Parallel Programming
in C with MPI and OpenMP

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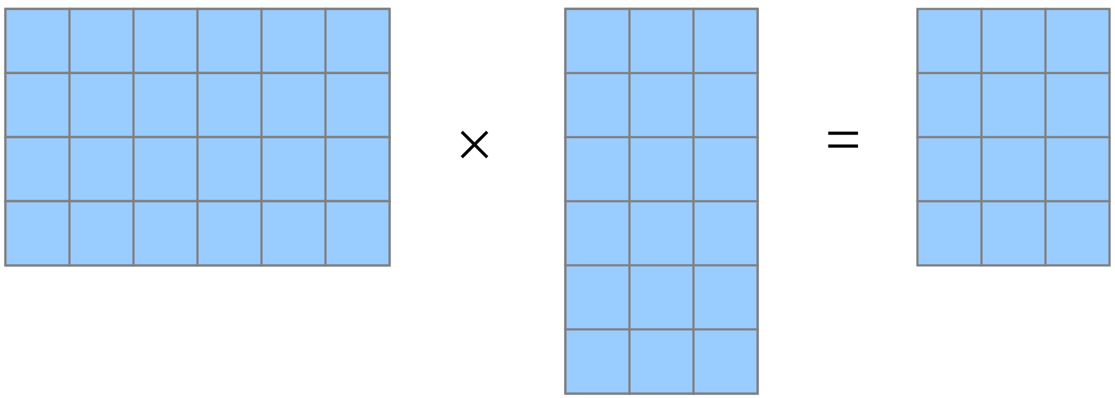
Chapter 11
Matrix Multiplication

Outline

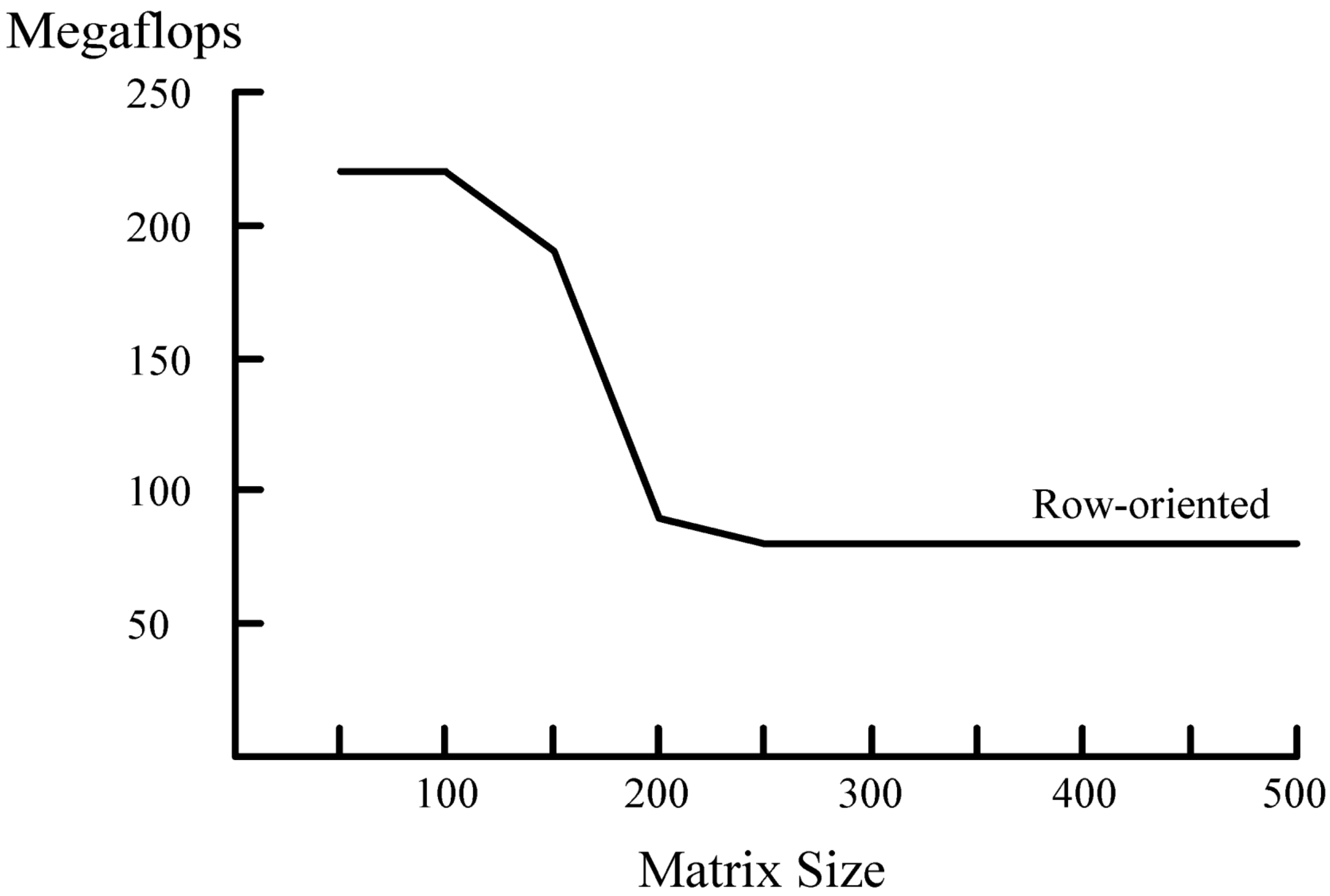
- Sequential algorithms
  - Iterative, row-oriented
  - Recursive, block-oriented
- Parallel algorithms
  - Rowwise block striped decomposition
  - Cannon's algorithm

