Parallel Programming
in C with MPI and OpenMP

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Chapter 8
Matrix-vector Multiplication

Chapter Objectives

- Review matrix-vector multiplication
- Propose replication of vectors
- Develop three parallel programs, each based on a different data decomposition

Outline

- Sequential algorithm and its complexity
- Design, analysis, and implementation of three parallel programs
  - Rowwise block striped
  - Columnwise block striped
  - Checkerboard block

Sequential Algorithm

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 1 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 14 \\ 19 \\ 11 \end{bmatrix}$$

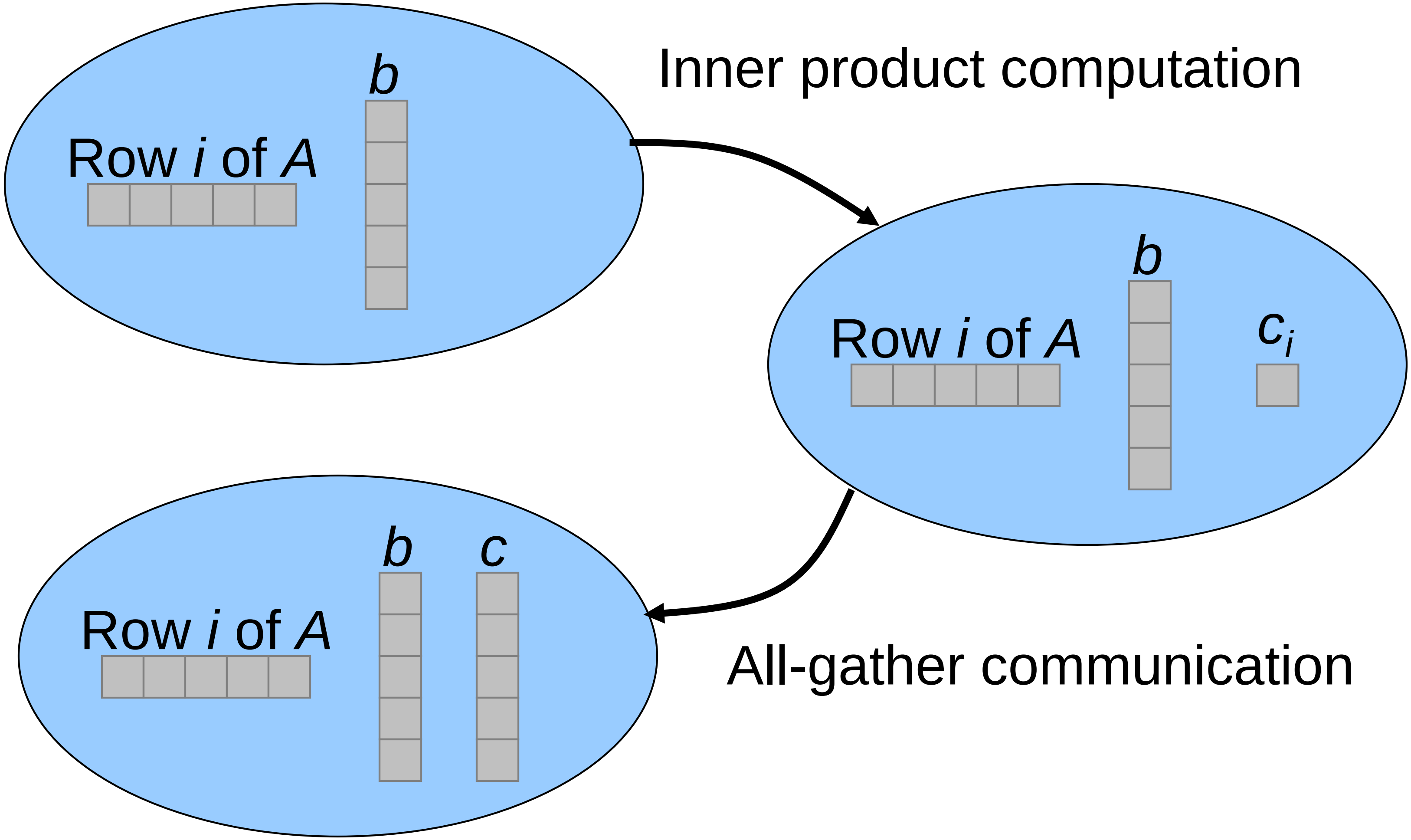
Storing Vectors

- Divide vector elements among processes
- Replicate vector elements
- Vector replication acceptable because vectors have only n elements, versus n^2 elements in matrices

Rowwise Block Striped Matrix

- Partitioning through domain decomposition
- Primitive task associated with
  - Row of matrix
  - Entire vector

Phases of Parallel Algorithm



Agglomeration and Mapping

- Static number of tasks
- Regular communication pattern (all-gather)
- Computation time per task is constant
- Strategy:
  - Agglomerate groups of rows
  - Create one task per MPI process

Complexity Analysis

- Sequential algorithm complexity: $\Theta(n^2)$
- Parallel algorithm computational complexity: $\Theta(n^2/p)$
- Communication complexity of all-gather: $\Theta(\log p + n)$
- Overall complexity: $\Theta(n^2/p + n + \log p)$

