Chapter 3 Higher Order Linear ODEs

3.1Higher Order Linear Differential Equations

*n*th order differential equation

$$F(x,\ y,\ y',......,\ y^{(n)}) = 0$$

$$\text{this eq. may be } \begin{cases} \text{homogeneous or nonhomogeneous} \\ \text{linear or nonlinear} \end{cases}$$

Linear: $y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = r(x)$ (1)

Homogeneous: $y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = 0$ (2)

Theorem 1: For the **homogeneous** linear differential equation (2), sum and constant multiples of solution on some open interval *I* are again solution of (2) on *I*.

General solution: a general solution of (2) on an open interval *I* is of the form $y(x) = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x)$ (3)

$y_1(x), y_2(x),..., y_n(x)$: basis of solutions of (2)

$c_1, c_2,..., c_n$: arbitrary constants

Linear independence: n functions $y_1(x), y_2(x),..., y_n(x)$ are called **linearly independent** on some interval *I* where they are defined if the equation

$$k_1y_1(x) + k_2y_2(x) + ... + k_ny_n(x) = 0$$

implies that all $k_1, k_2,..., k_n$ are zero. These functions are called **linearly dependent** on *I* if this equation also holds on *I* for some $k_1, k_2,..., k_n$ not all zero.

Govern eq.: $y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = 0$ (2)

Initial conditions: $y(x_0) = K_0,\quad y'(x_0) = K_1,...., y^{(n-1)}(x_0) = K_{n-1}$ (4)

(A) & (B) : **Initial value problem**

Theorem 2 : if $p_0(x),..., p_{n-1}(x)$ are continuous functions on some open interval *I* and x_0 is in *I*, then the initial value problem (2), (4) has a unique solution *y*(*x*) on the interval *I*.

Theorem 3: Suppose that the coefficients $p_0(x),..., p_{n-1}(x)$ of (2) are continuous on some open interval *I*. Then *n* solutions $y_1(x), y_2(x),..., y_n(x)$ of (2) on *I* are linearly dependent on *I* if and only if their Wronskian is zero for some $x = x_0$ in *I*. Furthermore, if $W = 0$ for some $x = x_0$, then $W \equiv 0$ on *I*; hence if there is an x_1 in *I* at which $W \neq 0$, then $y_1(x), y_2(x),..., y_n(x)$ are linearly independent on *I*.

Theorem 4: If the coefficients $p_0(x),..., p_{n-1}(x)$ of (2) are continuous on some open interval *I*. Then (2) Has a general solution on *I* .

Theorem 5: If (2) has continuous coefficients $p_0(x),..., p_{n-1}(x)$ on some interval *I*. Then every solution $y = Y(x)$ of (2) on *I* is of the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) + ... + C_n y_n(x)$$

where $y_1(x), y_2(x),..., y_n(x)$ is a basis of solutions of (2) on *I* and $C_1, C_2,..., C_n$ are suitable constants.

3.2 Higher order Homogeneous Equation with Constant Coefficients

Consider a nth order linear homogeneous equation:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \tag{1}$$

try $y = e^{\lambda x}$, obtains the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0 \tag{2}$$

※ Distinct Real roots

If all the n roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of (2) are real and different, then the n solutions $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$,......, $y_n = e^{\lambda_n x}$ constitute a basis for all x. Then the corresponding general solution of (1) is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots\dots + c_n e^{\lambda_n x} \tag{3}$$

Are the $y_1, y_2, \dots\dots, y_n$ linearly independent ?

$W(y_1, y_2, \dots\dots, y_n) \neq 0 \quad \Rightarrow$ linearly independent

e.g. $n = 3$

$$\Rightarrow \quad W(y_1, y_2, \dots\dots, y_n) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & e^{\lambda_3 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \lambda_3 e^{\lambda_3 x} \\ \lambda_1^2 e^{\lambda_1 x} & \lambda_2^2 e^{\lambda_2 x} & \lambda_3^2 e^{\lambda_3 x} \end{vmatrix}$$

$$= e^{(\lambda_1+\lambda_2+\lambda_3)x} \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix}$$ Vandermonde or Cauchy determinant

