Chapter 6: **Momentum Analysis of Flow Systems**

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Introduction

- Most engineering problems can be analyzed using one of three basic approaches: differential, experimental, and control volume.
  - ***Differential approaches***: the problem is formulated accurately using differential quantities, but the solution is usually relied on the use of numerical methods.
  - ***Experimental approaches****:* complemented with dimensional analysis are highly accurate, but they are typically time consuming and expensive.
  - ***Finite control volume approach****:* described in this chapter is remarkably fast and simple and usually gives answers that are sufficiently accurate for most engineering purposes.
- The linear momentum and angular momentum equations for control volumes were developed and use them to determine the forces and torques associated with fluid flow.

NEWTON'S LAWS AND CONSERVATION OF MOMENTUM

Newton's Law

- Newton's first law
- Newton's second law.
- Newton's third law.

For a rigid body of mass *m*, Newton's second law is expressed as

$$\vec{F} = m\vec{a} = m\,\frac{d\vec{V}}{dt} = \frac{d(m\vec{V})}{dt}$$

NEWTON'S LAWS AND CONSERVATION OF MOMENTUM

- The product of the mass and the velocity of a body is called the *linear momentum* or just the *momentum* of the body.
- Therefore, Newton's second law can also be stated as *the rate of change of the momentum of a body is equal to the net force acting on the body*
- Newton's second law $\Rightarrow$ the *linear momentum equation in fluid mechanics*
- The momentum of a system is conserved when it remains constant $\Rightarrow$ the *conservation of momentum principle*.
- Momentum is a vector. Its direction is the direction of velocity.

NEWTON'S LAWS AND CONSERVATION OF MOMENTUM

- Newton's second law for rotating rigid bodies is expressed as

$$\vec{M} = I\vec{\alpha}$$

- where $\vec{M}$ is the net moment or torque applied on the body, *I* is the moment of inertia of the body about the axis of rotation, and $\vec{\alpha}$ is the angular acceleration.

NEWTON'S LAWS AND CONSERVATION OF MOMENTUM

- The rate of change of angular momentum is

$$\vec{M} = I\vec{\alpha} = I\,\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt}$$

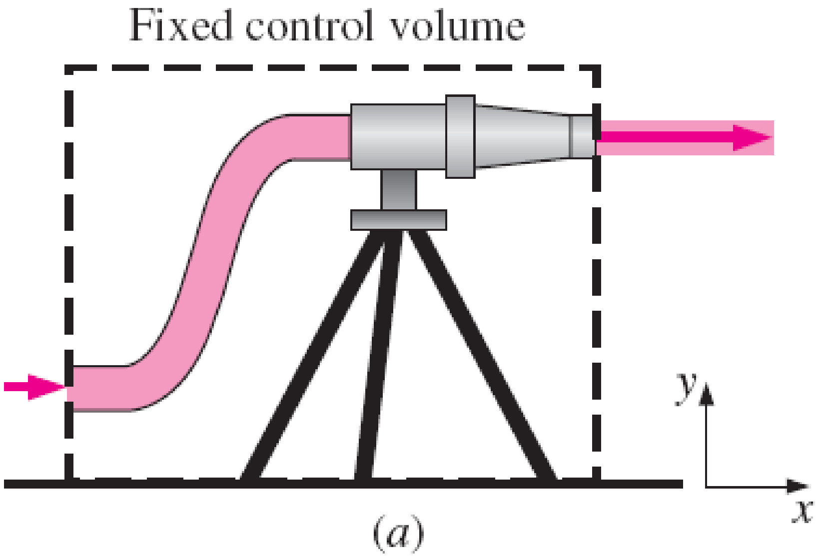
Where $\vec{\omega}$ is the angular velocity.

- *The rate of change of the angular momentum of a body is equal to the net torque acting on it*
- The *conservation of angular momentum principle* is hold as

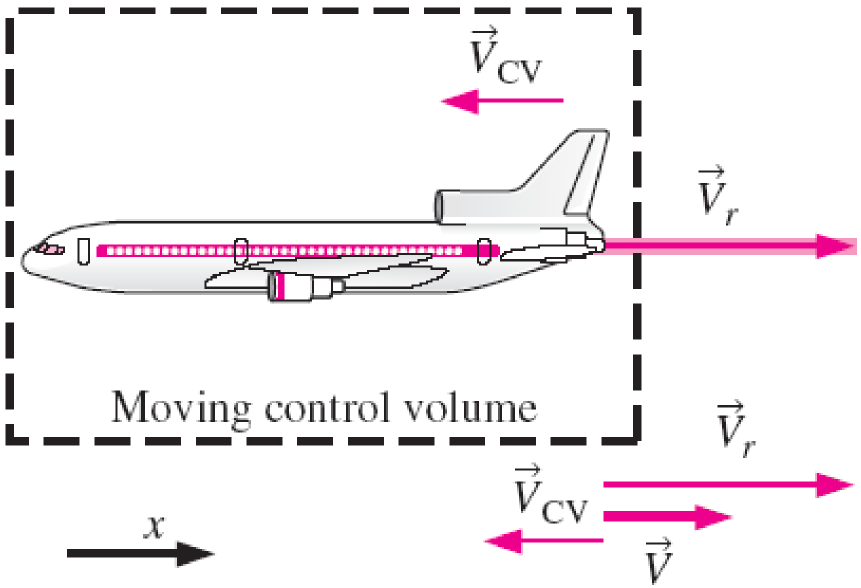
$$I\omega = \text{constant}$$

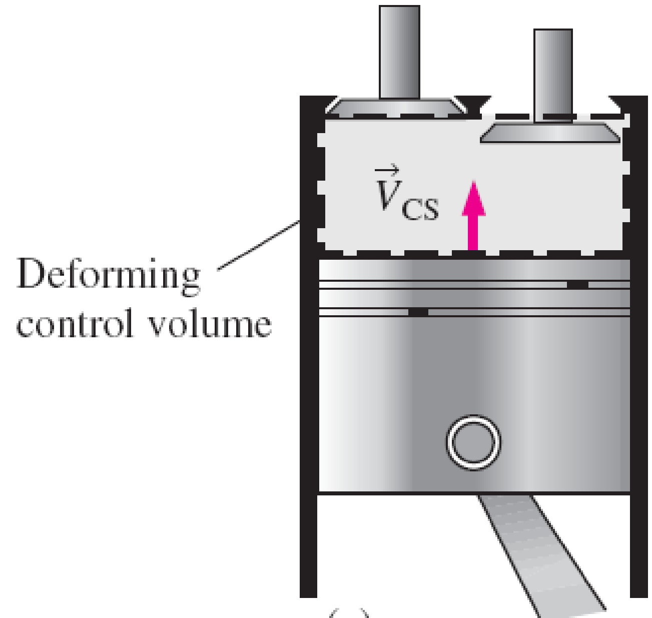
CHOOSING A CONTROL VOLUME

- How to *wisely* select a control volume?
- A control volume can be selected as any arbitrary region in space through which fluid flows.
- A control volume and its bounding control surface can be fixed, moving, and even deforming during flow.



CHOOSING A CONTROL VOLUME





(c)

$$\vec{V}_r = \vec{V} - \vec{V}_{CV}$$

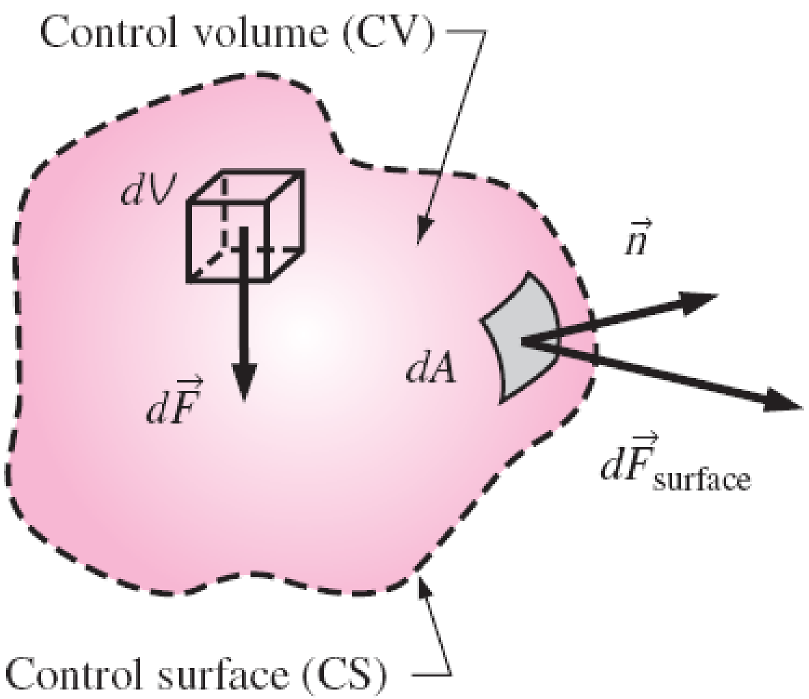
$\vec{V}_{CS} = \vec{V}_{CV}$ for moving but nondeforming control volumes