Iterative, Row-oriented Algorithm

- Series of inner product (dot product) operations

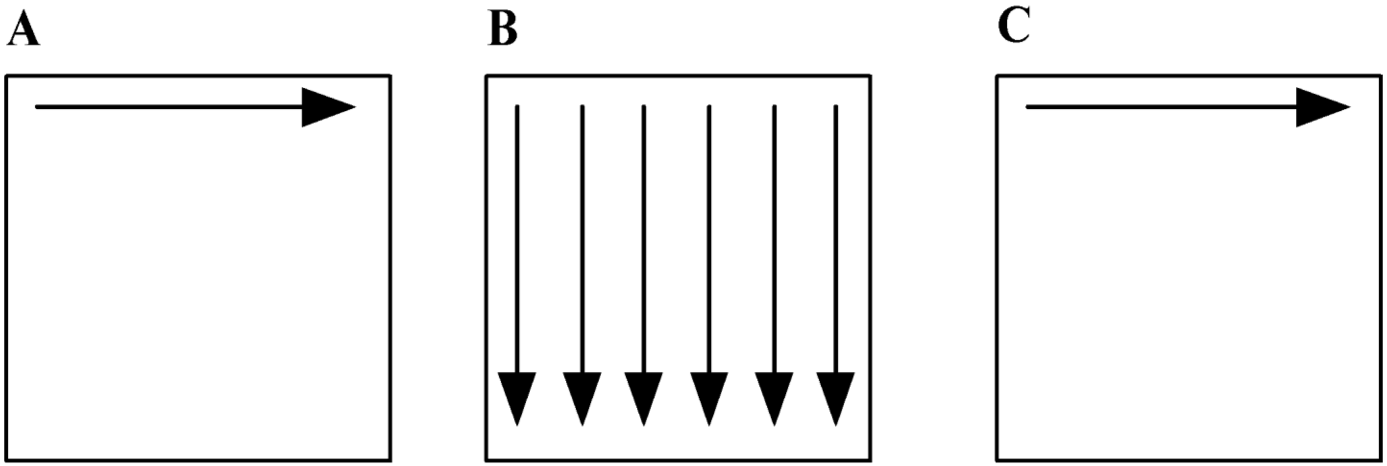


Performance as *n* Increases



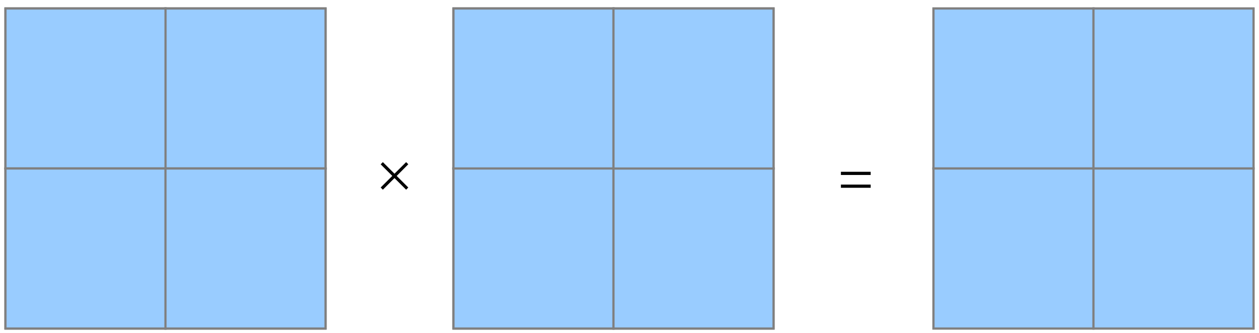
Reason: Matrix B Gets Too Big for Cache

- Computing a row of C requires accessing every element of B

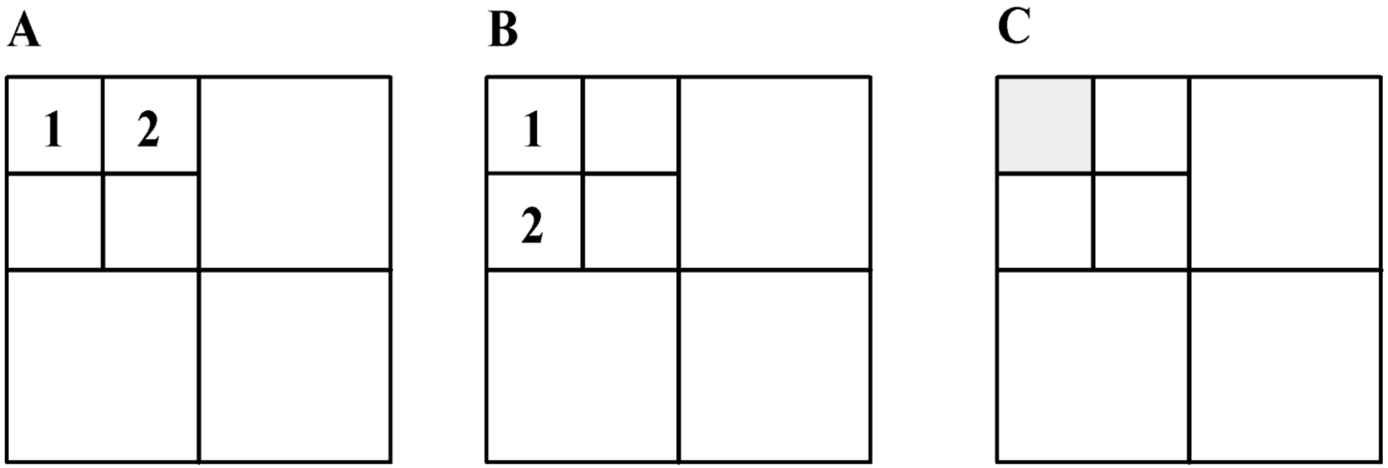


Block Matrix Multiplication

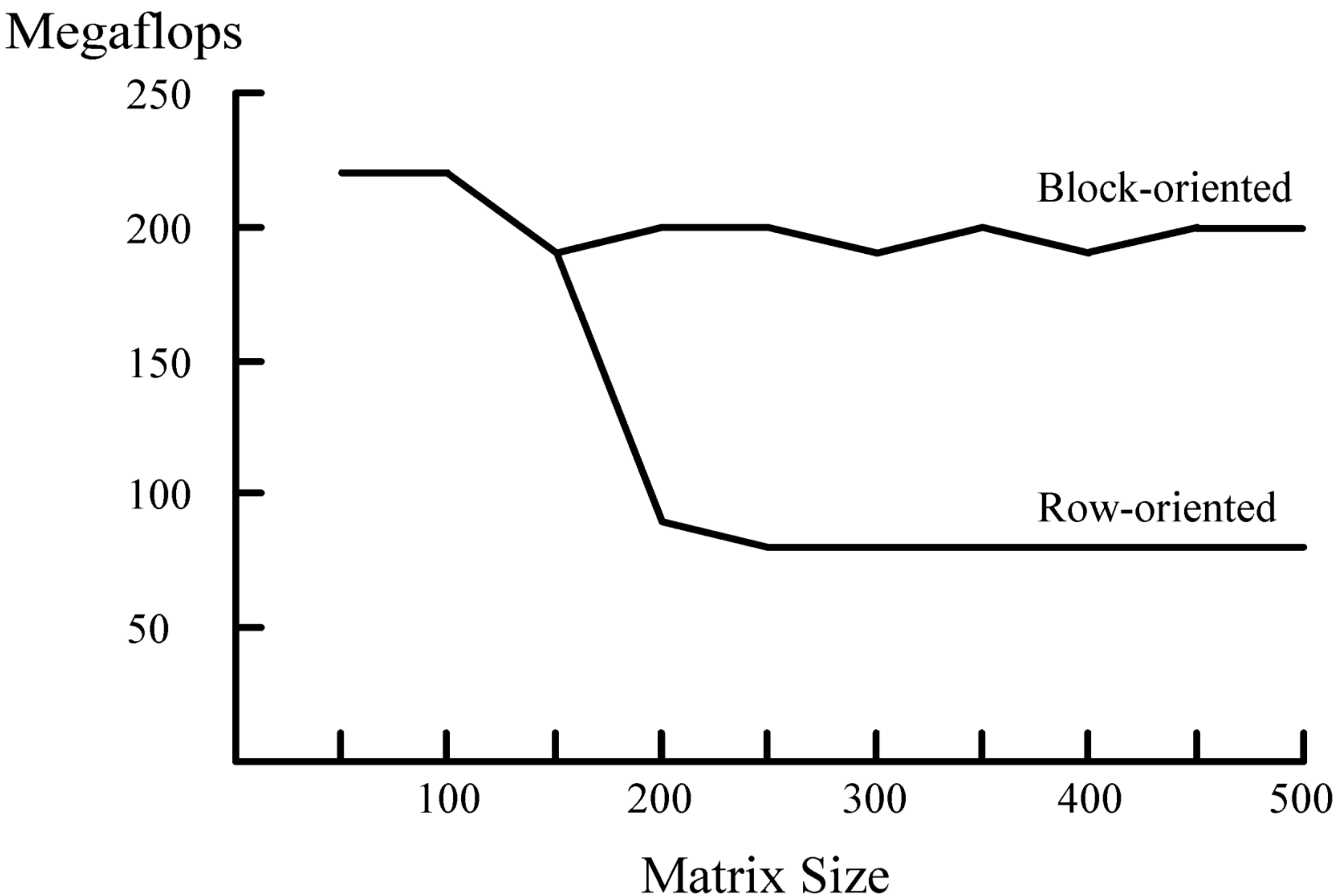
- Replace scalar multiplication with matrix multiplication
- Replace scalar addition with matrix addition



