Contents

31 Electromagnetic Oscillations and Alternating Current

31.1 LC Oscillations, Qualitatively

In RC and RL circuits the charge, current, and potential difference grow and decay exponentially.

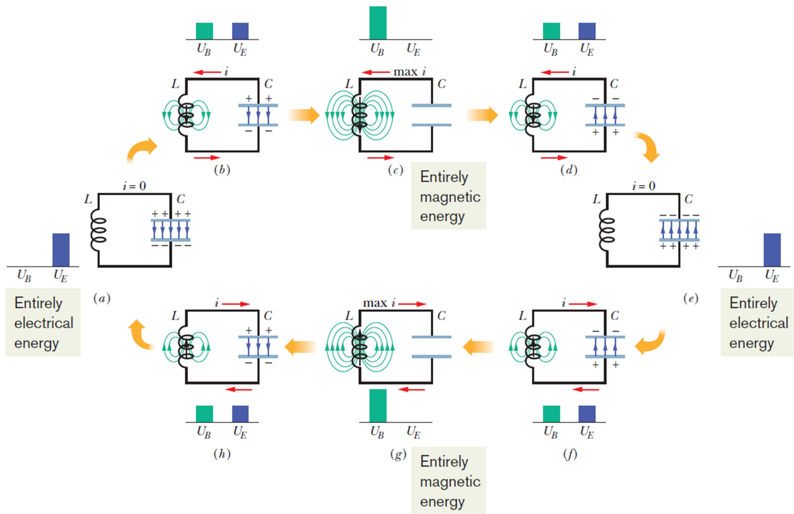
On the contrary, in an LC circuit, the charge, current, and potential difference vary sinusoidally with period T and angular frequency ω_0.

The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be electromagnetic oscillations.

The energy stored in the electric field of the capacitor at any time is $U_E = \frac{q^2}{2C}$ where q is the charge on the capacitor at that time.

The energy stored in the magnetic field of the inductor at any time is $U_B = \frac{Li^2}{2}$ where i is the current through the inductor at that time.

As the circuit oscillates, energy shifts back and forth from one type of stored energy to the other, but the total amount is conserved.



The time-varying potential difference (or voltage) V_C that exists across the capacitor C is

$$V_C = \frac{q}{C}$$

To measure the current, we can connect a small resistance R in series with the capacitor and inductor and measure the time-varying potential difference V_R across it:

$$V_R = iR$$

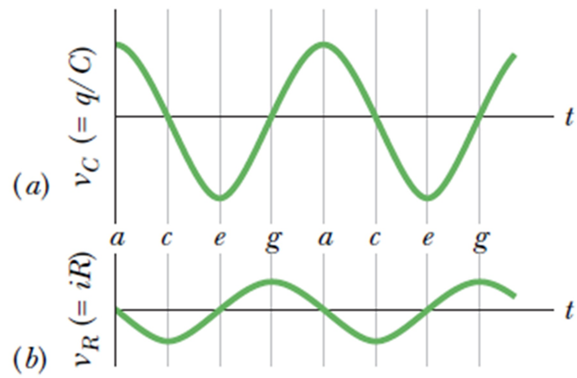


Fig. 31-2 (*a*) The potential difference across the capacitor of the circuit of Fig. 31-1 as a function of time. This quantity is proportional to the charge on the capacitor. (*b*) A potential proportional to the current in the circuit of Fig. 31-1. The letters refer to the correspondingly labeled oscillation stages in Fig. 31-1.

31.2 The Electrical Mechanical Analogy

One can make an analogy between the oscillating LC system and an oscillating block–spring system.

Two kinds of energy are involved in the block–spring system. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving block.

Here we have the following analogies:

Comparison of the Energy in Two Oscillating System

Block-Spring		LC Oscillator	
Element	Energy	Element	Energy
Spring	Potential, $\frac{1}{2}kx^2$	Capacitor	Electric, $\frac{1}{2}\left(\frac{1}{C}\right)q^2$
Block	Kinetic, $\frac{1}{2}mv^2$	Inductor	Magnetic, $\frac{1}{2}Li^2$
	$v = \frac{dx}{dt}$		$i = \frac{dq}{dt}$

With the correspondences:

$$(x, v) \Longleftrightarrow (q, i)$$

$$(k, m) \Longleftrightarrow \left(\frac{1}{C}, L\right)$$

The angular frequency for the block-spring system is

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Therefore, the angular frequency for the LC Oscillator corresponds to

$$\omega_0 = \sqrt{\frac{1}{LC}}.$$

The Block-Spring Oscillator:

$$E = U_b + U_s = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$0 = \frac{dE}{dt} = mv\frac{dv}{dt} + kx\frac{dx}{dt}$$

$\Longrightarrow$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\begin{aligned} x(t) &= x_0 \cos(\omega_0 t + \phi) = \mathrm{Re}\left[x_0 e^{i\phi} e^{i\omega_0 t}\right] \\ &= \mathrm{Re}\left[X_0 e^{i\omega_0 t}\right] = \mathrm{Re}\left[X(t)\right] \end{aligned}$$

where

$$X(t) = X_0 e^{i\omega_0 t}.$$

Note that the complex function $X(t)$ satisfies

$$\frac{d^2X}{dt^2} + \omega_0^2 X = \frac{d^2X}{dt^2} + \frac{k}{m}X = 0$$

The LC Oscillator: (To avoid confusion of current i with the imaginary $i \equiv \sqrt{-1}$, we will represent $\sqrt{-1}$ by j, $j = \sqrt{-1}$, from now on in this chapter.)

