Chapter 7: DIMENSIONAL ANALYSIS AND MODELING

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Objectives

- Understand dimensions, units, and dimensional homogeneity
- Understand benefits of dimensional analysis
- Know how to use the method of repeating variables
- Understand the concept of similarity and how to apply it to experimental modeling

DIMENSIONS AND UNITS

- A **dimension** is a measure of a physical quantity (without numerical values), while a **unit** is a way to assign a *number* to that dimension.

Primary dimensions and their associated primary SI and English units

Dimension	Symbol*	SI Unit	English Unit
Mass	m	kg (kilogram)	lbm (pound-mass)
Length	L	m (meter)	ft (foot)
Time†	t	s (second)	s (second)
Temperature	T	K (kelvin)	R (rankine)
Electric current	I	A (ampere)	A (ampere)
Amount of light	C	cd (candela)	cd (candela)
Amount of matter	N	mol (mole)	mol (mole)

Note: All nonprimary dimensions can be formed by some combination of the seven primary dimensions.

DIMENSIONAL HOMOGENEITY

- The **law of dimensional homogeneity,** stated as
  Every additive term in an equation must have the same dimensions.
  For example,

Change of total energy of a system: $\Delta E = \Delta U + \Delta \mathrm{KE} + \Delta \mathrm{PE}$

$$\Delta U = m(u_2 - u_1) \qquad \Delta \mathrm{KE} = \frac{1}{2}m(V_2^2 - V_1^2) \qquad \Delta \mathrm{PE} = mg(z_2 - z_1)$$

An equation that is not dimensionally homogeneous is a sure sign of an error.

$$\{\Delta \mathrm{KE}\} = \left\{\mathrm{Mass}\,\frac{\mathrm{Length}^2}{\mathrm{Time}^2}\right\} \quad \rightarrow \quad \{\Delta \mathrm{KE}\} = \{\mathrm{mL}^2/\mathrm{t}^2\}$$

$$\{\Delta \mathrm{PE}\} = \left\{\mathrm{Mass}\,\frac{\mathrm{Length}}{\mathrm{Time}^2}\,\mathrm{Length}\right\} \quad \rightarrow \quad \{\Delta \mathrm{PE}\} = \{\mathrm{mL}^2/\mathrm{t}^2\}$$

Nondimensionalization of Equations

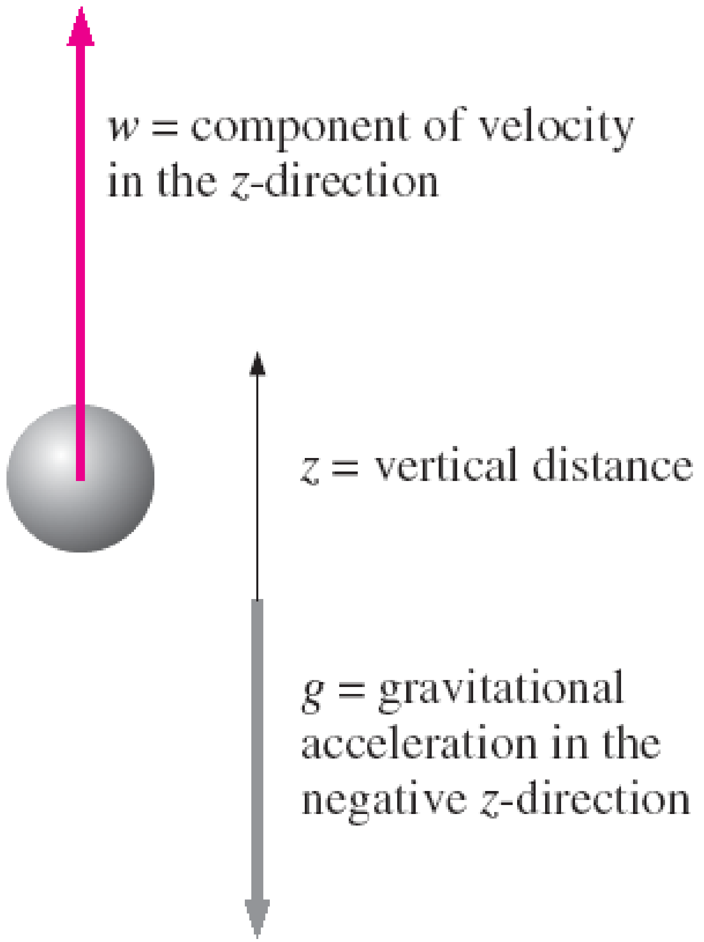
- Dimensional homogeneity $\Rightarrow$ every term in an equation has the same dimensions.
- **nondimensional** $\Rightarrow$ divide each term in the equation by a collection of variables and constants whose product has those same dimensions.
- If the nondimensional terms in the equation are of order unity $\Rightarrow$ called **normalized.**
- Normalization is thus more restrictive than nondimensionalization. (often used interchangeably).
- **Nondimensional parameters** are named after a notable scientist or engineer (e.g., the Reynolds number and the Froude number). This process is referred to by some authors as **inspectional analysis.**

Nondimensionalization of Equations — example

- An object falling by gravity through a vacuum (no air drag). The initial location of the object is z_0 and its initial velocity is w_0 in the z-direction.
- Equation of motion:

$$\frac{d^2z}{dt^2} = -g$$

- Two dimensional variables: z and t.
- Dimensional constant: g
- Two additional dimensional constants are z_0 and w_0.



Nondimensionalization of Equations — example

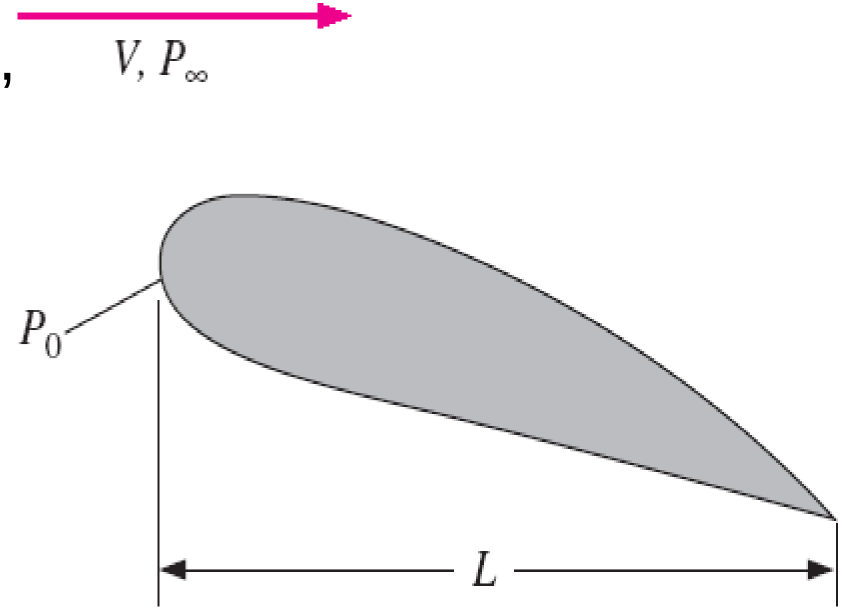
- The dimensional result is an expression for elevation *z* at any time *t*:

$$z = z_0 + w_0 t - \frac{1}{2} g t^2$$

- The constant 1/2 and the exponent 2 are called **pure constants.**
- **Nondimensional** (or **dimensionless**) **variables** are defined as quantities that change or vary in the problem, but have no dimensions.
- The term **parameters** for the combined set of dimensional variables, nondimensional variables, and dimensional constants in the problem.

