Parallel Programming
in C with MPI and OpenMP

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Chapter 5.
The Sieve of Eratosthenes

Eratosthenes

- Born: 276 BC in Cyrene, North Africa (now Shahhat, Libya)
- Died: 194 BC in Alexandria, Egypt

Chapter Objectives

- Analysis of block allocation schemes
- Function **`MPI_Bcast`**
- Performance enhancements

Outline

- Sequential algorithm
- Sources of parallelism
- Data decomposition options
- Parallel algorithm development, analysis
- MPI program
- Benchmarking
- Optimizations

Sequential Algorithm

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60

$$\text{Complexity} : \Theta(n \ln \ln n)$$

Pseudocode

1. Create list of unmarked natural numbers 2, 3, …, n
2. $k \leftarrow 2$
3. Repeat
   (a) Mark all multiples of k between k^2 and n
   (b) $k \leftarrow$ smallest unmarked number $> k$
   until $k^2 > n$
4. The unmarked numbers are primes

Sources of Parallelism

- Domain decomposition
  - Divide data into pieces
  - Associate computational steps with data
- One primitive task per array element

Making 3(a) Parallel

- Mark all multiples of k between k^2 and n

$\Rightarrow$

- for all j where $k^2 \le j \le n$ do
    if (j mod k = 0) then
        mark j (it is not a prime)
    endif
  endfor

Making 3(b) Parallel

- Find smallest unmarked number > *k*

$\Rightarrow$

- Min-reduction (to find smallest unmarked number > *k*)
- Broadcast (to get result to all tasks)

Agglomeration Goals

- Consolidate tasks
- Reduce communication cost
- Balance computations among processes

Data Decomposition Options

- Interleaved (cyclic)
  - Easy to determine “owner” of each index
  - Leads to load imbalance *for this problem*
- Block
  - Balances loads
  - More complicated to determine owner if *n* not a multiple of *p*

Block Decomposition Options

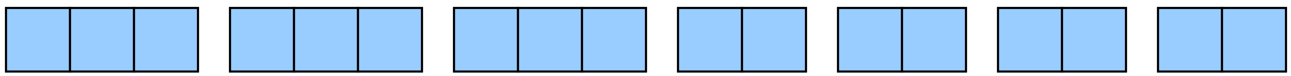
- Want to balance workload when n not a multiple of p
- Each process gets either $\lceil n/p \rceil$ or $\lfloor n/p \rfloor$ elements
- Seek simple expressions
  - Find low, high indices given an owner
  - Find owner given an index

Method #1

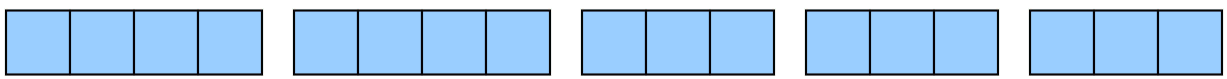
- Let $r = n \bmod p$
- If $r = 0$, all blocks have same size
- Else
  - First r blocks have size $\lceil n/p \rceil$
  - Remaining $p - r$ blocks have size $\lfloor n/p \rfloor$

Examples

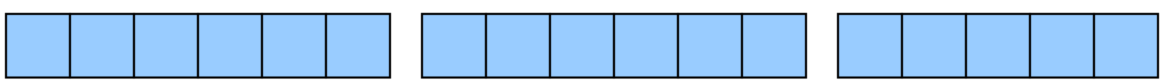
17 elements divided among 7 processes



17 elements divided among 5 processes



17 elements divided among 3 processes



Method #1 Calculations

- First element controlled by process *i*

$$i\lfloor n/p \rfloor + \min(i, r)$$

- Last element controlled by process *i*

$$(i+1)\lfloor n/p \rfloor + \min(i+1, r) - 1$$

- Process controlling element *j*

$$\min\left(\left\lfloor \frac{j}{\lfloor n/p \rfloor + 1} \right\rfloor, \left\lfloor \frac{(j-r)}{\lfloor n/p \rfloor} \right\rfloor\right)$$