$$U = U_B + U_E = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}$$

$$0 = \frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt}$$

$\Longrightarrow$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

$$\begin{aligned} q(t) &= q_0 \cos(\omega_0 t + \phi) = \mathrm{Re}\left[q_0 e^{j\phi} e^{j\omega_0 t}\right] \\ &= \mathrm{Re}\left[Q_0 e^{j\omega_0 t}\right] = \mathrm{Re}\left[Q(t)\right] \end{aligned}$$

Note that the complex function

$$Q(t) = Q_0 e^{j\omega_0 t}.$$

satisfies

$$\frac{d^2Q}{dt^2} + \omega_0^2 Q = \frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

Taking the time derivative of the identity $q(t) = \mathrm{Re}[Q(t)]$, we get

$$i(t) = \frac{dq}{dt} = \mathrm{Re}\left[\frac{dQ(t)}{dt}\right] = \mathrm{Re}\left[I(t)\right].$$

The real current $i(t)$ can also be expressed as the real part of a complex current, if $I(t)$ is identified as

$$\begin{aligned} I(t) &= \frac{dQ(t)}{dt} = \frac{d}{dt}\left(Q_0 e^{j\omega_0 t}\right) \\ &= j\omega_0 Q_0 e^{j\omega_0 t} = j\omega_0 Q(t) \end{aligned}$$

Angular Frequencies:

The real charge q satisfies

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0,$$

so does the complex charge $Q(t)$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

which has the general solution

$$Q(t) = Ae^{j\omega_0 t} + Be^{-j\omega_0 t}$$

where $A = a_1 + ia_2$ and $B = b_1 + ib_2$ can be any complex numbers (a_1,a_2,b_1,b_2 are real). Thus

$$\begin{aligned} q(t) &= \mathrm{Re}[Q(t)] = (a_1 + b_1)\cos\omega_0 t - (a_2 - b_2)\sin\omega_0 t \\ &= q_0 \cos(\omega_0 t + \phi) \end{aligned}$$

with

$$q_0 = \sqrt{(a_1 + b_1)^2 + (a_2 - b_2)^2}$$

$$\tan\phi = \frac{a_2 - b_2}{a_1 + b_1}$$

The electric energy stored in the LC circuit at time t is

$$U_E = \frac{q^2}{2C} = \frac{q_0^2}{2C}\cos^2(\omega_0 t + \phi)$$

The magnetic energy stored is

$$U_B = \frac{Li^2}{2} = \frac{\omega_0^2 L q_0^2}{2}\sin^2(\omega_0 t + \phi) = \frac{q_0^2}{2C}\sin^2(\omega_0 t + \phi)$$

$$U = U_E + U_B = \frac{q_0^2}{2C} = \frac{\omega_0^2 L q_0^2}{2}$$

31.3 Damped Oscillations in an RLC Circuit

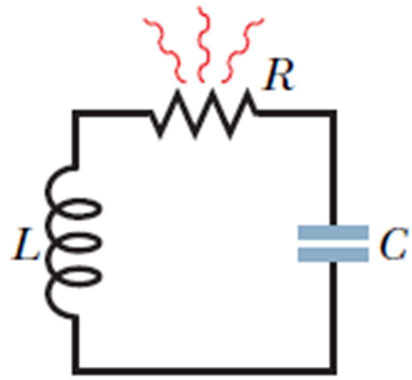


Fig. 31-5 A series *RLC* circuit. As the charge contained in the circuit oscillates back and forth through the resistance, electromagnetic energy is dissipated as thermal energy, damping (decreasing the amplitude of) the oscillations.

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$

$\Longrightarrow$

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2 R$$

$\Longrightarrow$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \quad \text{(RLC circuit)} \tag{1}$$

To find a solution, assume $Q(t) = e^{\alpha t}$. By requiring $Q(t)$, which may be complex, to satisfy

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \tag{2}$$

and $\frac{dQ}{dt} = \alpha Q$, $\frac{dQ}{dt} = \alpha^2 Q$, gives us

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \left(L\alpha^2 + R\alpha + \frac{1}{C}\right) Q = 0$$

or

$$\alpha^2 + \frac{R}{L}\alpha + \omega_0^2 = 0$$

where $\omega_0^2 = \frac{1}{LC}$. When $\omega_0^2 \geq \left(\frac{R}{2L}\right)^2$, α is equal to

$$\alpha = -\frac{R}{2L} \pm j\sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$$

which leads to a complex solution $Q(t)$

$$Q(t) = e^{-\frac{R}{2L}t} e^{\pm j\omega t}$$

where

$$\omega = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$$

The real part of (2) yields

$$L\frac{d^2\,\mathrm{Re}[Q]}{dt^2} + R\frac{d\,\mathrm{Re}[Q]}{dt} + \frac{\mathrm{Re}[Q]}{C} = 0$$

and we may identify the real q as $\mathrm{Re}[Q]$.

$$\begin{aligned}\mathrm{Re}[Q] &= \mathrm{Re}\left[e^{-\frac{R}{2L}t} e^{\pm j\omega t}\right] \\ &= e^{-\frac{R}{2L}t}\cos\omega t \qquad (3)\end{aligned}$$