Nondimensionalization of Equations

- To nondimensionalize equation, we need to select **scaling parameters** (Usually chosen from dimensional constants)**,** based on the primary dimensions contained in the original equation.



- In fluid flow problems there are typically at least *three* scaling parameters, e.g., L, V, and P_0 - P_∞, since there are at least three primary dimensions in the general problem (e.g., mass, length, and time).

Nondimensionalization of Equations — example

- In the case of the falling object, there are only two primary dimensions, length and time, and thus we are limited to selecting only *two* scaling parameters.
- We have some options in the selection of the scaling parameters since we have three available dimensional constants g, z_0, and w_0. We choose z_0 and w_0. You are invited to repeat the analysis with other combinations.
- Nondimensionalizing the dimensional variables z and t.
- The first step is to list the *Primary dimensions of all parameters*:

$$\{z\} = \{\mathrm{L}\} \qquad \{t\} = \{\mathrm{t}\} \qquad \{z_0\} = \{\mathrm{L}\} \qquad \{w_0\} = \{\mathrm{L/t}\} \qquad \{g\} = \{\mathrm{L/t^2}\}$$

Nondimensionalization of Equations — example

- The second step is to use our two scaling parameters to nondimensionalize *z* and *t* (by inspection) into nondimensional variables *z** and *t**.

$$z^* = \frac{z}{z_0} \qquad t^* = \frac{w_0 t}{z_0}$$

- Using these nondimensional variables in our equation, then we will get the desired nondimensional equation.

$$\frac{d^2 z}{dt^2} = \frac{d^2(z_0 z^*)}{d(z_0 t^*/w_0)^2} = \frac{w_0^2}{z_0}\frac{d^2 z^*}{dt^{*2}} = -g$$

$$\rightarrow \qquad \frac{w_0^2}{g z_0}\frac{d^2 z^*}{dt^{*2}} = -1$$

Nondimensionalization of Equations — example

- The grouping of dimensional constants in equation is the square of a well-known **nondimensional parameter** called the **Froude number,**

$$\mathrm{Fr} = \frac{w_0}{\sqrt{g z_0}}$$

- The Froude number can be thought of as the ratio of inertial force to gravitational force. Sometimes, Fr is defined as the *square* of the parameter.

Nondimensionalization of Equations — example

- The eq of motion can be rewritten as

$$\frac{d^2 z^*}{dt^{*2}} = -\frac{1}{\mathrm{Fr}^2}$$

- This equation can be solved easily by integrating twice. The result is

$$z^* = 1 + t^* - \frac{1}{2\mathrm{Fr}^2} t^{*2}$$

- If you redimensionalize the equation, you will get the same equation as

$$z = z_0 + w_0 t - \frac{1}{2} g t^2$$

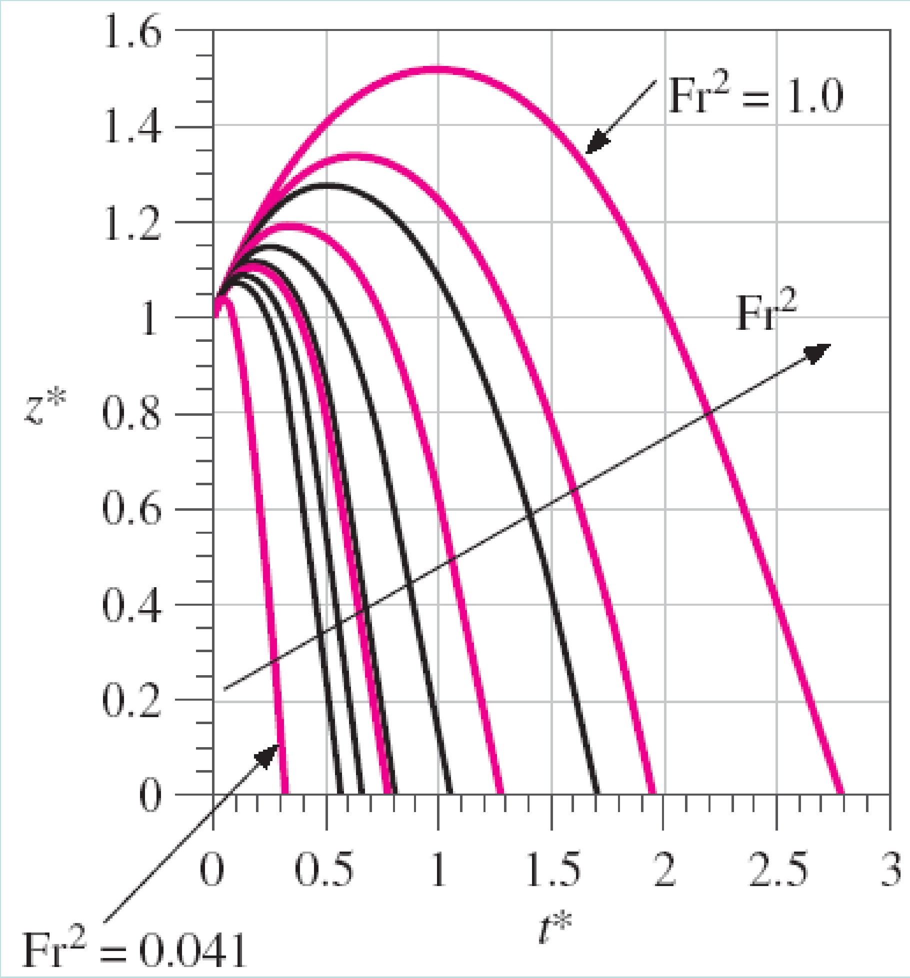
Nondimensionalization of Equations — example

- *What then is the advantage of nondimensionalizing the equation?*

- There are two key advantages of nondimensionalization.
  - First, *it increases our insight about the relationships between key parameters*. for example, that doubling w_0 has the same effect as decreasing z_0 by a factor of 4.

