Chapter 2 Linear Differential equation

2.1 Homogeneous Linear Equation of second order

$y''+p(x)y'+q(x)y=r(x) \quad \Rightarrow$ second-order linear diff. eq.

P, q, r are function of x

$r(x) = 0 \rightarrow$ homogeneous equation (A)

$r(x) \neq 0 \rightarrow$ non-homogeneous equation (A')

Example: $y''-y=0$

A function $y=e^{x}$ and $y=e^{-x}$ are the solutions. If arbitrary constant, say, 3, -8, are multiplied, then take the sum $y=3e^{x}+8e^{-x}$ is also a solution.

For a homogeneous linear differential equation a new solution can always obtain from know solutions by multiplication by constant and by addition.

y_1 and y_2 are solutions $\Rightarrow$ $y=c_1y_1+c_2y_2$ is also a solution

(**superposition principle**)

Theorem: for a homogeneous linear diff. equation, any linear combination of two solutions is again a solution of the eq.(A)

Proof: suppose y_1 and y_2 are solutions of (A)

Let $y=c_1y_1+c_2y_2$, c_1, c_2 are arbitrary constants

Substitute into (A)

$$\Rightarrow (c_1y_1+c_2y_2)''+p(x)(c_1y_1+c_2y_2)'+q(x)(c_1y_1+c_2y_2)$$

$$=c_1\underbrace{(y_1''+p(x)y_1'+q(x)y_1)}+c_2\underbrace{(y_2''+p(x)y_2'+q(x)y_2)}=0$$

$\Rightarrow$ y is also a solution of (A)

Caution: this theorem is not hold for non-homogeneous linear equation or nonlinear equation

※ general solution, basis

A **general solution** of second order homogeneous linear diff. eq. is of the form $y = c_1 y_1 + c_2 y_2$

y_1 and y_2 are not proportional solutions (linear independent)

c_1, c_2 are arbitrary constants

If y_1 and y_2 are linear independent $\rightarrow$ y_1 and y_2 are called a **basis** of (A)

※ Linear independent

$\rightarrow$ if $k_1 y_1 + k_2 y_2 = 0$ implies $k_1 = 0$ & $k_2 = 0$

Then $y_1(x), y_2(x)$ are said to be linearly independent.

Linear dependent

$\rightarrow$ if $k_1 y_1 + k_2 y_2 = 0$ but k_1 and k_2 are not both zero

$$\Rightarrow k_1 \neq 0 \ \ or \ \ k_2 \neq 0 \ \Rightarrow y_1 = -\frac{k_2 y_2}{k_1} \ \ or \ \ y_2 = -\frac{k_1 y_1}{k_2}$$

$y_1(x), y_2(x)$ are said to be linear dependent

e.g. $y_1 = e^x; y_2 = x$

since $k_1 e^x + k_2 x = 0$ is only satisfied when $k_1 = k_2 = 0$

$\rightarrow$ $y_1; y_2$ are linear independent

e.g. $y_1 = e^x; y_2 = 4e^x$

since $k_1 e^x + k_2 (4e^x) = 0$ is satisfied when $k_1 = 4; k_2 = -1$

$\rightarrow$ $y_1; y_2$ are linear dependent i.e. y_1, y_2 are proportional.

2.2 Homogeneous equation with constant coefficients

Standard form: $y''+ay'+by=0$ $\quad a, b$ constants $\qquad$ (A)

try : $y=e^{\lambda x}$ $\qquad$ λ: constant but unknown

$$(A) \Rightarrow (\lambda^2+a\lambda+b)\,e^{\lambda x}=0 \Rightarrow \lambda^2+a\lambda+b=0 \quad \text{characteristic eq.}$$

$$\Rightarrow \quad \lambda_1=\frac{1}{2}\left(-a+\sqrt{a^2-4b}\right) \;,\quad \lambda_2=\frac{1}{2}\left(-a-\sqrt{a^2-4b}\right)$$

$$\Rightarrow \quad y_1=e^{\lambda_1 x} \;,\quad y_2=e^{\lambda_2 x} \qquad \text{solutions}$$

but $\quad a^2-4b \begin{cases} >0 & \textit{two real roots} \\ =0 & \textit{a real double root} \\ <0 & \textit{complex roots} \end{cases}$

※ case I : $a^2-4b>0 \quad \Rightarrow \quad y_1=e^{\lambda_1 x}$, $y_2=e^{\lambda_2 x}$

the solution is $\quad y=c_1y_1+c_2y_2$

example: $y''-y=0 \quad \Rightarrow \quad \lambda^2-1=0 \quad \Rightarrow \quad \lambda=1, \quad -1$

$$\Rightarrow \quad y=c_1e^{x}+c_2e^{-x}$$

※ case II : $a^2-4b=0 \quad \Rightarrow \quad \lambda=\lambda_1=\lambda_2=-\dfrac{a}{2}$

$y_1=e^{-\frac{a}{2}x}$ $\quad$, but second solution ?

※ Method of reduction order

If y_1 is known, then solution y can be obtained by set $y=uy_1$

Consider $\quad y''+\underline{p(x)}y'+\underline{q(x)}y=0 \qquad y_1$ is a known solution

Set $\quad y=uy_1 \quad \Rightarrow y'=uy_1'+u'y_1 \quad \& \quad y''=uy_1''+2u'y_1'+u''y_1$

$$y''+p(x)y'+q(x)y=u\,[y_1''+p(x)y_1'+q(x)y_1]+y_1u''+[2y_1'+p(x)y_1)u'=0$$

It means that this method is suitable for both constant coefficient and variable coefficient linear differential equations

If it equals to zero, then y is a solution and

$$u''y_1+[2y_1'+p(x)y_1]\,u'=0$$

Let $\quad W=u' \quad \Rightarrow \quad W'y_1+[2y_1'+p(x)y_1]W=0$

$$\frac{dW}{W}+2\frac{y_1'}{y_1}dx+p(x)dx=0 \quad \Rightarrow \ln|W|+2\ln|y_1|+\int p(x)dx+c=0$$

$$\Rightarrow \ln\left|Wy_1{}^2\right|=-\int p(x)dx+c \quad \Rightarrow Wy_1{}^2=c_1e^{-\int p(x)dx}$$

$$\Rightarrow W=u'=c_1\frac{e^{-\int p(x)dx}}{y_1^2} \quad \Rightarrow u=c_1\int\frac{e^{-\int p(x)dx}}{y_1^2}dx+c_2$$

$$\therefore\ y=uy_1=c_1y_1(x)\int\frac{e^{-\int p(x)dx}}{y_1^2}dx+c_2y_1(x)$$

$$\therefore\ y_2=y_1(x)\int\frac{e^{-\int p(x)dx}}{y_1^2}dx \qquad (y_1,\ y_2\ \text{ are linear independent})$$

The second solution of case II can be obtained by

$$\therefore\ y_2=e^{-\frac{a}{2}x}\int\frac{e^{-\int a dx}}{e^{-ax}}dx=x\,e^{-\frac{a}{2}x}$$