Similarly the imaginary part of Q also satisfies

$$L\frac{d^2\,\mathrm{Im}[Q]}{dt^2} + R\frac{d\,\mathrm{Im}[Q]}{dt} + \frac{\mathrm{Im}[Q]}{C} = 0$$

and we have another real solution

$$\begin{aligned}\mathrm{Im}[Q] &= \mathrm{Im}\left[e^{-\frac{R}{2L}t} e^{j\omega t}\right] \\ &= e^{-\frac{R}{2L}t}\sin\omega t \qquad (4)\end{aligned}$$

The general solution is a linear combination of (3) and (4).

$$\begin{aligned}q(t) &= e^{-\frac{R}{2L}t}\left(a\cos\omega t + b\sin\omega t\right) \\ &= q_0 e^{-\frac{R}{2L}t}\cos(\omega t + \phi) \qquad (5)\end{aligned}$$

31.3.1 Example, Damped RLC Circuit

A series RLC circuit has inductance L, capacitance C, and resistance R and begins to oscillate at time $t = 0$.

(a) At what time t will the amplitude of the charge oscillations in the circuit be 50% of its initial value?

The charge amplitude at any time t is $q_0 e^{-\frac{R}{2L}t}$, in which q_0 is the amplitude at time $t = 0$. We want the time when the charge amplitude has decreased to $0.50q_0$, that is, when

$$\begin{aligned} q_0 e^{-\frac{R}{2L}t} &= 0.50q_0 \\ e^{-\frac{R}{2L}t} &= 0.50 \\ -\frac{R}{2L}t &= \ln(0.50) \\ t = -\frac{2L}{R}\ln(0.50) &= \frac{2L}{R}\ln 2.0 \end{aligned}$$

(b) How many oscillations are completed within this time t?

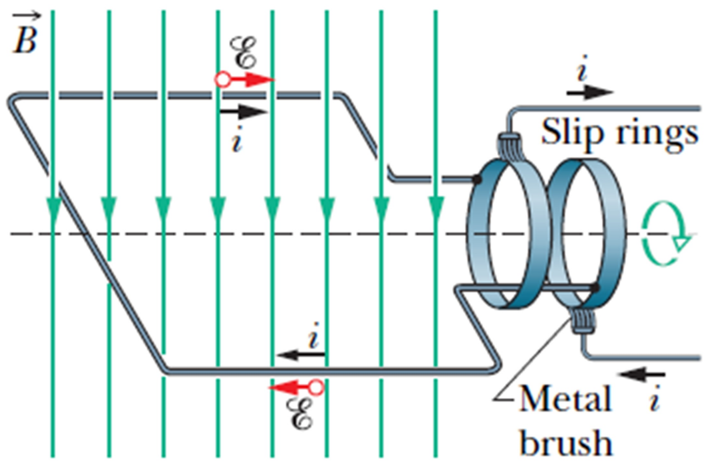
The time for one complete oscillation is the period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}$$

The number of complete oscillations is

$$\frac{t}{T} = \frac{\frac{2L}{R}\ln 2.0}{2\pi}\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

31.4 Alternating Current



The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field.

$$\xi(t) = \xi_0 \cos(\omega_d t)$$

$$i(t) = i_0 \cos(\omega_d t + \phi)$$

ω_d is called the driving angular frequency, and i_0 is the amplitude of the driven current.

31.5 Three Simple Circuits

31.5.1 A Resistive Load

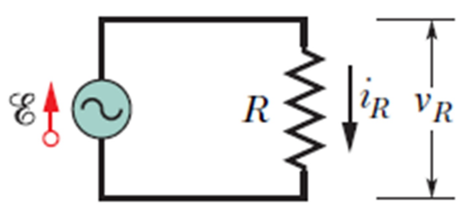


Fig. 31-8 A resistor is connected across an alternating-current generator.

$$iR = \xi(t) = \xi_0 \cos(\omega_d t) = \mathrm{Re}[\xi_c(t)]$$

where we have defined the complex emf $\xi_c(t) = \xi_0 e^{j\omega_d t}$. The real emf ξ is the real part of ξ_c.

$$\xi(t) = \mathrm{Re}[\xi_c(t)]$$

Since

$$i(t) = \mathrm{Re}[I(t)]$$

we have

$$\mathrm{Re}[RI(t)] = \mathrm{Re}[\xi_c(t)]$$

which suggests that we may relate the complex current I to the complex ξ_c by

$$\xi_c = RI \tag{6}$$

Example In Fig. 31-8, resistance R is 200Ω and the sinusoidal alternating emf device operates at amplitude $\xi_0 = 36.0V$ and frequency $f_d = 60Hz$.
(a) What are the potential difference $v_R(t)$ across the inductance and the amplitude of v_R?

$$\begin{aligned} v_R(t) &= \xi(t) = \xi_0 \cos(\omega_d t) \\ &= \xi_0 \cos(2\pi f_d t) = (36.0V)\cos(120\pi t) \end{aligned}$$

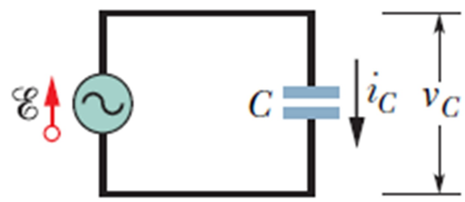
The amplitude v_{R0} is the same as $\xi_0 = 36.0V$.
(b) What are the current $i_R(t)$ in the circuit as a function of time and the amplitude of i_R?