Recurse Until B Small Enough



Comparing Sequential Performance



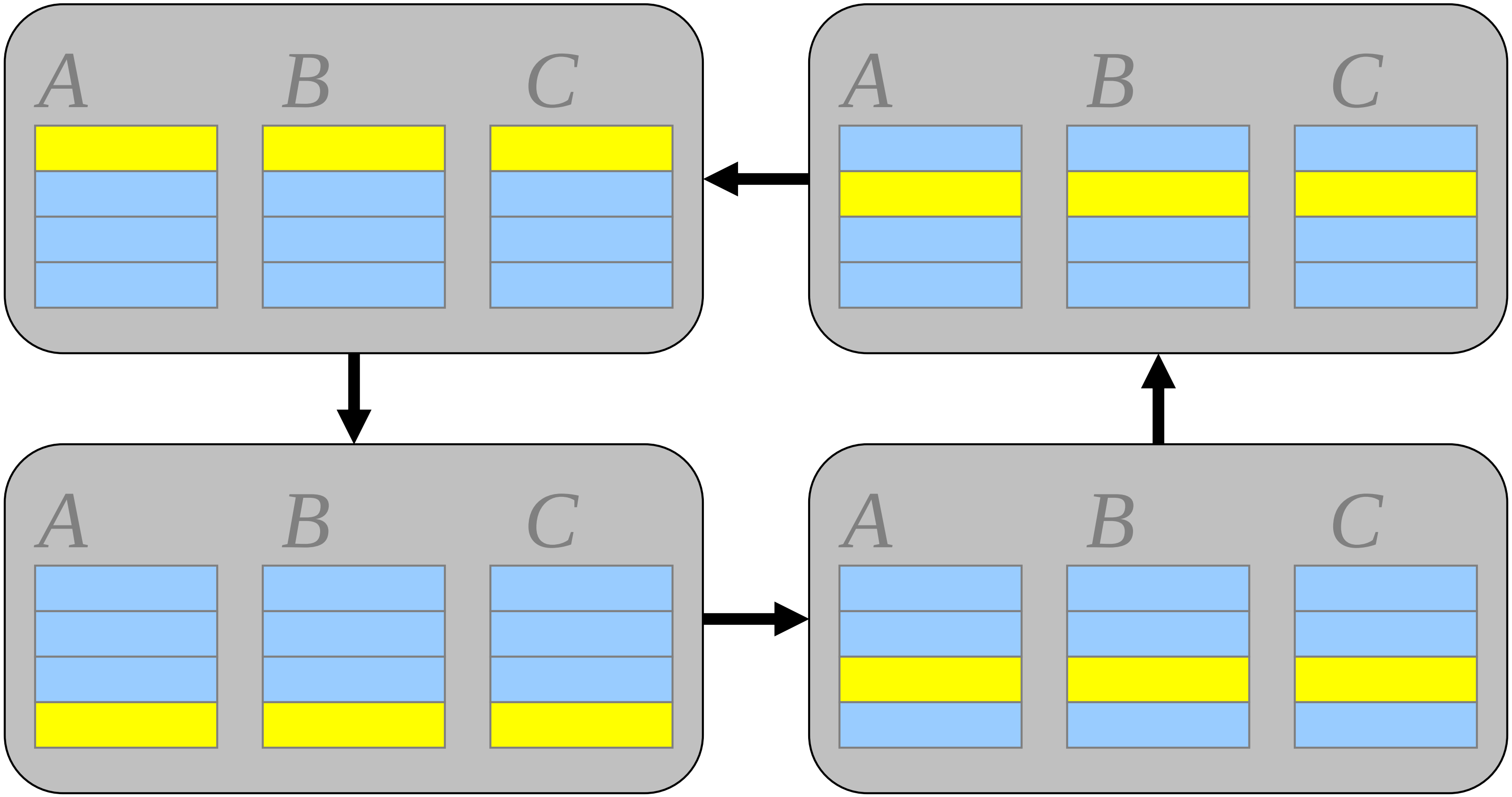
First Parallel Algorithm

- Partitioning
  - Divide matrices into rows
  - Each primitive task has corresponding rows of three matrices
- Communication
  - Each task must eventually see every row of B
  - Organize tasks into a ring