The corresponding general solution is

$$\therefore\ y=(c_1+c_2x)e^{-\frac{a}{2}x}$$

Example: $\quad y''+8y'+16y=0$

Characteristic equation $\Rightarrow \quad \lambda^2+8\lambda+16=0 \Rightarrow \lambda=-4,-4$

$$\therefore\ y=c_1\,e^{-4x}+c_2\,x\,e^{-4x}$$

case III , $a^2 - 4b < 0$

$$\Rightarrow \quad \lambda_1 = -\frac{1}{2}a + i\omega, \quad \lambda_2 = -\frac{1}{2}a - i\omega, \quad \omega = \sqrt{b - \frac{1}{4}a^2}$$

$$\Rightarrow y_1 = e^{(-\frac{1}{2}a + i\omega)x}, \quad y_2 = e^{(-\frac{1}{2}a - i\omega)x} \qquad \text{complex}$$

real solution is desired $\Rightarrow \quad y = c_1 y_1 + c_2 y_2$

$$= c_1(e^{-\frac{1}{2}ax} e^{i\omega x}) + c_2(e^{-\frac{1}{2}ax} e^{-i\omega x})$$

$$= e^{-\frac{1}{2}ax}[c_1(\cos\omega\, x + i\sin\omega\, x) + c_2\,(\cos\omega\, x - i\sin\omega\, x)]$$

$$= e^{-\frac{1}{2}ax}[(c_1 + c_2)\cos\omega\, x + i(c_1 - c_2)\sin\omega\, x]$$

$$= e^{-\frac{1}{2}ax}[A\cos\omega\, x + B\sin\omega\, x] \qquad A, B \quad \text{arbitrary constants}$$

※ Euler's formula $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$$z = x + iy \quad \Rightarrow \quad e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

Example: $\quad y'' + 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 5$

Let $y = e^{\lambda x} \quad \Rightarrow$ characteristic equation $\Rightarrow \lambda^2 + 2\lambda + 5 = 0$

$$\Rightarrow \lambda = -1 \pm 2i \quad \Rightarrow y(x) = e^{-x}(A\cos 2x + B\sin 2x)$$

I.C. $\quad y(0) = 1 \quad \Rightarrow \quad 1 = A$

$\quad\quad y'(0) = 5 \quad \Rightarrow \quad 5 = -A + 2B \Rightarrow \quad B = 3$

$$\Rightarrow \quad y(x) = e^{-x}(\cos 2x + 3\sin 2x)$$

2.3 Differential Operators

D: denote differentiation with respect to x $D \equiv \frac{d}{dx}$

i.e. $Dy = y' = \frac{dy}{dx}$ D: operator

second derivative $D(Dy) = D^2 y = Dy' = y''$

$\therefore \quad Dy = y', \quad D^2 y = y'', \quad D^3 y = y''', \cdots$

combination: $L = p(D) = D^2 + aD + b$ second order linear differential operator

$$\Rightarrow L(y) = (D^2 + aD + b)y = y'' + ay' + b$$

L : linear operator $\Rightarrow L(\alpha y + \beta W) = \alpha L(y) + \beta L(W)$

Consider $y'' + ay' + by = 0 \quad \Rightarrow L(y) = p(D)(y) = 0$

e.g. $L(y) = (D^2 + D - 6)y = y'' + y' - 6y = 0$

try solution $y = e^{\lambda x}$

since $D[e^{\lambda x}] = \lambda e^{\lambda x}, \quad D^2[e^{\lambda x}] = \lambda^2 e^{\lambda x}$

$\Rightarrow p(D)[e^{\lambda x}] = (\lambda^2 + a\lambda + b)e^{\lambda x} = p(\lambda)e^{\lambda x} = 0$

$\therefore e^{\lambda x}$ is a solution if and only if $p(\lambda) = 0$

If $p(\lambda) = 0 \quad \Rightarrow$ two different roots $\Rightarrow$ two independent solutions

$\Rightarrow$ a double root $\Rightarrow$ only one solution

to obtain the second solution, differentiate $p(D)[e^{\lambda x}] = p(\lambda)e^{\lambda x}$

with respect to λ

$$\Rightarrow p(D)[x e^{\lambda x}] = p'(\lambda)e^{\lambda x} + p(\lambda)x e^{\lambda x} \qquad \text{(A)}$$

for a double root $\Rightarrow p'(\lambda) = p(\lambda) = 0$

$\Rightarrow (A) = 0 \quad \Rightarrow \quad p(D)[\, x\, e^{\lambda x}\,] = 0 \;\Rightarrow\; x\, e^{\lambda x}$ is also a solution

$\therefore\, p(D)[e^{\lambda x}] = p(\lambda)\, e^{\lambda x}$ $\quad p(D)$ can be treated just like an algebraic quantity

Example: $\quad p(D) = D^2 + D - 6 \quad Solve \quad p(D)y = 0$

$\because D^2 + D - 6 = (D+3)(D-2)$, $\quad$ by definition $(D\text{-}2)y = y'\text{-}2y$

$\Rightarrow (D+3)(D-2)y = (D+3)[y' - 2y] = y'' - 2y' + 3y' - 6y = y'' + y' - 6y$
factorization is permissible

the solutions are $\quad (D + 3)y = 0 \qquad y = e^{-3x}$
$\qquad\qquad (D - 2)y = 0 \qquad y = e^{2x}$

$D^2 + D - 6 = (D+3)(D-2) = (D-2)(D+3)$

Note:1. Factors of a differential operator with constant coefficient commute.

2. Differential operators with variable coefficient generally do not commute.
such as $(D+x)$ $(D\text{-}x)$ $\quad$ or

$[D + f(x)][D + g(x)] \neq [D + g(x)][D + f(x)]$

Example: $\quad y = 3x^2 + 2x$

$$(D + x)Dy \neq D(D + x)y$$

$$6x^2 + 2x + 6 \neq 9x^2 + 4x + 6$$

2.4 Mass-Spring System (free oscillations of undamped system)

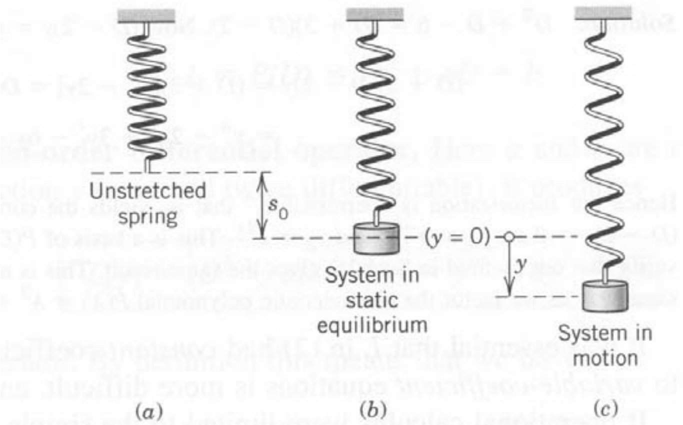


Fig. 39. Mechanical system under consideration

(b) Hooke's law: spring force =- kS_0 , gravity force = + mg

System in equilibrium $\rightarrow$ spring force + gravity force=0

$\rightarrow$ $kS_0 = mg$

(c) : note m in motion

spring force: $-k(y+S_0)$ gravity force = + mg

the net force applied at m is $-k(y+S_0)$ + mg= -ky

the mass will be accelerated by the force
Newton's second law: $F = ma$

Since the displacement is y $\rightarrow$ the acceleration is $\dfrac{d^2 y}{d t^2} = y''$

$$\therefore -ky = m \cdot \frac{d^2 y}{d t^2} \Rightarrow my'' + ky = 0 \quad \text{(free vibration without damping)}$$

$$\Rightarrow \quad y'' + \frac{k}{m} y = 0 \qquad set \quad y = e^{\lambda t}$$

$$\Rightarrow \quad \lambda^2 + \frac{k}{m} = 0 \quad \Rightarrow \quad \lambda = \pm \sqrt{\frac{k}{m}}\, i = \pm\, \omega_0\, i$$

$$\Rightarrow \quad y = A\cos\omega_0 t + B\sin\omega_0 t \qquad \text{harmonic motion}$$

or $\quad y = C\cos(\omega_0 t - \delta), \qquad C = \sqrt{A^2 + B^2}, \quad \tan\delta = \dfrac{B}{A}$

$\Rightarrow$ It is a harmonic motion.

