a^y \lim_{u \to 0} \frac{a^u - 1}{u}$

$= a^y \log a.$

05/02/22

Continous function:-

[If its graph can be drawn without lifting our pen from our paper]

A function $f(x)$ is said to be continous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

~~f(x) is~~

1. Find $\frac{dy}{dx}$ if (i) $y = \sin^3 x$, (ii) $y = \sqrt{1+3x}$
(iii) $y = \tan^{-1}(\sin x)$ (iv) $y = e^{\sqrt{x}}$.
(v) $y = \sin x \cos 2x$ (vi) $y = \frac{x^2+8}{2x+3}$
(vii) $y = 2^x$ (viii) $y = \log \sqrt{\frac{\cos x + \sin x}{\cos x - \sin x}}$
(ix) $y = (x+\sqrt{x^2+a^2})^n + (-x+\sqrt{x^2+a^2})^n$
(x) ~~$y = 2a \sin x$~~ (x) $y = \sqrt{\sin h \sqrt{x}}$.

Soln:-

(i) $y = \sin^3 x$.

$$\frac{dy}{dx} = 3\sin^2 x \times \cos x$$

$$\frac{dy}{dx} = 3\sin^2 x \cos x$$

(ii) $y = \sqrt{1+3x}$

$$\frac{dy}{dx} = \frac{1}{2}(1+3x)^{\frac{1}{2}-1} \cdot 3 \qquad \boxed{(1+3x)^{1/2} \quad \frac{1}{2}(1+3x)^{1/2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(1+3x)^{-1/2} \cdot 3$$

$$\frac{dy}{dx} = \frac{3}{2}(1+3x)^{-1/2}$$

$$\frac{dy}{dx} = \frac{3}{2\sqrt{1+3x}}$$

(iii) $y = \tan^{-1}(\sin x)$

$$\frac{dy}{dx} = \frac{1}{1+\sin^2 x} \cdot \cos x \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{\cos x}{1+\sin^2 x}$$

(iv) $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \quad \text{(or)} \quad e^{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2}$$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

(v) $y = \underset{u}{\sin x}\ \underset{v}{\cos 2x}$. $\qquad d(uv) = u'v + v'u$.

$$\frac{dy}{dx} = \cancel{\cos x} \cancel{(-\sin 2x)}\ \cancel{2}.$$

$$\frac{dy}{dx} = \cos x(\cos 2x) + \sin x(-\sin 2x \cdot x)$$

$$\frac{dy}{dx} = \cos x \cos 2x - 2\sin x \sin 2x //$$

vi) $y = \frac{x^2+8}{2x+3}$. $\frac{u}{v}$ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$= \frac{(2x+3)(2x)-(x^2+8)(2)}{(2x+3)^2}$

$=$

vii) $y = 2^x$

$\frac{dy}{dx} \Rightarrow \log y = \log 2^x$

$\log y = x \log 2$

Diff both sides

$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \log 2$

$\frac{dy}{dx} = y(\log 2)$

$\frac{dy}{dx} = 2^x(\log 2)$

$\frac{d}{dx}$

viii) $y = \log\sqrt{\frac{\cos x+\sin x}{\cos x - \sin x}}$

$\cos x =$

$\sin x =$

$\frac{\sin x}{\cos x} = \tan x$

$y = \log\sqrt{\frac{\cancel{\cos x}\left(1+\frac{\sin x}{\cos x}\right)}{\cancel{\cos x}\left(1-\frac{\sin x}{\cos x}\right)}}$

$y = \log\sqrt{\frac{1+\tan x}{1-\tan x}}$

$y = \log\sqrt{\frac{\tan \pi/4 + \tan x}{1 \cancel{\tan \pi/4} - 1 \cdot \tan x}}$

$y = \log\sqrt{\frac{\tan \pi/4 + \tan x}{1 - \tan \pi/4 \tan x}}$

$y = \log\sqrt{\tan(\pi/4 + x)}$

$y = \log(\tan(\pi/4 + x))^{1/2}$

$y = \frac{1}{2}\log(\tan(\pi/4 + x))$.

