Contents

3 Vectors

3.1 Vectors and Scalars

A scalar is a number which may be complex but is real if it represents physical quantity such as temperature, pressure, energy, mass and time. On the other hand, a vector has magnitude as well as direction. Geometrically, a vector can be described by an arrowed line pointing from a beginning point to an end point.

In flat Euclidean space, two vectors are considered equal if they both have the same magnitude and point to the same direction. (For a general curved space in advanced geometry, vectors may depend where the beginning point resides.) A point particle moving in three dimensional space is specified by a vector that points from the origin of the coordinate system to the position of the particle in question. The vector is called the position vector $\vec{x}$. Note in this course, a vector is always denoted by an arrow on top. The position vector of a moving particle is a function of time and is written as $\vec{x}\,(t)$.

3.1.1 Displacement vector

If a particle changes its position by moving from A to B, we say that it undergoes a displacement represented by the vector from A to B or $\overrightarrow{AB}$.

3.1.2 Scalar multiplication on a vector

For a vector $\vec{a}$, its magnitude is often denoted by $|\vec{a}|$ or $\|\vec{a}\|$. The multiplication of a scalar s to a vector $\vec{a}$ still gives us a vector $s\vec{a}$. $s\vec{a}$ points in the same direction as $\vec{a}$ if $s > 0$ or in the opposite direction of $\vec{a}$ if $s < 0$. Furthermore, the magnitude

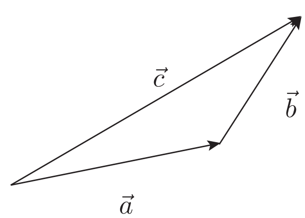
$$|s\vec{a}| = |s|\,|\vec{a}|$$

3.2 Adding Vectors Geometrically

The sum of vectors $\vec{a}$ and $\vec{b}$

$$\vec{c} = \vec{a} + \vec{b}$$

is the vector from the beginning of $\vec{a}$ to the end of $\vec{b}$ as illustrated below:



Vector addition is commutative

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

and associative

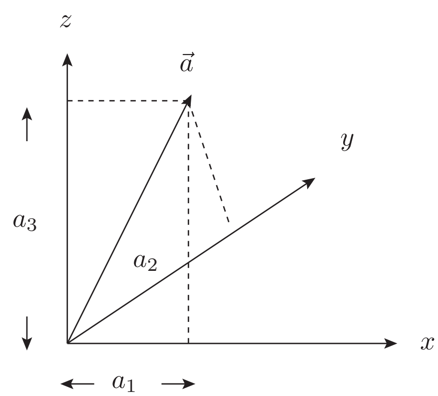
$$\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$$

Define the subtraction

$$\vec{a} - \vec{b} = \vec{a} + (-1)\,\vec{b}$$

3.3 Components of Vectors

A component of a vector is the projection of the vector on an axis. The projection of a vector on an x axis is its x component, the projection on the y axis is the y component and the projection on the z axis is the z component.



If vector $\vec{a}$ has a_1, a_2, a_3 as the x, y, z components, then we also identity

$$\vec{a} = (a_1, a_2, a_3)$$

3.4 Unit Vectors

A unit vector is a vector that has a magnitude of exactly 1 and points in a particular direction. The unit vectors in the positive x, y, and z direction are labeled $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ (or $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$). In terms of component notations, we have

$$\begin{aligned} \hat{\imath} &= \hat{e}_1 = (1, 0, 0) \\ \hat{\jmath} &= \hat{e}_2 = (0, 1, 0) \\ \hat{k} &= \hat{e}_3 = (0, 0, 1) \end{aligned}$$

Unit vectors are very useful for expressing other vectors; for example

$$\vec{a} = (a_1, a_2, a_3) = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = \sum_{i=1}^{3} a_i\hat{e}_i$$

The length of $\vec{a}$, expressed in terms its components, is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