Method #2

- Scatters larger blocks among processes
- First element controlled by process *i*

$$\left\lfloor \frac{in}{p} \right\rfloor$$

- Last element controlled by process *i*

$$\left\lfloor \frac{(i+1)n}{p} \right\rfloor - 1$$

- Process controlling element *j*

$$\left\lfloor \frac{p(j+1)-1}{n} \right\rfloor$$

$$j = \left\lfloor \frac{in}{p} \right\rfloor$$

$$j \le \frac{in}{p} < j+1$$

$$\frac{pj}{n} \le i < \frac{p(j+1)}{n} \le i+1 < \frac{p(j+2)}{n}$$

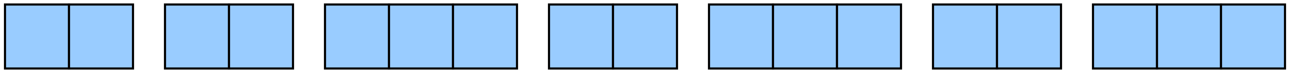
$$i < \frac{p(j+1)}{n} \le i+1$$

$$i \le \frac{p(j+1)-1}{n} < i+1$$

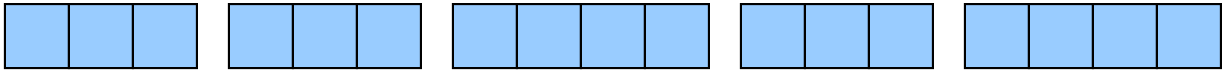
$$i = \left\lfloor \frac{p(j+1)-1}{n} \right\rfloor$$

Examples

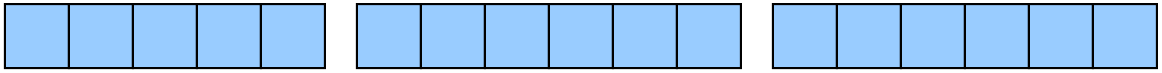
17 elements divided among 7 processes



17 elements divided among 5 processes



17 elements divided among 3 processes



Comparing Methods

Our choice

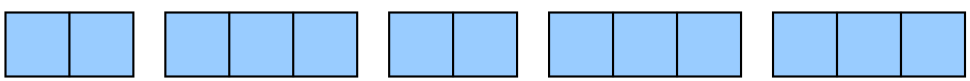
Operations	Method 1	Method 2
Low index	4	2
High index	6	4
Owner	7	4

Assuming no operations for “floor” function

Pop Quiz

- Illustrate how block decomposition method #2 would divide 13 elements among 5 processes.