$\vec{V}_{CS} = \vec{V}_{CV}$ = 0 for fixed ones

FORCES ACTING ON A CONTROL VOLUME

- The forces include:
  - Body forces: act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces)
  - Surface forces: act on the control surface (such as pressure and viscous forces and reaction forces at points of contact).
- *Total force acting on control volume* is expressed as

$$\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}}$$



Body Forces

- Body force: the only body force considered in this text is gravity

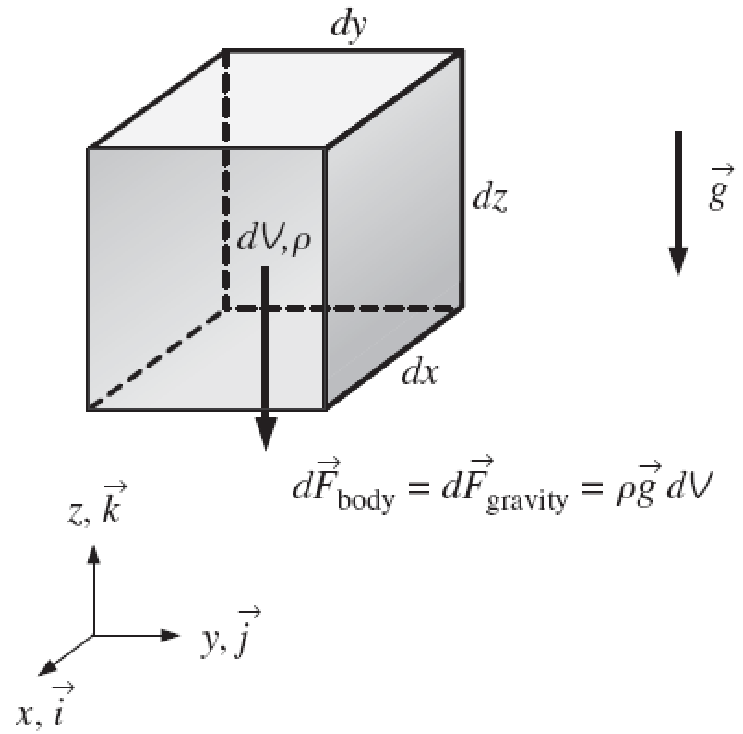
$$d\vec{F}_{\text{body}} = d\vec{F}_{\text{gravity}} = \rho\vec{g}\, dV$$

where

$$\vec{g} = -g\vec{k}$$

Therefore, the total body force is

$$\sum \vec{F}_{\text{body}} = \int_{\text{CV}} \rho\vec{g}\, dV = m_{\text{CV}}\vec{g}$$

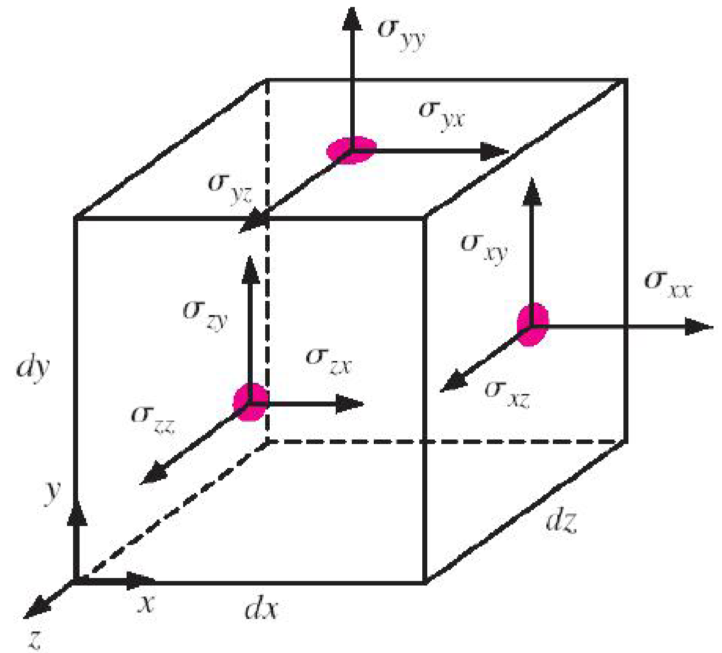


On earth at sea level, the gravitational constant *g* is equal to 9.807 m/s^2.

Surface Forces

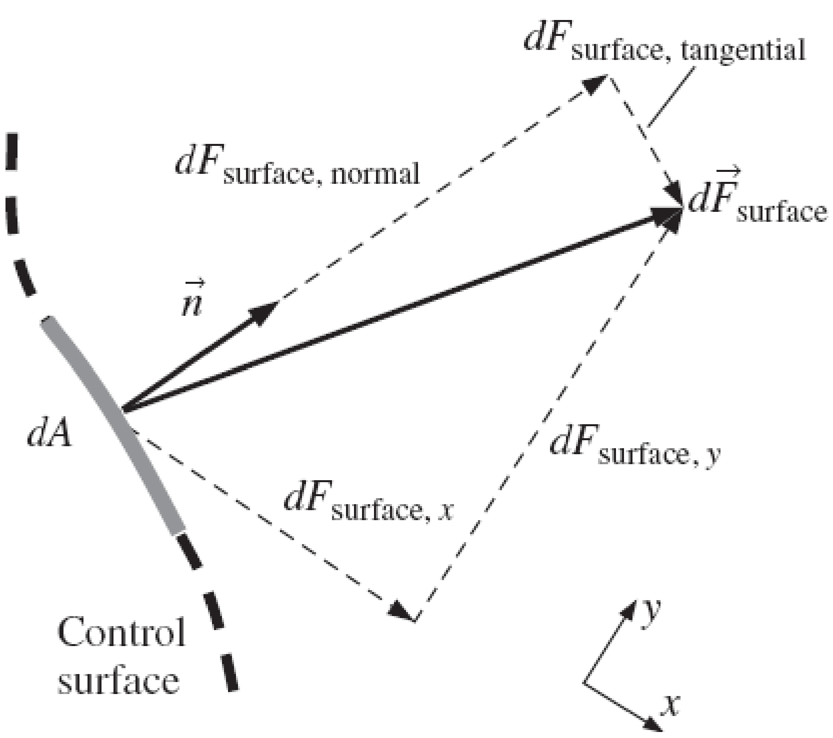
- Surface forces: are not as simple to analyze since they consist of both *normal* and *tangential* components.
- Diagonal components σ_{xx} , σ_{yy} , σ_{zz} are called **normal stresses** and are due to pressure and viscous stresses.
- Off-diagonal components σ_{xy}, σ_{xz}, etc. are called **shear stresses** and are due solely to viscous stresses.

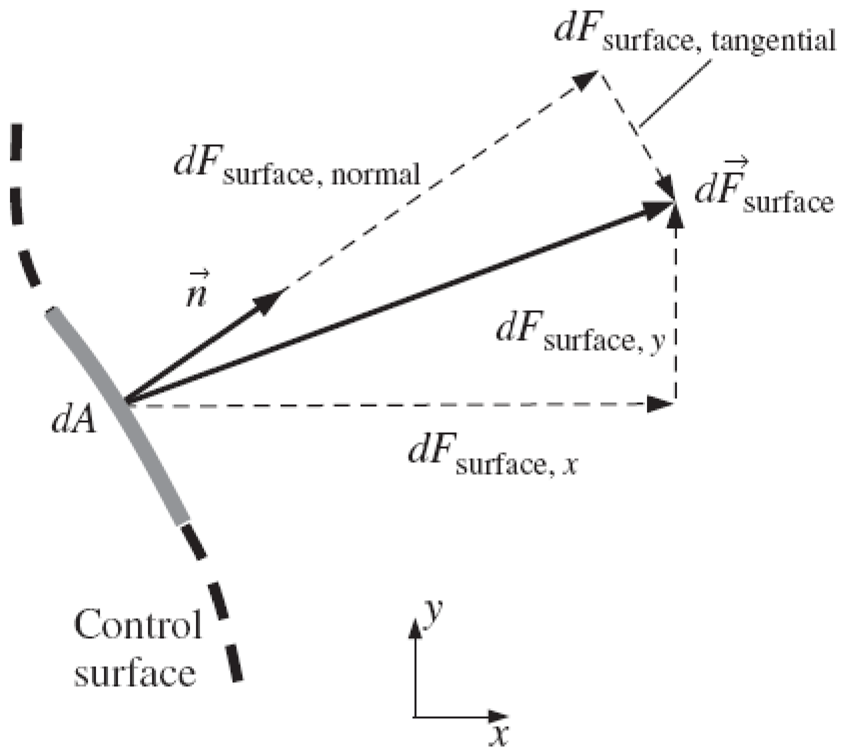
$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$



Surface Forces

The physical force acting on a surface is independent of orientation of the coordinate axes.





