# Chapter 4: Fluid Kinematics

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# Overview

- Fluid Kinematics deals with the motion of fluids without necessarily considering the forces and moments which create the motion.
- Items discussed in this Chapter.
  - Material derivative and its relationship to Lagrangian and Eulerian descriptions of fluid flow.
  - Flow visualization.
  - Plotting flow data.
  - Fundamental kinematic properties of fluid motion and deformation.
  - Reynolds Transport Theorem

# Lagrangian Description

- Two ways to describe motion are Lagrangian and Eulerian description
- Lagrangian description of fluid flow tracks the position and velocity of individual particles. (eg. Brilliard ball on a pooltable.)
- Motion is described based upon Newton's laws.
- Difficult to use for practical flow analysis.
  - Fluids are composed of *billions* of molecules.
  - Interaction between molecules hard to describe/model.
- However, useful for specialized applications
  - Sprays, particles, bubble dynamics, rarefied gases.
  - Coupled Eulerian-Lagrangian methods.
- Named after Italian mathematician Joseph Louis Lagrange (1736-1813).

# Eulerian Description

- Eulerian description of fluid flow: a **flow domain** or **control volume** is defined by which fluid flows in and out.
- We define **field variables** which are functions of space and time.
  - Pressure field, P=P(x,y,z,t)
  - Velocity field, $\vec{V}=\vec{V}(x,y,z,t)$

$$\vec{V}=u(x,y,z,t)\vec{i}+v(x,y,z,t)\vec{j}+w(x,y,z,t)\vec{k}$$

  - Acceleration field, $\vec{a}=\vec{a}(x,y,z,t)$

$$\vec{a}=a_x(x,y,z,t)\vec{i}+a_y(x,y,z,t)\vec{j}+a_z(x,y,z,t)\vec{k}$$

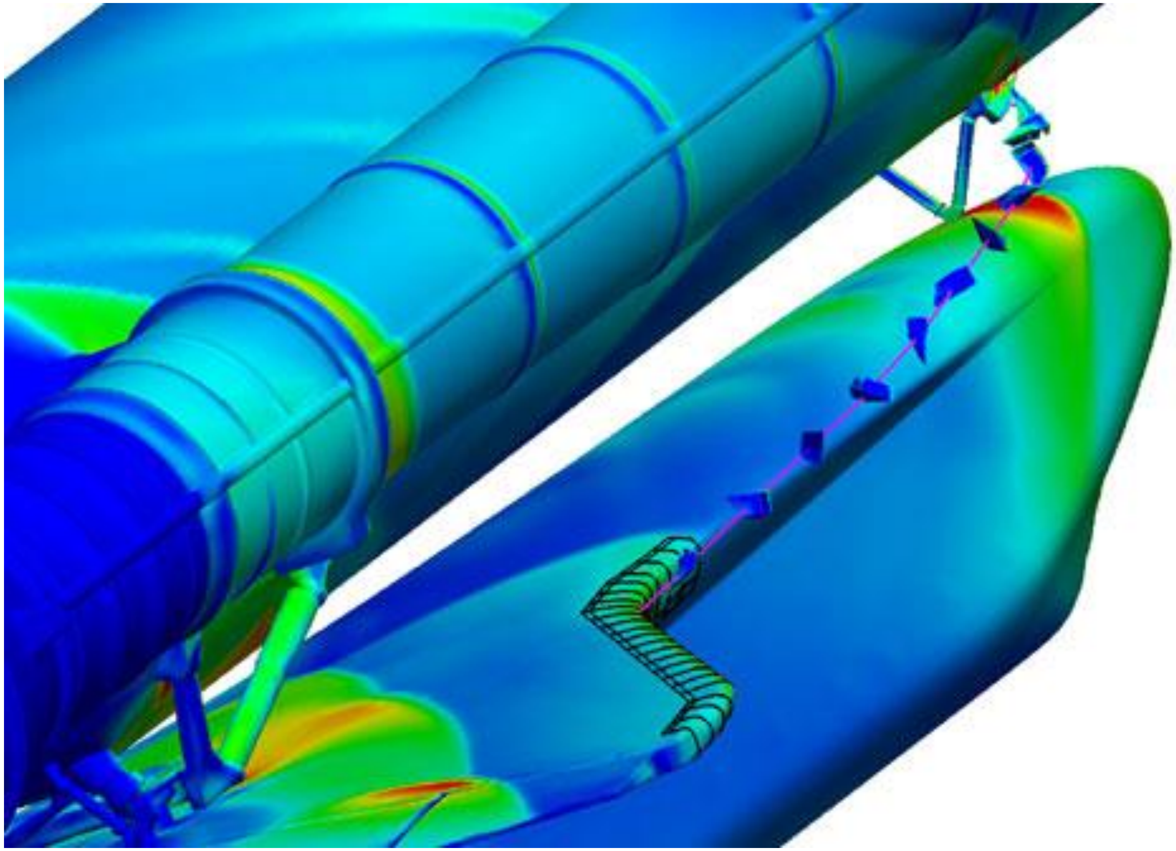
  - These (and other) field variables define the **flow field**.
- Well suited for formulation of initial boundary-value problems (PDE's).
- Named after Swiss mathematician Leonhard Euler (1707-1783).

# Example: Coupled Eulerian-Lagrangian Method

- Global Environmental MEMS Sensors (GEMS)
- Simulation of micron-scale airborne probes. The probe positions are tracked using a Lagrangian particle model embedded within a flow field computed using an Eulerian CFD code.

http://www.ensco.com/products/atmospheric/gem/gem_ovr.htm

# Example: Coupled Eulerian-Lagrangian Method



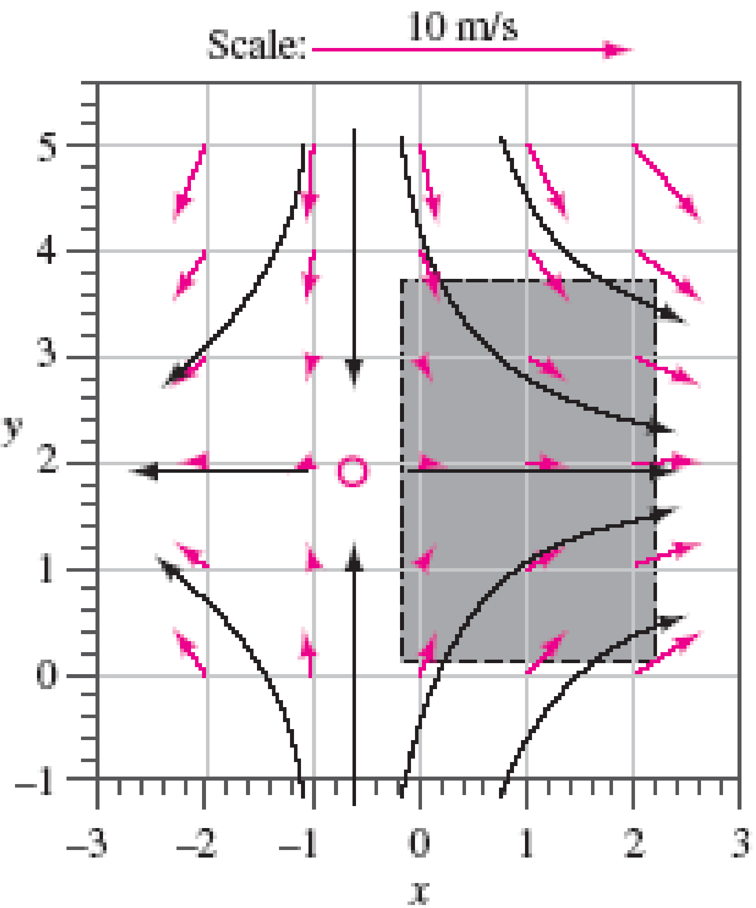
Forensic analysis of Columbia accident: simulation of shuttle debris trajectory using Eulerian CFD for flow field and Lagrangian method for the debris.

# *EXAMPLEL A*: A Steady Two-Dimensional Velocity Field

- A steady, incompressible, two-dimensional velocity field is given by

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j}$$

A **stagnation point** is defined as *a point in the flow field where the velocity is identically zero*. (*a*) Determine if there are any stagnation points in this flow field and, if so, where? (*b*) Sketch velocity vectors at several locations in the domain between $x$ = - 2 m to 2 m and $y$ = 0 m to 5 m; qualitatively describe the flow field.



# Acceleration Field

- Consider a fluid particle and Newton's second law,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

- The acceleration of the particle is the time derivative of the particle's velocity.

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

- However, particle velocity at a point at any instant in time t is the same as the fluid velocity,

$$\vec{V}_{particle} = \vec{V}\left(x_{particle}(t), y_{particle}(t), z_{particle}(t), t\right)$$

- To take the time derivative of, chain rule must be used.

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_{particle}}{dt}$$

# Acceleration Field

Where $\partial$ is the **partial derivative operator** and *d* is the **total derivative operator.**

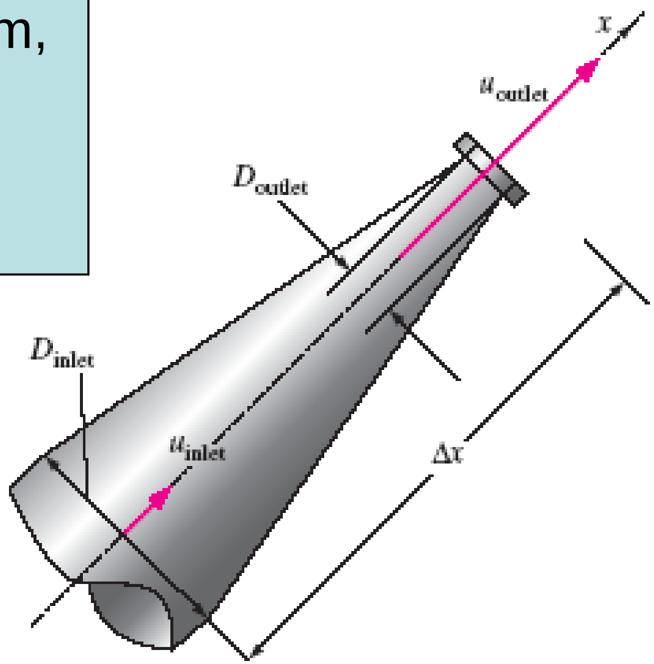
- Since $$\frac{dx_{particle}}{dt}=u,\frac{dy_{particle}}{dt}=v,\frac{dz_{particle}}{dt}=w$$
$$\vec{a}_{particle}=\frac{\partial\vec{V}}{\partial t}+u\frac{\partial\vec{V}}{\partial x}+v\frac{\partial\vec{V}}{\partial y}+w\frac{\partial\vec{V}}{\partial z}$$
- In vector form, the acceleration can be written as
$$\vec{a}\left(x,y,z,t\right)=\frac{d\vec{V}}{dt}=\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\bullet\vec{\nabla}\right)\vec{V}$$
- First term is called the **local acceleration** and is nonzero only for unsteady flows.
- Second term is called the **advective acceleration** and accounts for the effect of the fluid particle moving to a new location in the flow, where the velocity is different.