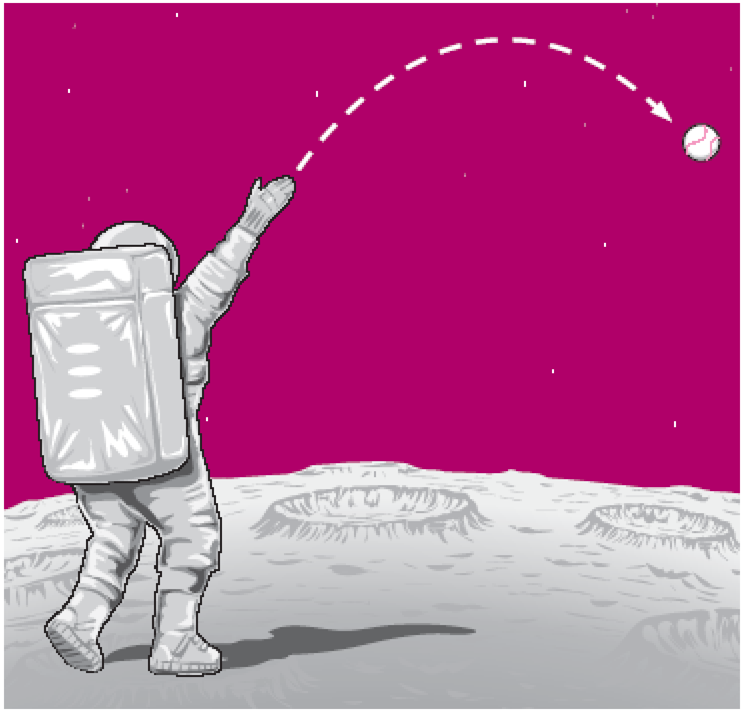
  - Second, *it reduces the number of parameters in the problem*. For example, original problem contains one *z*; one *t*; and *three* additional dimensional constants, g, w_0, and z_0. The nondimensionalized problem contains one z^*; one t^*; and only *one* additional parameter, Fr.

Nondimensionalization of Equations — example 7-3



EXAMPLE 7–4 Extrapolation of Nondimensionalized Data

- The gravitational constant at the surface of the moon is only about 1/6 of that on earth. An astronaut on the moon throws a baseball at an initial speed of 21.0 m/s at a 5° angle above the horizon and at 2.0 m above the moon's surface. (*a*) Using the dimensionless data of Example 7–3, predict how long it takes for the baseball to fall to the ground. (*b*) Do an *exact* calculation and compare the result to that of part (*a*).



EXAMPLE 7–4 Extrapolation of Nondimensionalized Data

- Solution: (*a*) The Froude number is calculated based on the value of g_{moon} and the vertical component of initial speed,

$$w_0 = (21.0 \text{ m/s}) \sin(5°) = 1.830 \text{ m/s}$$

$$\text{Fr}^2 = \frac{w_0^2}{g_{\text{moon}} z_0} = \frac{(1.830 \text{ m/s})^2}{(1.63 \text{ m/s}^2)(2.0 \text{ m})} = 1.03$$

- From Fig. 7-13, we can find t* = 2.75, Converting back to dimensional variables, we can get

$$t = \frac{t^* z_0}{w_0} = \frac{2.75(2.0 \text{ m})}{1.830 \text{ m/s}} = \mathbf{3.01 \text{ s}}$$

- *Exact time to strike the ground*:

$$t = \frac{w_0 + \sqrt{w_0^2 + 2z_0 g}}{g} = \mathbf{3.05 \text{ s}}$$

DIMENSIONAL ANALYSIS AND SIMILARITY

- Nondimensionalization of an equation is useful only when the equation is known!
- In many real-world flows, the equations are either unknown or too difficult to solve.
  - *Experimentation* is the only method of obtaining reliable information
  - In most experiments, geometrically–scaled models are used (time and money).
  - Experimental conditions and results must be properly scaled so that results are meaningful for the full-scale prototype. Therefore,
  - **Dimensional Analysis**

- Primary purposes of dimensional analysis
  - To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in reporting of results.
  - To obtain scaling laws so that prototype performance can be predicted from model performance.
  - To predict trends in the relationship between parameters.

The concept of dimensional analysis—the principle of **similarity.**

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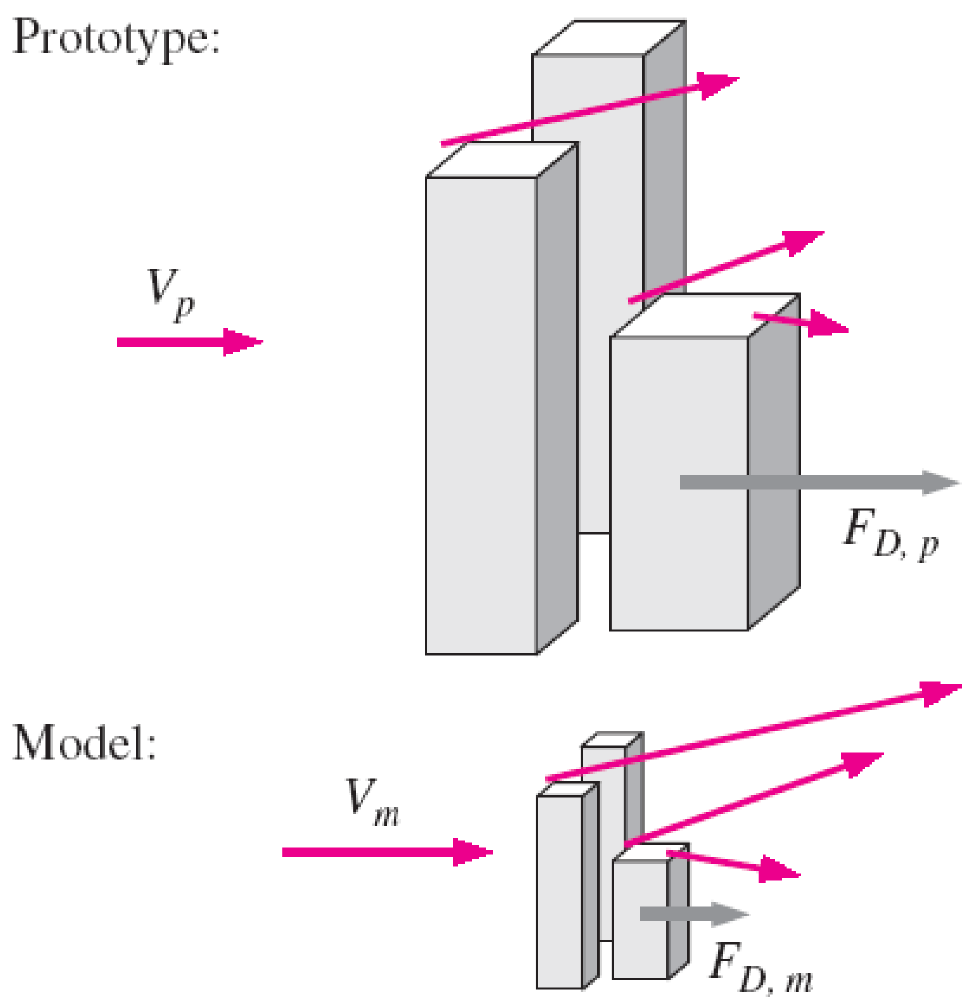


