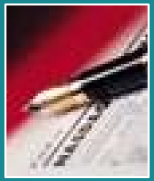
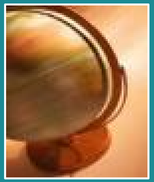




Chapter 3
Fourier series Representation of Periodic Signals

3.0 Introduction

- In this chapter, we focus on the representation of continuous-time and discrete-time periodic signals referred to as the Fourier series. In Chapters 4and 5, we extend the analysis to the Fourier transform representation of broad classes of aperiodic, finite energy signals. Together,

  本章將焦點置於連續時間與離散時間週期訊號的傅立葉級數表示法，第4及5章再將它展至非週期訊號

3.0 Introduction

- These representations provide one of the most powerful and important sets of tools and insights for analyzing, designing, and understanding signals and LTI systems, and we devote considerable attention in this and subsequent chapters to exploring the uses of Fourier methods.

這些表示法將是我們對訊號與LTI系統在分析、設計和理解上極有用而重要的工具。

3.1 A Historical Perspective

- We will see that if the input to an LTI system is expressed as a linear combination of periodic complex exponentials or sinusoids, the output can also be expressed in this form, with coefficients that are related in a straightforward way to those of the input.

  若一個LTI系統的輸入可表為數個週期性的複指數或弦波訊號的線性組合 ，則其(穩態)總輸出亦可利用各輸入相對的輸出透過相同的係數組合。

3.1 A Historical Perspective

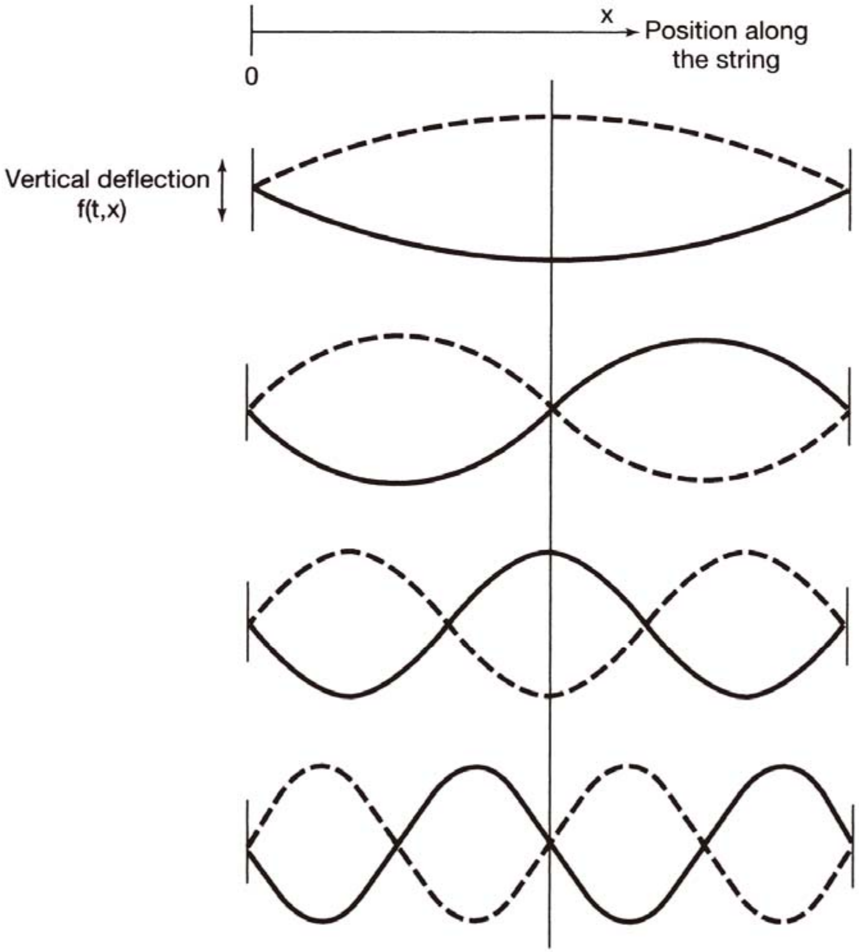


Figure 3.1 Normal modes of a vibrating string. (Solid lines indicate the configuration of each of these modes at some fixed instant of time, *t*.)