$$\frac{dy}{dx} = \frac{1}{2} \frac{1}{\tan\left(\frac{\pi}{4}+x\right)} \cdot \sec^2 x(\pi/4+x)$$ $\boxed{\tan x = \sec^2 x}$

$$\frac{dy}{dx} = \frac{1}{2} \frac{\sec^2(\underline{\pi/4+x})}{\tan(\underline{\pi/4+x})} \begin{matrix} \rightarrow \sec^2 x \\ \rightarrow \tan x. \end{matrix}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{\sec^2(\pi/4+x)}{\frac{\sin}{\cos}(\pi/4+x)}\right] = \frac{1}{2}\left[\frac{\sec^{\cancel{2}}(\pi/4+x)}{\sin(\pi/4+x)} \times \cos(\pi/4+x)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{\sec(\pi/4+x)}{\sin(\pi/4+x)}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{\underset{2\sin\theta}{\sin\left(\frac{\pi}{4}+x\right)}\underset{\cos\theta}{\cos(\pi/4+x)}}\right]$$

$$\frac{dy}{dx} = \frac{\cancel{1}}{\cancel{2}} \frac{1}{\sin\left(\frac{\pi}{2}+2x\right)}$$

$$\frac{dy}{dx} = \frac{1}{\cos 2x}$$ $\frac{1}{\cos x} = \sec x$

$$\boxed{\frac{dy}{dx} = \sec 2x}$$

(ix) $y = (x+\sqrt{x^2+a^2})^n + (-x + \sqrt{x^2+a^2})^{-n}$

Alter:

$$y = \log\sqrt{\frac{\cos x+\sin x}{\cos x-\sin x}}$$

$$y = \frac{1}{2}\log\frac{\cos x+\sin x}{\cos x-\sin x}$$

$$= \frac{1}{2}\left[\log(\cos x+\sin x) - \log(\cos x - \sin x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\cos x+\sin x}(-\sin x+\cos x) - \frac{1}{\cos x-\sin x}(-\sin x-\cos x)\right]$$

$$= \frac{1}{2}\left[\frac{\cos x-\sin x}{\cos x+\sin x} + \frac{\sin x+\cos x}{\cos x-\sin x}\right]$$

$$= \frac{1}{2}\left[\frac{\cos x - \sin x(\cos x - \sin x) + \sin x + \cos x(\cos x + \sin x)}{(\cos x + \sin x)(\cos x - \sin x)}\right]$$

$$= \frac{1}{2}\left[\frac{(\cos x - \sin x)^2 + (\cos x + \sin x)^2}{(\cos x + \sin x)(\cos x - \sin x)}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(\cos^2 x + \sin^2 x)}{\cos^2 x - \sin^2 x}\right]$$

$$= 1\left[\frac{1}{\cos 2x}\right]$$

$$\frac{dy}{dx} = \sec 2x$$

(ix) $y = (x + \sqrt{x^2 + a^2})^n + (-x + \sqrt{x^2+a^2})^{-n}$.

$$\frac{dy}{dx} = n(x + \sqrt{x^2+a^2})^{n-1}\left(1 + \frac{1}{\cancel{2}}(x^2+a^2)^{-1/2}(\cancel{2}x)\right).$$

$$+ (-n)(-x + \sqrt{x^2+a^2})^{-n-1}\left(-1 + \frac{1}{\cancel{2}}(x^2+a^2)^{-1/2}(\cancel{2}x)\right)$$

$$\frac{dy}{dx} = n(x + \sqrt{x^2+a^2})^{n-1}\left(1 + \frac{x}{\sqrt{x^2+a^2}}\right) - n(-x + \sqrt{x^2+a^2})^{-n-1}$$

$$\left(-1 + \frac{x}{\sqrt{x^2+a^2}}\right).$$

$$\frac{dy}{dx} = n(x + \sqrt{x^2+a^2})^{n-1}\left(\frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2}}\right)$$