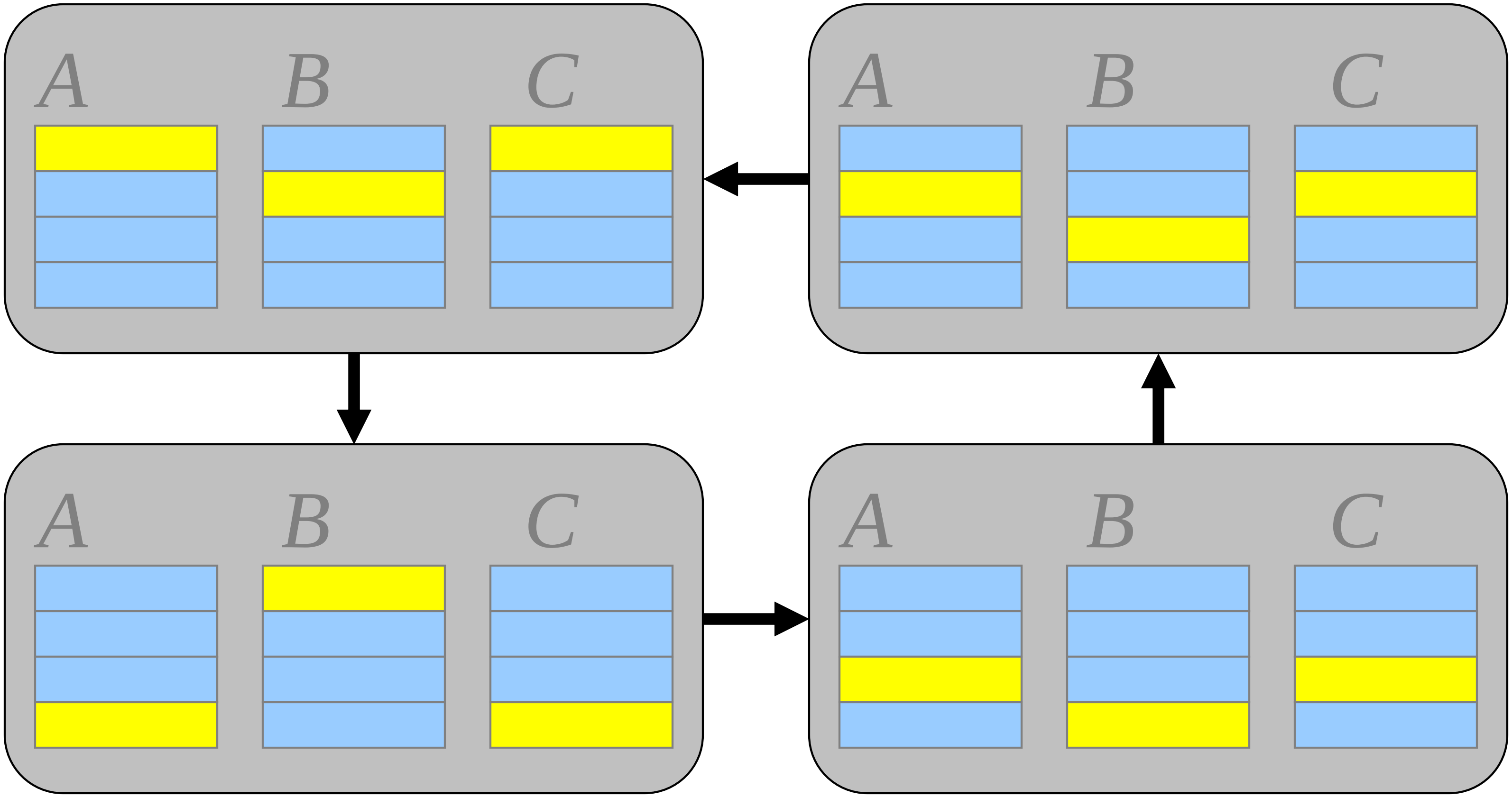
First Parallel Algorithm (cont.)

- Agglomeration and mapping
  - Fixed number of tasks, each requiring same amount of computation
  - Regular communication among tasks
  - Strategy: Assign each process a contiguous group of rows

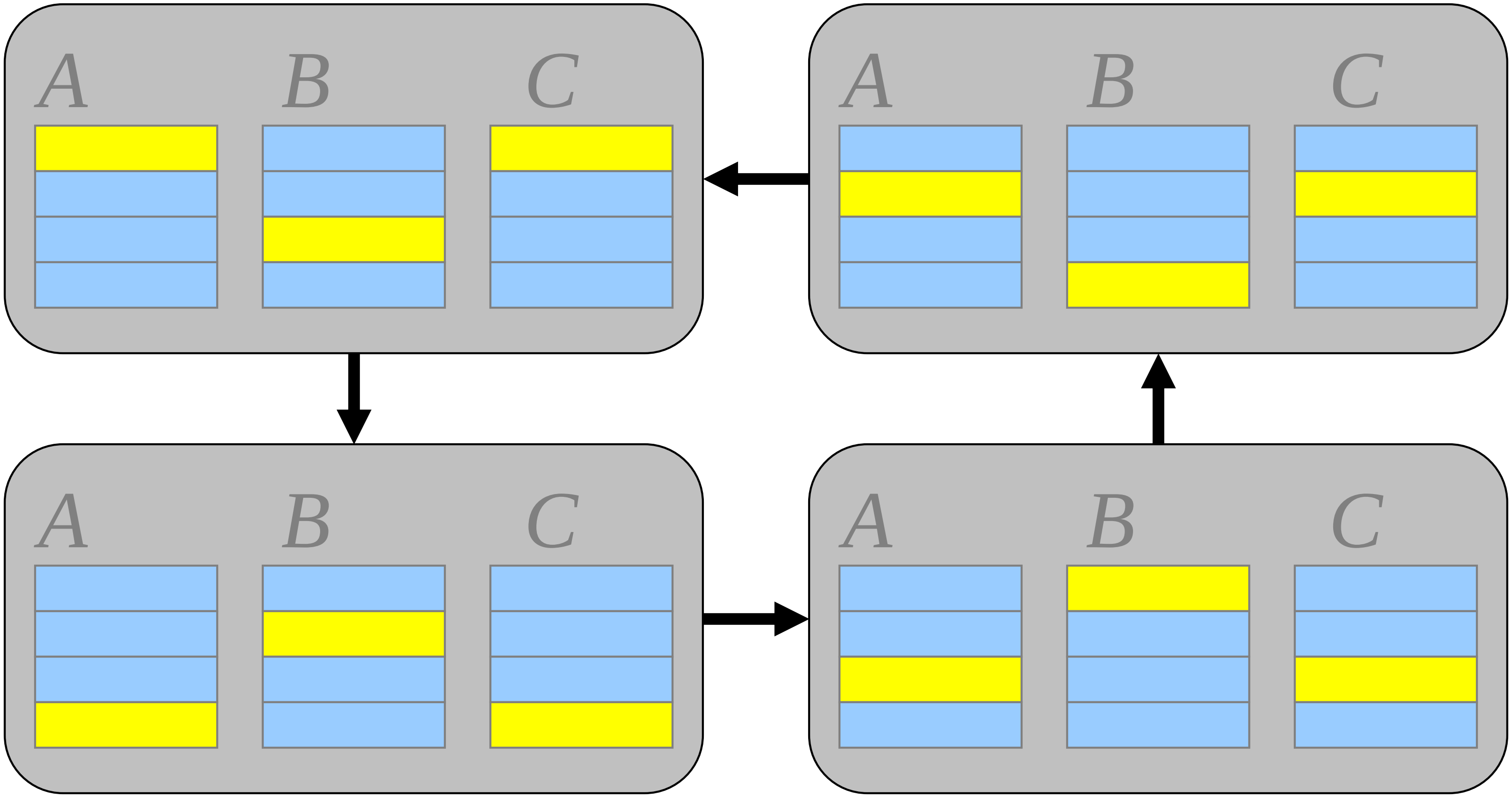
Communication of B (1)