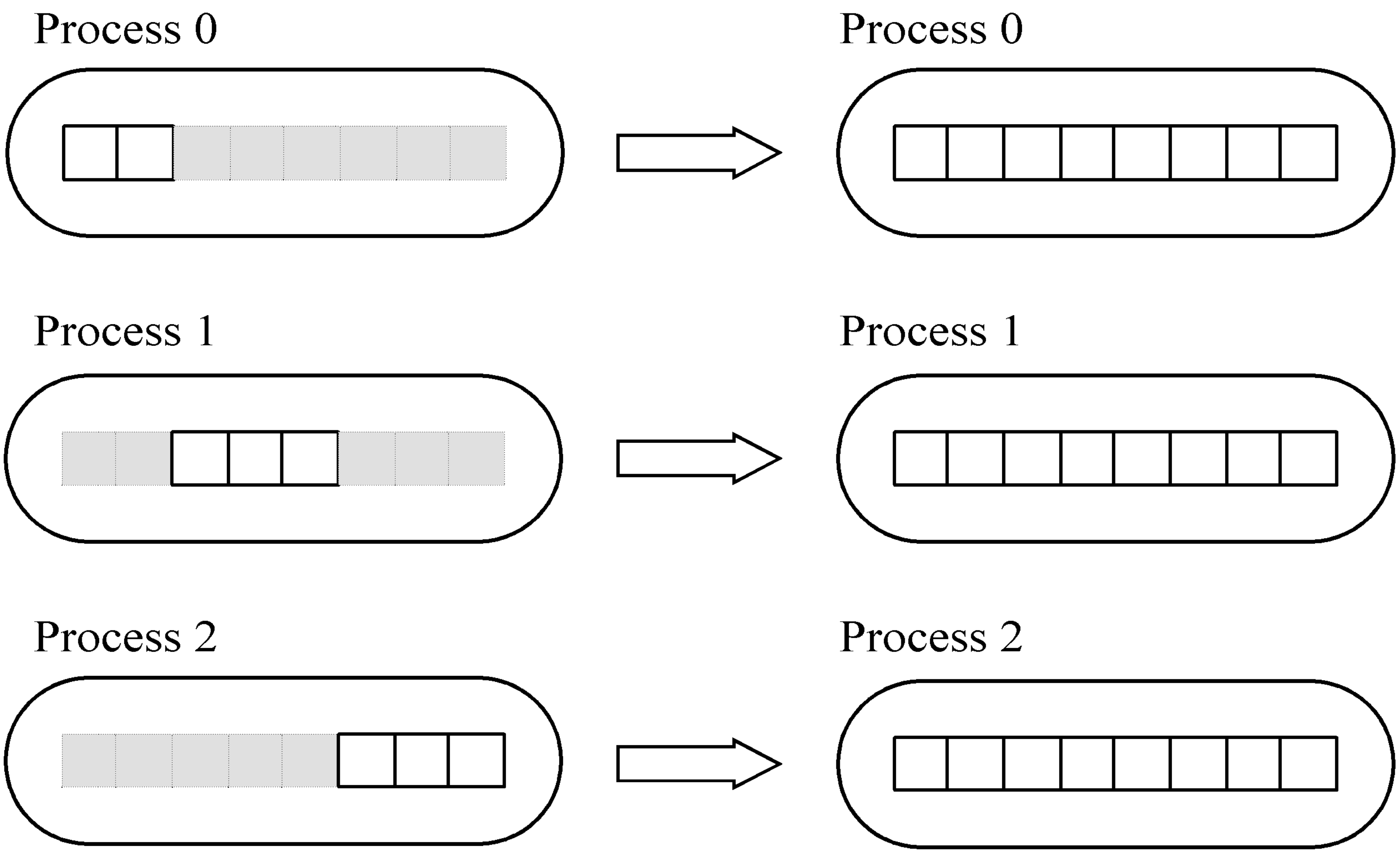
Isoefficiency Analysis

- Sequential time complexity: $\Theta(n^2)$
- Only parallel overhead is all-gather
  - When *n* is large, message transmission time dominates message latency
  - Parallel communication time: $\Theta(n)$
- $n^2 \geq Cpn \Rightarrow n \geq Cp$ and $M(n) = n^2$

