Chapter 5: Mass, Bernoulli, and Energy Equations

Chao-Lung Ting
Department of Engineering Science and Ocean Engineering
National Taiwan University

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Introduction

- This chapter deals with 3 equations commonly used in fluid mechanics
  - *The mass equation* is an expression of the conservation of mass principle.
  - *The Bernoulli equation* is concerned with the conservation of kinetic, potential, and flow energies of a fluid stream and their conversion to each other.
  - *The energy equation* is a statement of the conservation of energy principle. (*mechanical energy balance)*

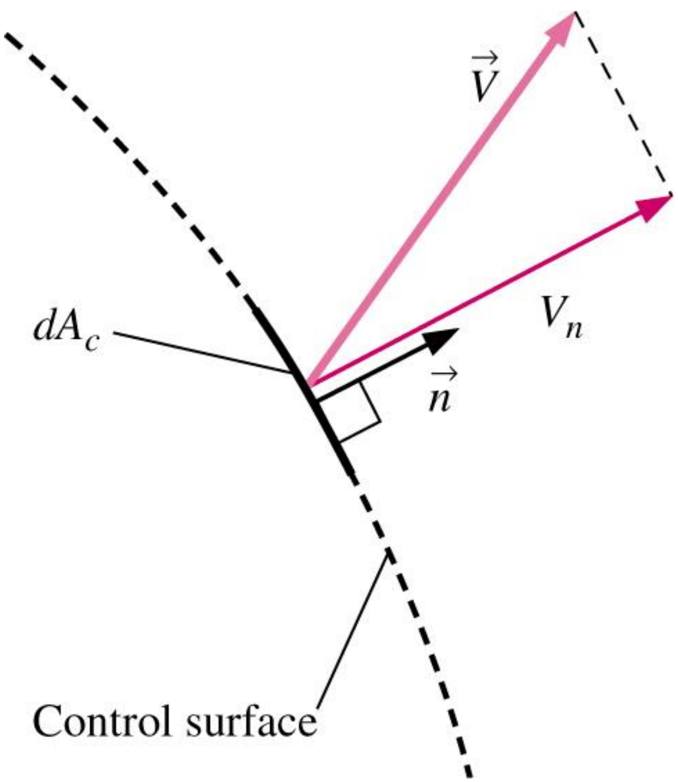
Objectives

- After completing this chapter, you should be able to
  - Apply the mass equation to balance the incoming and outgoing flow rates in a flow system.
  - Recognize various forms of mechanical energy, and work with energy conversion efficiencies.
  - Understand the use and limitations of the Bernoulli equation, and apply it to solve a variety of fluid flow problems.
  - Work with the energy equation expressed in terms of heads, and use it to determine turbine power output and pumping power requirements.

Conservation of Mass

- Conservation of mass principle is one of the most fundamental principles in nature.
- Mass, like energy, is a conserved property, and it cannot be created or destroyed during a process. (However, mass *m* and energy *E* can be converted to each other according to the well-known formula proposed by Albert Einstein (1879–1955), $E = mc^2$)
- For *closed systems* mass conservation is implicit since the mass of the system remains constant during a process.
- For *control volumes*, mass can cross the boundaries which means that we must keep track of the amount of mass entering and leaving the control volume.

Mass and Volume Flow Rates



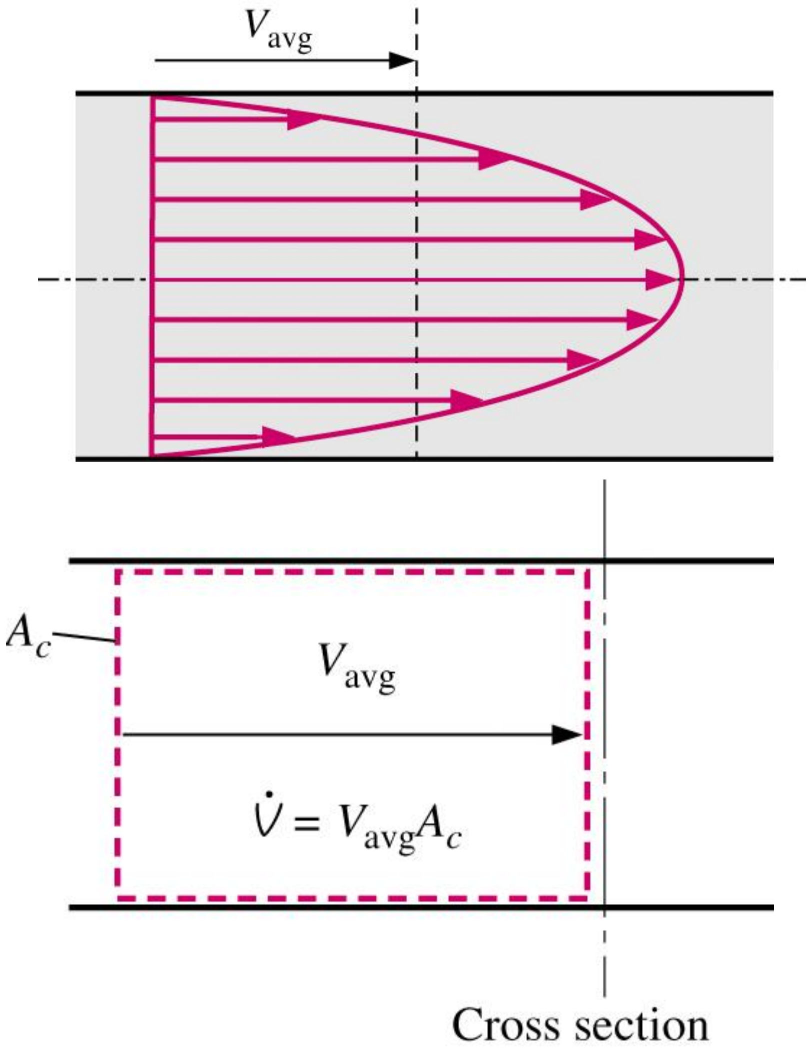
$$\delta \dot{m} = \rho V_n \, dA_c$$

- The amount of mass flowing through a control surface per unit time is called the **mass flow rate** and is denoted $\dot{m}$
- The dot over a symbol is used to indicate *time rate of change*.
- Flow rate across the entire cross-sectional area of a pipe or duct is obtained by integration

$$\dot{m} = \int_{A_c} \delta m = \int_{A_c} \rho V_n dA_c$$

- While this expression for $\dot{m}$ is exact, it is not always convenient for engineering analyses. (Express mass flow rate in terms of average values)

Average Velocity and Volume Flow Rate



- Integral in $\dot{m}$ can be replaced with average values of ρ and V_n

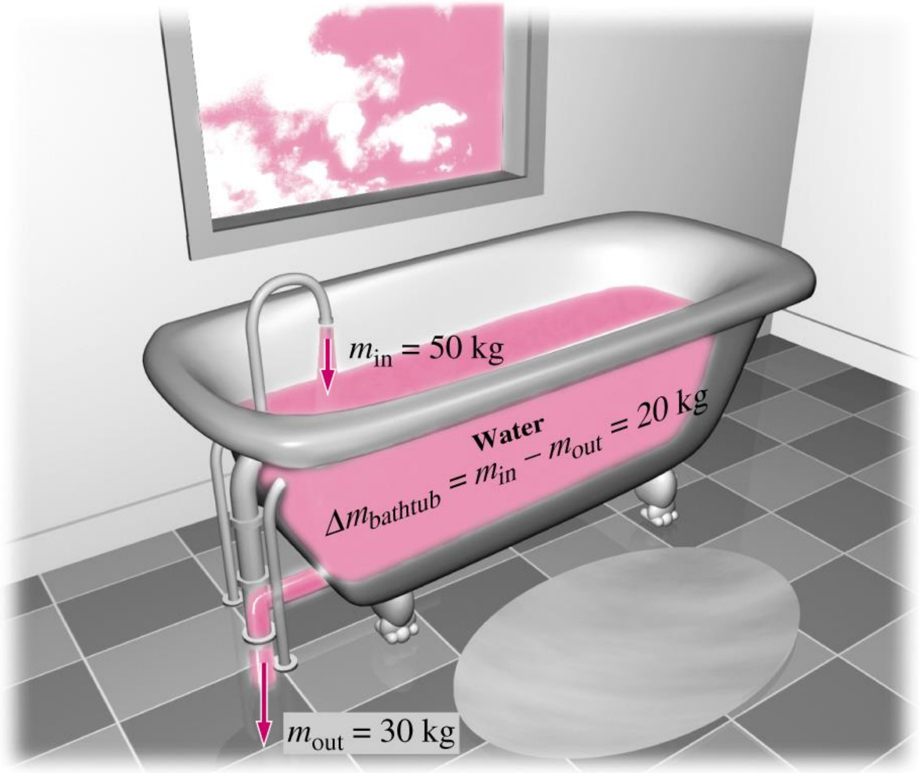
$$V_{avg} = \frac{1}{A_c} \int_{A_c} V_n dA_c$$

- For many flows variation of ρ is very small: $\dot{m} = \rho V_{avg} A_c$
- Volume flow rate $\dot{V}$ is given by

$$\dot{V} = \int_{A_c} V_n dA_c = V_{avg} A_c = V A_c$$

- Note: many textbooks use Q instead of $\dot{V}$ for volume flow rate.
- Mass and volume flow rates are related by $\dot{m} = \rho \dot{V}$

Conservation of Mass Principle

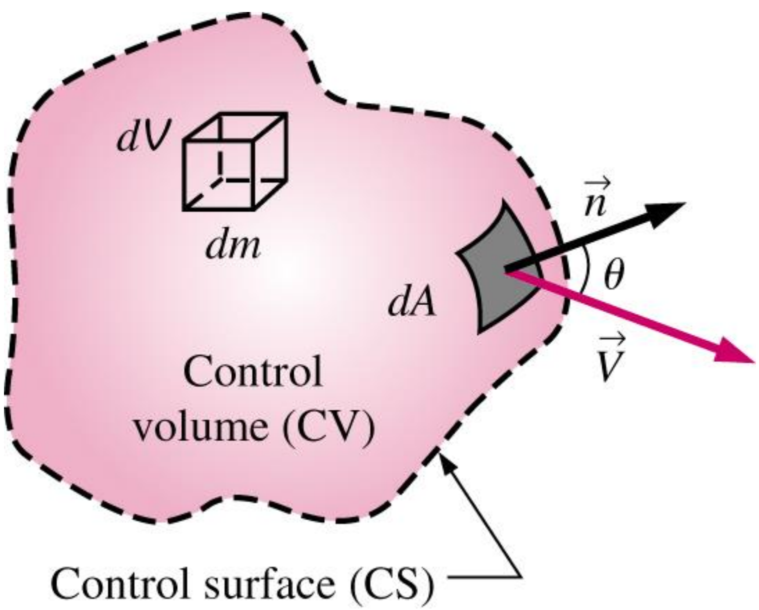


- The **conservation of mass principle** can be expressed as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

- Where $\dot{m}_{in}$ and $\dot{m}_{out}$ are the total rates of mass flow into and out of the CV, and dm_{CV}/dt is the rate of change of mass within the CV.