$\omega_0 = \sqrt{\dfrac{k}{m}}$ (rad/sec) is a special characteristic for the system,

$\dfrac{\omega_0}{2\pi}$: natural frequency

To solve A and B , the initial conditions
$y(0)$: initial displacement
and
$y'(0)$: initial velocity
should be given.

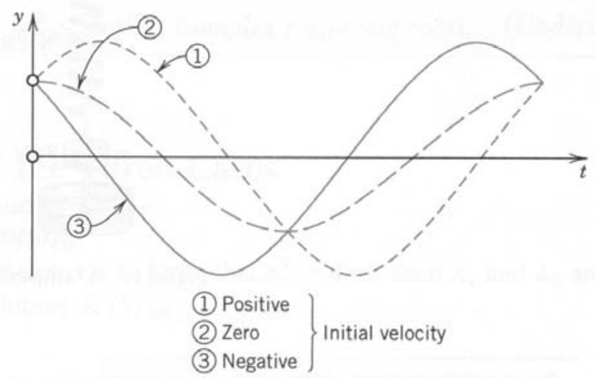


Fig. 40. Harmonic oscillations

Damped system:

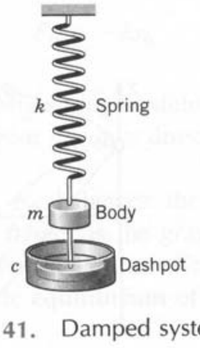


Fig. 41. Damped system

Damped force:

1. direction opposite to the instantaneous motion
2. the magnitude is assumed to be proportional to velocity

damping force = $-c\dfrac{dy}{dt}$ $\quad c$: damping constant

net force applied on m is

$$-k(y+S_0)-c\frac{dy}{dt}+mg=-ky-c\frac{dy}{dt} \qquad \text{net force}$$

Newton's second law:

$$-ky-c\frac{dy}{dt}=m\frac{d^2y}{dt^2} \quad \Rightarrow \quad my''+cy'+ky=0$$

Governing equation of free vibrating system with damping

Characteristic equation: $\quad \lambda^2+\dfrac{c}{m}\lambda+\dfrac{k}{m}=0$

$$\lambda_{1,2}=-\frac{c}{2m}\pm\frac{1}{2m}\sqrt{c^2-4mk}=-\alpha\pm\beta$$

where $\quad \alpha=\dfrac{c}{2m}, \quad \beta=\dfrac{1}{2m}\sqrt{c^2-4mk}$

Case I: $\quad c^2-4mk>0 \quad \Rightarrow$ overdamping

λ_1, λ_2 real distinct roots $\quad \Rightarrow \quad y(t)=c_1e^{-(\alpha-\beta)t}+c_2e^{-(\alpha+\beta)t}$

not oscillate

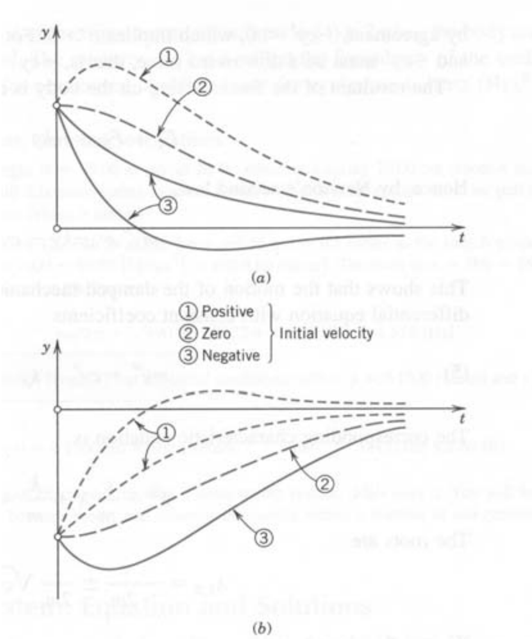


Fig. 42. Typical motions (7) in the overdamped case
(a) Positive initial displacement
(b) Negative initial displacement

Case II: critical damping. $c^2 - 4mk = 0 \quad \Rightarrow \beta = 0, \lambda_1 = \lambda_2 = -\alpha$

$$\Rightarrow \quad y(t) = (c_1 + c_2 t)e^{-\alpha t} \qquad \text{not oscillate}$$

$$c_c = \sqrt{4mk} = 2\sqrt{mk} : \text{critical damping}$$

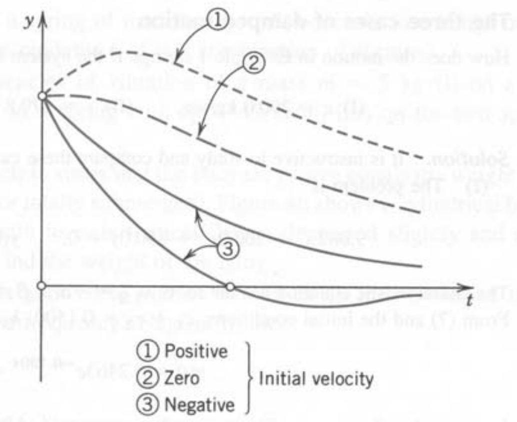


Fig. 43. Critical damping [see (8)]

Case III: underdamping, $c^2 - 4mk < 0 \quad \Rightarrow \beta = i\,\omega^*$ a pure imaginary value

$$\omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\omega_0{}^2 - \alpha^2}\,, \quad \alpha = \frac{c}{2m}$$

$$\therefore \lambda_1 = -\alpha + i\,\omega^* \qquad \lambda_2 = -\alpha - i\,\omega^*$$

$$\therefore y(t) = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t)$$

or $\quad y(t) = Ce^{-\alpha t}\cos(\omega^* t - \delta)$

$Ce^{-\alpha t}$: damped amplitude

$\omega^* = \sqrt{\omega_0{}^2 - \alpha^2}$ quasi-frequency(damped natural frequency)

ω_0 :(undamped) natural frequency

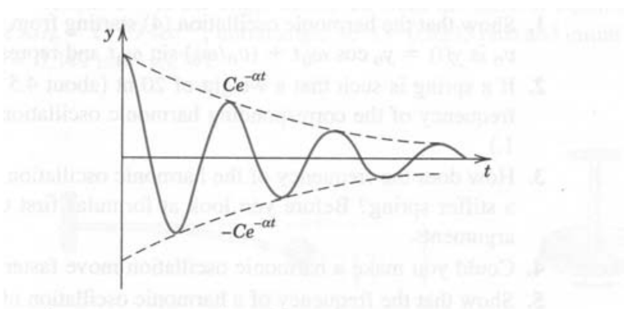


Fig. 44. Damped oscillation in Case III [see (10)]

