Chapter 5 Series Solutions of Differential Equations

5.1 Power series method

Standard basic method for solving linear differential equations with variable coefficients

Power series in power of $(x-x_o)$:

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\cdots$$

in particular, when $x_o = 0 \rightarrow$ power series in power of x

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

※ Idea of the power series method

Consider $y'' + p(x)y' + q(x)y = 0 ------(1)$

Firstly expand $p(x)$ and $q(x)$ in power series of x (or $x - x_o$)

Then assume the solution $y(x)$ in the power series form :

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots\cdots ------(2)$$

$$\Rightarrow \left\{ \begin{aligned} y' &= \sum_{m=1}^{\infty} a_m\, m\, x^{m-1} \\ y'' &= \sum_{m=2}^{\infty} a_m\, m(m-1)\, x^{m-2} \end{aligned} \right\} ---------(3)$$

substitute (2), (3) into the equation (1) $\Rightarrow$ compare the coefficient of the same power of x $\Rightarrow$ the unknown coefficients a_m's can be determined.

Example: $y' = 2xy$

assume $y = \sum_{m=0}^{\infty} a_m x^m$

then $y' = \sum_{m=1}^{\infty} a_m\, m\, x^{m-1}$

the equation becomes:

$$\sum_{m=1}^{\infty} a_m\, m\, x^{m-1} = 2x \cdot \sum_{m=0}^{\infty} a_m \cdot x^m$$

$$\Rightarrow \sum_{m=1}^{\infty} a_m\, m\, x^{m-1} = \sum_{m=0}^{\infty} 2 \cdot a_m \cdot x^{m+1}$$

$$\Rightarrow \sum_{m=0}^{\infty} a_{m+1} \cdot (m+1) \cdot x^m = \sum_{m=1}^{\infty} 2 \cdot a_{m-1} \cdot x^m$$

$$\Rightarrow a_1 + \sum_{m=1}^{\infty} \left[a_{m+1} \cdot (m+1) - 2 \cdot a_{m-1}\right] x^m = 0$$

$$\Rightarrow \begin{cases} a_1 = 0 \\ a_{m+1} = \dfrac{2}{(m+1)} \cdot a_{m-1} \qquad \text{for } \ m = 1,2,3,4... \end{cases}$$

$$\Rightarrow$$

$$a_1 = a_3 = a_5 = \cdots = 0$$

$$\begin{cases} a_2 = \dfrac{2}{2} a_0 = a_0 \\ a_4 = \dfrac{2}{4} a_2 = \dfrac{1}{2!} a_0 \\ a_6 = \dfrac{2}{6} a_4 = \dfrac{1}{3!} a_0 ,.... \end{cases} .$$

$$\therefore y = (a_0 + a_0 x^2 + \frac{a_0}{2!} x^4 + \frac{a_0}{3!} x^6 + \cdots)$$

$$= a_0 (1 + x^2 + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \cdots)$$

Where a_0 is any arbitrary constant

The exact solution is $y = c \cdot e^{x^2}$

5.2 Theory of the power series method

Consider a function $S(x)$, when expressed by a power series centered at x_0 :

$$S(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

$$\equiv a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots --- (1)$$

let

$$S_n(x) \equiv a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n$$

→n terms partial sum

$$R_n(x) \equiv a_{n+1}(x-x_0)^{n+1} + a_{n+2}(x-x_0)^{n+2} + \cdots$$

→Remainder

If for some $x = x_1$, $\lim_{n \to \infty} S_n(x_1) = S(x_1)$

The series of (1) is said convergent at $x = x_1$. $S(x_1)$ is the value of series at x_1, i.e.

$$S(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$$

$S(x_1) = S_n(x_1) + R_n(x_1)$ $\Leftrightarrow$ for any positive ε, there is an number N, such that

$$|R_n(x_1)| = |S(x_1) - S_n(x_1)| < \varepsilon \quad \text{for all } n > N$$

※ **Radius of Convergence:**

1. At $x = x_0$, (1) always converges, since only a_0 exists.
2. For $x \neq x_0$,If there is some value R such that the series (1) converge for all x inside the interval $|x - x_0| < R$, R is called the " radius of convergence " of this series.

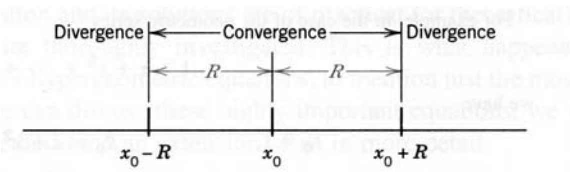


Fig. 100. Convergence interval (6) of a power series with center x_0

※ **Existence of power series solutions**

Definition: Analytic function

A function $f(x)$ is said to be analytic at point $x = x_0$ if it can be represented by a power series in power of $(x - x_0)$ with radius of convergence $R > 0$.

Consider $\quad y'' + p(x)y' + q(x)y = r(x) ------(2)$

And $\quad \tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = \tilde{r}(x) ------(2A)$

Thm: If $p(x), q(x), r(x)$ in (2) are analytic at $x = x_0$, then every solution of (2) is analytic at $x = x_0$ i.e. it can be represented by a power series in power of $(x - x_0)$ with radius of convergence $R > 0$. The same is true for (2A) provided that $\tilde{h}(x_0) \neq 0$.