Surface Forces

- The dot product of a second-order tensor and a vector yields a second vector whose direction is the direction of the surface force itself.
- *Surface force acting on a differential surface element*:

$$d\vec{F}_{\text{surface}} = \sigma_{ij} \cdot \vec{n}\, dA$$

- Total surface force acting on CS

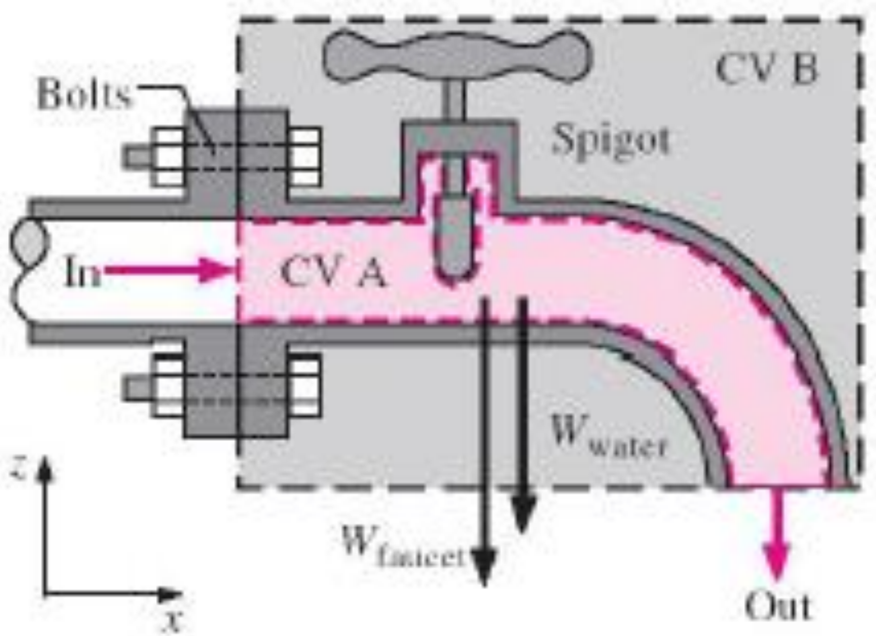
$$\sum \vec{F}_{surface} = \int_{CS} \sigma_{ij} \cdot \vec{n}\, dA$$

FORCES ACTING ON A CONTROL VOLUME

- *Total force*:

$$\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}} = \int_{CV} \rho \vec{g}\, dV + \int_{CS} \sigma_{ij} \cdot \vec{n}\, dA$$

- The control volume is drawn similar to drawing a free-body diagram in your statics and dynamics classes.
- Which one (CV A and CV B) is a wise choice if we want to calculate the force on the flange?



THE LINEAR MOMENTUM EQUATION

- Newton's second law for a system of mass *m* subjected to a force $\vec{F}$ is expressed as

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(m\vec{V}\right)$$

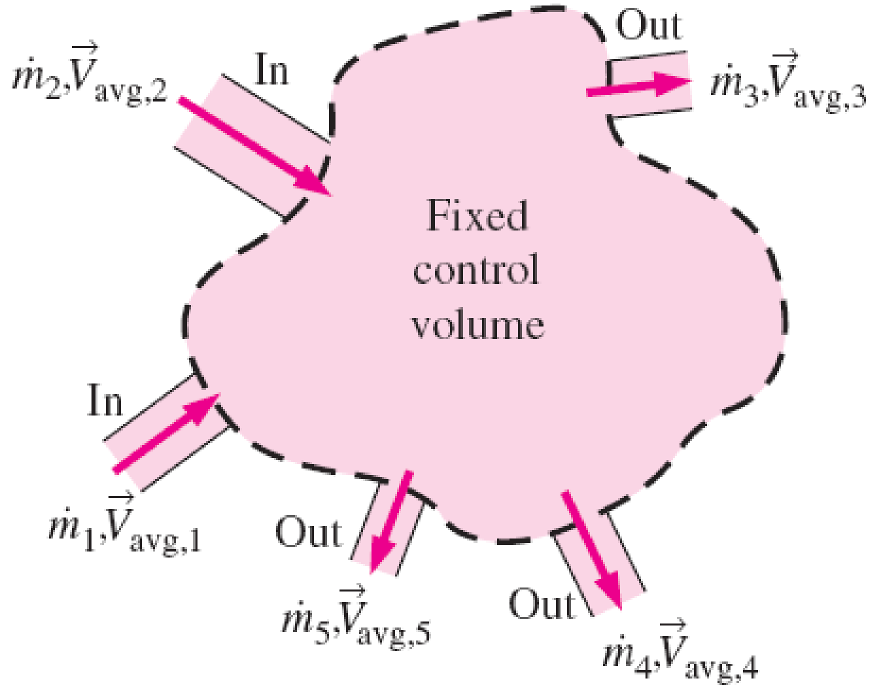
- Use RTT with $b = \vec{V}$ and $B = m\vec{V}$ to shift from system formulation to the control volume formulation

$$\frac{d\left(m\vec{V}\right)_{sys}}{dt} = \frac{d}{dt}\int_{CV}\rho\vec{V}\,d\mathbb{V} + \int_{CS}\rho\vec{V}\left(\vec{V}_r\cdot\vec{n}\right)\,dA$$

since $\sum\vec{F} = \dfrac{d\left(m\vec{V}\right)_{sys}}{dt}$

$$\sum\vec{F} = \frac{d}{dt}\int_{CV}\rho\vec{V}\,d\mathbb{V} + \int_{CS}\rho\vec{V}\left(\vec{V}_r\cdot\vec{n}\right)\,dA$$

- During *steady flow*, the amount of momentum within the control volume remains constant. The linear momentum equation becomes



$$\sum \vec{F} = \int_{CS} \rho \vec{V} \, (\vec{V}_r \cdot \vec{n}) \, dA$$

Momentum flow rate across a uniform inlet or outlet in algebraic form:

$$\int_{A_c} \rho \vec{V} (\vec{V} \cdot \vec{n}) \, dA_c = \rho V_{avg} \, A_c \vec{V}_{avg} = \dot{m} \vec{V}_{avg}$$

Momentum-Flux Correction Factor, β

- Since the velocity across most inlets and outlets is *not* uniform, the momentum-flux correction factor, β, is used to patch-up the error in the algebraic form equation. Therefore,

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}$$

Momentum flux across an inlet or outlet:

$$\int_{A_c} \rho \vec{V}(\vec{V}\cdot\vec{n})\, dA_c = \beta \dot{m} \vec{V}_{avg}$$

Momentum-flux correction factor:

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{avg}}\right)^2 dA_c$$

EXAMPLE: Momentum-Flux Correction Factor for Laminar Pipe Flow

Solution:

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{\text{avg}}}\right)^2 dA_c = \frac{4}{\pi R^2}\int_0^R \left(1 - \frac{r^2}{R^2}\right)^2 2\pi r\, dr$$

$$\beta = -4\int_1^0 y^2\, dy = -4\left[\frac{y^3}{3}\right]_1^0 = \frac{4}{3}$$

Note: For turbulent flow β may have an insignificant effect at inlets and outlets, but for laminar flow β may be important and should not be neglected.

