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UNIT - 4.

COMBINATORICS.

1. Basic counting principles (Permitivity / combination)
2. Pigeon hole principle.
3. Recurrence relations / Difference equations.
4. Generating functions.
5. Mathematical Induction.

$\lfloor n$ n factorial

$\lfloor n = n \lfloor n-1$

$0! = 1.$

$F_n = \lfloor n.$

$F_n = n \cdot F_{n-1}.$

$F_0 = 1.$

System — continous → Differential eqns < ODE, ~~ode~~, PDE

System — Discrete → Recurrence relation or difference eqn.s

$S(n) = H.S + P.S$

Homogeneous soln + Particular

$nC_r = \dfrac{\lfloor n}{\lfloor r \lfloor n-r}$

$\dfrac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10$

$5C_3 = \dfrac{\lfloor 5}{\lfloor 3 \lfloor 2}$ (or) $5C_3 = \dfrac{5 \times 4 \times 3}{1 \times 2 \times 3} = 10$

$$nC_r = \frac{n(n-1)(n-2)\cdots(n-(r-1))}{\lfloor r}$$

PASCAL'S TRIANGLE:-

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1.
1 8 28 65 70 55 28 8 1.
56.

$$\boxed{\begin{array}{l} nC_0 = 1 = nC_n \\ nC_1 = n \\ nC_r = nC_{n-r} \end{array}}$$

$5C_3$ $5C_2$
$10 \rightarrow$ 35
100

Eg: $10C_7 \longrightarrow$ Pascal's triangle

$\longrightarrow \lfloor 10 / \lfloor 7 \lfloor 3 \longrightarrow$

$\longrightarrow 10C_{10-7} = 10C_3 = \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$

$$\boxed{nC_0 + nC_1 + nC_2 + \cdots + nC_n = 2^n}$$ Formula

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

If $(a-b)$ means alternative

$+ \; - \; + \; - \; + \; - \; \cdots$

$a^4 b^0$
$a^3 b^0$ $a^3 b^1$
$a^2 b$ $a^2 b^2$
$a b^2$ $a^1 b^3$
$a^0 b^3$ $a^0 b^4$

COMBINATORICS.

Sum rule — Product rule.

Eg:- 5 Red balls $5C_1 + 3C_1$ $5C_2 \times 3C_1$.
3 white balls

Sum rule:-

Let the disjoint events $E_1, E_2, \ldots, E_k$ be happened in $n_1, n_2, n_3, \ldots n_k$ ways respectively then any one of the k events can happen in $n_1 + n_2 + \ldots + n_k$ ways.

Product rule:-

Suppose that a procedure can be broken down into the sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways for doing the first task there are n_2 ways to do the 2nd task. Then there are $n_1 \times n_2$ ways to do the procedure.

Permutation:-

An ordered arrangements of r elements from a set containing n elements is called r permutation of n elements.

It is denoted by the 'nP_r' or '$P(n;r)$' when $r \leq n$,

where $nP_r = \frac{\lfloor n}{\lfloor n-r}$ (or) $nP_r = \frac{n!}{n-r!}$

---

1. How many 6 digit number can be formed using the digits 0, 1, 2, ..., 9.

(i) without repetition. (^nP_r)

(ii) with repetition. (n^r)

Soln:

(i) 9 No possibilities, 10-2, 10-3, 10-4, 10-5

9	9	8	7	6	5

0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

we can't put '0'

$9 \times 9 \times 8 \times 7 \times 6 \times 5$.

$= 136080$.

6-digits number so we should put 6 boxes

(ii)

9	10	10	10	10	10

$9 \times 10 \times 10 \times 10 \times 10 \times 10$

$= 900000$.

The no. of 6 digits no's that can be formed using the digits without repetition $= 9 \times 9 \times 8 \times 7 \times 6 \times 5$
$= 136080$.

the no. of 6 digits no's that can be formed with repitions $= 9 \times 10 \times 10 \times 10 \times 10 \times 10$
$= 900000$.

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2. How many 5-digits numbers can be formed by using the digits (0, 1, 2 ... 9) without repetion if the no's are.
(i) divisible by 2 (ie, how many even no's can be formed)
(ii) divisible by 5.

8	8	7	6	4

0, 2, 4, 6, 8

8	8	7	6	4

Soln:

Note:

Without repetition.

(i)

9	8	7	6	0

→ 5 possibilities (ie 1, 2, 4, 6, 8)

provided '0' is in last position.

If '0' is not provided in last position

8	8	7	6	4

$9 \times 8 \times 7 \times 6 \times 1 + 8 \times 8 \times 7 \times 6 \times 4$

$= 336 (41)$

$= 13776$.

(ii) $\boxed{9}\boxed{8}\boxed{7}\boxed{6}\boxed{1}$ + $\boxed{8}\boxed{8}\boxed{7}\boxed{6}\boxed{1}$

$9 \times 8 \times 7 \times 6 \times 1 + 8 \times 8 \times 7 \times 6 \times 1.$

$= 5712$ //

~~3.~~ ~~Sol~~ With repetition means:

(i) $9 \times 10 \times 10 \times 10 \times 1 + 8 \times 10 \times 10 \times 10 \times 4,$

$= 41000.$

(ii) $9 \times 10 \times 10 \times 10 \times 1 + 8 \times 10 \times 10 \times 10 \times 1$

$= 17000$

---

3. How many 5 letter words that can be formed by using the alphabets A, B, C, D, E if

(i) No alphabet is repeated.

(ii) No restrictions.

$5 \times 4 \times 3 \times 2 \times 1$

Soln:

(i) $5P_5$ (or) $\lfloor 5 = 120$

(ii) $5^5 \rightarrow \boxed{5}\boxed{5}\boxed{5}\boxed{5}\boxed{5}$

5^5

---

4. How many 5 letter words that can be formed by using the letters of the word FACTS if no letter is repeated? Also find the position of the word ~~ACTST~~ ASTCF if the above are arranged in lexicographic order?

Soln: FACTS if no letter is repeated.

Permutation with repetitions:

The no. of arrangements of n objects of which n_1 are of first kind, n_2 are of second kind ... n_k are of kth kind is given by

$$\frac{\lfloor n}{\lfloor n_1 \lfloor n_2 \cdots \lfloor n_k} \quad \text{where } n_1 + n_2 + \cdots + n_k = n.$$

---

1. How many ways ~~arrangements~~ all ~~these~~ the letters of the word MISSISSIPPI can be arranged.

Soln.

MISSISSIPPI

S - 4
I - 4
M - 1
P - 2.

$n = 11.$

$n_1 = 1,\ n_2 = 4,\ n_3 = 4,\ n_4 = 2$

$\downarrow$ M, $\downarrow$ I, $\downarrow$ S, $\downarrow$ P.

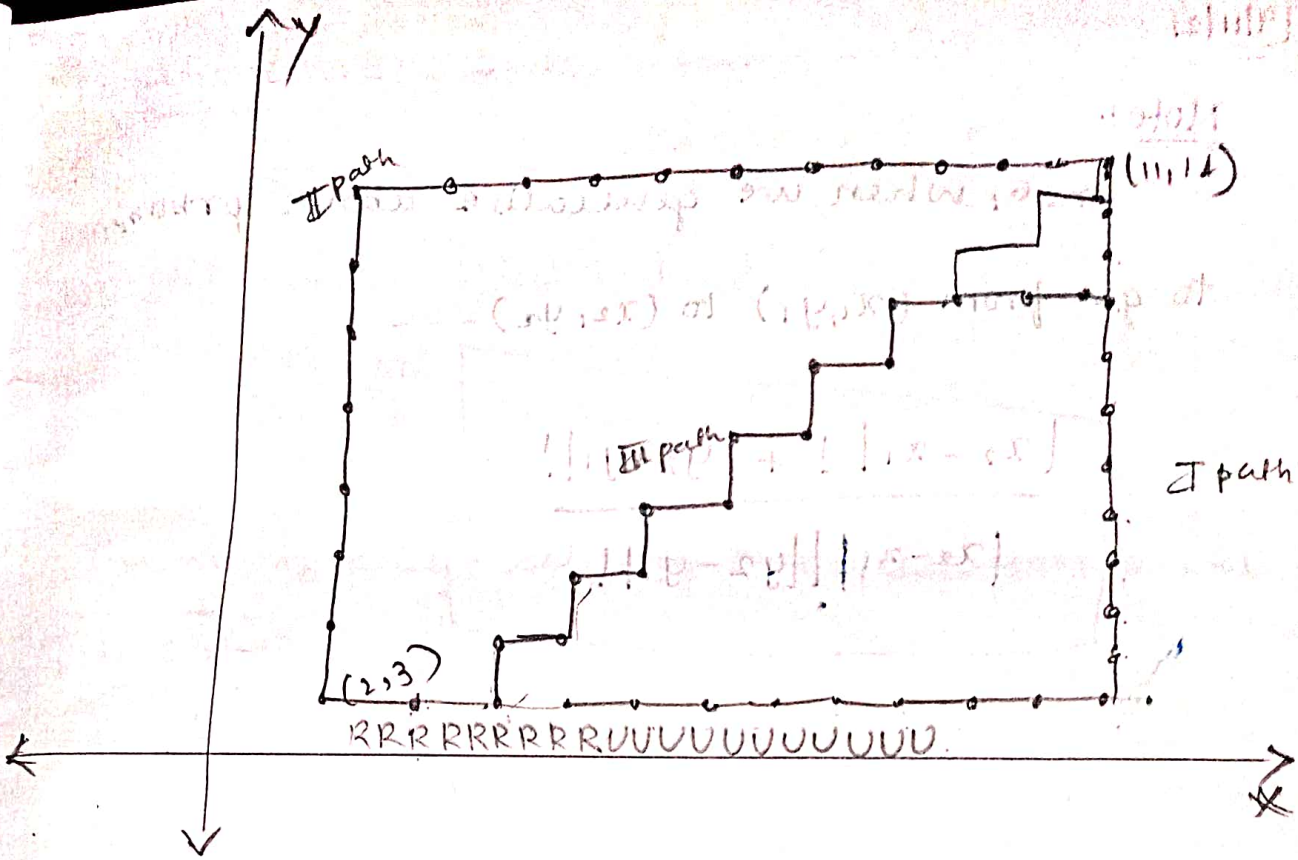
$n_1 + n_2 + n_3 + n_4 = 11.$

$$\frac{\lfloor 11}{\lfloor 1 \lfloor 4 \lfloor 4 \lfloor 2} = \frac{1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11}{1\times1\times2\times3\times4\times1\times2\times3\times4\times1\times2}$$

$$= \frac{5\times3\times2\times3\times5\times11\times7}{1\times1}$$

$$= 34650$$

PTO



From (2,3), we have to move to (11,14),

How many paths same like this we have?

Che:- given 3 paths, find its similarity.

$$\frac{\lfloor 20}{\lfloor 9 \, \lfloor 11} \qquad \frac{\lfloor 20}{\lfloor 9 \, \lfloor 11}$$

$$= \frac{12 \times 13 \times \overset{2}{\cancel{14}} \times \cancel{15} \times \overset{2}{\cancel{16}} \times 17 \times \overset{2}{\cancel{18}} \times 19 \times \overset{5}{\cancel{20}}}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times \cancel{6} \times \cancel{7} \times \cancel{8} \times \cancel{9}}$$

$= 8840 \times 19$

$= 167960$

1st, max. no. of right steps possible

$= 11 - 2 = 9$

2nd, max. no of up steps is

$= 14 - 3 = 11$.

Total steps possible $= 11 + 9 = 20$

$$\frac{20!}{9! \, 11!}$$

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Note:

→ so, when we generalise above prblm,

to go from (x_1, y_1) to (x_2, y_2) ---

$$\boxed{\frac{|x_2 - x_1|! + |y_2 - y_1|!}{|x_2 - x_1|!\,|y_2 - y_1|!}}$$

combination: order doesn't matter.

Permutation - orders matter

eg: abc

We want to select 2 objects.

ab ba
bc ca → order doesn't matter
ca cb

∴ The no. of ways selecting r objects from a set of n objects is nCr

$$nCr = \frac{\lfloor n}{\lfloor r \lfloor n-r}$$

$nC0 = 1 = nCn$

$nC1 = n = nCn-1.$

$nCr = nC(n-r)$

$$\boxed{nCr = \frac{n(n-1)(n-2)\cdots(n-(r-1))}{\lfloor r}}$$

PASCAL'S TRIANGLE:

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1.