$13(0)/5 = 0$ $\quad 13(2)/5 = 5$ $\quad 13(4)/5 = 10$



$13(1)/5 = 2$ $\quad 13(3)/5 = 7$

Block Decomposition Macros

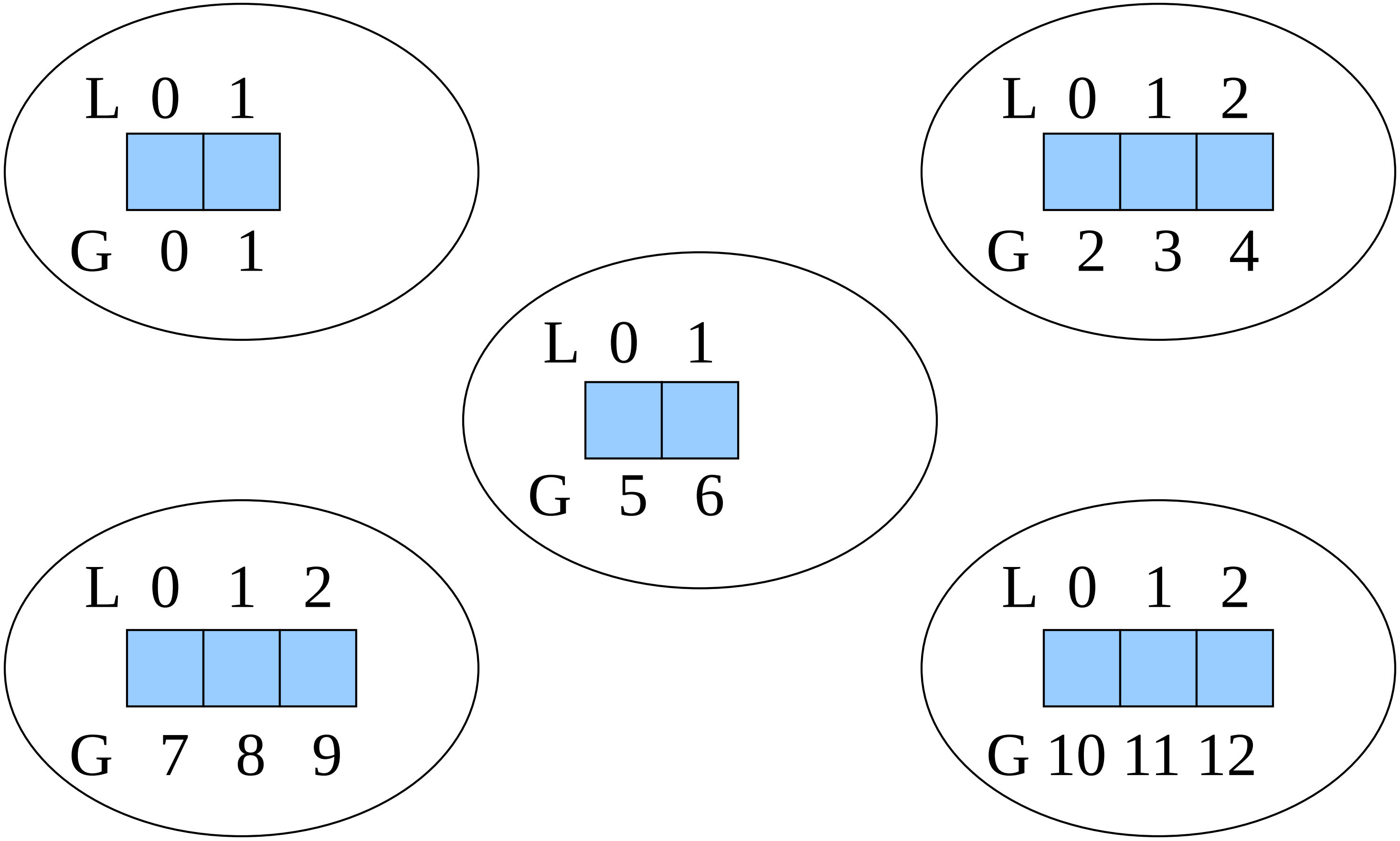
```
#define BLOCK_LOW(id,p,n)  ((id)*(n)/(p))

#define BLOCK_HIGH(id,p,n) \
        (BLOCK_LOW((id)+1,p,n)-1)

#define BLOCK_SIZE(id,p,n) \
        (BLOCK_HIGH(id,p,n)-BLOCK_LOW(id,p,n)+1)

#define BLOCK_OWNER(index,p,n) \
        (((p)*(index)+1)-1)/(n))
```

Local vs. Global Indices



Looping over Elements

- Sequential program

```
for (i = 0; i < n; i++) {
    …
}
```

Index *i* on this process…

- Parallel program

```
size = BLOCK_SIZE (id,p,n);
for (i = 0; i < size; i++) {
    gi = i + BLOCK_LOW(id,p,n);
}
```

…takes place of sequential program's index *gi*

Decomposition Affects Implementation

- Largest prime used to sieve is $\sqrt{n}$
- First process has $\lfloor n/p \rfloor$ elements
- It has all sieving primes if $p < \sqrt{n}$
- First process always broadcasts next sieving prime
- No reduction step needed

Fast Marking

- Block decomposition allows same marking as sequential algorithm:
- j, $j + k$, $j + 2k$, $j + 3k$, …

instead of

- for all j in block
  if $j \bmod k = 0$ then mark j (it is not a prime)

Original Algorithm

1. Create list of unmarked natural numbers 2, 3, …, n
2. $k \leftarrow 2$
3. Repeat
   (a) Mark all multiples of k between k^2 and n
   (b) $k \leftarrow$ smallest unmarked number > k

   until $k^2 > n$
4. The unmarked numbers are primes

Parallel Algorithm Development

1. Create list of unmarked natural numbers 2, 3, …, n
   Each process creates its share of list
2. $k \leftarrow 2$
   Each process does this
3. Repeat
   - (a) Mark all multiples of k between k^2 and n
     Each process marks its share of list
   - (b) $k \leftarrow$ smallest unmarked number $> k$
     Process 0 only
   - **(c) Process 0 broadcasts k to rest of processes**