5.3 Legendre's Equation

Standard form: $(1 - x^2)y'' - 2xy' + n(n+1)y = 0 ------(1)$

Where n is a given real number(parameter)

Since $h(x) \equiv 1 - x^2$; $p(x) \equiv -2x$; $q(x) \equiv n(n+1)$; all are analytic at $x = 0$, and $h(0) = 1 \neq 0$

$\rightarrow$ the solution of (1) is analytic at $x = 0$, i.e. it can be expressed by a power series centered at $x = 0$.

try $\quad y = \sum_{m=0}^{\infty} a_m x^m$,

when $n(n+1)$ is replaced by k, (1) becomes:

$$(1 - x^2) \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \underbrace{\sum_{m=2}^{\infty}}_{m \to m+2} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2 \sum_{m=1}^{\infty} m a_m x^m + k \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2 \sum_{m=1}^{\infty} m a_m x^m + k \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow (2a_2 + ka_0) + (6a_3 - 2a_1 + ka_1)x + \sum_{m=2}^{\infty}[(m+2)(m+1)a_{m+2} - m(m-1)a_m - 2ma_m + ka_m]x^m = 0$$

$$\Rightarrow \begin{cases} x^0: \quad 2a_2 + ka_0 = 0 ----------(2) \\ x^1: \quad 6a_3 - 2a_1 + ka_1 = 0 --------(3) \\ x^m: \quad (m+2)(m+1)a_{m+2} - m(m-1)a_m - 2ma_m + n(n+1)a_m = 0 \\ a_{m+2} = -\dfrac{(n-m)(m+n+1)}{(m+2)(m+1)}a_m \qquad \text{m} = 2,3,4\ldots ---(4) \end{cases}$$

$$(2) \Rightarrow \quad a_2 = -\frac{n(n+1)}{2}a_0,$$

$$(3) \Rightarrow \quad a_3 = \frac{2-n(n+1)}{6}a_1 = -\frac{(n+2)(n-1)}{3!}a_1$$

$$(4) \Rightarrow \quad a_4 = -\frac{(n-2)(n+3)}{4\cdot 3}a_2 = \frac{(n-2)\,n\,(n+1)\,(n+3)}{4\,!}a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{5\cdot 4}a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5\,!}a_1$$

$$\ldots\ldots\ldots$$

$$\Rightarrow \quad y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2\,!}x^2 + \frac{(n-2)\,n\,(n+1)\,(n+3)}{4\,!}x^4 + \cdots ---(5)$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3\,!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5\,!}x^5 + \cdots --(6)$$

since at $x = \pm 1 \quad \rightarrow h(x) = 0$, the solution is not analytic at $x = \pm 1$,

i.e. The solution converges for $|x| < 1$

If n is a non-negative integer, (4)→ $a_{n+2} = a_{n+4} = \ldots = 0$

Then one of $y_1(x)$ or $y_2(x)$ reduces to a polynomial of finite terms, this polynomials is called the Legendre's polynomials.

Legendre's polynomials:

If n is nonnegative integer and when $m = n+2$, right hand side of (4) is zero →

when n is even → y_1 reduces to an even power polynomial of x of degree n

when n is odd → y_2 reduces to an odd power polynomial of x of degree n

these polynomials are called **Legendre's polynomials** and denoted by $P_n(x)$.

since a_0(and a_1) in $P_n(x)$ are arbitrary constant

if we choose a_0, a_1 such that :

$$n: even \quad a_0 = (-1)^{\frac{n}{2}} \frac{n!}{2^n \left(\frac{n}{2}!\right)^2} ------(7)$$

$$n: odd \quad a_1 = (-1)^{\frac{n-1}{2}} \frac{(n+1)!}{2^n \left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!} ------(8)$$

then the **Legendre's polynomials** has the property $P_n(1) = 1$

Thus we have the standard form of $P_n(x)$:

$$\begin{cases} n = 0 & y_1(x) = 1 = P_0(x) \\ n = 1 & y_2(x) = x = P_1(x) \\ n = 2 & y_1(x) = \frac{1}{2}(3x^2 - 1) = P_2(x) \\ n = 3 & y_2(x) = \frac{1}{2}(5x^3 - 3x) = P_3(x) \\ n = 4 & y_1(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) = P_4(x) \\ \vdots & \vdots \end{cases}$$

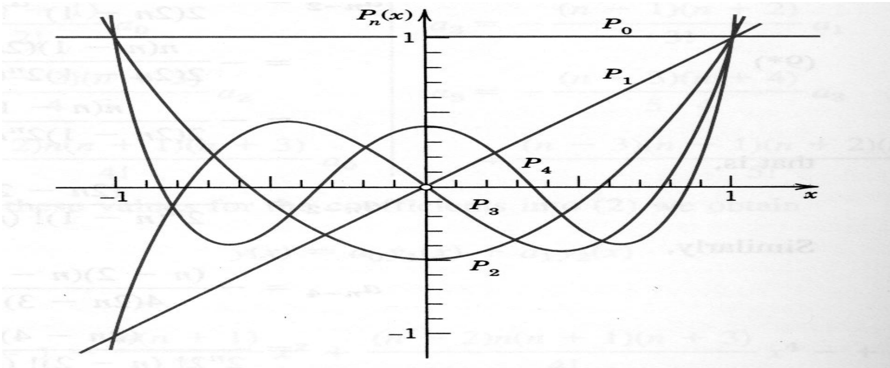


Fig. 101. Legendre polynomials

From the definition of (7),(8) and (4) $\Rightarrow$ the coefficient highest power of x in each series is equal to $\dfrac{(2n)!}{2^n (n!)^2}$