---

1. In how many ways one can select 3 white balls and 2 red balls from a bag containing 10 white balls and 7 red balls.

we need to select.

Soln: 10 white Balls → 3W 10W
7 Red Balls → 2R 7R

$10C_3 \times 7C_2$

$nC_r \times nC_r$

$$= \frac{\lfloor 10}{\lfloor 3 \lfloor 7} \times \frac{\lfloor 7}{\lfloor 2 \lfloor 5}$$

$$= \frac{6 \times 7 \times 8 \times 9 \times 10}{\lfloor 3 \lfloor 2}$$

$= 2520.$

2- 10 WB, 7 RB. We want to select 4 balls.

$10C_4 \times 7C_0$

W	R	
4	0	$\rightarrow 10C_4 \times 7C_0$
3	1	$\rightarrow 10C_3 \times 7C_1$
2	2	$\rightarrow 10C_2 \times 7C_2$
1	3	$\rightarrow 10C_1 \times 7C_3$
0	4	$\rightarrow 10C_0 \times 7C_4$

$$\Rightarrow \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} \times 1 + \frac{10 \times 9 \times 8}{1 \times 2 \times 3} \times \frac{7 \times 1}{1 \times 2} + \frac{10 \times 9}{1 \times 2} \times \frac{7 \times 6}{1 \times 2}$$

$$+ \frac{10}{1} \times \frac{7 \times 6 \times 5}{1 \times 2 \times 3} + \frac{1 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4}.$$

$\Rightarrow 210 + 840 + 945 + 350 + 35.$

$\Rightarrow 2380$ //

$$\begin{array}{r} 210 \\ 840 \\ 945 \\ 350 \\ 35 \\ \hline 80 \end{array}$$

$\lfloor C_1 \rfloor \quad \lfloor C_2 \rfloor \quad \lfloor C_3 \rfloor$ 6 cookies.

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T1	T2	T3
6	0	0
0	6	0
0	0	6
1	5	0
2	1	3
1	4	1
2	0	4
1	3	2
1	2	3
1	1	4
1	0	5
2	4	0
2	3	1
2	2	2
.		----

$r = 6$ → 6

$n = 3$

$(n+r-1)C_r$

$= (3+6-1)C_6$

$= 8C_6 \rightarrow 8C_2$

$= \frac{8 \times 7}{1 \times 2} = 28$

T1	T2
0	4
4	0
1	3
2	2
3	1
.	

4 cookies.

5 possibilities

$n = 2$

$r = 4$

$(n+r-1)C_r$

$= (2+4-1)C_4$

$= 5C_4 \rightarrow 5C_1$

$= 5$

$n+r-1$

$2+4-1$

$6-1$

$5C_4$

Combinations with repetition:-

There are $(n+r-1)C_r$ ways to select r objects from a set having n elements when repetition is permitted.

Types of cookies.

1. How many integer soln. does the eqn. $x_1 + x_2 + x_3 + x_4 = 7$ have? if $x_i \geq 0$, $i = 1, 2, 3, 4$. (Same as cookies prob).

Soln: $x_1 + x_2 + x_3 + x_4 = 7$

We have 4 cookies out of 4 → 7 cookies want to select.

$$\boxed{(n+r-1)C_r}$$

Soln:

There are $(n+r-1)C_r$ integer soln's.

$(n+r-1)C_r$

$10C_3$

$x_1 + x_2 + x_3 + x_4 = 7$

where $x_i \geq 0$ $i = 1, 2, 3, 4$.

Here $n = 4$, $r = 7$

$\therefore$ There are $(n+r-1)C_r$ integer soln's

$$(4+7-1)C_7 = 10C_7 = 10C_3$$

$$= \frac{10 \times 9 \times 8}{1 \times 2 \times 3}$$

$$= 120$$

---

2. How many integer soln's does the eqn. $x_1 + x_2 + x_3 + x_4 = 7$ have? if $x_1 > 0$ and $x_2, x_3, x_4 \geq 0$.

Soln:

$$\left.\begin{array}{l} x_1 + x_2 + x_3 + x_4 = 7. \\ x_1 > 0 \ \& \ x_2, x_3, x_4 \geq 0. \end{array}\right\} - (1)$$

Put $x_1 = u + 1$, $u \geq 0$

$$u + 1 + x_2 + x_3 + x_4 = 7$$

$$u + x_2 + x_3 + x_4 = 6.$$

Now (1) becomes $u + x_2 + x_3 + x_4 = 6$

Where $u, x_2, x_3, x_4 \geq 0$.

Here $n=4$; $r=6$

$(n+r-1)C_r$

$(4+6-1)C_6$

$= 9C_6 \rightarrow 9C_3 = \frac{9\times8\times7}{1\times2\times3}$

$= 84.$

---

3. How many integer soln's does the eqn.
$x_1+x_2+x_3+x_4=7$ have? if $x_1, x_2, x_3, x_4 > 0$.

Soln:-

Let $x_i = u_i+1$, $u_i \geq 0$, $i=1,2,3,4$.

then $u_1+u_2+u_3+u_4=3$

Here $n=4$, $r=3$.

$(n+r-1)C_r$

$= 6C_3 = \frac{6\times5\times4}{1\times2\times3}$

$= 20$

---

4. How many integer soln's does the eqn.
$x_1+x_2+x_3+x_4=7$ have? if $x_1 \geq 2$ and $x_2, x_3, x_4 \geq 0$

Soln:-

Put $x_1 = u+2$, where $u \geq 0$.

$u+x_2+x_3+x_4=5.$

$n=4$, $r=5$

$(n+r-1)C_r.$

$=(4+5-1)C_5$

$= 8C_5 \rightarrow 8C_3$

$= \frac{8\times7\times6}{1\times2\times3} = 56.$

5. How many ~~ways~~ integer soln's does the eqn. $x_1 + x_2 + x_3 + x_4 = 7$ have? if $0 \le x_1 \le 2$, $x_2, x_3, x_4 \ge 0$.

Soln:

$x_1 + x_2 + x_3 + x_4 = 7$, where

$0 \le x \le 2$, $x_2, x_3, x_4 \ge 0$ — (1)

Put $x_1 = u + 3$, where $u \ge 0$.

The (1) becomes;

$u + x_2 + x_3 + x_4 = 7 - 3$.

(ie;) $u + x_2 + x_3 + x_4 = 4$.

then $n = 4$; $r = 4$.

$(n+r-1)C_r$

$= (4+4-1)\,C_4$

$= 7C_4 \rightarrow 7C_3$

$= \dfrac{7 \times 6 \times 5}{1 \times 2 \times 3}$

The no. of integer soln; for the given eqn. $x_1 + x_2 + x_3 + x_4 = 7$, if $0 \le x_1 \le 2$, $x_2, x_3, x_4 \ge 0$.

= the no. of integer solns, for the given eqn. $x_1 + x_2 + x_3 + x_4 = 7$. if $x_i \ge 0$.

- The no. of integer soln for the given eqn. $x_1 + x_2 + x_3 + x_4 = 7$ if $x_1 \le 2$ at $x_2, x_3, x_4 \ge 0$

$= 120 - 35$

$= 85$

$^{4+7-1}C_4$

$^{10}C_4$

$^{10}C_6$

$\dfrac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4}$

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1(a) In how many ways 6 boys and 4 girls can sit in a row?

(b) In how many ways can they sit in a row if the boys are sit together and girls are to sit together?

(c) In how many ways can they sit in a row if the girls are to sit together?

2. How many bit strings of length 10 contain

(i) Exactly 4 1's (iii) atleast 4 1's

(ii) atmost 4 1's. (iv) equal no. of 0's & 1's.

3. How many permutations of letters A, B, C, D, E, F, G contain.

(i) the string BCF (ii) the string CFGA.

(iii) the strings BA and GF. (iv) the strings ABC and DE.

4. In how many ways can 2 letters be selected from the set {a, b, c, d} when the repetition of the letters allowed if

(i) order matters (ii) order doesn't matter.

ANSWERS

1. (a) 10!.

$10! \times 9 \times 8 \times 7 \times 6$

210

630

$10 \times 9 \times 8 \times 7$

$10 \times 9 \times 8 \times 7 \times 3$

1 (b) ~~7!~~ $2! \times 6! \times 4!$

(c) $7! \times 4!$

2. (a) Exactly 4 1's.

$$\frac{\cancel{10}^{5}!}{4! \times \cancel{6!}_{3}} = \frac{5!}{4! \times 3!}$$

$$= \frac{(5 \times 9 \times 8 \times 7)}{4!}$$

$$= 210.$$

(b) at most 4 1's.

0's	1's		
0	10	$\frac{10!}{0! \times 10!} = 1$	$\frac{10!}{3! \times 7!} = 120$
1	9		
2	8	$\frac{10!}{1! \times 9!} = 10$	$\frac{10!}{4! \times 6!} = 210$
3	7		
4	6	$\frac{10!}{2! \times 8!} = 45$	

$$1 + 10 + 45 + 120 + 210 = 386.$$

(c) at least 4 1's.

$$1024 - 1 - 10 - 45 - 120$$

$$= 848.$$

0's	1's	
		$C(10,0)$
		$+$
10	0	$C(10,1)$
		$+$
10	1	$C(10,2)$
		$+$
10	2	$C(10,3)$
		$+$
10	3	

(d) Equal no. of 0's and 1's?

$$\frac{10!}{5! \times 5!} = 252.$$

3. (a) String BCF.

$n = 5$
$r = 5.$

$$\frac{5!}{0!} = 5!$$
$$= 120$$

(b) String CFGA.

$n = 4$

$r = 4$

$$\frac{4!}{0!} = 4! = 24$$

(c) String BA and GF.

$n = 5$
$r = 5$

$$\frac{5!}{0!} = 5! = 120.$$

(d) Strings ABC and DE.

$n = 4$

$r = 4$

$$\frac{4!}{0!} = 4! = 24.$$

4. (a) order matters.

In 1st case a b c d we can fill one

1 do 4 ways → ~~4 × 4 = 16.~~ 4.

In 2nd case same as 1st one → 4

$4 \times 4 = 16$ ways.

(or)

$4P_1 \times 4P_1 = 4 \times 4 = 16$

(b) Selection with repetition

$^{(n+r-1)}C_r$

$n = 4, r = 2$

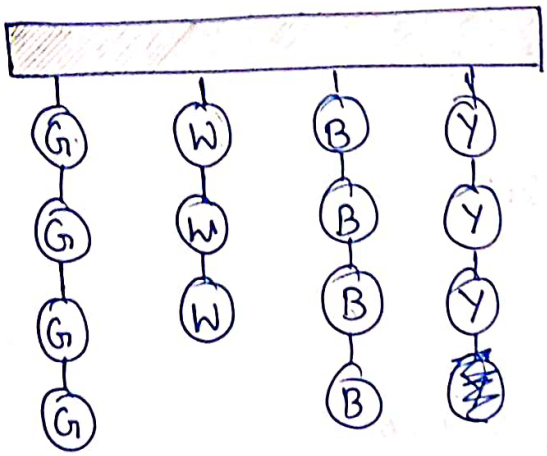
$^{(4+2-1)}C_2$

$^5C_2 = 10$

23/11/24.

Green (G), White (W), Black (B), Yellow (Y)

1.



How many ways we can remove the balls.

ie; G W B Y.

GG WW BB YYY GWBBG

GGGG WWW BBBYYY.

GGGG YYYYWWW BBBB.

GWBY GWBY GWBYGW

4 possibilities.

$\underset{I}{\underbrace{\overset{n_1}{GGGG}}}\ \underset{II}{\underbrace{\overset{n_2}{WWW}}}\ \underset{III}{\underbrace{\overset{n_3}{BBBB}}}\ \underset{IV}{\underbrace{\overset{n_4}{YYY}}}.$

$n = 14$.

$$\frac{^{n}14!}{4!\,3!\,4!\,3!}$$

$n_1 \quad n_2 \quad n_3 \quad n_4$

$$= \frac{\cancel{1\times2\times3\times4}\times5\times\cancel{6}\times7\times\cancel{8}\times\cancel{9}\times10\times11\times\cancel{12}\times13\times\cancel{14}}{\cancel{1\times2\times3\times4}\times\cancel{1\times2\times3}\times\cancel{1\times2\times3\times4}\times\cancel{1\times2\times3}}$$

$= 5\times7\times3\times10\times11\times4\times13\times7$

$= 105\times10\times44\times91$

$= 105\times4004\times10.$

$= 4204200.$

---

2 1 2 3 4. 5 6. 7 8 9.