# *EXAMPLE:* Acceleration of a Fluid Particle through a Nozzle

How to apply this equation to the problem,

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

volume flow rate through the garden hose (and through the nozzle) is 0.841 gal/min (0.00187 ft3/s), and the flow is steady. Estimate the magnitude of the acceleration of a fluid particle moving down the centerline of the nozzle.




$$a_x \cong \frac{\Delta u}{\Delta t} = \frac{u_{outlet} - u_{inlet}}{\Delta x / u_{avg}} = \frac{u_{outlet} - u_{inlet}}{2\,\Delta x / (u_{outlet} + u_{inlet})} = \frac{u^2_{outlet} - u^2_{inlet}}{2\,\Delta x}$$

# Material Derivative

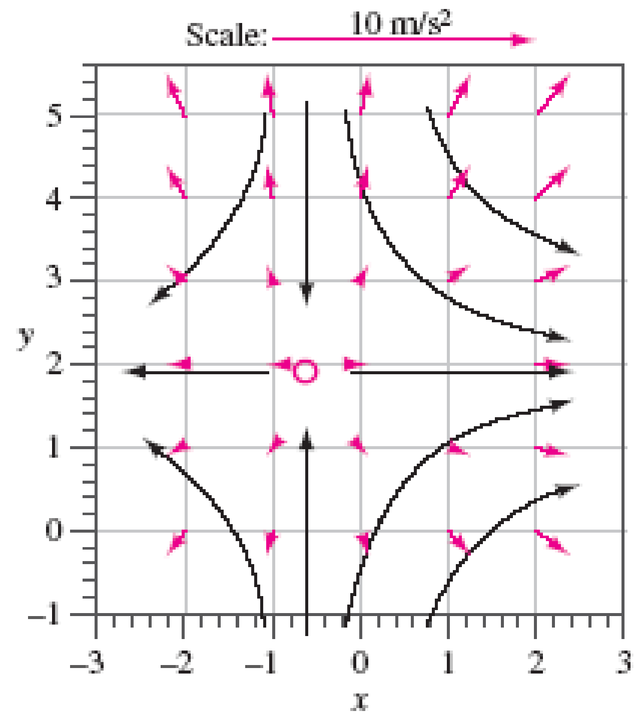
- The total derivative operator d/dt is called the **material derivative** and is often given special notation, D/Dt.

$$\frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

- Advective acceleration is nonlinear: source of many phenomenon and primary challenge in solving fluid flow problems.
- Provides ``transformation'' between Lagrangian and Eulerian frames.
- Other names for the material derivative include: **total, particle, Lagrangian, Eulerian,** and **substantial** derivative.

# *EXAMPLE B:* Material Acceleration of a Steady Velocity Field

- Consider the same velocity field of Example A. (*a*) Calculate the material acceleration at the point ($x$ = 2 m, $y$ = 3 m). (*b*) Sketch the material acceleration vectors at the same array of $x$- and $y$ values as in Example A.




$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
$$= 0 + \overbrace{(0.5 + 0.8x)(0.8)} + \overbrace{(15 - 0.8y)(0)} + 0 = (0.4 + 0.64x)\ \text{m/s}^2$$

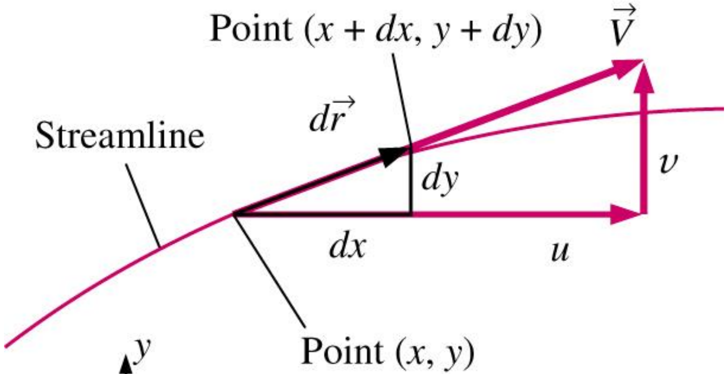
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$
$$= 0 + \overbrace{(0.5 + 0.8x)(0)} + \overbrace{(1.5 - 0.8y)(-0.8)} + 0 = (-1.2 + 0.64y)\ \text{m/s}^2$$

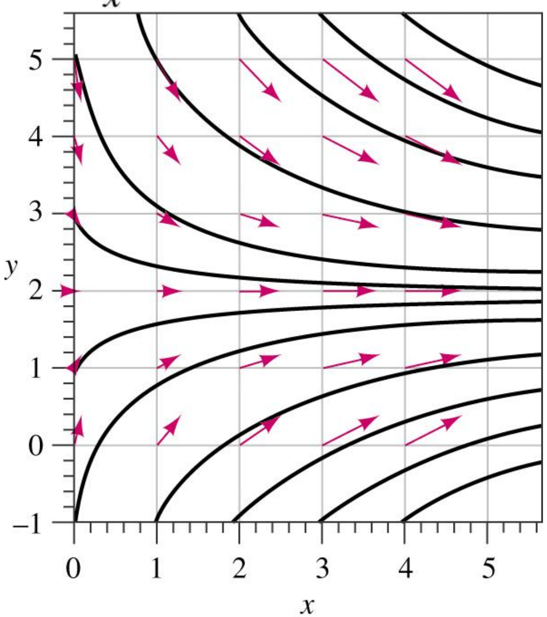
# Flow Visualization

- Flow visualization is the visual examination of flow-field features.
- Important for both physical experiments and numerical (CFD) solutions.
- Numerous methods
  - Streamlines and streamtubes
  - Pathlines
  - Streaklines
  - Timelines
  - Refractive techniques
  - Surface flow techniques

While quantitative study of fluid dynamics requires advanced mathematics, much can be learned from **flow visualization**

# Streamlines




- A **Streamline** is a curve that is everywhere tangent to the *instantaneous* local velocity vector.
- Consider an arc length

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

- $d\vec{r}$ must be parallel to the local velocity vector

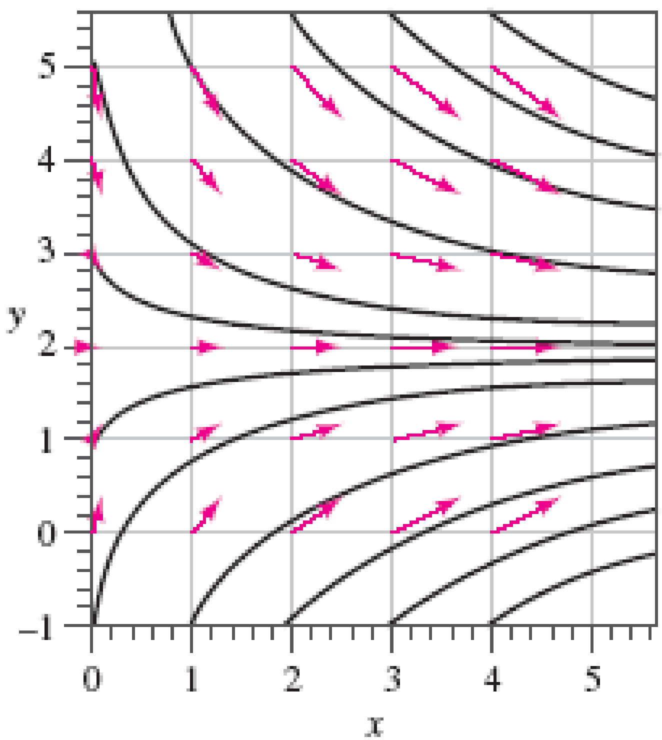
$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- Geometric arguments results in the equation for a streamline

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

# *EXAMPLE C:* Streamlines in the *xy* Plane—An Analytical Solution

For the same velocity field of Example A, plot several streamlines in the right half of the flow ($x > 0$) and compare to the velocity vectors.




$$\frac{dy}{dx} = \frac{1.5 - 0.8y}{0.5 + 0.8x}$$

$$\frac{dy}{1.5 - 0.8y} = \frac{dx}{0.5 + 0.8x}$$

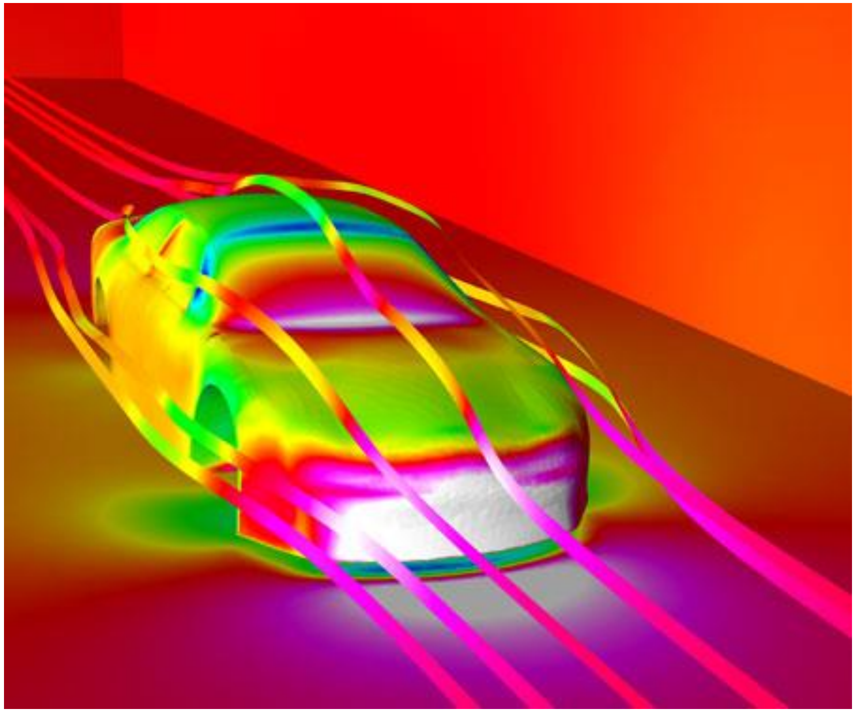
$$\rightarrow \int \frac{dy}{1.5 - 0.8y} = \int \frac{dx}{0.5 + 0.8x}$$

$$y = \frac{C}{0.8(0.5 + 0.8x)} + 1.875$$

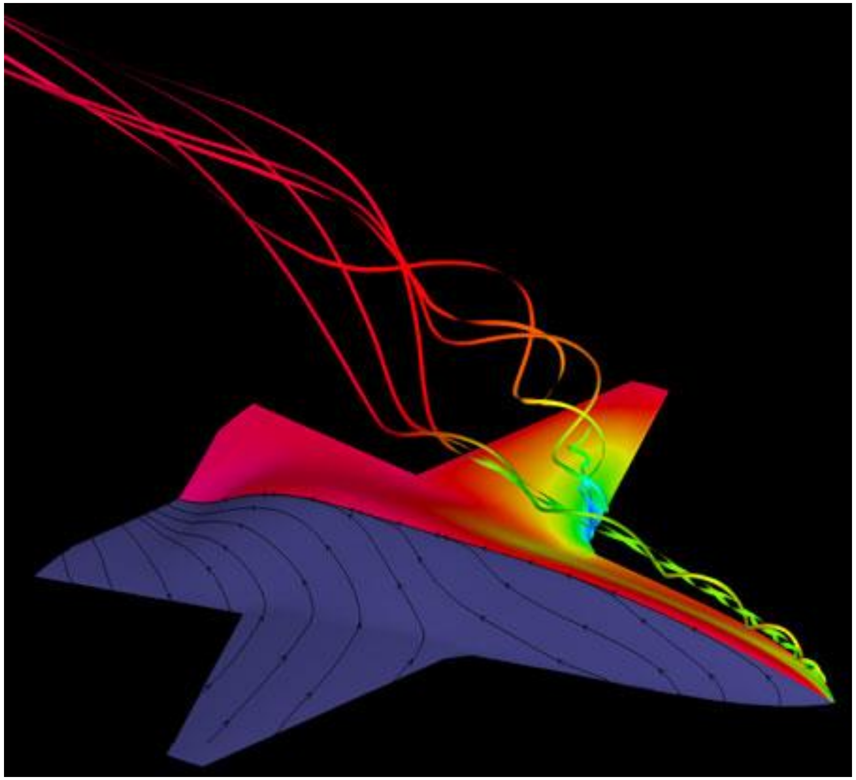
where $C$ is a constant of integration that can be set to various values in order to plot the streamlines.