3.5 Adding Vectors by Components

Assume

$$\vec{a} = (a_1, a_2, a_3) = \sum_{i=1}^{3} a_i \hat{e}_i, \vec{b} = (b_1, b_2, b_3) = \sum_{i=1}^{3} b_i \hat{e}_i$$

Then

$$\begin{aligned} \vec{a} + \vec{b} &= \left(\sum_{i=1}^{3} a_i \hat{e}_i \right) + \left(\sum_{i=1}^{3} b_i \hat{e}_i \right) = \sum_{i=1}^{3} (a_i + b_i) \hat{e}_i \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \end{aligned}$$

3.6 Multiplying Vectors

3.6.1 Multiplying a vector by a scalar s

$$\vec{a} = (a_1, a_2, a_3)$$

$$s\vec{a} = (sa_1, sa_2, sa_3)$$

There are two ways to multiply a vector by another vector: one way to produce a scalar (called the scalar product, dot product, or inner product), and the other way to produce a new vector (called the vector product, cross product, or outer product).

3.6.2 Scalar Product

The scalar or inner product of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is written as $\vec{a} \cdot \vec{b}$ and defined to be

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i \tag{1}$$

The scalar product is commutative

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

and distributive

$$\vec{a} \cdot \left(\vec{b} + \vec{c} \right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

The magnitude of $\vec{a}$ is given by

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$

For $i = 1, 2, 3$; $j = 1, 2, 3$; define δ_{ij}, the *Kronecker delta* function as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Then

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

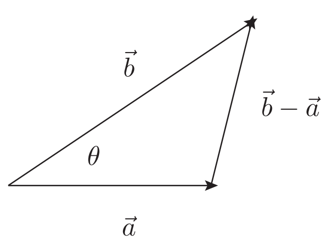
and

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \left(\sum_{i=1}^{3} a_i \hat{e}_i \right) \cdot \left(\sum_{j=1}^{3} b_j \hat{e}_j \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \left(\hat{e}_i \cdot \hat{e}_j \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \delta_{ij} = \sum_{i=1}^{3} a_i b_i \end{aligned}$$

By trigonometry, we have

$$\begin{aligned} \left| \vec{b} - \vec{a} \right|^2 &= \left| \vec{b} \right|^2 \sin^2 \theta + \left(\left| \vec{b} \right| \cos \theta - |\vec{a}| \right)^2 \\ &= |\vec{a}|^2 + \left| \vec{b} \right|^2 - 2 \, |\vec{a}| \left| \vec{b} \right| \cos \theta \end{aligned}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ as shown in the following figure:



But

$$\left| \vec{b} - \vec{a} \right|^2 = \left(\vec{b} - \vec{a} \right) \cdot \left(\vec{b} - \vec{a} \right) = |\vec{a}|^2 + \left| \vec{b} \right|^2 - 2 \vec{a} \cdot \vec{b}$$

Thus,

$$\vec{a} \cdot \vec{b} = |\vec{a}| \left| \vec{b} \right| \cos \theta$$

The above expression instead of (1) can be taken as the definition of scalar product.

Two vectors are orthogonal if their inner product vanishes. (zero vector is considered to be orthogonal to any vector.)

3.6.3 Vector Product

The vector product of $\vec{a}$ and $\vec{b}$ is written as $\vec{a} \times \vec{b}$ and defined to be

$$\begin{aligned}
&\vec{a} \times \vec{b} \\
&= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{e}_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{e}_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{e}_3 \\
&= (a_2 b_3 - a_3 b_2)\, \hat{e}_1 + (a_3 b_1 - a_1 b_3)\, \hat{e}_2 + (a_1 b_2 - a_2 b_1)\, \hat{e}_3
\end{aligned}$$

Since a determinant changes sign under the interchange of any two rows

$$\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix},$$

the vector product is anti-commutative,

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

Now

$$\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The vector is distributive

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Similarly, we also have

$$\left(\vec{a} + \vec{b}\right) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