Conservation of Mass Principle



- For CV of arbitrary shape,
  - rate of change of mass within the CV

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho dV$$

  - net mass flow rate

$$\dot{m}_{net} = \int_{CS} \delta\dot{m} = \int_{CS} \rho V_n dA = \int_{CS} \rho\left(\vec{V}\cdot\vec{n}\right)dA$$

$$V_n = V\cos\theta = \vec{V}\cdot\vec{n}$$

$$\delta\dot{m} = \rho V_n\, dA = \rho(V\cos\theta)\, dA = \rho(\vec{V}\cdot\vec{n})\, dA$$

Outflow (θ < 90) positive
Inflow (θ >90) negative

Conservation of Mass Principle

- Therefore, general conservation of mass for a fixed CV is:

$$\frac{d}{dt}\int_{CV}\rho\, dV + \int_{CS}\rho\left(\vec{V}\cdot\vec{n}\right)dA = 0$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV}\rho b\, dV + \int_{CS}\rho b(\vec{V}\cdot\vec{n})\,dA$$

$$B = m \qquad b = 1 \qquad b = 1$$

$$\frac{dm_{sys}}{dt} = \frac{d}{dt}\int_{CV}\rho\, dV + \int_{CS}\rho(\vec{V}\cdot\vec{n})\,dA$$

Using RTT

Conservation of Mass Principle

Change the surface integral into summation, then we can get the following expression:

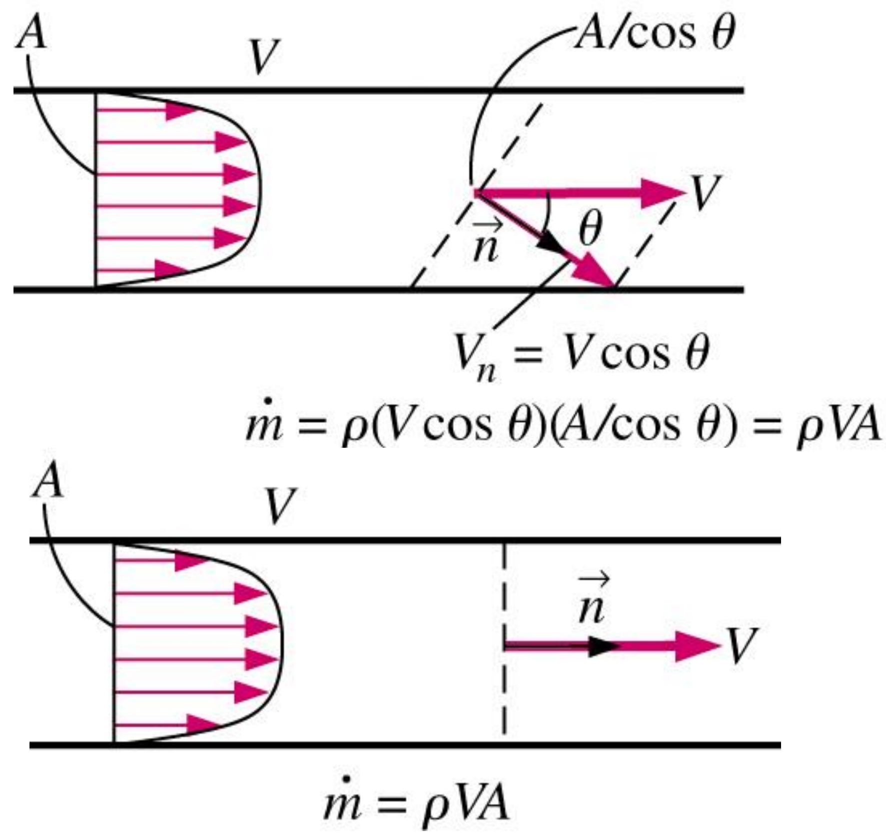
$$\frac{d}{dt}\int_{CV}\rho\, dV = \sum_{in}\dot{m} - \sum_{out}\dot{m}$$

or

$$\frac{dm_{CV}}{dt} = \sum_{in}\dot{m} - \sum_{out}\dot{m}$$

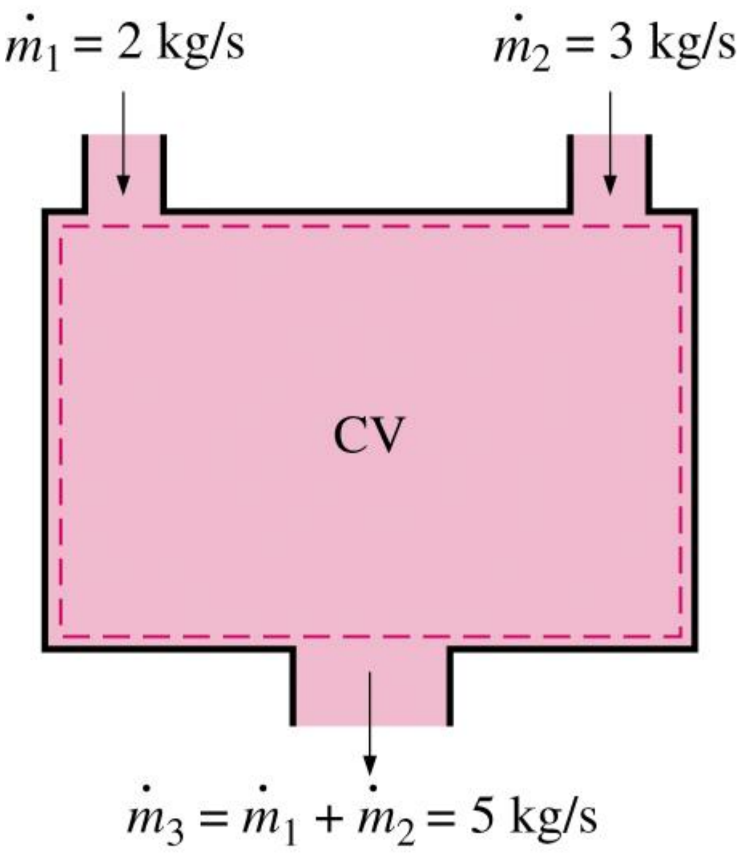
For a moving CV, just change V to V_r in the equation where V_r equal to

$$\vec{V}_r = \vec{V} - \vec{V}_{CV}$$



Proper choice of a control volume

Steady—Flow Processes



- For steady flow, the total amount of mass contained in CV is constant.
- Total amount of mass entering must be equal to total amount of mass leaving

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

- for *single-stream steady-flow systems*, $\dot{m}_1 = \dot{m}_2$

$$\rightarrow \quad \rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

- For incompressible flows (ρ = constant),

$$\sum_{in} V_n A_n = \sum_{out} V_n A_n$$

EXAMPLE: Discharge of Water from a Tank

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

$$(\dot{m}_{in} = 0)$$

$$\dot{m}_{out} = (\rho VA)_{out} = \rho\sqrt{2gh}A_{jet}$$

$$m_{CV} = \rho V = \rho A_{tank} h$$

$$-\rho\sqrt{2gh}A_{jet} = \frac{d(\rho A_{tank} h)}{dt}$$

$$\rightarrow -\rho\sqrt{2gh}(\pi D_{jet}^2/4) = \frac{\rho(\pi D_{tank}^2/4)\, dh}{dt}$$

$$dt = -\frac{D_{tank}^2}{D_{jet}^2}\frac{dh}{\sqrt{2gh}}$$

$$\int_0^t dt = -\frac{D_{tank}^2}{D_{jet}^2\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}}$$

$$\rightarrow \quad t = \frac{\sqrt{h_0} - \sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{tank}}{D_{jet}}\right)^2$$

$$t = \frac{\sqrt{4 \text{ ft}} - \sqrt{2 \text{ ft}}}{\sqrt{32.2/2 \text{ ft/s}^2}}\left(\frac{3 \times 12 \text{ in}}{0.5 \text{ in}}\right)^2$$

$$= 757 \text{ s} = \mathbf{12.6 \text{ min}}$$

Mechanical Energy

- **Mechanical energy** can be defined as *the form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device such as an ideal turbine.*
- Flow P/ρ, kinetic $V^2/2$, and potential gz energy are the forms of mechanical energy $e_{mech} = P/\rho + V^2/2 + gz$
- Mechanical energy change of a fluid during incompressible flow becomes

$$\Delta e_{mech} = \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right)$$

- In the absence of loses, Δe_{mech} represents the work supplied to the fluid ($\Delta e_{mech}>0$) or extracted from the fluid ($\Delta e_{mech}<0$).

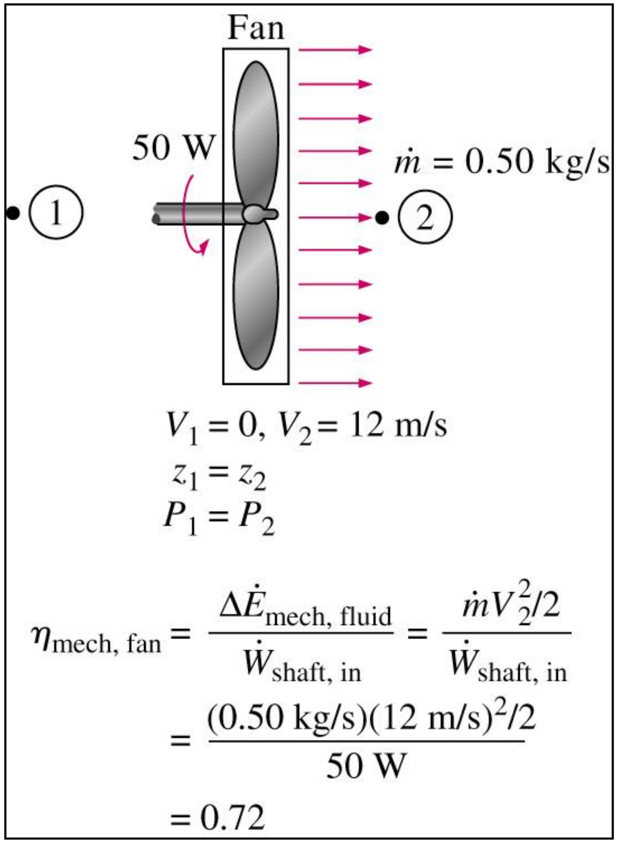
Efficiency

- Transfer of e_{mech} is usually accomplished by a rotating shaft: *shaft work*
- Pump, fan, propulsion: receives shaft work (e.g., from an electric motor) and transfers it to the fluid as mechanical energy
- Turbine: converts e_{mech} of a fluid to shaft work.
- In the absence of irreversibilities (e.g., friction), **mechanical efficiency** of a device or process can be defined as

$$\eta_{mech} = \frac{E_{mech,out}}{E_{mech,in}} = 1 - \frac{E_{mech,loss}}{E_{mech,in}}$$

- If η_{mech} < 100%, losses have occurred during conversion.