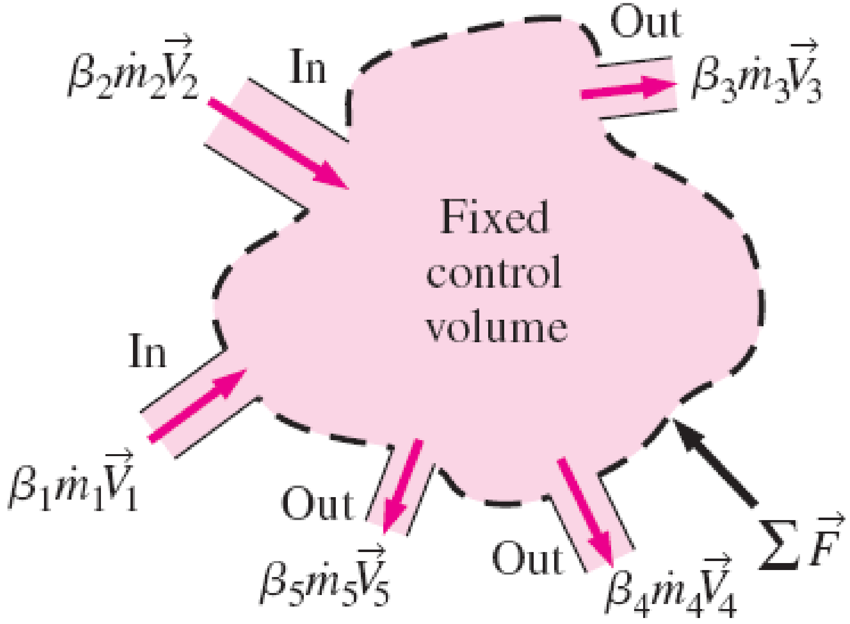
Steady linear momentum equation

- *The net force acting on the control volume during steady flow is equal to the difference between the rates of outgoing and incoming momentum flows*. Therefore,

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

One inlet and one outlet:

$$\sum \vec{F} = \dot{m} \, (\beta_2 \vec{V}_2 - \beta_1 \vec{V}_1)$$



Flow with No External Forces

- This is a common situation for space vehicles and satellites.
- For a control volume with multiple inlets and outlets, the linear momentum equation is

$$0 = \frac{d(m\vec{V})_{\text{CV}}}{dt} + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

- This is an expression of the conservation of momentum principle.
- If *m* remains nearly constant, then

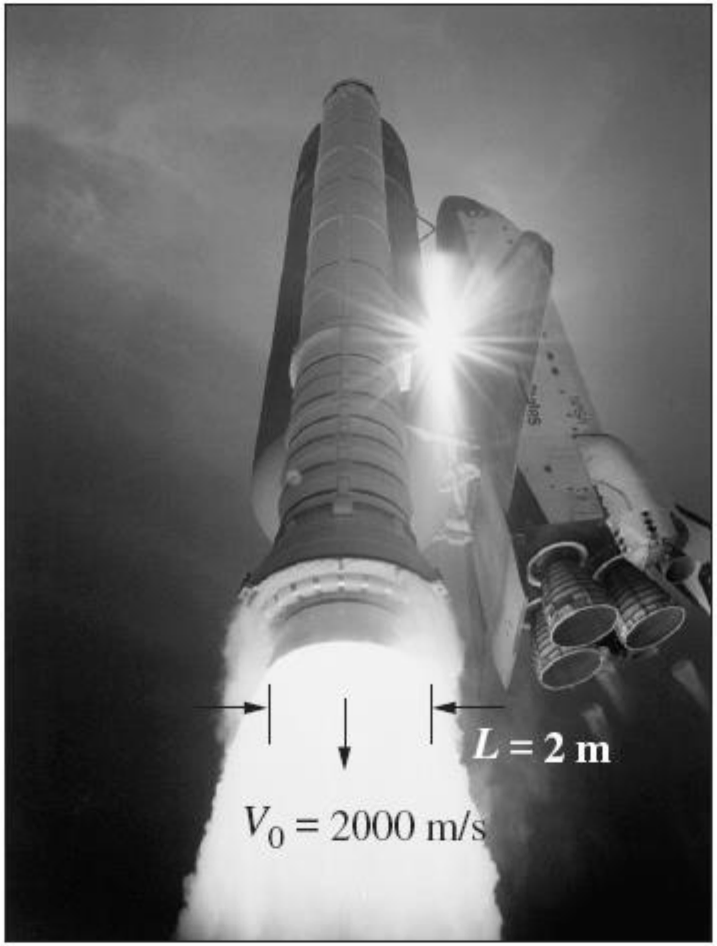
$$\frac{d(m\vec{V})_{\text{CV}}}{dt} = m_{\text{CV}} \frac{d\vec{V}_{\text{CV}}}{dt} = (m\vec{a})_{\text{CV}}$$

Flow with No External Forces

- In this case, the control volume can be treated as a solid body, with a **thrust** of

$$\vec{F}_{\text{body}} = m_{\text{body}}\vec{a} = \sum_{\text{in}} \beta \dot{m} \vec{V} - \sum_{\text{out}} \beta \dot{m} \vec{V}$$

- This approach can be used to determine the linear acceleration of space vehicles when a rocket is fired.



EXAMPLE: The Force to Hold a Reversing Elbow in Place

Solution: The vertical component of the anchoring force at the connection of the elbow to the pipe is zero, since weight is neglected. Only the ***F**$_{Rx}$* is considered.

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(0.0113 \text{ m}^2)} = 1.24 \text{ m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(7 \times 10^{-4} \text{ m}^2)} = 20.0 \text{ m/s}$$

$$P_1 - P_2 = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2 - z_1\right)$$

$$P_{1,\text{ gage}} = 202.2 \text{ kN/m}^2 = \mathbf{202.2\ kPa}$$

$$F_{Rx} + P_{1,\text{ gage}}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

$$F_{Rx} = -\beta\dot{m}(V_2 + V_1) - P_{1,\text{ gage}}A_1 = \mathbf{-2591\ N}$$

Where β is 1.03.

EXAMPLE: Repositioning of a Satellite

Solution:

$$0 = \frac{d(m\vec{V})_{\mathrm{CV}}}{dt} + \sum_{\mathrm{out}} \beta \dot{m} \vec{V} - \sum_{\mathrm{in}} \beta \dot{m} \vec{V} \quad \rightarrow \quad m_{\mathrm{sat}} \frac{d\vec{V}_{\mathrm{sat}}}{dt} = -\dot{m}_f \vec{V}_f$$

$$\frac{dV_{\mathrm{sat}}}{dt} = \frac{\dot{m}_f}{m_{\mathrm{sat}}} V_f = \frac{m_f/\Delta t}{m_{\mathrm{sat}}} V_f$$

$$a_{\mathrm{sat}} = \frac{dV_{\mathrm{sat}}}{dt} = \mathbf{30\ m/s^2}$$

$$dV_{\mathrm{sat}} = a_{\mathrm{sat}}\, dt \quad \rightarrow \quad \Delta V_{\mathrm{sat}} = a_{\mathrm{sat}}\, \Delta t = \mathbf{60\ m/s}$$

The thrust exerted on the satellite is

$$F_{\mathrm{sat}} = 0 - \dot{m}_f(-V_f) = \mathbf{150\ kN}$$

REVIEW OF ROTATIONAL MOTION AND ANGULAR MOMENTUM

- The motion of a rigid body:

  (Translation of + Rotation about) the center of mass.
- The translational motion can be analyzed using the linear momentum equation.
- Rotational motion is described with angular quantities such as the angular distance θ, angular velocity ϖ, and angular acceleration α.

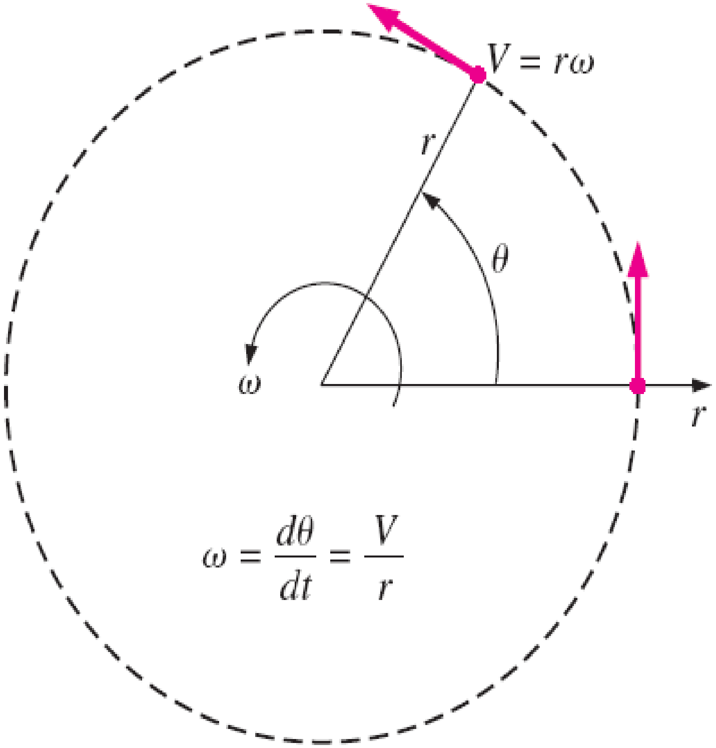
REVIEW OF ROTATIONAL MOTION AND ANGULAR MOMENTUM

$$l = \theta r$$

$$\omega = \frac{d\theta}{dt} = \frac{d(l/r)}{dt} = \frac{1}{r}\frac{dl}{dt} = \frac{V}{r}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{dV}{dt} = \frac{a_t}{r}$$

$V = r\omega$ and $a_t = r\alpha$



where V is the linear velocity and a_t is the linear acceleration in the tangential direction for a point located at a distance r from the axis of rotation.

