

12.8 Modeling: Membrane, Two-Dimensional Wave Equation

Since the modeling here will be similar to that of Sec. 12.2, you may want to take another look at Sec. 12.2.

The vibrating string in Sec. 12.2 is a basic one-dimensional vibrational problem. Equally important is its two-dimensional analog, namely, the motion of an elastic membrane, such as a drumhead, that is stretched and then fixed along its edge. Indeed, setting up the model will proceed almost as in Sec. 12.2.

Physical Assumptions

1. The mass of the membrane per unit area is constant (“homogeneous membrane”). The membrane is perfectly flexible and offers no resistance to bending.
2. The membrane is stretched and then fixed along its entire boundary in the xy -plane. The tension per unit length T caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.
3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

Although these assumptions cannot be realized exactly, they hold relatively accurately for small transverse vibrations of a thin elastic membrane, so that we shall obtain a good model, for instance, of a drumhead.

Derivation of the PDE of the Model (“Two-Dimensional Wave Equation”) from Forces.

As in Sec. 12.2 the model will consist of a PDE and additional conditions. The PDE will be obtained by the same method as in Sec. 12.2, namely, by considering the forces acting on a small portion of the physical system, the membrane in Fig. 301 on the next page, as it is moving up and down.

Since the deflections of the membrane and the angles of inclination are small, the sides of the portion are approximately equal to Δx and Δy . The tension T is the force per unit length. Hence the forces acting on the sides of the portion are approximately $T\Delta x$ and $T\Delta y$. Since the membrane is perfectly flexible, these forces are tangent to the moving membrane at every instant.

Horizontal Components of the Forces. We first consider the horizontal components of the forces. These components are obtained by multiplying the forces by the cosines of the angles of inclination. Since these angles are small, their cosines are close to 1. Hence the horizontal components of the forces at opposite sides are approximately equal. Therefore, the motion of the particles of the membrane in a horizontal direction will be negligibly small. From this we conclude that we may regard the motion of the membrane as transversal; that is, each particle moves vertically.

Vertical Components of the Forces. These components along the right side and the left side are (Fig. 301), respectively,

$$T\Delta y \sin \beta \quad \text{and} \quad -T\Delta y \sin \alpha.$$

Here α and β are the values of the angle of inclination (which varies slightly along the edges) in the middle of the edges, and the minus sign appears because the force on the

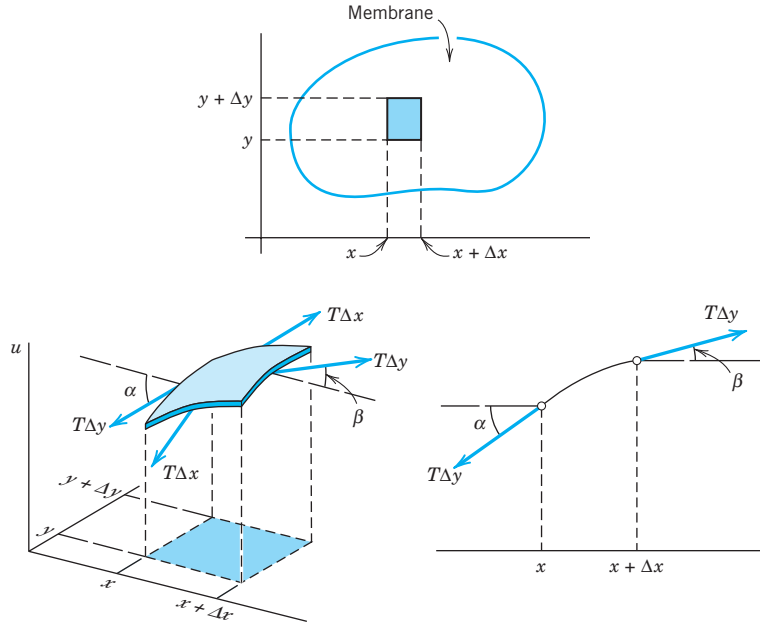


Fig. 301. Vibrating membrane

left side is directed downward. Since the angles are small, we may replace their sines by their tangents. Hence the resultant of those two vertical components is