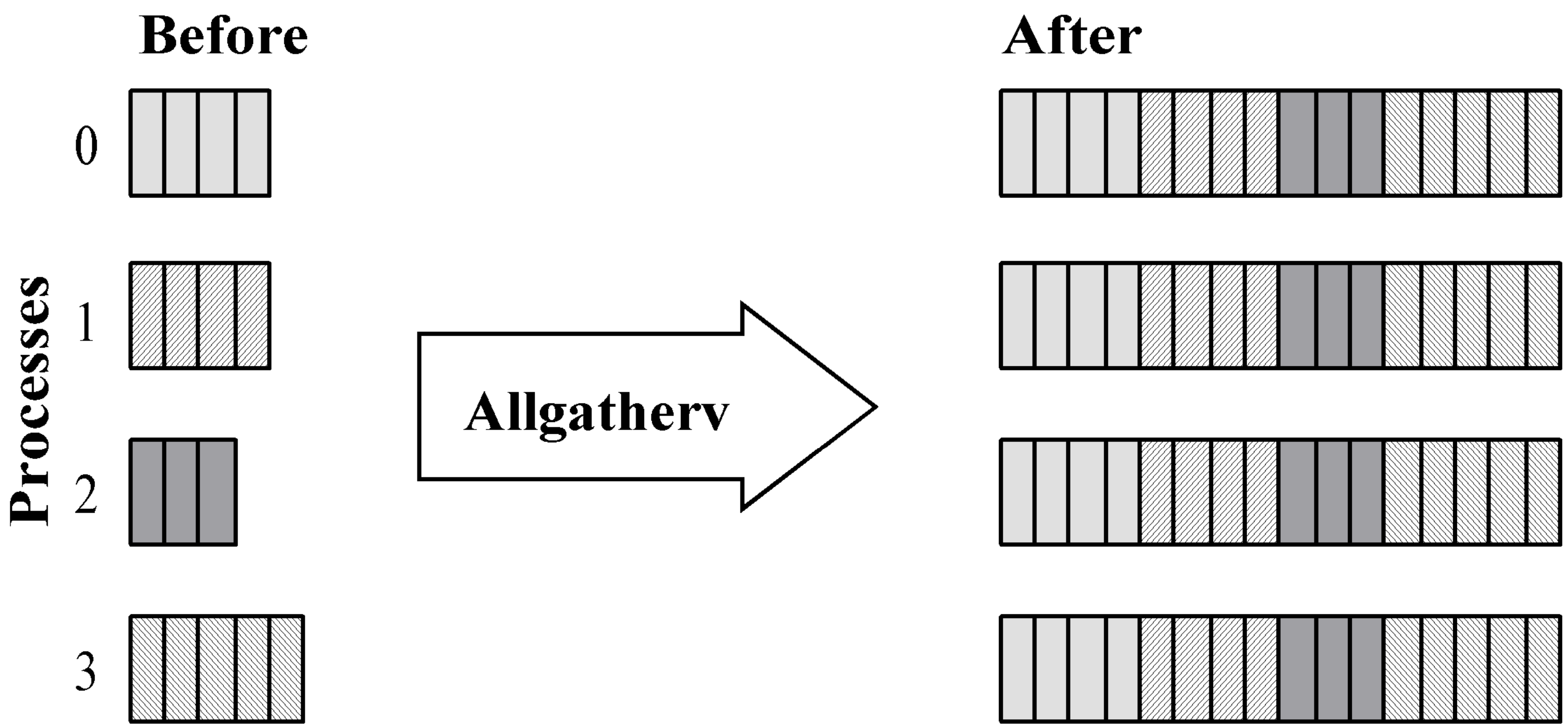
$$M(Cp)/p = C^2 p^2 / p = C^2 p$$

- System is not highly scalable

Block-to-replicated Transformation



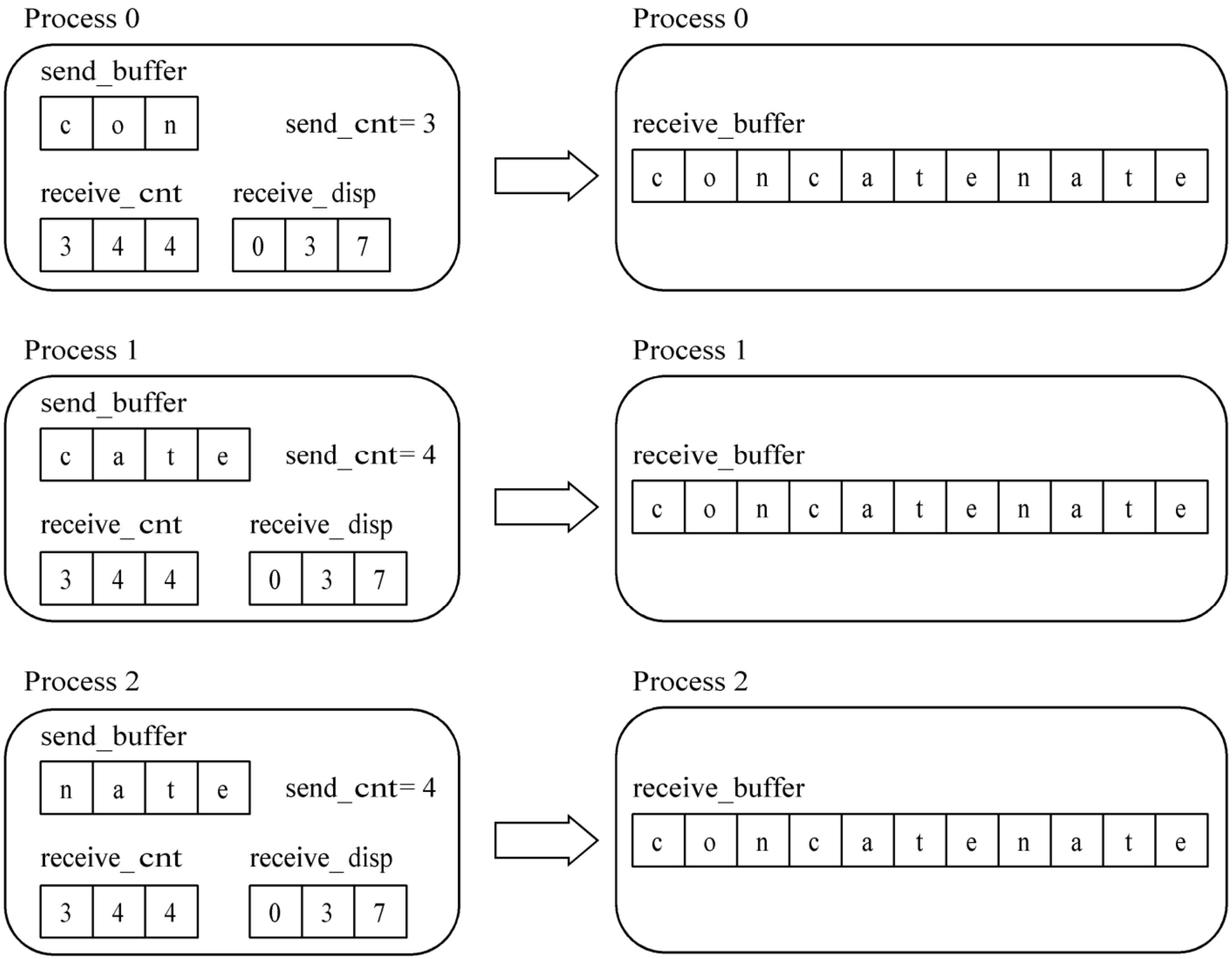
MPI_Allgatherv



MPI_Allgatherv

```
int MPI_Allgatherv (
   void          *send_buffer,
   int            send_cnt,
   MPI_Datatype   send_type,
   void          *receive_buffer,
   int           *receive_cnt,
   int           *receive_disp,
   MPI_Datatype   receive_type,
   MPI_Comm       communicator
)
```

MPI_Allgatherv in Action



Function create_mixed_xfer_arrays

- First array
  - How many elements contributed by each process
  - Uses utility macro BLOCK_SIZE
- Second array
  - Starting position of each process’ block
  - Assume blocks in process rank order

Function replicate_block_vector

- Create space for entire vector
- Create “mixed transfer” arrays
- Call **`MPI_Allgatherv`**

Function read_replicated_vector

- Process $p - 1$
  - Opens file
  - Reads vector length
- Broadcast vector length (root process = $p - 1$)
- Allocate space for vector
- Process $p - 1$ reads vector, closes file
- Broadcast vector

Function print_replicated_vector

- Process 0 prints vector
- Exact call to **`printf`** depends on value of parameter **`datatype`**

Run-time Expression

- χ: inner product loop iteration time
- Computational time: $\chi\, n\lceil n/p \rceil$
- All-gather requires $\lceil \log p \rceil$ messages with latency λ
- Total vector elements transmitted:
  $(2^{\lceil \log p \rceil} - 1) / 2^{\lceil \log p \rceil}$ (recall $(p - 1)/p$)
- Total execution time:
  $\chi\, n\lceil n/p \rceil + \lambda\lceil \log p \rceil + (2^{\lceil \log p \rceil} - 1) / (2^{\lceil \log p \rceil}\beta)$

Benchmarking Results

	Execution Time (msec)			
p	*Predicted*	*Actual*	*Speedup*	*Mflops*
1	63.4	63.4	1.00	31.6
2	32.4	32.7	1.94	61.2
3	22.3	22.7	2.79	88.1
4	17.0	17.8	3.56	112.4
5	14.1	15.2	4.16	131.6
6	12.0	13.3	4.76	150.4
7	10.5	12.2	5.19	163.9
8	9.4	11.1	5.70	180.2
16	5.7	7.2	8.79	277.8

Columnwise Block Striped Matrix

- Partitioning through domain decomposition
- Task associated with
  - Column of matrix
  - Vector element

Matrix-Vector Multiplication

$$
\begin{aligned}
c_0 &= a_{0,0}\,b_0 + a_{0,1}\,b_1 + a_{0,2}\,b_2 + a_{0,3}\,b_3 + a_{0,4}\,b_4 \\
c_1 &= a_{1,0}\,b_0 + a_{1,1}\,b_1 + a_{1,2}\,b_2 + a_{1,3}\,b_3 + a_{1,4}\,b_4 \\
c_2 &= a_{2,0}\,b_0 + a_{2,1}\,b_1 + a_{2,2}\,b_2 + a_{2,3}\,b_3 + a_{2,4}\,b_4 \\
c_3 &= a_{3,0}\,b_0 + a_{3,1}\,b_1 + a_{3,2}\,b_2 + a_{3,3}\,b_3 + b_{3,4}\,b_4 \\
c_4 &= a_{4,0}\,b_0 + a_{4,1}\,b_1 + a_{4,2}\,b_2 + a_{4,3}\,b_3 + a_{4,4}\,b_4
\end{aligned}
$$