Since

$$m = n, \quad \Rightarrow a_n = (-1)^{\frac{n}{2}} \frac{2\cdot 4\cdot 6\cdots(n-4)(n-2)n(n+1)(n+3)\cdots(n+n-1)}{n!} a_0$$

$$= (-1)^{\frac{n}{2}} \frac{2\cdot 4\cdot 6\cdots(n-4)(n-2)n(n+1)(n+3)\cdots(n+n-1)}{n!} \cdot \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}(-1)^{\frac{n}{2}}$$

$$= \frac{1\cdot 3\cdot 5\cdots(n-1)(n+1)(n+3)\cdots(n+n-1)}{n!} \times \frac{2\cdot 4\cdot 6\cdots(n+n)}{2\cdot 4\cdot 6\cdots(n+n)}$$

$$= \frac{(2n)!}{n! \quad (2\cdot 4\cdot 6\cdots 2n)} = \frac{(2n)!}{n! \quad 2^n (n)!}$$

$$\therefore \quad a_n = \frac{(2n)!}{2^n (n!)^2} ----------(9)$$

From (4) if let $m = n - 2$

$$\Rightarrow \quad a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = -\frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2}$$

$$= -\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)!\, n(n-1)(n-2)!}$$

$$\therefore \quad a_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!\,(n-2)!}$$

similar $\Rightarrow \quad a_{n-4} = -\dfrac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$

$$\therefore \quad a_{n-4} = -\frac{(2n-4)!}{2^n\ 2!\,(n-2)!\,(n-4)!}$$

$$\therefore \quad a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n\ m!\,(n-m)!\,(n-2m)!}$$

the **Legendre's polynomials** can be expressed in the general form:

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n\ m!\,(n-m)!\,(n-2m)!}\, x^{n-2m}$$

$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n\ 1!(n-1)!(n-2)!} x^{n-2} + \cdots$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$

Properties of Legendre function

1. $P_n(x) = (-1)^n P_n(-x)$
2. $P_n(+1) = 1$
3. $P_n(-1) = (-1)^n$
4. $\begin{cases} P_n(0) = 0, & \text{n} = 1, 3, 5, \ldots.. \\ \frac{dP_n(x)}{dx}(0) = 0, & \text{n} = 2, 4, 6, \ldots \end{cases}$
5. Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

6. Another form of Legendre equation:

$$\sin\theta \frac{d^2 y}{d\,\theta^2} + \cos\theta \frac{d\,y}{d\theta} + n(n+1)\sin\theta\ y = 0$$

It can be transformed to $(1-x^2)y''-2xy'+n(n+1)y = 0$

by set $x = \cos\theta$

7. $\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} \frac{2}{2n+1} & \text{when m = n} \\ 0 & \text{when m} \neq \text{n} \end{cases}$

5.4 Frobenius method:

Consider the equation: $y''+p(x)y'+q(x)y=0$

If $p(x), q(x)$ are analytic at $x=x_0$

x_0 is called a regular point of the equation. Otherwise, it is called a singular point.

Theorem: (Frobenius method)

A second order linear differential equation of the form:

$$y''+\frac{b(x)}{x}y'+\frac{c(x)}{x^2}y=0 --------(1)$$

if the functions $b(x)$ and $c(x)$ are analytic at $x=0$, then eq.(1) has at least one solution that can be represented in form

$$y(x)=\sum_{m=0}^{\infty}a_m x^{m+r}=x^r(a_0+a_1x+a_2x^2+\cdots) ----- (2)$$

where r may be any number (real or complex or zero), which is chosen such that $a_0\neq 0$.

The equation has a second solution that may be similar to (2) (with a different r and different coefficient) or may contain a logarithmic term.

Rewite (1): $\Rightarrow$ $x^2\cdot y''+b(x)\cdot x\cdot y'+c(x)y=0 ----- (3)$

Expand $b(x)$ and $c(x)$ in power series of x:

$$b(x)=b_0+b_1x+b_2x^2+\cdots \qquad c(x)=c_0+c_1x+c_2x^2+\cdots$$

$$y'(x)=\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}=x^{r-1}[ra_0+(r+1)a_1x+\cdots]$$

$$y''(x)=\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}$$

$$=x^{r-2}[r(r-1)a_0+(r+1)r\,a_1x+\cdots]$$

then (3) becomes:

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r}+(b_0+b_1x+b_2x^2+...)\sum_{m=0}^{\infty}(m+r)a_m x^{m+r}$$

$$+(c_0+c_1x+c_2x^2+\cdots)\sum_{m=0}^{\infty}a_m x^{m+r}=0$$

since the coefficient of x^r should be zero, we have :

if we require $a_0 \neq 0$ →

$$\boxed{r^2 + (b_0 - 1)r + c_0 = 0}$$

→ Indicial equation of the differential equation

Solve this indicial equation, we have two roots of r, say r_1 and r_2.