3.1 A Historical Perspective

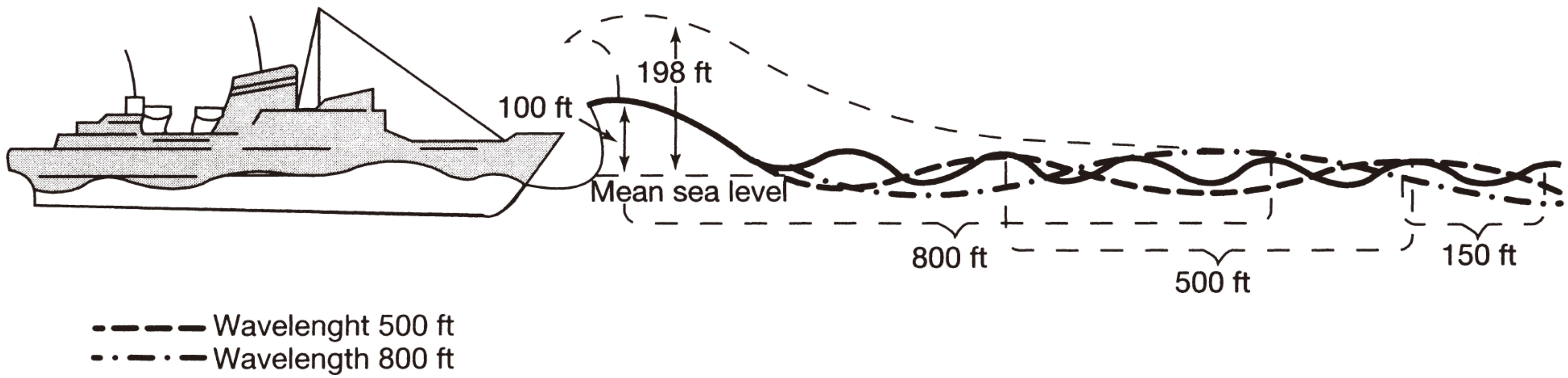


Figure 3.3 Ship encountering the superposition of three wave trains, each with a different spatial period. When these waves reinforce one another, a very large wave can result. In more severe seas, a giant wave indicated by the dotted line could result. Whether such a reinforcement occurs at any location depends upon the relative phases of the components that are superposed. [Adapted from an illustration by P. Mion in "Nightmare Waves Are All Too Real to Deepwater Sailors," by P. Britton, *Smithsonian* 8 (February 1978), pp. 64–65].

3.2 The Response of LTI Systems to Complex Exponentials

- As we indicated in Section 3.0, it is advantageous in the study of LTI systems to represent signals as linear combinations of basic signals that possess the following two properties:

1. The set of basic signals can be used to construct a broad and useful class of signals.
2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals.

3.2 The Response of LTI Systems to Complex Exponentials

- The importance of complex exponentials in the study of LTI systems stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is,

continuous time: $$e^{st} \rightarrow H(s)e^{st} , \qquad (3.1)$$

discrete time: $$z^{n} \rightarrow H(z)z^{n} , \qquad (3.2)$$

3.2 The Response of LTI Systems to Complex Exponentials

- LTI系統對於一複指數輸入所引起的響應為型式相同但振幅(係數)不同的複指數。

連續時間： $e^{st} \rightarrow H(s)e^{st}$

離散時間： $z^{n} \rightarrow H(z)z^{n}$

where the complex amplitude factor H(s) or H(z) will in general be a function of the complex variable s or z.

3.2 The Response of LTI Systems to Complex Exponentials

- To show that complex exponentials are indeed eigenfunctions of LTI systems, let us consider a continuous-time LTI system with impulse response $h(t)$. For an input $x(t)$, we can determine the output through the use of the convolution integral, so that with $x(t) = e^{st}$

一個訊號對系統的輸出正好是此輸入訊號乘以某一常數，則此訊號(函數)為系統的「特徵函數」，且振幅的因數為系統的「特徵值」。

令 $x(t) = e^{st}$

3.2 The Response of LTI Systems to Complex Exponentials

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau. \tag{3.3}$$

Expressing $e^{s(t-\tau)}$ as $e^{st}e^{-st}$, and noting that e^{st} can be moved outside the integral, we see that eq.(3.3) becomes

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-st}d\tau. \tag{3.4}$$

3.2 The Response of LTI Systems to Complex Exponentials

The response to e^{st} is of the form

$$y(t) = H(s)e^{st} \tag{3.5}$$

Where H(s) is a complex constant whose value depends on s ant which is related to the system impulse response by

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau \tag{3.6}$$

其中H(s)為如(3.6)式的複數常數。

3.2 The Response of LTI Systems to Complex Exponentials

Hence, we have shown that complex exponentials are eigenfunctions of LTI systems. The constant *H*(s) for a specific value of *s* is then the eigenvalue associated With the eigenfunction e^{st} .

則此LTI系統的特徵函數為複指數 e^{st}。*H*(s)與*s*參數有關，但對時間*t* 而言為一常數

Suppose that ant LTI system with impluse response $h[n]$ has as its input the sequence

同理，令 $$x[n] = z^n \tag{3.7}$$

3.2 The Response of LTI Systems to Complex Exponentials

Where z is a complex number. Then the output of the system can be determined from the convolution sum as

$$y[n]=\sum_{-\infty}^{+\infty}h[k]x[n-k] \tag{3.8}$$

$$=\sum_{-\infty}^{+\infty}h[k]z^{n-k}=z^{n}\sum_{-\infty}^{+\infty}h[k]z^{-k}.$$

3.2 The Response of LTI Systems to Complex Exponentials

The output is the same complex exponential multiplied by a constant that depends on thevalue of z. That is,

可得

$$y[n] = H(z)z^n, \tag{3.9}$$

where

其中

$$H(z) = \sum_{-\infty}^{+\infty} h[k]z^{-k}. \tag{3.10}$$

3.2 The Response of LTI Systems to Complex Exponentials

Consequently, as in the continuous-time case, complex exponentials are eigenfunctionsof discrete-time LTI systems. The constant $H(z)$ for a specified value of z is the eigenvalue associated with the eigenfunction z^n.

則此離散時間LTI系統的特徵函數為複指數 z^n 。*H(z)*與參數*z*有關，但對時間*k*而言為一常數。

3.2 The Response of LTI Systems to Complex Exponentials

Let *x(t)* correspond to a linear combination of three complex exponentials; that is,

設輸入

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}. \qquad (3.11)$$

From the eigenfunction property, the response to each separately is

各輸入項

相對的輸出

$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t},$$

$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t},$$

$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t},$$

3.2 The Response of LTI Systems to Complex Exponentials

and from the superposition property the reponse to the sum is the sum of the responses, so that

由重疊性質可知，輸出

$$y(t) = a_1 H(s_1)e^{s_1 t} + a_2 H(s_2)e^{s_2 t} + a_3 H(s_3)e^{s_3 t}. \qquad (3.12)$$