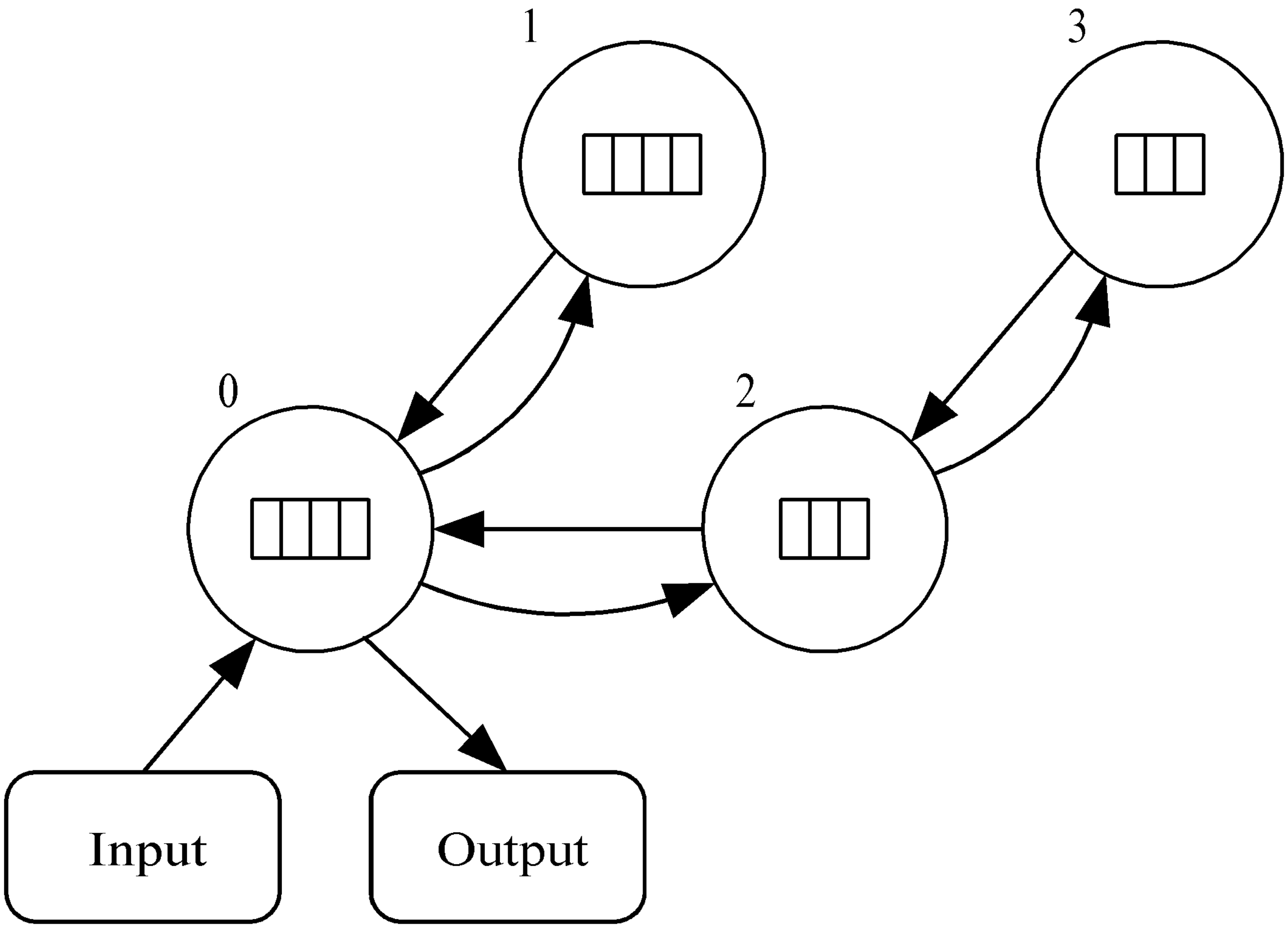
   until $k^2 > n$
4. The unmarked numbers are primes
5. **Reduction to determine number of primes**