# Streamlines

NASCAR surface pressure contours and streamlines



Airplane surface pressure contours, volume streamlines, and surface streamlines

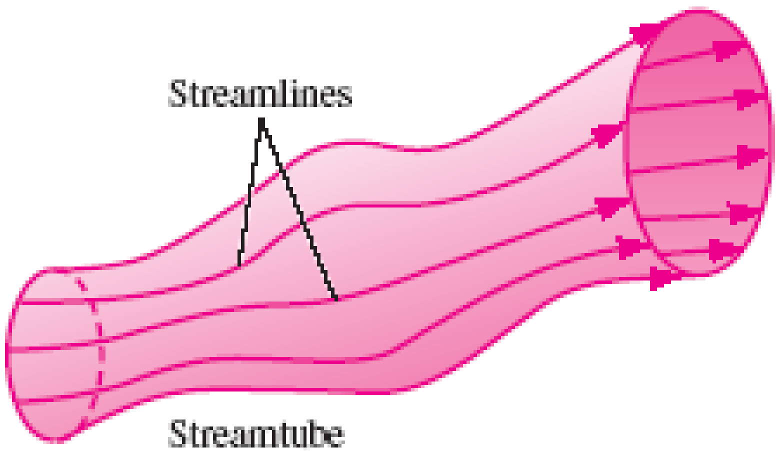


# Streamtube

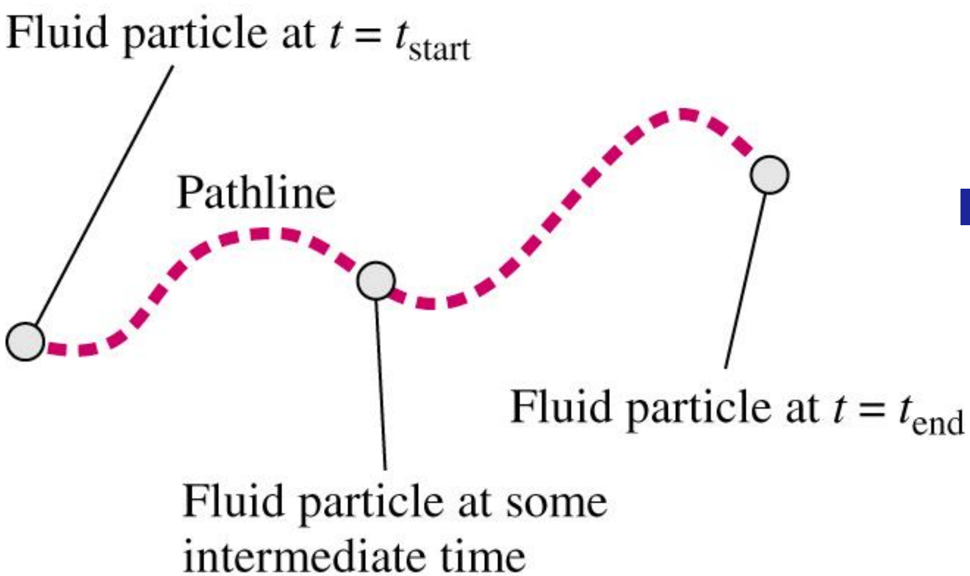
■A **streamtube** consists of a bundle of streamlines (Both are instantaneous quantities).

■ *Fluid within a streamtube must remain there and cannot cross the boundary of the streamtube*.

■In an *unsteady* flow, the streamline pattern may change significantly with time.⇒ the mass flow rate passing through any cross-sectional slice of a given streamtube must remain the same.



# Pathlines



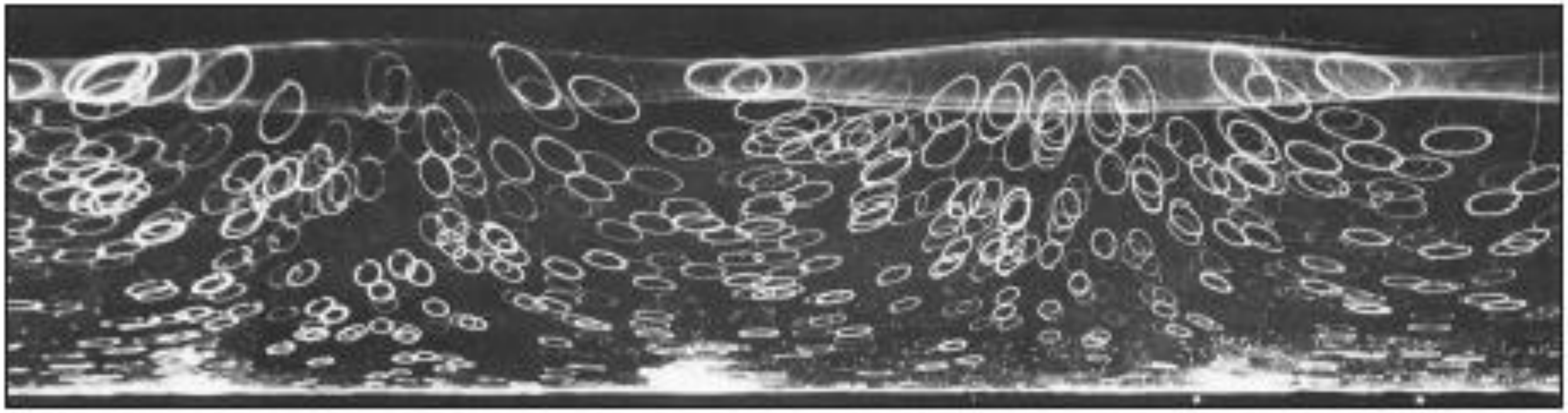

- A **Pathline** is the actual path traveled by an individual fluid particle over some time period.
- Same as the fluid particle's material position vector

$$\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$

- Particle location at time t:

$$\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V} dt$$

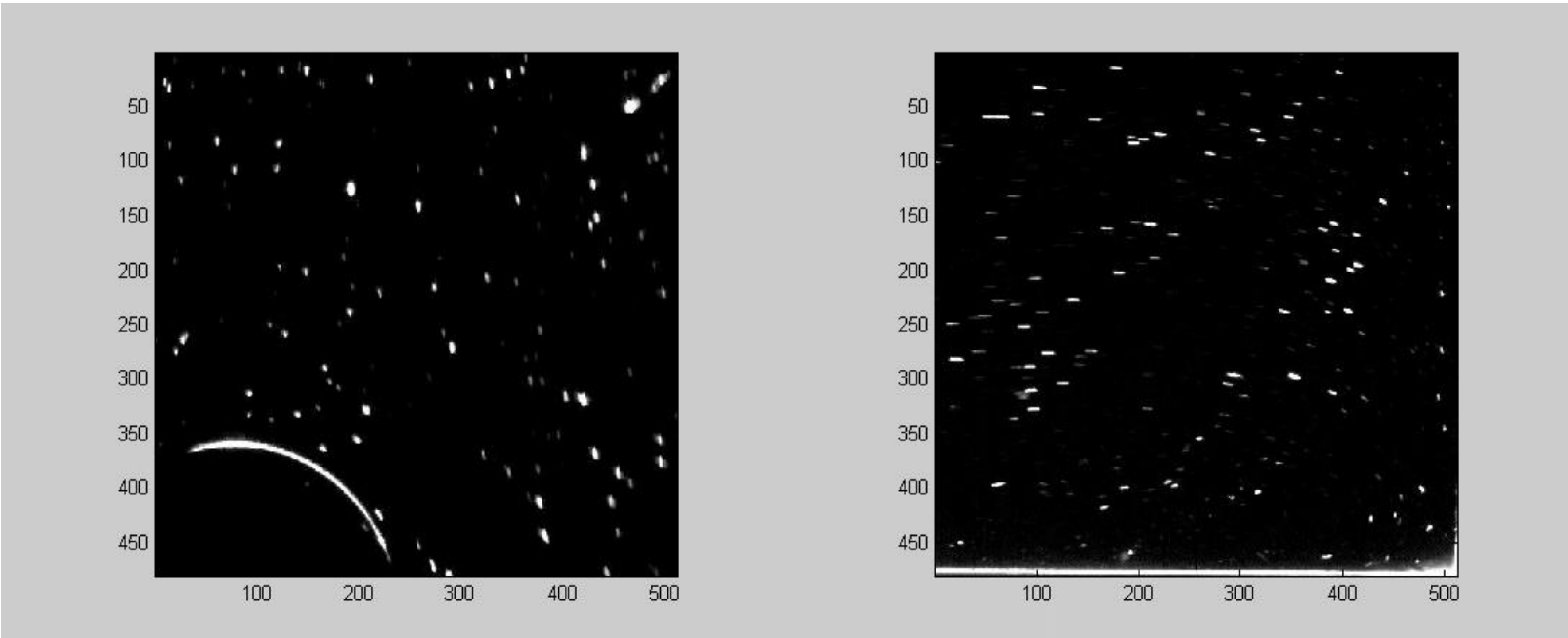
# Pathlines



A modern experimental technique called **particle image velocimetry (PIV)** utilizes (tracer) particle pathlines to measure the velocity field over an entire plane in a flow (Adrian, 1991).

# Pathlines

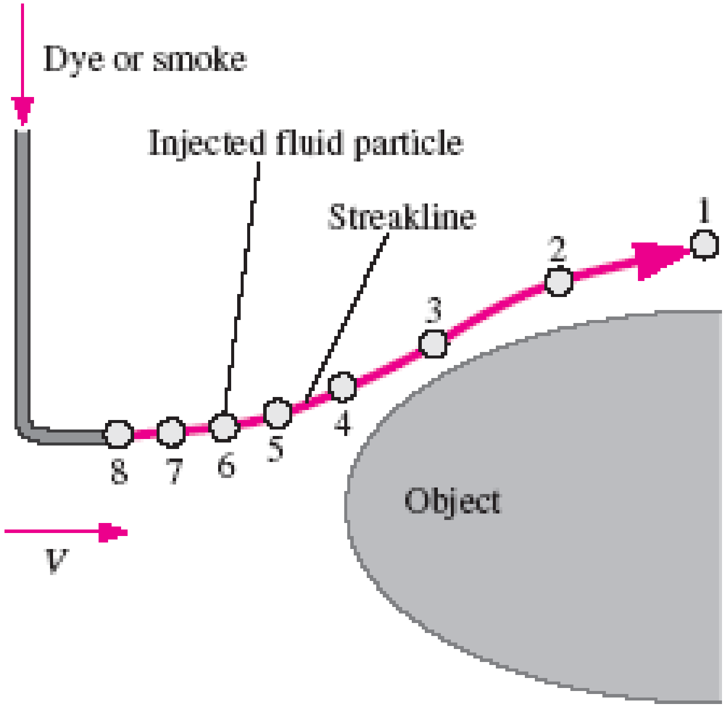
Flow over a cylinder




Top View

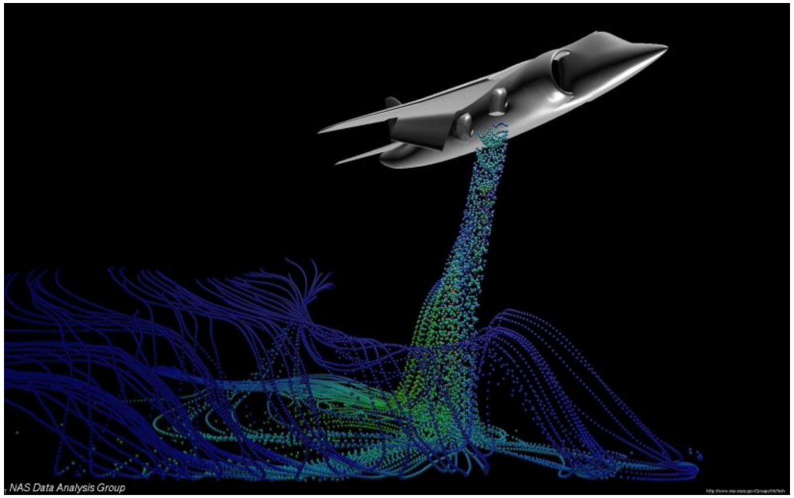
Side View

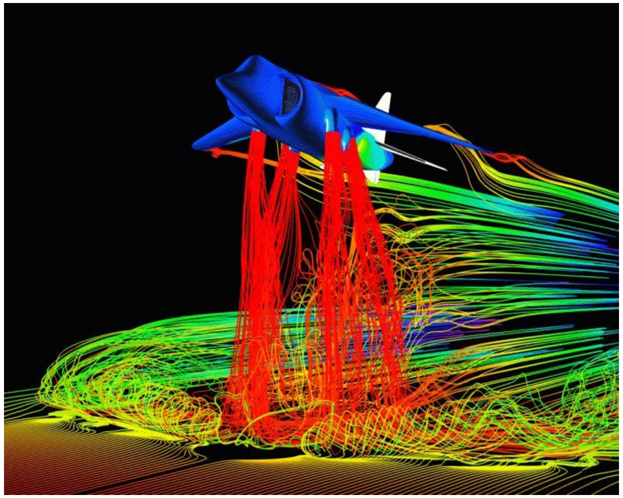
# Streaklines




- A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Easy to generate in experiments: dye in a water flow, or smoke in an airflow.