Proc 4

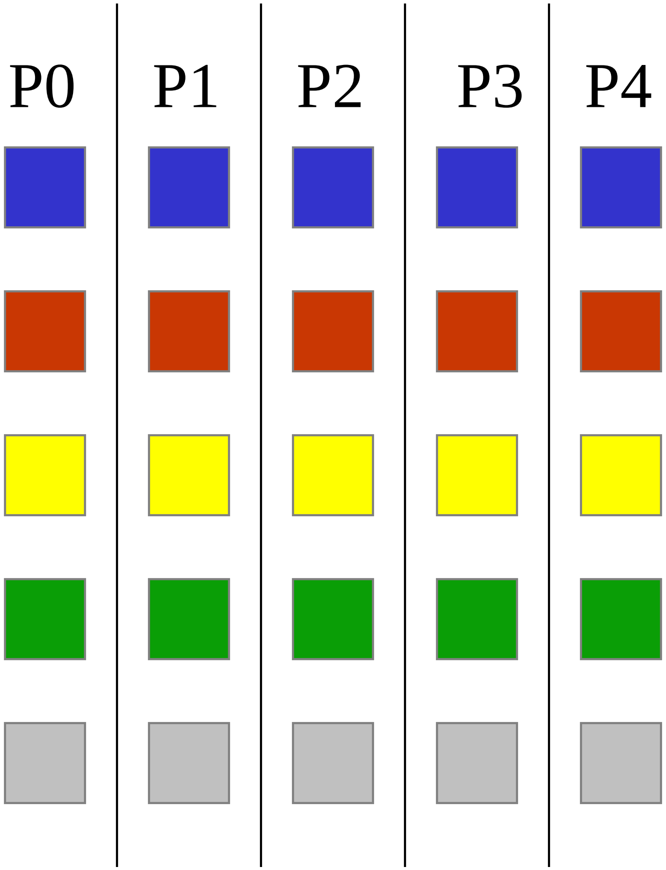
Proc 3

Proc 2

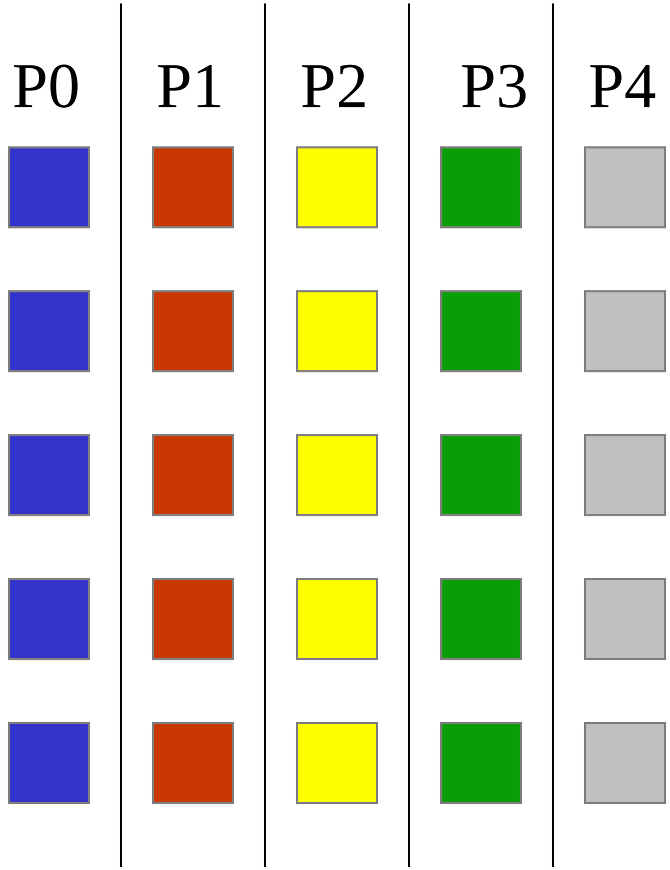
Processor 1’s initial computation

Processor 0’s initial computation

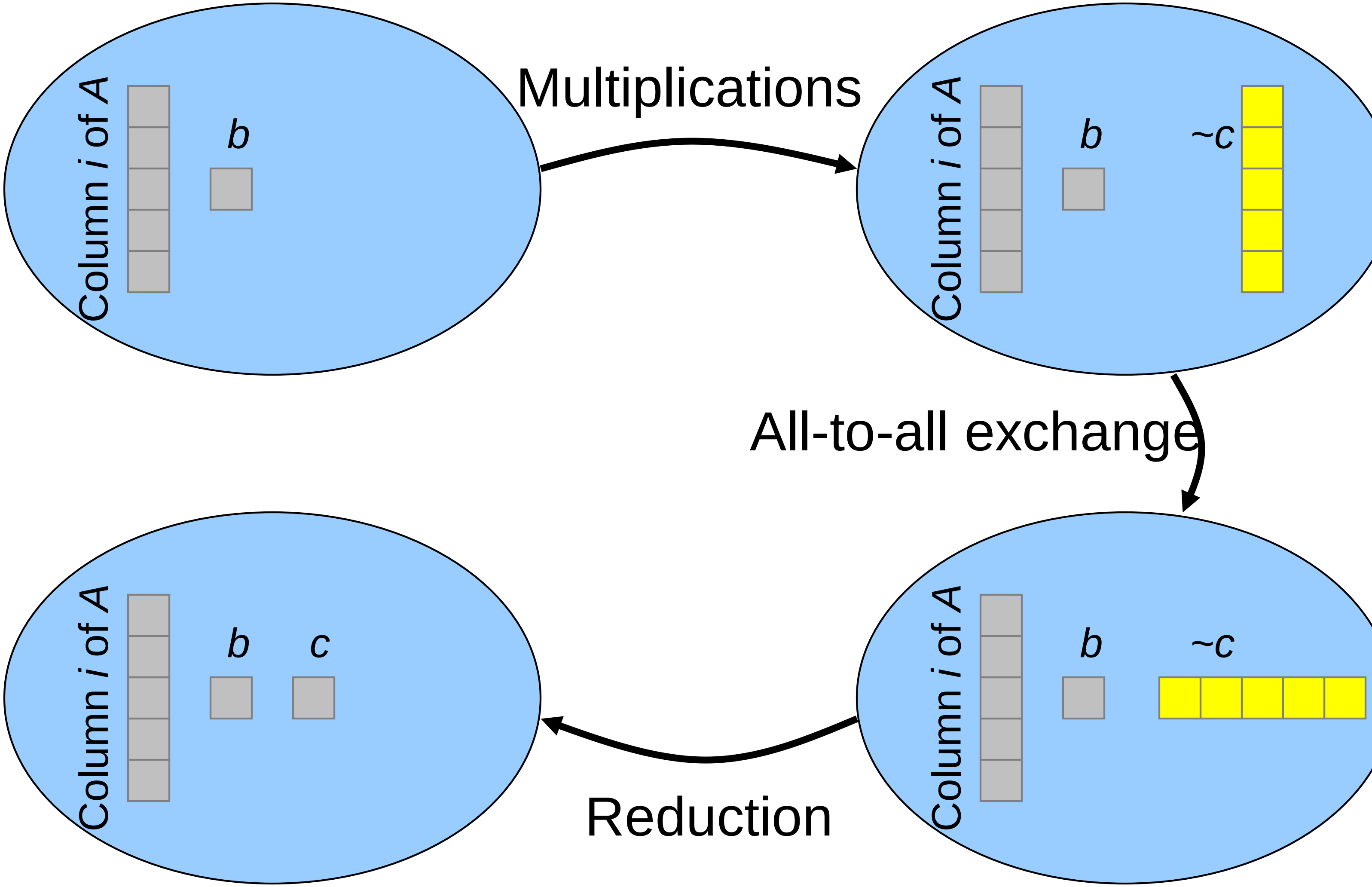
All-to-all Exchange (Before)



All-to-all Exchange (After)



Phases of Parallel Algorithm



Agglomeration and Mapping

- Static number of tasks
- Regular communication pattern (all-to-all)
- Computation time per task is constant
- Strategy:
  - Agglomerate groups of columns
  - Create one task per MPI process

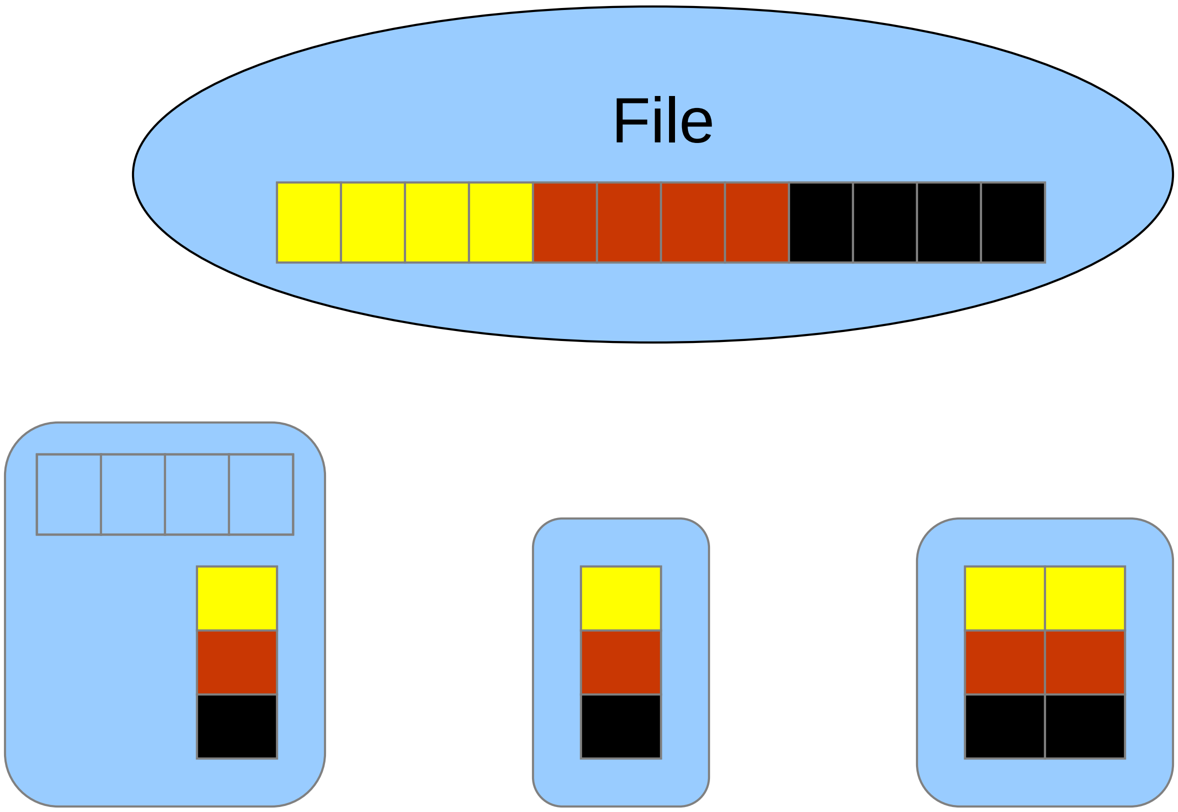
Complexity Analysis

- Sequential algorithm complexity: $\Theta(n^2)$
- Parallel algorithm computational complexity: $\Theta(n^2/p)$
- Communication complexity of all-to-all:
  - $\Theta(n \log p)$ or $\Theta(p + n)$
- Overall complexity:
  - $\Theta(n^2/p + n \log p)$ or $\Theta(n^2/p + p + n)$