Pump and Turbine Efficiencies

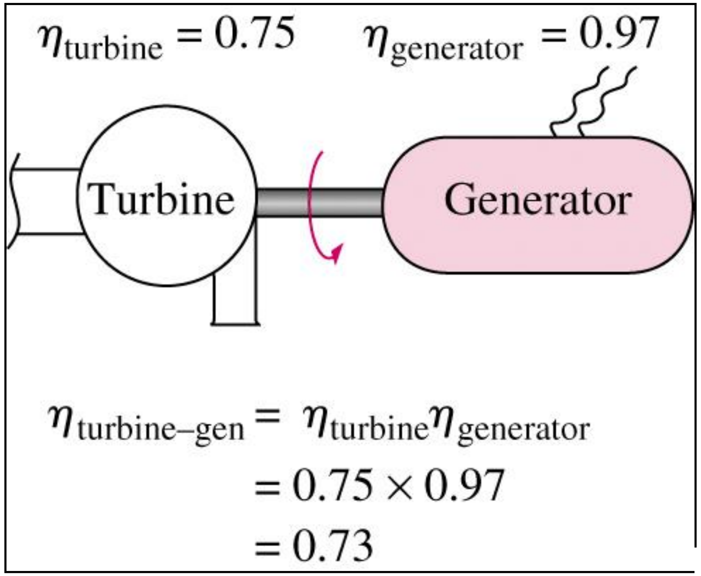


- In fluid systems, we are usually interested in increasing the pressure, velocity, and/or elevation of a fluid.
- In these cases, efficiency is better defined as the ratio of (supplied or extracted work) vs. rate of increase in mechanical energy

$$\eta_{pump} = \frac{\Delta \dot{E}_{mech,\,fluid}}{\dot{W}_{shaft,in}}$$

$$\eta_{turbine} = \frac{\dot{W}_{shaft,out}}{\left|\Delta \dot{E}_{mech,\,fluid}\right|}$$

Pump and Turbine Efficiencies



- Overall efficiency must include motor or generator efficiency.

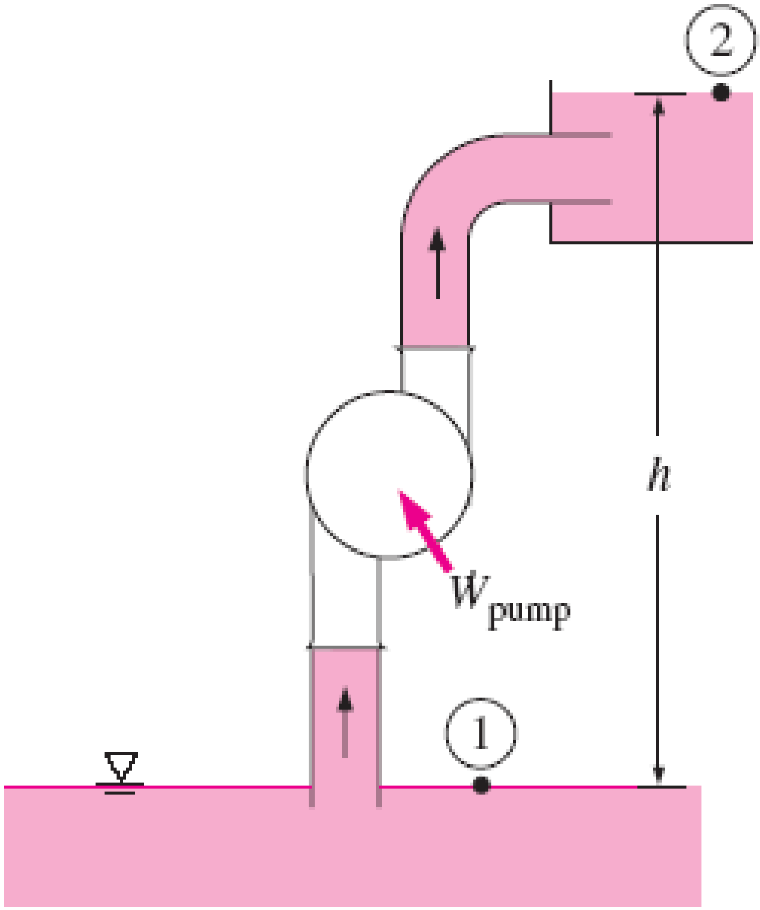
$$\eta_{motor} = \frac{\text{Mechanical power output}}{\text{Electric power input}} = \frac{\dot{W}_{shaft,\,out}}{\dot{W}_{elect,\,in}}$$

$$\eta_{generator} = \frac{\text{Electric power output}}{\text{Mechanical power input}} = \frac{\dot{W}_{elect,\,out}}{\dot{W}_{shaft,\,in}}$$

$$\eta_{pump-motor} = \eta_{pump}\,\eta_{motor} = \frac{\dot{W}_{pump,\,u}}{\dot{W}_{elect,\,in}} = \frac{\Delta \dot{E}_{mech,\,fluid}}{\dot{W}_{elect,\,in}}$$

$$\eta_{turbine-gen} = \eta_{turbine}\,\eta_{generator} = \frac{\dot{W}_{elect,\,out}}{\dot{W}_{turbine,\,e}} = \frac{\dot{W}_{elect,\,out}}{|\Delta \dot{E}_{mech,\,fluid}|}$$

Mechanical energy balance.



Steady flow

$$V_1 = V_2$$

$$z_2 = z_1 + h$$

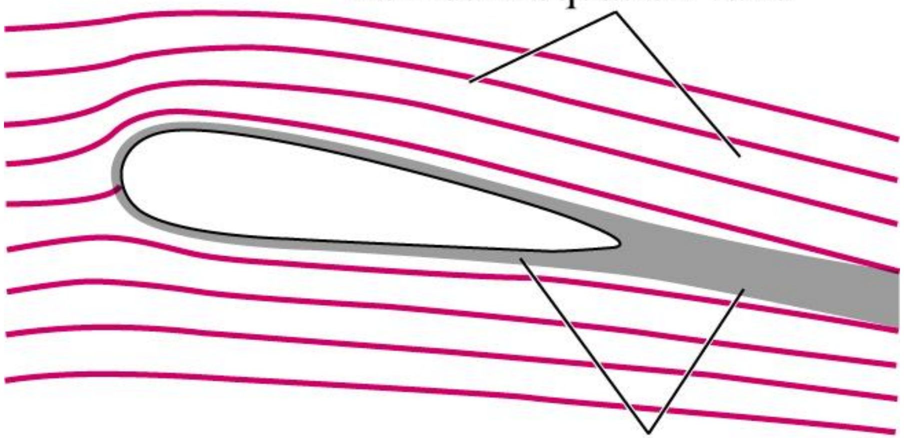
$$P_1 = P_2 = P_{\text{atm}}$$

$$\dot{E}_{\text{mech, in}} = \dot{E}_{\text{mech, out}} + \dot{E}_{\text{mech, loss}}$$

$$\dot{W}_{\text{pump}} + \dot{m}gz_1 = \dot{m}gz_2 + \dot{E}_{\text{mech, loss}}$$

$$\dot{W}_{\text{pump}} = \dot{m}gh + \dot{E}_{\text{mech, loss}}$$

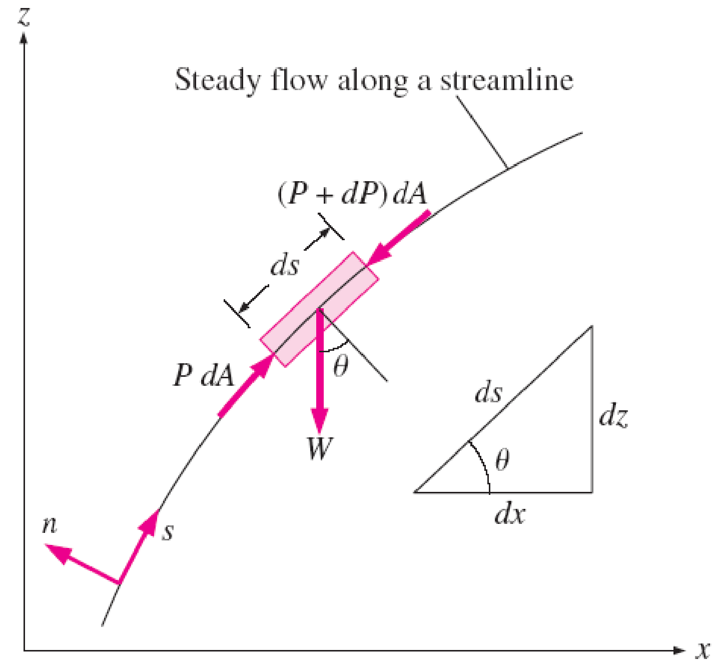
The Bernoulli Equation



- The **Bernoulli equation** is an *approximate relation between pressure, velocity*, and *elevation and is valid in regions of steady, incompressible flow where net frictional forces are negligible.*
- Equation is useful in flow regions outside of boundary layers and wakes, where the fluid motion is governed by the combined effects of pressure and gravity forces.

Acceleration of a Fluid Particle

- Describe the motion of a particle in terms of its distance *s* along a streamline together with the radius of curvature along the streamline. The velocity of a particle along a streamline is $V = V(s, t) = ds/dt$
- The acceleration can be decomposed into two components: *streamwise acceleration* a_s along the streamline and *normal acceleration* a_n in the direction normal to the streamline, which is given as $a_n = V^2/R$.