# Streaklines

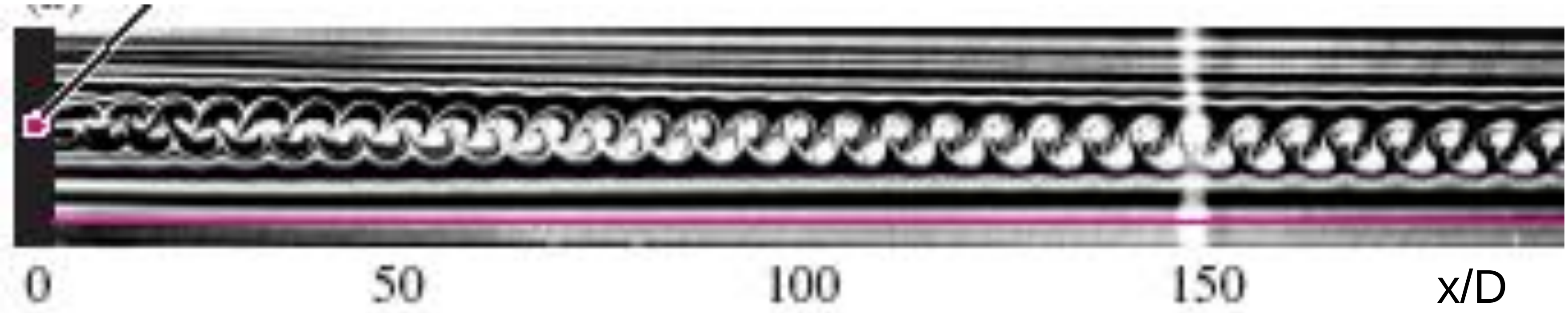


# Streaklines

Karman Vortex street

Cylinder



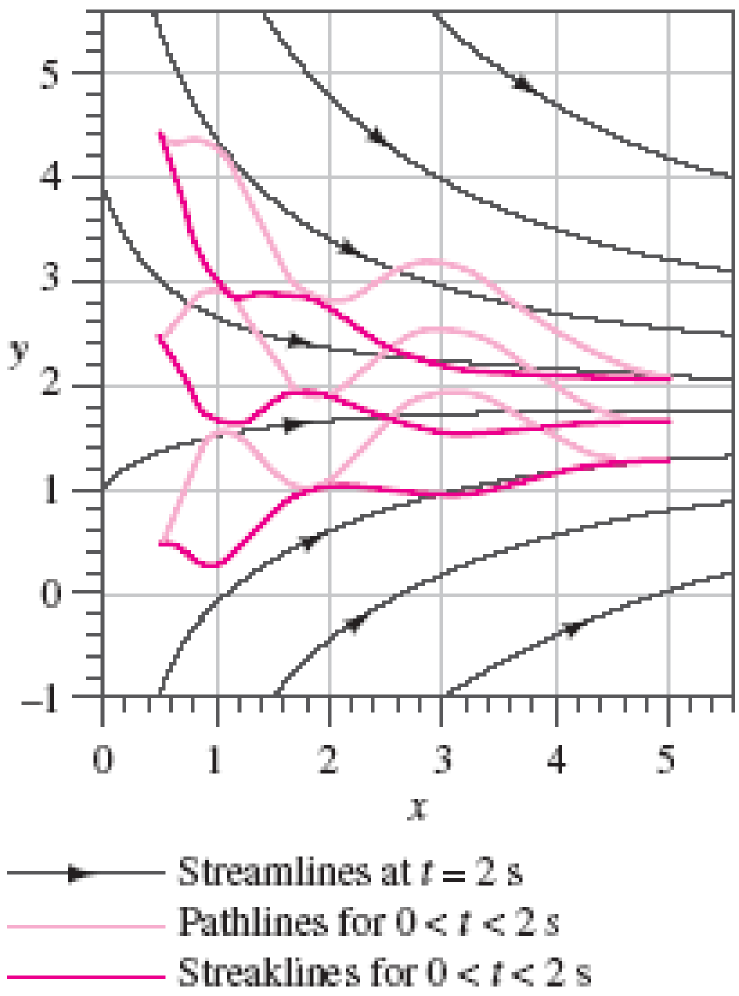

x/D

A smoke wire with mineral oil was heated to generate a rake of Streaklines

# Comparisons

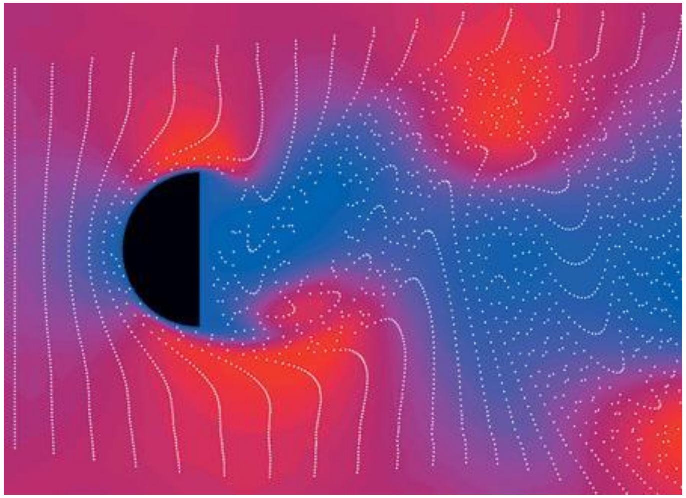
- For steady flow, streamlines, pathlines, and streaklines are identical.
- For unsteady flow, they can be very different.
  - Streamlines are an instantaneous picture of the flow field
  - Pathlines and Streaklines are flow patterns that have a time history associated with them.
  - Streakline:  instantaneous snapshot of a time-integrated flow pattern.
  - Pathline:  time-exposed flow path of an individual particle.

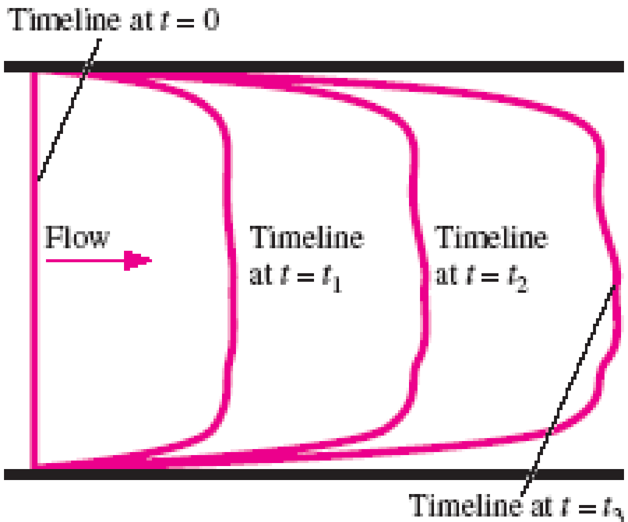
# Comparisons




$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i}$$
$$+ (1.5 + 2.5\sin(\omega t) - 0.8y)\vec{j}$$

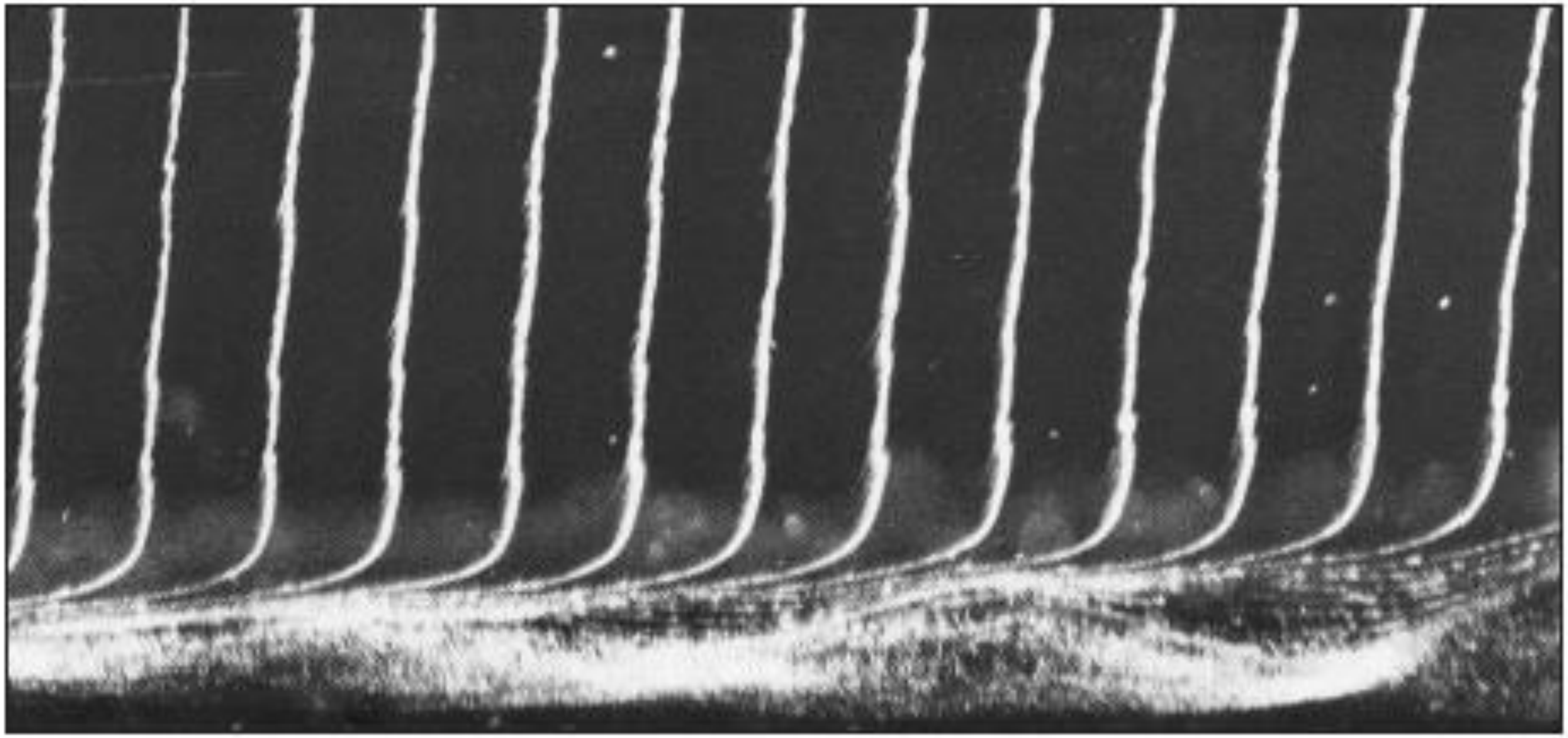
# Timelines






- A **Timeline** is a set of adjacent fluid particles that were marked at the same (earlier) instant in time.
- Timelines can be generated using a hydrogen bubble wire.