DIMENSIONAL ANALYSIS AND SIMILARITY

- Complete similarity is ensured if the model and prototype must be geometrically similar and all independent Π groups are the same between model and prototype.
- What is Π ?
- We let uppercase Greek letter Π denote a nondimensional parameter, e.g., Reynolds number *Re*, Froude number *Fr* , Drag coefficient, C_D, etc.
- In a general dimensional analysis problem, there is one Π that we call the **dependent** Π, giving it the notation Π_1. The parameter Π_1 is in general a function of several other Π's, which we call **independent** Π's. The functional relationship is

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_k)$$

DIMENSIONAL ANALYSIS AND SIMILARITY

- Consider automobile experiment
- Drag force is $F = f\,(V, \rho, \mu, L)$
- Through dimensional analysis, we can reduce the problem to

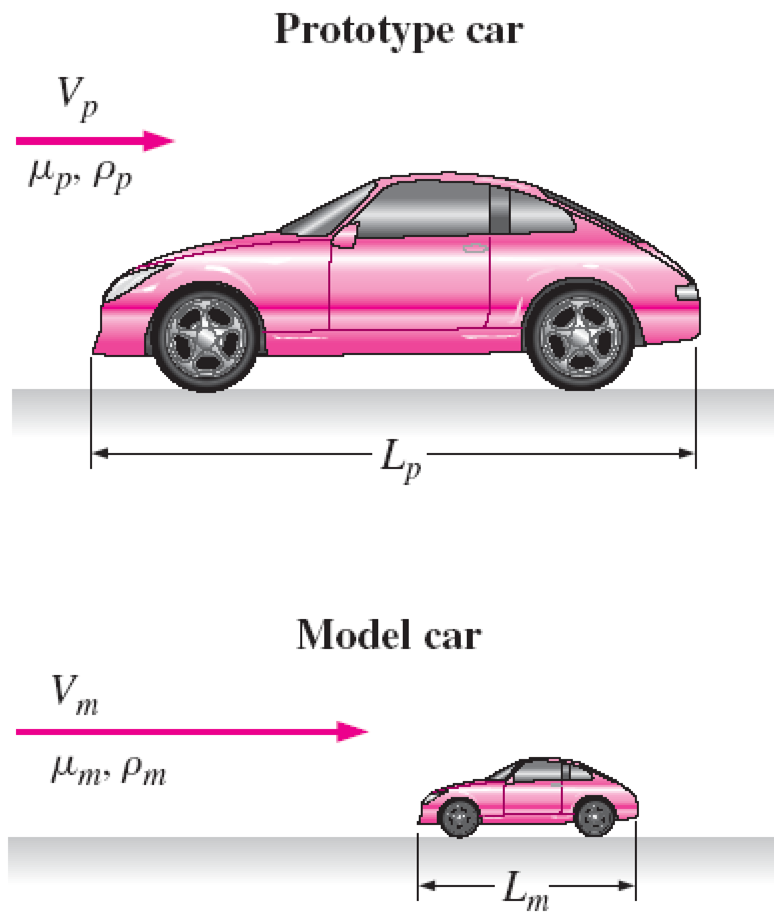
$$\Pi_1 = f\,(\Pi_2)$$

- where

$$\Pi_1 = \frac{F_D}{\rho V^2 L^2} = C_D$$

and

$$\Pi_2 = \frac{\rho V L}{\mu} = \mathrm{Re}$$



The Reynolds number is the most well known and useful dimensionless parameter in all of fluid mechanics.

EXAMPLE A: Similarity between Model and Prototype Cars

Solution

$$\Pi_{2,\,m} = \mathrm{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \Pi_{2,\,p} = \mathrm{Re}_p = \frac{\rho_p V_p L_p}{\mu_p}$$

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (50.0\ \mathrm{mi/h})\left(\frac{1.754\times10^{-5}\ \mathrm{kg/m\cdot s}}{1.849\times10^{-5}\ \mathrm{kg/m\cdot s}}\right)\left(\frac{1.184\ \mathrm{kg/m^3}}{1.269\ \mathrm{kg/m^3}}\right)(5) = \mathbf{221\ mi/h}$$

Discussion This speed is quite high (about 100 m/s), and the wind tunnel may not be able to run at that speed. Furthermore, the incompressible approximation may come into question at this high speed.

EXAMPLE B: Prediction of Aerodynamic Drag Force on the Prototype Car

Solution

$$\Pi_{1,m} = \frac{F_{D,m}}{\rho_m V_m^2 L_m^2} = \Pi_{1,p} = \frac{F_{D,p}}{\rho_p V_p^2 L_p^2}$$

$$F_{D,p} = F_{D,m}\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 = 25.3 \text{ lbf}$$

DIMENSIONAL ANALYSIS AND SIMILARITY

- In Examples A and B use a water tunnel instead of a wind tunnel to test their one-fifth scale model. Using the properties of water at room temperature (20°C is assumed), the water tunnel speed required to achieve similarity is easily calculated as

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (50.0 \text{ mi/h})\left(\frac{1.002 \times 10^{-3} \text{ kg/m}\cdot\text{s}}{1.849 \times 10^{-5} \text{ kg/m}\cdot\text{s}}\right)\left(\frac{1.184 \text{ kg/m}^3}{998.0 \text{ kg/m}^3}\right)(5) = 16.1 \text{ mi/h}$$

- The required water tunnel speed is much lower than that required for a wind tunnel using the same size model.

Method of Repeating Variables

- Nondimensional parameters Π can be generated by several methods.
- We will use the **Method of Repeating Variables** popularized by Edgar Buckingham (1867–1940) and first published by the Russian scientist Dimitri Riabouchinsky (1882–1962) in 1911.
- Six steps
  - List the parameters in the problem and count their total number *n*.
  - List the primary dimensions of each of the *n* parameters
  - Set the *reduction j* as the number of primary dimensions. Calculate *k*, the expected number of Π's, $k = n - j$ (**Buckingham Pi theorem**).
  - Choose *j repeating parameters*.
  - Construct the *k* Π's, and manipulate as necessary.
  - Write the final functional relationship and check algebra.

Method of Repeating Variables

- The best way to learn the method is by example and practice.

- As we go through each step of the method of repeating variables, we explain some of the subtleties of the technique in more detail using the falling ball as an example.