In how many ways can we select 6 no's from the set $\{1,2,3,4,5,6,7,8,9\}$?

Soln: $n=9$, $r=6$.

$nCr = nC_{n-r}$

$9C_6 \Rightarrow 9C_{9-6} \rightarrow 9C_3.$

$= \cancel{3}\times\frac{9\times8\times7}{1\times2\times3}$

$= 84.$

$9C_6$ *

(ii) If the multiples

Select both select non

$\boxed{7C_4} + \boxed{7C_6}$

$= 35 + 7$

$= 245$

---

3. Find the no. of arrangements of all the letters of the word MISSISSIPPI. if

(1) All I's are come together.

(ii) no two I's are adjacent.

Soln: this is with restrictions.

M - 1 , I - 4 , S - 4 , P - 2

$n \neq r$

(i) Treat all I's as a single object

MISSSSPP.

$\frac{\lfloor 8}{\lfloor 1 \lfloor 1 \lfloor 4 \lfloor 2}$

$= \frac{8 \times \cancel{1 \times 2 \times 3 \times 4} \times 5 \times \cancel{6}^{3} \times 7 \times 8}{1 \times 1 \times \cancel{1 \times 2 \times 3 \times 4} \times 1 \times \cancel{2}}$

$= 5 \times 3 \times 7 \times 8$

$= 15 \times 56$

$= 840.$

Without restriction

$n = 11$

$n_1 = 1$

$n_2 = 4$

$n_3 = 4$

$n_4 = 2.$

$\frac{\lfloor 11}{\lfloor 1 \lfloor 4 \lfloor 4 \lfloor 2}$

11 - 4

$\begin{array}{r} 15 \\ 56 \quad 3 \\ \hline 90 \quad 2 \\ 75 \\ \hline 840 \end{array}$

(ii) Remove all I's

MSSSSPP.

$n = 7.$

$n_1 = 1$ (M), $n_2 = 4$ (S), $n_3 = 2$ (P)

$\frac{\lfloor 7}{\lfloor 1 \lfloor 4 \lfloor 2} \times {}^{8}C_4$ → place → I's (4)

M S S S S P P

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

In these 8 positions we can arrange a

$\frac{\cancel{1 \times 2 \times 3 \times 4} \times 5 \times \cancel{6}^{3} \times 7}{1 \times \cancel{1 \times 2 \times 3 \times 4} \times 1 \times \cancel{2}} \times \frac{\cancel{8}^{4} \times 7 \times \cancel{6}^{2} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}}$

$= 15 \times 7 \times 7 \times 2 \times 5$

$= 15 \times 49 \times 10 \quad = 7350$

$\begin{array}{r} 15 \\ 49 \quad 4 \\ \hline 135 \quad 2 \\ 60 \end{array}$

Eg: Ball.

$\frac{\lfloor 4}{\lfloor 1 \lfloor 1 \lfloor 2}$ without any restrict

$\frac{\cancel{1 \times 2} \times 3 \times 4}{1 \times 1 \times \cancel{1 \times 2}} = 12$

4. In how many ways can we arrange all the letters of the word PERMUTATIONS if the difference in position b/w P & S is 4?

Left to Right (P to S)

Soln:

$$\underset{1}{P}\ \underbrace{E\ R\ M\ U\ T\ A\ T\ I\ O\ N}_{\text{difference}}\ \underset{12}{S}$$

[∘ If I fix 'P' in 1st position then 'S' shld b in 5th position ∘ If I fix 'P' in 2nd position then 'S' shld be in 6th position ∘ If I fix 'P' in 3rd position then 'S' shld be in 7th position. If I fix 'P' in 4th position then 'S' 8th position - - - - - -].

(1,5) -
(2,6) -
(3,7) -
(4,8) -
(5,9) -
(6,10) -
(7,11) -
(8,12) -

} 8

We can place 'P' & 'S' in this 8 possibilities.

If we remove P & S

then n=10

E R M U T A T I O N

$$\frac{\lfloor 10}{\lfloor 1\ \lfloor 1\ \lfloor 1\ \lfloor 1\ \lfloor 2\ \lfloor 1\ \lfloor 1\ \lfloor 1\ \lfloor 1} \times {}^8C_1 \times 2.$$

E R M U T A O N

$= \lfloor 10 \times 8$ //

27/11/21

Revision for CIA I

1. PT in a Boolean algebra $x+x=x$, $x\cdot x=x$ (idepotent)

2. PT in a Boolean algebra $(x+y)'=x'\cdot y'$ and $(x\cdot y)'=x'+y'$ (De-morgon's law)

3. PT in a Boolean algebra $x+(x\cdot y)=x$ and $x\cdot(x+y)=x$. (Absorption law)

4. Obtain the sum-of-product canonical form for the following Boolean functions (I) using truth table (II) without using truth table

   (a) $xy+y'z+x'z$

   (b) $(x+y)\cdot(y'+z')\cdot(x+z)$

5. Obtain the product-of-sum canonical form for the following Boolean functions (I) using TT (II) without using TT

   (a) $(x+y)(x'+y')(x'+z')$

   (b) $xy+yz+xz$

6. Simplify the Boolean function without using k-maps.

   (I) $F(x,y,z)=xyz+x'yz+xy$

   (II) $G(x,y,z)=(x'+y')(x+z)(y+z)$

7. Simplify the following using k-map and draw the logic circuit for the simplified exp

   (a) $f(x,y)=xy+x'y$ (b) $f(x,y,z)=\Sigma(0,1,2,7)$

   (c) $f(x,y,z)=\Sigma(1,3,5,7)$ (d) $f(a,b,c,d)=\Sigma(0,1,3,5,9,11,13)$

Answers:

1. Idepotent law

$\Rightarrow$ i. $x + x = x$

$x = x + 0$ (By identity)

$x + (x \cdot x')$ (By complement)

$= (x + x) \cdot (x + x')$ (By distribute)

$= (x + x) \cdot 1$ (By complement) $[\because x + x' = 1]$

$= x + x$.

$x + x = x$ //

$\Rightarrow x \cdot x = x$

$x = x \cdot 1$ (By identity)

$= x \cdot (x + x')$ (complement)

$= x \cdot x + (x \cdot x')$ (Distributive)

$= x \cdot x + 0$ (complement) $[x \cdot x' = 0]$

$= x \cdot x$ (identity).

$x \cdot x = x$ //

---

2. $\Rightarrow (x + y)' = x' \cdot y'$

$\overset{\text{I}}{(x + y)} + \overset{\text{II}}{(x' \cdot y')}$

1 2 3

$= ((x + y) + x') \cdot ((x + y) + y')$ (Distributive)

$= ((y + x) + x') \cdot ((x + y) + y')$ (commutative)

$= (y + (x + x')) \cdot (x + (y + y'))$ (associative)

$= (y + 1) \cdot (x + 1)$ (By complement)

$= 1 \cdot 1 = 1$ $[\because 1 \cdot 1 = 1]$

$\therefore (x+y)+(x'\cdot y')=1$ — ①

Now; $(x+y)\cdot(x'\cdot y') = (x\cdot(x'\cdot y'))+(y\cdot(x'\cdot y'))$

(Distributive)

$= [(x\cdot x')\cdot y']+[y\cdot(x'\cdot y')]$ (associative)

$= [0\cdot y']+(y\cdot(y'\cdot x'))$ (commutative & complement)

$= \cancel{0+y}\ 0+0\cdot x'$

$= 0+0$

$= 0$ //

$(x+y)'=x'\cdot y' \Rightarrow (x+y)\cdot(x'\cdot y')=0.$

$\Rightarrow (x\cdot y)'=x'+y'$

3. Absorption law:-

$\Rightarrow x + (x \cdot y) = x.$

$= x + (x \cdot y) = x \cdot 1 + x \cdot y$ (identity)

$= x \cdot (1 + y)$ (distributive)

$= x \cdot (1)$ $[x + 1 = 1]$

$= x \cdot 1 \rightarrow x$ (identity)

$\therefore x + (x \cdot y) = x$ //

$\Rightarrow x \cdot (x + y) = x.$

$x \cdot (x + y) = x \cdot x + x \cdot y$ (Distributive)

$= x + x \cdot y$ (idempotent)

$= x \cdot 1 + x \cdot y$ (identity)

$= x \cdot (1 + y)$ (Distributive)

$= x \cdot 1$ $(\because x + 1 = x)$

$= x \cdot 1 \rightarrow x$ (identity)

$\therefore x \cdot (x + y) = x$ //

---

4. sum - of - product. [Minterms]. In place 0 we should take that.

(1) using TT.

(a) $xy + y'z + x'z.$

x	y	z	x'	y'	xy	$y'z$	$x'z$	$xy + y'z + x'z.$
0	0	0	1	1	0	0	0	0
0	0	1	1	1	0	1	1	1 ✓ m_1
0	1	0	1	0	0	0	0	0
0	1	1	1	0	0	0	1	1 ✓ m_3

1	0	0	0	1	0	0	0	0	
1	0	1	0	1	0	1	0	1 ✓	m_5
1	1	0	0	0	1	0	0	1 ✓	m_6
1	1	1	0	0	1	0	0	1 ✓	m_7

Minterms:-

$x'y'$ m_0

$x'y'$ m_1

$x'y$ m_2

$x'y$ m_3

xy' m_4

xy' m_5

xy m_6

xy m_7

∴ the sum-of-product of canonical form of F is m_1, m_3, m_5, m_6, m_7

(or) $x'y'z + x'yz + xy'z + xyz' + xyz$

(or) $\sum(1,3,5,6,7)$. $+ xyz$

sum of product canonical particular case

(ii) without using TT.

$F = xy + y'z + x'z$.

$= xy \cdot 1 + y'z \cdot 1 + x'z \cdot 1$ (identity)

$= xy \cdot (z + z') + y'z \cdot (x + x') + x'z \cdot (y + y')$

$= xyz + xyz' + xy'z$

$= xyz + xyz' + xy'z + x'y'z + x'yz$

∴ the sum of product canonical form for the given Boolean fnc. is

$\sum(1,3,5,6,7)$

(b) $(x+y)\cdot(y'+z')\cdot(x+z)$.

(i) with using TT.

x	y	z	y'	z'	$x+y$	$y'+z'$	$x+z$	$(x+y)\cdot(y+z')\cdot(x+z)$	Minterms (OR) m_i
0	0	0	1	1	0	1	0	0	$x'y'$ m_0
0	0	1	1	0	0	1	1	0	$x'y'$ m_1
0	1	0	0	1	1	1	0	0	$x'y$ m_2
0	1	1	0	0	1	0	1	0	$x'y$ m_3
1	0	0	1	1	1	1	1	1 ✓	xy' m_4
1	0	1	1	0	1	1	1	1 ✓	xy' m_5
1	1	0	0	1	1	1	1	1 ✓	xy m_6
1	1	1	0	0	1	0	1	0	xy m_7

∴ The sum-of-product of canonical form of F is. m_4, m_5, m_6.

(or) $\Sigma(4,5,6)$.

(or) ~~$xy'+xy'+xy$~~.

$xy'z+xy'z'+xyz'$

(ii) without using TT.

$(x+y)\cdot(y'+z')(x+z)$

$= (x+(yz))(y'+z')$

$= x\cdot(y'+z') + (yz\cdot(y'+z'))$

$= xy'+xz'+\underline{yzy'}+\underline{yzz'}$. $(zz'=0)$ $(yy'=0)$

$= xy'+xz'+0+0$.

$= xy'+xz'$.