From the graph, the maximum and minimum occur at t_n for which

$$\cos(\omega^* t - \delta) = \pm 1 \quad \Rightarrow \quad \omega^* t_n - \delta = n\pi, \quad n = 0, 1, 2, \cdots$$

$$\Rightarrow \quad t_n = \frac{n\pi + \delta}{\omega^*}, \quad n = 0, 1, 2, \cdots$$

$$\Rightarrow \quad t_{n+1} - t_n = \frac{\pi}{\omega^*} \qquad\qquad \Rightarrow \quad t_{n+2} = \frac{2\pi}{\omega^*} + t_n$$

time between maximum and minimum quasi period

$$y(t_{n+2}) = C\, e^{-\alpha t_{n+2}} \cos(\omega^* t_{n+2} - \delta)$$

$$= C\, e^{-\alpha (t_n + \frac{2\pi}{\omega^*})} \cos[\omega^* (t_n + \frac{2\pi}{\omega^*}) - \delta]$$

$$= e^{-\alpha \frac{2\pi}{\omega^*}} C\, e^{-\alpha t_n} \cos[\omega^* t_n - \delta]$$

$$y(t_{n+2}) = e^{-\alpha \frac{2\pi}{\omega^*}} y(t_n) \quad \Rightarrow \quad \frac{y(t_n)}{y(t_{n+2})} = e^{\alpha \frac{2\pi}{\omega^*}}$$

$$\Delta = \ln\left[\frac{y(t_n)}{y(t_{n+2})}\right] = \alpha \frac{2\pi}{\omega^*} = \frac{2\pi\,\alpha}{\sqrt{\omega_0{}^2 - \alpha^2}} \quad \text{logarithmic decrement}$$

this equation can be used to determine α and then damping coefficient c when k, m are known and $\left[\frac{y(t_n)}{y(t_{n+2})}\right]$ are measured.

2.5 Euler-Cauchy Equation

Equation form: $x^2 y'' + axy' + by = 0$ a and b constants (1)

General form: $a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 xy' + a_0 y = g(x)$

Euler-Cauchy equation (equip-dimensional equation)

Try $y = x^m$ $\Rightarrow (1) \Rightarrow$ $x^2 m(m-1)x^{m-2} + a\,x\,m\,x^{m-1} + b\,x^m = 0$

$\Rightarrow$ $x^m[m^2 + (a-1)m + b] = 0$

$\therefore m^2 + (a-1)m + b = 0$ auxiliary eq.

Case I: distinct real root $m_1 \neq m_2$ $\Rightarrow y_1(x) = x^{m_1}$, $y_2(x) = x^{m_2}$

Thus general solution is $y(x) = c_1 x^{m_1} + c_2 x^{m_2}$

Compare: $y(x) = c_1\, e^{\lambda_1 x} + c_2\, e^{\lambda_2 x}$

Case II: double root $m_1 = \dfrac{1-a}{2}$ $\Rightarrow y_1(x) = x^{\frac{1-a}{2}}$, but y_2 = ?

Remember: method of reduction prder $y = uy_1$

$$\Rightarrow \quad y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx$$

here $p(x) = \dfrac{a}{x}$ $(\because y'' + p(x)y' + q(x)y = 0)$

$$\Rightarrow \quad y_2(x) = y_1(x)\int \frac{e^{-\int \frac{a}{x}dx}}{x^{1-a}}dx = y_1(x)\int \frac{e^{-a\ln x}}{x^{1-a}}dx = y_1(x)\int \frac{x^{-a}}{x^{1-a}}dx$$

$$= y_1(x)\int \frac{1}{x}dx = \ln x\; y_1(x) = x^{\frac{1-a}{2}} \ln x$$

the general solution is $y(x) = (c_1 + c_2 \ln x)\; x^{\frac{1-a}{2}}$

Compare: $y(x) = (c_1 + c_2 x)\, e^{\lambda_1 x}$

Case III: complex conjugate roots, m_1, m_2 are complex

Say $m_1 = \mu + i\upsilon$, $m_2 = \mu - i\upsilon$ $\Rightarrow y_1 = x^{\mu+i\upsilon}$, $y_2 = x^{\mu-i\upsilon}$

General solution is $y = c_1\ x^{\mu+i\upsilon} + c_2\ x^{\mu-i\upsilon}$

$$\because \quad x^{i\beta} = \left(e^{\ln x}\right)^{i\beta} = e^{i\beta \ln x} = \cos(\beta \ln x) + i\sin(\beta \ln x)$$

$$\therefore y = c_1\ x^{\mu+i\upsilon} + c_2\ x^{\mu-i\upsilon} = x^{\mu}(c_1 x^{i\upsilon} + c_2 x^{-i\upsilon})$$

$$= x^{\mu}[c_1 \cos(\upsilon \ln x) + c_1 i\sin(\upsilon \ln x) + c_2 \cos(\upsilon \ln x) - c_2 i\sin(\upsilon \ln x)]$$

$$= x^{\mu}[(c_1 + c_2)\cos(\upsilon \ln x) + (c_1 i - c_2 i)\sin(\upsilon \ln x)]$$

the solution is $y(x) = x^{\mu}[A\cos(\upsilon \ln x) + B\sin(\upsilon \ln x)]$

Compare: $y(x) = x^{-ax}[A\cos\omega\ x + B\sin\omega\ x]$

Example:

$$x^2 y'' - 2.5xy' - 2y = 0 \quad \Rightarrow \quad y = x^m, \quad \Rightarrow m^2 - 3.5m - 2 = 0$$

$$\Rightarrow m = -0.5, \quad 4 \quad \Rightarrow \quad y = c_1 x^4 + c_2 x^{-0.5} = c_1 x^4 + c_2 \frac{1}{\sqrt{x}}$$

Example: $x^2 y'' - 3xy' + 4y = 0 \quad \Rightarrow \quad y = x^m, \quad \Rightarrow m^2 - 4m + 4 = 0$

$$\Rightarrow m = 2, \quad 2 \quad \Rightarrow \quad y = (c_1 + c_2 \ln x)x^2$$

Example: $x^2 y'' + 7xy' + 13y = 0 \quad \Rightarrow \quad y = x^m, \quad \Rightarrow m^2 + 6m + 13 = 0$

$$\Rightarrow m = -3 \pm 2i \quad \Rightarrow \quad y = x^{-3}[A\cos(2\ln x) + B\sin(2\ln x)]$$

2.6 Existence and uniqueness theory

Consider an initial value problem

$$y''+p(x)y'+q(x)y=0 \tag{1}$$

$$y(x_0)=K_0\ ,\quad y'(x_0)=K_1 \tag{2}$$

Theorem: If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is in I, then the initial value problem has unique solution $y(x)$ on the interval I.

Def: Linear dependent

A set of function $f_1(x),\ f_2(x),\ f_3(x),\cdots f_n(x)$ is said to be linearly dependent on an interval I if there exist constants $c_1, c_2,\cdots c_n$ not all zero, such that $c_1 f_1(x)+c_2 f_2(x)+c_3 f_3(x)+\cdots+c_n f_n(x)=0$ for every x in the interval.

Def: Linear independent

A set of function $f_1(x),\ f_2(x),\ f_3(x),\cdots f_n(x)$ is said to be linearly independent if it is not linearly dependent on an interval.

$\Longrightarrow$ If it is linear independent, then all the constants $c_1, c_2,\cdots c_n$ must equal to zero.

Theorem 1: Suppose $f_1(x),\ f_2(x),\ f_3(x),\cdots f_n(x)$ possess at least n-1 derivatives. If the determinant

$$\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix} \neq 0$$

for at least one point in the interval I, then the functions $f_1(x),\ f_2(x),\ f_3(x),\cdots f_n(x)$ are linearly independent on the interval.

The determinant is denoted by $W\left(f_1,\ f_2,\ f_3,\cdots f_n\right)$ and is called **Wronskian** of the functions.

Theorem 2: Solutions y_1 and y_2 of homogeneous linear differential equation (1) on I are linearly dependent on I if and only if their Wronskian W is zero at some x_0 in I. Furthermore, if W =

0 for $x = x_0$, then $W = 0$ on I; hence if there is an x_1 in I at which $W \neq 0$, then y_1 and y_2 are linearly independent on I.