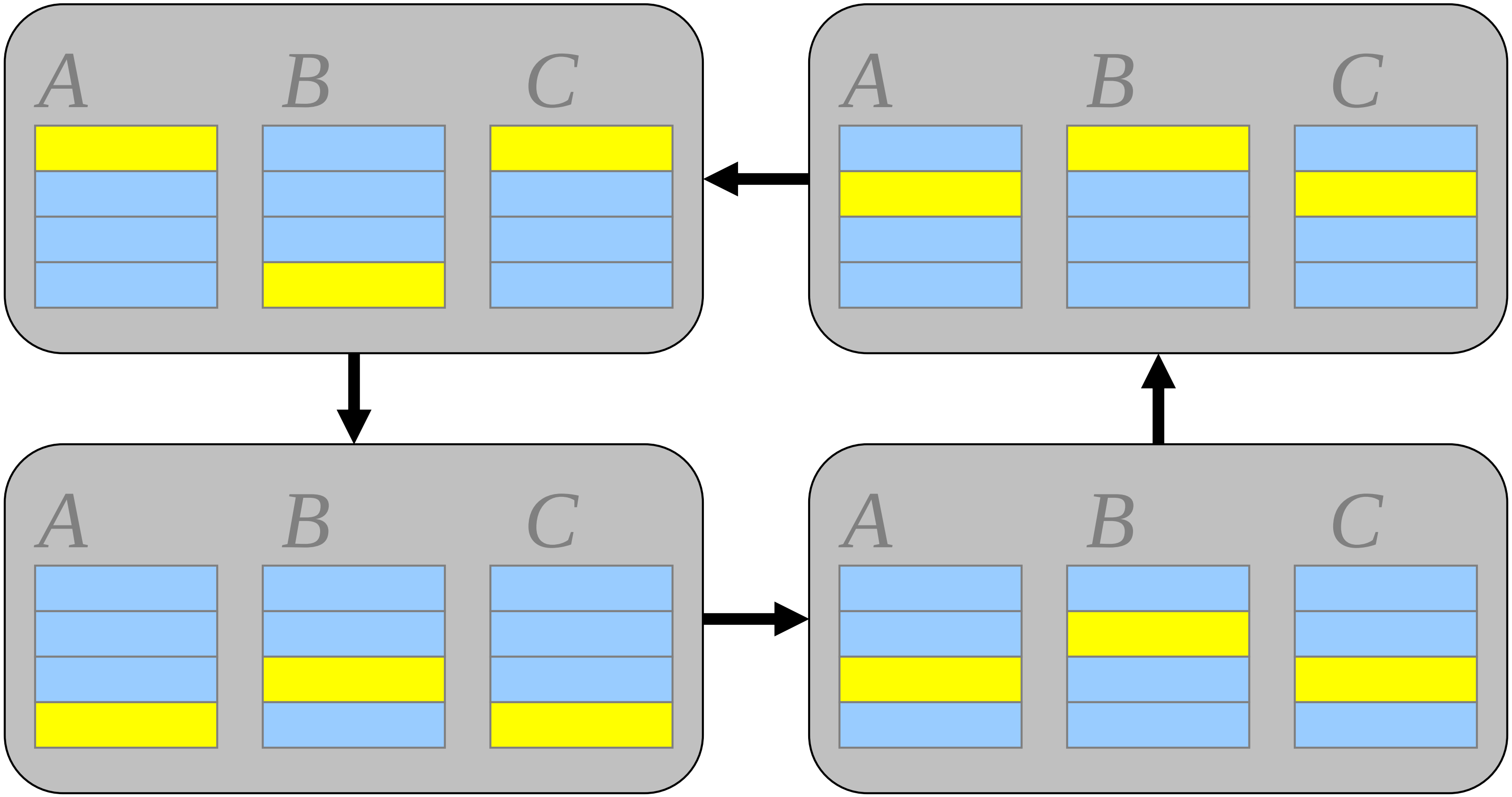
Communication of B (2)



Communication of B (3)



Communication of B (4)



Complexity Analysis

- Algorithm has p iterations
- During each iteration a process multiplies $(n / p) \times (n / p)$ block of A by $(n / p) \times n$ block of B: $\Theta(n^3 / p^2)$
- Total computation time: $\Theta(n^3 / p)$
- Each process ends up passing $(p - 1)n^2/p = \Theta(n^2)$ elements of B

Isoefficiency Analysis

- Sequential algorithm: $\Theta(n^3)$
- Parallel overhead: $\Theta(pn^2)$
  Isoefficiency relation: $n^3 \geq Cpn^2 \;\Rightarrow\; n \geq Cp$

$$M(Cp)/p = C^2 p^2 / p = C^2 p$$

- This system does not have good scalability

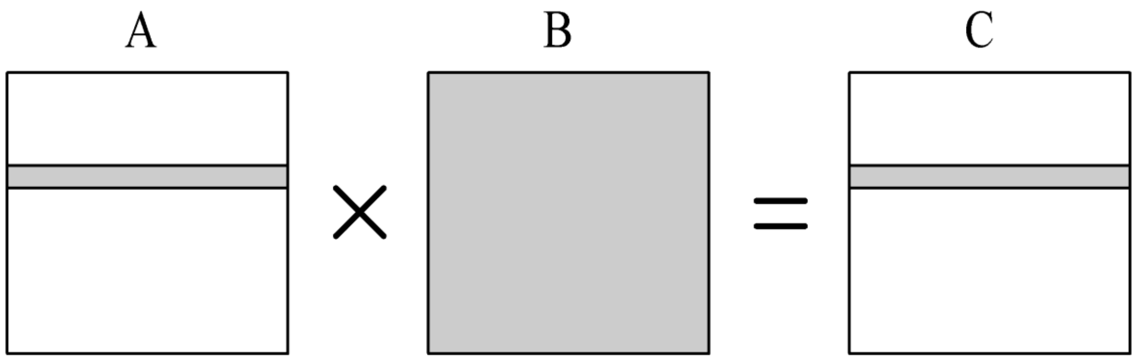
Weakness of Algorithm 1

- Blocks of B being manipulated have p times more columns than rows
- Each process must access every element of matrix B
- Ratio of computations per communication is poor: only $2n / p$

Parallel Algorithm 2 (Cannon's Algorithm)