$= xy'\cdot 1 + xz'\cdot 1$ (identity)

$= xy'\cdot(z+z') + xz'\cdot(y+y')$

(b) $xy + yz + xz$

(i) with using TT.

x	y	z	xy	yz	xz	$xy+yz+xz$	maxterms
0	0	0	0	0	0	0 ✓	$x+y+z$ M_0
0	0	1	0	0	0	0 ✓	$x+y+z'$ M_1
0	1	0	0	0	0	0 ✓	$x+y'+z$ M_2
0	1	1	0	1	0	1	$x+y'+z'$ ✓ M_3
1	0	0	0	0	0	0 ✓	$x'+y+z$ M_4
1	0	1	0	0	1	1	$x'+y+z'$ ✓ M_5
1	1	0	1	0	0	1	$x'+y'+z$ ✓ M_6
1	1	1	1	1	1	1	$x'+y'+z'$ ✓ M_7

∴ the sum of product of canonical form of F

m_0, m_1, m_2, m_4 (or) $\sum(0,1,2,4)$ (or)

$(x+y'+z')(x'+y+z')(x'+y'+z)(x'+y'+z')$

(ii) without using TT:

$xy + yz + xz = (xy + yz) + xz$ (Ass)

$(xy+0) + (yz+0) + (xz+0)$

$= ((x+z)\cdot y) + xz$ (Distributive)

$= ((x+z) + xz)\cdot(y + xz)$

$= (x + (z + xz))\cdot((y+x)\cdot(y+z))$

$= (x+z)\cdot(y+x)\cdot(y+z)$ (Absorption)

$= (x+z+0)\cdot(x+y+0)\cdot(y+z+0)$

$= (x+z+(y\cdot y'))\cdot(x+y+(z\cdot z'))(y+y+xx')$

$= (x+y+z)(x+y'+z)(x+y+z')(x'+y+z)$

7. K-map.

(a) $f(x,y) = xy + x'y$.

$\underline{0\ 0}\quad 1\ 0$

$0 + 2$

$f(x,y) = \Sigma(0,2)$.

$x \backslash y$	0	1
0	m_0 1	m_1
1	m_2 1	m_3

$f(x,y) = y'$.

$xy + x'y$.

$= y(x + x')$

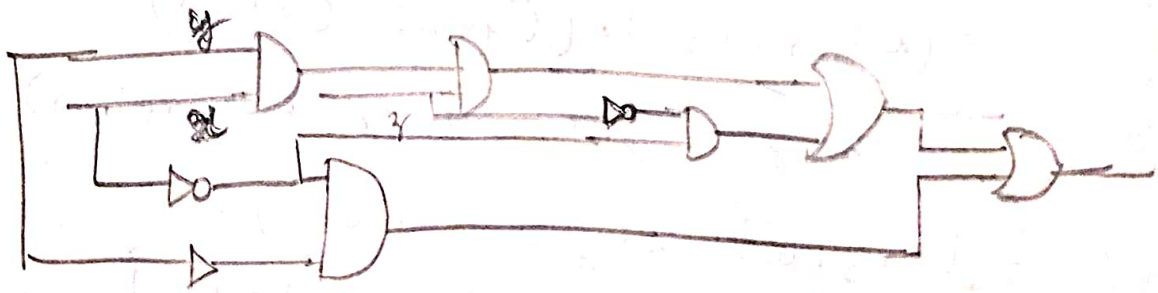
$= y' \cdot (1)$

$= y'$

(b) $f(x,y,z) = \Sigma(0,1,2,7)$.

$x \backslash yz$	00	01	11	10
0	m_0 1	m_1 1	m_3	m_2 1
1	4	5	7 1	6

$f(x,y,z) = x'y' + x'z' + xyz$



(c) $f(x,y,z) = \Sigma(1,3,5,7)$

$x \backslash yz$	00	01	11	10
0	0	1 (1)	3 1	2
1	4	5 1	7 1	6

$f(x,y,z) = z.$

001 ✓
011
101
111

001
011
$x'z$

z

101
111
xz

$x'z + xz$

$z(x'+x)$

$z \cdot (1)$

$= z$

(d) $f(a,b,c,d) = \Sigma(0,1,3,5,9,11,13)$

$ab \backslash cd$	00	01	11	10
00	0 1	1 1	3 1	2
01	4	5 1	7	6
11	12	13 1	15	14
10	8	9 1	11 1	10

0001 ✓
0011
$a'b'd$

0000
0001
$x'y'z'$

0001 ✓
0001
1101
1001
$c'd'$

0001 ✓
0011
1011
1001
d

$f(a,b,c,d) = \cancel{a'b'+c'd}.\ a'b'c'+c'd+b'd.$

5. (a) $(x+y)(x'+y')(x'+z')$

(ii) without using ++.

$= (x+y+0)(x'+y'+0)(x'+0+z')$

$= ((x+y)+(zz'))((x'+y')+(zz'))(x'+z'+yy')$

$= (x+y+z)\cdot(x+y+z')\cdot(x'+y'+z)(x'+y'+z')$

$(x'+y+z')(\cancel{x'+y'+z'})$

$\therefore$ the P-O-S canonical form is $\cancel{5+6+7}$.

$\Pi\,(0,1,5,6,7)$

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6. (i) $F(x,y,z) = xyz + x'yz + xy$

$= xy + x'yz$ (Absorption

$= y(x + x'z) = y\cdot((x+x')\cdot(x+z))$

$= y\cdot(1\cdot(x+z))$

$= y\cdot(x+z)$

$F(x,y,z) = y(x+z)$.

(ii) $G(x,y,z) = (x'+y')(x+z)(y+z)$

$= (x'+y')(xy+z)$

$= (x'+y')\cdot xy + (x'+y')\cdot z$

$= x'\cdot xy + y'\cdot xy + (x'+y')\cdot z$

$= (x'\cdot x)\cdot y + (y'\cdot y)\cdot x + (x'+y')z$

$= 0\cdot y + 0\cdot x + (x'+y')z$

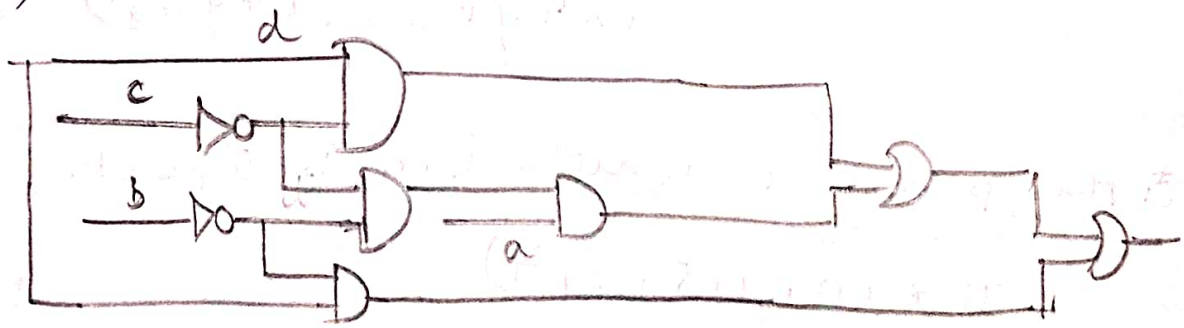
$= (x'+y')z$

$G(x,y,z) = (x'+y')z$

---

(7) logic circuits for 7th sum:

(d) $F(a,b,c,d) = c'd + a'b'c' + b'd$.



(4)

8. Simplify using k-map

$F(x, y, z) = \Pi(0, 1, 3, 5)$.

x \ yz	00	01	11	10
0	0	0	0	
1		0		

000 001 → $x'y'$

001 011 → $x'z$

001 101 → $y'z$

If it is in Π (P.O.S) → we should put 0.

If it is in Σ (S.O.P) → we should put 1

$f(x, y, z) = yz' + xy + xz'$.

Numba zero potomna we should write in '0' → not prime and in '1', we should write prime

---

9. Find the complement of the Boolean fnc's.

(I) $(BC' + A'D)(AB' + CD')$.

Soln:

$((B'+C)\cdot(A+D')) + ((A'+B)\cdot(C'+D))$.

(II) $(AB' + C'D')(AB + C'D)$

Soln:

$((A'+B)\cdot(C+D)) + ((A'+B)\cdot(C+D'))$.

30/11/21 Continuation of Permutations

Pigeonhole Principle:-

If k is a (+)ve integer and $k+1$ or more objects are placed into k boxes, then there is atleast one box containing two or more objects.

Eg:1

Among 367 people there must be atleast two with the same birth day.

Jan 1 - - - - Dec 31 365 or 366.

Eg:2

In any group of 27 english words there must be atleast two that begin with the same letter.

A, B, C, - - - , Z Pigeonholes (26)

27 words (pigeons)

1. How many students must be in a class to generalise that at least two students will have the same score in the final exam if the exam is graded on a scale from 0 to 100 points?

Ans:

No of pigeon holes 101

No. of pigeon = no. of students = 102

Generalised pigeonhole principle:

If N boxes are placed into k boxes then there is atleast one box containing at least $\lceil N/k \rceil$ object

NOTE:

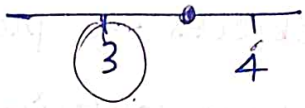
$\lceil x \rceil$ = The least integer greater than or equal to x.

ceiling function

$\lfloor x \rfloor$ = The greatest integer less than or equal to x

floor function

Eg: $x = 3.74$



$\lfloor x \rfloor = 3.$ $\quad \lfloor\ \rfloor$ $\qquad \lceil N/K \rceil.$

$\lceil x \rceil = 4$ $\quad \lceil\ \rceil$ $\qquad \lfloor\ \rfloor$

---

1. What is the minimum no. of students required in the DM class to be sure that atleast six will recive the same grade, if there are five possible grades? (A, B, C, D, E).

Soln:

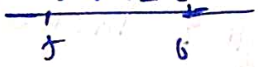
Here $N = ?$, $K = 5$, $\lceil N/K \rceil = 6$

ie; $\lceil N/5 \rceil = 6$. (The least integer $> N/5$)

$5 < \frac{N}{5} \leq 6.$

$\lceil x \rceil = 6$

$5 < x \leq 6.$



$N = 5 \cdot 5 + 1$

$N = 26.$

∴ There are 20 students in DM class to guarantee that atleast 6 will receive the same grade.

A	B	C	D	E
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

$\boxed{26}$

04/12/21

<u>K+1</u> or more objects are placed into k boxes then atleast one box containing 2 or more objects.

<u>Q PHP:</u>

If N objects are placed into k boxes then there is atleast one box containing $\lceil N/k \rceil$ objects.

<u>NOTE:</u>

1. What is the least no. of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone no's? (Assume that telephone no's are of the form N X X - N X X - X X X X, where the 1st three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit)

$\boxed{NXX} - \boxed{NXX - XXXX}$ 2+09.

9	10	10	10	10	10	10
N	X	X	X	X	X	X

$= 80,00,000.$

25 million phones.

$\therefore$ 8 million phone no's.

25 million objects among 8 million boxes.

$$\left\lceil \frac{25,00,00,00}{8000000} \right\rceil = \left\lceil \frac{25}{8} \right\rceil$$

$$= \lceil 3.025 \rceil = 4.$$

we need 4 area codes to guarantee that 25 m phones in a state can be assigned different telephone numbers.

---

2. During a month with 30 days, a baseball team plays atleast one game a day, but no more than 45 games, show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Soln:

Let a_j be the numbers of games played on or before j^{th} day of the month.

then $a_1, a_2, a_3, \ldots, a_{30}$ is an increasing sequence of +ve integers with $1 \le a_j \le 45$.

$\Rightarrow a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ is also

an ~~f~~ing sequence of '+'ve integers with ~~is~~

$15 \le a_j + 14 \le 59$.

$1 \to 3$
$a_1 = 3$
$a_2 = 4$
$a_3 =$

The 60 (+ve) integers $a_1, a_2, \ldots a_{30}, a_1 + 14, \ldots, a_{30} + 14$.

are all less than or equal to 59.

$\Rightarrow$ By pigeon hole principle two of the integers are equal.

Pigeons
$1, 2, 3, \ldots, 59$
$a_1, a_2, \ldots, a_{30}, a_{30}+14$
objects

$\Rightarrow$ there must be indices i and j.

ST $\boxed{a_i = a_j + 14}$

$\Rightarrow$ there is exactly 14 games are played b/w $j+1^{th}$ day to i^{th} day.

$a_1 \ldots a_{30}, a_1 + 14, \ldots$
$a_{30} + 14$
$i = 15$
$j = 4$
59

No. of games played till 15th day = No. of games that till 9th day + 14.

06/12/21

Mathematical induction:

To prove that $P(n)$ is ~~true~~ true for all (+)ve integers n, where $P(n)$ is a propositional function, we computed two steps.

Basic steps: verify that $P(1)$ is true

If $P(k)$ then $P(k+1)$

Inductive step: we show that $P(k) \to P(k+1)$ is true for all (+)ve integers k.