Proof: (a) y_1 , y_2 are linear dependent $\rightarrow$ $W = 0$

$\because$ y_1 and y_2 are linear dependent $\Rightarrow \begin{cases} y_2 = ky_1, & k \neq 0 \\ y_1 = \ell y_2, & \ell \neq 0 \end{cases}$

$$\therefore \quad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & ky_1 \\ y_1' & ky_1' \end{vmatrix} = ky_1y_1' - ky_1y_1' = 0$$

(b) $W(y_1, y_2) = 0$ for some $x = x_0$ $\rightarrow$ y_1 , y_2 are linear dependent

consider the linear system equations

$$\left.\begin{aligned} k_1y_1(x_0) + k_2y_2(x_0) = 0 \\ k_1y_1'(x_0) + k_2y_2'(x_0) = 0 \end{aligned}\right\} \quad k_1, k_2 \text{ unknowns} \qquad \text{(A)}$$

the Wronskian is $W[y_1(x_0), y_2(x_0)] = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$

since it is zero, thus the solution for k_1 and k_2 exist

$\Longrightarrow$ k_1 and k_2 not both zero.

If we define a function $y(x) = k_1y_1(x) + k_2y_2(x)$

From (A) $\Rightarrow y(x_0) = 0, \quad y'(x_0) = 0$ (B)

Thus the zero function $y(x) \equiv 0$ satisfy the differential equation

$y'' + p(x)y' + q(x)y = 0$ and the initial conditions (B). And also by theorem 1, the solution is unique

$$\Rightarrow k_1y_1(x) + k_2y_2(x) \equiv 0 \quad \Rightarrow \quad y_2(x) = -\frac{k_1}{k_2}y_1(x)$$

thus y_1 and y_2 are linearly dependent.

(c) $W \neq 0$ for some $x = x_1$ $\rightarrow$ y_1 , y_2 are linearly independent

From (b), if $W(y_1, y_2) = 0$ at x_0 $\rightarrow$ y_1 , y_2 are linear dependent

By (a), y_1 , y_2 are linear dependent $\rightarrow W(y_1, y_2) = 0$

$\therefore$ $W \neq 0$ at x_1 in I cannot happen in the case of linear dependence

$\Rightarrow$ $W \neq 0$ at x_1 implies linear independence

Theorem 3: If the coefficients $p(x)$ and $q(x)$ of (1) are continuous on some open interval I, then (1) has a general solution on I. (existence)

Theorem 4: Suppose that (1) has continuous coefficients $p(x)$ and $q(x)$ on some open interval I, then every solution $y = Y(x)$ of (1) is of the form

$$Y(x) = \tilde{c}_1 y_1(x) + \tilde{c}_2 y_2(x)$$

where $y_1(x), y_2(x)$ form a basis of solution of (1) on I and $\tilde{c}_1, \tilde{c}_2$ are suitable constants.

(unique)

2.7 nonhomogeneous equations

consider $y''+p(x)y'+q(x)y=r(x)$ (1)

if $r(x)=0 \Rightarrow y''+p(x)y'+q(x)y=0$ homogeneous eq. (2)

Theorem 1: (a) The difference of two solution of (1) on some open interval I is a solution of (2) on I.

(b) The sum of a solution of (1) on I and a solution of (2) on I is a solution of (1) on I.

Proof: (a) $y''+p(x)y'+q(x)y=r(x) \Rightarrow L(y)=r(x)$

Let y and $\tilde{y}$ be any solutions of (1)

$\therefore L(y)=r(x)$ and $L(\tilde{y})=r(x)$

$\therefore L(y-\tilde{y})=L(y)-L(\tilde{y})=r(x)-r(x)=0$

Thus $y-\tilde{y}$ is a solution of (2)

(b) assume y is a solution of (1) and y^* is a solution of (2)

$\therefore L(y+y^*)=L(y)+L(y^*)=r(x)+0=r(x)$

Thus $y+y^*$ is also a solution of (1)

Def: general solution

A general solution of nonhomogeneous eq. (1) is a solution of the form

$$y(x)=y_h(x)+y_p(x) \qquad (3)$$

$y_h(x)=c_1y_1(x)+c_2y_2(x)$ is a general solution of (2)

$y_p(x)$: is any solution of (1) containing no arbitrary constants

Example: $y''-4y'+3y=10e^{-2x}, \quad y(0)=1, \quad y'(0)=-3$

Characteristic equation of the homogeneous equation

$$\lambda^2 - 4\lambda + 3 = 0 \quad \Rightarrow \quad \lambda = 1, \quad 3 \quad \Rightarrow y_h = c_1 e^x + c_2 e^{3x}$$

try $\quad y_p(x) = ce^{-2x}$ substitute into original equation.

$$4ce^{-2x} - 4\left(-2ce^{-2x}\right) + 3ce^{-2x} = 10e^{-2x}$$

$$\Rightarrow \quad c = \frac{2}{3} \quad \Rightarrow \quad y_p = \frac{2}{3}e^{-2x}$$

$\therefore \quad y(x) = y_h + y_p = c_1 e^x + c_2 e^{3x} + \dfrac{2}{3} e^{-2x}$ general solution

$$\left.\begin{aligned} y(0) = 1 \qquad & \Rightarrow c_1 + c_2 + \frac{2}{3} = 1 \\ y'(0) = -3 \qquad & \Rightarrow c_1 + 3c_2 - \frac{4}{3} = -3 \end{aligned}\right\} \Rightarrow c_1 = \frac{4}{3}, \quad c_2 = -1$$

$$\therefore \quad y(x) = \frac{4}{3}e^x - e^{3x} + \frac{2}{3} e^{-2x}$$

Method of undetermined coefficients (Solve y_p)

This method is limited to nonhomogeneous linear equation

- that has constant coefficients
- where $r(x)$ is a constant, polynomial function, exponential function, sine, cosine, or finite sum and products of those functions

Example 1 $y''-2y'-3y=-5------(A)$

$$y_h = c_1 e^{-x} + c_2 e^{3x}$$

Try $y_p = A------(B)$

Substitute (B) into (A):

$$-3A=-5 \Rightarrow A=\frac{5}{3}$$

$$\therefore y_p = \frac{5}{3}$$

Example 2 $y''-2y'-3y=4x------(A)$

Try $y_p = Ax+B------(B)$

Substitute (B) into (A):

$$-2A-3(Ax+B)=4x$$

$$\Rightarrow \begin{cases} -3A=4 \\ -2A-3B=0 \end{cases} \Rightarrow A=\frac{-4}{3}; B=\frac{8}{9}$$

$$\therefore y_p = -\frac{4}{3}x+\frac{8}{9}$$

Example 3 $y''-2y'-3y=6xe^{2x}------(A)$

Try $y_p = (Ax+B)e^{2x}------(B)$

Substitute (B) into (A):

$$\left\{4Ae^{2x}+4(Ax+B)e^{2x}\right\}-2\left\{Ae^{2x}+2(Ax+B)e^{2x}\right\}-3\left\{(Ax+B)e^{2x}\right\}=6xe^{2x}$$

$$\Rightarrow \{2A-3B\}e^{2x}-3Axe^{2x}=6xe^{2x}$$

$$\Rightarrow \begin{cases} -3A=6 \\ 2A-3B=0 \end{cases} \Rightarrow A=-2; B=-\frac{4}{3}$$

$$\therefore y_p = -(2x+\frac{4}{3})e^{2x}$$

Example: $y''-2y'-3y=4x-5+6xe^{2x}$

First the associated homogeneous equation $\Rightarrow$ $y''-2y'-3y=0$

$\Rightarrow \lambda^2-2\lambda-3=0 \quad \Rightarrow m=-1, 3 \quad \Rightarrow y_h=c_1e^{-x}+c_2e^{3x}$

