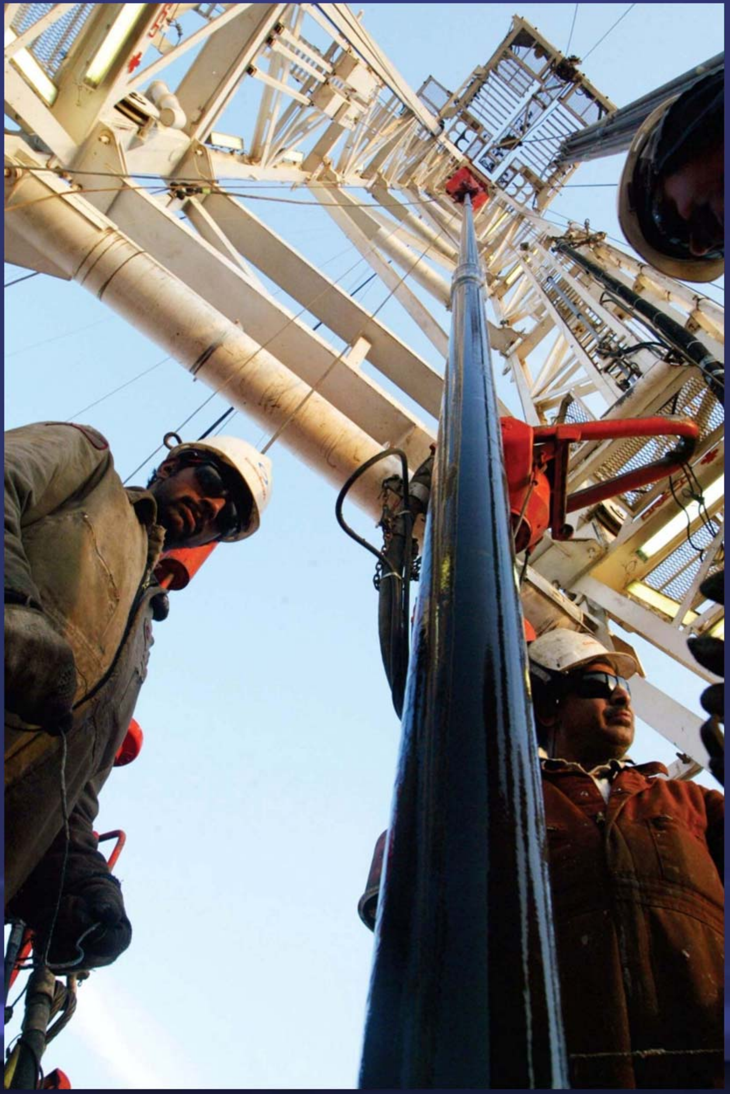
Chapter 2
Axially Loaded Members



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2.1 INTRODUCTION

Axially Loaded Members.

Examples:

- Truss members,
- Connecting rods in engines,
- Spokes in bicycle wheels,
- Columns in buildings,
- Struts in aircraft engine mounts

2.2 CHANGES IN LENGTHS OF AXIALLY LOADED MEMBERS

- *Tension*
- *Compression*

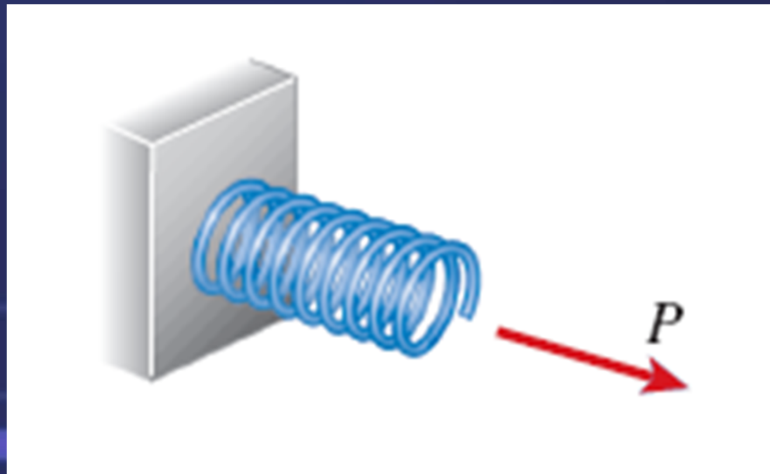


FIG. 2-1 Spring subjected to an axial load P

If the material of the spring is linearly elastic, the load and elongation will be proportional:

$$P = k\delta \qquad \delta = fP \qquad \text{(2-1a,b)}$$

FIG. 2-2 Elongation of an axially loaded.

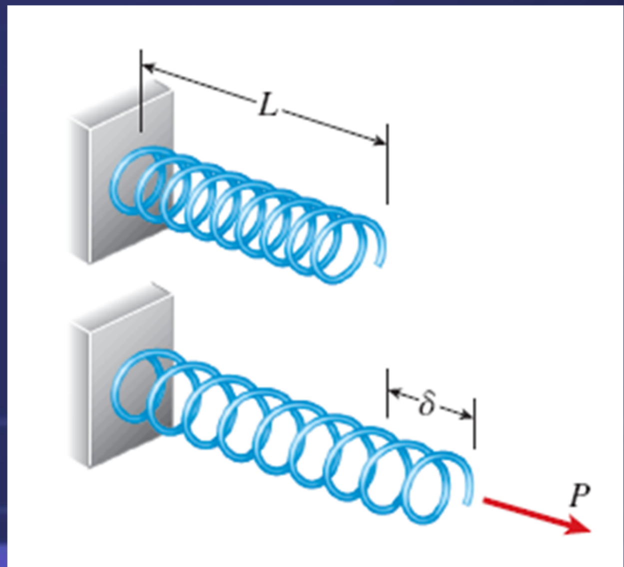
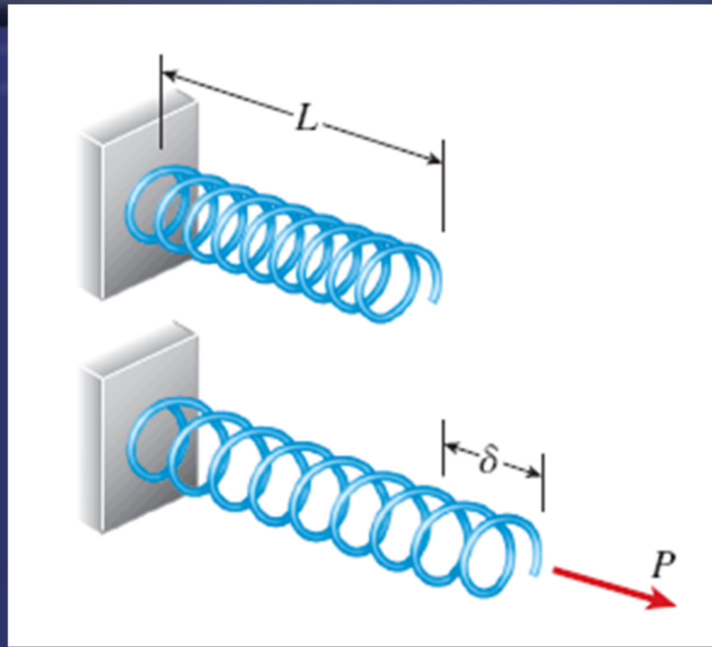


FIG. 2-2 Elongation of an axially loaded.



k : stiffness, $k = P/\delta$.

f : flexibility, $f = \delta/P$.

Prismatic Bars

Prismatic bar: a structural member having a straight longitudinal axis and constant cross section throughout its length.

Elongation of the bar:

$$\delta = \frac{PL}{EA} \qquad (2\text{-}3)$$

The product *EA* is known as the axial rigidity of the bar.