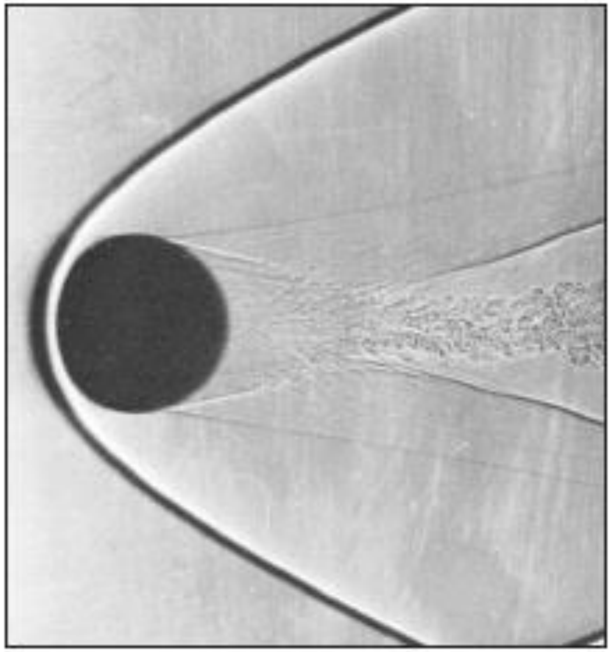
# Timelines



Timelines produced by a hydrogen bubble wire are used to visualize the boundary layer velocity profile shape.

# Refractive Flow Visualization Techniques

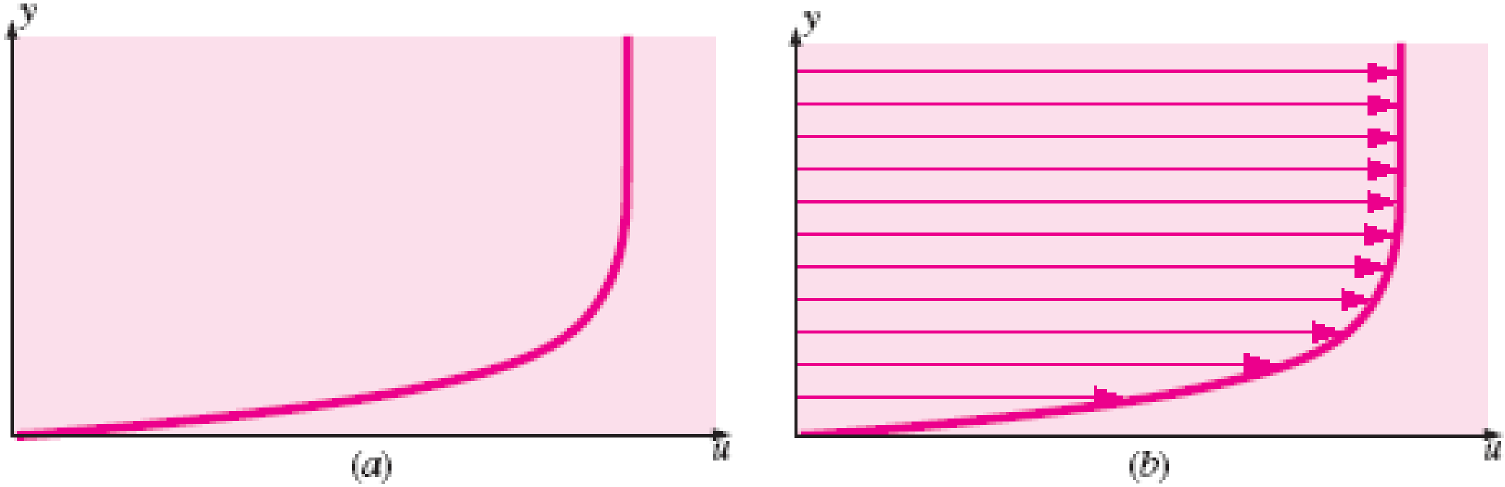
- Based on the **refractive property** of light waves in fluids with different index of refraction, one can visualize the flow field: **shadowgraph technique and** s**chlieren technique.**



# Plots of Flow Data

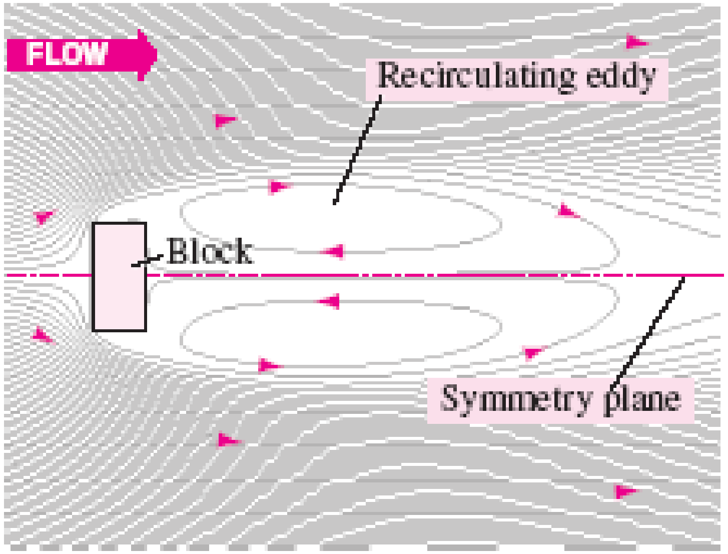
- Flow data are the presentation of the flow properties varying in time and/or space.
- A **Profile plot** indicates how the value of a scalar property varies along some desired direction in the flow field.
- A **Vector plot** is an array of arrows indicating the magnitude and direction of a vector property at an instant in time.
- A **Contour plot** shows curves of constant values of a scalar property for the magnitude of a vector property at an instant in time.

# Profile plot

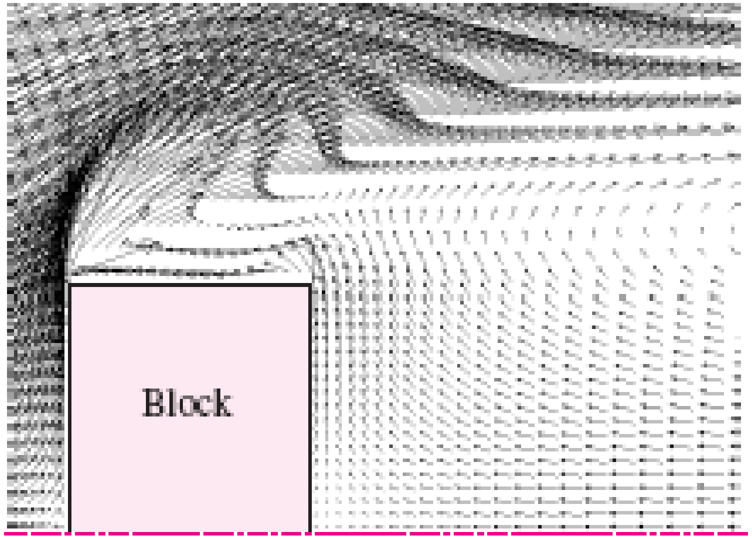



*Profile plots* of the horizontal component of velocity as a function of vertical distance; flow in the boundary layer growing along a horizontal flat plate.

# Vector plot

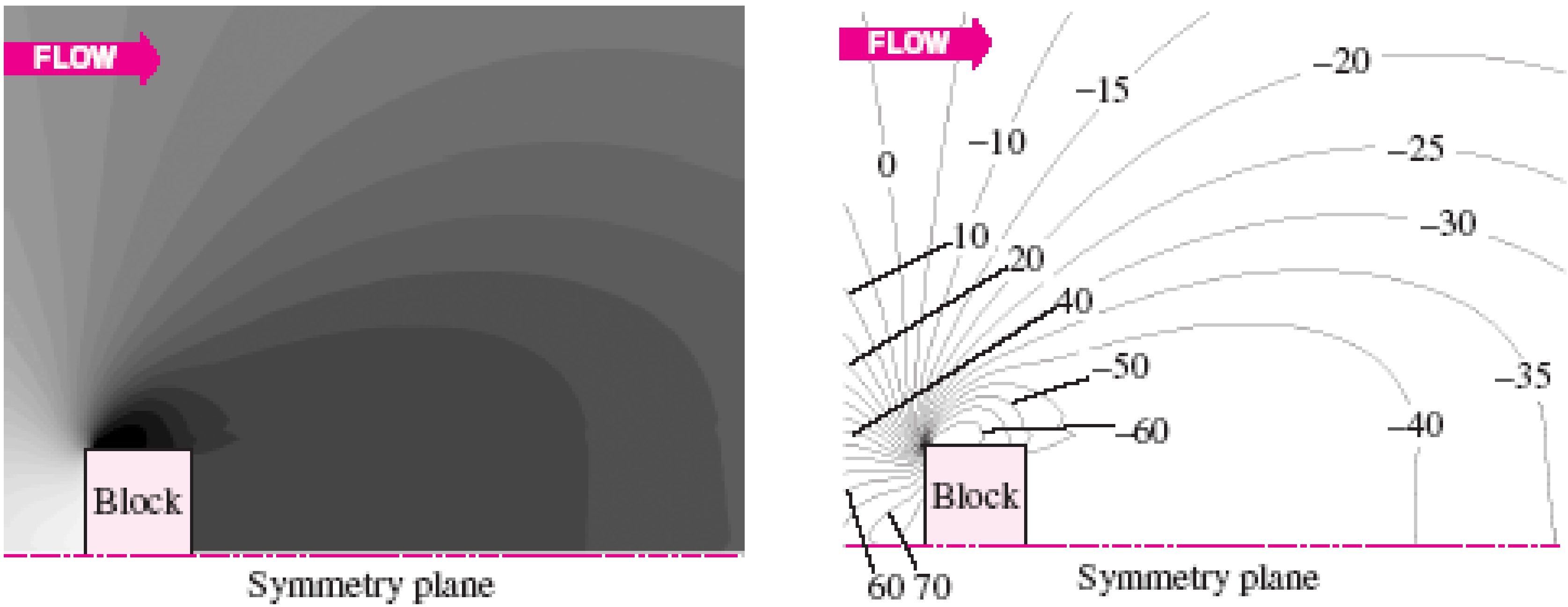


(a)



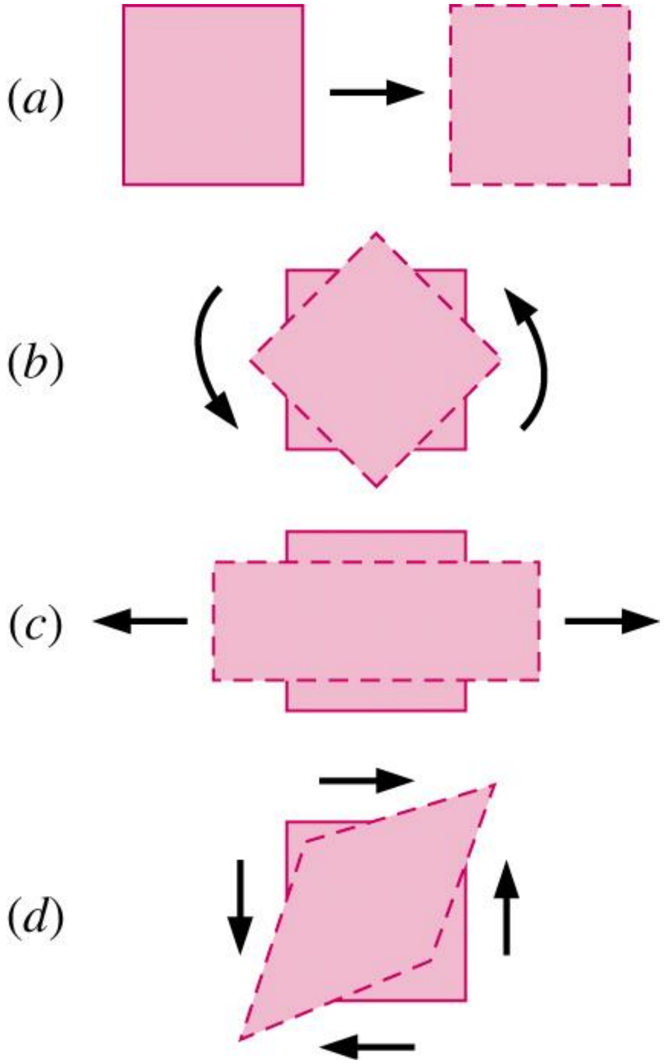
(c)
Symmetry plane

# Contour plot




Contour plots of the pressure field due to flow impinging on a block.