Next: $4x-5$ in $r(x)$ is a polynomial $\Rightarrow$ try $Ax+B$

$xe^{2x} \Rightarrow x\cdot e^{2x} \Rightarrow$ try (polynomial $Cx+D)\cdot e^{2x}$

$\therefore$ try $y_p(x)=(Ax+B)+(Cx+D)\cdot e^{2x}$

Eq. $\Rightarrow -3Ax-2A-3B-3Cxe^{2x}+(2C-3D)e^{2x}=4x-5+6xe^{2x}$

$$\begin{array}{r} x^0: \\ x: \\ e^{2x}: \\ xe^{2x}: \end{array} \begin{cases} -2A-3B=-5 \\ -3A=4 \\ 2C-3D=0 \\ -3C=6 \end{cases} \Rightarrow \begin{array}{l} A=-4/3 \\ B=23/9 \\ C=-2 \\ D=-4/3 \end{array}$$

$$\therefore \quad y(x)=c_1e^{-x}+c_2e^{3x}-\frac{4}{3}x+\frac{23}{9}-(+2x+\frac{4}{3})e^{2x}$$

Summary:

$r(x)$		Form of $y_p(x)$
constant	$\Rightarrow$	A
$x^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0$	$\Rightarrow$	$K_nx^n+K_{n-1}x^{n-1}+\cdots+K_1x+K_0$
$\cos\omega x$	$\Rightarrow$	$A\cos\omega x+B\sin\omega x$
$\sin\omega x$	$\Rightarrow$	$A\cos\omega x+B\sin\omega x$
$e^{\gamma x}$	$\Rightarrow$	$Ae^{\gamma x}$
$x^ne^{\gamma x}$	$\Rightarrow$	$(K_nx^n+K_{n-1}x^{n-1}+\cdots+K_1x+K_0)e^{\gamma x}$
$e^{\gamma x}\sin\omega x$	$\Rightarrow$	$e^{\gamma x}(A\cos\omega x+B\sin\omega x)$
$e^{\gamma x}\cos\omega x$	$\Rightarrow$	$e^{\gamma x}(A\cos\omega x+B\sin\omega x)$
$\begin{cases} x^n\sin\omega x \\ x^n\cos\omega x \end{cases}$	$\Rightarrow$	$\begin{cases} (K_nx^n+K_{n-1}x^{n-1}+\cdots+K_1x+K_0)\sin\omega x+ \\ (C_nx^n+C_{n-1}x^{n-1}+\cdots+C_1x+C_0)\cos\omega x \end{cases}$
$xe^{3x}\cos 4x$	$\Rightarrow$	$(Ax+B)e^{3x}\sin 4x+(Cx+D)e^{3x}\cos 4x$

Example: $y''-3y'+2y=e^x$

Characteristic eq. $\lambda^2-3\lambda+2=0 \quad \Rightarrow \quad \lambda=1,2$

$y_h(x)=c_1e^x+c_2e^{2x}$

to find y_p, try $y_p(x)=Ae^x$ since $r(x)=e^x$

$\Rightarrow Ae^x-3Ae^x+2Ae^x=e^x \quad \Rightarrow \quad 0=e^x$ that's impossible

Thus $y_p(x)=Ae^x$ is not a solution. Why ?

Since $y_p(x)=Ae^x$ is the same as c_1e^x

It is a solution of the homogeneous equation.

※Modification Rule: If a term in your choice for y_p happens to be solution of the homogeneous equation, then multiply your choice of y_p by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous equation).

Example: $y''-2y'+y=e^x$

Characteristic eq. $\lambda^2-2\lambda+1=0 \quad \Rightarrow \quad \lambda=1,1$

$y_h(x)=c_1e^x+c_2\,xe^x$

according to the modification rule try $y_p(x)=Ax^2e^x$

$$\Rightarrow Ax^2e^x+4Axe^x+2Ae^x-4Axe^x-2Ax^2e^x+Ax^2e^x=e^x$$

$$\Rightarrow 2Ae^x=e^x \quad \Rightarrow \quad A=\frac{1}{2}$$

Thus $y(x)=c_1e^x+c_2\,xe^x+\frac{1}{2}x^2e^x$

2.10 Variation of parameters

When $r(x)$ in the nonhomogeneous equation is not of the form as mentioned above, the solution of y_p = ?

$\Rightarrow$ variation of parameters [not only for constant coefficients but also for variable coefficients $p(x)$, $q(x)$]

Consider $a_2(x)y''(x)+a_1(x)y'(x)+a_0(x)y(x)=r(x)$

At first divide by a_2(x) $\Rightarrow y''(x)+p(x)y'(x)+q(x)y(x)=f(x)$

This is the standard equation form.

If y_1 and y_2 is the basis solutions of the associated homogeneous equation,

i.e., $y_1''(x)+p(x)y_1'(x)+q(x)y_1(x)=0$

$$y_2''(x)+p(x)y_2'(x)+q(x)y_2(x)=0$$

$$\therefore \quad y_h(x)=c_1y_1(x)+c_2y_2(x)$$

To get $y_p(x)$, Now replace c_1 and c_2 by the “variable parameters” $u_1(x)$ and $u_2(x)$ $\Rightarrow$ $y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$ particular solution

If $u_1(x)$ and $u_2(x)$ are determined, then $y_p(x)$ is obtained

$$\because \quad y_p=u_1y_1+u_2y_2 \quad \Rightarrow \quad y_p'=u_1y_1'+y_1u_1'+u_2y_2'+y_2u_2' \qquad (1)$$

If we make the demand that $u_1(x)$ and $u_2(x)$ be functions of which

$$y_1u_1'+y_2u_2'=0 \qquad (2)$$

then (1)$\Rightarrow$ $y_p'=u_1y_1'+u_2y_2'$ $\quad\therefore\quad$ $y_p''=u_1y_1''+y_1'u_1'+u_2y_2''+u_2'y_2'$

hence $y\ \ ''(x)+p(x)y\ \ '(x)+q(x)y\ \ (x)$

$$=u_1y_1''+u_1'y_1'+u_2y_2''+u_2'y_2'+pu_1y_1'+pu_2y_2'+qu_1y_1+qu_2y_2$$

$$=u_1(y_1''+py_1'+qy_1)+u_2(y_2''+py_2'+qy_2)+u_1'y_1'+u_2'y_2'=f(x)$$

i.e., $u_1(x)$ and $u_2(x)$ must be functions that also satisfy

$$u_1'y_1'+u_2'y_2'=f(x) \qquad (3)$$

(2) & (3) $\quad y_1u_1'+y_2u_2'=0$

$$u_1'y_1'+u_2'y_2'=f(x)$$

by Cramer’s Rule:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 f(x)}{W(y_1, y_2)} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f(x)}{W(y_1, y_2)}$$

$$\therefore u_1 = \int \frac{-y_2 f(x)}{W(y_1, y_2)} dx \qquad u_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)} dx$$

$$\Rightarrow \quad y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

Example: $y'' + 2y' + y = \dfrac{e^{-x}}{x}$

$$y_h = c_1 e^{-x} + c_2 x e^{-x} \qquad y_1 = e^{-x}, \quad y_2 = x e^{-x}$$

$$W(y_1, y_2) = e^{-2x}$$

$$\therefore u_1 = \int \frac{-y_2 f(x)}{W(y_1, y_2)} dx = \int \frac{-xe^{-x} \frac{e^{-x}}{x}}{e^{-2x}} dx = \int (-1) dx = -x$$