3.2 The Response of LTI Systems to Complex Exponentials

If the input to a continuous-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x(t) = \sum_k a_k e^{s_k t}, \tag{3.13}$$

then the output will be

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{3.14}$$

3.2 The Response of LTI Systems to Complex Exponentials

If

$$x[n] = \sum_k a_k z_k^n, \tag{3.15}$$

then the output will be

$$y[n] = \sum_k a_k H(z_k) z_k^n. \tag{3.16}$$

Example 3.1

As an illustration of eqs. (3.5) and (3.6), consider an LTI system for which the input *x(t)* and output *y(t)* are related by a time shift of 3,i.e.,

$$y(t) = x(t-3). \tag{3.17}$$

If the input to this system is the complex exponential signal $x(t) = e^{j2t}$, then, from eq.(3.17),

$$y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}. \tag{3.18}$$

Example 3.1

Equation (3.18) is in the form of eq. (3.5), as we would expect, since e^{j2t} is an eigenfunction. The associated eigenvalue is $H(j2) = e^{-j6}$. It is straightforward to confirm eq. (3.6)for this example. Specifically, from eq. (3.17), the impulse response of the system is $h(t) = \delta(t-3)$. Substituting into eq. (3.6), we obtain $H(s) = \int_{-\infty}^{+\infty} \delta(\tau - 3) e^{-st} d\tau = e^{-3s}$, so that $H(j2) = e^{-j6}$.

Example 3.1

As a second example, in this case illustrating eqs. (3.11) and (3.12), consider the input signal $x(t) = \cos(4t) + \cos(7t)$. From eq. (3.17), *y(t)* will of course be

$$y(t) = \cos(4(t-3)) + \cos(7(t-3)). \tag{3.19}$$

To see that this will also result from eq. (3.12), we first expand *x(t)* using Euler's relation:

$$x(t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t}. \tag{3.20}$$

Example 3.1

From eqs. (3.11) and (3.12),

$$y(t) = \frac{1}{2}e^{-j12}e^{j4t} + \frac{1}{2}e^{j12}e^{-j4t} + \frac{1}{2}e^{-j21}e^{j7t} + \frac{1}{2}e^{j21}e^{-j7t},$$

or

$$y(t) = \frac{1}{2}e^{j4(t-3)} + \frac{1}{2}e^{-j4(t-3)} + \frac{1}{2}e^{j7(t-3)} + \frac{1}{2}e^{-j7(t-3)}$$

$$= \cos(4(t-3)) + \cos(7(t-3)).$$

Example 3.1

For this simple example, multiplication of each periodic exponential component of *x(t)*—for example, $\frac{1}{2}e^{j4t}$—by the corresponding eigenvalue—e.g., $H(j4) = e^{-j12}$ — effectively causes the input component to shift in time by 3. Obviously, in this case we can determine *y(t)* in eq. (3.19) by inspection rather than by employing eqs. (3.11) and (3.12) not only allows us to calculate the responses of more complex LTI systems, but also provides the basis for the frequency domain representation and analysis of LTI systems.

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

As defined in Chapter 1, a signal is periodic if, for some positive value of T,

$$x(t) = x(t+T) \quad \text{for all t.} \tag{3.21}$$

若週期訊號週期為正數T，則必滿足：

$$x(t) = x(t+T) \quad \text{對所有t。}$$

The fundamental period of *x(t)* is the minimum positive, nonzero value of T for which eq. (3.21) is satisfied, and the value $\omega_0 = 2\pi/T$ is referred to as the fundamental frequency.

基本頻率 $\omega_0 = 2\pi/T$

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

Two basic periodic signals, the sinusoidal signal

弦波訊號 $$x(t) = \cos\omega_0 t \qquad (3.22)$$

and the periodic complex exponential

週期複指數 $$x(t) = d^{j\omega_0 t}. \qquad (3.23)$$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T = 2\pi/\omega_0$. Associated with the signal in eq. (3.23) is the set of harmonically related complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, k = 0, \pm 1, \pm 2, \ldots\ldots$$

均具有相同的基本頻率 ω_0 及基本週期 $T = 2\pi/\omega_0$ 。

連續時間諧波相關的複指數函數型式

基本頻率為 ω_0 的整數倍

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

A linear combination of harmonically related complex exponentials of the form

$$x(t)=\sum_{-\infty}^{+\infty} a_k e^{jk\omega_0 t}=\sum_{-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \qquad (3.25)$$

is also periodic with period *T*. In eq. (3.25), the term for k=0 is a constant. The terms for k= +1 and k= -1 both have fundamental frequency equal to ω_0 and are collectively referred to as the *fundamental components* for the *first harmonic components*.

亦為週期訊號且基本週期為*T*。

上述k= ±1時，基本頻率為 ω_0 ，則 $\phi_{-1}(t)$ 及 $\phi_1(t)$ 稱為「基本分量」或「一次諧波分量」。

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

More generally, the components for k= +N and k= -N are referred to as the *N*th harmonic components.

依此類推，當k= ±*N* 時之分量稱為「*N*次諧波分量」。

Example 3.2

Consider a periodic signal *x(t)*, with fundamental frequency 2π , that is expressed in the form of eq. (3.25) as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}, \qquad (3.26)$$

where

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

Suppose that x(t) is real and can be represented in the form of eq. (3.25). Then, since $x^*(t) = x(t)$, we obtain

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}.$$

Replacing k by – k in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t},$$

by comparison with eq. (3.25), requires that $a_k = a_{-k}^*$, or equivalently, that

若x(t)為實數訊號，則 $$a_k^* = a_{-k}. \qquad (3.29)$$

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

To derive the alternative forms of the Fourier series, we first rearrange the summation in eq. (3.25) as

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right]$$

Substituting a_k^* for a_{-k} from eq. (3.29), we obtain

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right].$$

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

Since the two terms inside the summation are complex conjugates of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\left\{a_k e^{jk\omega_0 t}\right\}. \tag{3.30}$$