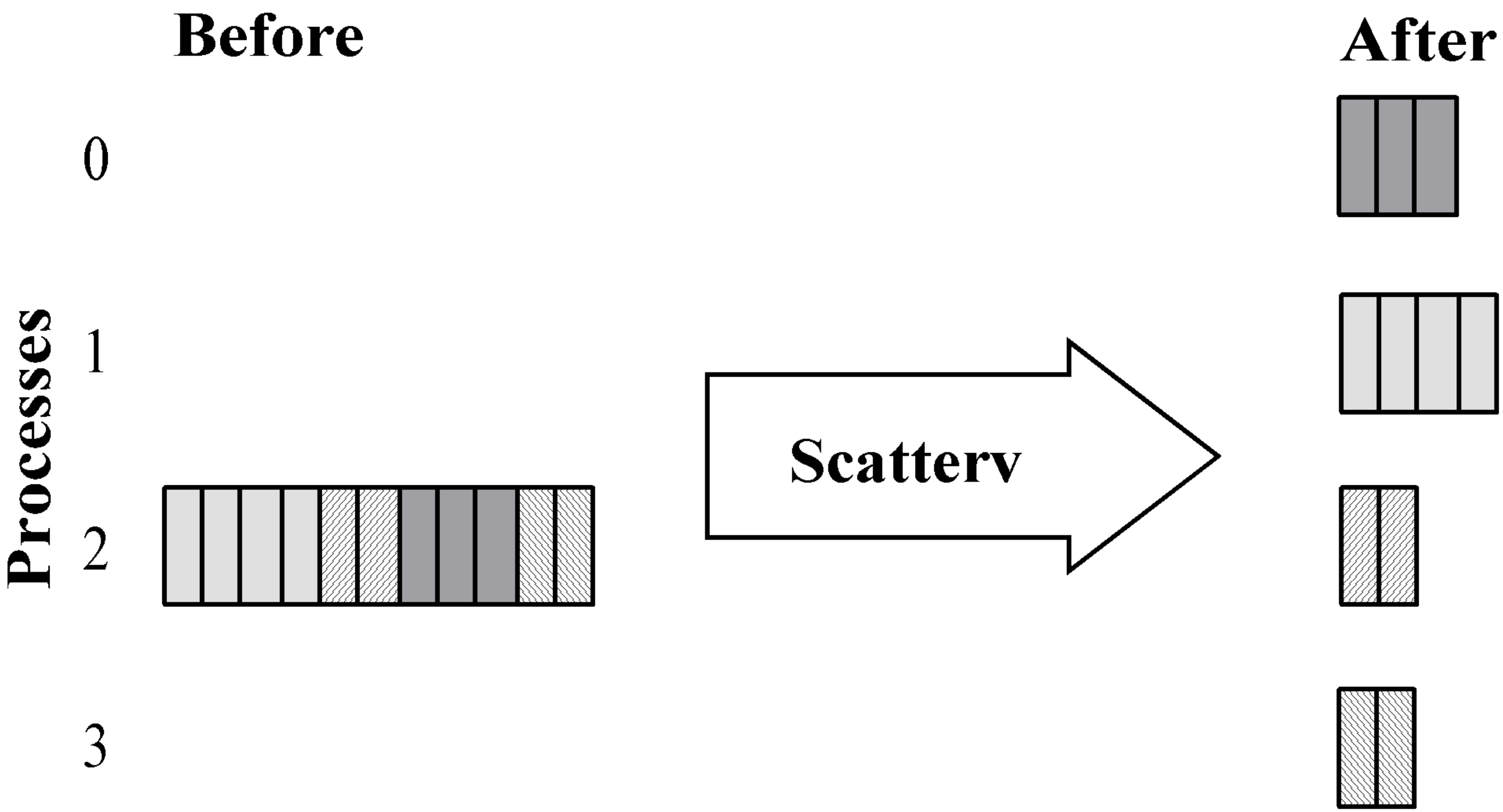
Isoefficiency Analysis

- Sequential time complexity: $\Theta(n^2)$
- Only parallel overhead is all-to-all
  - When $\boldsymbol{n}$ is large, message transmission time dominates message latency
  - Parallel communication time: $\Theta(n)$
- $n^2 \geq Cpn \Rightarrow n \geq Cp$
- Scalability function same as rowwise algorithm: C^2p

Reading a Block-Column Matrix



MPI_Scatterv



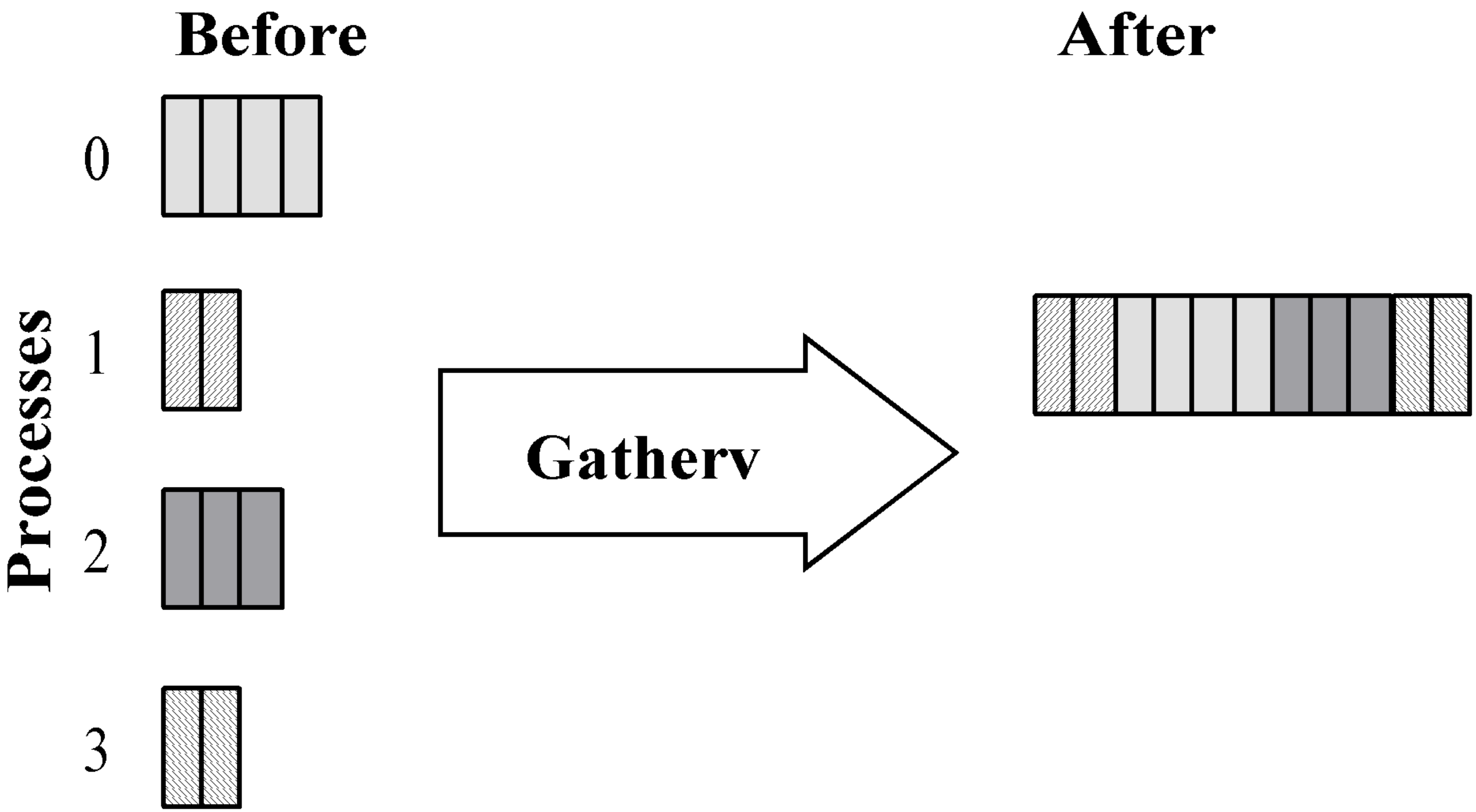
Header for MPI_Scatterv

```
int MPI_Scatterv (
   void             *send_buffer,
   int              *send_cnt,
   int              *send_disp,
   MPI_Datatype      send_type,
   void             *receive_buffer,
   int               receive_cnt,
   MPI_Datatype      receive_type,
   int               root,
   MPI_Comm          communicator
)
```