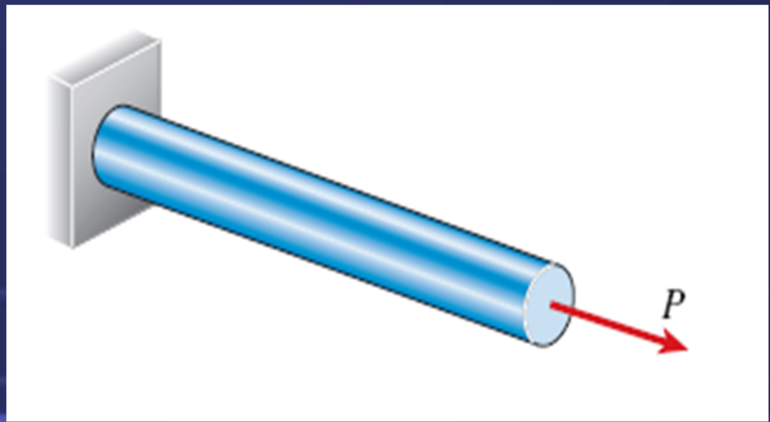
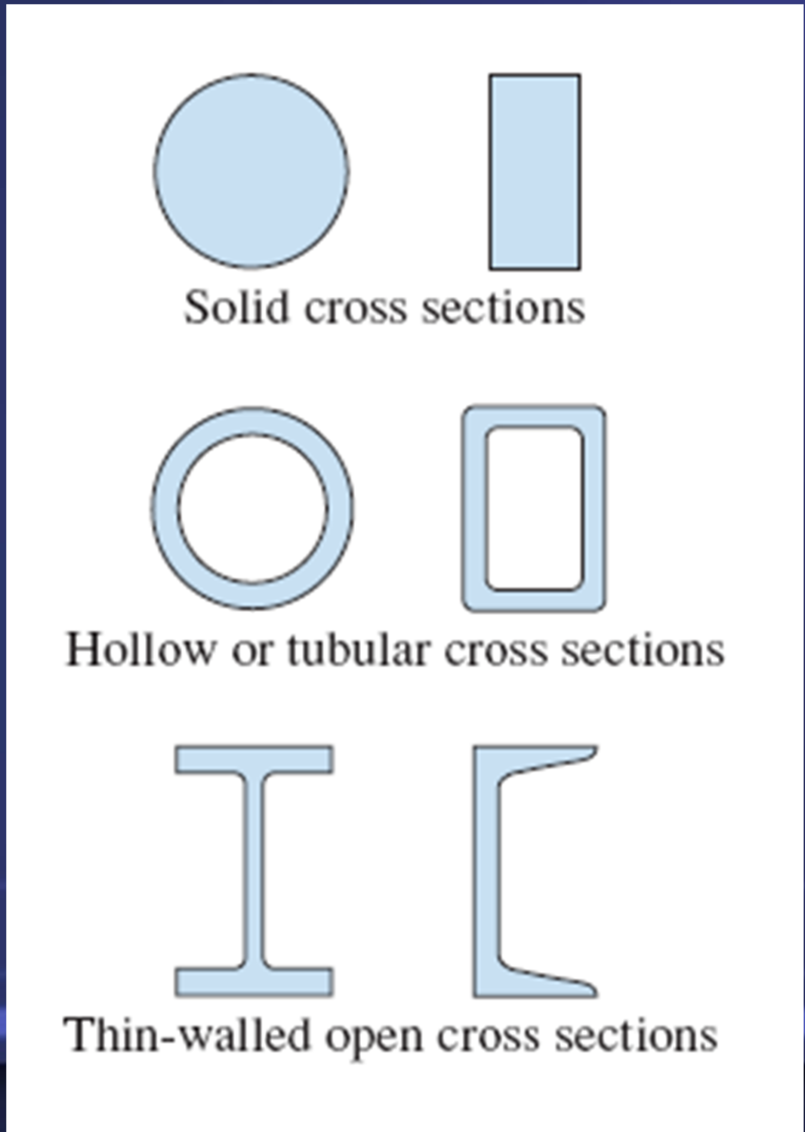


FIG. 2-3 Prismatic bar of circular cross-section

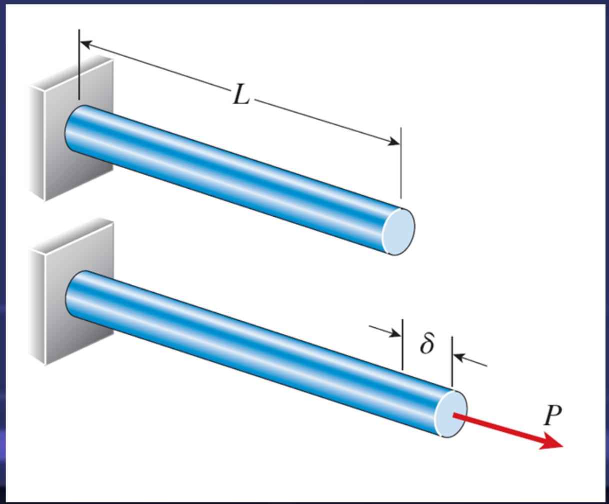
FIG. 2-4 Typical cross sections of structural members



The equation of stiffness and flexibility of a prismatic bar are, respectively,

$$k = \frac{EA}{L} \qquad f = \frac{L}{EA} \qquad \text{(2-4a,b)}$$

FIG. 2-5 Elongation of a prismatic bar in tension



2.3 CHANGES IN LENGTHS UNDER NONUNIFORM CONDITIONS

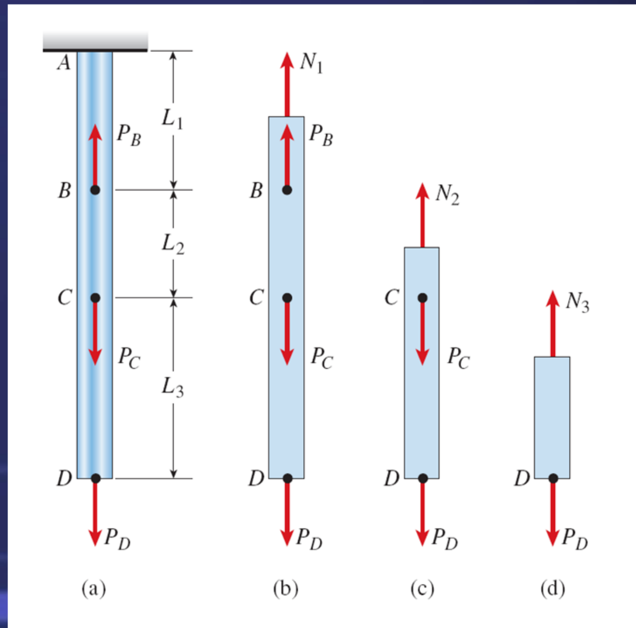
By summing forces in the vertical direction, we obtain the following expressions for the axial forces:

$$N_1 = -P_B + P_C + P_D \qquad N_2 = P_C + P_D \qquad N_3 = P_D$$

Determine the changes in the lengths of the segments

$$\delta_1 = \frac{N_1 L_1}{EA} \qquad \delta_2 = \frac{N_2 L_2}{EA} \qquad \delta_3 = \frac{N_3 L_3}{EA}$$

$$\delta = \sum_{i=1}^{n} \frac{N_i L_i}{E_i A_i} \qquad \text{(2-5)}$$



The elongation *dδ of the differential element (Fig. 2-11c) may be* obtained

$$d\delta = \frac{N(x)dx}{EA(x)} \quad \text{(2-6)}$$

$$\delta = \int_0^L d\delta = \int_0^L \frac{N(x)dx}{EA(x)} \quad \text{(2-7)}$$

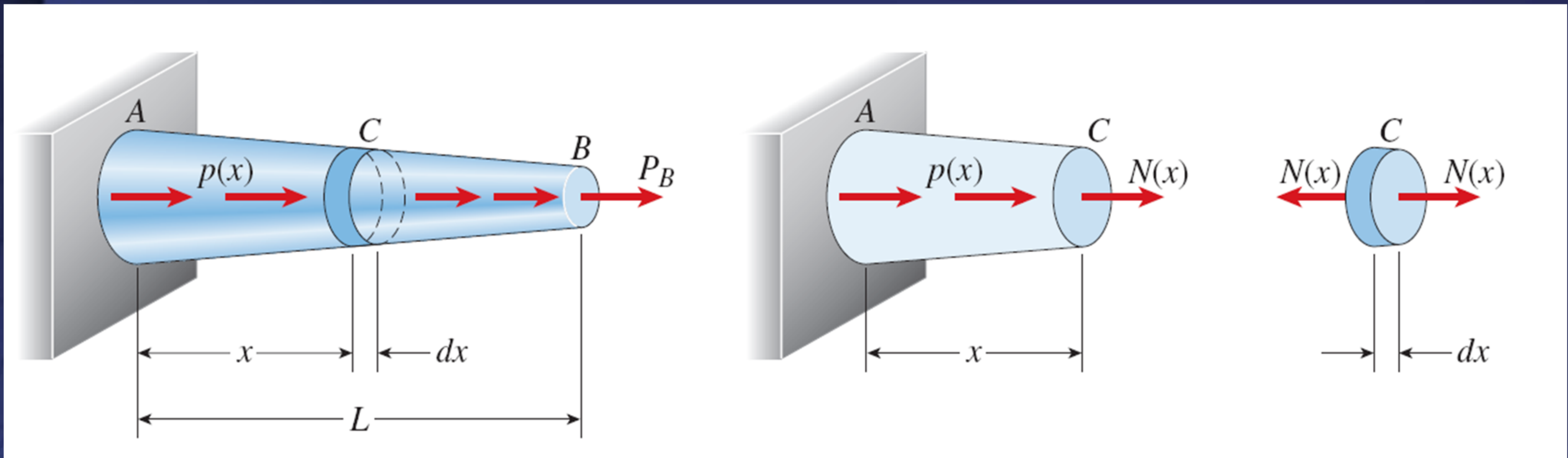


FIG. 2-11 Bar with varying cross-sectional area and varying axial force

Example 2-4

A tapered bar AB of solid circular cross section and length L (Fig. 2-13a) is supported at end B and subjected to a tensile load P at the free end A. The diameters of the bar at ends A and B are d_A and d_B, respectively. Determine the elongation of the bar due to the load P, assuming that the angle of taper is small.