If a_k is expressed in polar form as $a_k = A_k e^{j\theta_k}$,

then eq. (3.30) becomes

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\left\{A_k e^{j(k\omega_0 t + \theta_k)}\right\}$$

That is,

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k). \tag{3.31}$$

實數週期訊號的結合三角函數型傅立葉級數

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

Another form is obtained by writing a_k in rectangular form as

$$a_k = B_k + jC_k$$

where B_k and C_k are both real. With this expression for a_k , eq. (3.30) takes the form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}\left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]. \quad (3.32)$$

實數週期訊號的三角函數型傅立葉級數

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

Multiplying both sides of eq. (3.25) by $e^{-jn\omega_0 t}$, we obtain

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}.$$

Integrating both sides from 0 to $T = 2\pi / \omega_0$, we have

$$\int_0^T x(t)e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt.$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

T is the fundamental period of *x(t)*, and consequently. Interchanging the order of integration and summation yields

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}dt\right]. \qquad (3.34)$$

Rewriting this integral using Euler's formula, we obtain

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos(k-n)\omega_0 t dt + j\int_0^T \sin(k-n)\omega_0 t dt. \qquad (3.35)$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

For k ≠ n, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic sinusoids with fundamental period (T/｜k-n｜). Therefore, in eq. (3.35). For k = n, the integrand on the left-hand side of eq. (3.35) equals 1, and thus, the integral equals *T*. In sum, we then have

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, k=n \\ 0, k \neq n \end{cases}$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

the right-hand side of eq. (3.34) reduces to Ta_n

Therefore,

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt, \qquad (3.36)$$

決定傅立葉係數的公式

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

if we denote integration over any interval of length *T* by $\int_T$, we have

$$\int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k=n \\ 0, & k\neq n \end{cases},$$

and consequently,

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t} dt. \qquad (3.37)$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

defines the Fourier series of a periodic continuous-time signal:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \qquad (3.38)$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt. \qquad (3.39)$$

連續時間週期訊號*x(t)*的傅立葉級數

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

The set of coefficients $\{a_k\}$ are often called the Fourier series coefficients or the spectral coefficients of *x(t).*

The coefficient a_0 is the dc or constant component of *x(t)* and is given by eq. (3.39) with k = 0. That is,

$$a_0 = \frac{1}{T}\int_T x(t)dt, \quad (3.40)$$

x(t) 的直流分量

3.4 Convergence of the Fourier Series

signal *x(t)* by a linear combination of a finite number of harmonically related complex exponentials—that is, by a finite series of the form

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}. \qquad (3.47)$$

Let $e_N(t)$ denote the approximation error; that is,

$$e_N(t) = (t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \qquad (3.48)$$

3.4 Convergence of the Fourier Series

The criterion that we will use is the energy in the error over one period:

$$E_N = \int_T |e_N(t)|^2 dt. \qquad (3.49)$$

As shown in Problem 3.66, the particular choice for the coefficients in eq. (3.47) that minimize the energy in the error is

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt. \qquad (3.50)$$

3.4 Convergence of the Fourier Series

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period, i.e., signals for which

$$\int_T \left|x(t)\right|^2 dt \langle \infty. \tag{3.51}$$

$x_N(t)$ be the approximation to *x(t)* obtained by using these coefficients for $|k| \le N$:

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \tag{3.52}$$

3.4 Convergence of the Fourier Series

$N \rightarrow \infty$. That is, if we define

$$e(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \tag{3.53}$$

then

$$\int_T |e(t)|^2 dt = 0. \tag{3.54}$$

signal x(t) and is Fourier series representation

$$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \tag{3.55}$$

迪利斯雷(Dirichlet)條件：

Condition 1. Over an period, *x(t)* must be *absolutely integrable*; that is,

$$\int_T |x(t)|dt \langle \infty. \qquad (3.56)$$

條件1:在任何時間區間上，*x(t)*必須為絕對可積分。

迪利斯雷(Dirichlet)條件：

this guarantees that each coefficient will be finite, since

$$|a_k| \le \frac{1}{T}\int_T \left|x(t)e^{-jk\omega_0 t}\right| dt = \frac{1}{T}\int_T |x(t)|dt.$$

So if

$$\int_T |x(t)|dt \langle \infty,$$

then

$$|a_k| \langle \infty.$$

A periodic signal that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \qquad 0 \langle t \le 1;$$

迪利斯雷(Dirichlet)條件：

Condition 2. In any finite interval of time, *x(t)* is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

條件2：在任何有限的時間區間中，x(t)為有界變化；亦即，在訊號的任何單一區間內，具有有限數量的極大值與極小值。

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \qquad 0 \langle t \leq 1, \tag{3.57}$$

as illustrated in Figure 3.8(b). For this function, which is periodic with *T*=1,