Suppose r_1 is the larger (magnitude)one ,

Case I: $r_1 \neq r_2$ and $r_1 - r_2 \neq$ integer

There exist two linearly independent solutions of the form:

$$y_1(x) = \sum_{m=0}^{\infty} a_m x^{m+r_1} = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots), \quad a_0 \neq 0$$

$$y_2(x) = \sum_{m=0}^{\infty} A_m x^{m+r_2} = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots), \quad A_0 \neq 0$$

Case II: $r_1 = r_2$

There exist two linearly independent solutions of the form:

$$y_1(x) = \sum_{m=0}^{\infty} a_m x^{m+r_1} = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots), \quad a_0 \neq 0$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}(A_1 x + A_2 x^2 + \cdots)$$

Case III: $r_1 \neq r_2$ and $r_1 - r_2 =$ positive integer.

There exist two linearly independent solutions of the form:

$$y_1(x) = \sum_{m=0}^{\infty} a_m x^{m+r_1} = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots), \quad a_0 \neq 0$$

$$y_2(x) = k y_1(x)\ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots), \quad A_0 \neq 0$$

where k is a constant which may be zero.

Example 1: Euler–Cauchy equation

$$x^2 y'' + b_0 x y' + c_0 y = 0$$

Try $y = x^r \Rightarrow r(r-1) + b_0 r + c_0 = 0$

$$r^2 + (b_0 - 1)r + c_0 = 0$$

if $r_1 \neq r_2$ we get $y_1 = c_1 \cdot x^{r_1}, \; y_2 = c_2 \cdot x^{r_2}$

if $r_1 = r_2$ we get $y_1 = c_1 \cdot x^{r_1}, y_2 = c_2 \cdot x^{r_2} \ln x$

Example 2: (Double root) $x(x-1)y'' + (3x-1)y' + y = 0$

Try $y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$ substitute into the above equation

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}$$
$$+3\sum_{m=0}^{\infty}(m+r)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\Rightarrow \sum_{s=0}^{\infty}(s+r)(s+r-1)a_s x^{s+r} - \sum_{s=-1}^{\infty}(s+1+r)(s+r)a_{s+1} x^{s+r}$$
$$+3\sum_{s=0}^{\infty}(s+r)a_s x^{s+r} - \sum_{s=-1}^{\infty}(s+1+r)a_{s+1} x^{s+r} + \sum_{s=0}^{\infty} a_s x^{s+r} = 0$$

$$\Rightarrow \sum_{s=0}^{\infty}(s+r)(s+r-1)a_s x^{s+r} - \sum_{s=0}^{\infty}(s+r+1)(s+r)a_{s+1} x^{s+r}$$
$$+3\sum_{s=0}^{\infty}(s+r)a_s x^{s+r} - \sum_{s=0}^{\infty}(s+r+1)a_{s+1} x^{s+r} + \sum_{s=0}^{\infty} a_s x^{s+r}$$
$$+a_0[-r(r-1)-r]x^{r-1} = 0$$

The coefficient of the x^{r-1}: $\Rightarrow [-r(r-1)-r]a_0 = 0$

Since we require $a_0 \neq 0$, we have $[r(r-1)+r] = 0$

i.e. $r_1 = r_2 = 0$ (double root)

The coefficient of the the x^{s+r}, $s = 0,1,2,3.... \Rightarrow$

$$(s+r)(s+r-1)a_s - (s+r+1)(s+r)a_{s+1}$$
$$+3(s+r)a_s - (s+r+1)a_{s+1} + a_s = 0$$

since $r = r_1 = r_2 = 0$,

$$\Rightarrow s(s-1)a_s - (s+1)sa_{s+1} + 3sa_s - (s+1)a_{s+1} + a_s = 0$$

$$\Rightarrow (s+1)^2(a_{s+1} - a_s) = 0$$

since $(s+1) \neq 0$ for s=0.1.2.3.…→ we have

$$a_{s+1} = a_s \quad \text{i.e.} \quad a_0 = a_1 = a_2 = \cdots$$

The first solution is

$$y_1(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$= \sum_{m=0}^{\infty} a_0 x^m = a_0\left[1 + x + x^2 + x^3 + ...\right]$$

$$= a_0 \cdot \frac{1}{1-x} \qquad \text{converges for } |x| < 1$$

For the second solution, we should try the form:

$$y_2(x) = y_1(x)\ln x + x^{r_1}(A_1x + A_2x^2 + \cdots)$$

ie. Try

$$y_2(x) = a_0\left[1 + x + x^2 + x^3 + ...\right]\ln x + (A_1x + A_2x^2 + \cdots)$$

However, since y_1 can be expressed in exact form, we can find y_2 by the method of variation of parameter (see p.2-6, sec.2.3)

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx$$

since the equation is

$$x(x-1)y'' + (3x-1)y + y = 0 \rightarrow \qquad y'' + \frac{(3x-1)}{x(x-1)}y + \frac{1}{x(x-1)}y = 0$$

$$\rightarrow p(x) = \frac{(3x-1)}{x(x-1)}, \quad \text{thus}$$

$$-\int p(x)dx = -\int \frac{(3x-1)}{x(x-1)}dx = -\int\left(\frac{2}{x-1} + \frac{1}{x}\right)dx$$

$$= -2\ln(x-1) - \ln x = \ln\frac{1}{(x-1)^2 x}$$

$$Exp(-\int p(x)dx) = \frac{1}{(x-1)^2 x}$$

$$y_2(x) = y_1(x)\int \frac{(x-1)^2}{(x-1)^2 x}dx = y_1(x)\ln x = \frac{\ln x}{1-x}$$

the general solution is then

$$y = (c_1 + c_2 \ln x)\frac{1}{1-x}$$

Example: $(x^2 - x)y'' - xy' + y = 0$

Try $y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$ substitute into the above equation

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}$$

$$- \sum_{m=0}^{\infty}(m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}(m+r-1)^2 a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} = 0$$

$$\Rightarrow \sum_{s=0}^{\infty}(s+r-1)^2 a_s x^{s+r} - \sum_{s=-1}^{\infty}(s+r+1)(s+r)a_{s+1} x^{s+r} = 0$$

$$\Rightarrow \sum_{s=0}^{\infty}[(s+r-1)^2 a_s - (s+r+1)(s+r)a_{s+1}]x^{s+r} - r(r-1)a_0 x^{r-1} = 0$$

x^{r-1} : $r(r-1) = 0 \quad \Rightarrow \quad r_1 = 1 \; and \; r_2 = 0$

x^{s+r} : $(s+r-1)^2 a_s - (s+r+1)(s+r)a_{s+1} = 0 \quad s = 0,1,2,3...$

First solution: when $r = r_1 = 1$

$$s^2 a_s - (s+2)(s+1)a_{s+1} = 0$$

$$\Rightarrow a_{s+1} = \frac{s^2}{(s+2)(s+1)}a_s \qquad \text{s} = 0,1,2,3...$$

Thus $a_1 = a_2 = a_3 = ... = 0$

→the first solution is $y_1(x) = a_0 x$

the second solution: $y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx$

since the equation becomes $y'' - \frac{x}{x^2 - x}y' + \frac{1}{x^2 - x}y = 0$

$$\Rightarrow p(x) = -\frac{x}{x^2 - x} = -\frac{1}{x-1}$$

$$\therefore \quad -\int p(x)dx = \int \frac{1}{x-1}dx = \ln(x-1)$$

$$y_2(x) = y_1\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx = y_1\int \frac{x-1}{x^2}dx$$

$$= y_1\int\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = y_1\left(\ln x + \frac{1}{x}\right) = x\ln x + 1$$

the general solution is:

$y = c_1 x + c_2 (x\ln x + 1)$

5.5 Bessel's Equation

Bessel's differential equation form

$$\left.\begin{array}{l} x^2y''+xy'+(x^2-v^2)y=0 \\ y''+\dfrac{1}{x}y'+\dfrac{x^2-v^2}{x^2}y=0 \end{array}\right\} ------(1)$$

where v is a given parameter

By Frobenius method try $y=\sum_{m=0}^{\infty}a_m x^{m+r},\quad a_0\neq 0$

$$\Rightarrow \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r}+\sum_{m=0}^{\infty}(m+r)a_m x^{m+r}$$

$$+\sum_{m=0}^{\infty}a_m x^{m+r+2}-v^2\sum_{m=0}^{\infty}a_m x^{m+r}=0$$

$$\Rightarrow \sum_{m=0}^{\infty}\left[(m+r)^2-v^2\right]a_m x^{m+r}+\sum_{m=0}^{\infty}a_m x^{m+r+2}=0$$

$$\Rightarrow \sum_{m=0}^{\infty}\left[(m+r)^2-v^2\right]a_m x^{m+r}+\sum_{m=2}^{\infty}a_{m-2}\cdot x^{m+r}=0$$

$$\Rightarrow \sum_{m=2}^{\infty}\{[(m+r)^2-v^2]a_m-a_{m-2}\}\cdot x^{m+r}$$

$$+a_0(r^2-v^2)\cdot x^r+a_1[(r+1)^2-v^2]\cdot x^{r+1}=0$$

$$\Rightarrow\begin{cases} r^2-v^2=0 ----------------------------(2) \\ [(r+1)^2-v^2]a_1=0 --------------------------(3) \\ \left[(s+r+2)^2-v^2\right]a_{s+2}+a_s=0 \quad s=0,1,2,3...--------(4) \end{cases}$$