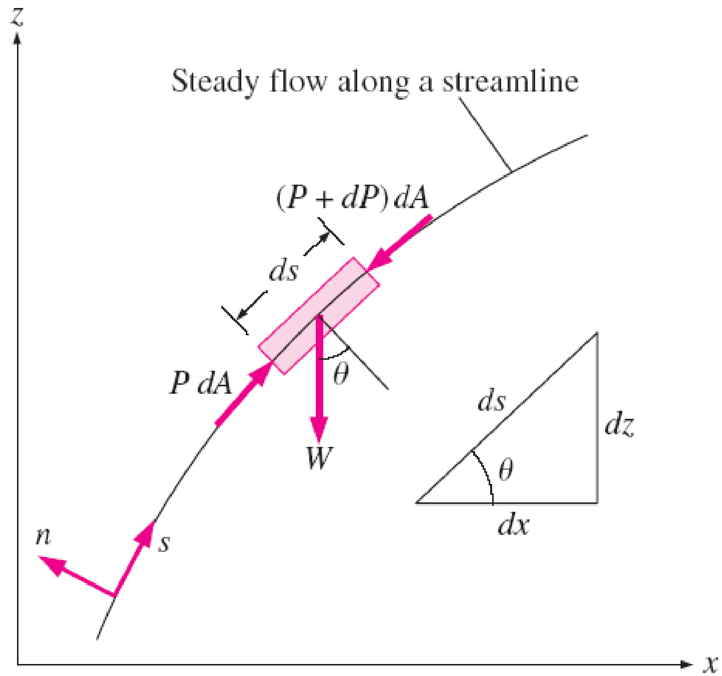
Acceleration of a Fluid Particle

- Note that streamwise acceleration is due to a change in speed along a streamline, and normal acceleration is due to a change in direction.
- The time rate change of velocity is the acceleration

$$\frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t}$$

In steady flow, the acceleration in the *s* direction becomes

$$a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} = \frac{\partial V}{\partial s}V = V\frac{dV}{ds}$$



(Proof on Blackboard)

Derivation of the Bernoulli Equation

Applying Newton's second law in the *s*-direction on a particle moving along a streamline in a steady flow field gives

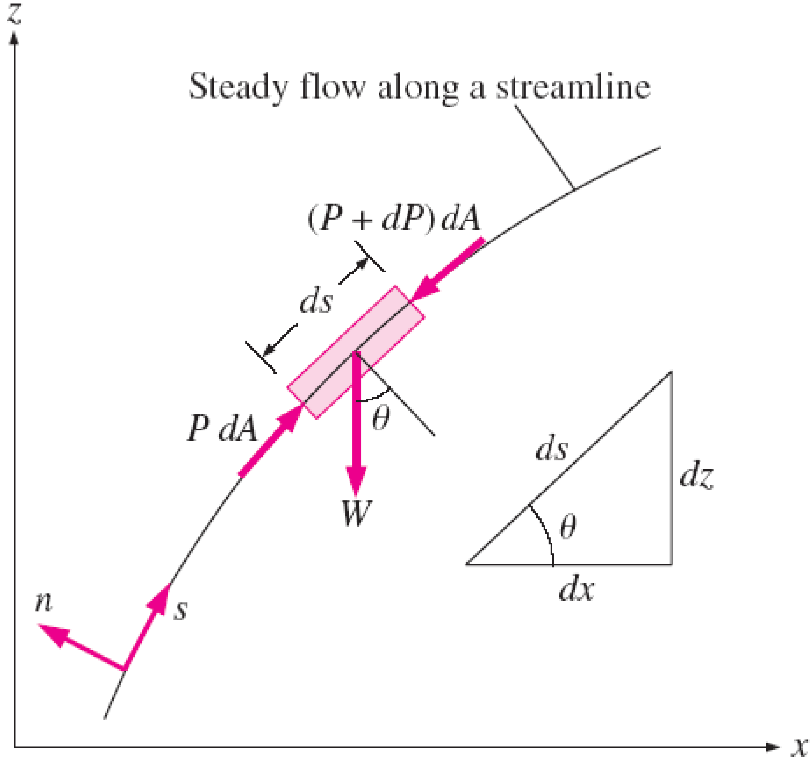
$$\sum F_s = ma_s$$

The force balance in *s* direction gives

$$P\,dA - (P + dP)\,dA - W \sin\theta = mV\,\frac{dV}{ds}$$

where

$W = mg = \rho g\,dA\,ds$ and $\sin\theta = dz/ds$



Derivation of the Bernoulli Equation

Therefore,

$$\frac{dP}{\rho} + \tfrac{1}{2}\,d(V^2) + g\,dz = 0$$

Integrating steady flow along a streamline

$$\int \frac{dP}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Steady, Incompressible flow

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

$\Rightarrow$ This is the famous **Bernoulli equation.**

The Bernoulli Equation

- Without the consideration of any losses, two points on the same streamline satisfy

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2$$

- where P/ρ as *flow energy,* $V^2/2$ as *kinetic energy, and gz* as *potential energy,* all per unit mass.
- The Bernoulli equation can be viewed as an expression of *mechanical energy balance*
- Was first stated in words by the Swiss mathematician Daniel Bernoulli (1700–1782) in a text written in 1738.

The Bernoulli Equation

Force Balance across Streamlines

A force balance in the direction *n* normal to the streamline for steady, incompressible flow:

$$\frac{P}{\rho} + \int \frac{V^2}{R} dn + gz = \text{constant}$$

For flow along a straight line, $R \to \infty$, then equation becomes

$$P/\rho + gz = \text{constant}$$

which is an expression for the variation of hydrostatic pressure as same as that in the stationary fluid

The Bernoulli Equation

Bernoulli equation for *unsteady, compressible flow* is

$$\int \frac{dP}{\rho} + \int \frac{\partial V}{\partial t} ds + \frac{V^2}{2} + gz = \text{constant}$$

Static, Dynamic, and Stagnation Pressures

The Bernoulli equation

$$P + \rho \frac{V^2}{2} + \rho g z = \text{constant (along a streamline)}$$

- *P* is the **static pressure**; it represents the actual thermodynamic pressure of the fluid. This is the same as the pressure used in thermodynamics and property tables.
- $\rho V^2/2$ is the **dynamic pressure;** it represents the pressure rise when the fluid in motion.
- $\rho g z$ is the **hydrostatic pressure,** depends on the reference level selected.

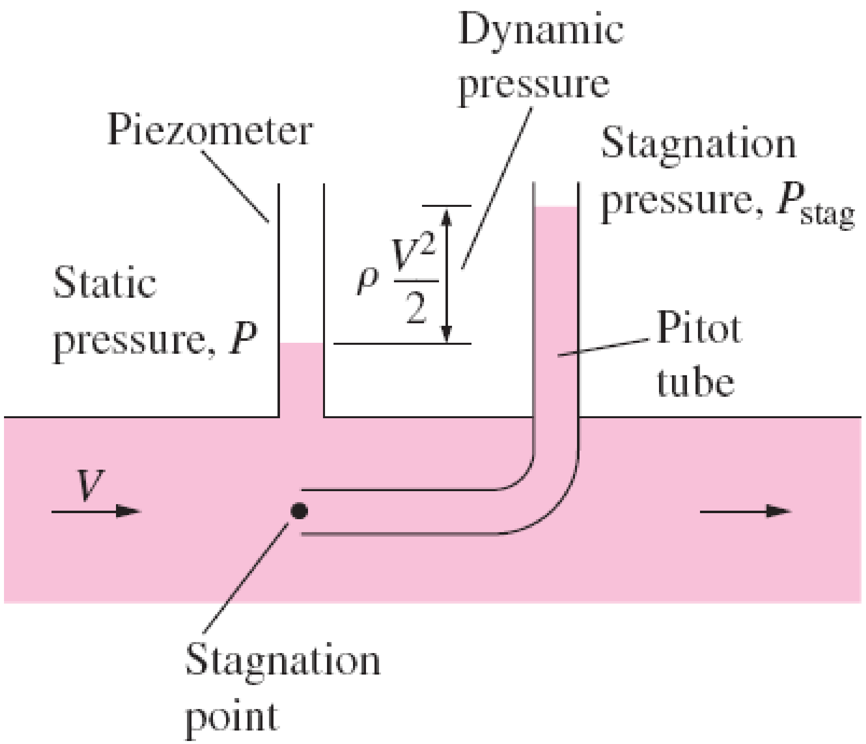
Static, Dynamic, and Stagnation Pressures

- The sum of the static, dynamic, and hydrostatic pressures is called the **total pressure (a constant along a streamline).**
- The sum of the static and dynamic pressures is called the **stagnation pressure,**

$$P_{\text{stag}} = P + \rho \frac{V^2}{2} \qquad \text{(kPa)}$$

The fluid velocity at that location can be calculated from

$$V = \sqrt{\frac{2(P_{\text{stag}} - P)}{\rho}}$$

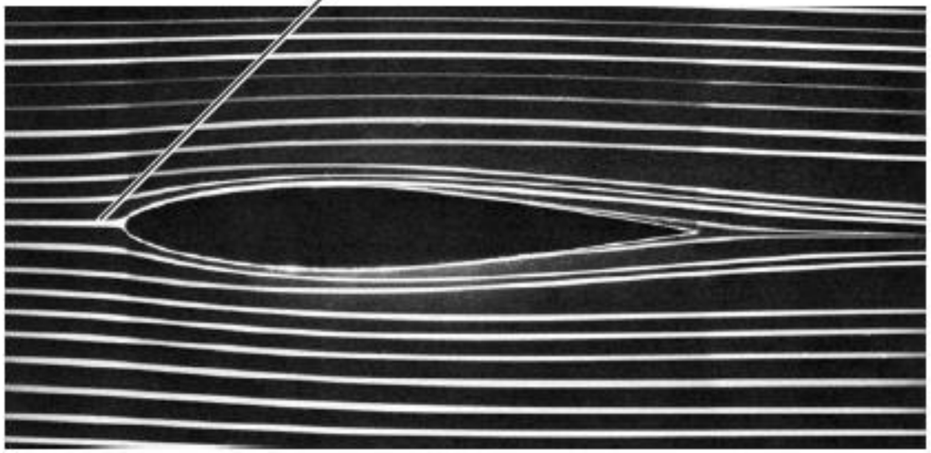


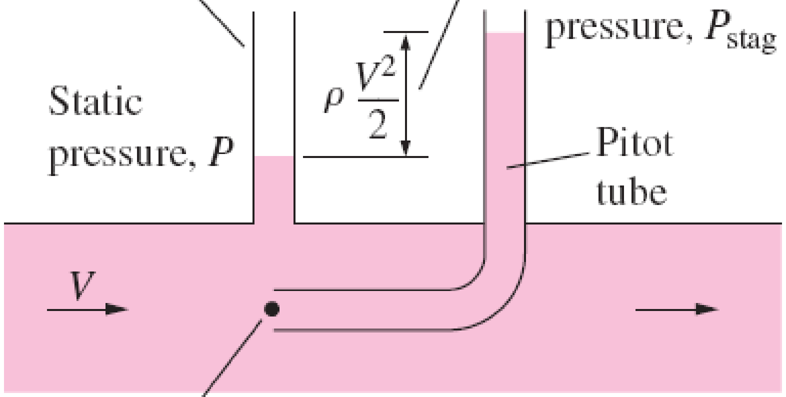
Pitot-static probe

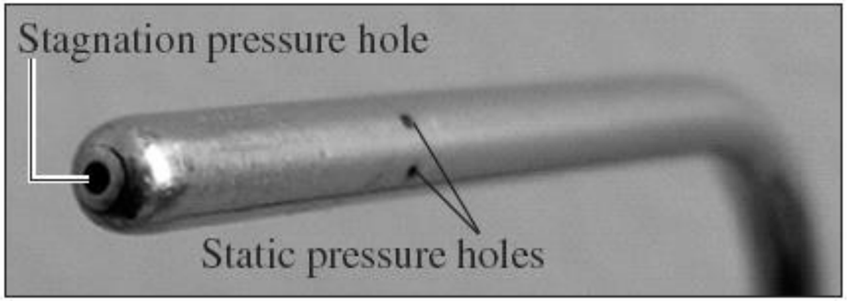
The fluid velocity at that location can be calculated from

$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$

A piezometer measures static pressure.