1 rad = $360/(2\pi) \cong 57.3°$

Moment or Torque

- Newton's second law requires that there must be a force acting in the tangential direction to cause angular acceleration.
- The strength of the rotating effect, called the *moment* or *torque*, is proportional to the magnitude of the force and its distance from the axis of rotation.
- The perpendicular distance from the axis of rotation to the line of action of the force is called the *moment arm.*
- The torque *M* acting on a point mass *m* at a normal distance *r* from the axis of rotation is expressed as

$$M = rF_t = rma_t = mr^2\alpha$$

Moment or Torque

- The total torque acting on a rotating rigid body about an axis can be determined by

$$M = \int_{\text{mass}} r^2 \alpha \, dm = \left[\int_{\text{mass}} r^2 \, dm \right] \alpha = I\alpha$$

  - where *I* is the *moment of inertia* of the body about the axis of rotation, which is a measure of the inertia of a body against rotation.
  - Note that unlike mass, the rotational inertia of a body also depends on the distribution of the mass of the body with respect to the axis of rotation.

Analogy between corresponding linear and angular quantities.

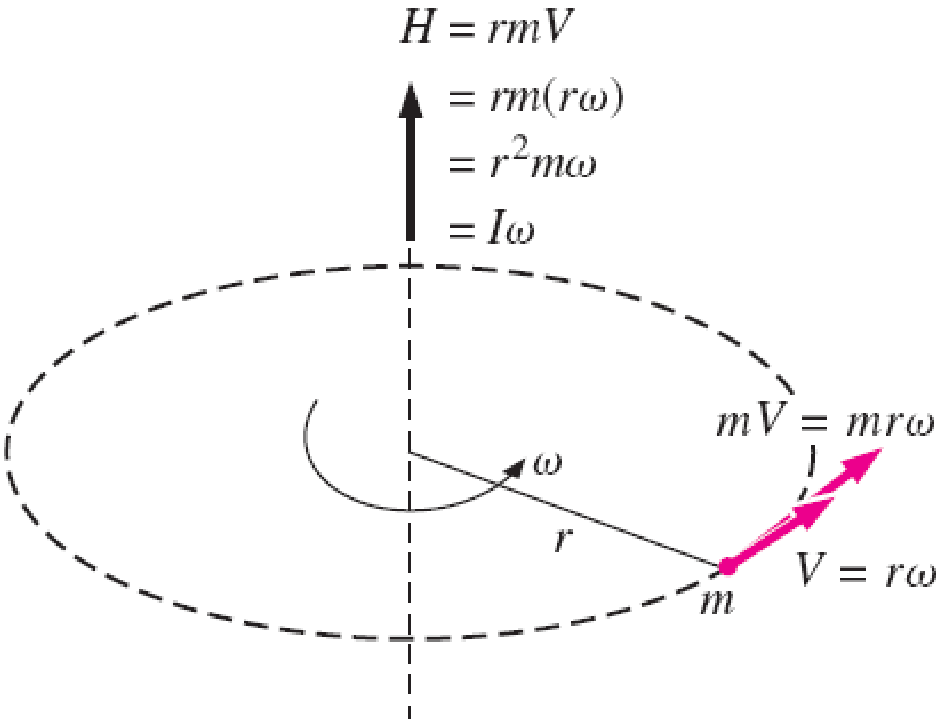
Mass, m	$\longleftrightarrow$	Moment of inertia, I
Linear acceleration, $\vec{a}$	$\longleftrightarrow$	Angular acceleration, $\vec{\alpha}$
Linear velocity, $\vec{V}$	$\longleftrightarrow$	Angular velocity, $\vec{\omega}$
Linear momentum	$\longleftrightarrow$	Angular momentum
$m\vec{V}$	$\longleftrightarrow$	$I\vec{\omega}$
Force, $\vec{F}$	$\longleftrightarrow$	Torque, M
$\vec{F} = m\vec{a}$	$\longleftrightarrow$	$\vec{M} = I\vec{\alpha}$
Moment of force, $\vec{M}$	$\longleftrightarrow$	Moment of momentum, $\vec{H}$
$\vec{M} = \vec{r} \times \vec{F}$	$\longleftrightarrow$	$\vec{H} = \vec{r} \times m\vec{V}$

Angular momentum

- The moment of momentum, called the **angular momentum,** of a point mass *m* about an axis can be expressed as

$$H = rmV = r^2 m\omega$$

- the total angular momentum of a rotating rigid body can be determined by integration to be



$$H = \int_{\text{mass}} r^2\omega\, dm = \left[\int_{\text{mass}} r^2\, dm\right]\omega = I\omega$$

Angular momentum

- The vector form of angular momentum can be expressed as

$$\vec{H} = I\vec{\omega}$$

- Note that the angular velocity is the same at every point of a rigid body.
- The moment, the rate of change of angular momentum, is

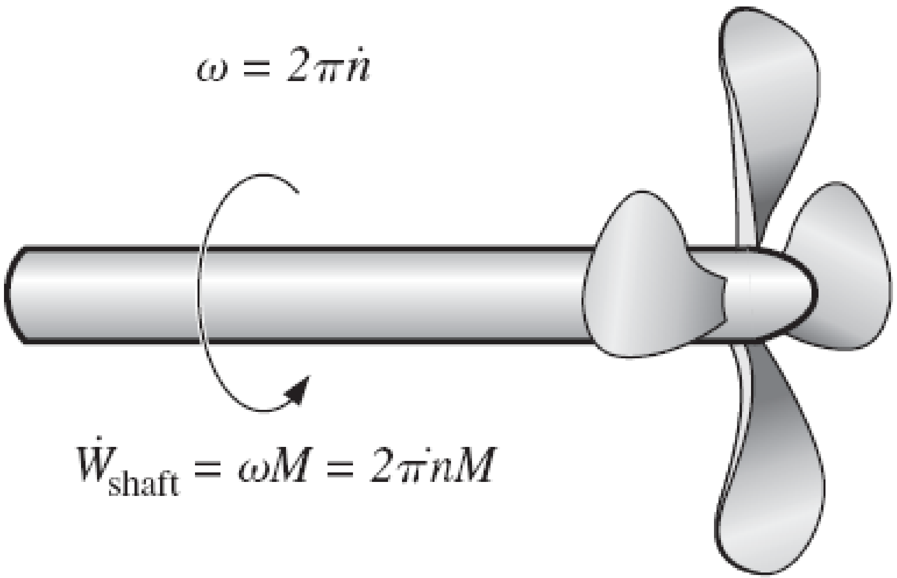
$$\vec{M} = I\vec{\alpha} = I\,\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt}$$

Shaft power

- The angular velocity of rotating machinery is typically expressed in rpm and denoted by $\dot{n}$.
- The angular velocity of rotating machinery is

$$\omega = 2\pi\dot{n} \quad \text{(rad/min)}$$

- The power transmitted by a shaft rotating at an rpm $\dot{n}$ of under the influence of an applied torque *M* is



$$FV = Fr\omega = \dot{W}_{shaft} = \omega M = 2\pi\dot{n}M \qquad \text{(W)}$$

Rotational kinetic energy

- The rotational kinetic energy of a body of mass *m* at a distance *r* from the axis of rotation is

$$\mathrm{KE} = \tfrac{1}{2} m r^2 \omega^2$$

- The total rotational kinetic energy of a rotating rigid body about an axis can be determined by

$$\mathrm{KE}_r = \tfrac{1}{2} I \omega^2$$

Centripetal acceleration and force

- During rotational motion, the direction of velocity changes even when its magnitude remains constant.
- The **centripetal acceleration** changes the direction of the velocity. Its magnitude is

$$a_r = \frac{V^2}{r} = r\omega^2$$

- Centripetal acceleration is directed toward the axis of rotation. The **centripetal force,** which induces the acceleration, is