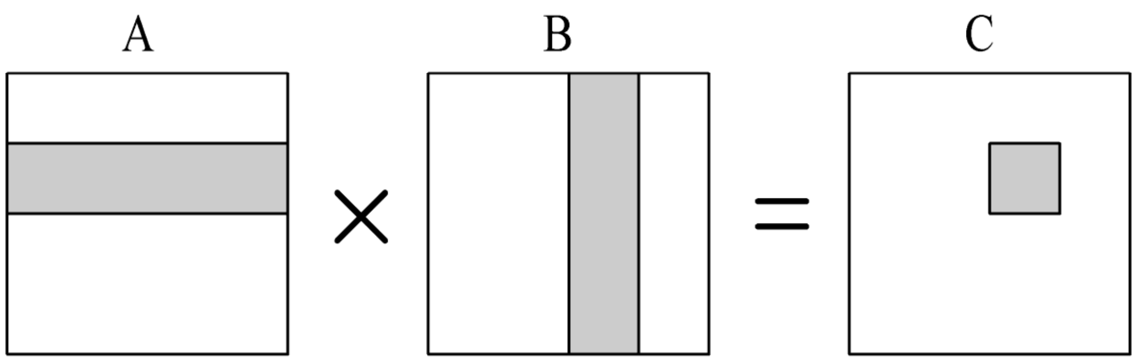
- Associate a primitive task with each matrix element
- Agglomerate tasks responsible for a square (or nearly square) block of C
- Computation-to-communication ratio rises to $n / \sqrt{p}$

Elements of A and B Needed to Compute a Process's Portion of C



(a)

Algorithm 1



(b)

Cannon's Algorithm

Blocks Must Be Aligned

$A_{0,0}$ $B_{0,0}$	$A_{0,1}$ $B_{0,1}$	$A_{0,2}$ $B_{0,2}$	$A_{0,3}$ $B_{0,3}$
$A_{1,0}$ $B_{1,0}$	$A_{1,1}$ $B_{1,1}$	$A_{1,2}$ $B_{1,2}$	$A_{1,3}$ $B_{1,3}$
$A_{2,0}$ $B_{2,0}$	$A_{2,1}$ $B_{2,1}$	$A_{2,2}$ $B_{2,2}$	$A_{2,3}$ $B_{2,3}$
$A_{3,0}$ $B_{3,0}$	$A_{3,1}$ $B_{3,1}$	$A_{3,2}$ $B_{3,2}$	$A_{3,3}$ $B_{3,3}$

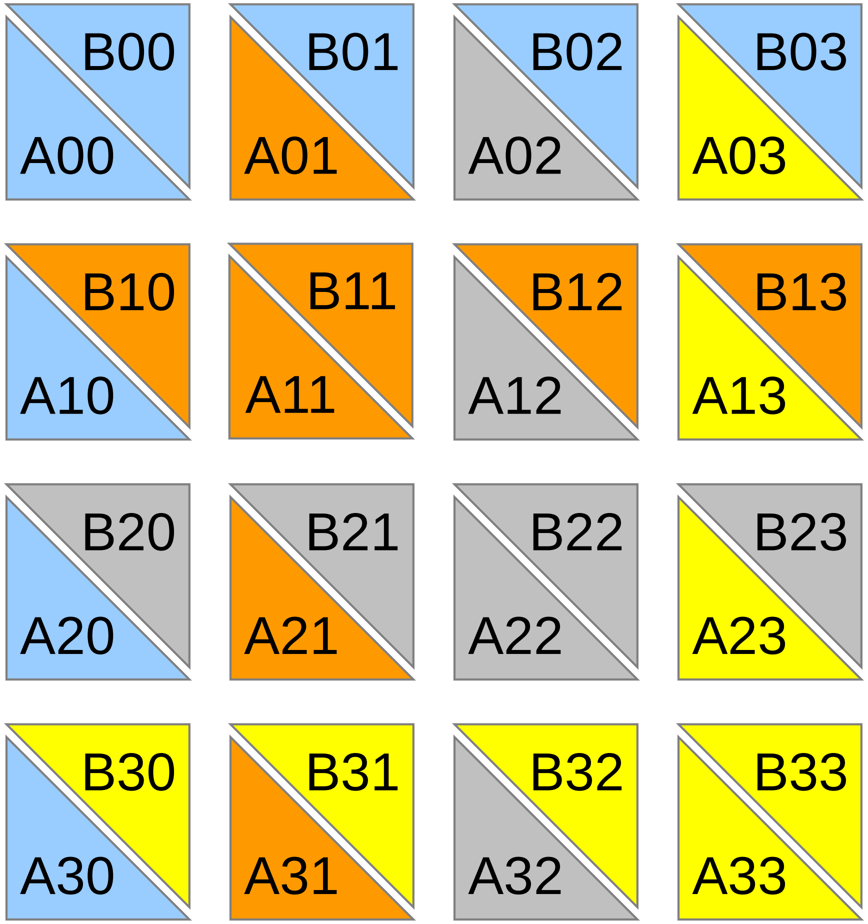
Before

$A_{0,0}$ $B_{0,0}$	$A_{0,1}$ $B_{1,1}$	$A_{0,2}$ $B_{2,2}$	$A_{0,3}$ $B_{3,3}$
$A_{1,1}$ $B_{1,0}$	$A_{1,2}$ $B_{2,1}$	$A_{1,3}$ $B_{3,2}$	$A_{1,0}$ $B_{0,3}$
$A_{2,2}$ $B_{2,0}$	$A_{2,3}$ $B_{3,1}$	$A_{2,0}$ $B_{0,2}$	$A_{2,1}$ $B_{1,3}$
$A_{3,3}$ $B_{3,0}$	$A_{3,0}$ $B_{0,1}$	$A_{3,1}$ $B_{1,2}$	$A_{3,2}$ $B_{2,3}$