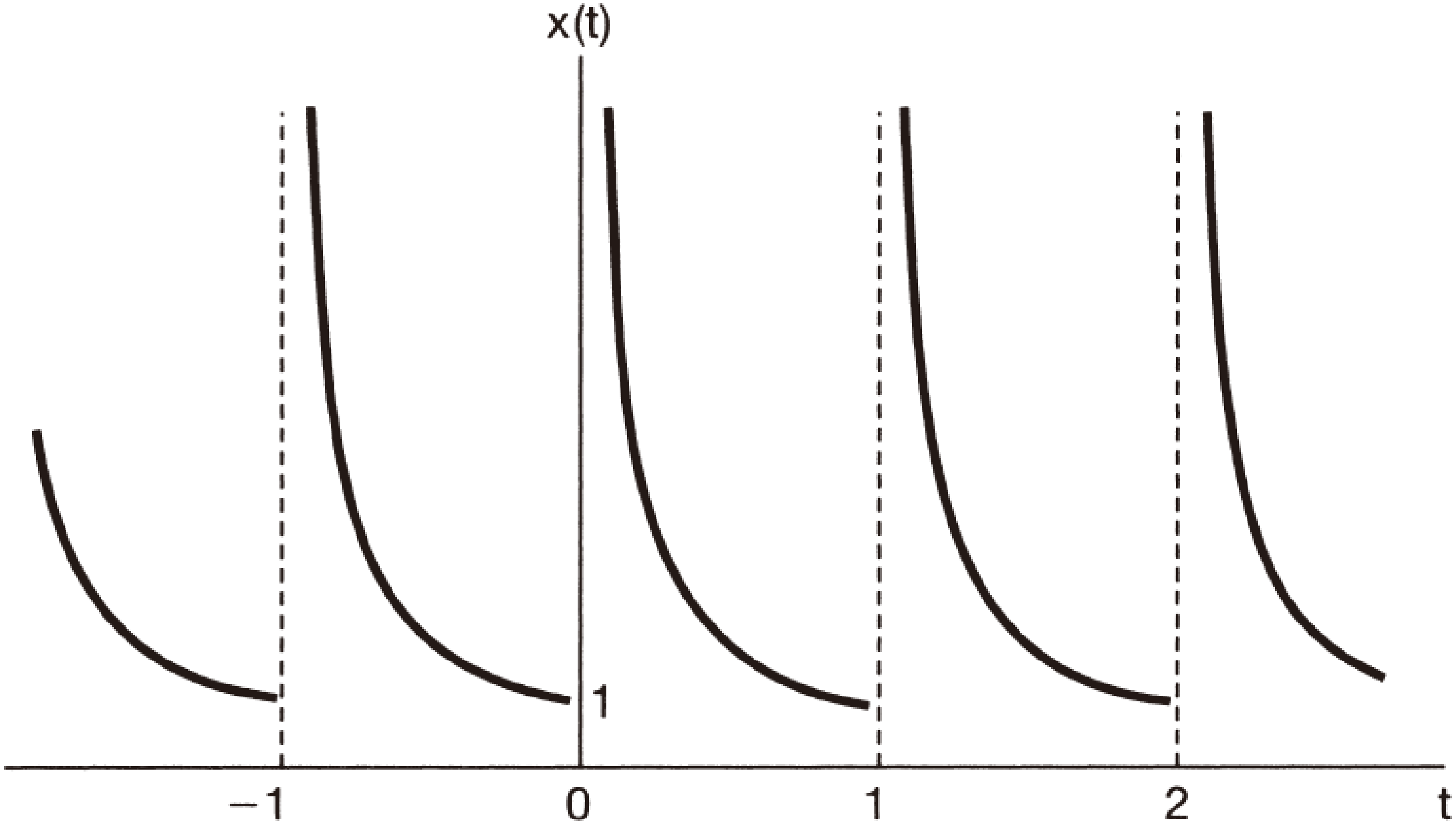
$$\int_0^1 \left| x(t) \right| dt \langle 1.$$

迪利斯雷(Dirichlet)條件：

Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

條件3：在任何有限的時間區間中，只含有有限個不連續點，而且每個不連續點均為有限值。

3.4 Convergence of the Fourier Series



(a)