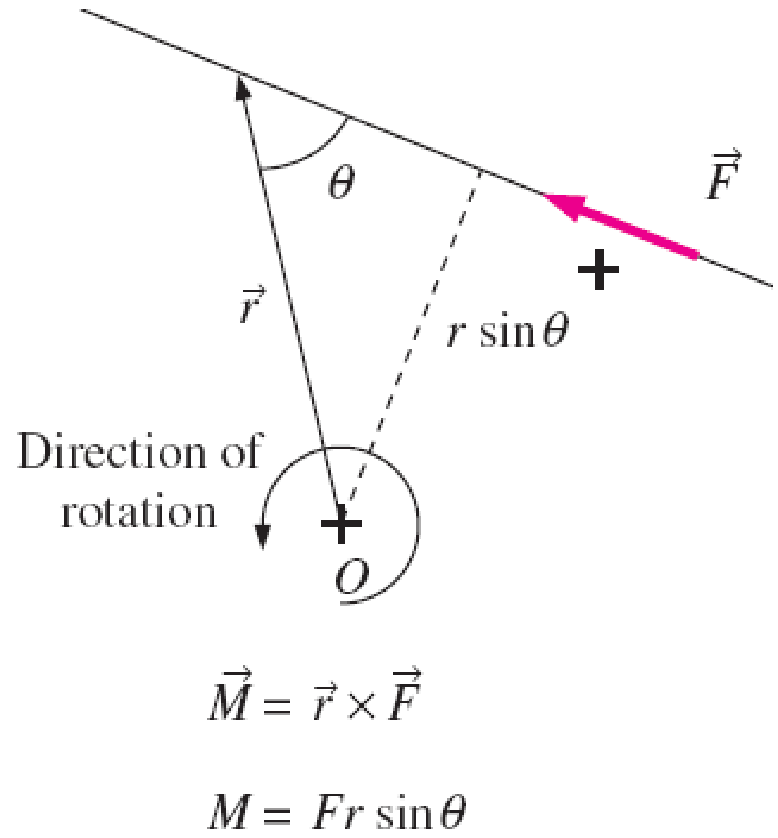
$$F_r = mV^2/r$$

- Tangential and radial accelerations are perpendicular to each other, and the total linear acceleration is determined by their vector sum, $\vec{a} = \vec{a}_t + \vec{a}_r$

THE ANGULAR MOMENTUM EQUATION

- Many engineering problems involve the moment of the linear momentum of flow streams, and the rotational effects caused by them, which are best analyzed by the *angular momentum equation,*
- The *moment of a force* $\vec{F}$ about a point *O* is the vector (or cross) product*.*

$$\vec{M} = \vec{r} \times \vec{F}$$



- Whose magnitude is $M = Fr\sin\theta$

THE ANGULAR MOMENTUM EQUATION

- The sense of the moment vector $\vec{M}$ is determined by the right-hand rule
- Replacing the vector $\vec{F}$ by the momentum vector $m\vec{V}$ gives the *moment of momentum*, also called the *angular momentum*

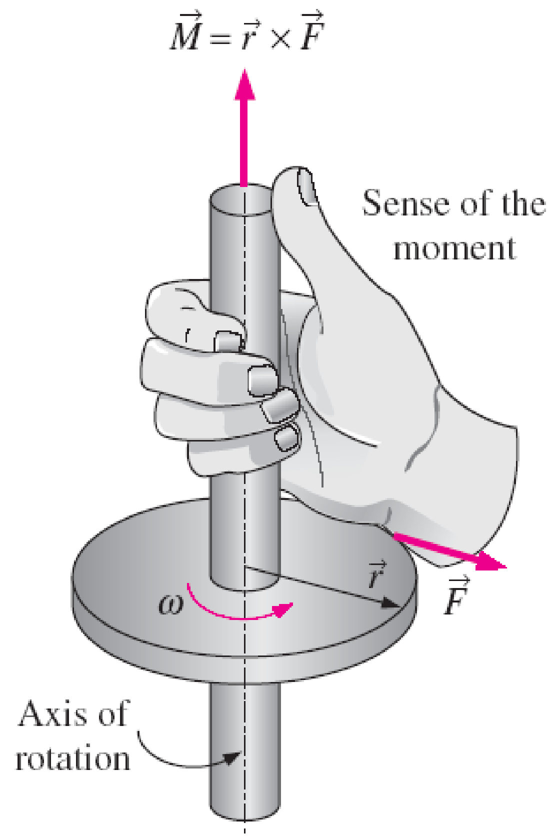
$$\vec{H} = \vec{r} \times m\vec{V}$$

- The angular momentum of a differential mass *dm* is

$$(\vec{r} \times \vec{V})\rho\, dV$$

- *Moment of momentum* (*system*):

$$\vec{H}_{\text{sys}} = \int_{\text{sys}} (\vec{r} \times \vec{V})\rho\, dV$$



THE ANGULAR MOMENTUM EQUATION

- *Rate of change of moment of momentum*:

$$\frac{d\vec{H}_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{sys}} (\vec{r} \times \vec{V})\rho\, dV$$

- *The rate of change of angular momentum of a system is equal to the net torque acting on the system* (valid for a fixed quantity of mass and an inertial reference frame).

$$\sum \vec{M} = \frac{d\vec{H}_{\text{sys}}}{dt}$$

THE ANGULAR MOMENTUM EQUATION

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b (\vec{V}_r \cdot \vec{n})\, dA$$

$$B = \vec{H} \qquad b = \vec{r} \times \vec{V} \qquad b = \vec{r} \times \vec{V}$$

$$\frac{dH_{sys}}{dt} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

Applying the RTT

General: $$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

Fixed CV: $$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V} \cdot \vec{n})\, dA$$

THE ANGULAR MOMENTUM EQUATION - Special Cases

- *Steady Flow*: $$\sum \vec{M} = \int_{\mathrm{CS}} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$
- In many practical applications, an approximate form of the angular momentum equation in terms of average properties at inlets and outlets becomes

$$\sum \vec{M} = \frac{d}{dt}\int_{\mathrm{CV}} (\vec{r} \times \vec{V})\rho\, dV + \sum_{\mathrm{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\mathrm{in}} \vec{r} \times \dot{m}\vec{V}$$

- No correction factor is introduced since it varies from problem to problem and the induced error is small.
- Steady Flow $$\sum \vec{M} = \sum_{\mathrm{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\mathrm{in}} \vec{r} \times \dot{m}\vec{V}$$

Flow with No External Moments

- When there are no external moments applied, the angular momentum equation reduces to

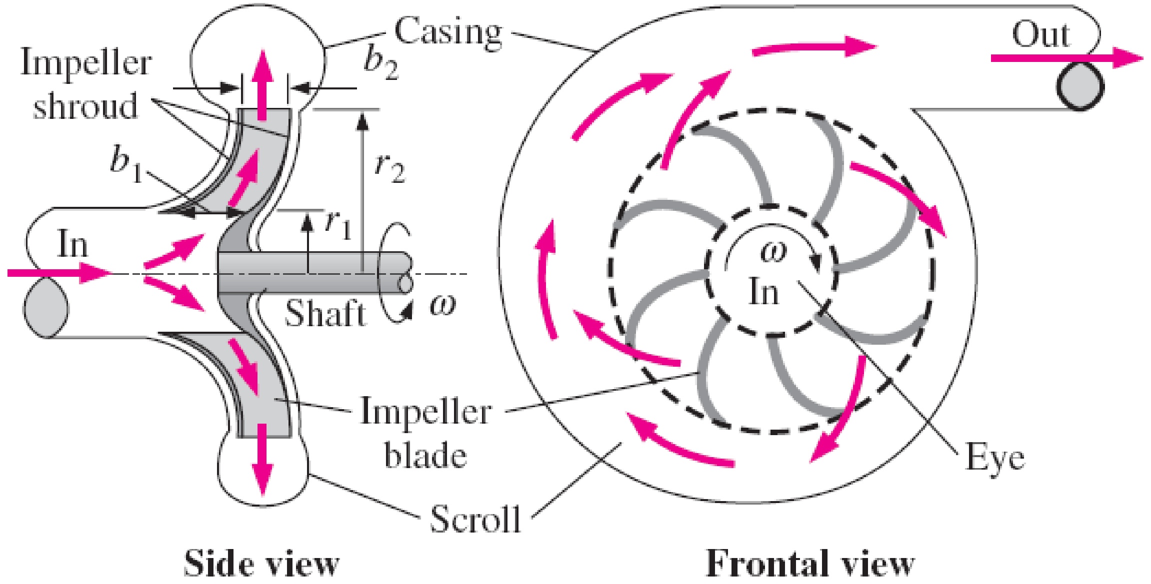
$$0 = \frac{d\vec{H}_{CV}}{dt} + \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V}$$