$$u_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)} dx = \int \frac{e^{-x} \frac{e^{-x}}{x}}{e^{-2x}} dx = \int (\frac{1}{x}) dx = \ln|x|$$

$$\Rightarrow \quad y_p(x) = -xe^{-x} + xe^{-x} \ln|x|$$

$$\therefore \quad y = y_h + y_p = c_1 e^{-x} + c_2 x e^{-x} + x e^{-x} \ln|x|$$

2.8 Forced Oscillations

Free motion (no applied force)

$$my'' + cy' + ky = 0$$

inertia force damping force spring force

Forced motion (force applied on the system)

$$my'' + cy' + ky = r(t)$$

$r(t)$: input, driving force

$y(t)$: output, response of the system to the driving force

If $r(t) = F_0 \cos \omega t$

$$\Rightarrow \quad my'' + cy' + ky = F_0 \cos \omega t \qquad (1)$$

Solve: y_h is known in section 2.5 $\Rightarrow y_h(t) = C e^{-\alpha t} \cos(\omega^* t - \delta)$

y_p is assuming of the form $y_p(t) = a \cos \omega t + b \sin \omega t$

$$(1) \Rightarrow \left[\left(k - m\omega^2\right)a + \omega c b\right]\cos \omega t + \left[-\omega c a + \left(k - m\omega^2\right)b\right]\sin \omega t = F_0 \cos \omega t$$

$$\Rightarrow \begin{cases} \left[\left(k - m\omega^2\right)a + \omega c b\right] = F_0 \\ -\omega c a + \left(k - m\omega^2\right)b = 0 \end{cases}$$

$$a = F_0 \frac{k - m\omega^2}{\left(k - m\omega^2\right)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{\left(k - m\omega^2\right)^2 + \omega^2 c^2}$$

$\because \; k/m = \omega_0^2$: natural frequency of the system (undamped)

$$\Rightarrow a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}$$

$$\Rightarrow y(t) = C e^{-\alpha t} \cos(\omega^* t - \delta) + F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2} \cos \omega t + F_0 \frac{\omega c}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2} \sin \omega t \qquad (2)$$

Case I: undamped forced oscillation

★ If $c = 0$ and assume that $\omega_0^2 \neq \omega^2$

then $\alpha = 0, \quad b = 0, \quad \omega_0^* = \omega_0$

$$(2) \Rightarrow y(t) = C \cos(\omega_0 t - \delta) + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t \qquad (3)$$

the maximum amplitude of y_p is $a_0 = \dfrac{F_0}{m\left(\omega_0^2 - \omega^2\right)} = \dfrac{F_0}{k} \dfrac{1}{1 - \left(\omega / \omega_0\right)^2}$

If $\omega = \omega_0 \Rightarrow a_0 \uparrow$ tend to infinity

This phenomenon of large oscillations by matching the input and the natural frequency is known as " **resonance** "

★ when $\omega = \omega_0$ i.e. at resonance, (1) $\Rightarrow$ $y'' + \omega_0^2 y = \dfrac{F_0}{m} \cos \omega_0 \ t$

One should try $y_p = t(a \cos \omega_0 \ t + b \sin \omega_0 t)$

(Since $y_h = c_1 \cos \omega_0 \ t + c_2 \sin \omega_0 t)$

$\Rightarrow a = 0, \quad b = F_0 / 2m\omega_0$

$\therefore \quad y_p = \dfrac{F_0}{2m\omega_0} t \sin \omega_0 t$ as $t \to \infty, \ y_p \to \infty$ $\Rightarrow$ pure resonance

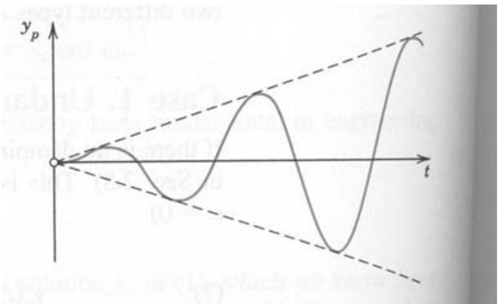


Fig. 58. Particular solution in the case of resonance

★ If c = 0 and $\omega_0^2 \neq \omega^2$ but ω very close to ω_0 and assume initially rest, i.e., $y(0) = 0$, $y'(0) = 0$.

Then (3) $\Rightarrow y(t) = \dfrac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos \omega t - \cos \omega_0 t\right)$

$$= \frac{2F_0}{m(\omega_0^2 - \omega^2)} \left(\sin \frac{\omega + \omega_0}{2} t \sin \frac{\omega_0 - \omega}{2} t \right)$$

If let $\omega_0 - \omega = 2\varepsilon, \quad \varepsilon : small, \quad \omega_0 \approx \omega$

$$y(t) \cong \frac{2F_0}{m(\omega_0 + \omega)(\omega_0 - \omega)} \left(\sin \frac{\omega + \omega_0}{2} t \sin \frac{\omega_0 - \omega}{2} t \right)$$

$$\cong \frac{F_0}{2m\,\varepsilon\,\omega_0}\left(\sin\omega_0 t \;\; \sin\varepsilon\; t\right) \qquad \text{“beating”}$$

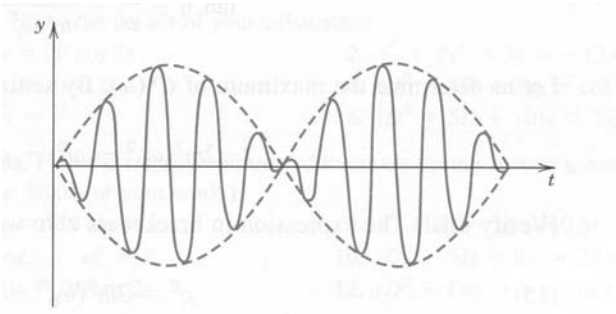


Fig. 59. Forced undamped oscillation when the difference of the input and natural frequencies is small (**“beats”**)

Case II: damped forced oscillation ($c \neq 0$)

(2) $\Rightarrow y = y_h + y_p$

$$y_h(t) = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t)$$

transient solution

$$y_p = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}\cos\omega t + F_0 \frac{\omega c}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}\sin\omega t \quad (4)$$

steady state solution

in undamped case, the amplitude of $y_p \to \infty$, as $\omega \to \omega_0$

in damped case , the amplitude of y_p is finite as $\omega \to \omega_0^*$

Amplitude of y_p:

(4) in form of $y_p(t) = C^* \cos(\omega t - \eta)$

where $C^* = \dfrac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}}$, $\qquad \tan\eta = \dfrac{\omega c}{m(\omega_0^2 - \omega^2)}$

the maximum of C^* occurs at $\dfrac{dC^*}{d\omega} = 0$

$$\Rightarrow \left[-2m^2(\omega_0^2 - \omega^2) + c^2\right]\omega = 0$$

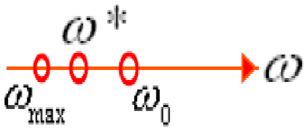
$$\Rightarrow \omega = 0 \quad or \quad \omega = \sqrt{\omega_0^2 - \frac{c^2}{2\,m^2}} \qquad \Leftarrow \omega_{\max}$$

$$\text{if} \quad c^2 > 2m^2\omega^2 \quad \Rightarrow \quad \textit{no solution} \quad \Rightarrow \quad C^* \text{ decrease (no maximum)}$$

$$\text{if} \quad c^2 \le 2m^2\omega^2 \quad \Rightarrow \quad \omega_{\max}\ \textit{exist} \quad \Rightarrow \quad C^*{}_{\max} = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}$$

$$c \text{ decrease } \Rightarrow C^*{}_{\max} \text{ increase}$$

Note: $$\begin{cases} \omega_{\max} = \sqrt{\omega_0^2 - \dfrac{c^2}{2m^2}} \\ \omega^* = \sqrt{\omega_0^2 - \dfrac{c^2}{4m^2}} \end{cases}$$