3.4 Convergence of the Fourier Series

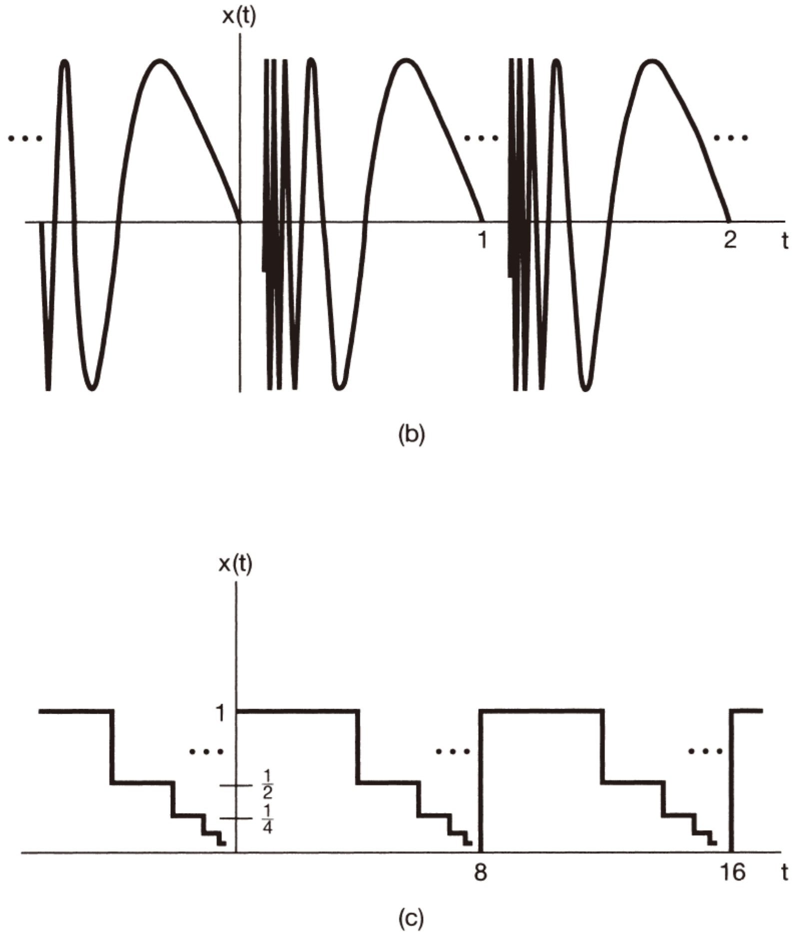


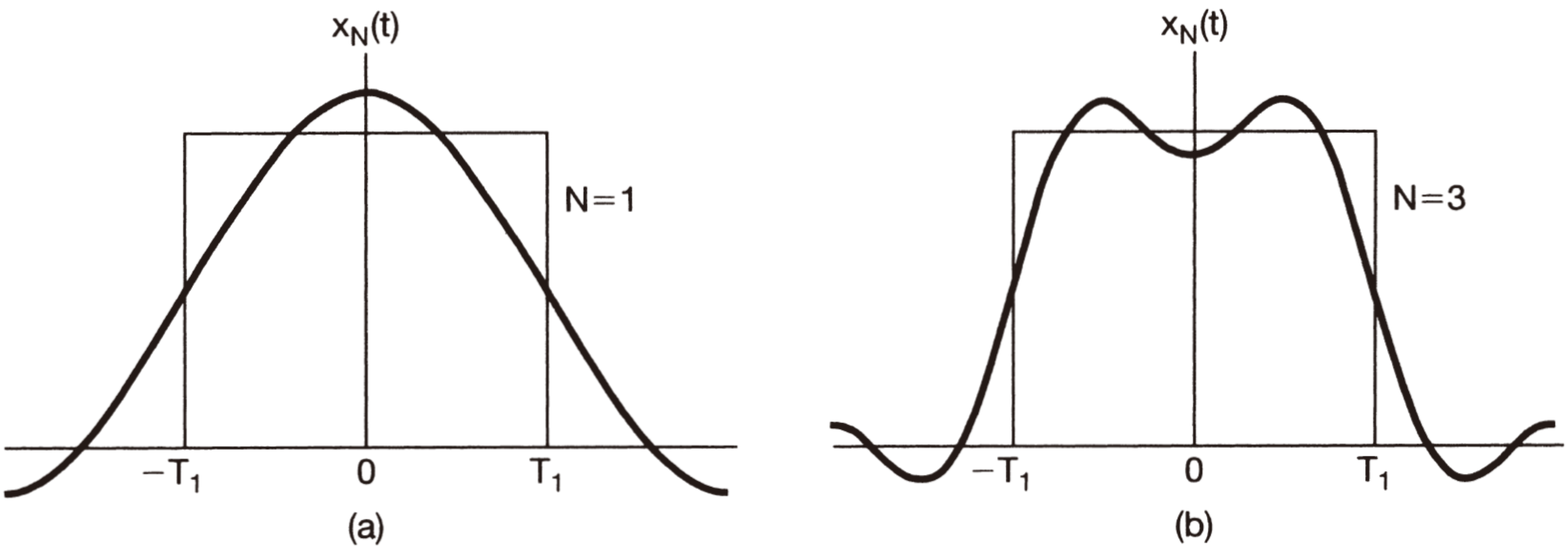
Figure 3.8 Signals that violate the Dirichlet conditions: (a) the signal $x(t) = 1/t$ for $0 < t \leq 1$, a periodic signal with period 1 (this signal violates the first Dirichlet condition); (b) the periodic signal of eq. (3.57), which violates the second Dirichlet condition; (c) a signal periodic with period 8 that violates the third Dirichlet condition [for $0 \leq t < 8$, the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to 8 decreases by a factor of 2; that is, $x(t) = 1,\ 0 \leq t < 4,\ x(t) = 1/2,\ 4 \leq t < 6,\ x(t) = 1/4,\ 6 \leq t < 7,\ x(t) = 1/8,\ 7 \leq t < 7.5$, etc.].

3.4 Convergence of the Fourier Series

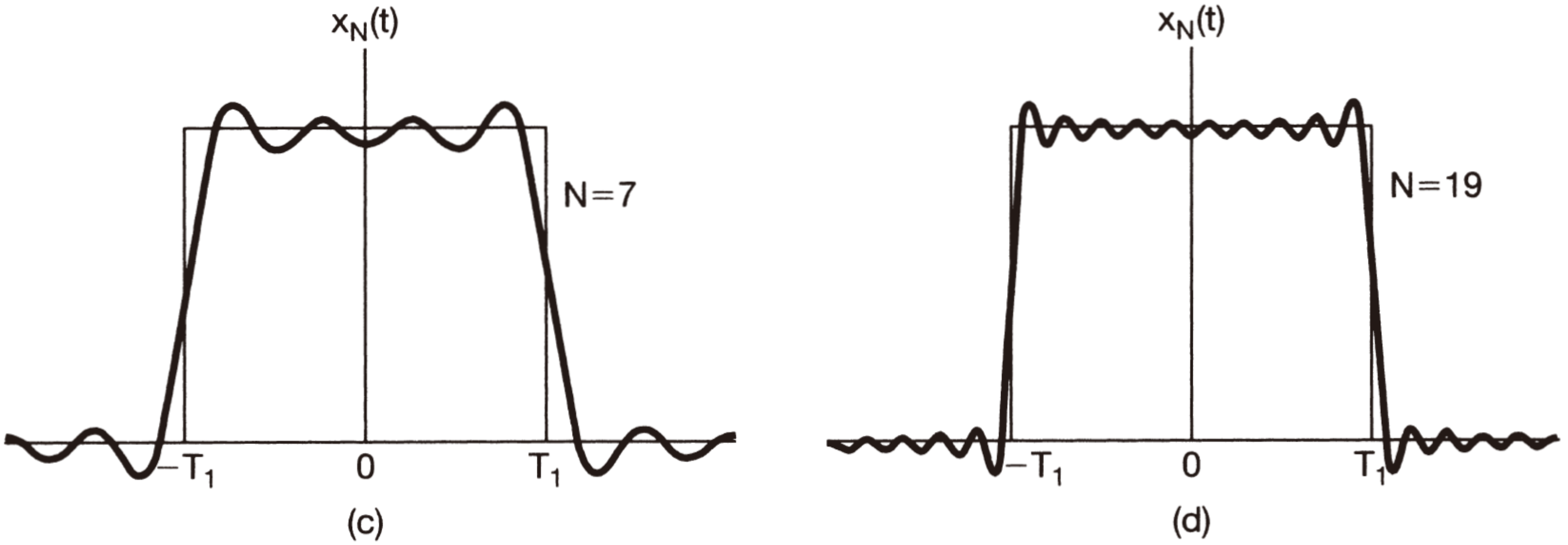
We see from the figure that this is in fact the case, since for any N, $x_N(t)$ has exactly that value at the discontinuities. Furthermore, for any other value of t, say, $t = t_1$, we are guaranteed that