- When the moment of inertia *I* of the control volume remains constant, then

$$\vec{M}_{\text{body}} = I_{\text{body}}\vec{\alpha} = \sum_{\text{in}} (\vec{r} \times \dot{m}\vec{V}) - \sum_{\text{out}} (\vec{r} \times \dot{m}\vec{V})$$

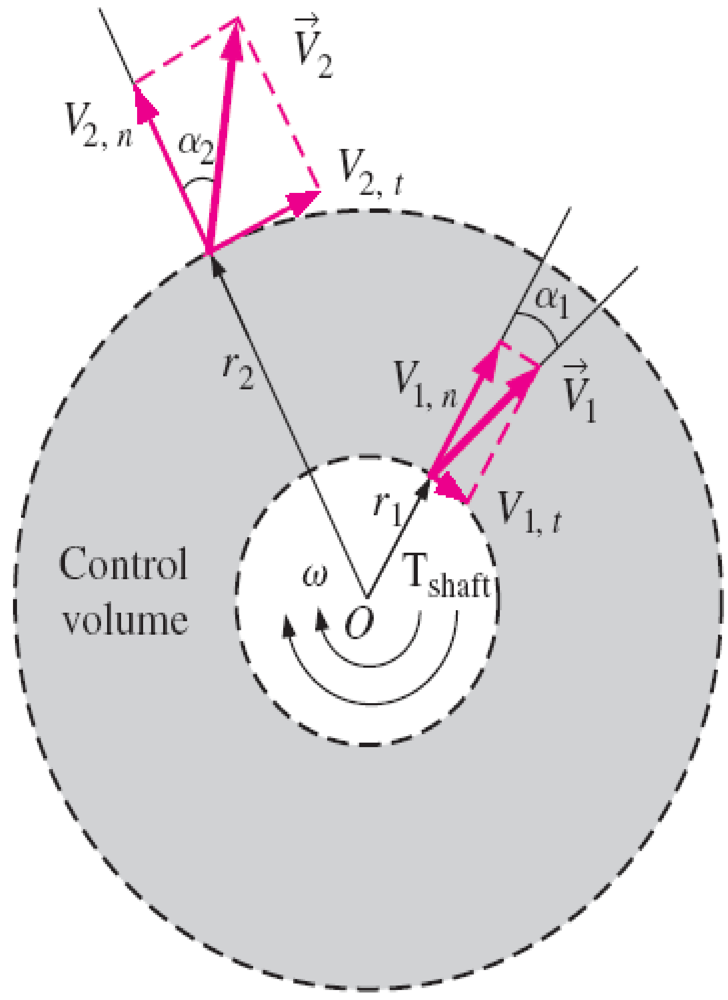
Radial-Flow Devices

- Flow in the radial direction normal to the axis of rotation and are called *radial flow devices.*
- In a centrifugal pump, the fluid enters the device in the axial direction through the eye of the impeller, and is discharged in the tangential direction.



Radial-Flow Devices

- Consider a centrifugal pump. The impeller section is enclosed in the control volume.
- The average flow velocity, in general, has normal and tangential components at both the inlet and the outlet of the impeller section.
- when the shaft rotates at an angular velocity of ϖ, the impeller blades have a tangential velocity of ϖr_1 at the inlet and ϖr_2 at the outlet.



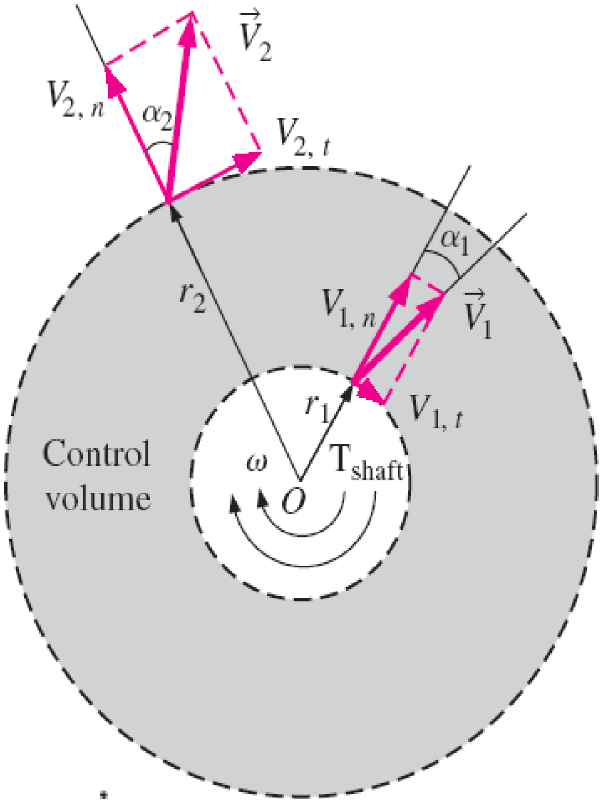
Radial-Flow Devices

- The conservation of mass equation tells

$$\dot{V}_1 = \dot{V}_2 = \dot{V} \qquad \rightarrow$$

$$(2\pi r_1 b_1)V_{1,n} = (2\pi r_2 b_2)V_{2,n}$$

- where b_1 and b_2 are the flow widths at the inlet and outlet.
- Then the average normal components are



$$V_{1,n} = \frac{\dot{V}}{2\pi r_1 b_1} \qquad \text{and} \qquad V_{2,n} = \frac{\dot{V}}{2\pi r_2 b_2}$$

Radial-Flow Devices

- The normal velocity components and pressure act through the shaft center and contribute no torque. Only the tangential velocity components contribute to the angular momentum equation, which gives the famous **Euler's turbine formula.**

$$\mathrm{T}_{\text{shaft}} = \dot{m}(r_2 V_{2,t} - r_1 V_{1,t})$$

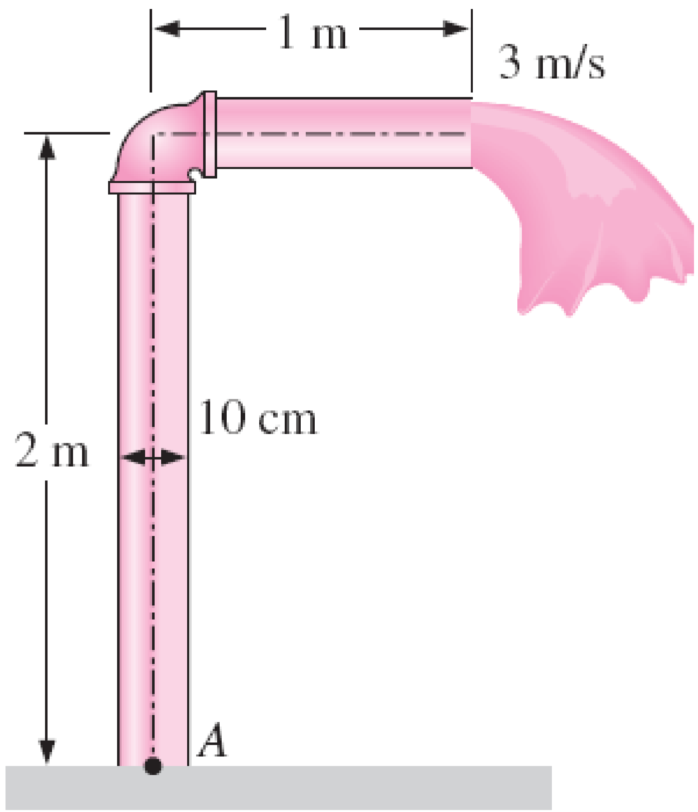
$$\mathrm{T}_{\text{shaft}} = \dot{m}(r_2 V_2 \sin\alpha_2 - r_1 V_1 \sin\alpha_1)$$

- In the idealized case, $\mathrm{T}_{\text{shaft, ideal}} = \dot{m}\omega(r_2^2 - r_1^2)$
- The shaft power

$$\dot{W}_{\text{shaft}} = \omega \mathrm{T}_{\text{shaft}} = 2\pi\dot{n}\mathrm{T}_{\text{shaft}}$$

EXAMPLE: Bending Moment Acting at the Base of a Water Pipe

- Underground water is pumped to a sufficient height through a 10-cm diameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point *A*) and the required length of the horizontal section that would make the moment at point *A* zero.