$$= e^{(\lambda_1+\lambda_2+\lambda_3)x} \left[\begin{vmatrix} \lambda_2 & \lambda_3 \\ \lambda_2^2 & \lambda_3^2 \end{vmatrix} - \lambda_1 \begin{vmatrix} 1 & 1 \\ \lambda_2^2 & \lambda_3^2 \end{vmatrix} + \lambda_1^2 \begin{vmatrix} 1 & 1 \\ \lambda_2 & \lambda_3 \end{vmatrix} \right]$$

$$= e^{(\lambda_1+\lambda_2+\lambda_3)x} \left[(\lambda_3^2\lambda_2 - \lambda_3\lambda_2^2) - \lambda_1(\lambda_3^2 - \lambda_2^2) + \lambda_1^2(\lambda_3 - \lambda_2) \right]$$

$$= e^{(\lambda_1+\lambda_2+\lambda_3)x} (\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \neq 0$$

$\therefore y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}, y_3 = e^{\lambda_3 x}$ are linearly independent

Thus if $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct

$\Rightarrow y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}, \dots, y_n = e^{\lambda_n x}$ are linearly independent

the determinant for n components can be shown that it equals

$$(-1)^{n(n-1)/2}V$$

V is the product of all factors $\lambda_j - \lambda_k$, with $j < k$

※ Simple complex roots

The same as that in section 2.3

If $\lambda = \gamma + i\omega$ then the conjugate $\lambda = \gamma - i\omega$ exists

$$\Rightarrow \quad y_1 = e^{\gamma x}\cos\omega x, \qquad y_2 = e^{\gamma x}\sin\omega x$$

※ Multiple real roots

Double root (as in section 2.3) $\Rightarrow \lambda = \lambda_1 = \lambda_2$ then

$$y_1(x) = e^{\lambda_1 x} \text{ and } y_2(x) = xy_1(x) = x e^{\lambda_1 x}$$

y_1 and xy_1 are linearly independent solutions

Triple root $\Rightarrow \lambda = \lambda_1 = \lambda_2 = \lambda_3$, then

$$y_1(x) = e^{\lambda_1 x},\ y_2(x) = xy_1(x) = x e^{\lambda_1 x} \text{ and } y_3 = x^2 y_1 = x^2 e^{\lambda_1 x}$$

$\vdots$

root of order n $\Rightarrow \lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_n$, then

$$y_1(x) = e^{\lambda_1 x},\ y_2 = xy_1,\ y_3 = x^2 y_1, \cdots, y_n = x^{n-1} y_1$$

that is the n corresponding linearly independent solutions are

$$e^{\lambda_1 x},\ xe^{\lambda_1 x},\ x^2 e^{\lambda_1 x},\ x^3 e^{\lambda_1 x},\ ......, x^{n-1} e^{\lambda_1 x},$$

※ Multiple complex roots

If $\lambda = \gamma + i\omega$ is a complex double root, so is the conjugate $\lambda = \gamma - i\omega$

Thus the linearly independent solutions are

$$e^{\gamma x}\cos\omega x, \quad e^{\gamma x}\sin\omega x, \quad x\,e^{\gamma x}\cos\omega x, \quad x\,e^{\gamma x}\sin\omega x$$

If λ is a triple root, then the linearly independent solutions are

$$e^{\gamma x}\cos\omega x, \quad e^{\gamma x}\sin\omega x, \quad x\,e^{\gamma x}\cos\omega x, \quad x\,e^{\gamma x}\sin\omega x$$

$$x^2 e^{\gamma x}\cos\omega x, \quad x^2 e^{\gamma x}\sin\omega x$$

3.3 Higher Order Nonhomogeneous Equations

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y'(x) + p_0(x)y = r(x) \qquad (1)$$

general solution is $y = y_h(x) + y_p(x)$

$y_p(x)$: is the particular solution

$y_h(x)$: is a general solution of the corresponding homogeneous eq.

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y'(x) + p_0(x)y = 0 \qquad (2)$$

and can be obtained by the method in section 2.14.