$$\begin{aligned} i_R &= i_0 \cos(\omega_d t) = \mathrm{Re}\left[\frac{V_R}{R}\right] \\ &= \mathrm{Re}\left[\frac{\xi_0}{R} e^{j\omega_d t}\right] = \frac{\xi_0}{R}\cos(\omega_d t) \end{aligned}$$

The amplitude

$$i_0 = \frac{\xi_0}{R} = \frac{36.0V}{200\Omega} = 0.180A$$

31.5.2 A Capacitive Load



$$\frac{q(t)}{C} = \xi(t) = \xi_0 \cos(\omega_d t) = \mathrm{Re}\left[\xi_c(t)\right]$$

Now that

$$q(t) = \mathrm{Re}\left[Q(t)\right]$$

and

$$I(t) = \frac{dQ(t)}{dt} = j\omega_d Q(t)$$

or

$$Q(t) = \frac{I(t)}{j\omega_d}$$

We thus have

$$\mathrm{Re}\left[\frac{Q(t)}{C}\right] = \mathrm{Re}\left[\frac{1}{j\omega_d C} I(t)\right] = \mathrm{Re}\left[\xi_c(t)\right]$$

which suggests that we may relate the complex current I to the complex ξ_c by

$$\xi_c = \frac{1}{j\omega_d C} I \tag{7}$$

Example In the previous figure, capacitance C is $15.0\mu F$ and the sinusoidal alternating emf device operates at amplitude $\xi_0 = 36.0V$ and frequency $f_d = 60Hz$.
(a) What are the potential difference $v_C(t)$ across the inductance and the amplitude of v_C?

$$\begin{aligned} v_C(t) &= \xi(t) = \xi_0 \cos(\omega_d t) \\ &= \xi_0 \cos(2\pi f_d t) = (36.0V)\cos(120\pi t) \end{aligned}$$

The amplitude v_{C0} is the same as $\xi_0 = 36.0V$.

(b) What are the current $i_C(t)$ in the circuit as a function of time and the amplitude of i_C?

$$\begin{aligned} i_C &= i_0 \cos(\omega_d t + \phi_C) = \mathrm{Re}\left[\frac{V_C}{1/j\omega_d C}\right] \\ &= \mathrm{Re}\left[j\omega_d C \xi_0 e^{j\omega_d t}\right] = \mathrm{Re}\left[\omega_d C \xi_0 e^{j\left(\omega_d t + \frac{\pi}{2}\right)}\right] \\ &= \omega_d C \xi_0 \cos\left(\omega_d t + \frac{\pi}{2}\right) = -\omega_d C \xi_0 \sin(\omega_d t) \end{aligned}$$

The amplitude

$$i_0 = \omega_d C \xi_0 = (2\pi)(60HZ)(15.0\mu F)\,36.0V = 0.203A$$

31.5.3 An Inductive Load

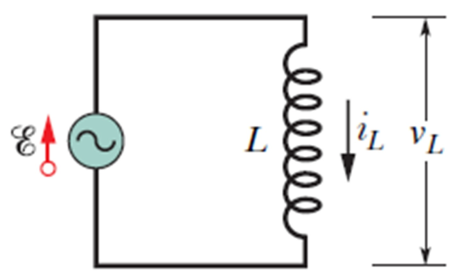


Fig. 31-12 An inductor is connected across an alternating-current generator.

$$L\frac{di}{dt} = \xi(t) = \mathrm{Re}[\xi_c(t)]$$

Now that

$$i = \mathrm{Re}[I]$$

and

$$\frac{dI(t)}{dt} = j\omega_d I(t).$$

We thus have

$$\mathrm{Re}\left[L\frac{dI\left(t\right)}{dt}\right]=\mathrm{Re}\left[j\omega_{d}LI\left(t\right)\right]=\mathrm{Re}\left[\xi_{c}\left(t\right)\right]$$

which suggests that we may relate the complex current I to the complex ξ_c by

$$\xi_{c}=j\omega_{d}LI \tag{8}$$

Example In Fig. 31-12, inductance L is $230mH$ and the sinusoidal alternating emf device operates at amplitude $\xi_0 = 36.0V$ and frequency $f_d = 60Hz$.
(a) What are the potential difference $v_L(t)$ across the inductance and the amplitude of v_L?

$$\begin{aligned}v_{L}\left(t\right)&=\xi\left(t\right)=\xi_{0}\cos\left(\omega_{d}t\right)\\&=\xi_{0}\cos\left(2\pi f_{d}t\right)=\left(36.0V\right)\cos\left(120\pi t\right)\end{aligned}$$

The amplitude v_{L0} is the same as $\xi_0 = 36.0V$.
(b) What are the current $i_L(t)$ in the circuit as a function of time and the amplitude of i_L?

$$\begin{aligned}i_{L}&=i_{0}\cos\left(\omega_{d}t+\phi_{L}\right)=\mathrm{Re}\left[\frac{V_{L}}{j\omega_{d}L}\right]\\&=\mathrm{Re}\left[\frac{\xi_{0}}{j\omega_{d}L}e^{j\omega_{d}t}\right]=\mathrm{Re}\left[\frac{\xi_{0}}{\omega_{d}L}e^{j\left(\omega_{d}t-\frac{\pi}{2}\right)}\right]\\&=\frac{\xi_{0}}{\omega_{d}L}\cos\left(\omega_{d}t-\frac{\pi}{2}\right)=\frac{\xi_{0}}{\omega_{d}L}\sin\left(\omega_{d}t\right)\end{aligned}$$