Function MPI_Bcast

```
int MPI_Bcast (
    void *buffer, /* Addr of 1st element */
    int count,    /* # elements to broadcast */
    MPI_Datatype datatype, /* Type of elements */
    int root,     /* ID of root process */
    MPI_Comm comm /* Communicator */
)
```

```
MPI_Bcast (&k, 1, MPI_INT, 0, MPI_COMM_WORLD);
```

Task/Channel Graph



Analysis

- χ is time needed to mark a cell
- Sequential execution time: $\chi n \ln \ln n$
- Number of broadcasts: $\sqrt{n} / \ln \sqrt{n}$
- Broadcast time: $\lambda \lceil \log p \rceil$
- Expected execution time:

$$\chi n \ln \ln n / p + \left(\sqrt{n} / \ln \sqrt{n} \right) \lambda \lceil \log p \rceil$$

Code (1/4)

```
#include <mpi.h>
#include <math.h>
#include <stdio.h>
#include "MyMPI.h"
#define MIN(a,b)  ((a)<(b)?(a):(b))

int main (int argc, char *argv[])
{
   ...
   MPI_Init(&argc, &argv);
   MPI_Barrier(MPI_COMM_WORLD);
   elapsed_time = -MPI_Wtime();
   MPI_Comm_rank(MPI_COMM_WORLD, &id);
   MPI_Comm_size(MPI_COMM_WORLD, &p);
   if (argc != 2) {
      if (!id) printf("Command line: %s <m>\n", argv[0]);
      MPI_Finalize();
      exit (1);
   }
```

Code (2/4)

```
n = atoi(argv[1]);
low_value = 2 + BLOCK_LOW(id, p, n-1);
high_value = 2 + BLOCK_HIGH(id, p, n-1);
size = BLOCK_SIZE(id, p, n-1);
proc0_size = (n-1)/p;
if ((1 + proc0_size) < (int) sqrt((double) n)) {
   if (!id) printf("Too many processes\n");
   MPI_Finalize();
   exit (1);
}

marked = (char *) malloc(size);
if (marked == NULL) {
   printf("Cannot allocate enough memory\n");
   MPI_Finalize();
   exit (1);
}
```

Code (3/4)

```
for (i = 0; i < size; i++) marked[i] = 0;
if (!id) index = 0;
prime = 2;
do {
   if (prime * prime > low_value)
      first = prime * prime - low_value;
   else {
      if (!(low_value % prime)) first = 0;
      else first = prime - (low_value % prime);
   }
   for (i = first; i < size; i += prime) marked[i] = 1;
   if (!id) {
      while (marked[++index]);
      prime = index + 2;
   }
   MPI_Bcast(&prime,  1, MPI_INT, 0, MPI_COMM_WORLD);
} while (prime * prime <= n);
```

Code (4/4)

```
   count = 0;
   for (i = 0; i < size; i++)
      if (!marked[i]) count++;
   MPI_Reduce (&count, &global_count, 1, MPI_INT, MPI_SUM,
      0, MPI_COMM_WORLD);
   elapsed_time += MPI_Wtime();
   if (!id) {
      printf ("%d primes are less than or equal to %d\n",
         global_count, n);
      printf ("Total elapsed time: %10.6f\n", elapsed_time);
   }
   MPI_Finalize ();
   return 0;
}
```