$$\lim_{N \to \infty} x_N(t_1) = x(t_1).$$

3.4 Convergence of the Fourier Series



3.4 Convergence of the Fourier Series



3.4 Convergence of the Fourier Series

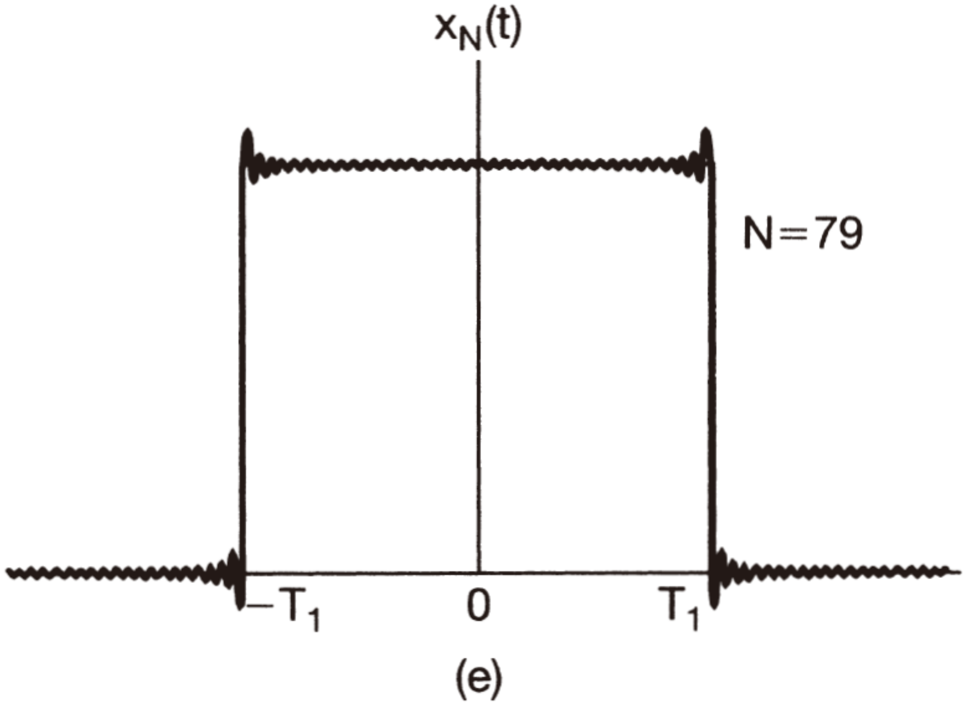


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

3.5 Properties of Continuous-time Fourier series

If the Fourier series coefficients of *x(t)* are denoted by a_k , we will use the notation

$$x(t) \xleftrightarrow{\mathfrak{F}S} a_k$$

to signify the pairing of a periodic signal with its Fourier series coefficients.

訊號與其傅立葉係數相互對應的記號。

3.5.1 Linearity

Let *x(t)* and *y(t)* denote two periodic signals with period T and which have Fourier Series coefficients denoted by a_k and b_k , respectively. That is,

$$x(t) \xleftrightarrow{\mathfrak{F}S} a_k \, ,$$

$$y(t) \xleftrightarrow{\mathfrak{F}S} b_k \, .$$

3.5.1 Linearity

the Fourier series coefficients of the linear combination of *x(t)* and *y(t)*, *z(t) =Ax(t) + By(t)*

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\Im S} c_k = Aa_k + Bb_k.$$

傅立葉級數的線性性質：訊號線性組合，則傅立葉係數亦線性組合。

The proof of this follows directly from the application of eq. (3.39). We also note that the linearity property is easily extended to a linear combination of an arbitrary number of signals with period *T*.

3.5.2 Time Shifting

The fourier series coefficients b_k of the resulting signal $y(t) = x(t - t_0)$ may be expressed as

$$b_k = \frac{1}{T}\int_T x(t - t_0)e^{-jk\omega_0 t}dt.$$

3.5.2 Time Shifting

Letting $\tau = t - t_0$ in the integral, and noting that the new variable τ will also range over an interval of duration *T*, we obtain

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau \qquad (3.60)$$

$$= e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k,$$

3.5.2 Time Shifting

where a_k is the *k*th Fourier series coefficient of *x(t).* That is, if

$$x(t) \xleftrightarrow{\mathfrak{F}S} a_k,$$

then

$$x(t-t_0) \xleftrightarrow{\mathfrak{F}S} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

時間移位性質

3.5.3 Time Reversal

To determine the Fourier series coefficients of *y(t) = x(-t)*, let us consider the effect of time reversal on the synthesis equation (3.38):