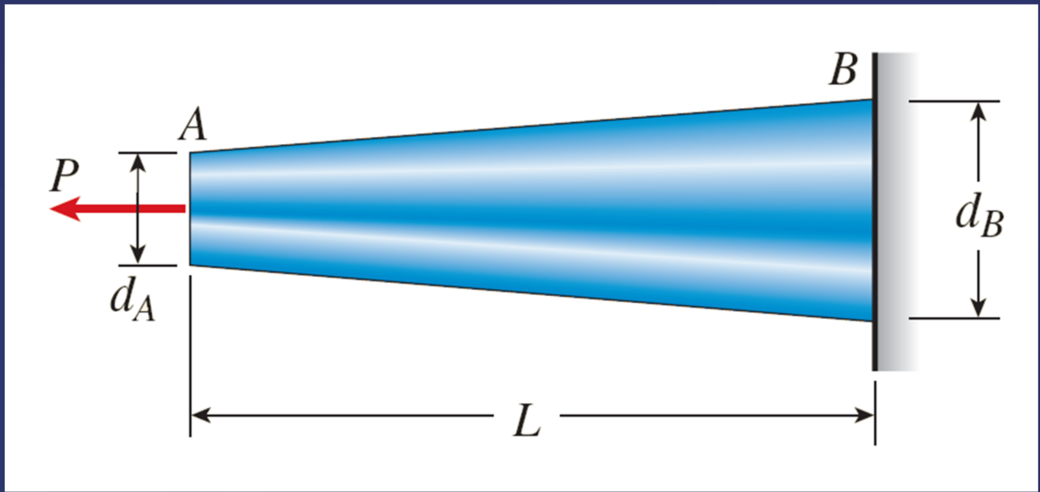
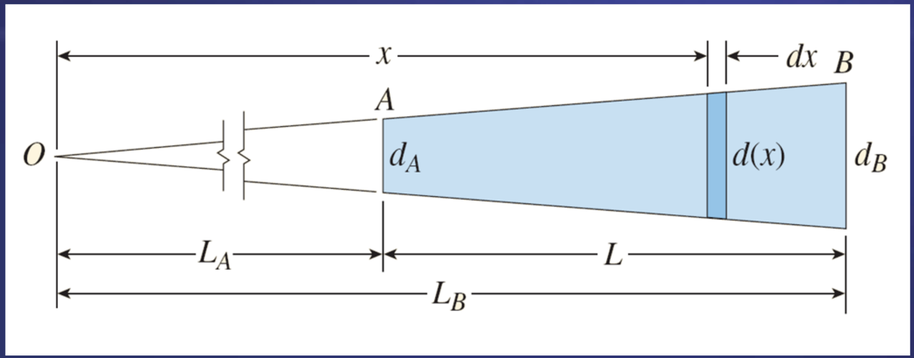


FIG. 2-13 Example 2-4. Change in length of a tapered bar of solid circular cross section.

Solution

Cross-sectional area



The distances L_A and L_B from the origin O to ends A and B, respectively, are in the ratio

$$\frac{L_A}{L_B} = \frac{d_A}{d_B} \qquad \text{(a)}$$

The ratio of the diameter d(x) at distance x from the origin to the diameter d_A at the small end of the bar:

$$\frac{d(x)}{d_A} = \frac{x}{L_A} \quad \text{or} \quad d(x) = \frac{d_A x}{L_A} \qquad \text{(b)}$$

Therefore, the cross-sectional area at distance x from the origin is

$$A(x) = \frac{\pi\left[d(x)\right]^2}{4} = \frac{\pi d_A^2 x^2}{4L_A^2} \qquad \text{(c)}$$

Change in length. The elongation δ:

$$\delta = \int \frac{N(x)dx}{EA(x)} = \int_{L_A}^{L_B} \frac{Pdx(4L_A^2)}{E(\pi d_A^2 x^2)} = \frac{4PL_A^2}{\pi E d_A^2}\int_{L_A}^{L_B} \frac{dx}{x^2} \qquad \text{(d)}$$

By performing the integration, we get

$$\delta = \frac{4PL_A^2}{\pi E d_A^2}\left[-\frac{1}{x}\right]_{L_A}^{L_B} = \frac{4PL_A^2}{\pi E d_A^2}\left(\frac{1}{L_A} - \frac{1}{L_B}\right) \qquad \text{(e)}$$

This expression for δ *can be simplified by noting that*

$$\frac{1}{L_A} - \frac{1}{L_B} = \frac{L_B - L_A}{L_A L_B} = \frac{L}{L_A L_B} \qquad \text{(f)}$$

δ *becomes*

$$\delta = \frac{4PL}{\pi E d_A^2}\left(\frac{L_A}{L_B}\right) \qquad \text{(g)}$$

We substitute $L_A / L_B = d_A / d_B$ and obtain

$$\delta = \frac{4PL}{\pi E d_A d_B} \qquad \text{(2-8)}$$

2.4 STATICALLY INDETERMINATE STRUCTURES

Structures of this type are classified as statically determinate.

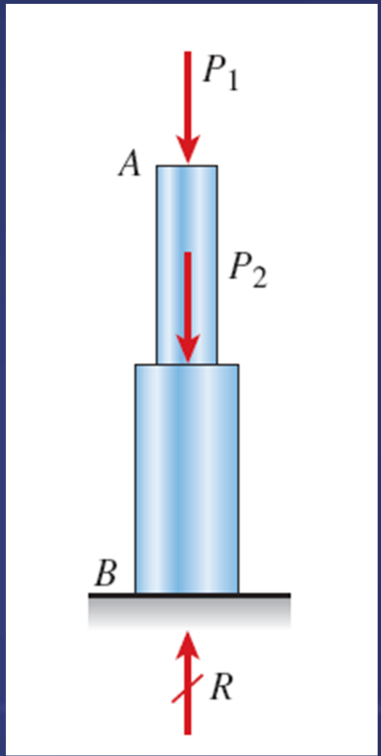


FIG. 2-14 Statically determinate bar

Structures of this kind are classified as statically indeterminate.

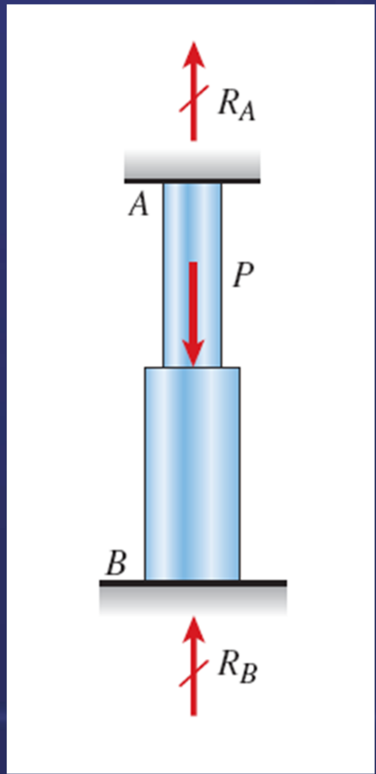


FIG. 2-15 Statically indeterminate bar

The prismatic bar *AB* is attached to rigid supports at both ends and is axially loaded by a force *P* at an intermediate point *C*.