Printing a Block-Column Matrix

- Data motion opposite to that we did when reading the matrix
- Replace “scatter” with “gather”
- Use “v” variant because different processes contribute different numbers of elements

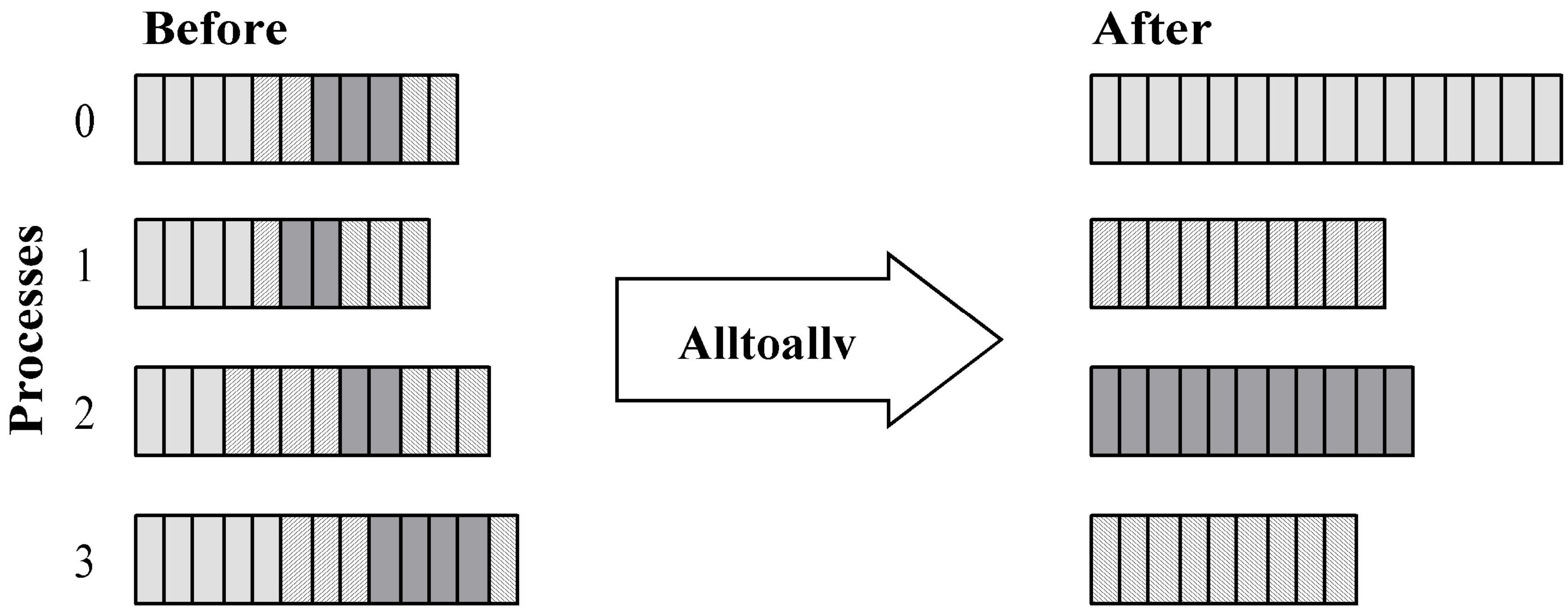
Function MPI_Gatherv



Header for MPI_Gatherv

```
int MPI_Gatherv (
   void           *send_buffer,
   int             send_cnt,
   MPI_Datatype    send_type,
   void           *receive_buffer,
   int            *receive_cnt,
   int            *receive_disp,
   MPI_Datatype    receive_type,
   int             root,
   MPI_Comm        communicator
)
```

Function MPI_Alltoallv



Header for MPI_Alltoallv

```
int MPI_Alltoallv (
   void           *send_buffer,
   int            *send_cnt,
   int            *send_disp,
   MPI_Datatype    send_type,
   void           *receive_buffer,
   int            *receive_cnt,
   int            *receive_disp,
   MPI_Datatype    receive_type,
   MPI_Comm        communicator
)
```

Count/Displacement Arrays

- MPI_Alltoallv requires two pairs of count/displacement arrays
- First pair for values being sent
  - send_cnt: number of elements
  - send_disp: index of first element

`create_mixed_xfer_arrays` builds these

- Second pair for values being received
  - recv_cnt: number of elements
  - recv_disp: index of first element

Function create_uniform_xfer_arrays

- First array
  - How many elements received from each process (always same value)
  - Uses ID and utility macro block_size
- Second array
  - Starting position of each process' block
  - Assume blocks in process rank order

Run-time Expression

- χ: inner product loop iteration time
- Computational time: $\chi\, n \lceil n/p \rceil$
- All-gather requires $p - 1$ messages, each of length about n/p
- 8 bytes per element
- Total execution time:
  $\chi\, n \lceil n/p \rceil + (p - 1)(\lambda + 8n/(p\beta))$

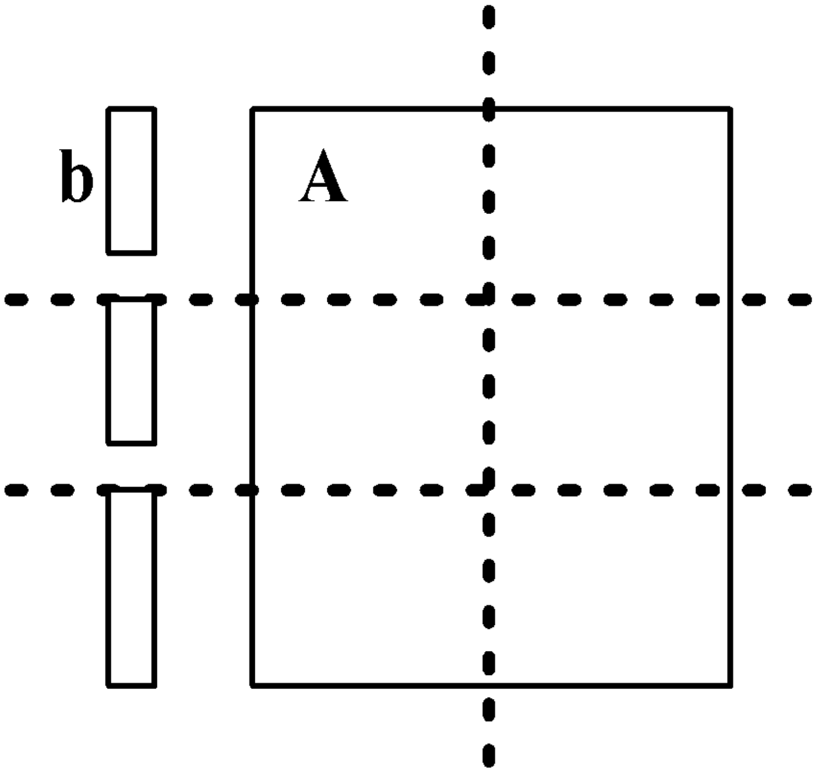
Benchmarking Results

	Execution Time (msec)			
p	*Predicted*	*Actual*	*Speedup*	*Mflops*
1	63.4	63.8	1.00	31.4
2	32.4	32.9	1.92	60.8
3	22.2	22.6	2.80	88.5
4	17.2	17.5	3.62	114.3
5	14.3	14.5	4.37	137.9
6	12.5	12.6	5.02	158.7
7	11.3	11.2	5.65	178.6
8	10.4	10.0	6.33	200.0
16	8.5	7.6	8.33	263.2