$$- n(-x + \sqrt{x^2+a^2})^{-n-1}\left(\frac{-\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2}}\right)$$

$$\frac{dy}{dx} = n(-x + \sqrt{x^2+a^2})\left(\frac{-\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$$

$$\frac{dy}{dx} = \frac{n(x + \sqrt{x^2+a^2})^n}{\sqrt{x^2+a^2}} + \frac{n(-x + \sqrt{x^2+a^2})^{-n}}{\sqrt{x^2+a^2}}$$

$$= \frac{n}{\sqrt{x^2+a^2}}\left[(x+\sqrt{x^2+a^2})^n + (-x+\sqrt{x^2+a^2})^{-n}\right]$$

$$\therefore \frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$$

(x) $y = \sqrt{\sinh\sqrt{x}}$.

$$\frac{dy}{dx} = \frac{1}{2}(\sinh\sqrt{x})^{-1/2} \cdot (\cosh\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{2} \frac{(\cosh\sqrt{x})}{\sqrt{x}(\sinh\sqrt{x})^{1/2}}$$

$$\frac{dy}{dx} = \frac{1}{4}\frac{(\cosh\sqrt{x})}{\sqrt{x}\sqrt{\sinh\sqrt{x}}}$$

$$\frac{dy}{dx} = \frac{\cosh\sqrt{x}}{4\sqrt{x\sinh\sqrt{x}}}$$

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2. If $x = 2a\sin^{-1}\sqrt{\frac{y}{2a}} - \sqrt{2ay-y^2}$, then s.t

$$\frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$$

soln :-

(i) $x = 2a\sin^{-1}\sqrt{\frac{y}{2a}} - \sqrt{2ay-y^2}$ — ①

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ — ②

Diff ① w.r.t y

$$\frac{dx}{dy} = 2a \cdot \frac{1}{\sqrt{1-\frac{y^2}{4a^2}}}\left(\frac{1}{2a}\right) - \sqrt{2ay-y^2}$$

$$\frac{dy}{dx} = 2a \cdot \frac{1}{\sqrt{1-\frac{y^2}{4a^2}}}\left(\frac{1}{2a}\right) - \frac{1}{2}(2ay-y^2)^{-1/2}(2a-2y)$$

$$= \frac{1}{\sqrt{\frac{4a^2-y^2}{4a^2}}} - \frac{a-y}{\sqrt{2ay-y^2}}$$

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$u = \sin^{-1}x$

$\sin u = x$

$\cos u \frac{du}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\cos u}$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$= \frac{2a}{\sqrt{4a^2-y^2}} + \frac{(a-y)}{\sqrt{2ay-y^2}}$$

+

Soln:

3. Find $\frac{dy}{dx}$, if x and y are connected by the relation $x^y + y^x = 2$.

Soln:-

$$x^y + y^x = 2 \quad \text{— (1)}$$

Diff w.r.t. 'x'.

Let $u = x^y$ — (2)

$$\log u = y \log x.$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + \frac{y}{x}$$

$d(u+v) = du + dv$

$$\Rightarrow \frac{du}{dx} = u\left(\log x \frac{dy}{dx} + \frac{y}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = x^y \cdot \frac{y}{x} + x^y \log x \frac{dy}{dx} \quad -③$$

Let $v = y^x$ — ④.

$\log v = x \log y$.

$$\frac{1}{v}\cdot\frac{dv}{dx} = x\cdot\frac{1}{y} + \log y \ (1)$$

$$\frac{dv}{dx} = v\cdot\left[\log y + \frac{x}{y}\right]$$

$$\frac{dv}{dx} = y^x\left[\log y + \frac{x}{y}\right]$$

$$\frac{dv}{dx} = y^x \log y + y^x \cdot \frac{x}{y}\frac{dy}{dx} \quad -⑤$$

Now, diff eqn. ③ & ⑤ w.r.t. 'x'

$$\frac{d}{dx}(x^y) + \frac{d}{dx}(y^x) = 0.$$

$$\Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0$$

(i.e.) $\left(x^y\cdot\frac{y}{x} + x^y \log x \frac{dy}{dx}\right) + \left(y^x\cdot\log y + y^x \frac{x}{y}\frac{dy}{dx}\right) = 0.$