Method of Repeating Variables

- Step 1: List relevant parameters.

$$z = f(t, w_0, z_0, g) \Rightarrow n = 5$$

- Step 2: Primary dimensions of each parameter

$$\begin{array}{ccccc} z & t & w_0 & z_0 & g \\ \{L^1\} & \{t^1\} & \{L^1 t^{-1}\} & \{L^1\} & \{L^1 t^{-2}\} \end{array}$$

- Step 3: As a first guess, reduction j is set to 2 which is the number of primary dimensions (L and t). Number of expected Π's is $k = n - j = 5 - 2 = 3$
- Step 4: Choose repeating variables w_0 and z_0

Guidelines for choosing *Repeating parameters*

- Never pick the dependent variable. Otherwise, it may appear in all the Π's.
- Chosen repeating parameters must not *by themselves* be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the Π's.
- Chosen repeating parameters must represent *all* the primary dimensions.
- Never pick parameters that are already dimensionless.
- Never pick two parameters with the same dimensions or with dimensions that differ by only an exponent.
- Choose dimensional constants over dimensional variables so that only one Π contains the dimensional variable.
- Pick common parameters since they may appear in each of the Π's.
- Pick simple parameters over complex parameters.

Method of Repeating Variables

- Step 5: Combine repeating parameters into products with each of the remaining parameters, one at a time, to create the Π's.
  - $\Pi_1 = z w_0^{a_1} z_0^{b_1}$
    - a_1 and b_1 are constant exponents which must be determined.
    - Use the primary dimensions identified in Step 2 and solve for a_1 and b_1.
      $\{\Pi_1\} = \{L^0 t^0\} = \{z w_0^{a_1} z_0^{b_1}\} = \{L^1 (L^1 t^{-1})^{a_1} L^{b_1}\}$
      Time: $\{t^0\} = \{t^{-a_1}\} \rightarrow 0 = -a_1 \rightarrow a_1 = 0$
      Length: $\{L^0\} = \{L^1 L^{a_1} L^{b_1}\} \rightarrow 0 = 1 + a_1 + b_1 \rightarrow b_1 = -1 - a_1 \rightarrow b_1 = -1$
    - This results in $\Pi_1 = z w_0^0 z_0^{-1} = \frac{z}{z_0}$

Method of Repeating Variables

- Step 5, continued
  - Repeat process for Π_2 by combining repeating parameters with *t*.
    - $\Pi_2 = t w_0^{a_2} z_0^{b_2}$
    - $\{\Pi_2\} = \{L^0 t^0\} = \{t w_0^{a_2} z_0^{b_2}\} = \{t^1 (L^1 t^{-1})^{a_2} L^{b_2}\}$
      Time: $\{t^0\} = \{t^1 t^{-a_2}\} \rightarrow 0 = 1 - a_2 \rightarrow a_2 = 1$
      Length:
      $\{L^0\} = \{L^{a_2} L^{b_2}\} \rightarrow 0 = a_2 + b_2 \rightarrow b_2 = -a_2 \rightarrow b_2 = -1$
    - This results in $\Pi_2 = t w_0^1 z_0^{-1} = \frac{w_0 t}{z_0}$

Guidelines for manipulation of the Π's

- We may impose a constant (dimensionless) exponent on a Π or perform a functional operation on a Π.
- We may multiply a by a Π pure (dimensionless) constant.
- We may form a product (or quotient) of any Π with any other Π in the problem to replace one of the Π's.
- We may use any of guidelines 1 to 3 in combination.
- We may substitute a dimensional parameter in the Π with other parameter(s) of the same dimensions.

Method of Repeating Variables

- Step 6:
  - Double check that the Π's are dimensionless. Write the functional relationship between Π's.

$$\Pi_1 = f(\Pi_2, \Pi_3) \rightarrow \frac{z}{z_0} = f\left(\frac{w_0 t}{z_0}, \frac{w_0}{sqrt g z_0}\right)$$

  - Or, in terms of nondimensional variables,

$$z^* = f(t^*, Fr)$$

- Overall conclusion: Method of repeating variables properly predicts the functional relationship between dimensionless groups.
- However, the method cannot predict the exact mathematical form of the equation.

Some common established nondimensional parameters or Π's encountered in fluid mechanics and heat transfer*

Name	Definition	Ratio of Significance
Archimedes number	$\mathrm{Ar} = \dfrac{\rho_s g L^3}{\mu^2}(\rho_s - \rho)$	$\dfrac{\text{Gravitational force}}{\text{Viscous force}}$
Aspect ratio	$\mathrm{AR} = \dfrac{L}{W}$ or $\dfrac{L}{D}$	$\dfrac{\text{Length}}{\text{Width}}$ or $\dfrac{\text{Length}}{\text{Diameter}}$
Bond number	$\mathrm{Bo} = \dfrac{g(\rho_f - \rho_v)L^2}{\sigma_s}$	$\dfrac{\text{Gravitational force}}{\text{Surface tension force}}$
Cavitation number	Ca (sometimes σ_c) $= \dfrac{P - P_v}{\rho V^2}$ $\left(\text{sometimes } \dfrac{2(P - P_v)}{\rho V^2}\right)$	$\dfrac{\text{Pressure} - \text{Vapor pressure}}{\text{Inertial pressure}}$
Darcy friction factor	$f = \dfrac{8\tau_w}{\rho V^2}$	$\dfrac{\text{Wall friction force}}{\text{Inertial force}}$
Drag coefficient	$C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$	$\dfrac{\text{Drag force}}{\text{Dynamic force}}$

EXAMPLE: Pressure in a Soap Bubble

Solution

Reduction (second guess): $j = 2$

$k = n - j = 3 - 2 = 1$

need to choose two repeating parameters since $j = 2$.

our only choices are R and σ_s

Dependent Π: $\Pi_1 = \Delta P R^{a_1} \sigma_s^{b_1}$

$$\{\Pi_1\} = \{m^0 L^0 t^0\} = \{\Delta P R^{a_1} \sigma_s^{b_1}\} = \{(m^1 L^{-1} t^{-2}) L^{a_1} (m^1 t^{-2})^{b_1}\} \Rightarrow \begin{array}{l} b_1 = -1 \\ a_1 = 1 \end{array}$$

$$\Rightarrow \quad \Pi_1 = \frac{\Delta P R}{\sigma_s}$$

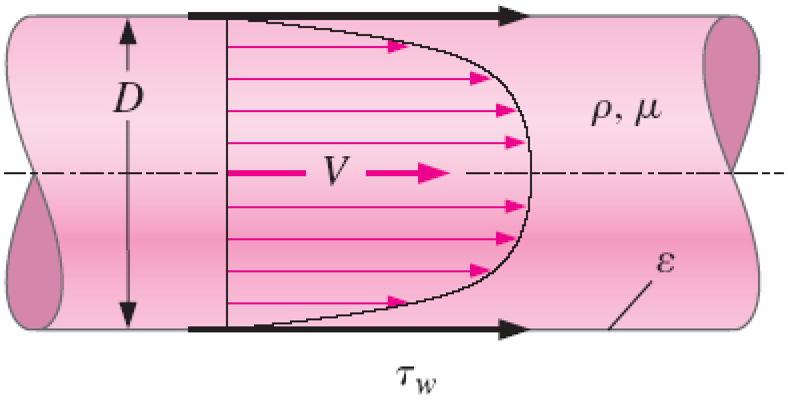
(Weber Number)