1. ST: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Soln:

Let $p(n) : 1+2+3+\sim+n = \frac{n(n+1)}{2}$

when $n=1$

$P(1) = 1 = \frac{1\cdot(1+1)}{2}$

$\Rightarrow 1 = 1.$

$\therefore P(1)$ is true

Let us assume that $P(k)$ is true for some arbitary +ve integer k.

$p(k) = 1+2+3+\cdots k = \frac{k(k+1)}{2}$

when $n=k+1$.

$P(k+1): 1+2+3+\cdots+k+(k+1) = (k+1)\frac{(k+1+1)}{2}$

$k(k+1)(k+1)$
$k(k+2)$

$\Rightarrow \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$

$\Rightarrow \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

$\Rightarrow \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$

$\therefore P(k+1)$ is true.

Now by induction hypothesis, we have

$1+2+3+\cdots+n = \frac{n(n+1)}{2}$

---

2. ST: $1+2+2^2+\cdots+2^n = 2^{n+1}-1$

Soln:

Let $P(n) = 1+2+2^2+\cdots+2^n = 2^{n+1}-1$

when $n=0$.

$P(0) = 1 = 2^{0+1} - 1$

$1 = 1 \quad 1 = 2^1 - 1$

$P(0)$ is true.

$P(k) = 2^k - 2^{k+1} - 1$

Let us assume that $P(k)$ is true where k is an arbitrary (+)ve integer.

(ie); $1 + 2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 1$ — ①.

when $n = k+1$

$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$

[By ①]

$= 2 - 2^{k+1} - 1$

$= 2^{k+2} - 1.$

$\therefore P(k+1)$ is true.

By induction hypothesis $P(k+1)$ is true.

Thus, $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$

---

3. P.T:- $n < 2^n$ for all (+)ve integers n.

Soln:

Let $P(n) = n < 2^n$

$P(1) = 1 < 2^1 = 2$

$\therefore P(1)$ is true.

Let us assume that $P(k)$ is true where k is some arbitrary (+)ve integer.

(ie;) $k < 2^k$ — ①

$5 < 10$

< 25

$k+1 = 2^{k+1}$

$a < b$

when $n = k+1$

$k+1 < 2^{k+1}.$

$< 2^k + 2^k \quad (\because 1 < 2^k)$

$= 2 \cdot 2^k$

$= 2^{k+1}$.

(ie;) $k+1 < 2^{k+1}$.

$\therefore P(k+1)$ is true.

Hence by induction hypothesis $P(n)$ is true

(ie;) $n < 2^n$

---

4. ST:: $2^n < \lfloor n$ for all (+)ve integers $n \geq 4$.

Soln:

Let $p(n) = 2^n < \lfloor n$

When $n=4$:

$P(4) = 2^4 = 16 < \lfloor 4 = 24$.

(ie;) $2^4 < \lfloor 4$.

$\therefore P(4)$ is true.

Let us assume that the result holds good for any arbitary +ive integer $k \geq 4$.

(ie;) $2^k < \lfloor k$ — ①

When $n = k+1$

$2^{k+1} = 2^k \cdot 2$

$< 2 \cdot \lfloor k$ [By ①]

$< (k+1) \cdot \lfloor k = \lfloor k+1$

(ie;) $2^{k+1} < \lfloor k+1$

$\therefore P(k+1)$ is true.

Thus by induction hypothesis $p(n)$ is true.

5. P.T:- $n^3 - n$ is divisible by 3 where n is a (+)ve integer.

Soln:

Let $p(n)$ be the statement that

" $n^3 - n$ is divisible by 3".

When $n = 1$

$P(1) = 1^3 - 1$ is divisible by 3

$\therefore P(1)$ is true.

Let us assume that $P(k)$ is true where k is an arbitary (+)ve integer.

(i.e.) $k^3 - k$ is dividible by 3

$\Rightarrow k^3 - k = 3r$ where r is some integer

$\hookrightarrow$ ①.

When $n = k+1$

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$$

$$= k^3 - k + \underline{3k^2 + 3k}$$

$$= 3r + 3(k^2 + k) \quad [\text{By } ①]$$

$$(k+1)^3 - (k+1) = 3[r + k^2 + k].$$

$\Rightarrow (k+1)^3 - (k+1)$ is divisible by 3.

$\therefore P(k+1)$ is true.

Thus by induction hypothesis $P(n)$ is true

Home Work:

1. PT. $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every non-negative integer n.

2. PT:- If a set A has n elements then A has 2^n subsets.

3. State and prove Gossipping problem.

---

1. PT. $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n = \frac{3(5^{n+1}-1)}{4}$, $n \geq 0$.

Soln:

$$P(n) = 3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n = \frac{3(5^{n+1}-1)}{4}$$

when $n=0$

$P(0) = 3 = 3$

$\therefore P(n)$ is true when $n=0$.

Let us assume that $P(k)$ is true.

(ie;) $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k = \frac{3(5^{k+1}-1)}{4}$ — ①

Put $n=k+1$.

$$3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k+3\cdot 5^{k+1}$$

$$= \frac{3(5^{k+1}-1)}{4} + 3\cdot 5^{k+1}.$$

$$= 3\left(\frac{5^{k+1}-1}{4}\right) + 5^{k+1}.$$

$5\cdot 5^{k+1}$

$$= 3\left[\frac{5^{k+1}-1+4\cdot 5^{k+1}}{4}\right]$$

$$= 3\left[\frac{5 \cdot 5^{k+1} - 1}{4}\right]$$

$$= \frac{3\left[5^{k+2} - 1\right]}{4}$$

$\therefore P(k+1)$ is true.

Thus, by induction principle P

2/ PT:- $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2$

$$= \frac{(n+1)(2n+1)(2n+3)}{3}, \quad n \geq 0$$

soln.

Let $P(n) :- 1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

when $n = 0$

$$P(0) = 1^2 = \frac{(0+1)(0+1)(0+3)}{3}$$

$$1^2 = 1.$$

$\therefore P(n)$ is true for $n = 0$.

Let us assume that $P(n)$ is true for $n = k$

(i.e.) $1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$ ↳ ①

when $n = k+1$.

$$1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2k+3)^2$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3} + \quad [\text{By } ①]$$

$$= \frac{(k+1)(2k+1)(2k+3) + 3(2k+3)^2}{3}$$

$$= \frac{(2k+3)\left[k(k+1)(2k+1) + 3(2k+3)\right]}{3}$$

$$= \frac{(2k+3)\left[2k^2 + k + 2k + 1 + 6k + 9\right]}{3}$$

$$= \frac{(2k+3)\left[2k^2 + 9k + 10\right]}{3} \geq 0$$

20

$+5$

$4+$

$$= \frac{(2k+3)\left[2k^2 + 4k + 5k + 10\right]}{3}$$

$$= \frac{(2k+3)\left[2k(k+2) + 5(k+2)\right]}{3}$$

$P(1) \Rightarrow P(0)$

---

3. PTS. $1^3 + 3^3 + 5^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2, \; n \geq 1.$

Soln.

Let $P(n)$: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \; n \geq 1.$

when $n = 1$;

$$1^3 = \left[1\left(\frac{1+1}{2}\right)\right]^2.$$

$$1 = 1$$

$\therefore P(n)$ is true when $n = 1$.

Let us assume that $P(n)$ is true when $n = k$.

(i.e.) $1^3+2^3+3^3+\cdots+k^3 = \left[\frac{k(k+1)}{2}\right]^2$ — ①

when $n=k+1$;

$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3.$$

$$= \frac{k^2(k+1)^2}{4}+(k+1)^3.$$

$$= (k+1)^2\left[\frac{k^2}{4}+(k+1)\right]$$

$$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$\therefore P(n)$ is true when $n=k$

Thus, by induction principle $p(n)$ is true.

---

4. PT:- $1\cdot1!+2\cdot2!+3\cdot3!+\cdots+n\cdot n! = (n+1)!-1,\ n\geq 1$

Soln:

When $n=1$;

$P(1): 1\cdot1! = (1+1)!-1.$

$1=1.$

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true for $n=k$

(i.e.) $1\cdot1!+2\cdot2!+3\cdot3!+\cdots+k\cdot k! = (k+1)!-1.$ — ①

When $n=k+1$;

By ①

$1\cdot1!+2\cdot2!+3\cdot3!+\cdots+k\cdot k!+(k+1)(k+1)! = (k+1)!-1+(k+1)(k+1)!$

$= (k+1)!\,[1+k+1]-1$

$= (k+2)(k+1)! - 1.$

$= (k+2)! - 1.$

$\therefore P(n)$ is true when $n=k$.

Thus, by induction principle $p(n)$ is true.

---

5. P.T n^3-n is divisible by 6 for $n \geq 0$.

6. P.T. 21 divides $4^{n+1}+5^{2n-1}$, whenever n is a +'ve integer.

7. PT:- $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1)$.

for $n \geq 1$.

$\boxed{\text{Home work sums}}$

~~7~~ $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1)$.

Soln:

Let $P(n) = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1)$

When $n=1$;

$P(1): 1 > 2(\sqrt{1+1}-1)$

$1 > 0.9.$

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true for $n=k$.

(i.e.) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}} \geq 2(\sqrt{k+1}-1)$. — (1).

when $n=k+1$;

$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > 2(\sqrt{k+1}-1)+\frac{1}{\sqrt{k+1}}$

$$= \frac{2\sqrt{k+1}\sqrt{k+1} - 2\sqrt{k+1}}{\sqrt{k+1}}$$

$$= \frac{2(k+1) - 2\sqrt{k+1}}{\sqrt{k+1}}$$

Home work sum.

5. $n^3 - n$ is divisible by 6, $n \geq 0$.

Let $p(n) = n^3 - n$.

when $n = 0$;

$0^3 - 0$

$1 = 0$.

$n^3 - n = 6r$.

Let us assume that $p(n)$ is true for $n = k$.

(i.e.) $k^3 - k = 6r$.

when $n = k+1$

$P(k+1): (k+1)^3 - (k+1) = 6r$.

$= k^3 - k + (k+1)^3 - k = 6r$.

$+ (n+1)^3 - n$.

$(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k$

$= k^3 + 3k^2 + 2k$.

(1) — $= 3 \cdot (k^2 + k) + (k^3 - k)$

$P(k+1)$ is true

Thus, by induction hypothesis $P(n)$ is true.

6. 21 divides $4^{n+1} + 5^{2n-1}$, whenever n is a (+)ve integer.

Soln:

$4^{n+1} + 5^{2n-1}$ is divisible by 21.

~~P(1)~~

when $n=1$,

$P(1) = 4^{(1)+1} + 5^{(2)1-1}$.

$= 4^2 + 5^1$

$= 16 + 5$

$= 21$, which is divisible by 21.

$\therefore P(1)$ is true.

Let us assume that $P(n)$ is true for $n=k$.

(ie;) $4^{k+1} + 5^{2k-1}$ — ①.

when $n=k+1$;

$= 4^{(k+1)+1} + 5^{2(k+1)-1}$.

$= 4^{k+2} + 5^{2k+2-1}$

$= 4^{k+2} + 5^{2k+1}$.

$= 4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1}$.

$= 4 \cdot 4^{k+1} + (21+4) \cdot 5^{2k-1}$

$= 4 \cdot 4^{k+1} + 21 \cdot 5^{2k-1} + 4 \cdot 5^{2k-1}$.

$= 4(4^{k+1} + 5^{2k-1}) + 21 \cdot 5^{2k-1}$

$= 4 \cdot 21A + 21 \cdot 5^{2k-1}$

$= 21 \cdot 4A + 21 \cdot 5^{2k-1}$.

$= 21(4r + 5^{2k-1})$ which is divisible by 21

$P(n)$ is true when $n = k$.

Thus, by inductive hypothesis $P(n)$ is true.

08/12/21

PT:- $7^{n+2} + 8^{2n+1}$ is divisible by 57 when $n \geq 0$.

soln.

$7^2 + 8 = 57$ $P(0)$.

$57 / 7^{k+2} + 8^{2k+1}$

(ie;) $57r = 7^{k+2} + 8^{2k+1}$ — ①.

$7^1 \cdot 7^{k+3} + 8^{2(k+1)+1} = 7^{k+3} + 8^{2k+3}$

$= 7(7^{k+2}) + 8^2 \cdot 8^{2k+1}$

$= 7(7^{k+2}) + 57 \cdot 8^{2k+1} + 7 \cdot 8^{2k+1}$

$= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}$

$= 7(57r) + 57 \cdot 8^{2k+1}$

$= 57(7r + 8^{2k+1})$

$7^{k+3} + 8^{2k+3}$ is divisible by 57//

Note:-

$|A| = n$ A has 2^n subsets

- $A = \{\}$ Empty set. (A contains no elements).