# Kinematic Description

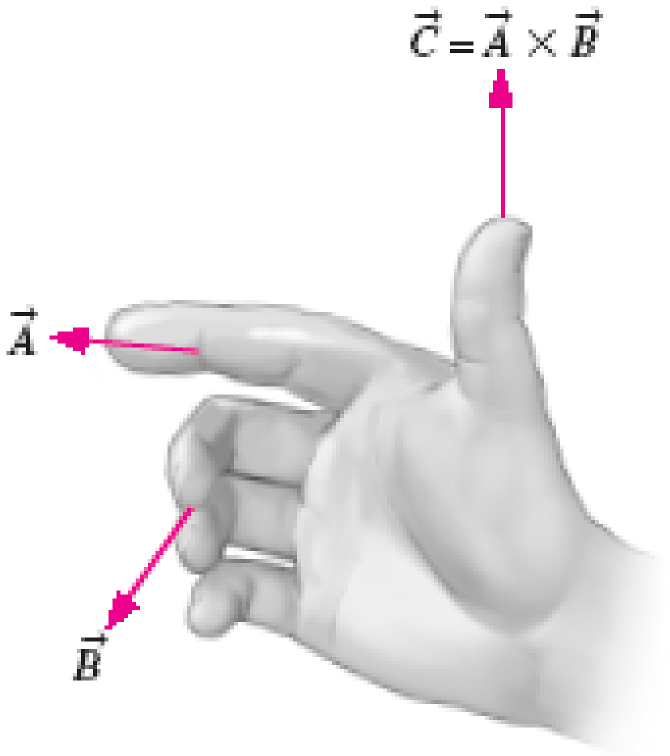



- In fluid mechanics, an element may undergo four fundamental types of motion.
  a) Translation
  b) Rotation
  c) Linear strain
  d) Shear strain
- Because fluids are in constant motion, motion and deformation is best described in terms of rates
  a) velocity: rate of translation
  b) angular velocity: rate of rotation
  c) linear strain rate: rate of linear strain
  d) shear strain rate: rate of shear strain

# Rate of Translation and Rotation

- To be useful, these rates must be expressed in terms of velocity and derivatives of velocity
- The **rate of translation vector** is described as the velocity vector. In Cartesian coordinates:

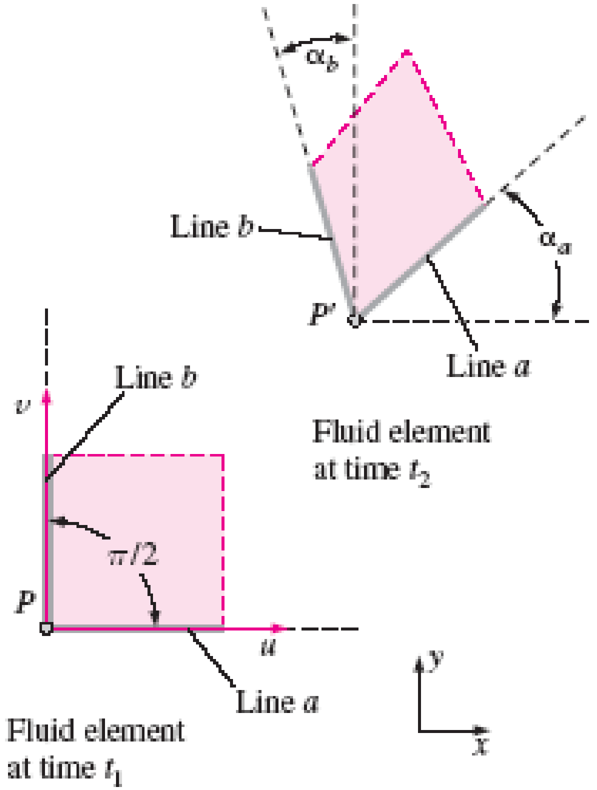
$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$




Rule of thumb for rotation

# Rate of Translation and Rotation

- **Rate of rotation** at a point is defined as the average rotation rate of two initially perpendicular lines that intersect at that point. The rate of rotation vector in Cartesian coordinates: (Proof on blackboard)




$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

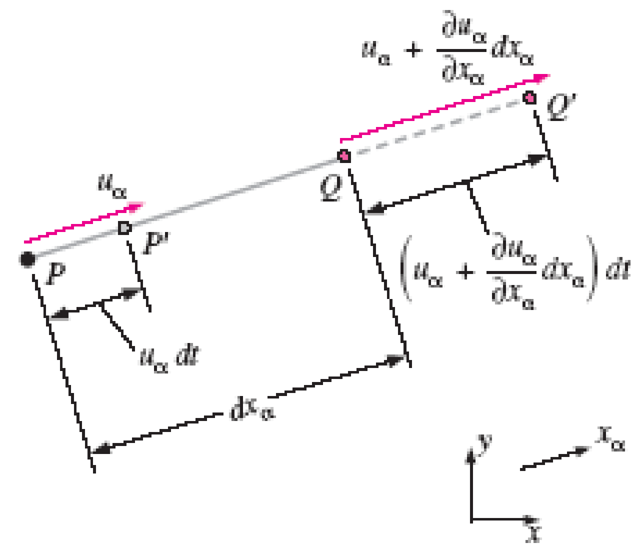
# Linear Strain Rate

- **Linear Strain Rate** is defined as the rate of increase in length per unit length.
- In Cartesian coordinates (Proof on blackboard)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

- Volumetric strain rate in Cartesian coordinates

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

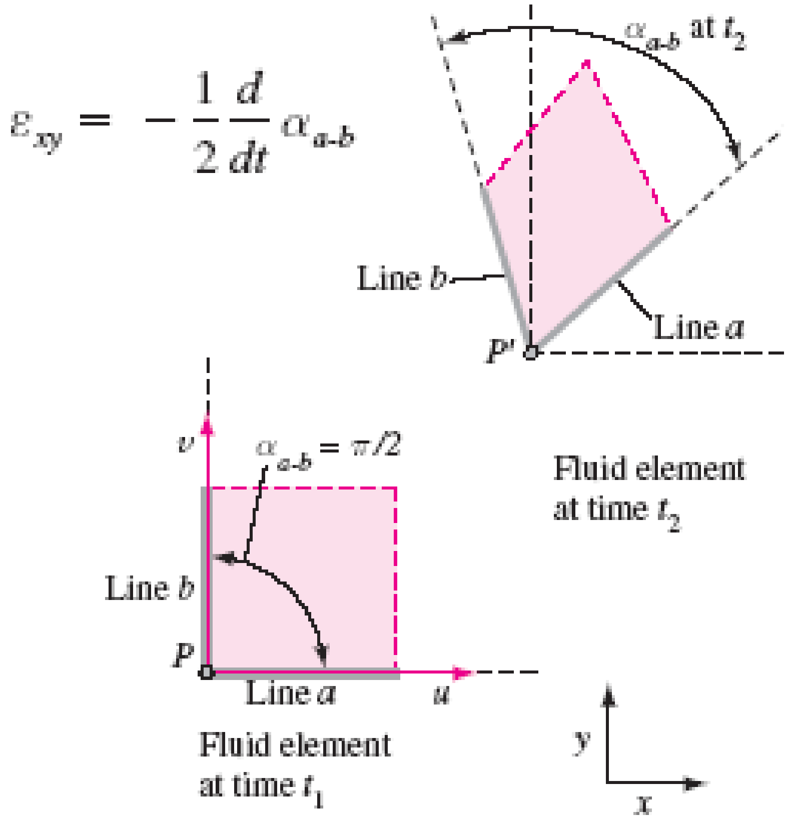



- Since the volume of a fluid element is constant for an incompressible flow, the volumetric strain rate must be zero.

# Shear Strain Rate

- **Shear Strain Rate** at a point is defined as *half of the rate of decrease of the angle between two initially perpendicular lines that intersect at a point*.

- Shear strain rate can be expressed in Cartesian coordinates as: (Proof on blackboard)




$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

# Shear Strain Rate

We can combine linear strain rate and shear strain rate into one symmetric second-order tensor called the **strain-rate tensor.**

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

# Shear Strain Rate

- Purpose of our discussion of fluid element kinematics:
  - Better appreciation of the inherent complexity of fluid dynamics
  - Mathematical sophistication required to fully describe fluid motion
- Strain-rate tensor is important for numerous reasons. For example,
  - Develop relationships between fluid stress and strain rate.

# Vorticity and Rotationality

- The **vorticity vector** is defined as the curl of the velocity vector $\vec{\zeta} = \vec{\nabla} \times \vec{V}$, *a measure of rotation of a fluid particle*.
- Vorticity is equal to twice the angular velocity of a fluid particle. $\vec{\zeta} = 2\vec{\omega}$

  Cartesian coordinates

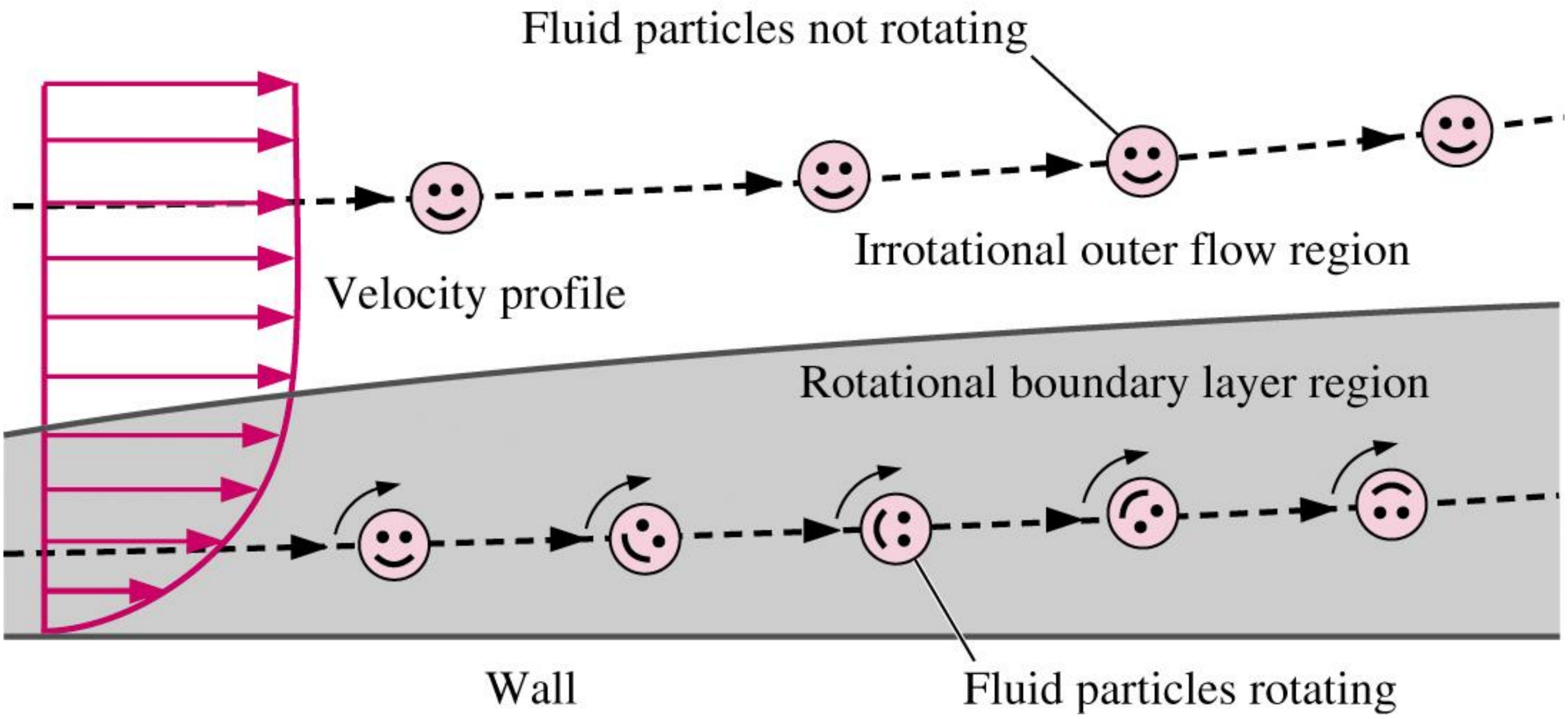
  $$\vec{\zeta} = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k}$$