$$\sum F_{\text{vert}} = 0 \qquad R_A - P + R_B = 0 \qquad \text{(a)}$$

The bar to change in length by an amount δ_{AB} , which must be equal to zero :

$$\delta_{AB} = 0 \qquad \text{(b)}$$

This equation, called an equation of compatibility

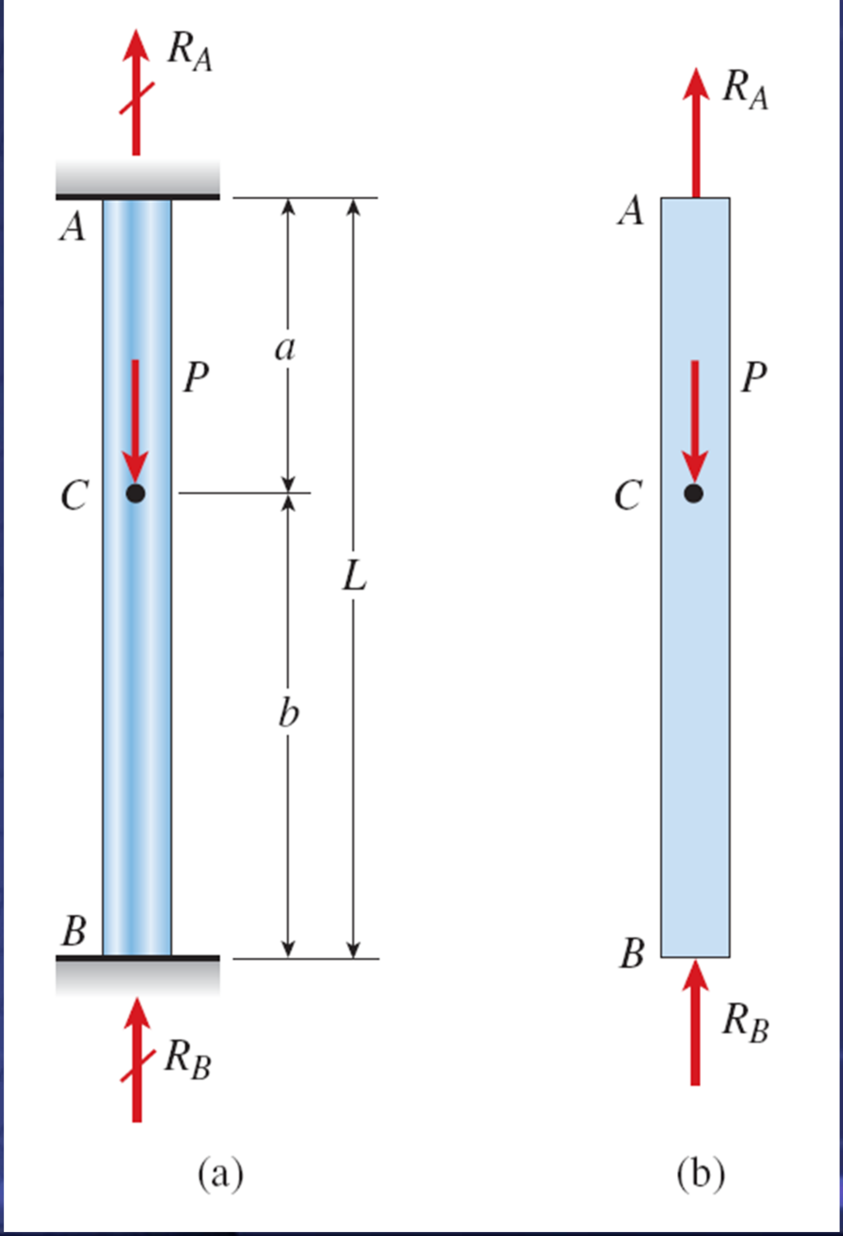


FIG. 2-16

Analysis of a statically indeterminate bar

Let us assume that the bar of Fig. 2-16 has cross-sectional area *A* and is made of a material with modulus *E*.

Then the changes in lengths of the upper and lower segments of the bar are, respectively,

$$\delta_{AC} = \frac{R_A a}{EA} \qquad \delta_{CB} = -\frac{R_B b}{EA} \qquad \text{(c,d)}$$

The change in length of the entire bar :

$$\delta_{AB} = \delta_{AC} + \delta_{CB} = \frac{R_A a}{EA} - \frac{R_B b}{EA} = 0 \qquad \text{(e)}$$

Thus, the equation of compatibility now becomes

$$\frac{R_A a}{EA} - \frac{R_B b}{EA} = 0 \qquad \text{(e)}$$

The results are

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L} \qquad \text{(2-9a,b)}$$

This displacement is equal to the elongation of segment *AC* :

$$\delta_C = \delta_{AC} = \frac{R_A a}{EA} = \frac{Pab}{LEA} \qquad \text{(2-10)}$$

Example 2-5

A solid circular steel cylinder *S* is encased in a hollow circular copper tube *C* (Figs. 2-17a and b). The cylinder and tube are compressed between the rigid plates of a testing machine by compressive forces *P*. The steel cylinder has cross-sectional area A_s and modulus of elasticity E_s, the copper tube has area A_c and modulus E_c, and both parts have length *L*.

Determine the following quantities :

(a) the compressive forces P_s in the steel cylinder and P_c in the copper tube ;

(b) the corresponding compressive stresses σ_s and σ_c ; and

(c) the shortening δ of the assembly.

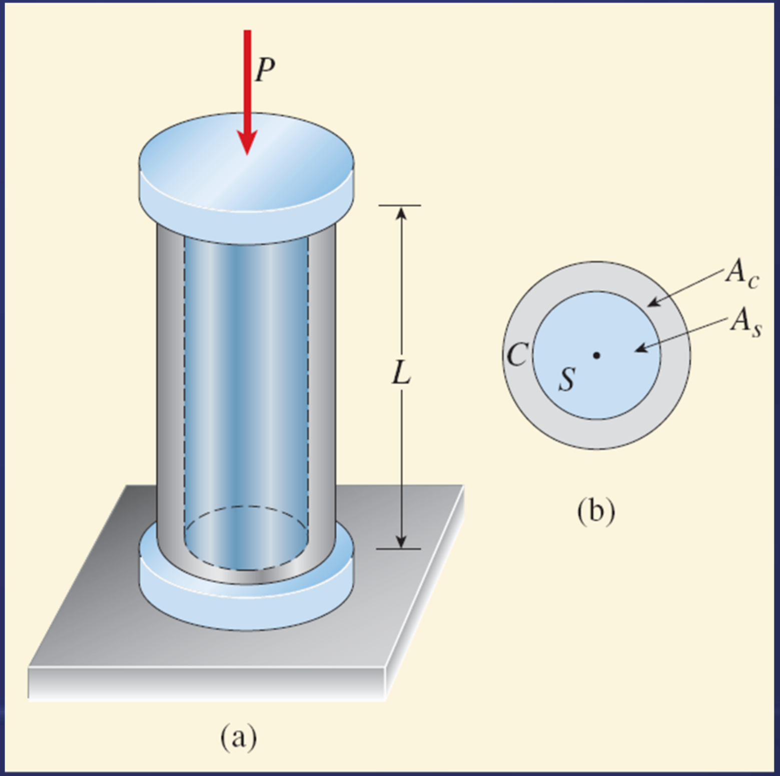
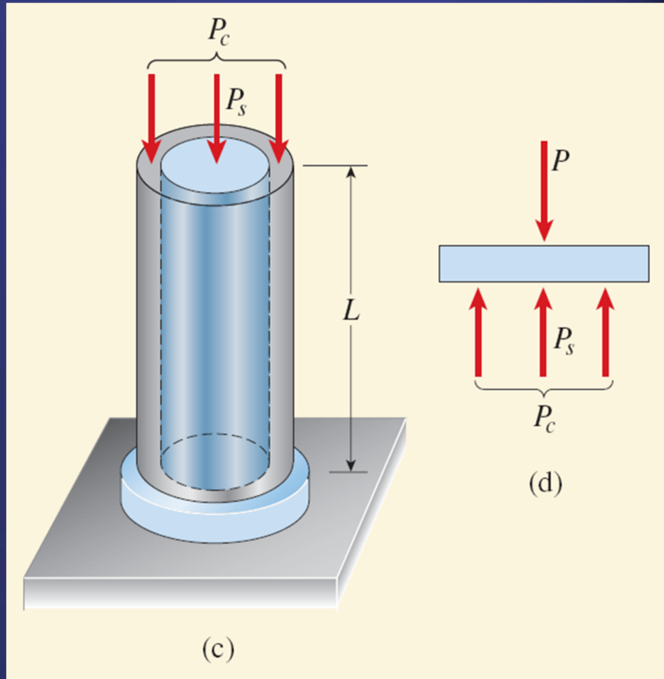


FIG. 2-17 Example 2-5. Analysis of a statically indeterminate structure

Solution

(a) Compressive forces in the steel cylinder and copper tube.

Compressive forces P_s and P_c acting on the steel cylinder and copper tube.



The equation of equilibrium is

$$\sum F_{\text{vert}} = 0 \qquad R_S + P_C - P = 0 \qquad \text{(f)}$$

Denoting the shortenings of the steel and copper parts by δ_s and δ_c, respectively, we obtain the following equation of compatibility:

$$\delta_S = \delta_C \qquad \text{(g)}$$

The force-displacement relations are

$$\delta_S = \frac{P_S L}{E_S A_S} \qquad \delta_C = \frac{P_S L}{E_C A_C} \qquad \text{(h,i)}$$

Substitute the force-displacement relations in the equation of compatibility, which gives

$$\frac{P_S L}{E_S A_S} = \frac{P_C L}{E_C A_C} \qquad \text{(j)}$$

Solution of equations.

We now solve simultaneously the equation of equilibrium and the preceding equation of compatibility and obtain the axial forces in the steel cylinder and copper tube:

$$P_S = P\left(\frac{E_S A_S}{E_S A_S + E_C A_C}\right) \qquad P_C = P\left(\frac{E_C A_C}{E_S A_S + E_C A_C}\right) \qquad \text{(2-11a,b)}$$

(b) *Compressive stresses in the steel cylinder and copper tube.*

$$\sigma_S = \frac{P_S}{A_S} = \frac{PE_S}{E_S A_S + E_C A_C} \qquad \sigma_C = \frac{P_C}{A_C} = \frac{PE_C}{E_S A_S + E_C A_C} \qquad \text{(2-12a,b)}$$

(c) From shortening of the assembly then we get

$$\delta = \frac{P_S L}{E_S A_S} = \frac{P_C L}{E_C A_C} = \frac{PL}{E_S A_S + E_C A_C} \qquad \text{(2-13)}$$

2.5 THERMAL EFFECTS

Thermal strain ϵ_T is proportional to the temperature change T; that is,

$$\epsilon_T = \alpha(\Delta T) \qquad (2\text{-}15)$$

α is called the coefficient of thermal expansion.

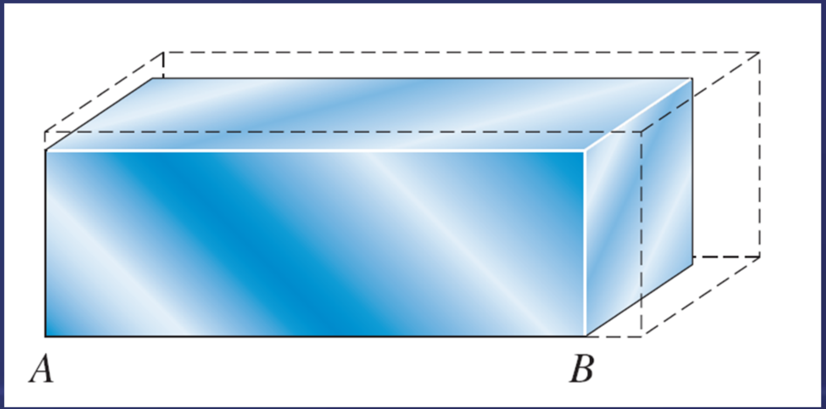


FIG. 2-19 Block of material subjected to an increase in temperature

$$\delta_T = \in_T L = \alpha(\Delta T)L \qquad (2\text{-}16)$$

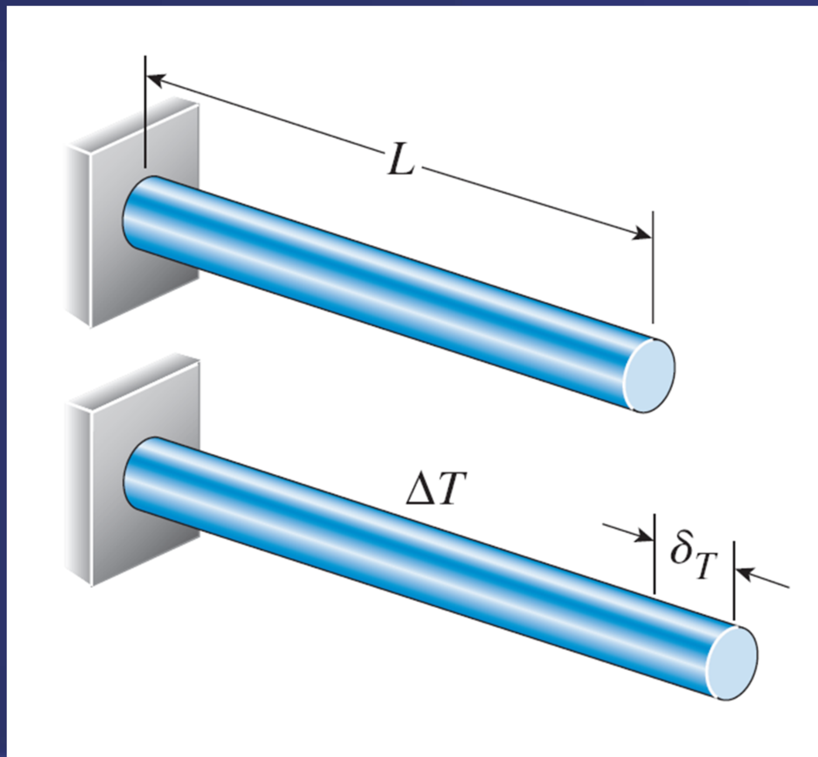
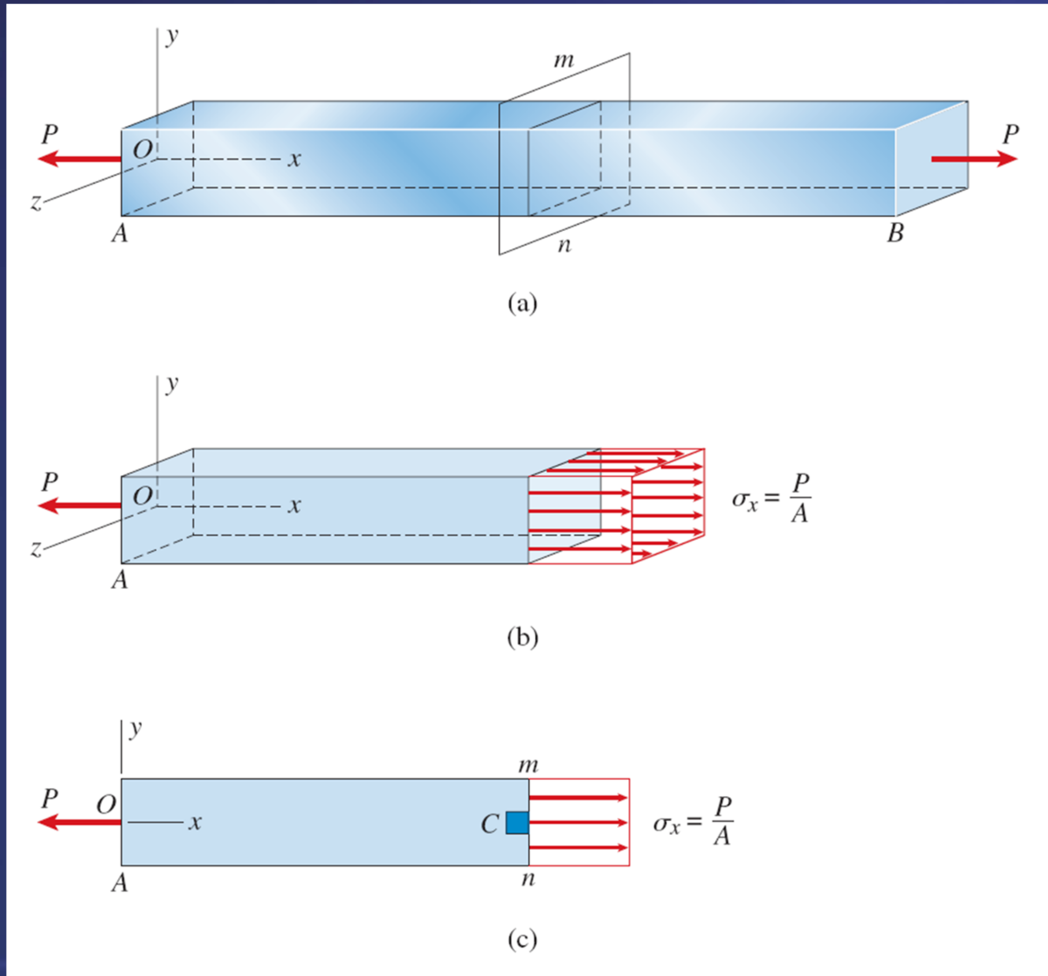


FIG. 2-20 Increase in length of a prismatic bar due to a uniform increase in temperature

2.6 STRESSES ON INCLINED SECTIONS



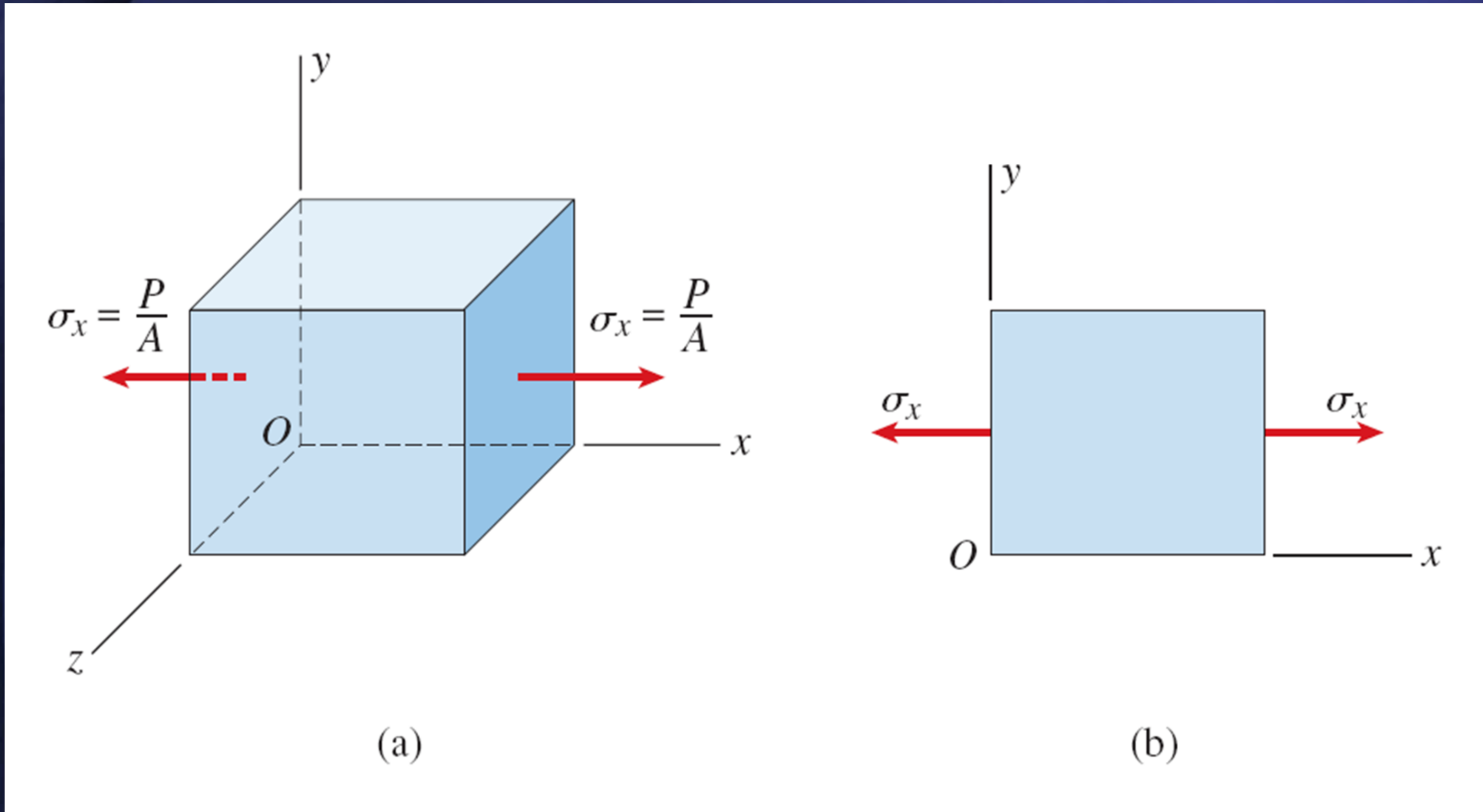


FIG. 2-31 Stress element at point C of the axially loaded bar shown in Fig. 2-30c:
(a) three-dimensional view of the element, and
(b) two-dimensional view of the element

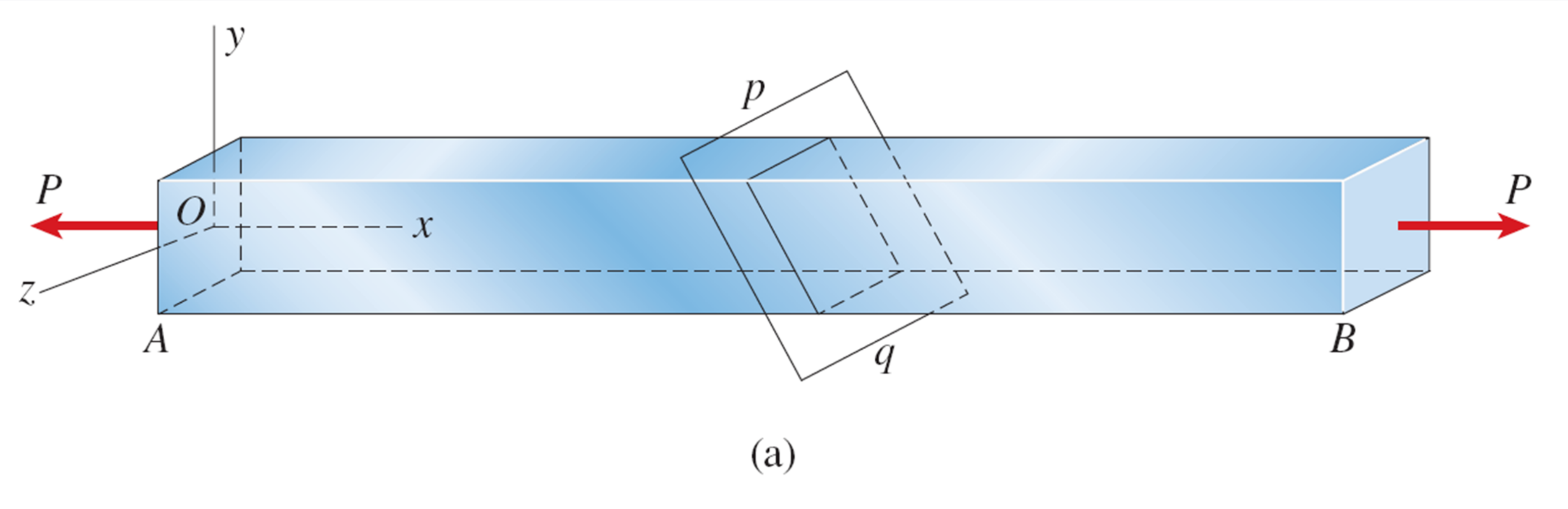


FIG. 2-32 Prismatic bar in tension showing the stresses acting on an inclined section *pq* :
(a) bar with axial forces *P*