The amplitude

$$i_{0}=\frac{\xi_{0}}{\omega_{d}L}=\frac{36.0V}{\left(2\pi\right)\left(60Hz\right)\left(230mH\right)}=0.415A$$

31.5.4 Summary

(6), (7) and (8) can be summarized as

$$V=XI$$

where V is the complex potential across the device R, C or L. $X = \frac{V}{I}$, the ratio between the complex potential and the complex current, is called the impedance of the device.

$$X = \begin{cases} R & \text{resistor} \\ \frac{1}{j\omega C} & \text{capacitor} \\ j\omega L & \text{inductor} \end{cases}$$

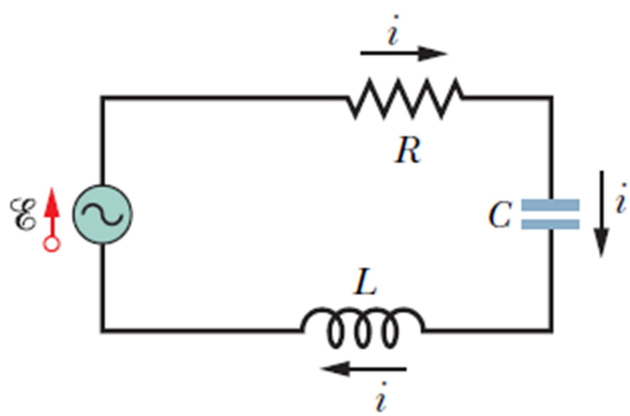
or

$$V_R = RI$$

$$V_C = \frac{1}{j\omega C} I$$

$$V_L = j\omega L I$$

31.6 Forced Oscillations



A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator, represented by a sine wave in a circle, produces an alternating emf that establishes an alternating current; the direction of the emf and current are indicated at only one instant.

Loop equation:

$$L\frac{di}{dt} + Ri + \frac{q}{C} = \xi(t) = \xi_0 \cos(\omega_d t) \tag{9}$$

with

$$i = \frac{dq}{dt}$$

The general solution for the 2nd order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \xi_0 \cos\left(\omega_d t\right)$$

can be expressed as a sum of

$$q\left(t\right) = q_h\left(t\right) + q_p\left(t\right)$$

where $q_p\left(t\right)$ is any particular solution of (9) and $q_h\left(t\right)$ is the solution satisfying the homogeneous equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

with solution given by (5). Note that $q_h\left(t\right) \to 0$ as $t \to \infty$ and therefore only q_p survives as the steady state solution

$$\lim_{t\to\infty} q\left(t\right) \to q_p\left(t\right).$$

Whatever the natural angular frequency ω of a circuit may be, we shall show that forced oscillations of charge, current, and potential difference in the circuit always occur at the driving frequency ω_d. Since $q = \text{Re}\left[Q\right]$, $i = \text{Re}\left[I\right]$ and $\xi = \text{Re}\left[\xi_c\right]$, Eq. (9) can be written as

$$\text{Re}\left[L\frac{dI}{dt} + RI + \frac{Q}{C}\right] = \text{Re}\left[\xi_c\right] = \text{Re}\left[\xi_0 e^{j\omega_d t}\right] \tag{10}$$

Thus, if

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = \xi_c = \xi_0 e^{j\omega_d t} \tag{11}$$

is satisfied, then Eq. (9) is automatically satisfied. Assuming

$$I = I_0 e^{j\omega_d t}, Q = Q_0 e^{j\omega_d t}$$

we have

$$I = \frac{dQ}{dt} = j\omega_d Q$$

and a particular solution of (11) can be found by requiring

$$\left(j\omega_d L + R + \frac{1}{j\omega_d C}\right) I = \xi_0 e^{j\omega_d t}$$

The above equation may also be obtained by noting that the impedance $X = \frac{\xi_c}{I}$ of the series RLC circuit is the sum of its three individual pieces.

$$X = X_L + X_R + X_C = j\omega_d L + R + \frac{1}{j\omega_d C}$$

$$\xi_c = XI = (X_L + X_R + X_C)\, I$$

$$I\,(t) = \frac{\xi_c}{X} = \frac{\xi_0}{j\omega_d L + R + \frac{1}{j\omega_d C}} e^{j\omega_d t} = i_0 e^{j\phi} e^{j\omega_d t} = i_0 e^{j(\omega_d t + \phi)}$$

where we have identified the complex number

$$\frac{\xi_0}{j\omega_d L + R + \frac{1}{j\omega_d C}} = i_0 e^{j\phi}$$

with

$$i_0 = \left| \frac{\xi_0}{j\omega_d L + R + \frac{1}{j\omega_d C}} \right| = \frac{\xi_0}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} \qquad (12)$$

and

$$\tan\phi = -\frac{\omega_d L - \frac{1}{\omega_d C}}{R}$$

The physical current is then equal to

$$\begin{aligned} i\,(t) &= \mathrm{Re}\,[I] = \mathrm{Re}\left[i_0 e^{j(\omega_d t + \phi)}\right] \\ &= i_0 \cos(\omega_d t + \phi) \end{aligned}$$