$$\begin{aligned}
 (1) \quad T\Delta y(\sin \beta - \sin \alpha) &\approx T\Delta y(\tan \beta - \tan \alpha) \\
 &= T\Delta y[u_x(x + \Delta x, y_1) - u_x(x, y_2)]
 \end{aligned}$$

where subscripts x denote partial derivatives and y_1 and y_2 are values between y and $y + \Delta y$. Similarly, the resultant of the vertical components of the forces acting on the other two sides of the portion is

$$(2) \quad T\Delta x[u_y(x_1, y + \Delta y) - u_y(x_2, y)]$$

where x_1 and x_2 are values between x and $x + \Delta x$.

Newton's Second Law Gives the PDE of the Model. By Newton's second law (see Sec. 2.4) the sum of the forces given by (1) and (2) is equal to the mass $\rho \Delta A$ of that small portion times the acceleration $\partial^2 u / \partial t^2$; here ρ is the mass of the undeflected membrane per unit area, and $\Delta A = \Delta x \Delta y$ is the area of that portion when it is undeflected. Thus

$$\begin{aligned}
 \rho \Delta x \Delta y \frac{\partial^2 u}{\partial t^2} &= T\Delta y[u_x(x + \Delta x, y_1) - u_x(x, y_2)] \\
 &\quad + T\Delta x[u_y(x_1, y + \Delta y) - u_y(x_2, y)]
 \end{aligned}$$

where the derivative on the left is evaluated at some suitable point (\tilde{x}, \tilde{y}) corresponding to that portion. Division by $\rho \Delta x \Delta y$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \left[\frac{u_x(x + \Delta x, y_1) - u_x(x, y_2)}{\Delta x} + \frac{u_y(x_1, y + \Delta y) - u_y(x_2, y)}{\Delta y} \right].$$

If we let Δx and Δy approach zero, we obtain the PDE of the model

$$(3) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad c^2 = \frac{T}{\rho}.$$

This PDE is called the **two-dimensional wave equation**. The expression in parentheses is the Laplacian $\Delta^2 u$ of u (Sec. 10.8). Hence (3) can be written

$$(3') \quad \frac{\partial^2 u}{\partial t^2} = c^2 \Delta^2 u.$$

Solutions of the wave equation (3) will be obtained and discussed in the next section.

12.9 Rectangular Membrane. Double Fourier Series

Now we develop a solution for the PDE obtained in Sec. 12.8. Details are as follows.

The model of the vibrating membrane for obtaining the displacement $u(x, y, t)$ of a point (x, y) of the membrane from rest ($u = 0$) at time t is

$$(1) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$(2) \quad u = 0 \text{ on the boundary}$$

$$(3a) \quad u(x, y, 0) = f(x, y)$$

$$(3b) \quad u_t(x, y, 0) = g(x, y).$$

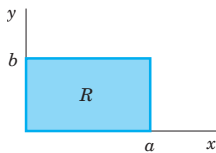


Fig. 302.
Rectangular
membrane

Here (1) is the **two-dimensional wave equation** with $c^2 = T/\rho$ just derived, (2) is the **boundary condition** (membrane fixed along the boundary in the xy -plane for all times $t \geq 0$), and (3) are the **initial conditions** at $t = 0$, consisting of the given *initial displacement* (initial shape) $f(x, y)$ and the given *initial velocity* $g(x, y)$, where $u_t = \partial u / \partial t$. We see that these conditions are quite similar to those for the string in Sec. 12.2.

Let us consider the **rectangular membrane** R in Fig. 302. This is our first important model. It is much simpler than the circular drumhead, which will follow later. First we note that the boundary in equation (2) is the rectangle in Fig. 302. We shall solve this problem in three steps: