12.5 Modeling: Heat Flow from a Body in Space. Heat Equation

After the wave equation (Sec. 12.2) we now derive and discuss the next "big" PDE, the **heat equation**, which governs the temperature u in a body in space. We obtain this model of temperature distribution under the following.

Physical Assumptions

1. The *specific heat* σ and the *density* ρ of the material of the body are constant. No heat is produced or disappears in the body.
2. Experiments show that, in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient (cf. Sec. 9.7) of the temperature; that is, the velocity $\mathbf{v}$ of the heat flow in the body is of the form

$$\mathbf{v} = -K \operatorname{grad} u \tag{1}$$

   where $u(x, y, z, t)$ is the temperature at a point (x, y, z) and time t.
3. The *thermal conductivity* K is constant, as is the case for homogeneous material and nonextreme temperatures.

Under these assumptions we can model heat flow as follows.

Let T be a region in the body bounded by a surface S with outer unit normal vector $\mathbf{n}$ such that the divergence theorem (Sec. 10.7) applies. Then

$$\mathbf{v} \cdot \mathbf{n}$$

is the component of $\mathbf{v}$ in the direction of $\mathbf{n}$. Hence $|\mathbf{v} \cdot \mathbf{n}\, \Delta A|$ is the amount of heat *leaving* T (if $\mathbf{v} \cdot \mathbf{n} > 0$ at some point P) or *entering* T (if $\mathbf{v} \cdot \mathbf{n} < 0$ at P) per unit time at some point P of S through a small portion ΔS of S of area ΔA. Hence the total amount of heat that flows across S from T is given by the surface integral

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dA.$$

Note that, so far, this parallels the derivation on fluid flow in Example 1 of Sec. 10.8.

Using Gauss's theorem (Sec. 10.7), we now convert our surface integral into a volume integral over the region T. Because of (1) this gives [use (3) in Sec. 9.8]

$$\begin{aligned} \iint_S \mathbf{v} \cdot \mathbf{n}\, dA &= -K \iint_S (\operatorname{grad} u) \cdot \mathbf{n}\, dA = -K \iiint_T \operatorname{div} (\operatorname{grad} u)\, dx\, dy\, dz \\ &= -K \iiint_T \nabla^2 u\, dx\, dy\, dz. \end{aligned} \tag{2}$$

Here,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

is the **Laplacian** of u.

On the other hand, the total amount of heat in T is

$$H = \iiint_T \sigma \rho u \, dx \, dy \, dz$$

with σ and ρ as before. Hence the time rate of decrease of H is

$$-\frac{\partial H}{\partial t} = -\iiint_T \sigma \rho \frac{\partial u}{\partial t} \, dx \, dy \, dz.$$

This must be equal to the amount of heat leaving T because no heat is produced or disappears in the body. From (2) we thus obtain

$$-\iiint_T \sigma \rho \frac{\partial u}{\partial t} \, dx \, dy \, dz = -K \iiint_T \nabla^2 u \, dx \, dy \, dz$$

or (divide by $-\sigma\rho$)

$$\iiint_T \left(\frac{\partial u}{\partial t} - c^2 \nabla^2 u \right) dx \, dy \, dz = 0 \qquad c^2 = \frac{K}{\sigma\rho}.$$

Since this holds for any region T in the body, the integrand (if continuous) must be zero everywhere. That is,

$$\text{(3)} \qquad \boxed{\frac{\partial u}{\partial t} = c^2 \nabla^2 u.} \qquad c^2 = K/\rho\sigma$$

This is the **heat equation**, the fundamental PDE modeling heat flow. It gives the temperature $u(x, y, z, t)$ in a body of homogeneous material in space. The constant c^2 is the *thermal diffusivity*. K is the *thermal conductivity*, σ the *specific heat*, and ρ the *density* of the material of the body. $\nabla^2 u$ is the Laplacian of u and, with respect to the Cartesian coordinates x, y, z, is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

The heat equation is also called the **diffusion equation** because it also models chemical diffusion processes of one substance or gas into another.

12.6 Heat Equation: Solution by Fourier Series. Steady Two-Dimensional Heat Problems. Dirichlet Problem

We want to solve the (one-dimensional) heat equation just developed in Sec. 12.5 and give several applications. This is followed much later in this section by an extension of the heat equation to two dimensions.