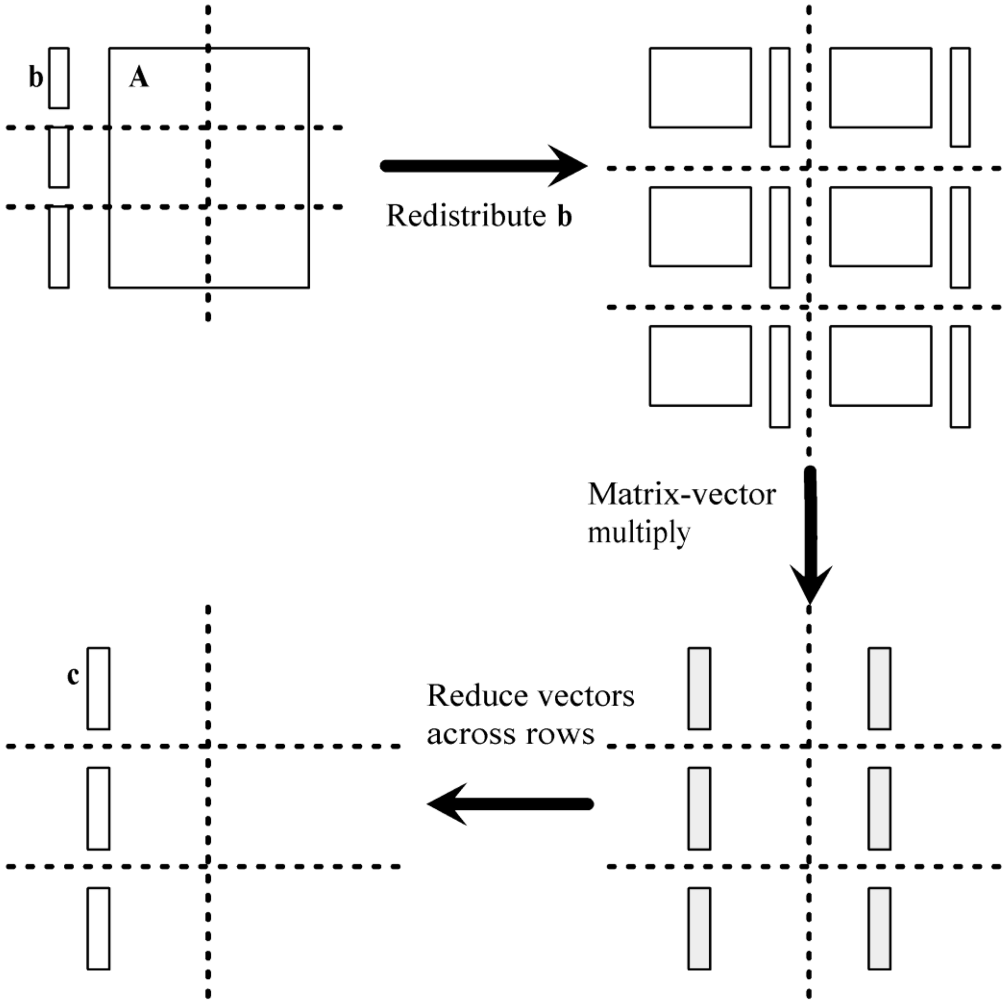
Checkerboard Block Decomposition

- Associate primitive task with each element of the matrix **A**
- Each primitive task performs one multiply
- Agglomerate primitive tasks into rectangular blocks
- Processes form a 2-D grid
- Vector **b** distributed by blocks among processes in first column of grid

Tasks after Agglomeration



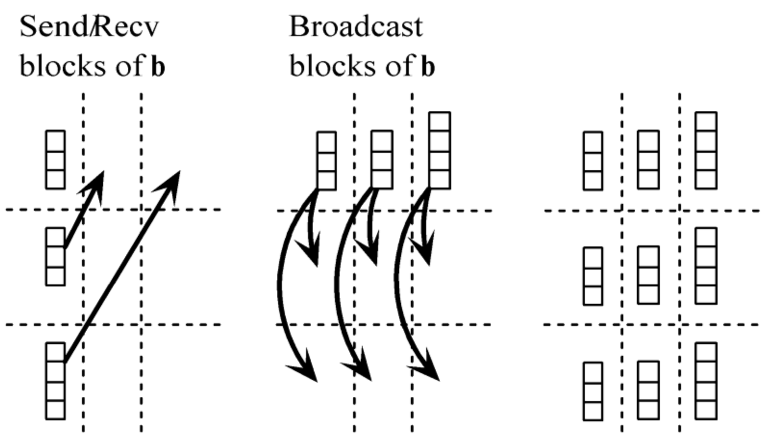
Algorithm's Phases



Redistributing Vector **b**

- Step 1: Move **b** from processes in first column to processes in first row
  - If p square
    - First column/first row processes send/receive portions of **b**
  - If p not square
    - Gather **b** on process 0, 0
    - Process 0, 0 scatter to first row procs
- Step 2: First row processes broadcast **b** within columns

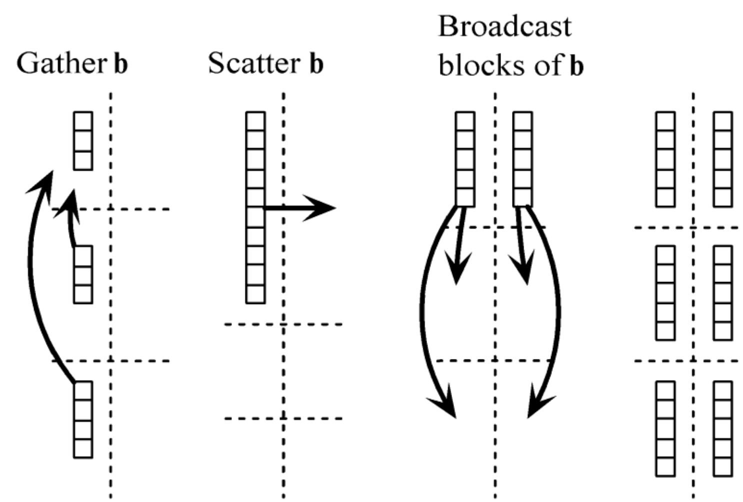
Redistributing Vector b



(a)

When p is a square number

When p is not a square number



(b)

Complexity Analysis

- Assume p is a square number
  - If grid is $1 \times p$, devolves into columnwise block striped
  - If grid is $p \times 1$, devolves into rowwise block striped

Complexity Analysis (continued)

- Each process does its share of computation: $\Theta(n^2/p)$
- Redistribute **b**: $\Theta(n / \sqrt{p} + \log \sqrt{p}(n / \sqrt{p})) = \Theta(n \log p / \sqrt{p})$
- Reduction of partial results vectors: $\Theta(n \log p / \sqrt{p})$
- Overall parallel complexity: $\Theta(n^2/p + n \log p / \sqrt{p})$

Isoefficiency Analysis

- Sequential complexity: $\Theta(n^2)$
- Parallel communication complexity: $\Theta(n \log p / \sqrt{p})$
- Isoefficiency function:
  $n^2 \geq Cn \sqrt{p} \log p \Rightarrow n \geq C \sqrt{p} \log p$
  $$M\left(C\sqrt{p}\log p\right)/\, p = C^2 p \log^2 p / p = C^2 \log^2 p$$
- This system is much more scalable than the previous two implementations

Creating Communicators

- Want processes in a virtual 2-D grid
- Create a custom communicator to do this
- Collective communications involve all processes in a communicator
- We need to do broadcasts, reductions among subsets of processes
- We will create communicators for processes in same row or same column

What's in a Communicator?