Triple Product The scalar $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$ is called the triple product of three vectors $\vec{a},\vec{b},\vec{c}$. By definition

$$\begin{aligned}\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} &= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\cdot\vec{c} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}\end{aligned}$$

So

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \left(\vec{b}\times\vec{c}\right)\cdot\vec{a} = \left(\vec{c}\times\vec{a}\right)\cdot\vec{b}$$

Since

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0, \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{a} = \left(\vec{a}\times\vec{b}\right)\cdot\vec{b} = 0$$

$\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$, and is thus perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

There are 9 possible cross products between two unit vectors $\hat{e}_i$ and $\hat{e}_j$:

$$\begin{aligned} &\hat{e}_1\times\hat{e}_1 = \hat{e}_2\times\hat{e}_2 = \hat{e}_3\times\hat{e}_3 = 0 \\ &\hat{e}_1\times\hat{e}_2 = -\hat{e}_2\times\hat{e}_1 = \hat{e}_3 \\ &\hat{e}_2\times\hat{e}_3 = -\hat{e}_3\times\hat{e}_2 = \hat{e}_1 \\ &\hat{e}_3\times\hat{e}_1 = -\hat{e}_1\times\hat{e}_3 = \hat{e}_2 \end{aligned} \tag{2}$$

For $i,j,k\in\{1,2,3\}$, the quantity

$$\epsilon_{ijk} = (\hat{e}_i\times\hat{e}_j)\cdot\hat{e}_k$$

is called the Levi-Civita tensor. There are $3^3 = 27$ possible i,j,k indices for ϵ_{ijk}. ϵ_{ijk} vanishes if any two of the three indices are identical. We are left with $3! = 6$ possible ϵ_{ijk} with $\{i, j, k\} = \{1, 2, 3\}$ that may be non-zero. In fact, from (2) we get

$$\epsilon_{ijk} = \begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \\ 0, \text{ otherwise} \end{cases}$$

ϵ_{ijk} changes sign if any two indices are exchanged:

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$$

With ϵ_{ijk}, it is straightforward to see that

$$\vec{a} \times \vec{b} = \sum_{i,j,k} \epsilon_{ijk} a_i b_j \hat{e}_k$$

Now, $\vec{a} \times \vec{b}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$. $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ is perpendicular to $\vec{a} \times \vec{b}$ and must be in the plane containing $\vec{a}$ and $\vec{b}$. Therefore $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ can be written as a linear combination of $\vec{a}$ and $\vec{b}$.

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = s\vec{a} + t\vec{b}$$

which must also be perpendicular to $\vec{c}$. Thus

$$0 = \left(\left(\vec{a} \times \vec{b}\right) \times \vec{c}\right) \cdot \vec{c} = 0 = s\vec{a} \cdot \vec{c} + t\vec{b} \cdot \vec{c}$$

So we may let

$$s = u\vec{b} \cdot \vec{c}, t = -u\vec{a} \cdot \vec{c}$$

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = u\left(\left(\vec{b} \cdot \vec{c}\right)\vec{a} - \left(\vec{a} \cdot \vec{c}\right)\vec{b}\right)$$

Furthermore, u must be a constant independent of the components of $\vec{a}, \vec{b}$, and $\vec{c}$. This is because if we scale $\vec{a} \to s\vec{a}$, then $\left(\vec{a} \times \vec{b}\right) \times \vec{c} \to s\left(\vec{a} \times \vec{b}\right) \times \vec{c}$ and $u\left(\left(\vec{b} \cdot \vec{c}\right)\vec{a} - \left(\vec{a} \cdot \vec{c}\right)\vec{b}\right) \to su\left(\left(\vec{b} \cdot \vec{c}\right)\vec{a} - \left(\vec{a} \cdot \vec{c}\right)\vec{b}\right)$. u is unchanged under $\vec{a} \to s\vec{a}$. Similarly, u is unchanged under $\vec{b} \to s\vec{b}$ or $\vec{c} \to s\vec{c}$. As a consequence, u is independent of the components of $\vec{a}, \vec{b}$, and $\vec{c}$. Let us choose $\vec{a} = \hat{e}_1, \vec{b} = \hat{e}_2$ and $\vec{c} = \hat{e}_1$. Then