(2)$\Rightarrow r=r_{1,2}=\pm v$

(3)$\Rightarrow$we may take $a_1=0$ no matter $r=+v$ or $r=-v$

if we take $r=+v$, then

$$(4) \Rightarrow a_{s+2} = -\frac{1}{(s+2)(s+2v+2)} a_s \qquad s = 0, 1, 2, 3, \ldots.$$

since $a_1 = 0, \quad \Rightarrow a_3 = a_5 = a_7 = \cdots = 0$

set $s = 2m - 2$

$$\Rightarrow a_{2m} = -\frac{a_{2m-2}}{2m(2m+2v)}$$

$$= -\frac{a_{2m-2}}{2^2 m(m+v)} \qquad m = 1, 2, 3, \ldots.$$

$$\Rightarrow \begin{cases} a_2 = -\dfrac{a_0}{2^2(v+1)} \\ a_4 = -\dfrac{a_2}{2^2 \cdot 2(v+2)} = \dfrac{a_0}{2^4 \cdot 2(v+2)(v+1)} \\ a_6 = \ldots. \end{cases}$$

$$\Rightarrow a_{2m} = \frac{(-1)^m}{2^{2m} m!(v+1)(v+2)\cdots(m+v)} a_0 \qquad m = 1, 2, 3, \ldots.$$

If $v = n$: an integer and choose the coefficient $a_0 = \dfrac{1}{2^n n!}$

Then $a_{2m} = \dfrac{(-1)^m}{2^{2m+n} m!(n+m)!}, \quad m = 0, 1, 2, 3, \ldots.$

The solution is denoted by $J_n(x)$,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}$$

Bessel function of the first kind **of order *n***

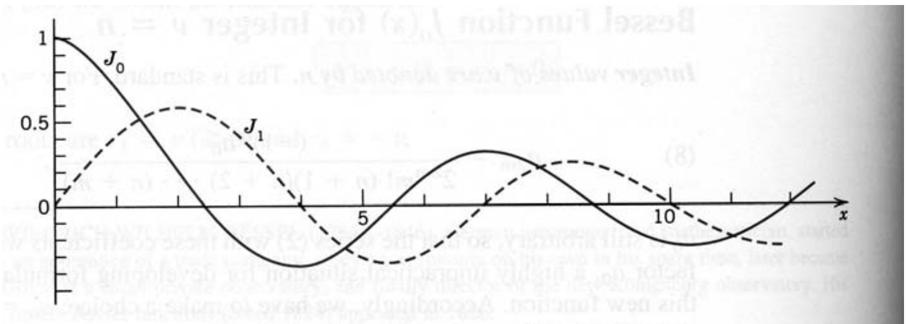
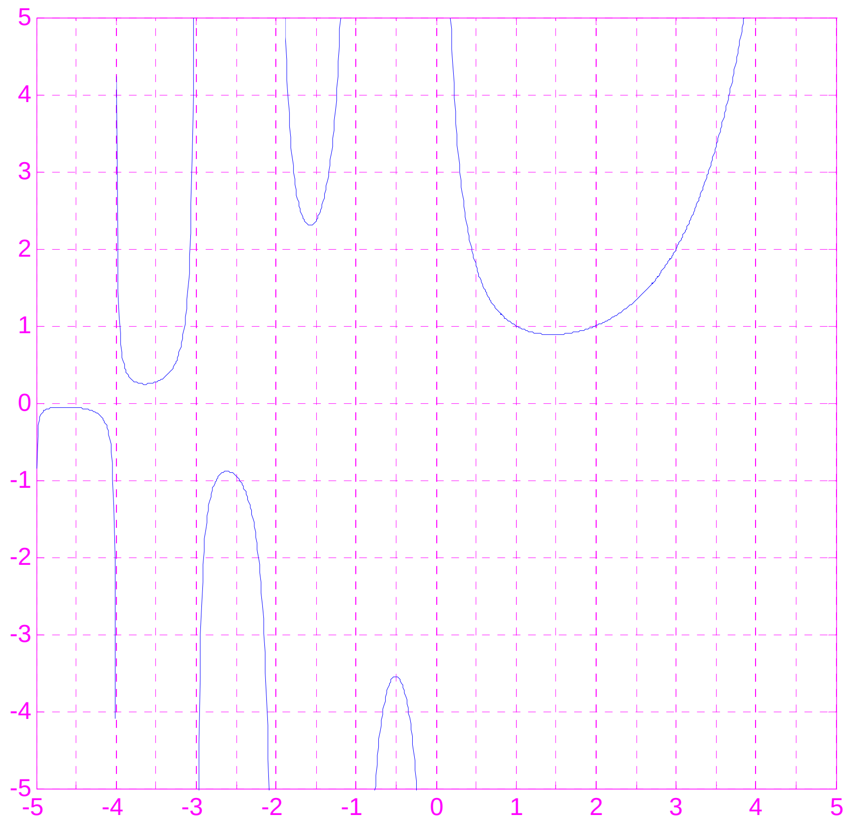


Fig. 103. Bessel functions of the first kind

※ Gamma function Γ

Define:

$$\Gamma(v)=\int_0^\infty e^{-t}\,t^{v-1}\,d\,t,\qquad v>0$$



$$\Gamma(v+1)=\int_0^\infty e^{-t}\,t^{v}\,d\,t=-e^{-t}t^{v}\Big|_0^\infty+v\int_0^\infty e^{-t}\,t^{v-1}\,d\,t\Rightarrow\Gamma(v+1)=v\Gamma(v)$$

$$\Gamma(1)=\int_0^\infty e^{-t}\;d\,t=1,\quad \Gamma(2)=1\cdot\Gamma(1)=1,\quad \Gamma(3)=2\cdot\Gamma(2)=2!$$