$P(A) = \{\{\}\}$ $|A| = 0$.

↳ A has 1 subset

- $A = \{1, 2\}$

$P(A) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$

A has 2^2 subsets. ie; A.

~~Assumption~~

when $n=0$;

$|A|=0$.

$\Rightarrow A = \{\}$

$\Rightarrow$ A has only one subset (ie) $\{\}$.

$P(0)$ is true.

$P(k)$ is true..

when $n=k+1$; $|A| = k$

$|A| = k+1$. A has 2^k subsets.

GOSSIP PROBLEM:

There are n people in a group each one knows a scandal, which no one else in the group knows about. If two people talk over telephone they can share the information about all the scandals each knows.

09/12/21

Strong Induction principle:

To prove $P(n)$ is true for all (+)ve integers n, where $P(n)$ is a propositional function, we complete two steps.

- Basis steps:

  we verify that $P(1)$ is true.

- Inductive steps:

  we show that

  $(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+)$ is true $\forall$ (+)ve integer k.

---

1. St if n is an integer greater that 1 then n can be written as a product of primes.

Soln:

Let $P(n)$ be "n can be written as product of primes".

When $n=2$

$P(2)$ is true because 2 can be written as the product of one prime (ie) itself.

Let us assume that $P(j)$ is true for all integers j with $2 \leq j \leq k$.

(ie;) j can be written as product of prime if $2 \leq j \leq k$

Now, TPT $k+1$ can be written as product of primes

if $k+1$ is a prime number then there is nothing to prove.

If $k+1$ is a composite numbers then $\exists$ integers $2 \le a, b < k$ such that $k+1 = ab$.

$\Rightarrow$ By Induction Hypothesis, a and b can be written as product of primes.

$\Rightarrow$ $k+1$ can be written as product of primes.

ie $P(k+1)$ is true.

Thus, by strong induction principle $P(n)$ is true.

---

2. PT every amount of postage of 12 rupees and above can be formed using just Rs 4/- and Rs 5/- stamps.

<u>soln:</u>

$\underline{n = 12}$

$P(12)$ is true

$/ \; | \; \backslash$

4 4 4

$P(j)$ is true $12 \le j \le k$.

$k \to$ 'a', 'b'.

<u>when $n = k+1$;</u>

case (i) :- If atleast one Rs. 4/- stamp is used

Replace one Rs. 4/- stamp by on Rs. 5/- stamp.

$a \ge 1$.

Rs. k/- $\to$ a Rs. 4/- and b Rs. 5/-

$k+1 = (a-1)(b+1)$

case (ii) :- There is no Rs. 4/- stamp.

$k - (b)$ Rs. 5/- $\quad b \geq 3$

$b < 3$

Replace 3 Rs. 5/- $\quad b = ?$ or $b \leq 2$.

by 4 Rs. 4/-

$k = 5b$

$k+1 = 5(b-3) + 4 \cdot 4$ $\quad a \geq 1$.

$k - 4 + 5 \quad k = 4a + 5b$

$k+1 \quad k+1 = 4(a-1) + 5(b+1)$

10/12/21. (special class)

Recurrance relation:

Sequence: $\{0, 1, 2, \ldots\} \to R$

It is a function from set of non $\ominus$ve integers

$\{1, 2, 3, \ldots\} = \{n\}_{n=1}^{\infty}$ $\qquad \{1, 1, 1, 1, \ldots\} = \{1^n\}_{n \geq 1}^{\infty}$

$\{1, -1, 1, -1, 1, -1, \ldots\} = \{(-1)^{n+1}\}_{n=1}^{\infty}$

Requirence relation:

A requirence relation for the sequence $\{a_n\}$ is a formula that express a_n in terms of one or more preceding terms of the sequence viz $a_0, a_1, a_2, \ldots a_{n-1}$ for $n \geq n_0$. Here n_0 a non-negative interger is used to define the initial condition.

Eg: ① $a_n = a_{n-1} + 2$, $n \geq 1$ $\quad a_0 = 1$.

② $a_n = a_{n-1} + a_{n-2} + n$, $n \geq 2$, $a_0 = 2$, $a_1 = 5$.

③ $nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1} \quad \forall n > 0.$

$nC_0 = 1 \; ; \; nC_n = 1.$

Soln:

A sequence is called as solution of a recurrence relation if its terms satisfy the recurrence relation.

$$\underset{F_n}{\underline{|n}} = n \underset{F_{n-1}}{\underline{|(n-1)}}$$

$$F_n = nF_{n-1}, \quad F_0 = 1$$

$\{1, 1, 2, 6, 24, 120, 720, \ldots\}$ $\boxed{n!}$

$= \{\underline{|n}\}_{n=0}^{\infty}$

$\rightarrow F_n = nF_{n-1}, \; n \geq 1, \; F_0 = 1.$

Definition:-

The order of a recurrence relation is defined to be the difference between highest and the lowest subscripts of the dependent variable appearing in the relation.

$(n+2)^{th}$ term = 5 times $(n+1)^{th}$ term − 6 times n^{th} term.

$\rightarrow a_{n+2} - 5a_{n+1} + 6a_n = 0.$

$\rightarrow a_n - 5a_{n-1} + 6a_{n-2} = 0.$

$a_{n+2} = 5a_{n+1} - 6a_{n+2}.$

$\downarrow$?

$a_k = 5a_{k-1} - 6a_{k-2}.$

$a_n = 5a_{n-1} - 6a_{n-2}.$

$n+2$ by k

$n+2 = k$

$n+1+1 = k$

$n+1 = k-1.$

---

② $a_n - 5a_{n-1} + 6a_{n-2} = 0; \; a_0 = 2, \; a_1 = 5.$

PTO

$n=2$

$$a_2 - 5a_1 + 6a_0 = 0.$$

$$a_2 - 5(5) + 6\times 2 = 0$$

$$a_2 - 25 + 12 = 0$$

$$\boxed{a_2 = 13}$$

$n=3$

$$a_3 = 5a_2 + 6a_1 = 0$$

$$a_3 = 5 - 13 - 6\times 5$$

$$a_3 = 65 - 30$$

$$\boxed{a_3 = 35}$$

$n=4$

$$a_4 = 5a_3 - 6a_2$$

$$a_4 = 5\times 35 - 6\times 13$$

$$a_4 = 175 - 78$$

$$\boxed{a_4 = 97}$$

$\vdots$

$\{2, 5, 13, 35, 97, \ldots\}$

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

$$a_{n-1} = 5a_{n-3} - 6a_{n-4}.$$

$\vdots$

$$a_3 = 5a_2 - 6a_1$$

$$a_2 = 5a_1 - 6a_0.$$

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{3x} + 6\cdot 4x.$$

$$\boxed{y = C.F. + P.I.}$$

Solving recurrence relation :-

consider a linear k^{th} order recurrance relation with constant co-efficients

$$C_0 a^r + C_1 a^{r-1} + C_2 a^{r-2} + \cdots + C_k a^{r-k} = f(r)$$

where $C_0, C_1, C_2, C_3, \ldots, C_k$ are all constants and $C_0 \neq 0$. — ①

The recurrence relation ① is.

Homogeneous if $f(r) = 0$

Non-homogeneous if $f(r) \neq 0$.

The soln. of ① is

$a_r = H.S + P.S. \rightarrow$ Particular soln.

↳ Homogeneous soln.

To find homogeneous soln:-

Step 1:- consider the homogeneous relation for ①.

Step 2:- Form the characteristic equation.

Step 3:- Find the characteristic roots.

Let $m_1, m_2, \ldots, m_k$ be the characteristic roots.

Step 4:- case (i):- If $m_1, m_2, \ldots, m_k$ are all distinct.

$$H.S = A_1 m_1^r + A_2 m_2^r + \cdots + A_k m_k^r$$

where $A_1, A_2, \ldots, A_k$ are arbitrary constant

case (ii):- If $m_1 = m_2 = \cdots = m_j = m$ (say)

$$H.S = (A_1 + A_2 r + A_3 r^2 + \cdots + A_j r^{j-1}) m^r$$

$$+ A_{j+1} m_{j+1}^r + A_{j+2} m_{j+2}^r + \cdots + A_k m_k^r$$

Note:

The arbitrary constant $A_1, A_2, \ldots A_k$ can be found out by using initial conditions.

To form characteristic equations:-

Replace a_r by m^k

a_{r-1} by m^{k-1}

a_{r-2} by m^{k-2}

$\vdots$

a_{r-k} by $m^0 = 1$

$$\boxed{\begin{aligned} a_3 - 7a_2 + a_1 &= 0 \\ m^2 - 7m + 1 &= 0 \end{aligned}}$$

---

1. Solve $a_n - 6a_{n-1} + 8a_{n-2} = 0$.

Soln.

$$a_n - 6a_{n-1} + 8a_{n-2} = 0 \quad \text{— (1)}$$

The characteristic eqn is

$$m^2 - 6m + 8 = 0 \quad \text{— (2)}.$$

$$m = \frac{6 \pm \sqrt{36-32}}{2} = \frac{6 \pm 2}{2} = 4, 2.$$

$$m = 2, 4$$

$$H.S = A_1 2^n + A_2 4^n.$$

$$\therefore a_n = A_1 2^n + A_2 4^n$$

---

2. Solve $a_n + 10a_{n-1} + 21a_{n-2} = 0$, given $a_0 = 1$, $a_1 = 2$.

Soln.

$a_n + 10a_{n-1} + 21a_{n-2} = 0$ — ①

The characteristic equation is

$m^2 + 10m + 21 = 0$ — ②

$$m = \frac{-10 \pm \sqrt{100 - 84}}{2} = \frac{-10 \pm 4}{2} = -3, -7$$

$m = -3, -7.$

$\therefore a_n = A_1(-3)^n + A_2(-7)^n$ — ③

Put $n = 0$ in ③, we get

$a_0 = A_1(-3)^0 + A_2(-7)^0$

$1 = A_1 + A_2$

(ie) $A_1 + A_2 = 1$ — ④

Put $n = 1$ in ③, we get

$a_1 = A_1(-3)^1 + A_2(-7)^1$

$-3A_1 - 7A_2 = 2$ — ⑤

④ × 3 + ⑤

$3A_1 + 3A_2 = 3.$

$-3A_1 - 7A_2 = 2$

$-4A_2 = 5$

$\boxed{A_2 = -5/4}$

Sub $A_2 = -5/4$ in ④

$A_1 - 5/4 = 1.$

$A_1 = 1 + 5/4.$

$\boxed{A_1 = 9/4}$

$\therefore a_n = 9/4\,(-3)^n - 5/4\,(-7)^n$.