Relationship between Π*'s:*

$$\Pi_1 = \frac{\Delta P R}{\sigma_s} = \mathbf{f}(\text{nothing}) = \text{constant} \quad \rightarrow \quad \boldsymbol{\Delta P = \text{constant}\,\frac{\sigma_s}{R}}$$

EXAMPLE: Friction in a Pipe

- Consider flow shown in Fig.; *V* is the average speed across the pipe cross section. The flow is **fully developed,** which means that the velocity profile also remains uniform down the pipe. Because of frictional forces between the fluid and the pipe wall, there exists a shear stress τ_w on the inside pipe wall. The shear stress is also constant down the pipe in the region. We assume some constant average roughness height, ε, along the inside wall of the pipe. In fact, the only parameter that is *not* constant down the length of pipe is the pressure, which must decrease (linearly) down the pipe in order to "push" the fluid through the pipe to overcome friction. Develop a nondimensional relationship between shear stress τ_w and the other parameters in the problem.



EXAMPLE: Friction in a Pipe

Solution

List of relevant parameters: $\tau_w = f(V, \varepsilon, \rho, \mu, D) \qquad n = 6$

primary dimensions

τ_w	V	ε	ρ	μ	D
$\{m^1L^{-1}t^{-2}\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{m^1L^{-3}\}$	$\{m^1L^{-1}t^{-1}\}$	$\{L^1\}$

Reduction: $j = 3 \quad \Rightarrow \quad k = n - j = 3$

Repeating parameters: V, D, and ρ

$$\Pi_1 = \tau_w V^{a_1} D^{b_1} \rho^{c_1} \quad \rightarrow \quad \Pi_1 = \frac{\tau_w}{\rho V^2}$$

Modified Π_1: $\Pi_{1,\,\text{modified}} = \dfrac{8\tau_w}{\rho V^2} = \text{Darcy friction factor} = f$

EXAMPLE: Friction in a Pipe

Solution

$$\Pi_2 = \mu V^{a_2} D^{b_2} \rho^{c_2} \quad \rightarrow \quad \Pi_2 = \frac{\rho V D}{\mu} = \text{Reynolds number} = \text{Re}$$

$$\Pi_3 = \varepsilon V^{a_3} D^{b_3} \rho^{c_3} \quad \rightarrow \quad \Pi_3 = \frac{\varepsilon}{D} = \text{Roughness ratio}$$

$$\Rightarrow \quad f = \frac{8\tau_w}{\rho V^2} = f\left(\text{Re}, \frac{\varepsilon}{D}\right)$$

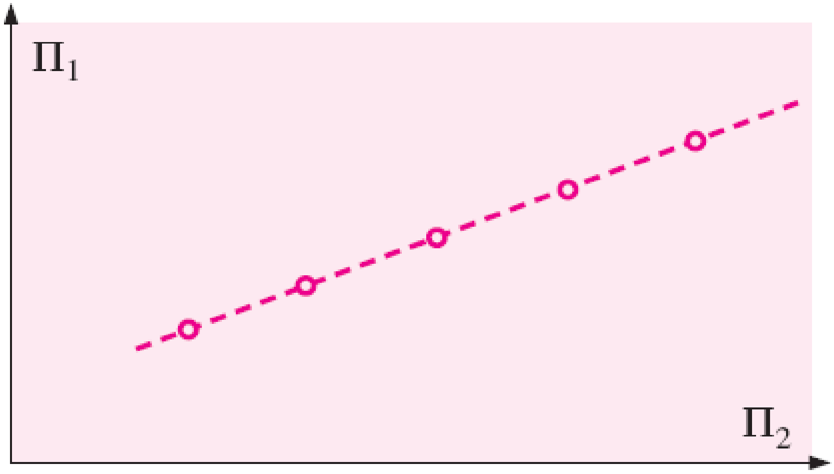
Experimental Testing and Incomplete Similarity

- On of the most useful applications of dimensional analysis is in designing physical and or numerical experiments, and in reporting the results.

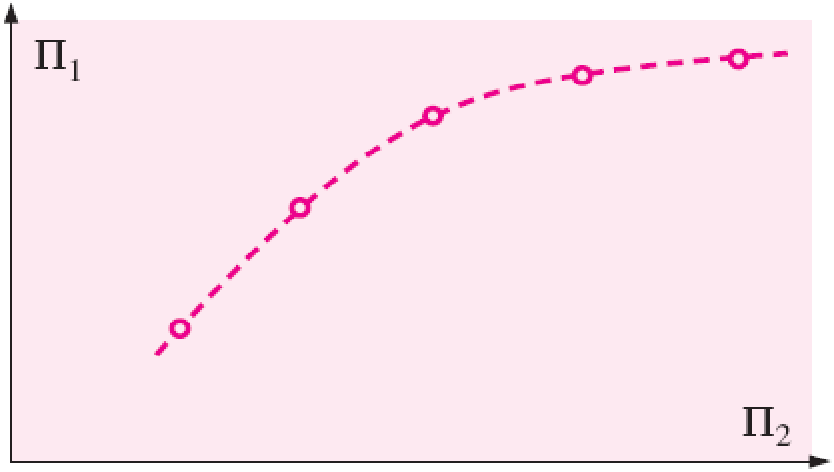
- Setup of an experiment and correlation of data
  - Consider a problem with 5 parameters: one dependent and 4 independent.
  - Full test matrix with 5 data points for each independent parameter would require $5^4 = 625$ experiments!!
  - If we can reduce to 2 Π's, the number of independent parameters is reduced from 4 to 1, which results in $5^1 = 5$ experiments vs. 625!!

Experimental Testing and Incomplete Similarity

- Discussion of a two-Π problem, once the experiments are complete, plot (Π_1) as a function of the independent dimensionless parameter (Π_2). Then determine the functional form of the relationship by performing a **regression analysis** on the data.
- More than two Π's in the problem need to set up a test matrix to determine the relationship between them. (How about only one Π Problem?)