$$\left(\hat{e}_1 \times \hat{e}_2\right) \times \hat{e}_1 = u\left(\left(\hat{e}_2 \cdot \hat{e}_1\right)\hat{e}_1 - \left(\hat{e}_1 \cdot \hat{e}_1\right)\hat{e}_2\right) = -u\hat{e}_2$$

So $u = -1$, and we have

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = (\vec{a} \cdot \vec{c})\,\vec{b} - \left(\vec{b} \cdot \vec{c}\right)\vec{a} \tag{3}$$

From the above identity, we get

$$\begin{aligned}\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{a} \times \vec{b}\right) &= \left(\left(\vec{a} \times \vec{b}\right) \times \vec{a}\right) \cdot \vec{b} = \left(|\vec{a}|^2\,\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{a}\right) \cdot \vec{b} \\ &= |\vec{a}|^2 \left|\vec{b}\right|^2 - \left(\vec{a} \cdot \vec{b}\right)^2 = |\vec{a}|^2 \left|\vec{b}\right|^2 \sin^2\theta\end{aligned}$$

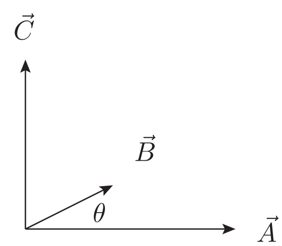
where $\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta$. Thus the magnitude of $\vec{a} \times \vec{b}$ is

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \left|\vec{b}\right| \sin\theta$$

The identity (3) enables us to write

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = -\left(\vec{b} \times \vec{c}\right) \times \vec{a} = (\vec{a} \cdot \vec{c})\,\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$$

Geometrical definition



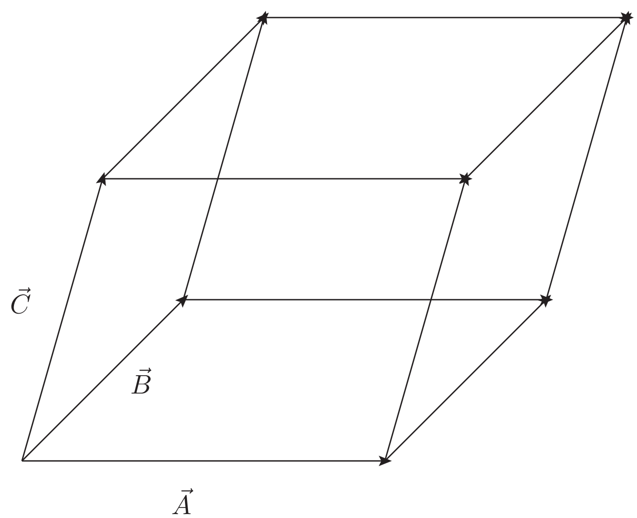
The vector product of $\vec{A} \times \vec{B}$ produces a vector whose magnitude is $\left|\vec{A}\right| \left|\vec{B}\right| \sin\theta$ where is the smaller of the two angles between $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta\,\hat{u}$$

In the above, the unit vector $\hat{u}$, in the direction of $\vec{A} \times \vec{B}$, is perpendicular to the plane that contains $\vec{A}$ and $\vec{B}$, with the direction determined by the right hand rule: Sweep vector $\vec{A}$ into $\vec{B}$ with figures of your right hand and then your outstretched thumb shows the direction of $\vec{A} \times \vec{B}$.

3.6.4 Triple Product

$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$ is ($\pm$) the volume of the parallelepiped



with three sides formed by vectors $\vec{A}, \vec{B}$, and $\vec{C}$.