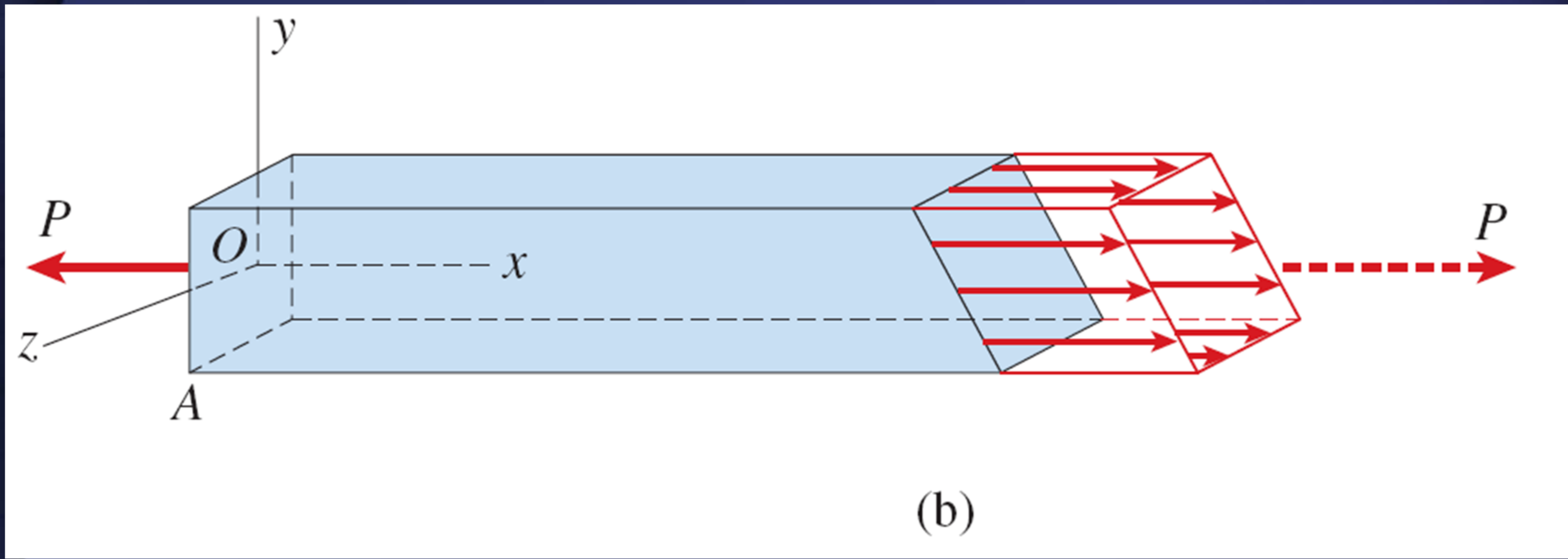


FIG. 2-32 (b) three-dimensional view of the cut bar showing the stresses

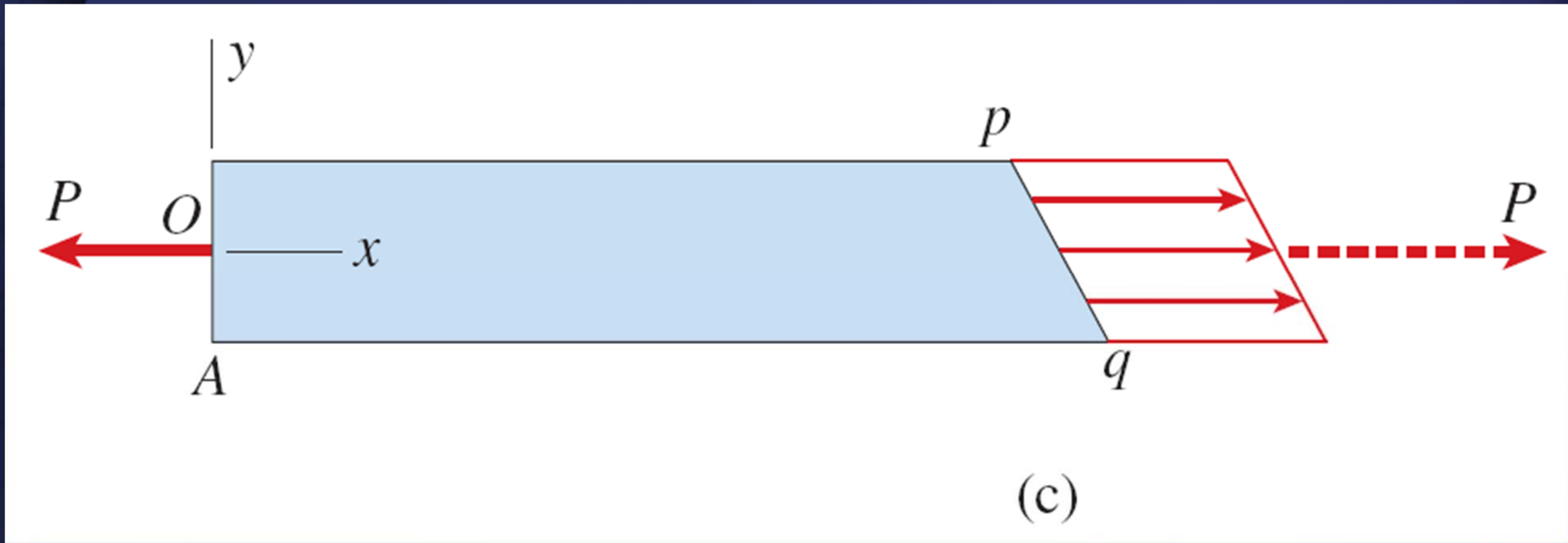


FIG. 2-32 (c) two-dimensional view

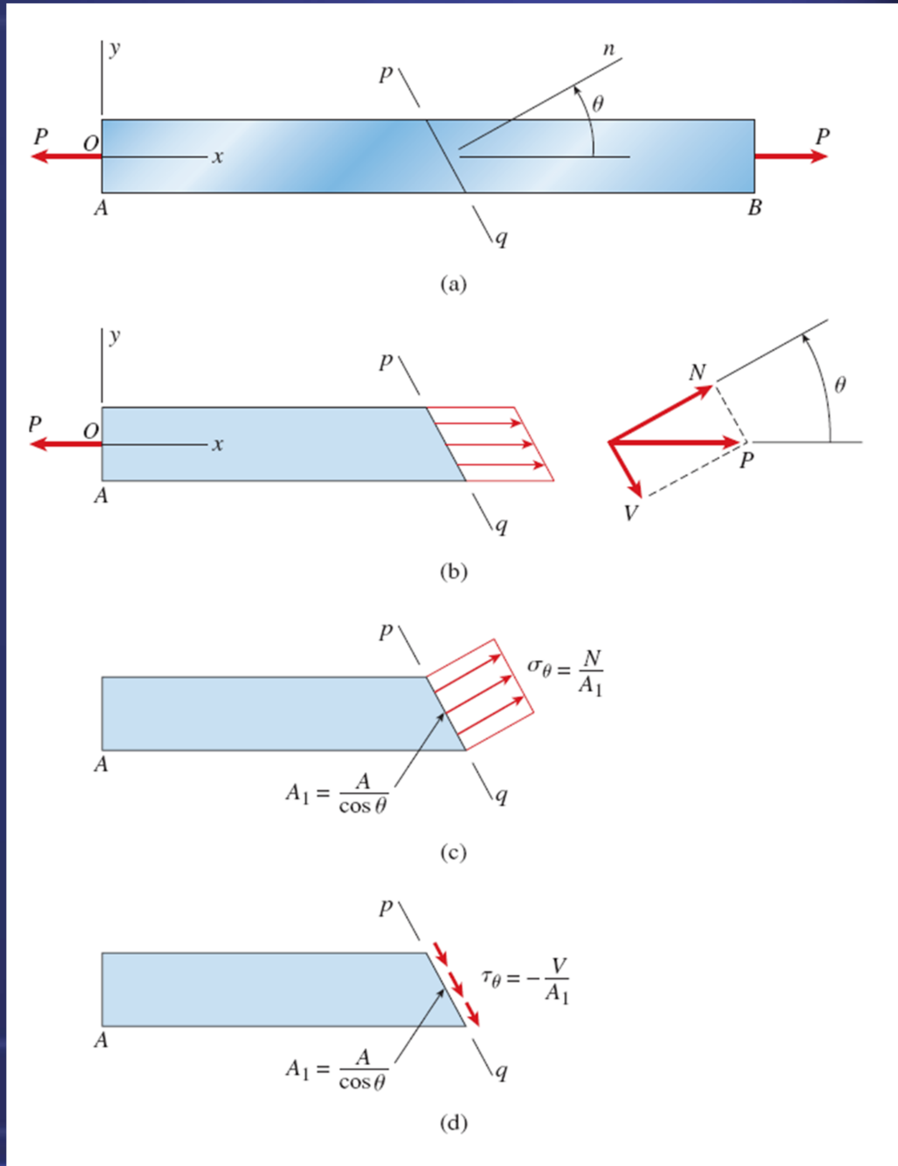


FIG. 2-33 Prismatic bar in tension showing the stresses acting on an inclined section *pq*

These force components are

$$N = P\cos\theta \quad V = P\sin\theta \qquad \text{(2-26a,b)}$$

The stresses are

$$\sigma = \frac{N}{A_1} \qquad \tau = \frac{V}{A_1} \qquad \text{(2-27a,b)}$$

The area of the inclined section :

$$A_1 = \frac{A}{\cos\theta} \qquad \text{(2-28)}$$

Normal stresses σ_u are positive in tension and shear stresses τ_u are positive when they tend to produce counterclockwise rotation of the material, as shown in Fig. 2-34.

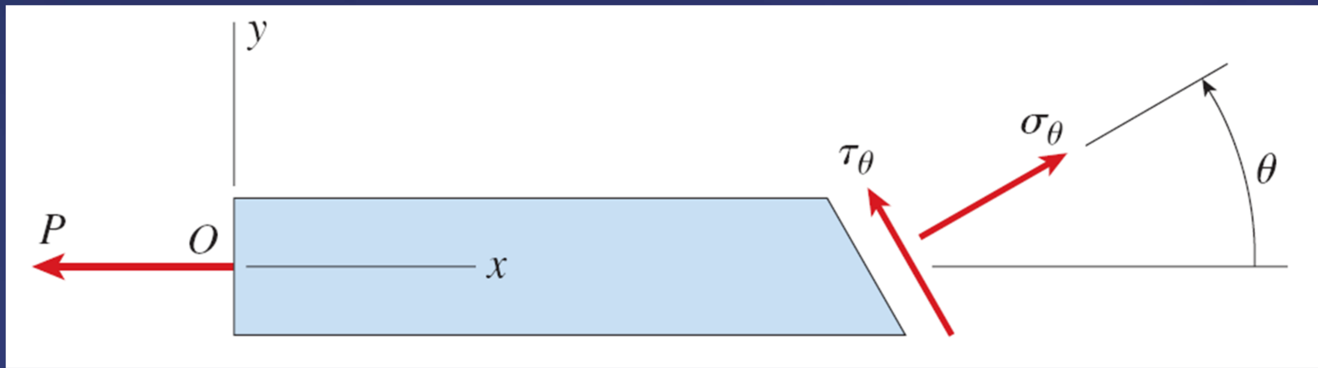


FIG. 2-34 Sign convention for stresses acting on an inclined section. (Normal stresses are positive when in tension and shear stresses are positive when they tend to produce counterclockwise rotation.)