(*a*)



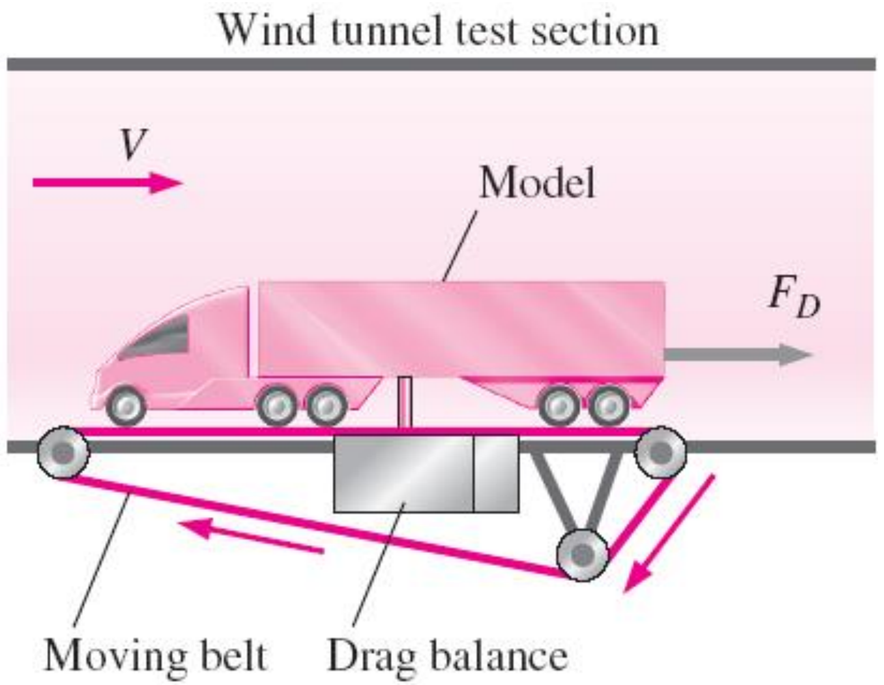
(*b*)

Experimental Testing and Incomplete Similarity

- It is not always possible to match *all* the Π's of a model to the corresponding 's of the prototype. This situation is called **incomplete similarity.**
- Fortunately, in some cases of incomplete similarity, we are still able to extrapolate model tests to obtain reasonable full-scale predictions.

Experimental Testing and Incomplete Similarity — Wind Tunnel Testing

- The problem of measuring the drag force on a model truck in a wind tunnel. Suppose a one-sixteenth geometrically similar scale model of a tractor-trailer rig is used. The model truck is 0.991 m long and to be tested in a wind tunnel that has a maximum speed of 70 m/s. The wind tunnel test section is enough without worrying about blockage effects.



The air in the wind tunnel is at the same temperature and pressure as the air flowing around the prototype. We want to simulate flow at V_p = 60 mi/h (26.8 m/s) over the full-scale prototype truck.

Experimental Testing and Incomplete Similarity — **Wind Tunnel Testing**

- The first thing we do is match the Reynolds numbers,

$$\mathrm{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \mathrm{Re}_p = \frac{\rho_p V_p L_p}{\mu_p}$$

The required wind tunnel speed for the model tests V_m is

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right) = (26.8\ \mathrm{m/s})(1)(1)\left(\frac{16}{1}\right) = 429\ \mathrm{m/s}$$

This speed is more than six times greater than the maximum achievable wind tunnel speed. Also, the flow would be *supersonic* (about 346 m/s). While the Mach number of the prototype (0.080) does not match the Mach number of the model (1.28). It is clearly not possible to match the model Reynolds number to that of the prototype with this model and wind tunnel facility.

What do we do?

Experimental Testing and Incomplete Similarity — **Wind Tunnel Testing**

- Several options to resolve the incomplete similarity:
  - **Use a bigger wind tunnel.** (Automobile manufacturers typically test with 3/8 scale model cars and with 1/8 scale model trucks and buses in very large wind tunnels.) However, it is more expensive. How big can a model be? A useful rule of thumb is that the **blockage** (ratio of the model frontal area to the cross sectional area of the test section) should be less than 7.5 percent.
  - **Use a different fluid for the model tests**. Water tunnels can achieve higher Reynolds numbers than can wind tunnels of the same size, but they are much more expensive to build and operate.
  - **Pressurize the wind tunnel and/or adjust the air temperature** to increase the maximum Reynolds number capability (limited).
  - **Run the wind tunnel at several speeds near the maximum speed**, and then extrapolate our results to the full-scale Reynolds number.

EXAMPLE: Model Truck Wind Tunnel Measurements

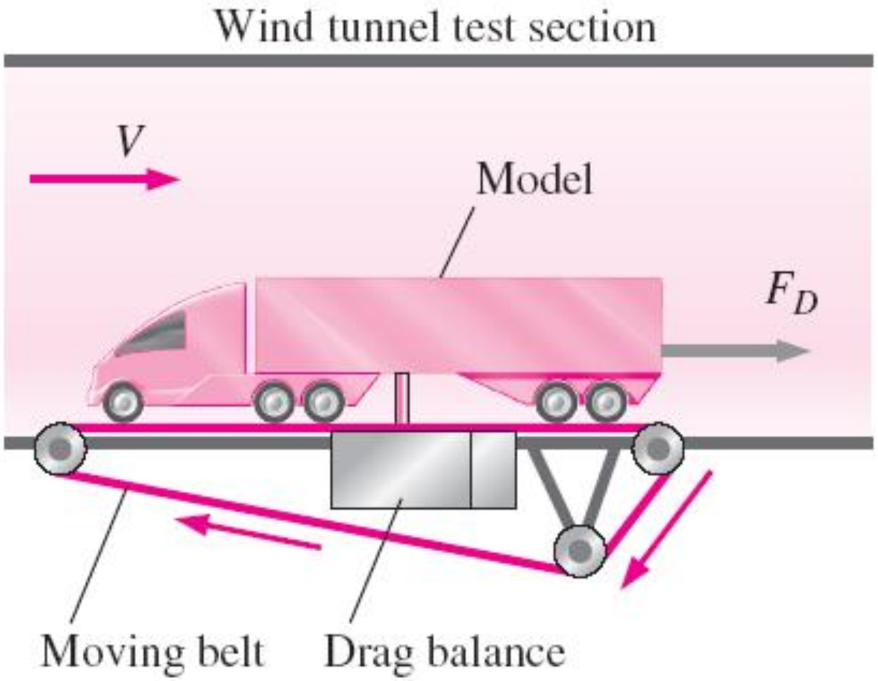
- A one-sixteenth scale model tractor-trailer truck is tested in a wind tunnel. The model truck is 0.991 m long, 0.257 m tall, and 0.159 m wide. Aerodynamic drag force F_D is measured as a function of wind tunnel speed; the experimental results are listed in Table 7–7. Plot the drag coefficient C_D as a function of Re, where the area used for the calculation of C_D is the frontal area of the model truck, and the length scale used for calculation of Re is truck width W. Have we achieved dynamic similarity? Have we achieved Reynolds number independence in our wind tunnel test? Estimate the aerodynamic drag force on the prototype truck traveling on the highway at 26.8 m/s. Assume that both the wind tunnel air and the air flowing over the prototype car are at 25°C and standard atmospheric pressure.