31.6.1 Resonance

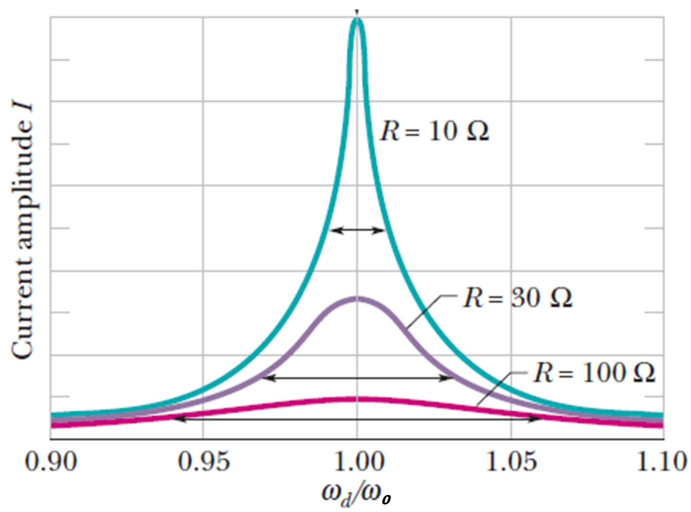
The amplitude for the oscillating current i given in (12) attains its maximum when

$$\omega_d L - \frac{1}{\omega_d C} = 0$$

$\Rightarrow$

$$\omega_d = \frac{1}{\sqrt{LC}} = \omega_0$$

That is the resonance occurs when the driving angular frequency ω_d matches the natural angular frequency $\frac{1}{\sqrt{LC}}$.



Resonance curves for the driven RLC circuit with three values of R. The current amplitude of the alternating current depends on how close the driving frequency ω_d is to the natural frequency $\omega_0 = 1/\sqrt{LC}$. The horizontal arrow on each curve measures the curve's half-width, which is the width at the half-maximum level and is a measure of the sharpness of the resonance.

The maximum level the th evalue of i_0 when $\omega_d = \omega_0$:

$$i_m = i_0|_{\omega_d=\omega_0} = \frac{\xi_0}{R}.$$

To attain the half-maximum level, we need

$$i_0 = \frac{\xi_0}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}} = \frac{1}{2} i_m = \frac{\xi_0}{2R},$$

or

$$\omega_d L - \frac{1}{\omega_d C} = \pm\sqrt{3}R.$$

Thus

$$\omega_d \mp \sqrt{3}\frac{R}{L} - \frac{1}{\omega_d LC} = 0.$$

$\rightarrow$

$$\omega_d^2 \mp \sqrt{3}\frac{R}{L}\omega_d - \omega_0^2 = 0.$$

This happens when

$$\omega_d = \pm\frac{\sqrt{3}R}{2L} \pm \sqrt{\omega_0^2 + \left(\frac{\sqrt{3}R}{2L}\right)^2}.$$

Only positive ω_d is meaningful, therefore we have two frequencies at which $i_0 = \frac{1}{2}i_m$.

$$\omega_\pm = \pm\frac{\sqrt{3}R}{2L} + \sqrt{\omega_0^2 + \left(\frac{\sqrt{3}R}{2L}\right)^2}$$

and the width

$$\omega_+ - \omega_- = \frac{\sqrt{3}R}{L}$$

is proportional to R. (Note i_m is proportional to $\frac{1}{R}$.)

31.7 Power in Alternating Current Circuits

The instantaneous rate at which energy is dissipated in a device is

$$P = i\left(t\right) v\left(t\right)$$

where $v\left(t\right)$ is the voltage drop in the direction of the current i across the device. Suppose $I\left(t\right) = i_0 e^{j(\omega_d t+\phi_1)}$ and $V\left(t\right) = v_0 e^{j(\omega_d t+\phi_2)}$ are the complex current and voltage corresponding to i and v with

$$i = \mathrm{Re}\left[I\right] = i_0 \cos\left(\omega_d t + \phi_1\right)$$

and

$$v = \mathrm{Re}\left[V\right] = v_0 \cos\left(\omega_d t + \phi_2\right).$$

Then

$$P = iv = i_0 v_0 \cos\left(\omega_d t + \phi_1\right) \cos\left(\omega_d t + \phi_2\right)$$

The average rate at which energy is dissipated is

$$\begin{aligned}\langle P\rangle &= i_0 v_0 \left\langle \cos\left(\omega_d t+\phi_1\right)\cos\left(\omega_d t+\phi_2\right)\right\rangle \\ &= i_0 v_0 \left\langle \begin{array}{c} \left(\cos\left(\omega_d t\right)\cos\phi_1 - \sin\left(\omega_d t\right)\sin\left(\phi_1\right)\right) \\ \times\left(\cos\left(\omega_d t\right)\cos\phi_2 - \sin\left(\omega_d t\right)\sin\left(\phi_2\right)\right)\end{array}\right\rangle \\ &= i_0 v_0 \left\langle \begin{array}{c} \cos^2\left(\omega_d t\right)\cos\phi_1\cos\phi_2 \\ +\sin^2\left(\omega_d t\right)\sin\phi_1\sin\phi_2 \\ -\cos\left(\omega_d t\right)\sin\left(\omega_d t\right)\cos\phi_1\sin\phi_2 \\ -\cos\left(\omega_d t\right)\sin\left(\omega_d t\right)\cos\phi_2\sin\phi_1 \end{array}\right\rangle\end{aligned}$$