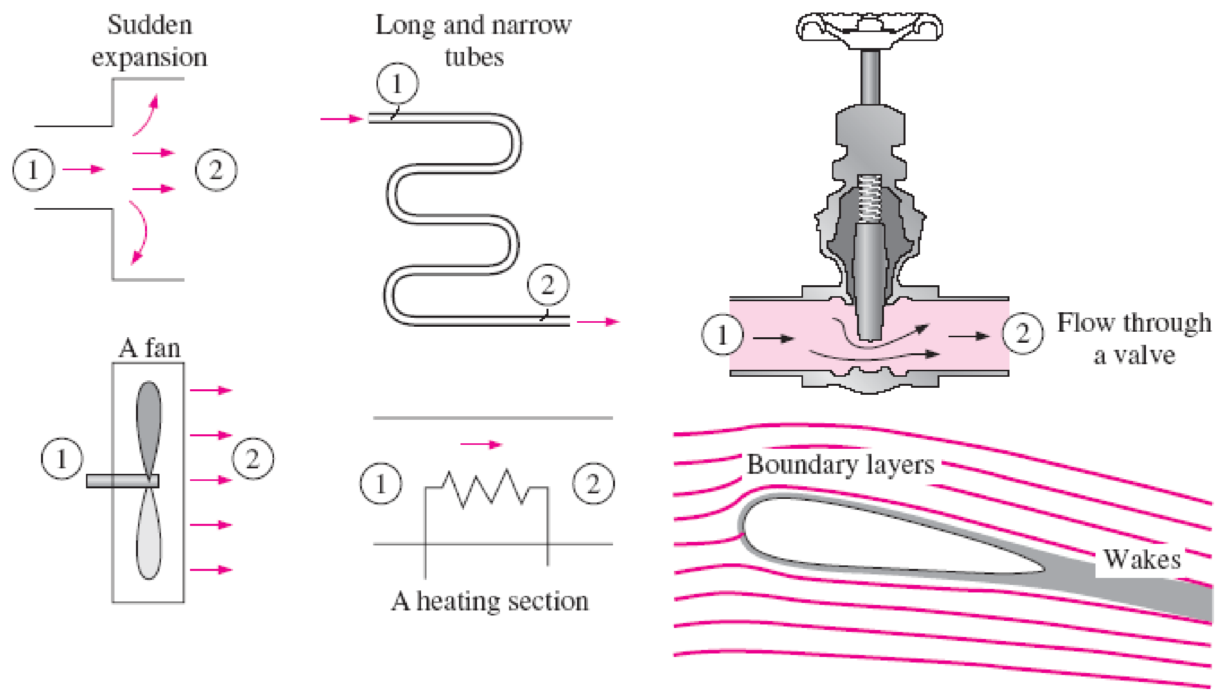




Limitations on the use of the Bernoulli Equation

- Steady flow: *d/dt* = 0, it should not be used during the transient start-up and shut-down periods, or during periods of change in the flow conditions.
- Frictionless flow

  : The flow conditions described by the right graphs can make the Bernoulli equation inapplicable.



Limitations on the use of the Bernoulli Equation

- No shaft work: $w_{pump}=w_{turbine}=0$. The Bernoulli equation can still be applied to a flow section prior to or past a machine (with different Bernoulli constants)
- Incompressible flow: ρ = constant (liquids and also gases at Mach No. less than about 0.3)
- No heat transfer: $q_{net,in}=0$
- Applied along a streamline: The Bernoulli constant C, in general, is different for different streamlines. But when a region of the flow is *irrotational*, and thus there is no *vorticity* in the flow field, the value of the constant C remains the same for all streamlines.

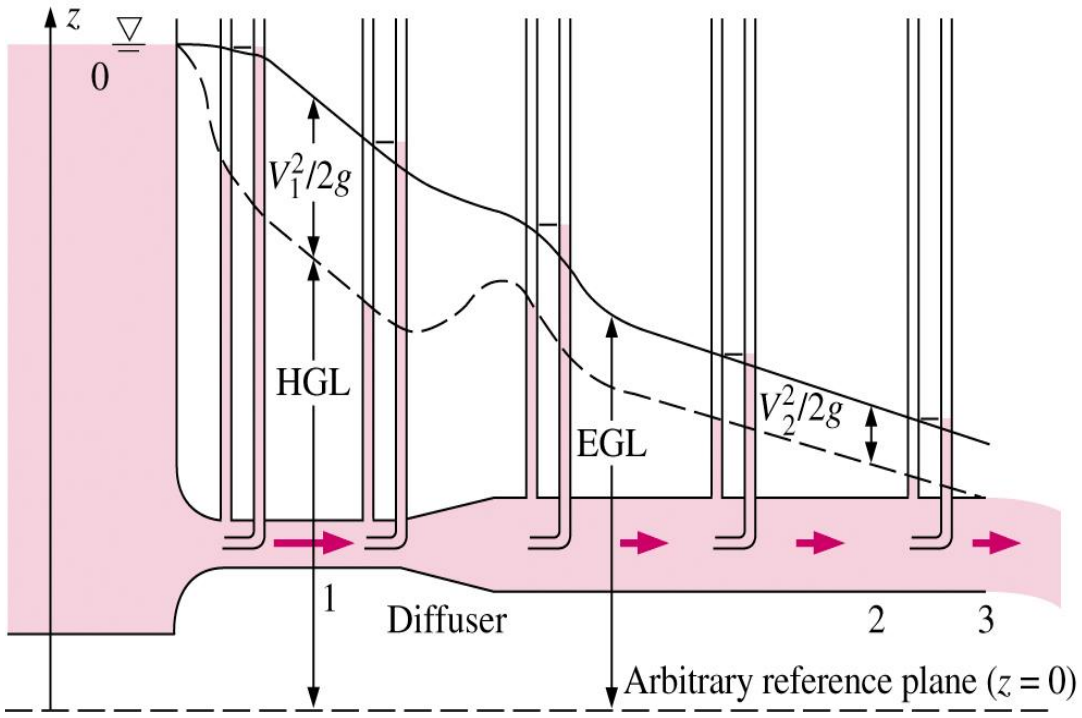
HGL and EGL

- It is often convenient to plot mechanical energy graphically using heights.

$$\frac{P}{\rho g} + \frac{V^2}{2g} + z = H = \text{constant}$$

- $P/\rho g$ is the **pressure head;** it represents the height of a fluid column that produces the static pressure P.
- $V^2/2g$ is the **velocity head;** it represents the elevation needed for a fluid to reach the velocity V during frictionless free fall.
- z is the **elevation head;** it represents the potential energy of the fluid.
- H is the total head.

HGL and EGL



- Hydraulic Grade Line (HGL)

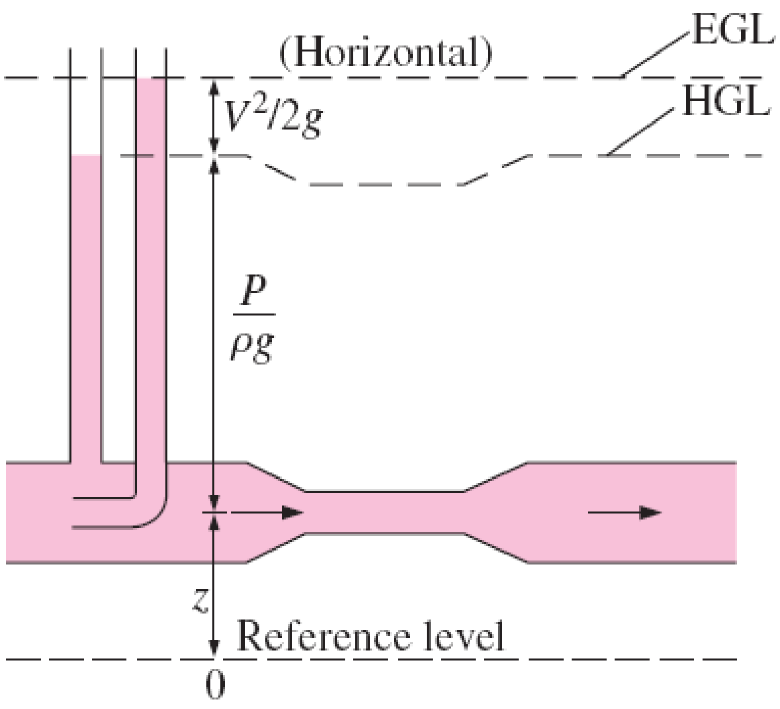
$$HGL = \frac{P}{\rho g} + z$$

- Energy Grade Line (EGL) (or total head)

$$EGL = \frac{P}{\rho g} + \frac{V^2}{2g} + z$$

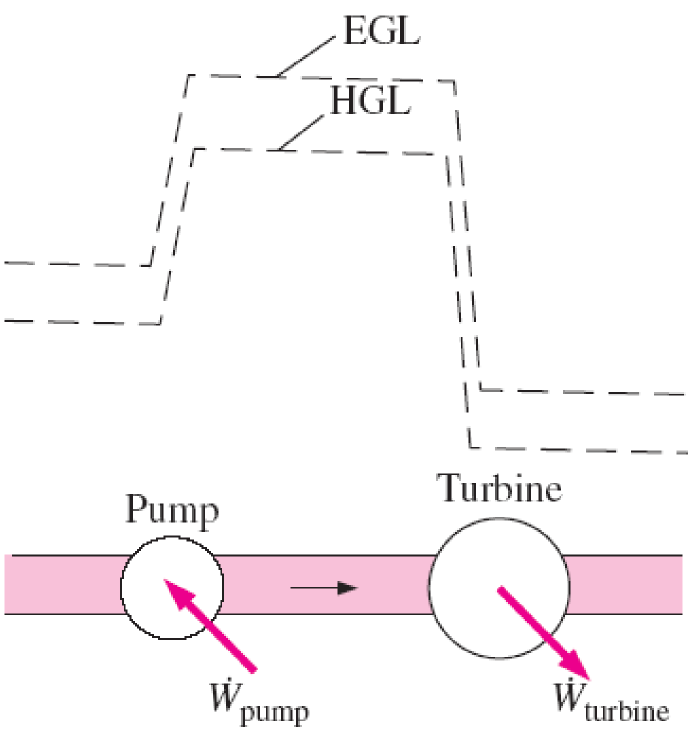
Something to know about HGL and EGL:

■For ***stationary bodies*** such as reservoirs or lakes, the EGL and HGL coincide with the free surface of the liquid, since the velocity is zero and the static pressure (gage) is zero.
■The EGL is always a distance $V2/2g$ above the HGL.
■In an ***idealized Bernoulli-type flow***, EGL is horizontal and its height remains constant. This would also be the case for HGL when the flow velocity is constant .
■For ***open-channel flow***, the HGL coincides with the free surface of the liquid, and the EGL is a distance $V^2/2g$ above the free surface.