(By substituting $A_1 = 9/4$ and $A_2 = -5/4$ in ③)

<u>Verification:</u>

$n = 0$,

$$a_0 = \frac{9}{4} - \frac{5}{4} = \frac{4}{4} = 1$$

$$a_1 = \frac{9}{4}(-3) - \frac{5}{4}(-7) = -\frac{27}{4} + \frac{35}{4} = \frac{8}{4} = 2$$

$$a_n + 10a_{n-1} + 21a_{n-2} = 0.$$

$$\left[\frac{9}{4}(-3)^n - \frac{5}{4}(-7)^n\right] + 10\left[\frac{9}{4}(-3)^{n-1} - \frac{5}{4}(-7)^{n-1}\right]$$

$$+ 21 = \frac{9}{4}(-3)^n - \frac{5}{4}(-7)^n$$

$$+ \frac{30}{4}(-3)^n + \frac{25}{14}(-7)^n + 21$$

$$a_n = \frac{9}{4}(-3)^n - \frac{5}{4}(-7)^n$$

$$a_{n-1} = \frac{9}{4}(-3)^{n-1} - \frac{5}{4}(-7)^{n-1}$$

$$a_{n-2} = \frac{9}{4}(-3)^{n-2} - \frac{5}{4}(-7)^{n-2}$$

$$a_n + 10a_{n-1} + 21a_{n-2} =$$

$$\frac{9}{4}(-3)^n - \frac{5}{4}(-7)^n$$

$$+ \frac{90}{4}(-3)^{n-1} - \frac{50}{4}(-7)^{n-1}$$

$$+\frac{189}{4}(-3)^{n-2} - \frac{105}{4}(-7)^{n-2}$$

$$= (-3)^n\left[\frac{9}{4} - \frac{30}{4} + \frac{21}{4}\right] - (-7)^n\left[\frac{5}{4} - \frac{50}{28} - \frac{7}{4}\right]$$

$$= 0$$

---

3. solve $a_n + 12a_{n-1} + 36a_{n-2} = 0$, given that $a_0 = 1 \mp a_1$

Soln:

$$a_n + 12a_{n-1} + 36a_{n-2} = 0 \quad - (1)$$

$$m^2 + 12m + 36 = 0 \Rightarrow m = \frac{-12 \pm \sqrt{144-144}}{2}$$

$$m = -6, -6. \qquad = -6, -6.$$

$$a_n = (A_1 + A_2 n)(-6)^n \quad - (2)$$

$n=0$

$$\cancel{a_0 = A_1(-6)^n}. \quad \therefore a_0 = A_1 \cdot 1.$$

$$\boxed{A_1 = 1} \quad (\because a_0 = 1)$$

$$a_1 = (A_1 + A_2)(-6)$$

$$1 = (1 + A_2)(-6)$$

$$1 = -6 - 6A_2.$$

$$\Rightarrow -6A_2 = 7.$$

$$\boxed{A_2 = -7/6}$$

$$a_n = \left(1 - \frac{7}{6}n\right)(-6)^n$$

---

4. solve $F_n = F_{n-1} + F_{n-2}$, $F_0 = 1 = F_1$

Soln:

$$F_n = F_{n-1} + F_{n-2} \quad - (1)$$

The characteristic eqn is $m^2 - m - 1 = 0$.

$$m = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$F_n = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$n=0$;

$$F_0 = A_1 + A_2 \Rightarrow A_1 + A_2 = 1. \quad - (3)$$

$$F_1 = A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\left(\frac{1+\sqrt{5}}{2}\right)A_1 + \left(\frac{1-\sqrt{5}}{2}\right)A_2 = 1. \quad - (4)$$

$$(3) \times \frac{1+\sqrt{5}}{2} \quad - (4)$$

$$\cancel{\frac{1+\sqrt{5}}{2}A_1} + \frac{1+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$$

$$\cancel{\frac{1+\sqrt{5}}{2}A_1} + \frac{1-\sqrt{5}}{2}A_2 = 1.$$

$(-)$ $(-)$ $(-)$

$$\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right]A_2 = \frac{1+\sqrt{5}}{2} - 1$$

$$\sqrt{5}\,A_2 = \frac{1+\sqrt{5}-2}{2}$$

$$\boxed{A_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}}$$

$$A_1 = 1 - \frac{\sqrt{5}-1}{2\sqrt{5}} = \frac{2\sqrt{5}-\sqrt{5}-1}{2\sqrt{5}}$$

$$A_1 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

sub $A_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}$, $A_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$ in (2)

$$F_n = \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^{n+1}\right).$$

HW sums:-

1. $s(n) - 6s(n-1) + 11s(n-2) - 6s(n-3) = 0$.
2. solve $s(k) - 9s(k-1) + 27s(k-2) - 27s(k-3) = 0$
3. solve $a_n + 12a_{n-1} + 48a_{n-2} - 64a_{n-3} = 0$.

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1. solve $a(k) - 11a(k-1) + 44a(k-2) - 76a(k-3) + 48a(k-4) = 0$.

given that $a_0 = 1$, $a_1 = 1$, $a_2 = 2$.

soln.

$$a(k) - 11a(k-1) + 44a(k-2) - 76a(k-3) + 48a(k-4) = 0 \quad - (1).$$

The characteristic function is

$$m^4 - 11m^3 + 44m^2 - 76m + 48 = 0 \quad - (2)$$

	1	-11	44	-76	48	
2	0	2	-18	52	-48	
2	1	-9	26	-24	0	
	0	2	-14	24		
	1	-7	12	0		

± 1 ± 8
± 2 ± 12
± 3 ± 16
± 4 ± 24
± 6 ± 48.

$$m^2 - 7m + 12 = 0.$$

$$m^2 - 7m + 12 = 0.$$

$$m = \frac{7 \pm \sqrt{49-48}}{2} = \frac{7 \pm 1}{2} = 4, 3.$$

$\therefore m = 2, 2, 3, 4.$

$$H.S = a(k) = (A_1 + A_2 k)2^k + A_3 3^k + A_4 4^k \quad - (3)$$

Put $k=0$ in (3) we get;

$$a(0) = (A_1 + A_2 0)2^0 + A_3 3^0 + A_4 \cdot 4^0$$

$$0 = A_1 + A_3 + A_4.$$

ie; $A_1 + A_3 + A_4 = 0 \rightarrow (4)$

Put $k=1$ in (3) we get;

$$a(1) = (A_1 + A_2 \cdot 1)2^1 + A_3 \cdot 3^1 + A_4 \cdot 4^1.$$

$$1 = 2A_1 + 2A_2 + 3A_3 + 4A_4.$$

ie; $2A_1 + 2A_2 + 3A_3 + 4A_4 = 1 \quad - (5)$

Put $k=2$ in (3). we get;

$$a(2) = (A_1 + A_2 \cdot 2)2^2 + A_3 \cdot 3^2 + A_4 \cdot 4^2.$$

$$2 = 4A_1 + 8A_2 + 9A_3 + 16A_4$$

$$4A_1 + 8A_2 + 9A_3 + 16A_4 = 2 \quad - (6)$$

Put $k=3$ in (3) we get;

$$a(3) = (A_1 + A_2 \cdot 3)2^3 + A_3 \cdot 3^3 + A_4 \cdot 4^3.$$

$$3 = 8A_1 + 24A_2 + 27A_3 + 64A_4$$

ie; $8A_1 + 24A_2 + 27A_3 + 64A_4 = 3 \quad - (7).$

On solving (4), (5), (6) and (7);

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 1 \\ 4 & 8 & 9 & 16 & 2 \\ 8 & 24 & 27 & 64 & 3 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 & -1 \\ 0 & 8 & 5 & 12 & -2 \\ 0 & 24 & 19 & 56 & -5 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - 8R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 & -1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 7 & -88 & 7 \end{bmatrix} \begin{matrix} \\ R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 12R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 & -1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & -116 & -7 \end{bmatrix} \begin{matrix} \\ \\ R_4 \to R_4 - 7R_3 \\ \\ \end{matrix}$$

$$A_1 + A_3 + A_4 = 1$$

$$2A_2 + A_3 + 2A_4 = -1$$

$$A_3 + 4A_4 = 2$$

$$-116A_4 = -7$$

$$\boxed{A_4 = \frac{7}{116}}$$

$$A_3 + \cancel{4}^{1} \times \frac{7}{\cancel{116}_{29}} = 2$$

$$A_3 = 2 - \frac{7}{29}$$

$$\boxed{A_3 = \frac{51}{29}}$$

$$2A_2 + A_3 + 2A_4 = -1$$

$$2A_2 + \frac{51}{29} + 2 \cdot \frac{7}{116} = -1$$

$$2A_2 = -1 - \frac{51}{29} - \frac{14}{116}$$

$$= \frac{-116 - 204 - 14}{116}$$

$$= -\frac{334}{116}$$

$$\boxed{A_2 = \frac{-167}{116}}$$

$$A_1 + A_3 + A_4 = 1.$$

$$A_1 + \frac{51}{29} + \frac{7}{116} = 1$$

$$A_1 = 1 - \frac{51}{29} - \frac{7}{116}$$

$$A_1 = \frac{116 - 204 - 7}{116}$$

$$\boxed{A_1 = -\frac{95}{116}}$$

$\therefore A_1 = -\frac{95}{116}$, $A_2 = \frac{-167}{116}$; $A_3 = \frac{51}{29}$; $A_4 = \frac{7}{116}$.

Sub these values ie; A_1, A_2, A_3, A_4 in ③.

$$\boxed{a(k) = \left(\frac{-95}{116} - \frac{167k}{116}\right)2^k + \frac{51}{29}3^k + \frac{7}{116}4^k}$$

<u>To find particular solution:</u>

consider $C_1 a_r + C_2 a_{r-1} + C_3 \cdot a_{r-3} + \cdots + C_k a_{r-k} = f(r)$

Type I: $f(r)$ = constant.

Then the particular soln can be obtained by taking the assumed soln $f(r) = p$, where p is a constant, if '1' is not a characteristic root

$$f(r) = \begin{cases} p, & \text{if '1' is not a characteristic root} \\ r^m p, & \text{if '1' is a characteristic root with multiplicity } m \end{cases}$$

---

1. solve $a_r - 5a_{r-1} + 6a_{r-2} = 3$.

Soln: The assumed

$$a_r - 5a_{r-1} + 6a_{r-2} = 3 \quad \text{— (1)}$$

The associated homogeneous relation for (1) is

$$a_r - 5a_{r-1} + 6a_{r-2} = 0 \quad \text{— (2)}$$

The characteristic eqn is $m^2 - 5m + 6 = 0$.

$$m = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} = 3, 2.$$

$m = 2, 3$.

Now, the H-S is

$$a_r^{(h)} = A_1 2^r + A_2 3^r \quad \text{— (3)}$$

To find particular soln:-

Let $a_r = p$

Then (1) $\rightarrow p - 5p + 6p = 3$.

$\rightarrow 2p = 3$

$\rightarrow \boxed{p = 3/2}$

$\therefore a_r^{(p)} = 3/2$ — (4)

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = A_1 2^r + A_2 3^r + \frac{3}{2}$$

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1. Solve $a_r - 5a_{r-1} + 6a_{r-2} = 3$.

Soln:

$$a_r - 5a_{r-1} + 6a_{r-2} = 3 \quad \text{— (1)}$$

the associated homogeneous relation to (1) is

$$a_r - 5a_{r-1} + 6a_{r-2} = 0 \quad \text{— (2)}$$

The characteristic eqn is.

$$m^2 - 5m + 6 = 0 \quad \text{— (3)}$$

$$m = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} = 3, 2.$$

$$m = 2, 3.$$

$$a_r^{(hs)} = A_1 2^r + A_2 3^r \quad \text{— (4)}$$

to find particular soln;

Put $a_r = p$ in (1), we get — (5)

$$p - 5p + 6p = 3.$$

$$2p = 3$$

$$\boxed{p = \frac{3}{2}}$$

$$\therefore a_r^{(ps)} = \frac{3}{2} \quad \text{— (6)}$$

Now, $a_r = a_r^{(hs)} + a_r^{(ps)}$

$$\boxed{\therefore a_r = A_1 2^r + A_2 3^r + \frac{3}{2}}$$ (By (4) & (6))