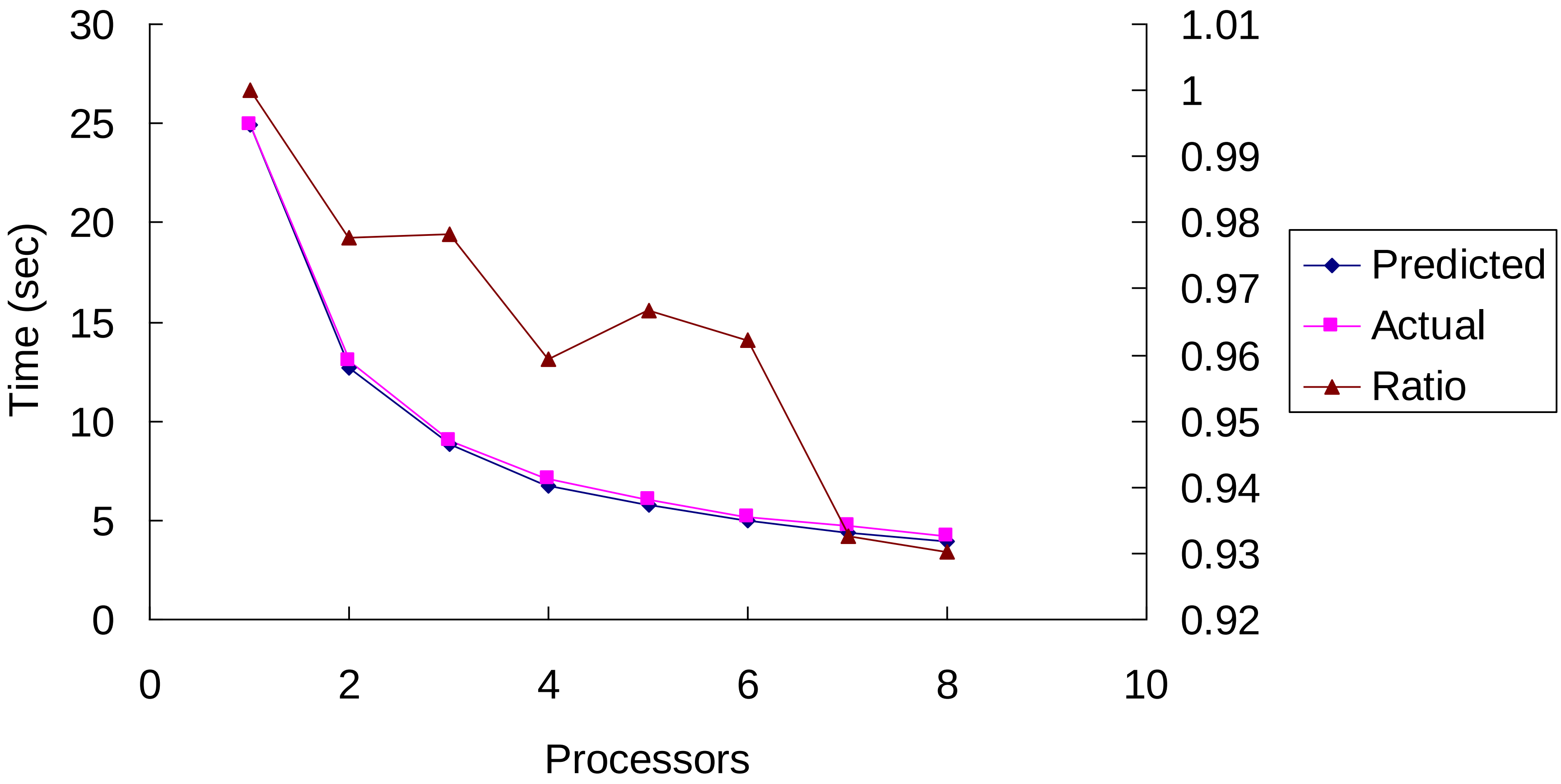
Benchmarking

- Execute sequential algorithm
- Determine χ = 85.47 nanosec
- Execute series of broadcasts
- Determine λ = 250 μsec

Execution Times (sec)

Processors	Predicted	Actual (sec)
1	24.900	24.900
2	12.721	13.011
3	8.843	9.039
4	6.768	7.055
5	5.794	5.993
6	4.964	5.159
7	4.371	4.687
8	3.927	4.222