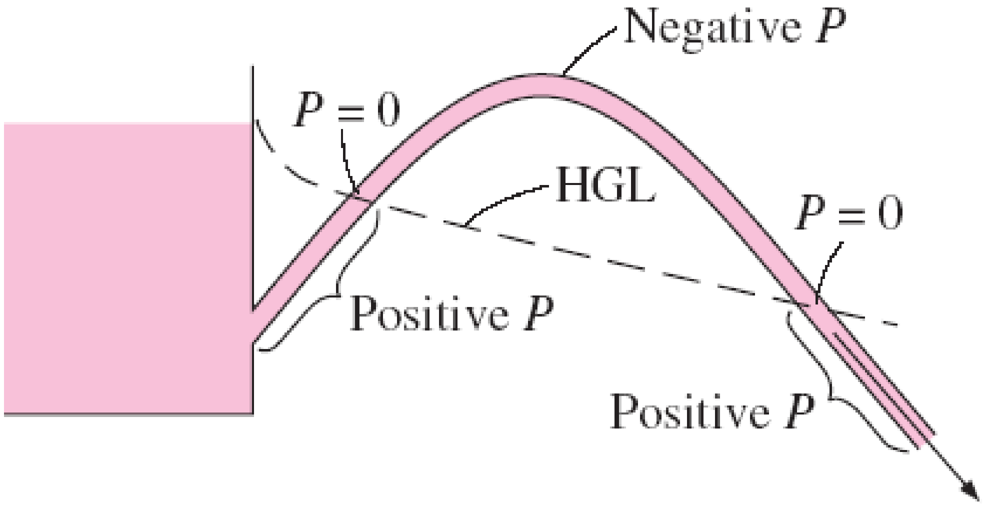
Something to know about HGL and EGL:

■At a ***pipe exit***, the pressure head is zero (atmospheric pressure) and thus the HGL coincides with the pipe outlet.
■The ***mechanical energy loss*** due to frictional effects (conversion to thermal energy) causes the EGL and HGL to slope downward in the direction of flow.
■A ***steep jump*** occurs in EGL and HGL whenever mechanical energy is added to the fluid. Likewise, a ***steep drop*** occurs in EGL and HGL whenever mechanical energy is removed from the fluid.



Something to know about HGL and EGL:

- The pressure (gage) of a fluid is zero at locations where the HGL *intersects* the fluid. The pressure in a flow section that lies above the HGL is negative, and the pressure in a section that lies below the HGL is positive.



APPLICATIONS OF THE BERNOULLI EQUATION

Assumptions: The flow exiting into the air is steady, incompressible, and irrotational (so that the Bernoulli equation is applicable). The velocity inside the hose is relatively low ($V_1 = 0$) and we take the hose outlet as the reference level ($z_1 = 0$). At the top of the water trajectory $V_2 = 0$, and atmospheric pressure pertains.

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g}^{\nearrow 0} + z_1^{\nearrow 0} = \frac{P_2}{\rho g} + \frac{V_2^2}{2g}^{\nearrow 0} + z_2$$

$$\rightarrow \qquad \frac{P_1}{\rho g} = \frac{P_{\text{atm}}}{\rho g} + z_2$$

$$z_2 = \frac{P_1 - P_{\text{atm}}}{\rho g} = \frac{P_{1,\,\text{gage}}}{\rho g} = \mathbf{40.8\ m}$$

EXAMPLE: Velocity Measurement by a Pitot Tube

Solution:

$$P_1 = \rho g(h_1 + h_2)$$

$$P_2 = \rho g(h_1 + h_2 + h_3)$$

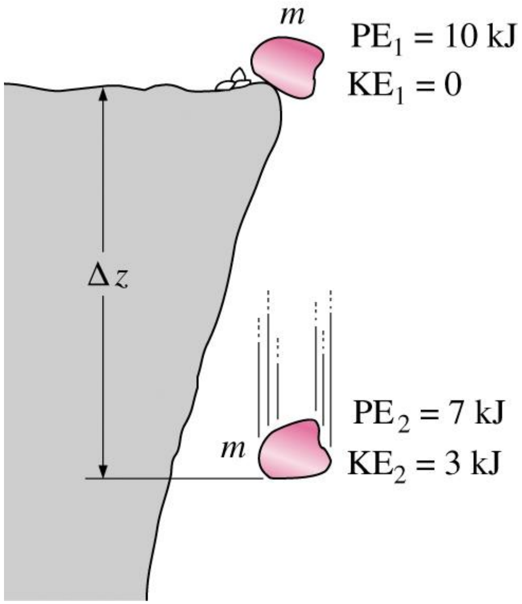
$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{P_2}{\rho g} + \frac{V_2^2}{2g}\nearrow^{0} + \cancel{z_2} \rightarrow \frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g}$$

$$\frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g} = \frac{\rho g(h_1 + h_2 + h_3) - \rho g(h_1 + h_2)}{\rho g} = h_3$$

$$V_1 = \sqrt{2gh_3} = \sqrt{2(9.81\ \text{m/s}^2)(0.12\ \text{m})} = \mathbf{1.53\ m/s}$$

General Energy Equation

- One of the most fundamental laws in nature is the **1st law of thermodynamics**, which is also known as the **conservation of energy principle**.
- It states that *energy can be neither created nor destroyed during a process; it can only change forms*

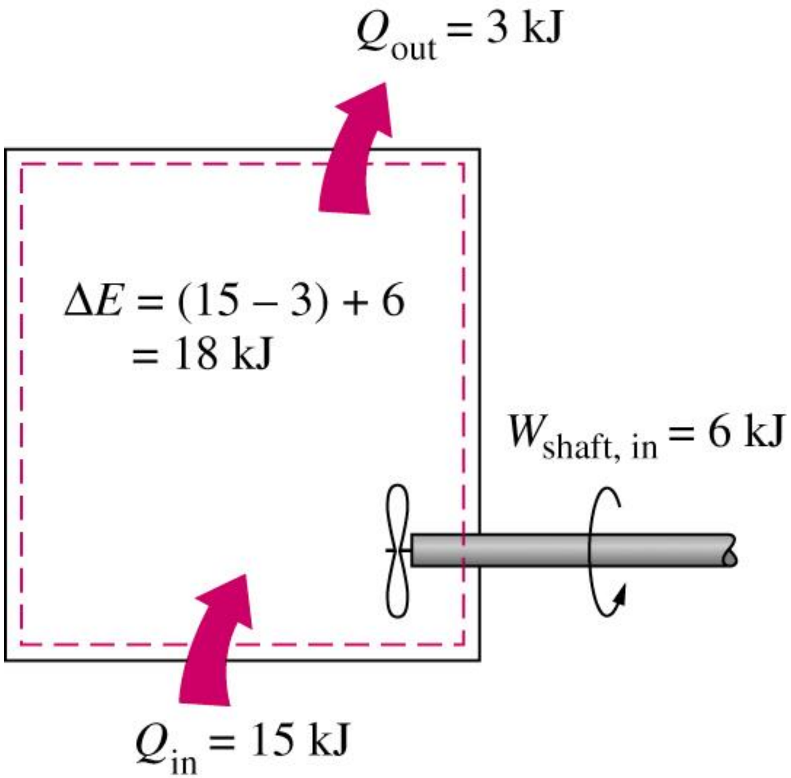


- Falling rock, picks up speed as PE is converted to KE.
- If air resistance is neglected,

  PE + KE = constant
- The conservation of energy principle

$$E_{in} - E_{out} = \Delta E$$

General Energy Equation



Where *e* is total energy per unit mass

$$e = u + \mathrm{ke} + \mathrm{pe} = u + \frac{V^2}{2} + gz$$

- The energy content of a closed system can be changed by two mechanisms: *heat transfer Q* and *work transfer W*.
- Conservation of energy for a closed system can be expressed in rate form as

$$\dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{dE_{sys}}{dt} = \frac{d}{dt}\int_{\mathrm{sys}} \rho e \, dV$$

- Net rate of heat transfer to the system:

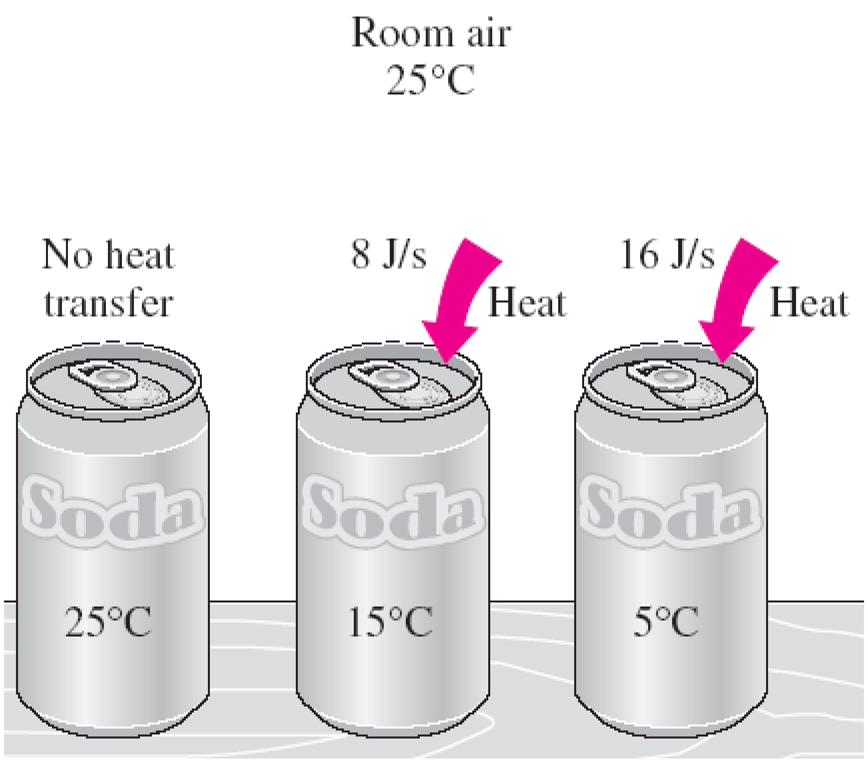
$$\dot{Q}_{net,in} = \dot{Q}_{in} - \dot{Q}_{out}$$

- Net power input to the system:

$$\dot{W}_{net,in} = \dot{W}_{in} - \dot{W}_{out}$$

Energy Transfer by Heat, *Q*

■We frequently refer to the sensible and latent forms of internal energy as ***heat***, or ***thermal energy***.
■For single phase substances, a change in the thermal energy ⇒ a change in temperature,
■The transfer of thermal energy as a result of a temperature difference is called **heat transfer.**
■A process during which there is no heat transfer is called an **adiabatic Process: insulated or same temperature**
■An **adiabatic** process ⇔ an **isothermal** process.