(By eqn ③ & ⑤)

$$\frac{dy}{dx}\left(x^y \log x + y^x\cdot\frac{x}{y}\right) = -x^y\cdot\frac{y}{x} - y^x \log y.$$

$$\frac{dy}{dx}\left(x^y \log x + xy^{x-1}\right) = -yx^{y-1} - y^x \log y.$$

$$\frac{dy}{dx} = -\frac{\left[y^x \log y + yx^{y-1}\right]}{\left(x^y \log x + xy^{x-1}\right)}$$

---

4. <u>Partial</u> <u>order</u> <u>derivative</u>

$u = \sin^{-1}x + \cos^{-1}y$, find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

soln: $u = \sin^{-1}x + \cos^{-1}y$.

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}} \; ; \; \frac{\partial u}{\partial y} = 0 + \left(\frac{-1}{\sqrt{1-y^2}}\right)$$

$$\frac{\partial u^2}{\partial x^2} = \frac{-1}{\cancel{2}}(1-x^2)^{-3/2} \cdot (-\cancel{2}x)$$

$$\frac{\partial u^2}{\partial x^2} = \frac{x}{(1-x^2)^{3/2}}$$

$$\frac{\partial u^2}{\partial y^2} = -\left(-\frac{1}{\cancel{2}}(1-y^2)^{-3/2} \cdot (-\cancel{2}y)\right)$$

$$\frac{\partial u^2}{\partial y^2} = -\frac{y}{(1-y^2)^{3/2}}$$

$$\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = \frac{x}{(1-x^2)^{3/2}} + \frac{(-y)}{(1-y^2)^{3/2}}$$

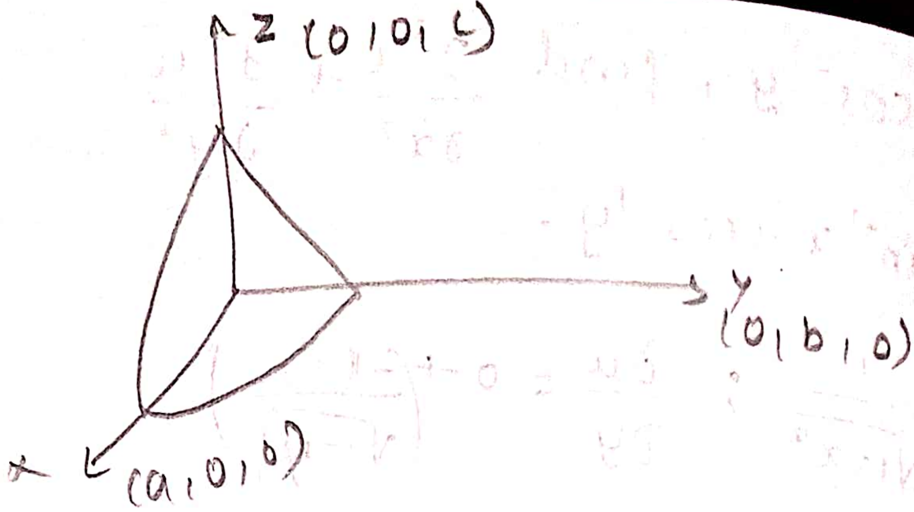
$$\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = \frac{x}{(1-x^2)^{3/2}} - \frac{y}{(1-y^2)^{3/2}}.$$

03/02/22 (B4 the page of 04/02/22 & after the sum of $\boxed{I = 9/2}$))

16 Evaluate $\iiint_V xyz \, dx\,dy\,dz$ where V is the region of space bounded by the planes $x=0$, $y=0$, $z=0$, and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

soln:- Let $I = \iiint_V xyz \, dx\,dy\,dz$ where V is the region of space bounded by the planes, $x=0, y=0$, $z=0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\boxed{\frac{x}{a^2} + \frac{y}{b^2} = 1}$



the limits

21/2/22

1. Evaluate $\int_0^1\int_3^4 (x+y)\,dx\,dy$ 2. $\int_{x=1}^{2}\int_{y=2}^{3} xy\,dy\,dx$.

3. $\iint_R xy\,dx\,dy$ where R is the region bounded by $x=0,\ y=0,\ x+y=4$.