Methods to determine particular solution $y_p(x)$

※ Method of Undetermined Coefficients

Rule is the same as that of second order (section 2.9)

Example: $y''' - 3y'' + 3y' - y = 30e^x \qquad (3)$

Try $y(x) = e^{\lambda x}$, characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \quad \Rightarrow \quad (\lambda - 1)^3 = 0 \Rightarrow \lambda = 1, 1, 1$$

$$y_h(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

To determine $y_p(x)$,

try $y_p(x) = Ae^x;\ Axe^x;\ Ax^2e^x$ can not get the solution according the rule of multiple root, $y_p(x)$ must be of the form $y_p(x) = Ax^3 e^x$

Then (3)

$$\Rightarrow (x^3 + 9x^2 + 18x + 6)A - 3(x^3 + 6x^2 + 6x)A$$
$$+ 3(x^3 + 3x^2)A - x^3 A = 30$$
$$\Rightarrow \quad A = 5$$
$$\therefore \quad y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + 5x^3 e^x$$

※ Method of Variation of Parameters

Assume $y_1, ..., y_n$ is a basis of the solutions of the homogeneous eq. (2) $\Rightarrow y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$

To determine $y_p(x)$, replace the parameter c's by function of $u_1(x), ..., u_n(x)$.

$$\Rightarrow y_p = u_1 y_1 + u_2 y_2 + ... + u_n y_n$$

$$y_p' = (u_1 y_1' + u_2 y_2' + ... + u_n y_n') + (u_1' y_1 + u_2' y_2 + ... + u_n' y_n)$$

choose $(u_1' y_1 + u_2' y_2 + ... + u_n' y_n) = 0$ (A)

then differentiate what is left

$$y_p'' = (u_1 y_1'' + u_2 y_2'' + ... + u_n y_n'') + (u_1' y_1' + u_2' y_2' + ... + u_n' y_n')$$

now choose $(u_1' y_1' + u_2' y_2' + ... + u_n' y_n') = 0$ (B)

$$\vdots$$

$$y_p^{(n-1)} = (u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + ... + u_n y_n^{(n-1)}) + (u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + ... + u_n' y_n^{(n-2)})$$

$$\Rightarrow \quad (u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + ... + u_n' y_n^{(n-2)}) = 0 \qquad \text{(C)}$$

$$y_p^{(n)} = (u_1 y_1^{(n)} + u_2 y_2^{(n)} + ... + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + ... + u_n' y_n^{(n-1)})$$

Substituting into (1)

$$\Rightarrow \quad (u_1 y_1^{(n)} + u_2 y_2^{(n)} + ... + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + ... + u_n' y_n^{(n-1)})$$
$$+ p_{n-1}(u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + ... + u_n y_n^{(n-1)}) + ... + p_0(u_1 y_1 + u_2 y_2 + ... + u_n y_n) = r(x)$$

$$\Rightarrow \quad (u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + ... + u_n' y_n^{(n-1)}) = r(x) \qquad \text{(D)}$$

Combination of (A), (B), …,(C) and (D)

$$\begin{aligned}
&u_1' y_1 + u_2' y_2 + ... + u_n' y_n = 0 \\
&u_1' y_1' + u_2' y_2' + ... + u_n' y_n' = 0 \\
&u_1' y_1'' + u_2' y_2'' + ... + u_n' y_n'' = 0 \\
&\vdots \\
&u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + ... + u_n' y_n^{(n-2)} = 0 \\
&u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + ... + u_n' y_n^{(n-1)} = r(x)
\end{aligned} \qquad \text{(E)}$$

$$u_1' = \frac{r(x)\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ 1 & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}}{W(y_1, y_2, \ldots, y_n)} = \frac{r(x)W_1}{W(y_1, y_2, \ldots, y_n)}$$

$$u_2' = \frac{r(x)\begin{vmatrix} y_1 & 0 & \cdots & y_n \\ y_1' & 0 & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & 1 & \cdots & y_n^{(n-1)} \end{vmatrix}}{W(y_1, y_2, \ldots, y_n)} = \frac{r(x)W_2}{W(y_1, y_2, \ldots, y_n)}$$

$$\vdots$$

$$\therefore u_1 = \int \frac{W_1}{W(y_1, y_2, \ldots, y_n)} r(x)dx, \quad u_2 = \int \frac{W_2}{W(y_1, y_2, \ldots, y_n)} r(x)dx, \ldots$$

$$y_p(x) = y_1(x)\int \frac{W_1}{W} r(x)dx + y_2(x)\int \frac{W_2}{W} r(x)dx + \ldots + y_n(x)\int \frac{W_n}{W} r(x)dx$$