Energy Transfer by Work, *W*

- An energy interaction is **work** if it is associated with a force acting through a distance.
- The time rate of doing work is called **power,** $\dot{W}$.
- A system may involve numerous forms of work, and the total work can be expressed as

$$W_{\text{total}} = W_{\text{shaft}} + W_{\text{pressure}} + W_{\text{viscous}} + W_{\text{other}}$$

- Where W_{other} is the work done by other forces such as electric, magnetic, and surface tension, which are insignificant and negligible in this text. Also, W_{viscous}, the work done by viscous forces, are neglected.

Energy Transfer by Work, *W*

- **Shaft Work:** The power transmitted via a rotating shaft is proportional to the shaft torque T_{shaft} and is expressed as

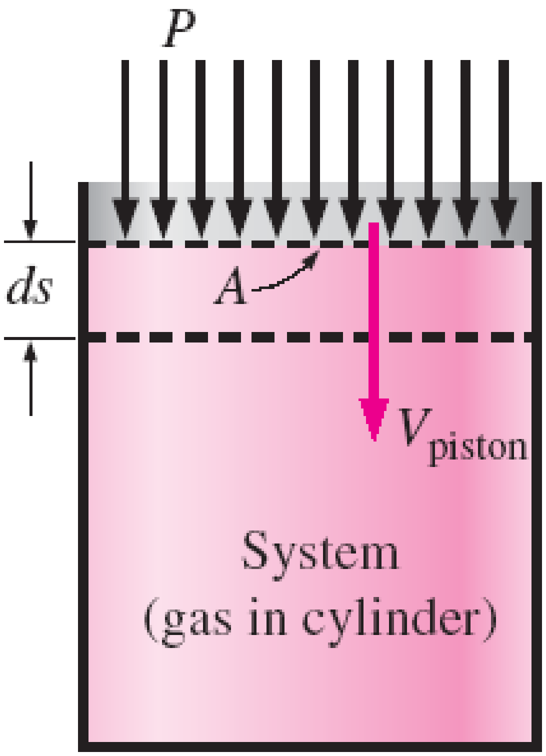
$$\dot{W}_{shaft} = \omega T_{shaft} = 2\pi \dot{n} T_{shaft}$$

- **Work Done by Pressure Forces:** the work done by the pressure forces on the control surface

$$\delta W_{boundary} = PA\ ds$$

  - The associated power is

$$\delta \dot{W}_{pressure} = \delta \dot{W}_{boundary} = PAV_{piston}$$

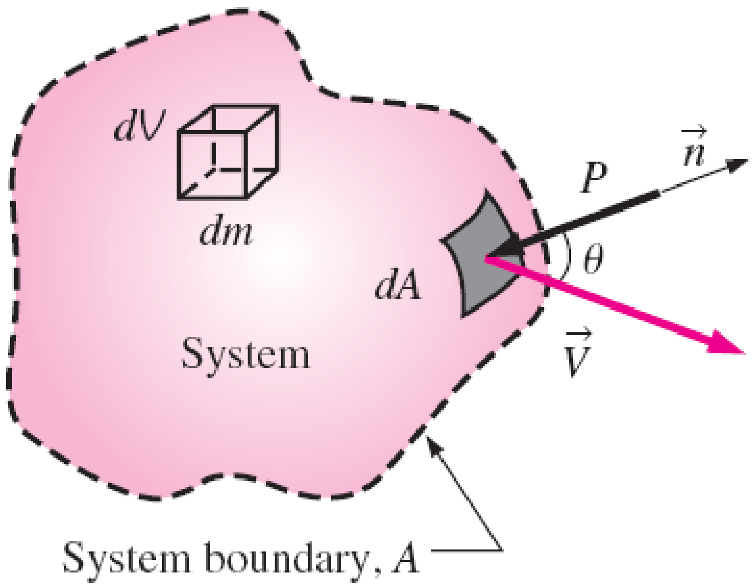


Work Done by Pressure Forces

- Consider a system shown in the right graph can deform arbitrarily. **What is the power done by pressure?**

$$V_n = V\cos\theta = \vec{V}\cdot\vec{n}$$

$$\delta\dot{W}_{\text{pressure}} = -P\,dA\,V_n = -P\,dA(\vec{V}\cdot\vec{n})$$



- Why is a negative sign at the right hand side?
- The total rate of work done by pressure forces is

$$\dot{W}_{\text{pressure, net in}} = -\int_A P(\vec{V}\cdot\vec{n})\,dA = -\int_A \frac{P}{\rho}\rho(\vec{V}\cdot\vec{n})\,dA$$

General Energy Equation

- Therefore, the net work in can be expressed by

$$\dot{W}_{net,in} = \dot{W}_{shaft,net,in} + \dot{W}_{pressure,net,in} = \dot{W}_{shaft,net,in} - \int P\left(\vec{V} \cdot \vec{n}\right) dA$$

- Then the rate form of the conservation of energy relation for a closed system becomes

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} + \dot{W}_{\text{pressure, net in}} = \frac{dE_{\text{sys}}}{dt}$$

General Energy Equation

- Recall general RTT

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b d\forall + \int_{CS} \rho b \left(\vec{V}_r \bullet \vec{n}\right) dA$$

- "Derive" energy equation using *B*=*E* and *b*=*e*

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net,in} + \dot{W}_{net,in} = \frac{d}{dt}\int_{CV} \rho e d\forall + \int_{CS} \rho e \left(\vec{V}_r \bullet \vec{n}\right) dA$$

$$\begin{pmatrix}\text{The net rate of energy}\\ \text{transfer into a CV by}\\ \text{heat and work transfer}\end{pmatrix} = \begin{pmatrix}\text{The time rate of}\\ \text{change of the energy}\\ \text{content of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{energy out of the control}\\ \text{surface by mass flow}\end{pmatrix}$$

General Energy Equation

- Moving integral for rate of pressure work to RHS of energy equation results in:

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \frac{d}{dt}\int_{\text{CV}} e\rho \, dV + \int_{\text{CS}} \left(\frac{P}{\rho} + e\right)\rho(\vec{V}_r \cdot \vec{n}) \, dA$$

  - For fixed control volume, then $V_r = V$
- Recall that P/ρ is the **flow work,** which is the work associated with pushing a fluid into or out of a CV per unit mass.

General Energy Equation

- As with the mass equation, practical analysis is often facilitated as averages across inlets and exits

$$Q_{net,in} + W_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e d\dot{V} + \sum_{out} \dot{m}\left(\frac{P}{\rho} + e\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + e\right)$$

$$\dot{m} = \int_{A_c} \rho(\vec{V} \cdot \vec{n})\, dA_c$$

- Since $e = u + ke + pe = u + V^2/2 + gz$

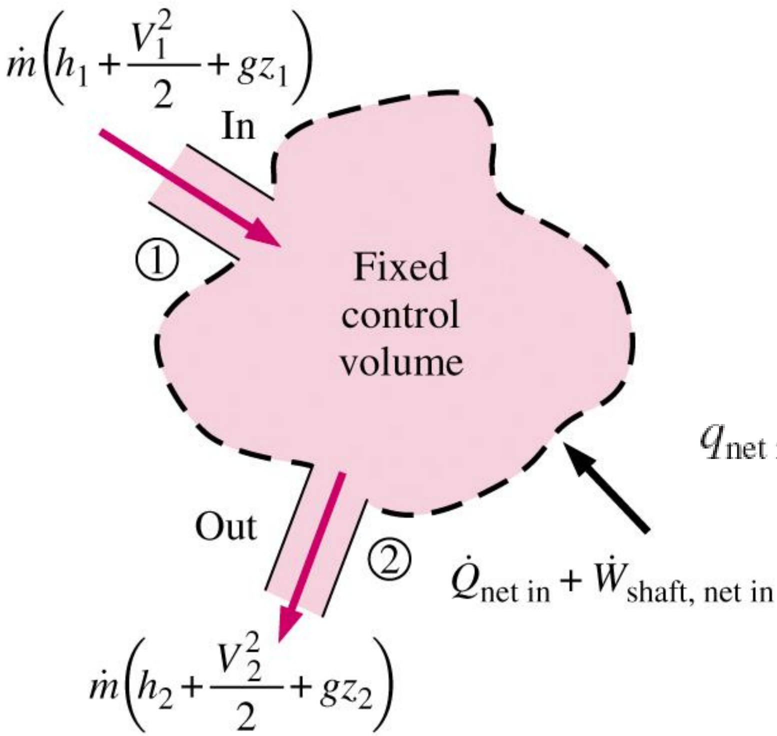
$$Q_{net,in} + W_{shaft,net,in} = \frac{d}{dt}\int_{CV} \rho e d\dot{V} + \sum_{out} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(\frac{P}{\rho} + u + \frac{V^2}{2} + gz\right)$$

Energy Analysis of Steady Flows

$$Q_{net,in} + W_{shaft,net,in} = \sum_{out} \dot{m}\left(h + \frac{V^2}{2} + gz\right) - \sum_{in} \dot{m}\left(h + \frac{V^2}{2} + gz\right)$$

- For steady flow, time rate of change of the energy content of the CV is zero.
- This equation states: *the net rate of energy transfer to a CV by heat and work transfers during steady flow is equal to the difference between the rates of outgoing* and incoming energy flows with mass.