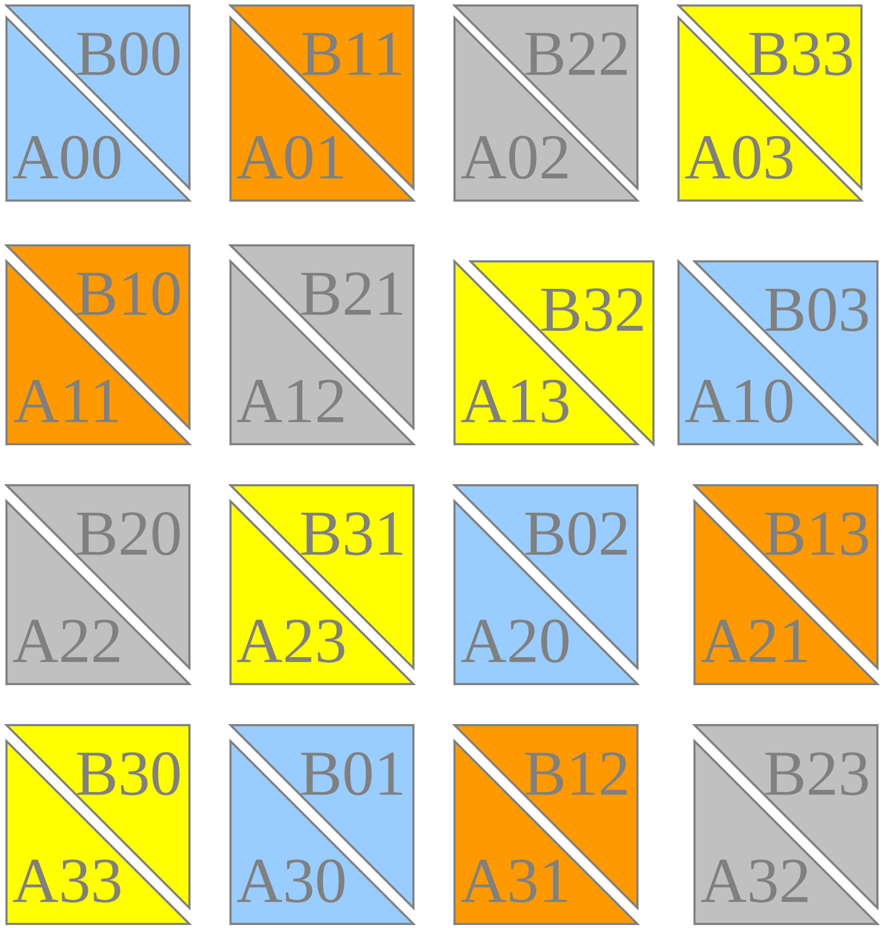
After

Blocks Need to Be Aligned

- Each triangle represents a matrix block
- Only same-color triangles should be multiplied