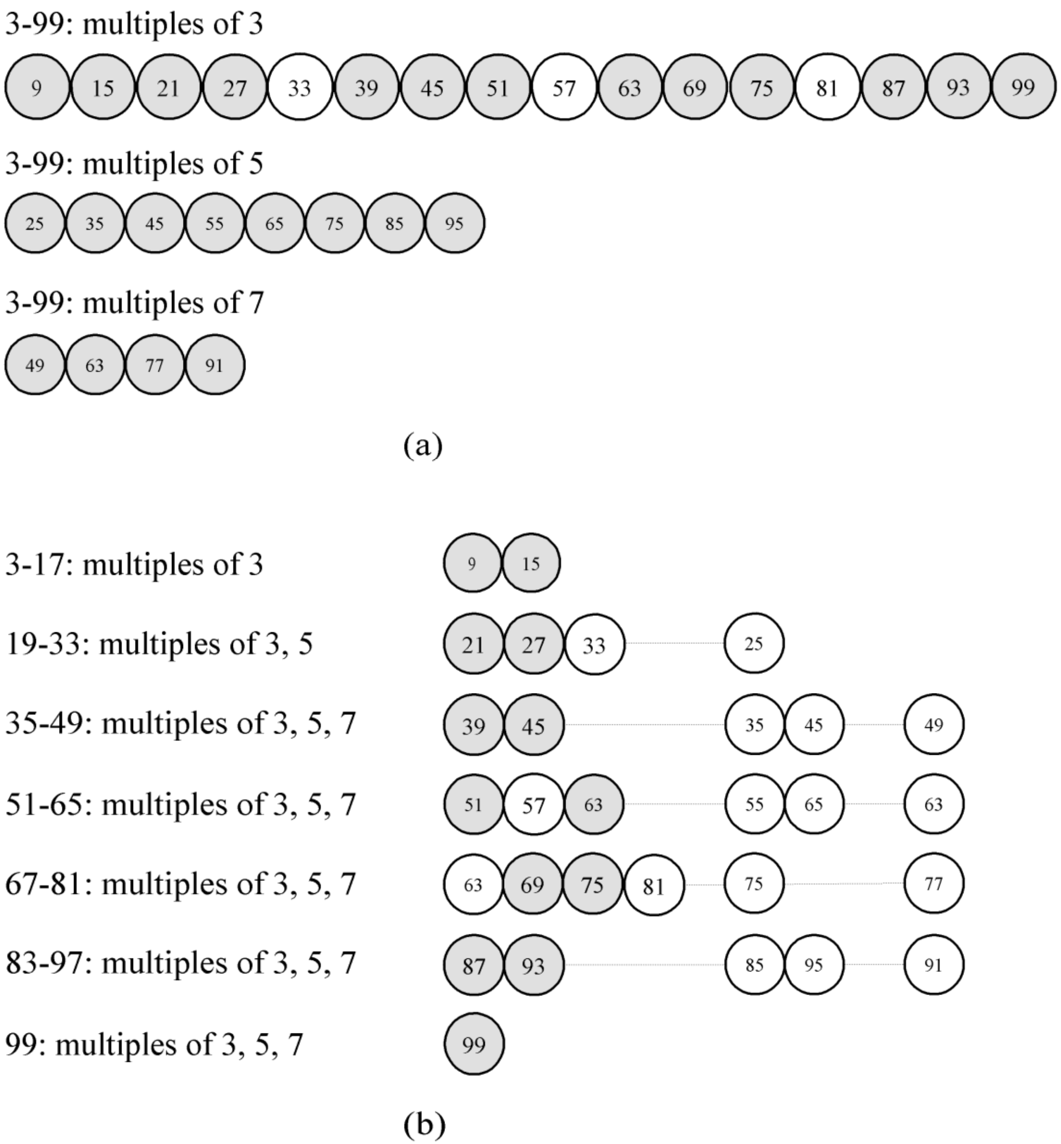
Execution Times and Speed Ratio



Improvements

- Delete even integers
  - Cuts number of computations in half
  - Frees storage for larger values of n
- Each process finds own sieving primes
  - Replicating computation of primes to $\sqrt{n}$
  - Eliminates broadcast step
- Reorganize loops
  - Increases cache hit rate

Reorganize Loops



Lower

↑

Cache hit rate

↓

Higher

Comparing 4 Versions

Procs	Sieve 1	Sieve 2	Sieve 3	Sieve 4
1	24.900	12.237	12.466	2.543
2	12.721	6.609	6.378	1.330
3	8.843	5.019	4.272	0.901
4	6.768	4.072	3.201	0.679
5	5.794	3.652	2.559	0.543
6	4.964	3.270	2.127	0.456
7	4.371	3.059	1.820	0.391
8	3.927	2.856	1.585	0.342

10-fold improvement

7-fold improvement

Summary

- Sieve of Eratosthenes: parallel design uses domain decomposition
- Compared two block distributions
  - Chose one with simpler formulas
- Introduced **`MPI_Bcast`**
- Optimizations reveal importance of maximizing single-processor performance

Exercises

- 5.9
- 5.11