EXAMPLE: Bending Moment Acting at the Base of a Water Pipe

- Solution

since A_c = constant

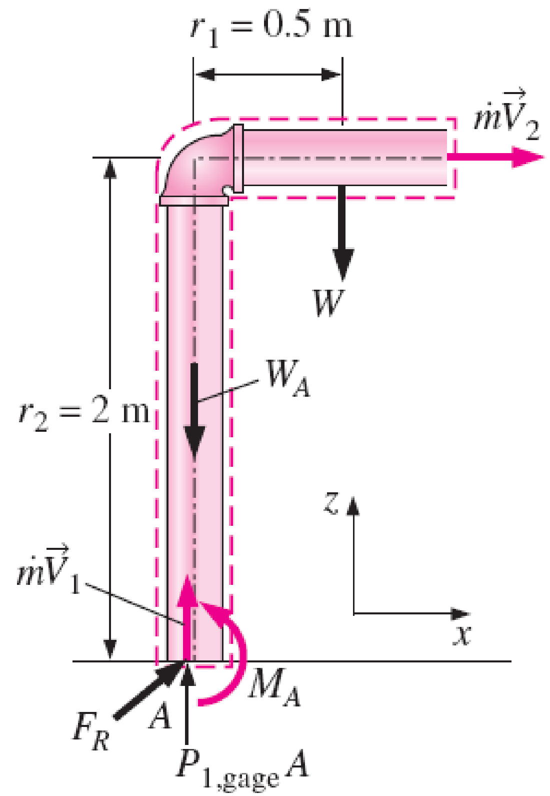
Conservation of mass gives

$$\dot{m}_1 = \dot{m}_2 = \dot{m}, \text{ and } V_1 = V_2 = V$$

Therefore, we can get

$$\dot{m} = \rho A_c V = 23.56 \text{ kg/s}$$

$$W = mg = 118 \text{ N}$$



Then the angular momentum equation about point *A* becomes

$$M_A - r_1 W = -r_2 \dot{m} V_2$$

EXAMPLE: Bending Moment Acting at the Base of a Water Pipe

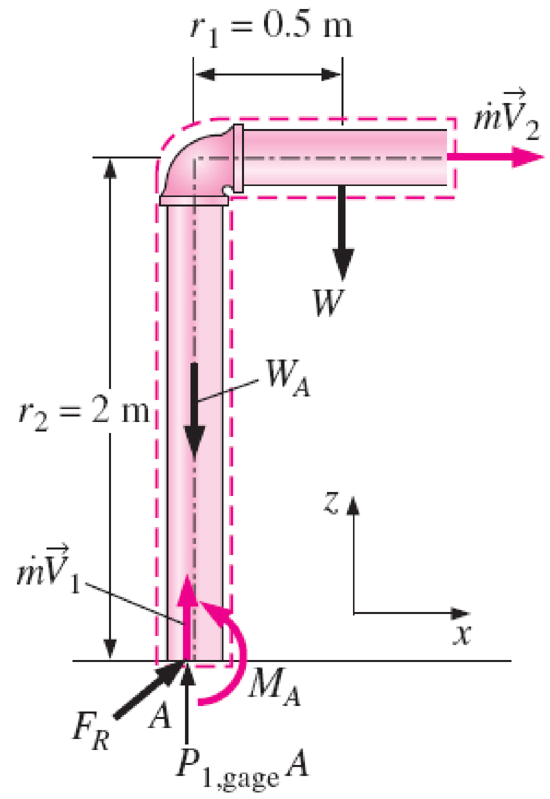
Therefore,

$$M_A = r_1 W - r_2 \dot{m} V_2 = -82.5 \text{ N} \cdot \text{m}$$

Setting $M_A = 0$, then we can get

$$0 = r_1 W - r_2 \dot{m} V_2$$

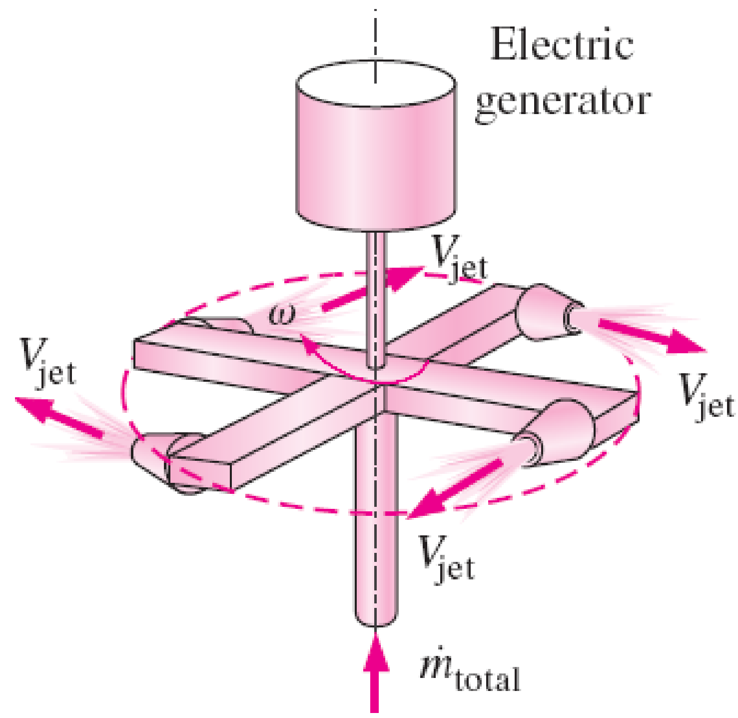
$$\rightarrow \qquad 0 = (L/2)Lw - r_2 \dot{m} V_2$$



$$L = \sqrt{\frac{2 r_2 \dot{m} V_2}{w}} = \sqrt{\frac{2 \times 141.4 \text{ N} \cdot \text{m}}{118 \text{ N/m}}} = 2.40 \text{ m}$$

EXAMPLE: Power Generation from a Sprinkler System

- A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.



EXAMPLE: Power Generation from a Sprinkler System

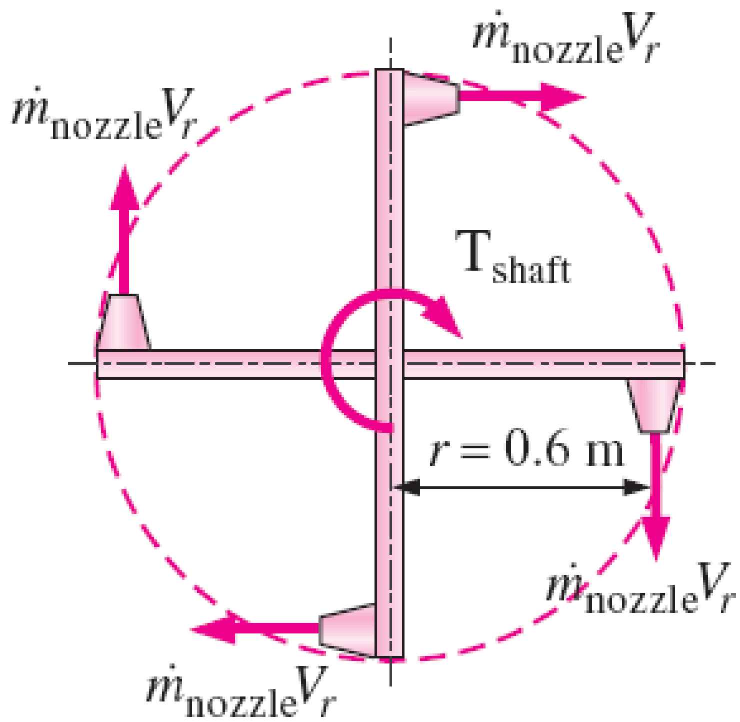
- Solution

$$\dot{V}_{nozzle} = \dot{V}_{total}/4 = 5 \text{ L/s}$$

$$V_{jet} = \frac{\dot{V}_{nozzle}}{A_{jet}} = 63.66 \text{ m/s}$$

$$\omega = 2\pi\dot{n} = 31.42 \text{ rad/s}$$

$$V_{nozzle} = r\omega = 18.85 \text{ m/s}$$



$$V_r = V_{jet} - V_{nozzle} = 63.66 - 18.85 = 44.81 \text{ m/s}$$

$$-T_{shaft} = -4r\dot{m}_{nozzle}V_r \quad \text{or} \quad T_{shaft} = r\dot{m}_{total}V_r$$

EXAMPLE: Power Generation from a Sprinkler System

$$T_{shaft} = 537.7\ \text{N} \cdot \text{m} \qquad \dot{W} = 2\pi\dot{n}T_{shaft} = \omega T_{shaft} = \mathbf{16.9\ kW}$$

- Discussion of two limiting cases