2. Solve $a_r - 9a_{r-1} + 26a_{r-2} - 24a_{r-3} = 7$

Soln:

$$a_r - 9a_{r-1} + 26a_{r-2} - 24a_{r-3} = 7 \quad - (1)$$

The associated homogeneous solution relation to (1) is

$$a_r - 9a_{r-1} + 26a_{r-2} - 24a_{r-3} = 0 \quad - (2)$$

The characteristic equation is:

$$m^3 - 9m^2 + 26m - 24 = 0 \quad - (3)$$

2	1	-9	26	-24
	0	2	-14	24
	1	-7	12	0

$$m^2 - 7m + 12 = 0$$

$$m = \frac{7 \pm \sqrt{49-48}}{2} = \frac{7 \pm 1}{2} = 4, 3$$

$$m = 3, 4, 2.$$

$$a_r^{(hs)} = A_1 2^r + A_2 3^r + A_3 4^r \quad - (4)$$

Put $a_r = p$ in (1) $\quad - (5)$

$$p - 9p + 26p - 24p = 7$$

$$-6p = 7$$

$$\boxed{p = -\frac{7}{6}}$$

$$\therefore a_r^{(ps)} = -\frac{7}{6} \quad - (6)$$

$$\boxed{\therefore a_r = A_1 2^r + A_2 3^r + A_3 4^r - \frac{7}{6}} \quad \text{(By (4) \& (6))}$$

$$m^2 - 7m + 12$$

$$= m^2 - 3m - 4m + 12$$

$$= m(m-3) - 4(m-3)$$

$$= (m-4)(m-3)$$

3. Solve $a_r - 6a_{r-1} + 11a_{r-2} - 6a_{r-3} = 4$.

Soln:

$$a_r - 6a_{r-1} + 11a_{r-2} - 6a_{r-3} = 4 \quad \text{— (1)}$$

The associated homogeneous relation to (1) is,

$$a_r - 6a_{r-1} + 11a_{r-2} - 6a_{r-3} = 0 \quad \text{— (2)}$$

The characteristic eqn is

$$m^3 - 6m^2 + 11m - 6 = 0 \quad \text{— (3)}$$

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$$m^2 - 5m + 6 = 0$$

$$m^2 - 2m - 3m + 6 = 0$$

$$m(m-2) - 3(m-2) = 0$$

$$(m-3)(m-2) = 0$$

$$m = 3, m = 2.$$

$\therefore m = 1, 2, 3.$

$$a_r^{(hs)} = A_1 + A_2 2^r + A_3 3^r \quad \text{— (4)}$$

$r^m p$

$a_r = rp$

$a_{r-1} = (r-1)p$

$a_{r-2} = (r-2)p$

$a_{r-3} = (r-3)p$

Put $a_r = rp$ in (1) $\hookrightarrow$ (5)

$$rp - 6(r-1)p + 11(r-2)p - 6(r-3)p = 4.$$

$$rp - 6rp + 6p + 11rp - 22p - 6rp + 18p = 4$$

$$2p = 4$$

$$p = \frac{4}{2}$$

$$\boxed{p = 2}$$

$$a_r^{(ps)} = 2r \quad \text{— (6)}$$

$$\therefore a_r = A_1 + A_2 2^r + A_3 3^r + 2r.$$

---

4. $a_r - 4a_{r-1} + 4a_{r-2} = 2$

$$m^2 - 4m + 4 = 0$$

$m = 2, 2$

$$a_r^{(hs)} = (A_1 + A_2 r)\, 2^r$$

$$a_r = p$$

$$p - \cancel{4p} + \cancel{4p} = 2$$

$$p = +2$$

$$\boxed{p = 2}$$

$$\boxed{a_r = (A_1 + A_2 r)\, 2^r + 2}$$

---

5. solve $a_r - 7a_{r-1} + 17a_{r-2} - 17a_{r-3} + 6a_{r-4} = 12$

Soln:- $m^4 - 7m^3 + 17m^2 - 17m + 6 = 0.$

	1	-7	17	-17	6
1	0	1	-6	11	-6
	1	-6	-11	-6	0
1	0	1	-5	6	
	1	-5	6	0	

$$m^2 - 5m + 6 = 0$$

$$m = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} = 3, 2.$$

$$m = 1, 1, 2, 3.$$

$$a_r^{(hs)} = (A_1 + A_2 r) + A_3 2^r + A_4 3^r.$$

Put $a_r = r^2 p$ in ①.

$$r^2 p - 7(r-1)^2 p + 17(r-2)^2 p - 17(r-3)^2 p + 6(r-4)^2 p = 12$$

$$r^2 p - 7p(r^2 - 2r + 1) + 17p(r^2 - 4r + 4) - 17p(r^2 - 6r + 9) + 6p(r^2 - 8r + 16) = 12$$

$$p\left[(r^2 - 7r^2 + 17r^2 - 17r^2 + 6r^2) + (14r - 68r + 102r - 48r) + (-7 + 68 - 153 + 96)\right] = 12$$

$$p(0+0+4) = 12$$

$$p(4) = 12$$

$$4p = 12$$

$$p = \frac{12}{4}$$

$$\boxed{p = 3}$$

$$a_r^{(ps)} = 3r^2$$

$$a_r = (A_1 + A_2 r) + A_3 2^r + A_4 3^r + 3r^2$$

To find a particular soln:

$$\boxed{c_0 a_r + c_1 a_{r-1} + \cdots + c_k a_{r-k} = f(r)}$$

Type II:

$$f(r) = \beta^r.$$

Assumed soln $a_r = \begin{cases} p\beta^r, & \text{if } \beta \text{ is not a characteristic root} \\ pr^m \beta^r, & \text{if } \beta \text{ is a characteristic root with multiplicity } m. \end{cases}$

---

1. Solve $a_r - 5a_{r-1} + 6a_{r-2} = 4^r$

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$$\frac{1}{2} 4^r - \frac{5}{2} 4^{r-1} + \frac{6}{2} 4^{r-2} = 4^r.$$

$$4^r \left[\frac{1}{2} - \frac{5}{8} + \frac{6}{32}\right] = 4^r.$$

$$\frac{16 - 20 + 6}{32}$$

$$p4^r - 5p4^{r-1} + 6p4^{r-2} = 4^r$$

$$p4^r \left[1 - \frac{5}{4} + \frac{6}{16}\right] = 4^r$$

$$p4^r \left[\frac{16 - 20 + 6}{16}\right] = 4^r$$

$\frac{p}{8} = 1 \Rightarrow \boxed{p = 8}$

$$8 \cdot 4^r - 5 \cdot 8\, 4^{r-1} + 6 \cdot 8\, 4^{r-2} = 4^r$$

$$\underline{8 \cdot 4^r} - 10\, 4^r + \underline{3 \cdot 4^r} = 4^r$$

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Type III:

$$f(r) = (d_1 r^t + d_2 r^{t-1} + \cdots + d_t)\beta^r.$$

Assumed soln is $a_r = \begin{cases} (p_1 r^t + p_2 r^{t-1} + \cdots + p_t)\beta^r & \text{if } \beta \text{ is not a chara. root} \\ (p_1 r^t + p_2 r^{t-1} + \cdots + p_t) r^m \beta^r & \text{if } \beta \text{ is a chra. root with multiplicity } m \end{cases}$

1. Solve $a_r - 7a_{r-1} + 12 a_{r-2} = r^3 + 3r^2 + r + 1$.

Soln

$$a_r^{(hs)} = A_1 3^r + A_2 4^r$$

Put $a_r = (p_1 r^3 + p_2 r^2 + p_3 r + p_4)$ in ①

$$(p_1 r^3 + p_2 r^2 + p_3 r + p_4) - 7(p_1 (r-1)^3 - p_2 (r-1)^2 + p_3 (r-1) + p_4)$$

$$+ 12 (p_1 (r-2)^3 + p_2 (r-2)^2 + p_3 (r-2) + p_4 (r-2)$$

$$= r^3 + 3r^2 + r + 1$$

$$p_1 r^3 + p_2 r^2 + p_3 r + p_4 - 7[p_1 (r^3 - 3r^2 + 3r - r^3) + p_2 (r^2 - 2r + 2) + p_3 (r-1) + p_4] = 0$$

$$+ 12 [p_1 (r^2 - 6r^2 + 12r - 8) + p_2 (r^2 - 4r + 4) + p_3 (r-2) + p_4 = r^3 - 3r^2 + r + 1$$

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Generating functions:

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers. then the power series or the infinite series.

$$G(x) \text{ (or) } G(a, x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called as generating function for the sequence $\{a_n\}_{n=0}^{\infty}$

$\{a(n)\}$

$$G(a, x) = a(0) + a(1)x + a(2)x^2 + \cdots$$

$\{s(n)\}$

$$G(s, x) = s_0 + s_1 x + s_2 x^2 + \cdots$$

$\{y_n\}$

$$G(y, x) = y_0 + y_1 x + y_2 x^2 + \cdots$$

Fibonacci sequence:

$$\{F_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$$

$$G(F, x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots$$

$$a_n = (-1)^n, \; n \geq 0.$$

$$G(a, x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 \cdots$$

---

1. Solve $a_r - 3a_{r-1} + 2a_{r-2} = 0$, $r \geq 2$, given that $a_0 = 2$ and $a_1 = 3$.

Soln:

$$a_r - 3a_{r-1} + 2a_{r-2} = 0, \; r \geq 2 \quad \text{--- (1)}$$

$$\Rightarrow a_r x^r - 3a_{r-1} x^r + 2a_{r-2} x^r = 0.$$

$$\Rightarrow \sum_{r=2}^{\infty} a_r x^r - 3 \sum_{r=2}^{\infty} a_{r-1} x^r + 2 \sum_{r=2}^{\infty} a_{r-2} x^r = 0$$

$$\Rightarrow [a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots].$$

$$-3[a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots].$$

$$+2[a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots] = 0$$

$$\Rightarrow [a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots].$$

$$-3x[a_1 x + a_2 x^2 + \cdots].$$

$$+2x^2[a_0 + a_1 x + a_2 x^2 + \cdots] = 0.$$

$$\Rightarrow [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots]$$

$$- a_0 - a_1 x$$

$$-3x\{[a_0 + a_1 x + a_2 x^2 + \cdots] - a_0\}.$$

$$+2x^2[a_0 + a_1 x + a_2 x^2 + \cdots] = 0.$$

$$\Rightarrow a(a, x) - 2 - 3x - 3x\, a(a, x) + 6x$$

$$+ 2x^2\, a(a, x) = 0$$

$$\Rightarrow (2x^2 - 3x + 1)\, a(a, x) + 3x - 2 = 0$$

$$\Rightarrow (2x^2 - 3x + 1)\, a(a, x) = 2 - 3x.$$

$$\Rightarrow a(a, x) = \frac{2 - 3x}{(2x^2 - 3x + 1)}$$

$$\Rightarrow a(a, x) = \frac{2 - 3x}{(2x - 1)(x - 1)} \quad \text{--- (2)}$$

Let us resolve $\dfrac{2 - 3x}{(2x - 1)(x - 1)}$ into partial functions.

$$\frac{2-3x}{(2x-1)(x-1)} = \frac{A}{2x-1} + \frac{B}{x-1} \quad - (3)$$

$$\Rightarrow \frac{2-3x}{(2x-1)(x-1)} = \frac{A(x-1) + B(2x-1)}{(2x-1)(x-1)}$$

$$\Rightarrow 2-3x = A(x-1) + B(2x-1) \quad - (4)$$

Put $x = \frac{1}{2}$ in (4);

$$2 - \frac{3}{2} = A\left(\frac{1}{2} - 1\right)$$

$$\frac{1}{2} = -\frac{1}{2} A.$$

$$\boxed{A = -1}$$

Put $x = 1$ in (4) we get.

$$-1 = B.$$

$$\boxed{B = -1}$$

sub $A = -1$ and $B = -1$ in (3).

$$\frac{2-3x}{(2x-1)(x-1)} = \frac{-1}{(2x-1)} - \frac{1}{(x-1)}.$$

$$\Rightarrow \frac{2-3x}{(2x-1)(x-1)} = \frac{1}{(1-2x)} + \frac{1}{(1-x)} \quad - (5)$$

sub (5) in (2), we get.

$$a(a, x) = \frac{1}{1-2x} + \frac{1}{1-x}$$

$$\Rightarrow a(a, x) = (1-2x)^{-1} + (1-x)^{-1}$$

$$\Rightarrow a(a, x) = [1 + 2x + (2x)^2 + (2x)^3 + \cdots] + [1 + x + x^2 + x^3 + \cdots]$$

$a_r = 2^r + 1$

$\therefore \boxed{a_r = 1 + 2^r}$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 + \cdots$

$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \cdots$

$\{a_n\} = \{1, 2, 3, 4, 5, 6 - \cdots\}$.

$G(a, x) = 1 + 2x + 3x^2 + \cdots$

$G(a, x) = (1-x)^{-2}$.

$G(a, x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$G(a, x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4$.

$\{n+1\}_{n=0}^{\infty}$.

---

2. Solve $a_{r+2} - 2a_{r+1} + a_r = 2^r$, given that $a_0 = 2$ and $a_1 = 1$.

Soln:

$a_{r+2} - 2a_{r+1} + a_r = 2^r$.

$\Rightarrow a_{r+2} x^r - 2a_{r+1} x^r + a_r x^r = 2^r x^r$.

$\Rightarrow \sum_{r=0}^{\infty} a_{r+2} x^r - 2 \sum_{r=0}^{\infty} a_{r+1} x^r + \sum_{r=0}^{\infty} a_r x^r$

$= \sum_{r=0}^{\infty} (2x)^r$.

$$\Rightarrow \frac{1}{x^2} \sum_{r=0}^{\infty}{}' a_{r+2} x^{r+2} - \frac{2}{x} \sum_{r=0}^{\infty}{}' a_{r+1} x^{r+1} +$$

$$\sum_{r=0}^{\infty}{}' a_r x^r$$

$$= \sum_{r=0}^{\infty}{}' (2x)^r.$$