These stresses are

$$\sigma_\theta = \frac{N}{A_1} = \frac{P}{A}\cos^2\theta \qquad \tau_\theta = -\frac{N}{A_1} = -\frac{P}{A}\sin\theta\cos\theta$$

Using the trigonometric relations

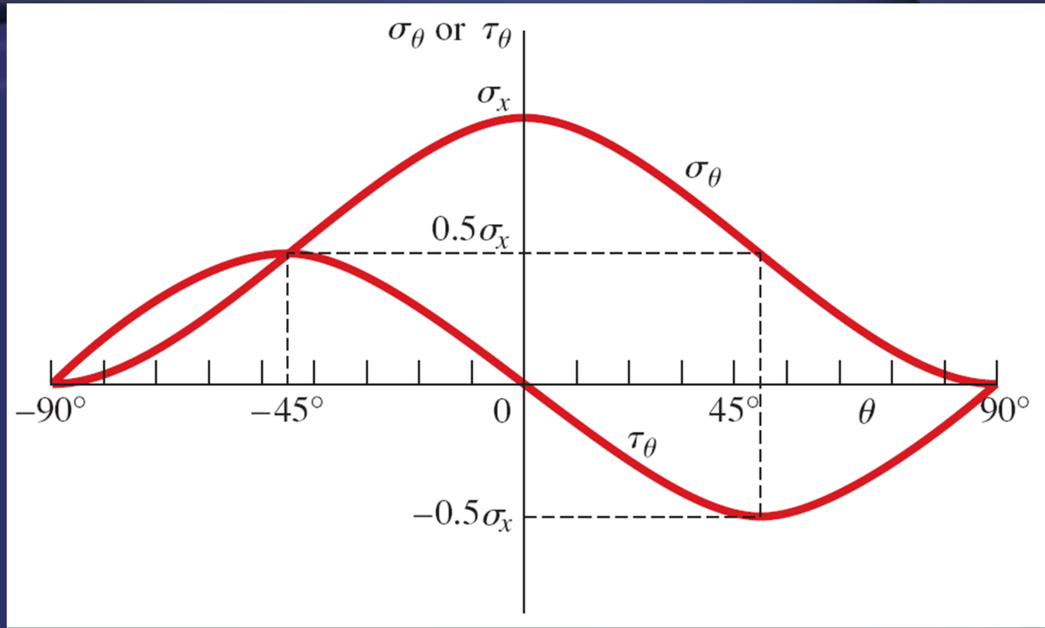
$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \qquad \sin\theta\cos\theta = \frac{1}{2}(\sin 2\theta)$$

We get

$$\sigma_\theta = \sigma_x\cos^2\theta = \frac{\sigma_x}{2}(1+\cos 2\theta) \qquad \text{(2-29a)}$$

$$\tau_\theta = -\sigma_x\sin\theta\cos\theta = -\frac{\sigma_x}{2}(\sin 2\theta) \qquad \text{(2-29b)}$$

FIG. 2-35 Graph of normal stress σ_θ and shear stress τ_θ versus angle θ of the inclined section (see Fig. 2-34 and Eqs. 2-29a and b)



The **maximum normal stress** occurs at θ = 0 and is

$$\sigma_{\max} = \sigma_x \quad \text{(2-30)}$$

These **maximum shear stresses** have the same magnitude:

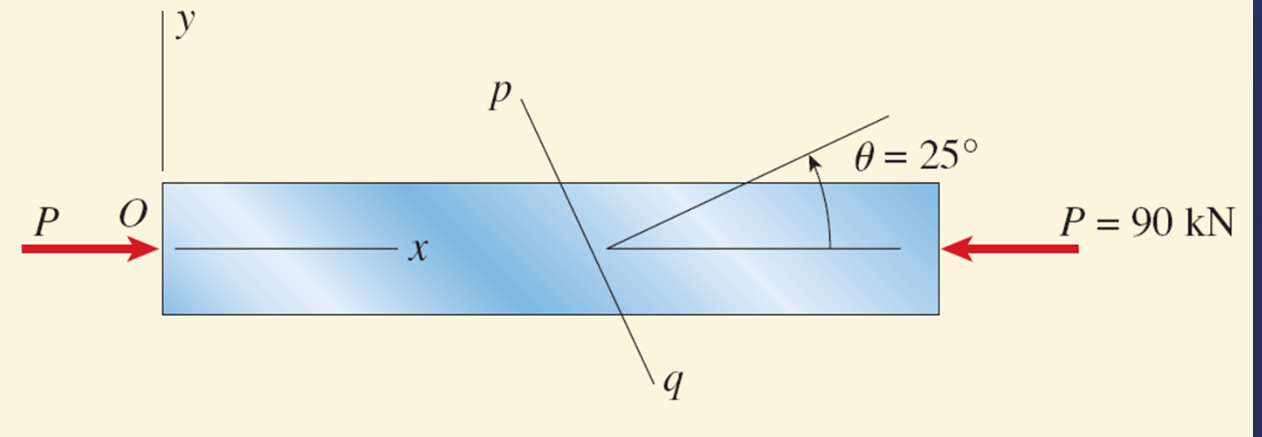
$$\tau_{\max} = \frac{\sigma_x}{2} \quad \text{(2-31)}$$

Example 2-10

A prismatic bar having cross-sectional area A = 1200 mm^2 is compressed by an axial load P = 90 kN (Fig. 2-39a).

(a) Determine the stresses acting on an inclined section *pq* cut through the bar at an angle $\theta = 25^\circ$.

(b) Determine the complete state of stress for $\theta = 25^\circ$ and show the stresses on a properly oriented stress element.



(a)

Solution

The normal stress σ_x acting on a cross section:

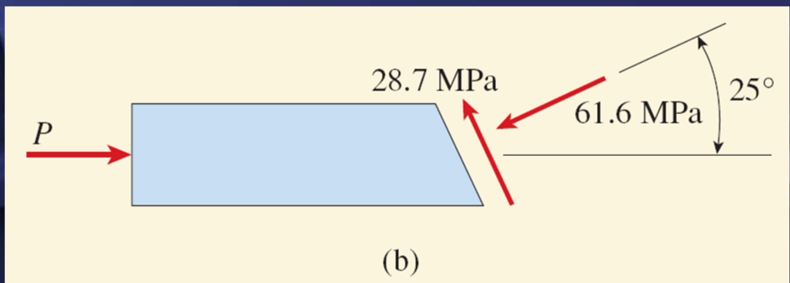
$$\sigma_x = -\frac{P}{A} = -\frac{90\text{kN}}{1200\text{mm}^2} = -75\text{MPa}$$

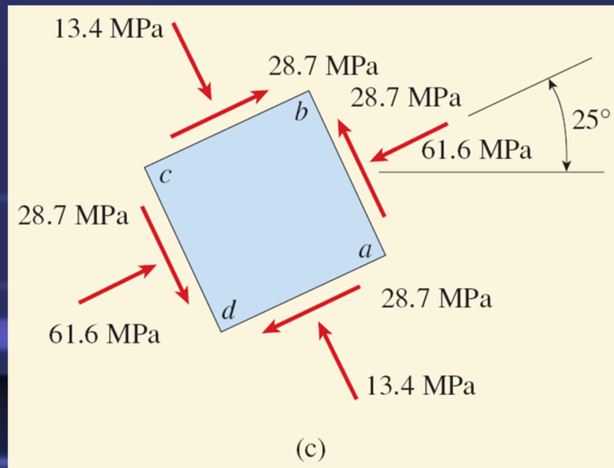
The normal and shear stresses from Eqs. (2-29a and b) with θ = 25°, as follows:

$$\sigma_\theta = \sigma_x \cos^2\theta = (-75\text{MPa})(\cos 25^\circ)^2 = -61.6\text{MPa}$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta = (75\text{MPa})(\sin 25^\circ)(\cos 25^\circ) = 28.7\text{MPa}$$

FIG. 2-39 Example 2-10. Stresses on an inclined section





The End of Chap. 2