Since the averages

$$\left\langle\cos^2\left(\omega_d t\right)\right\rangle = \left\langle\sin^2\left(\omega_d t\right)\right\rangle = \frac{\left\langle\cos^2\left(\omega_d t\right)+\sin^2\left(\omega_d t\right)\right\rangle}{2} = \frac{1}{2},$$

and

$$\left\langle\cos\left(\omega_d t\right)\sin\left(\omega_d t\right)\right\rangle = \frac{1}{2}\left\langle\sin\left(2\omega_d t\right)\right\rangle = 0,$$

we get

$$\begin{aligned}\langle P\rangle &= \frac{1}{2} i_0 v_0 \left\langle\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_2\right\rangle \\ &= \frac{1}{2} i_0 v_0 \cos\left(\phi_1-\phi_2\right) = \frac{1}{2}\operatorname{Re}\left[i_0 v_0 e^{j\left(\phi_1-\phi_2\right)}\right] \\ &= \frac{1}{2}\operatorname{Re}\left[i_0 v_0 e^{j\left(\omega_d t+\phi_1\right)} e^{-j\left(\omega_d t+\phi_2\right)}\right] \\ &= \frac{1}{2}\operatorname{Re}\left[i_0 e^{j\left(\omega_d t+\phi_1\right)}\left(v_0 e^{j\left(\omega_d t+\phi_2\right)}\right)^*\right] \\ &= \frac{1}{2}\operatorname{Re}\left[I V^*\right] = \frac{1}{2}\operatorname{Re}\left[\left(I V^*\right)^*\right] = \frac{1}{2}\operatorname{Re}\left[I^* V\right]\end{aligned}$$

Let X be the device impedance that satisfies

$$V = IX$$

Then we also have

$$\langle P\rangle = \frac{1}{2}\operatorname{Re}\left[I^* V\right] = \frac{1}{2}\operatorname{Re}\left[I^* I X\right] = \frac{1}{2}\left|I\right|^2\operatorname{Re}\left[X\right] = \frac{1}{2} i_0^2 \operatorname{Re}\left[X\right] \tag{13}$$

Only real part of X contributes the the average dissipation of energy.

For the current $i(t) = i_0 \cos(\omega_d t + \phi_1)$, define the root mean square i_{rms} as

$$i_{rms} = \sqrt{\langle i^2(t) \rangle} = i_0 \sqrt{\langle \cos^2(\omega_d t + \phi_1) \rangle} = \frac{i_0}{\sqrt{2}}$$

Similarly, the root mean square for the voltage $v(t) = \cos(\omega_d t + \phi_2)$ is

$$v_{rms} = \sqrt{\langle v^2(t) \rangle} = v_0 \sqrt{\langle \cos^2(\omega_d t + \phi_2) \rangle} = \frac{v_0}{\sqrt{2}}$$

For a resistance with $X = R$, and $v_0 = i_0 R$, we have $v(t) = Ri(t)$, $v_0 = Ri_0$, and

$$\langle P_R \rangle = \frac{1}{2} i_0^2 \operatorname{Re}[X] = \frac{1}{2} i_0^2 R = \left(\frac{i_0}{\sqrt{2}} \right)^2 R$$

$$= i_{rms}^2 R = \frac{\left(\frac{i_0 R}{\sqrt{2}} \right)^2}{R} = \frac{v_{rms}^2}{R}$$

For L and C, the impedances $X_L = j\omega_d L$ and $X_C = -j\frac{1}{\omega_d C}$. X_L and X_C are purely imaginary, the average power dissipated therefore vanishes

$$\langle P_L \rangle = \frac{1}{2} i_0^2 \operatorname{Re}[X_L] = 0$$

$$\langle P_C \rangle = \frac{1}{2} i_0^2 \operatorname{Re}[X_C] = 0$$

The L and C devices store and release energy but do not consume energy.

31.7.1 Example, Power Dissipation of Driven RLC circuit

A series RLC circuit, driven by $\xi_{rms} = 120V$ at frequency $f_d = \frac{\omega_d}{2\pi} = 60Hz$, contains a resistance $R = 200\Omega$, an inductance with $X_L = 80.0\Omega j$, and a capacitance with $X_C = -150\Omega j$.
(a) What is the average rate $\langle P \rangle$ at which energy is dissipated in the resistance?

The complex current $I(t)$ is related to the complex emf ξ_c by

$$I = \frac{\xi_c}{R + X_L + X_c}.$$

Let ξ_0 be the amplitude of the oscillating emf $\xi(t)$. We then have

$$\xi_0 = |\xi_c| = \sqrt{2}\xi_{rms}$$

According to (13),

$$
\begin{aligned}
\langle P\rangle &= \frac{1}{2}\left|I^2\right|\mathrm{Re}\left[X\right] = \frac{1}{2}\left|\frac{\xi_c}{R+X_L+X_c}\right|^2 R \\
&= \frac{1}{2}\frac{\left(\sqrt{2}\times 120V\right)^2}{\left|\left(200-70.0j\right)\Omega\right|^2}\times 200\Omega = \frac{120^2}{200^2+70.0^2}\times 200W \\
&= \frac{120^2}{200^2+70.0^2}\times 200W = 64.1W
\end{aligned}
$$

(b) What new capacitance C_{new} is needed to maximize $\langle P\rangle$ if the other parameters of this circuit are not changed?

$$X_C \to X_{C_{new}} = -\frac{1}{\omega_d C_{new}}j$$

$$\langle P\rangle \to \frac{1}{2}\left|\frac{\xi_c}{R+X_L+X_{C_{new}}}\right|^2 R$$