$$\Gamma(n+1)=n\;!$$

$$\Gamma(\frac{1}{2})=\int_0^\infty e^{-t}\,t^{-\frac{1}{2}}\,d\,t=2\int_0^\infty e^{-u^2}du=I$$

$$I^2=4\int_0^\infty e^{-u^2}du\quad\int_0^\infty e^{-v^2}dv=4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv=4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta=\pi$$

$$\therefore \quad \Gamma(\frac{1}{2}) = I = \sqrt{\pi}$$

in previous, $a_0 = \dfrac{1}{2^n n!} = \dfrac{1}{2^n \Gamma(n+1)}$

in general , $n \to v$ (any value, not necessary integer) i.e.,

$$a_0 = \frac{1}{2^v \Gamma(v+1)}$$

$$\therefore \quad a_{2m} = \frac{(-1)^m}{2^{2m+v} m!\,(v+1)(v+2)\cdots(m+v)\Gamma(v+1)} = \frac{(-1)^m}{2^{2m+v}\, m!\,\Gamma(v+m+1)}$$

$$\therefore \quad J_v(x) = x^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v}\, m!\,\Gamma(v+m+1)}$$

Bessel function of the first kind of order *v*

Similarly for $r = -v$

$$J_{-v}(x) = x^{-v} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-v}\, m!\,\Gamma(m-v+1)}$$

when v is not an integer, since $J_v(x)$ and $J_{-v}(x)$ are linearly independent →

If v is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is $$y(x) = c_1 J_v(x) + c_2 J_{-v}(x)$$

But if v is an integer, say, $v = n$, since $J_n(x) = (-1)^n J_{-n}(x)$, i.e. $J_n(x)$ and $J_{-n}(x)$ are dependent, in this case, we need to find one more independent solution to form the general solution of the Bessel's equation

→ Bessel's function of the second kind!

Proof of $J_n(x) = (-1)^n J_{-n}(x)$

Since $J_{-v}(x) = x^{-v} \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-v}\, m!\,\Gamma(m-v+1)}$

let ν approach a positive integer *n*. Then the Gamma function in the first *n* terms become infinite, the coefficient become zero, and the summation should actually starts with $m = n$. i.e.

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n}\, m!\,\Gamma(m-n+1)} = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n}\, m!\,(m-n)!}$$

$$\xrightarrow[m=n+s]{} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n}\,(n+s)!\,s!} = (-1)^n J_n(x)$$

※ properties of $J_\nu(x)$

1. $\dfrac{d}{dx}\left[x^{\nu} J_\nu(x)\right] = x^{\nu} J_{\nu-1}(x)$

2. $\dfrac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$

3. $J_{\nu-1}(x) + J_{\nu+1}(x) = \dfrac{2\nu}{x} J_\nu(x)$

4. $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu{}'(x)$

5. $J_{\frac{1}{2}}(x) = \sqrt{\dfrac{2}{x\pi}} \sin x$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi\, x}} \cos x$$

$$J_{2/3}(x) = \sqrt{\frac{2}{x\pi}} \left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{-2/3}(x) = -\sqrt{\frac{2}{x\pi}} \left(\frac{\cos x}{x} + \sin x\right)$$

5.6 Bessel function of second kind

when $\nu = n =$ integer , one solution of the Bessel's equation is $y_1 = J_n(x)$, we need to find one more independent solution.

Firstly we consider the case $\nu = n = 0$, and the equation becomes:

$$x\,y'' + y' + x\,y = 0 \text{-------} (1)$$

Thus $J_0(x)$ is a solution, According to Frobenius Theorem , the second solution must be of the form:(see p.4-11, case II)

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} A_m x^m$$

substitute into (1)→

$$2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0 \text{---} (A)$$

$$\because J_0(x) = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

$$\therefore J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2mx^{2m-1}}{2^{2m}(m!)^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!}$$

$(A) \Rightarrow$

$$\Rightarrow \underbrace{\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!}}_{x,x^3,x^5,\ldots} + \underbrace{\sum_{m=1}^{\infty} m^2 A_m x^{m-1}}_{1,x,x^2,x^3,\ldots} + \underbrace{\sum_{m=1}^{\infty} A_m x^{m+1}}_{x^2,x^3,x^4,\ldots} = 0$$

for x^0 term: $A_1 = 0 \text{-------} (2)$

for even power of x, i.e., x^{2s} : let $m = 2s+1$ (in the second series), $m = 2s-1$ (in the third series)

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0, \qquad s = 1, 2, 3, 4, \cdots \text{------} (3)$$

for odd power of x, i.e. x^{2s+1} : let $m = s+1$ (in the first series),

$m = 2s+2$ (in the second series), $m = 2s$ (in the third series)

for $s = 0$ $-1+4A_2 = 0 \quad \Rightarrow A_2 = \frac{1}{4}$ ------(4)

for $s = 1, 2, 3, \ldots$ $\frac{(-1)^{s+1}}{2^{2s}\, s!(s+1)!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$ ------(5)

(2) & (3) $\rightarrow$ $A_1 = A_3 = A_5 = \ldots. = 0, \ldots\ldots$

(5) $s = 1 \rightarrow 1/8 + 16 A_4 + A_2 = 0 \rightarrow A_4 = -3/128$

In general: (5) set $s= m+1$

$$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right), \quad m = 1, 2, 3, \cdots$$