Energy Analysis of Steady Flows



- For **single-stream devices**, mass flow rate is constant.

$$q_{\text{net in}} + w_{\text{shaft, net in}} = h_2 - h_1 + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1)$$

$$w_{\text{shaft, net in}} + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + (u_2 - u_1 - q_{\text{net in}})$$

Energy Analysis of Steady Flows

Rearranging

$$w_{\text{shaft, net in}} + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 =$$

$$\frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + (u_2 - u_1 - q_{\text{net in}})$$

- The left side of Eq. is the mechanical energy input, while the first three terms on the right side represent the mechanical energy output. If the flow is ideal with no loss, the total mechanical energy must be conserved, and the term in parentheses must equal zero.
- Any increase in u_2 - u_1 above $q_{\text{net in}}$ represents the mechanical energy loss

Energy Analysis of Steady Flows

The steady-flow energy equation on a unit-mass basis can be written as

$$e_{\text{mech, in}} = e_{\text{mech, out}} + e_{\text{mech, loss}}$$

or

$$w_{\text{shaft, net in}} + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + e_{\text{mech, loss}}$$

If $w_{\text{shaft, net in}} = w_{\text{shaft, in}} - w_{\text{shaft, out}} = w_{\text{pump}} - w_{\text{turbine}}$

Also multiplying the equation by the mass flow rate, then equation becomes

$$\dot{m}\left(\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1\right) + \dot{W}_{\text{pump}} = \dot{m}\left(\frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2\right) + \dot{W}_{\text{turbine}} + \dot{E}_{\text{mech, loss}}$$

Energy Analysis of Steady Flows

- where

$$\dot{E}_{\text{mech, loss}} = \dot{E}_{\text{mech loss, pump}} + \dot{E}_{\text{mech loss, turbine}} + \dot{E}_{\text{mech loss, piping}}$$

- In terms of heads, then equation becomes

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 + h_{\text{pump, }u} = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 + h_{\text{turbine, }e} + h_L$$

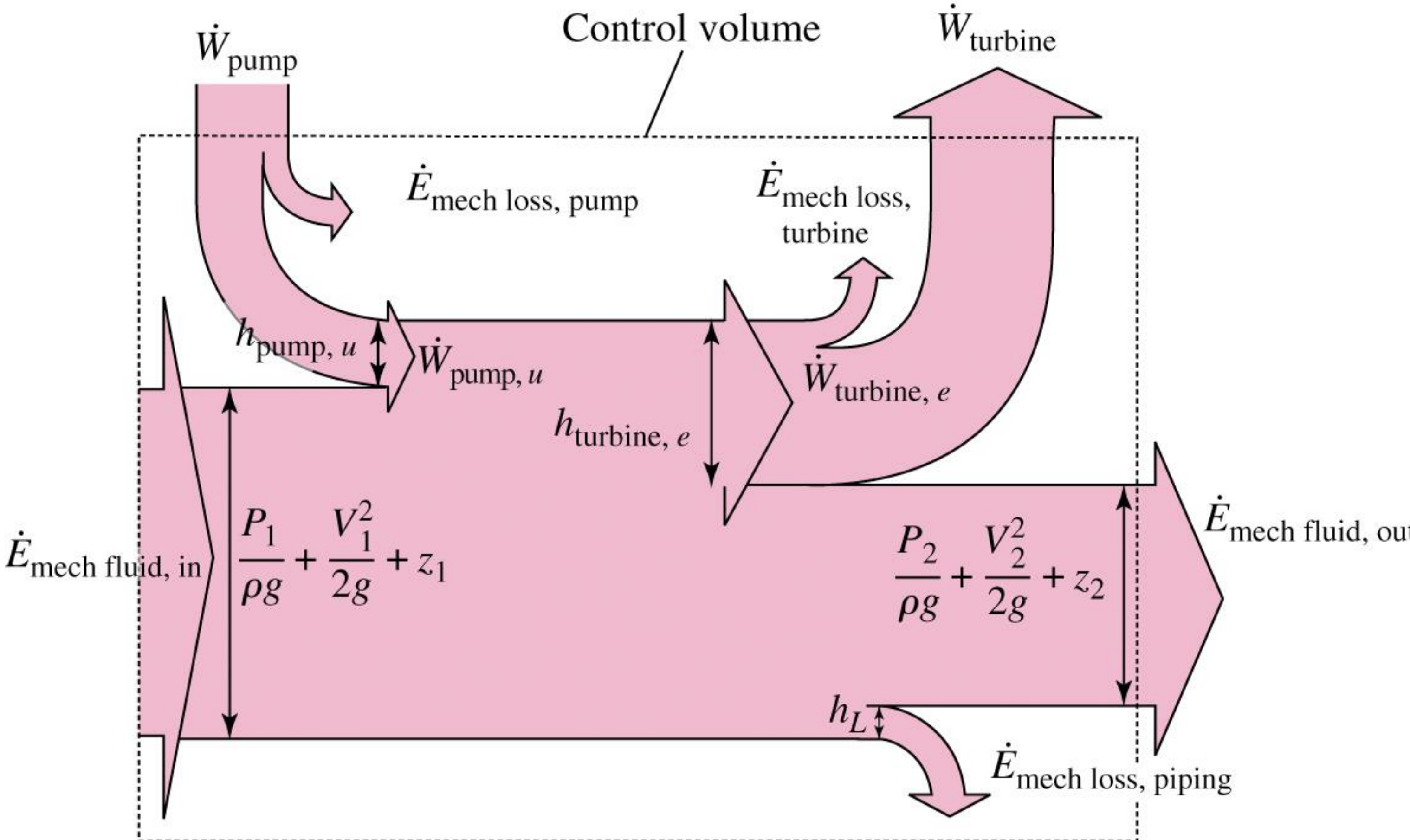
- where

$$h_{\text{pump, }u} = \frac{w_{\text{pump, }u}}{g} = \frac{\dot{W}_{\text{pump, }u}}{\dot{m}g} = \frac{\eta_{\text{pump}}\dot{W}_{\text{pump}}}{\dot{m}g}$$

$$h_{\text{turbine, }e} = \frac{w_{\text{turbine, }e}}{g} = \frac{\dot{W}_{\text{turbine, }e}}{\dot{m}g} = \frac{\dot{W}_{\text{turbine}}}{\eta_{\text{turbine}}\dot{m}g}$$

$$h_L = \frac{e_{\text{mech loss, piping}}}{g} = \frac{\dot{E}_{\text{mech loss, piping}}}{\dot{m}g}$$

Energy Analysis of Steady Flows



Mechanical energy flow chart for a fluid flow system that involves a pump and a turbine.

Energy Analysis of Steady Flows

- If no mechanical loss and no mechanical work devices, then equation becomes Bernoulli equation

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2$$

- **Kinetic Energy Correction Factor,** α

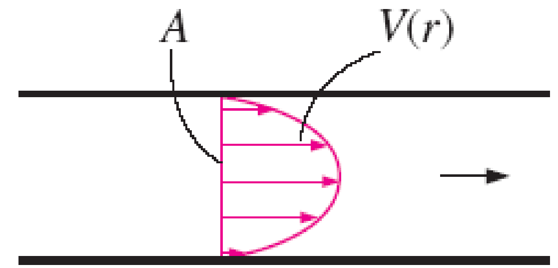
  Using the average flow velocity in the equation may cause the error in the calculation of kinetic energy; therefore, α, the **kinetic energy correction factor,** is used to correct the error by replacing the kinetic energy terms $V^2/2$ in the energy equation by $\alpha V_{avg}^2/2$.

α is 2.0 for fully developed laminar pipe flow, and it ranges between 1.04 and 1.11 for fully developed turbulent flow in a round pipe.

Energy Analysis of Steady Flows

- α is often ignored, since it is near one for turbulent flow and the kinetic energy contribution is small.
- the energy equations for *steady incompressible flow* become

$$\frac{P_1}{\rho g} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_{\text{pump}, u} =$$
$$\frac{P_2}{\rho g} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_{\text{turbine}, e} + h_L$$



$$\dot{m} = \rho V_{\text{avg}} A, \qquad \rho = \text{constant}$$

$$\dot{KE}_{\text{act}} = \int \text{ke}\,\delta\dot{m} = \int_A \frac{1}{2} V^2(r)[\rho V(r)\, dA]$$
$$= \frac{1}{2}\rho \int_A V^3(r)\, dA$$

$$\dot{KE}_{\text{avg}} = \frac{1}{2}\dot{m}V_{\text{avg}}^2 = \frac{1}{2}\rho A V_{\text{avg}}^3$$

$$\alpha = \frac{\dot{KE}_{\text{act}}}{\dot{KE}_{\text{avg}}} = \frac{1}{A}\int_A \left(\frac{V(r)}{V_{\text{avg}}}\right)^3 dA$$

EXAMPLE: Hydroelectric Power Generation from a Dam

Solution

$$\dot{m} = \rho\dot{V} = (1000\ \text{kg/m}^3)(100\ \text{m}^3/\text{s}) = 10^5\ \text{kg/s}$$

$$\cancel{\frac{P_1}{\rho g}} + \alpha_1 \cancel{\frac{V_1^2}{2g}} + z_1 + h_{\text{pump},\,u} \xrightarrow{} 0 =$$

$$\cancel{\frac{P_2}{\rho g}} + \alpha_2 \cancel{\frac{V_2^2}{2g}} + z_2 \xrightarrow{} 0 + h_{\text{turbine},\,e} + h_L$$

$$h_{\text{turbine},\,e} = z_1 - h_L$$

$$h_{\text{turbine},\,e} = z_1 - h_L = 120 - 35 = 85\ \text{m}$$

$$\dot{W}_{\text{turbine},\,e} = \dot{m}gh_{\text{turbine},\,e} = 83{,}400\ \text{kW}$$

$$\dot{W}_{\text{electric}} = \eta_{\text{turbine-gen}}\dot{W}_{\text{turbine},\,e} = \mathbf{66.7\ MW}$$

EXAMPLE: Head and Power Loss During Water Pumping

Solution

$$\dot{m} = \rho \dot{V} = (1000\ \mathrm{kg/m^3})(0.03\ \mathrm{m^3/s}) = 30\ \mathrm{kg/s}$$

$$\dot{m}\left(\cancel{\frac{P_1}{\rho}} + \alpha_1 \cancelto{0}{\frac{V_1^2}{2}} + \cancelto{0}{g z_1}\right) + \dot{W}_{\text{pump}} = \dot{m}\left(\cancel{\frac{P_2}{\rho}} + \alpha_2 \cancelto{0}{\frac{V_2^2}{2}} + g z_2\right) + \cancelto{0}{\dot{W}_{\text{turbine}}} + \dot{E}_{\text{mech, loss}}$$

$$\dot{E}_{\text{mech, loss}} = \dot{W}_{\text{pump}} - \dot{m} g z_2$$

$$\dot{E}_{\text{mech, loss}} = \mathbf{6.76\ kW}$$

$$h_L = \frac{\dot{E}_{\text{mech loss, piping}}}{\dot{m} g} = \mathbf{23.0\ m}$$