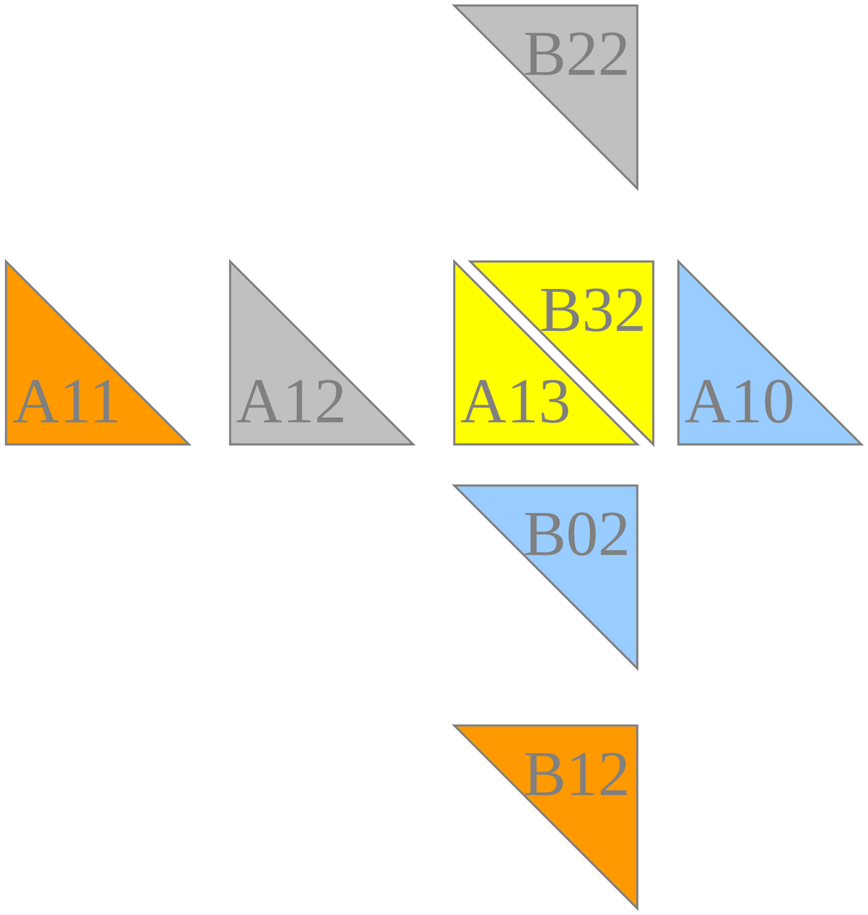
Rearrange Blocks



Block Aij cycles left i positions

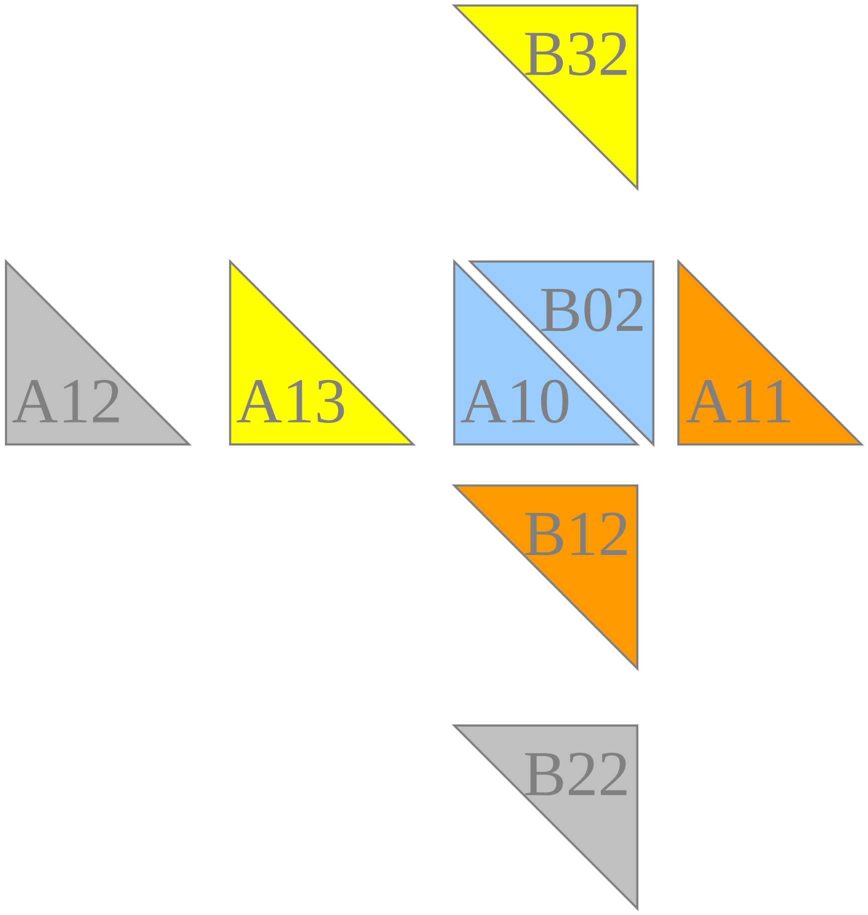
Block Bij cycles up j positions

Consider Process $P_{1,2}$



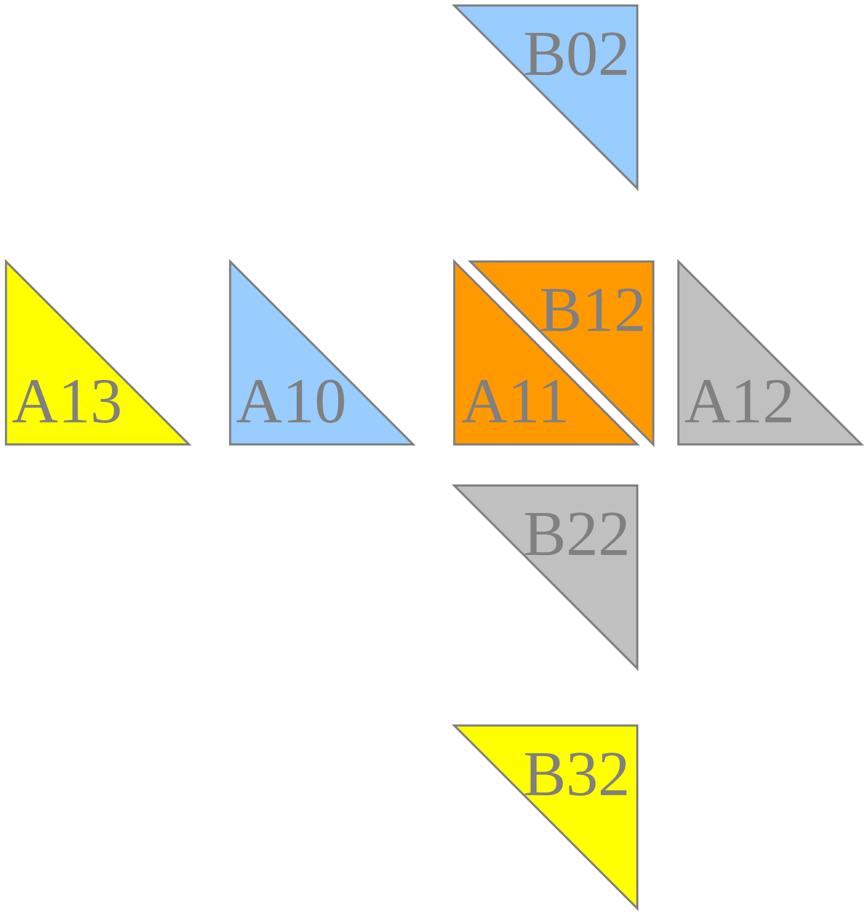
Step 1

Consider Process $P_{1,2}$



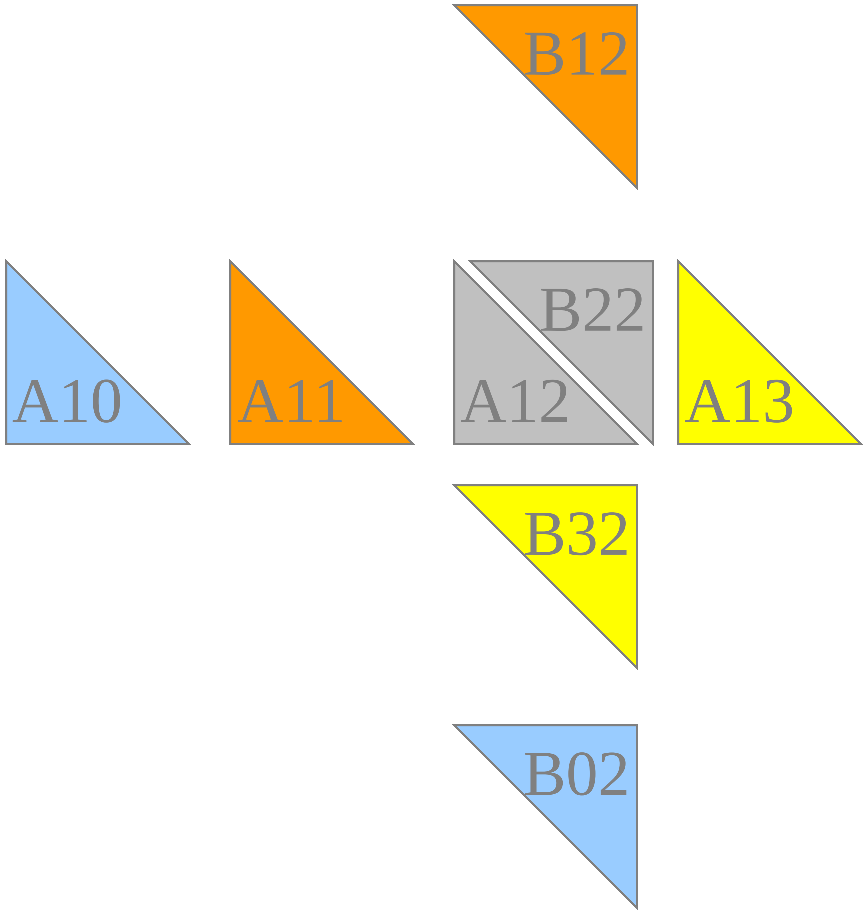
Step 2

Consider Process $P_{1,2}$



Step 3

Consider Process $P_{1,2}$



Step 4

Complexity Analysis

- Algorithm has $\sqrt{p}$ iterations
- During each iteration process multiplies two $(n / \sqrt{p}) \times (n / \sqrt{p})$ matrices: $\Theta(n^3 / p^{3/2})$
- Computational complexity: $\Theta(n^3 / p)$
- During each iteration process sends and receives two blocks of size $(n / \sqrt{p}) \times (n / \sqrt{p})$
- Communication complexity: $\Theta(n^2 / \sqrt{p})$

Isoefficiency Analysis

- Sequential algorithm: $\Theta(n^3)$
- Parallel overhead: $\Theta(\sqrt{p}n^2)$

Isoefficiency relation: $n^3 \geq C\sqrt{p}n^2 \Rightarrow n \geq C\sqrt{p}$

$$M\left(C\sqrt{p}\right)/p = C^2 p/p = C^2$$

- This system is highly scalable

Summary

- Considered two sequential algorithms
  - Iterative, row-oriented algorithm
  - Recursive, block-oriented algorithm
  - Second has better cache hit rate as n increases
- Developed two parallel algorithms
  - First based on rowwise block striped decomposition
  - Second based on checkerboard block decomposition
  - Second algorithm is scalable, while first is not