$\langle P\rangle$ is maximized when

$$\left|R+X_L+X_{C_{new}}\right|^2 = \left|200\Omega + 80.0j\Omega - \frac{1}{\omega_d C_{new}}j\right|^2$$

attains the minimum value, which occurs at

$$80.0j\Omega - \frac{1}{\omega_d C_{new}}j = 0$$

$\Rightarrow$

$$
\begin{aligned}
C_{new} &= \frac{1}{\omega_d\times 80.0\Omega} = \frac{1}{\left(2\pi\right)\left(60Hz\right)\left(80.0\Omega\right)} = 3.32\times 10^{-5}F \\
&= 33.2\mu F
\end{aligned}
$$

31.8 Transformers

In electrical power distribution systems it is desirable for reasons of safety and for efficient equipment design to deal with relatively low voltages at both the generating end (the electrical power plant) and the receiving end (the home or factory).

Nobody wants an electric toaster or a child's electric train to operate at, say, $10kV$.

On the other hand, in the transmission of electrical energy from the generating plant to the consumer, we want the lowest practical current (hence the largest practical voltage) to minimize i^2R losses (often called ohmic losses) in the transmission line.

A device with which we can raise and lower the ac voltage in a circuit, keeping the product current voltage essentially constant, is called the transformer.

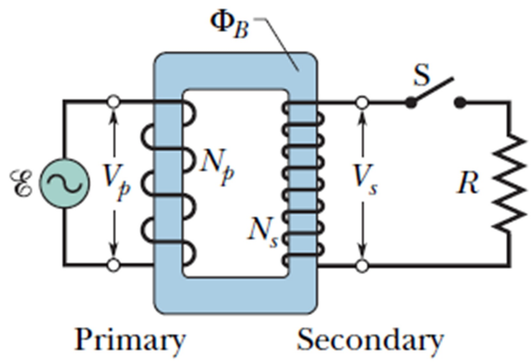


Fig. 31-18 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load R when switch S is closed.

The ideal transformer consists of two coils, with different numbers of turns, wound around an iron core.

In use, the primary winding, of N_p turns, is connected to an alternating-current generator whose emf at any time t is given by

$$\xi(t) = \xi_0 \cos(\omega) t$$

The secondary winding, of N_s turns, is connected to load resistance R, but its circuit is an open circuit as long as switch S is open.

The small sinusoidally changing primary current I_{mag} produces a sinusoidally changing magnetic flux B in the iron core.

Because B varies, it induces an emf ($-\frac{d\phi_B}{dt}$) in each turn of the secondary. This emf per turn ξ_{1turn} is the same in the primary and the secondary. Across the primary, the voltage $V_p = \xi_{1turn}N_p$. Similarly, across the secondary the voltage is $V_s = \xi_{1turn}N_s$. Thus

$$V_s = V_p\frac{N_s}{N_p} \quad \text{(transformation of voltage)}$$

If $N_s > N_p$, the device is a step-up transformer because it steps the primary's voltage V_p up to a higher voltage V_s. Similarly, if $N_s < N_p$, it is a step-down transformer.

If no energy is lost along the way, conservation of energy requires that

$$I_sV_s = I_pV_p$$

$\Rightarrow$

$$I_s = I_p\frac{V_p}{V_s} = I_p\frac{N_p}{N_s}$$

Now

$$I_s = \frac{V_s}{R} = V_p\frac{N_s}{N_p}\frac{1}{R}$$

Hence

$$I_p\frac{N_p}{N_s} = V_p\frac{N_s}{N_p}\frac{1}{R}$$

or

$$V_p = I_p\left(\frac{N_p}{N_s}\right)^2 R = I_pR_{eq}$$

Here R_{eq} is the load resistance as "seen" by the generator.

For maximum transfer of energy from an emf device to a resistive load, the resistance of the emf device must equal the resistance of the load. To see this, assume r is the internal resistance and R is the resistance of the load, then

$$P_R = i^2R = \left(\frac{\xi}{r+R}\right)^2 R.$$

Maximum of P_R occurs at

$$0 = \frac{dP_R}{dR} = \xi^2\frac{d}{dR}\left(\frac{R}{(r+R)^2}\right) = \xi^2\frac{R-r}{(r+R)^3}$$

i.e., at $R = r$. For ac circuits, for the same to be true, the impedance (rather than just the resistance) of the generator must equal that of the load.

31.8.1 Example

A transformer on a utility pole operates at $V_p = 8.5kV$ on the primary side and supply electrical energy to a number of nearby houses at $V_s = 120V$, both quantities being rms values. Assume an ideal step-down transformer, a purely resistive load.
(a) What is the turns ratio $\frac{N_p}{N_s}$ of the transformer?

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{8.5kV}{120V} = 71$$

(b) The average rate of energy consumption (or dissipation) in the houses served by the transformer is $78kW$. What are the rms currents in the primary and secondary of the transformer?

$$I_p = \frac{\langle P \rangle}{V_p} = \frac{78kW}{8.5kV} = 9.2A$$

$$I_s = \frac{\langle P \rangle}{V_s} = \frac{78kW}{120V} = 650A$$

(c) What is the resistive load R, in the secondary circuit?

$$R_s = \frac{V_s}{I_s} = \frac{120V}{650A} = 0.18\Omega$$

Similarly, for the primary circuit

$$R_p = \frac{V_p}{I_p} = \frac{8.5kV}{9.2A} = 930\Omega$$