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right) x^{2m}$$

$$= J_0(x)\ln x + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - \cdots$$

But it is customary to choose the form $a(y_2 + bJ_0)$

Where

$a = \frac{2}{\pi}$; $b = \gamma - \ln 2 = 0.57721566490$

and the γ (Euler constant) is defined as

$$\gamma \equiv 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{s}-\ln s, \qquad s \to \infty$$

The particular form solution is called Bessel function of second kind (or Neumann function) of order zero and denoted by $Y_0(x)$, i.e.,

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)(\ln\frac{x}{2}+\gamma) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right) x^{2m} \right]$$

When $\nu \neq 0$

The standard form of the second solution $Y_\nu(x)$ (for all υ) is:

$$\begin{cases} Y_\nu(x) = \dfrac{1}{\sin\nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right] \\ Y_n(x) = \lim\limits_{\nu\to n} Y_\nu(x) \qquad \textit{exist} \end{cases}$$

$$Y_n(x) = \frac{2}{\pi}J_n(x)(\ln\frac{x}{2}+\gamma) + \frac{x^n}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}(h_m+h_{m+n})}{2^{2m+n}m!(m+n)!}x^{2m}$$

$$-\frac{x^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{2^{2m-n}m!}x^{2m} \qquad \text{for all } x>0$$

where $\mathrm{h}_0 \equiv 0, \quad \mathrm{h}_\mathrm{m} \equiv 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\mathrm{m}}$ for $\mathrm{m}=1,2,3...$

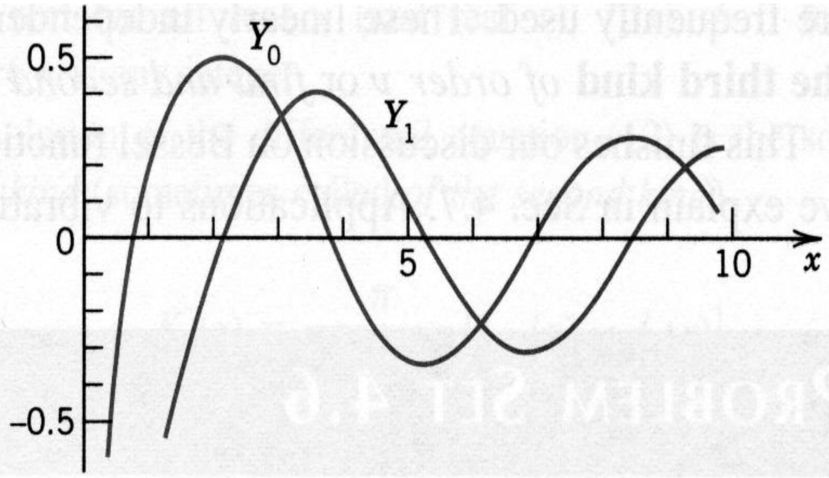


Fig. 105. Bessel functions of the second kind. (For a small table, see Appendix 5.)

$Y_\nu(x)$ and $J_\nu(x)$ are linearly independent
Thus the general solution of the Bessel's equation can be expressed as

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x) \qquad \nu: \text{for any number}$$

Other solution form :
If we defined:

$$H_\nu^{(1)}(x) \equiv J_\nu(x) + iY_\nu(x) \qquad H_\nu^{(2)}(x) \equiv J_\nu(x) - iY_\nu(x)$$

$H_\nu^{(1)}(x), H_\nu^{(2)}(x)$: are called the Hankel function (Bessel function of third kind) the general solution of the Bessel's equation can also be expressed as

$$y(x) = c_1 H_v^{(1)}(x) + c_2 H_v^{(2)}(x)$$

Summary:

the general solution of the Bessel's equation: $x^2 y'' + xy' + (x^2 - v^2) y = 0$ can be expressed as:

$$\begin{aligned}
&y(x) = c_1 J_v(x) + c_2 J_{-v}(x) && \text{when } v \neq \text{integer} \\
&\left.\begin{aligned} y(x) &= c_1 J_v(x) + c_2 Y_v(x) \\ y(x) &= c_1 H_v^{(1)}(x) + c_2 H_v^{(2)}(x) \end{aligned}\right\} && \text{For all } v
\end{aligned}$$

※ Bessel's equation of order v with parameter λ

Consider the equation:

$$x^2 y'' + xy' + (\lambda^2 x^2 - v^2) y = 0 ------(A)$$

If we let $t = \lambda x$

Then $y' = \dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \lambda \dot{y}$ $\quad y'' = \lambda \dfrac{d\dot{y}}{dt}\dfrac{dt}{dx} = \lambda^2 \ddot{y}$

(A)→

$$t^2 \ddot{y} + t\dot{y} + (t^2 - v^2) y = 0 ------(B)$$

the general solution of (B) is

$$y(t) = c_1 J_v(t) + c_2 Y_v(t) --------(C)$$

Therefore, the general solution of (A) is

$$y(x) = c_1 J_v(\lambda x) + c_2 Y_v(\lambda x) --------(D)$$

or

$$y(x) = c_1 H_v^{(1)}(\lambda x) + c_2 H_v^{(2)}(\lambda x) ------(D)$$