EXAMPLE: Model Truck Wind Tunnel Measurements

TABLE 7–7

Wind tunnel data: aerodynamic drag force on a model truck as a function of wind tunnel speed

V, m/s	F_D, N
20	12.4
25	19.0
30	22.1
35	29.0
40	34.3
45	39.9
50	47.2
55	55.5
60	66.0
65	77.6
70	89.9



EXAMPLE: Model Truck Wind Tunnel Measurements

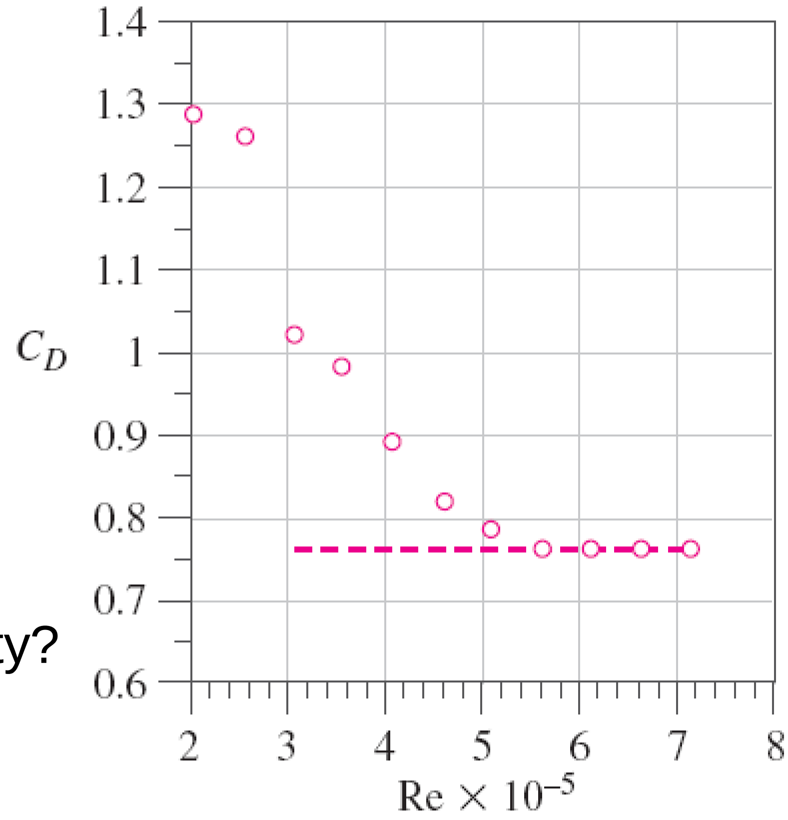
- Solution:

Calculate C_D and Re for the last data point listed in Table 7–7

$$C_{D,m} = \frac{F_{D,m}}{\frac{1}{2}\rho_m V_m^2 A_m} = 0.758$$

$$\mathrm{Re}_m = \frac{\rho_m V_m W_m}{\mu_m} = 7.13 \times 10^5$$

Repeat these calculations for all the data points in Table 7–7, and we plot C_D versus Re.



Have we achieved dynamic similarity?

$$\mathrm{Re}_p = \frac{\rho_p V_p W_p}{\mu_p} = 4.37 \times 10^6$$

EXAMPLE: Model Truck Wind Tunnel Measurements

- Solution:
- The prototype Reynolds number is more than six times larger than that of the model. Since we cannot match the independent Π's in the problem, **dynamic similarity has not been achieved.**
- Have we achieved Reynolds number independence? From the Fig. we see that **Reynolds number independence has indeed been achieved**—at Re greater than about 5×10^5, C_D has leveled off to a value of about 0.76 (to two significant digits).
- Since we have achieved Reynolds number independence, we can extrapolate to the full-scale prototype, assuming that C_D remains constant as Re is increased to that of the full-scale prototype.
- *Predicted aerodynamic drag on the prototype*:

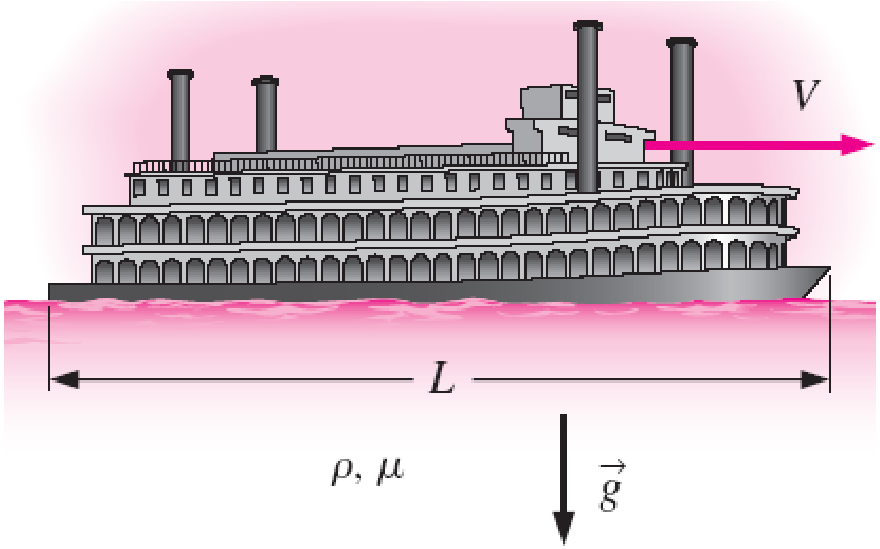
$$F_{D,p} = \frac{1}{2}\rho_p V_p^2 A_p C_{D,p} = 3400 \text{ N}$$

NOTE: No guarantee that the extrapolated results are correct.

Incomplete Similarity —— **Flows with Free Surfaces**

Incomplete Similarity —— Flows with Free Surfaces

- In many practical problems involving free surfaces, both the Reynolds number and Froude number appear as relevant independent Π groups in the dimensional analysis.
- It is difficult (often impossible) to match both of these dimensionless parameters simultaneously.



$$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$$

$$\mathrm{Fr} = \frac{V}{\sqrt{gL}}$$

Incomplete Similarity —— Flows with Free Surfaces

- For a free-surface flow, the Reynolds number and Froude number are matched between model and prototype when

$$\mathrm{Re}_p = \frac{V_p L_p}{\nu_p} = \mathrm{Re}_m = \frac{V_m L_m}{\nu_m} \quad \text{and} \quad \mathrm{Fr}_p = \frac{V_p}{\sqrt{g L_p}} = \mathrm{Fr}_m = \frac{V_m}{\sqrt{g L_m}}$$

To match both Re and Fr simultaneously, we require length scale factor L_m/L_p satisfy

$$\frac{L_m}{L_p} = \frac{\nu_m}{\nu_p}\frac{V_p}{V_m} = \left(\frac{V_m}{V_p}\right)^2 \quad \Rightarrow \quad \frac{\nu_m}{\nu_p} = \left(\frac{L_m}{L_p}\right)^{3/2}$$

From the results, we would need to use a liquid whose kinematic viscosity satisfies the equation. Although it is sometimes possible to find an appropriate liquid for use with the model, in most cases it is either impractical or impossible. (refer to example 7-11)