- Process group
- Context
- Attributes
  - Topology (lets us address processes another way)
  - Others we won't consider

Creating 2-D Virtual Grid of Processes

- MPI_Dims_create
  - Input parameters
    - Total number of processes in desired grid
    - Number of grid dimensions
  - Returns number of processes in each dim
- MPI_Cart_create
  - Creates communicator with Cartesian topology

MPI_Dims_create

```
int MPI_Dims_create (
   int nodes, /* Input - Procs in grid*/

   int dims,  /* Input - Number of dims*/

   int *size  /* Input/Output - Size of
                 each grid dimension*/
)
```

MPI_Cart_create

```
int MPI_Cart_create (
   MPI_Comm old_comm, /* Input - old communicator */

   int dims, /* Input - grid dimensions */

   int *size, /* Input - # procs in each dim */

   int *periodic,
       /* Input - periodic[j] is 1 if dimension j
          wraps around; 0 otherwise */

   int reorder,
       /* 1 if process ranks can be reordered */

   MPI_Comm *cart_comm
       /* Output - new communicator */
)
```

Using MPI_Dims_create and MPI_Cart_create

```
MPI_Comm cart_comm;
int p;
int periodic[2];
int size[2];
...
size[0] = size[1] = 0;
MPI_Dims_create(p, 2, size);
periodic[0] = periodic[1] = 0;
MPI_Cart_create(MPI_COMM_WORLD, 2, size,
  periodic, 1, &cart_comm);
```

Useful Grid-related Functions

- MPI_Cart_rank
  - Given coordinates of process in Cartesian communicator, returns process rank
- MPI_Cart_coords
  - Given rank of process in Cartesian communicator, returns process' coordinates

Header for MPI_Cart_rank

```
int MPI_Cart_rank (
   MPI_Comm comm,
      /* In - Communicator */

   int *coords,
      /* In - Array containing process'
                grid location */

   int *rank
      /* Out - Rank of process at
         specified coords */
)
```

Header for MPI_Cart_coords

```
int MPI_Cart_coords (
   MPI_Comm comm,
      /* In - Communicator */

   int rank,
      /* In - Rank of process */

   int dims,
      /* In - Dimensions in virtual grid */

   int *coords
      /* Out - Coordinates of specified
         process in virtual grid */
)
```

MPI_Comm_split

- Partitions the processes of a communicator into one or more subgroups
- Constructs a communicator for each subgroup
- Allows processes in each subgroup to perform their own collective communications
- Needed for columnwise scatter and rowwise reduce

Header for MPI_Comm_split

```
int MPI_Comm_split (
   MPI_Comm old_comm,
      /* In - Existing communicator */

   int partition, /* In - Partition number */

   int new_rank,
      /* In - Ranking order of processes
         in new communicator */

   MPI_Comm *new_comm
      /* Out - New communicator shared by
         processes in same partition */
)
```

Example: Create Communicators for Process Rows

```
MPI_Comm grid_comm;
         /* 2-D process grid */

MPI_Comm grid_coords[2];
         /* Location of process in grid */

MPI_Comm row_comm;
         /* Processes in same row */

MPI_Comm_split (grid_comm, grid_coords[0],
   grid_coords[1], &row_comm);
```

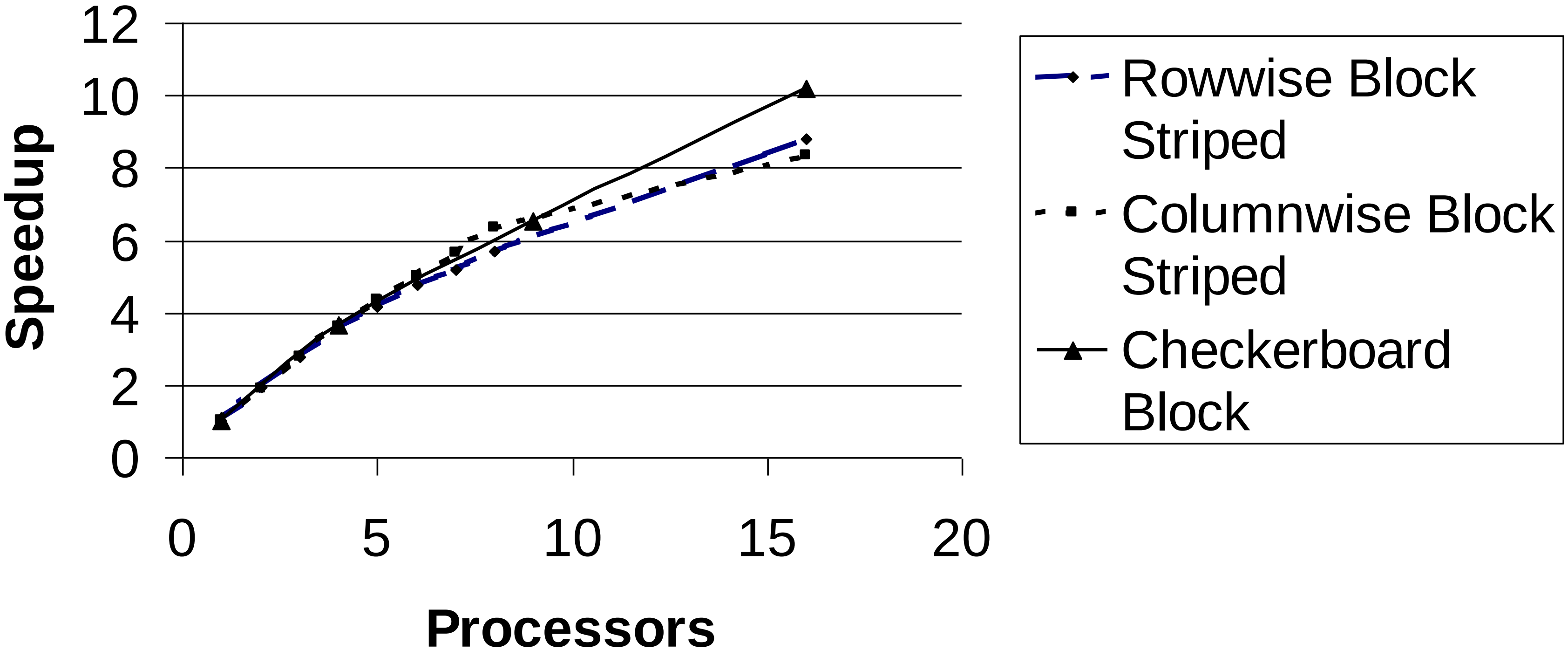
Run-time Expression

- Computational time: $\chi \lceil n/\sqrt{p} \rceil \lceil n/\sqrt{p} \rceil$
- Suppose p a square number
- Redistribute **b**
  - Send/Recv: $\lambda + 8 \lceil n/\sqrt{p} \rceil / \beta$
  - Broadcast: $\log \sqrt{p} \, (\lambda + 8 \lceil n/\sqrt{p} \rceil / \beta)$
- Reduce partial results:
  $\log \sqrt{p} \, (\lambda + 8 \lceil n/\sqrt{p} \rceil / \beta)$

Benchmarking

Procs	Predicted(msec)	Actual (msec)	Speedup	Megaflops
1	63.4	63.4	1.00	31.6
4	17.8	17.4	3.64	114.9
9	9.7	9.7	6.53	206.2
16	6.2	6.2	10.21	322.6

Comparison of Three Algorithms



Summary (1/3)

- Matrix decomposition $\Rightarrow$ communications needed
  - Rowwise block striped: all-gather
  - Columnwise block striped: all-to-all exchange
  - Checkerboard block: gather, scatter, broadcast, reduce
- All three algorithms: roughly same number of messages
- Elements transmitted per process varies
  - First two algorithms: $\Theta(n)$ elements per process
  - Checkerboard algorithm: $\Theta(n/\sqrt{p})$ elements
- Checkerboard block algorithm has better scalability

Summary (2/3)

- Communicators with Cartesian topology
  - Creation
  - Identifying processes by rank or coords
- Subdividing communicators
  - Allows collective operations among subsets of processes

Summary (3/3)

- Parallel programs and supporting functions much longer than C counterparts
- Extra code devoted to reading, distributing, printing matrices and vectors
- Developing and debugging these functions is tedious and difficult
- Makes sense to generalize functions and put them in libraries for reuse

MPI Application Development