  Cylindrical coordinate

  $$\vec{\zeta} = \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \vec{e}_r + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \vec{e}_\theta + \left( \frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \vec{e}_z$$

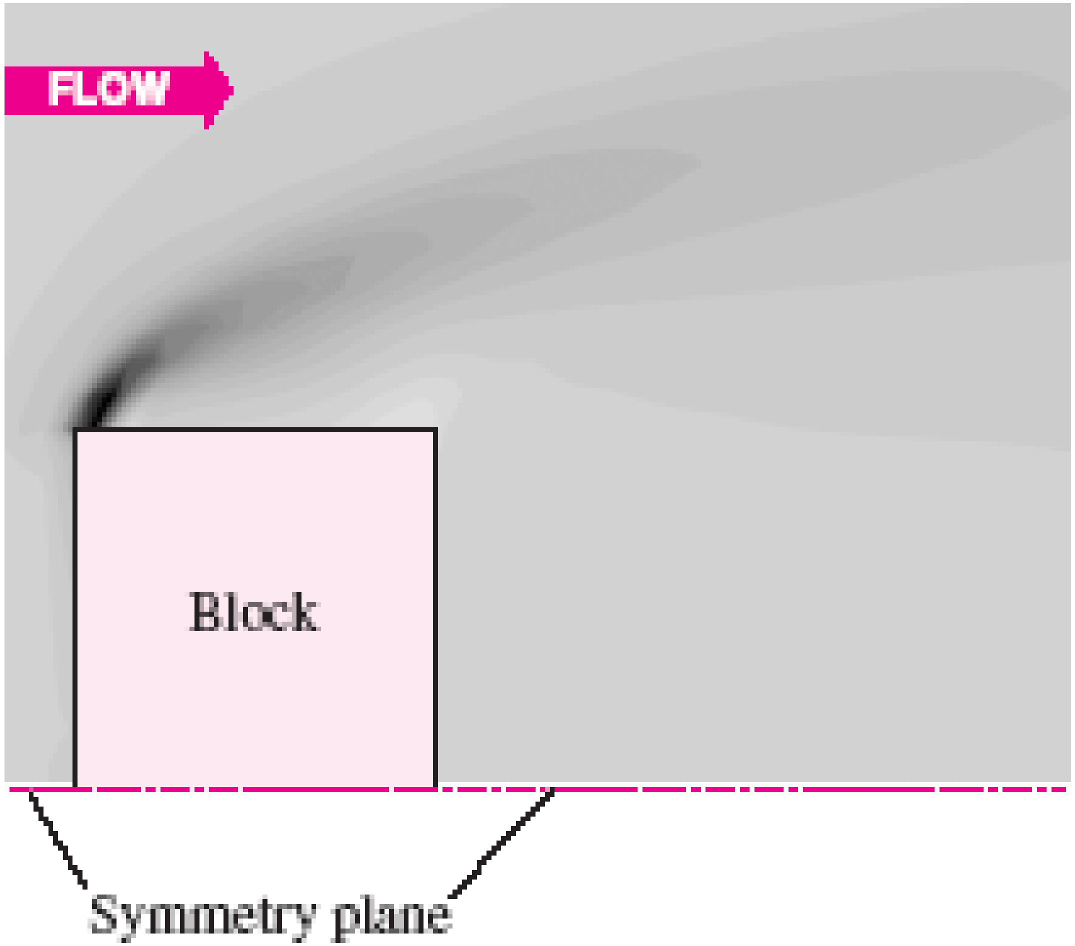
- In regions where $\zeta = 0$, the flow is called **irrotational.**
- Elsewhere, the flow is called **rotational.**

# Vorticity and Rotationality



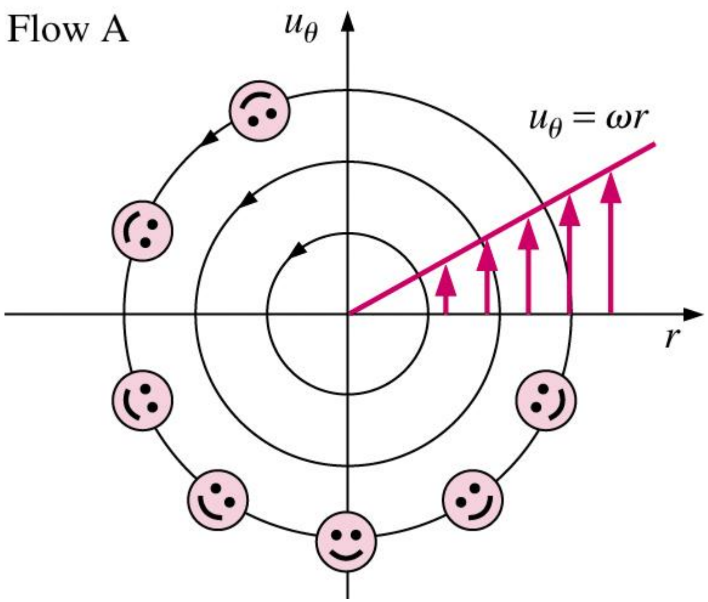
# Contour plot of the vorticity field $\zeta_z$

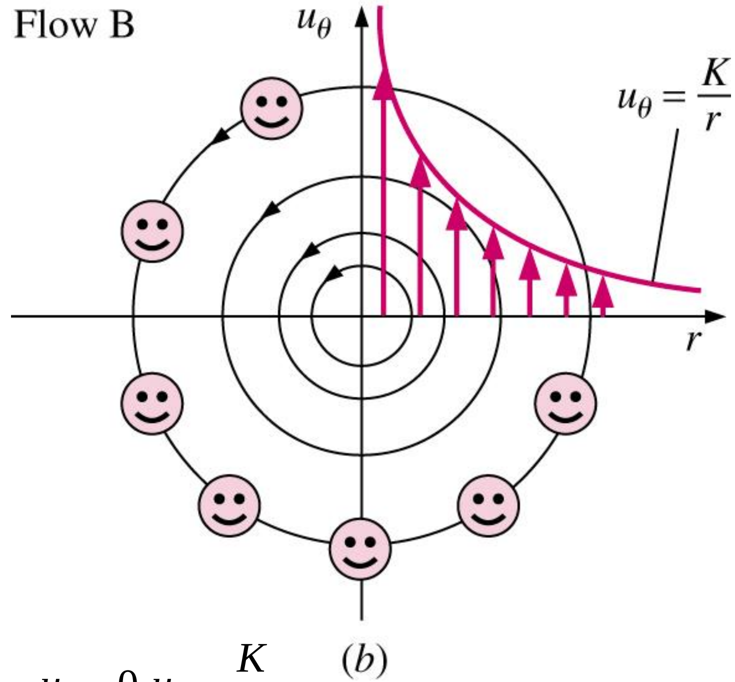
Dark regions represent large negative vorticity, and light regions represent large positive vorticity.



# Comparison of Two Circular Flows

Special case: consider two flows with circular streamlines




$$u_r = 0, u_\theta = \omega r$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{e}_z = 2\omega\vec{e}_z$$

$$u_r = 0, u_\theta = \frac{K}{r}$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

# Comparison

(a)

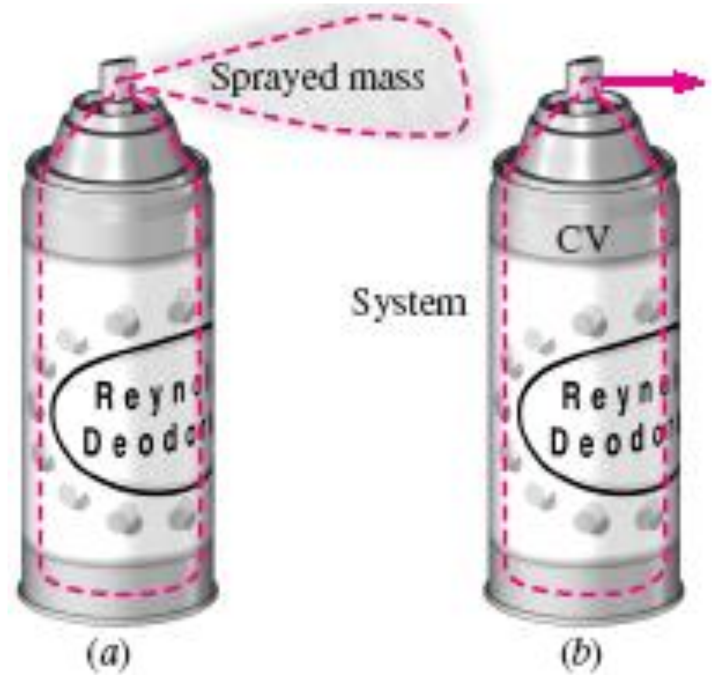
A merry-go-round or roundabout

(b)

A Ferris wheel

# Reynolds—Transport Theorem (RTT)

- A **system** is a quantity of matter of fixed identity. *No mass can cross a system boundary.*
- A **control volume** is a region in space chosen for study. Mass can cross a control surface.

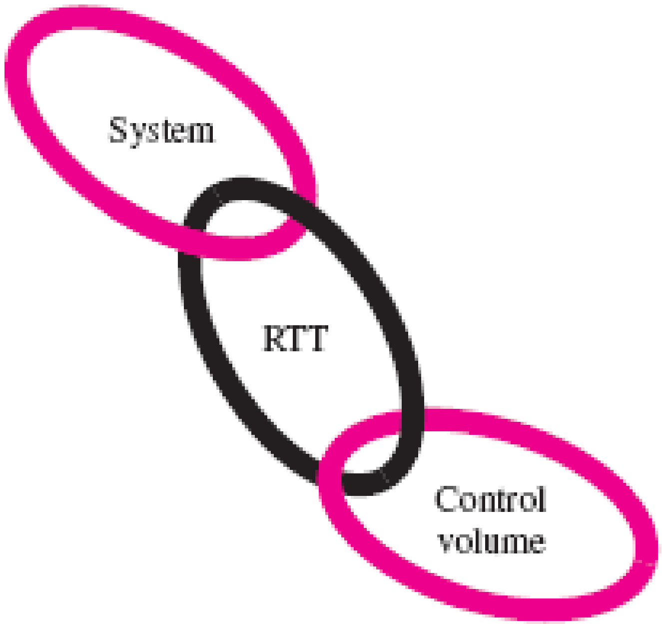



System deformable

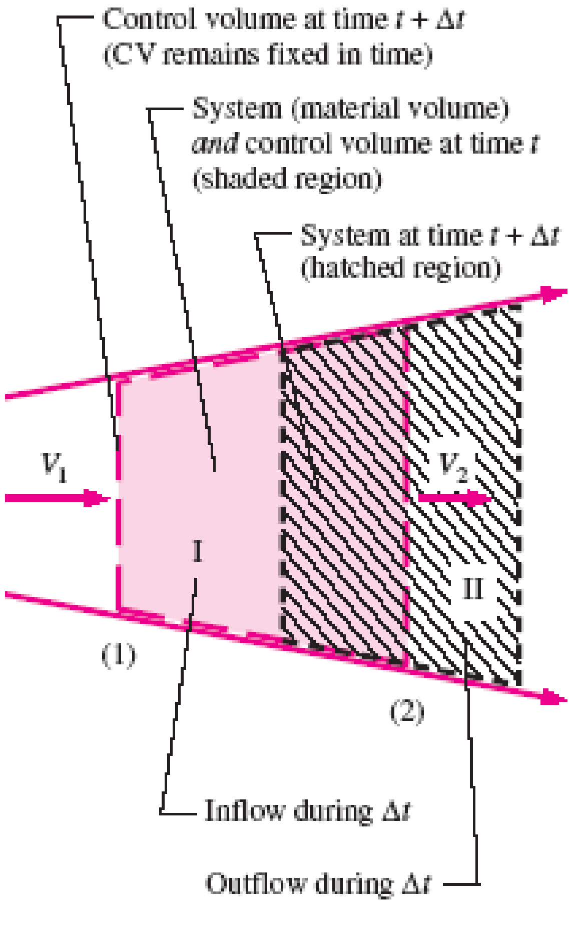
CV fixed, nondeformable

# Reynolds—Transport Theorem (RTT)

- The fundamental conservation laws (conservation of mass, energy, and momentum) apply directly to systems.
- However, in most fluid mechanics problems, control volume analysis is preferred over system analysis (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to transform the conservation laws from a system to a control volume. This is accomplished with the Reynolds transport theorem (RTT).