In general $\omega_0, \omega^*, \omega_{\max}$ are very close to each other.

$$\frac{C^*}{F_0} = \frac{1}{\sqrt{m^2\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 c^2}} = \frac{1}{m\omega_0^2\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_0}\right)^2\right]^2 + \dfrac{\omega^2 c^2}{m^2\omega_0^4}}}$$

$$= \frac{1}{k\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_0}\right)^2\right]^2 + 4\left(\dfrac{\omega}{\omega_0}\dfrac{c}{c_c}\right)^2}},$$

Where $c_c = 2\sqrt{km} \Rightarrow c_c{}^2 = 4km = 4m^2\omega_0^2$

The magnification factor is defined as:

$$\text{magnification ratio} \quad \frac{C^*}{F_0/k} = \frac{1}{\sqrt{\left[1 - \left(\dfrac{\omega}{\omega_0}\right)^2\right]^2 + \left(2\dfrac{\omega}{\omega_0}\dfrac{c}{c_c}\right)^2}}$$

It means (dynamic displacement) / (static displacement amplitude).

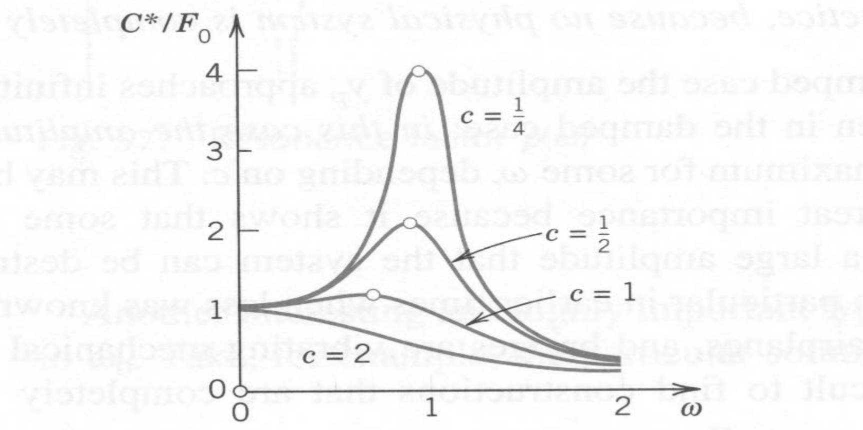


Fig. 60. Amplification C^*/F_0 as a function of ω for $m = 1$, $k = 1$, and various values of the damping constant c

phase angle:

$$\tan\eta = \frac{\omega c}{m(\omega_0^2 - \omega^2)}, \quad \eta : \text{phase angle, since} \quad y_p(t) = C^* \cos(\omega t - \eta)$$

$$\omega < \omega_0 \Rightarrow \quad \tan\eta > 0 \Rightarrow \eta < \frac{\pi}{2}$$

$$\omega > \omega_0 \Rightarrow \quad \tan\eta < 0 \Rightarrow \eta > \frac{\pi}{2}$$

$$\omega = \omega_0 \Rightarrow \quad \tan\eta \to \infty \Rightarrow \eta = \frac{\pi}{2}$$

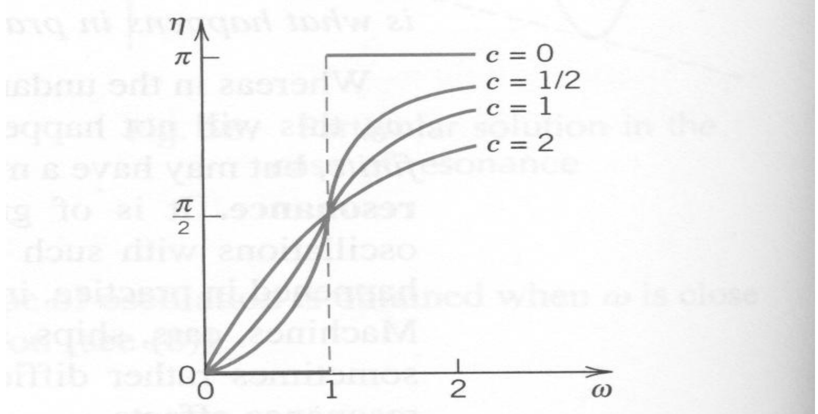


Fig. 61. Phase lag η as a function of ω for $m = 1$, $k = 1$, thus $\omega_0 = 1$, and various values of the damping constant c

2.9 Electric Circuits

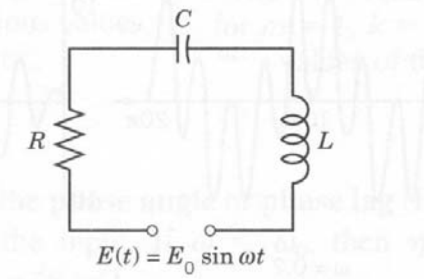


Fig. 63. *RLC*-circuit

Kirchhoff's law: $\quad L\,I' + RI + \dfrac{1}{C}\int I\,dt = E_0 \sin \omega t$

Differentiate with respect to t

$$L\,I'' + RI' + \frac{I}{C} = E_0 \omega \cos \omega t$$

compare: $\quad my'' + cy' + ky = F_0 \cos \omega t$

Electric circuit		mass-spring system
Inductance L	$\Leftrightarrow$	mass m
Resistance R	$\Leftrightarrow$	damping constant c
Reciprocal of capacitance $1/C$	$\Leftrightarrow$	spring constant k
Electromotive force $E_0 \omega \cos \omega t$	$\Leftrightarrow$	driving force $F_0 \cos \omega t$
Current I	$\Leftrightarrow$	displacement y

The phenomenon is the same as previous section

※ Complex method for particular solution

Consider $\quad L\,I'' + RI' + \dfrac{I}{C} = E_0 \omega \cos \omega t \qquad (1)$

Since $\cos \omega t$ is the real part of $e^{i\omega t}$

(1) $\Rightarrow$ complex equation : $L\,I'' + RI' + \dfrac{I}{C} = E_0 \omega\, e^{i\omega t} \qquad (2)$

the real part of the particular solution of (2) is the solution of (1).

Assume $\quad I = Ke^{i\omega t}$

(2) $\Rightarrow$ $(-\omega^2 L + i\omega R + \dfrac{1}{C})Ke^{i\omega t} = E_0 \omega e^{i\omega t}$

$$\Rightarrow K = \frac{E_0}{-(\omega L - \frac{1}{\omega C}) + i\,R} = \frac{E_0}{-S + i\,R} = \frac{-E_0(S + i\,R)}{R^2 + S^2}$$

where $\quad S = \omega L - \frac{1}{\omega C}$

$$\therefore I = Ke^{i\omega t} = \frac{-E_0}{R^2 + S^2}(S + iR)\,(\cos\omega t + i\sin\omega t)$$

the solution is the real part of $I = \dfrac{-E_0}{R^2 + S^2}(S\cos\omega t - R\sin\omega t)$

If the input at right hand side is $\sin\omega t$, then the solution will be imaginary part of *I*.