1. $\int_0^1\int_3^4 (x+y)\,dx\,dy$. Let $I = \int_0^1\int_3^4 (x+y)\,dx\,dy$.

$\cancel{\int_0^1\int_3^4} x = \int_0^1 \left[x\right]_3^4 dy = \int_0^1 \left(\frac{x^2}{2} + xy\right)_3^4 dy$

$= \int_0^1 \left[(8+4y) - (9/2 + 3y)\right] dy = \int_0^1 \left(y+\frac{7}{2}\right) - \left(\frac{9}{2}+3y\right) dy$

$= \cancel{\int_0^1} \ = \left(\frac{y^2}{2} + \frac{7y}{2}\right)_0^1 = \frac{1}{2} + \frac{7}{2} = 4$ //

2. $\int_{x=1}^{2}\int_{y=2}^{3} xy\,dy\,dx$

Let $I = \int_1^2\int_2^3 xy\,dy\,dx = \int_1^2 x\frac{y^2}{2}\Big|_2^3 dx$

$= \int_1^2 \left(\frac{xy^2}{2}\right)_2^3 dx$

$= \int_1^2 \left[\frac{9}{2}x - 2x\right] dx = \int_1^2 \frac{5}{2}x\,dx$

$= \cancel{\int} \frac{5}{2}\left[\frac{x^2}{2}\right]_1^2 = \frac{5}{2}\left[\frac{\cancel{4}}{2} - \frac{1}{2}\right]$

$= \frac{5}{2} \times \frac{3}{2} = \frac{15}{4}$ //

3. $\iint_R xy\,dy\,dx$, $x=0,\ y=0,\ x+y=4$.

x varies from 0 to $\cancel{y=}\ 4-y$

y " " " 0 to 4

$$I = \int_{y=0}^{4}\int_{0}^{4-y} xy\, dx\, dy$$

4. $$\int_0^4 \frac{x^2 y}{2}\Big|_0^{4-y} dy = \frac{1}{2}\int_0^4 (4-y)^2\, y\, dy.$$

$$= \frac{1}{2}\int_0^4 \ldots$$

5. $$\int_{\theta=0}^{\pi}\int_{r=0}^{1} r \sin^4\theta \cos\theta\, dr\, d\theta.$$

Let $$I = \int_{\theta=0}^{\pi}\int_{r=0}^{1} r\sin^9\theta \cos\theta\, dr\, d\theta.$$

$$= \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\sin^9\theta\cos\theta\right]_{r=0}^{1} d\theta.$$

Put $f(x) = t$

$= t^n$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} \sin^9\theta \cos\theta\, d\theta. = \frac{1}{2}\int_{\theta=0}^{\pi}\sin^9\theta\, d(\sin\theta).$$

$$= \frac{1}{2}\left[\frac{\sin^{10}\theta}{10}\right]_0^{\pi} = 0$$

6. Evaluate $$\int_0^2\int_0^{\sqrt{4-y^2}} (x^2+y^2)\, dx\, dy.$$

Let $$I = \int_0^2\int_0^{\sqrt{4-y^2}} (x^2+y^2)\, dx\, dy = \int_0^{2}\int_0^{\sqrt{4-y^2}} (x^2+y^2)\, dx\, dy$$

$$= \int_0^{\sqrt{4-y^2}} (x^2+y^2)\Big|_0^2 dx\, dy = \int_0^{\sqrt{4-y^2}} (\cancel{x^2}+y^2)$$

$$= \int_0^{\sqrt{4-y^2}}$$