# Reynolds—Transport Theorem (RTT)




At time $t$: Sys = CV
At time $t + \Delta t$: Sys = CV − I + II

$$B_{\mathrm{sys},\,t} = B_{\mathrm{CV},\,t} \qquad \text{(the system and CV concide at time } t\text{)}$$

$$B_{\mathrm{sys},\,t+\Delta t} = B_{\mathrm{CV},\,t+\Delta t} - B_{\mathrm{I},\,t+\Delta t} + B_{\mathrm{II},\,t+\Delta t}$$

$$\frac{B_{\mathrm{sys},\,t+\Delta t} - B_{\mathrm{sys},\,t}}{\Delta t} = \frac{B_{\mathrm{CV},\,t+\Delta t} - B_{\mathrm{CV},\,t}}{\Delta t} - \frac{B_{\mathrm{I},\,t+\Delta t}}{\Delta t} + \frac{B_{\mathrm{II},\,t+\Delta t}}{\Delta t}$$

$$\frac{dB_{\mathrm{sys}}}{dt} = \frac{dB_{\mathrm{CV}}}{dt} - \dot{B}_{\mathrm{in}} + \dot{B}_{\mathrm{out}}$$

$$B_{\mathrm{I},\,t+\Delta t} = b_1 m_{\mathrm{I},\,t+\Delta t} = b_1 \rho_1 V_{\mathrm{I},\,t+\Delta t} = b_1 \rho_1 V_1 \Delta t\, A_1$$

$$B_{\mathrm{II},\,t+\Delta t} = b_2 m_{\mathrm{II},\,t+\Delta t} = b_2 \rho_2 V_{\mathrm{II},\,t+\Delta t} = b_2 \rho_2 V_2 \Delta t\, A_2$$

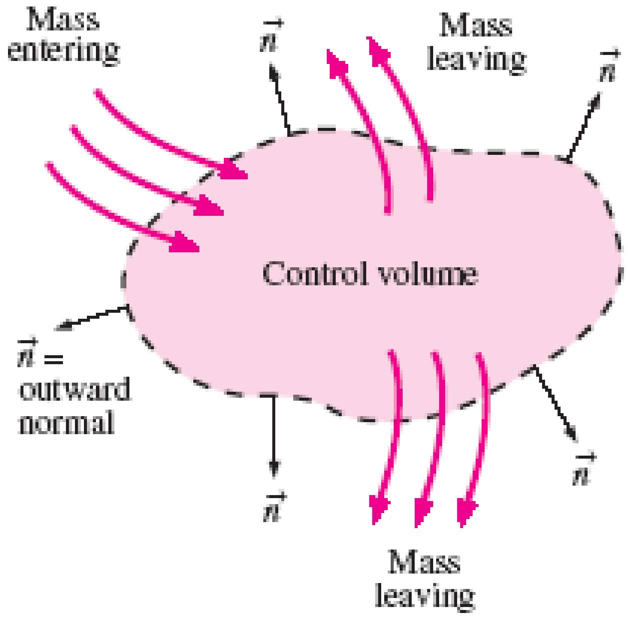
$$\dot{B}_{\mathrm{in}} = \dot{B}_{\mathrm{I}} = \lim_{\Delta t \to 0} \frac{B_{\mathrm{I},\,t+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{b_1 \rho_1 V_1 \Delta t\, A_1}{\Delta t} = b_1 \rho_1 V_1 A_1$$

$$\dot{B}_{\mathrm{out}} = \dot{B}_{\mathrm{II}} = \lim_{\Delta t \to 0} \frac{B_{\mathrm{II},\,t+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{b_2 \rho_2 V_2 \Delta t\, A_2}{\Delta t} = b_2 \rho_2 V_2 A_2$$

$$\frac{dB_{\mathrm{sys}}}{dt} = \frac{dB_{\mathrm{CV}}}{dt} - b_1 \rho_1 V_1 A_1 + b_2 \rho_2 V_2 A_2$$

# Reynolds—Transport Theorem (RTT)

- *the time rate of change of the property B of the system is equal to the time rate of change of B of the control volume plus the net flux of B out of the control volume by mass crossing the control surface*.




$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA \quad \text{(inflow if negative)}$$

# Reynolds—Transport Theorem (RTT)

The total amount of property *B* within the control volume must be determined by integration:

$$B_{\mathrm{CV}} = \int_{\mathrm{CV}} \rho b \, dV$$

Therefore, the *system-to-control- volume transformation* for a fixed control volume:

$$\frac{dB_{\mathrm{sys}}}{dt} = \frac{d}{dt} \int_{\mathrm{CV}} \rho b \, dV + \int_{\mathrm{CS}} \rho b \vec{V} \cdot \vec{n} \, dA$$

# Reynolds—Transport Theorem (RTT)

- Material derivative (differential analysis):

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \left(\vec{V} \bullet \vec{\nabla}\right) b$$

- General RTT, nonfixed CV (integral analysis):

$$\frac{dB_{\mathrm{sys}}}{dt} = \frac{d}{dt}\int_{\mathrm{CV}} \rho b \, dV + \int_{\mathrm{CS}} \rho b \vec{V}_r \cdot \vec{n} \, dA$$

| | **Mass** | **Momentum** | **Energy** | **Angular momentum** |
|---|---|---|---|---|
| B, Extensive properties | m | $m\vec{V}$ | E | $\vec{H}$ |
| b, Intensive properties | 1 | $\vec{V}$ | e | $\left(\vec{r} \times \vec{V}\right)$ |

- In Chaps 5 and 6, we will apply RTT to conservation of mass, energy, linear momentum, and angular momentum.

# Reynolds—Transport Theorem (RTT)

- Interpretation of the RTT:
  - Time rate of change of the property B of the system is equal to (Term 1) + (Term 2)
  - Term 1: the time rate of change of B of the control volume
  - Term 2: the net flux of B out of the control volume by mass crossing the control surface

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}\left(\rho b\right) d\forall + \int_{CS} \rho b \vec{V} \cdot \vec{n} dA$$

# RTT Special Cases

For **moving** and/or **deforming** control volumes,

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b \, dV + \int_{\text{CS}} \rho b \vec{V}_r \cdot \vec{n} \, dA$$

- Where the absolute velocity $V$ in the second term is replaced by the **relative velocity** $V_r = V - V_{CS}$
- $V_r$ is the fluid velocity expressed relative to a coordinate system moving **with** the control volume.

# RTT Special Cases

For steady flow, the time derivative drops out,

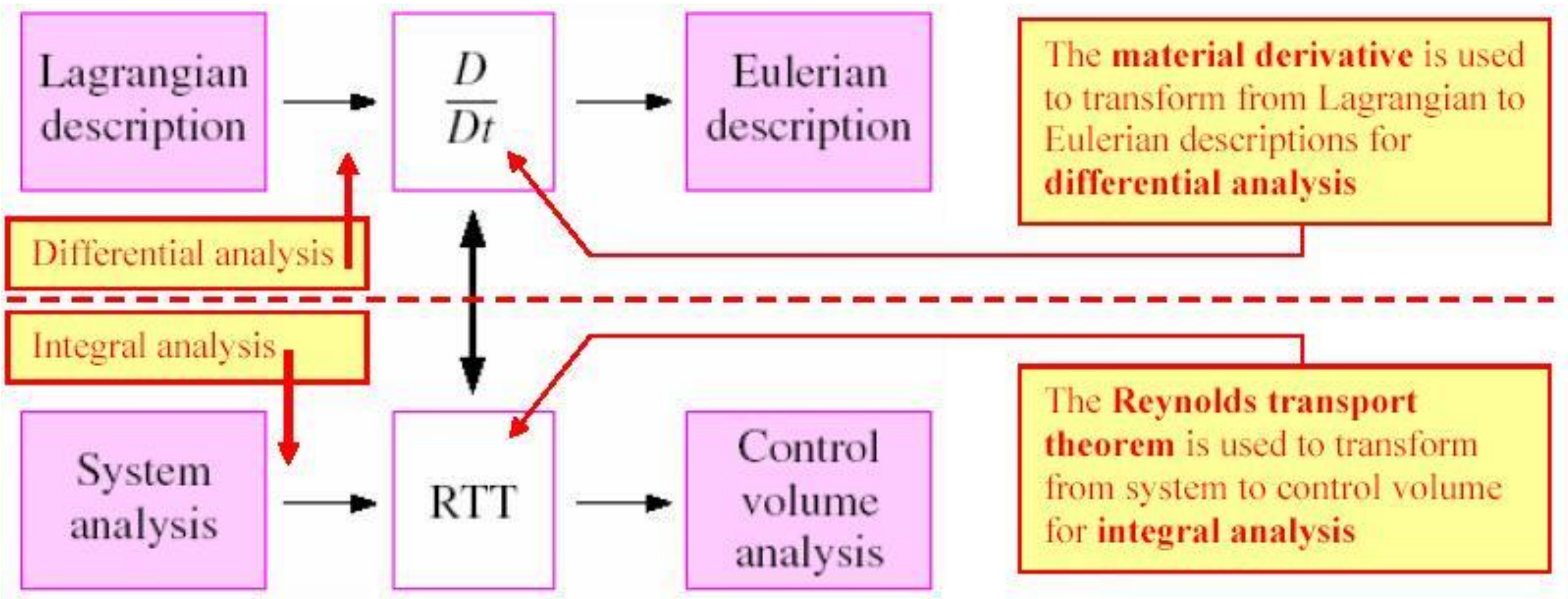
$$\frac{dB_{sys}}{dt} = \cancelto{0}{\int_{CV} \frac{\partial}{\partial t}(\rho b)\, d\mathcal{V}} + \int_{CS} \rho b \vec{V}_r \cdot \vec{n}\, dA = \int_{CS} \rho b \vec{V}_r \cdot \vec{n}\, dA$$

For control volumes with well-defined inlets and outlets

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, d\mathcal{V} + \sum_{out} \rho_{avg} b_{avg} V_{r,avg} A - \sum_{in} \rho_{avg} b_{avg} V_{r,avg} A$$

- Alternate Derivation (Leibnitz rule) of the Reynolds Transport Theorem is referred to the text book from pages 153 to 155.

# Reynolds—Transport Theorem (RTT)




There is a direct analogy between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and the transformation from systems to control volumes (for integral analysis using large, finite flow fields).