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \tag{3.61}$$

Making the substitution k = -m, we obtain

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}. \tag{3.62}$$

3.5.3 Time Reversal

where the Fourier series coefficients are

$$b_k = a_{-k}. \quad (3.63)$$

That is, if

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

then

$$x(-t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_{-k}.$$

時間倒轉性質

3.5.4 Time Scaling

if *x(t)* has the Fourier series representation in eq. (3.38), then

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

時間刻度變換性質

3.5.5 Multiplication

Suppose that *x(t)* and *y(t)* are both periodic with period *T* and that

$$x(t) \xleftrightarrow{\mathfrak{F}S} a_k,$$

$$y(t) \xleftrightarrow{\mathfrak{F}S} b_k.$$

expand it in a Fourier series with Fourier series coefficients expressed in terms of those for *x(t)* and *y(t).* The result is

$$x(t)y(t) \xleftrightarrow{\mathfrak{F}S} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \qquad (3.64)$$

訊號乘法性質

3.5.6 Conjugation and Conjugate Symmetry

Taking the complex conjugate conjugate of a periodic signal *x(t)* has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients. That is, if

$$x(t) \xleftrightarrow{\mathfrak{F}S} a_k ,$$

then

$$x^*(t) \xleftrightarrow{\mathfrak{F}S} a^*_{-k} . \qquad (3.65)$$

3.5.6 Conjugation and Conjugate Symmetry

see from eq. (3.65) that the Fourier series coefficients will be conjugate symmetric, i.e.,

$$a_{-k} = a_k^*, \tag{3.66}$$

若*x(t)*為實訊號，則傅立葉係數具有共軛對稱。

For example, from eq. (3.66), we see that if *x(t)* is real, then a_0 is real and

$$|a_k| = |a_{-k}|.$$

3.5.7 Parseval's Relation for Continuous-Time Periodic Signals

As shown n Problem 3.46, Parseval's relation for continuous-time periodic signals is

$$\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{+\infty} \left|a_K\right|^2, \qquad (3.67)$$

連續時間週期訊號的巴斯瓦關係式(定理)；訊號的平均功率等於各次諧波平均功率的總和。

3.5.7 Parseval's Relation for Continuous-Time Periodic Signals

Note that the left-hand side of eq. (3.67) is the average power in one period of the periodic signal *x(t).* Also,

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T \left|a_k\right|^2 dt = \left|a_k\right|^2, \tag{3.68}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES 表 3.1 連續時間傅立葉級數的性質

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

As defined in Chapter 1, a discrete-time signal *x[n]* is periodic with period *N* if

$$x[n] = x[n+N]. \qquad (3.84)$$

the set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{jw\omega_0 n} = e^{jk(2\pi/N)n}, k = 0, \pm 1, \pm 2, \ldots \qquad (3.85)$$

離散時間諧波相關的複指數型式

基本頻率為2 π/*N*的整數倍。

All of these signals have fundamental frequencies that are multiples of 2 π/*N* and thus are harmonically related.

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

This is a consequence of the fact that discrete-time complex exponentials which differ in frequency by a multiple of 2 π are identical. Specifically, $\phi_0[n]=\phi_N[n], \phi_1[n]=\phi_{N+1}[n]$, and, in general,

$$\phi_k[n]=\phi_{k+rN}[n]. \tag{3.86}$$

$\phi_k[n]$對*k*而言，亦為週期函數，週期為*N*。故$\phi_k[n]$對不同的*k*而言，只有*N*種不同的函數。

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

Such a linear combination has the form

$$x[n]=\sum_k a_k\phi_k[n]=\sum_k a_k e^{jk\omega_0 n}=\sum_k a_k e^{jk(2\pi/N)n}. \qquad (3.87)$$

$$x[n]=\sum_{k=\langle N\rangle} a_k\phi_k[n]=\sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}=\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}. \qquad (3.88)$$

離散時間傅立葉級數型式

注意：此級數並非無限項，而是有限的N項。

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

If we evaluate eq. (3.88) for N successive values of n corresponding to one period of x[n], we obtain

$$
\begin{aligned}
x[0] &= \sum_{k=\langle N\rangle} a_k, \\
x[1\} &= \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N}, \\
&\vdots \\
x[N-1] &= \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}.
\end{aligned}
\tag{3.89}
$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

The basis for this result is the fact, shown in Problem 3.54, that

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k=0,\pm N,\pm 2N,.... \\ 0, & otherwise \end{cases} \tag{3.90}$$

Now consider the Fourier series representation of eq. (3.88). Multiplying both sides by $e^{-jr(2\pi/N)n}$ and summing over N terms, we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N\rangle}\sum_{n=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n.} \tag{3.91}$$

Interchanging the order of summation on the right-hand side, we have

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} \sum_{k=\langle k\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}. \tag{3.92}$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

The right-hand side of eq. (3.92) then reduces to ,
and we have

$$a_r = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jr(2\pi/N)n}. \tag{3.93}$$

This provides a closed-form expression for obtaining the Fourier series coefficients, and we have the discrete-time Fourier series pair:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}, \tag{3.94}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}. \tag{3.95}$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

Referring to eq. (3.88), we see that if we take k in the range from 0 to $N - 1$, we have

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \ldots + a_{N-1}\phi_{N-1}[n]. \tag{3.96}$$

Similarly, if k ranges from 1 to N, we obtain

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \ldots + a_N\phi_N[n]. \tag{3.97}$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

By letting k range over any set of N consecutive integers and using eq. (3.86),

$$a_k = a_{k+N}. \qquad (3.98)$$

離散時間傅立葉係數具有週期性，其週期為N。

That is, if we consider more than N sequential values of k, the values a_k repeat periodically with period N.

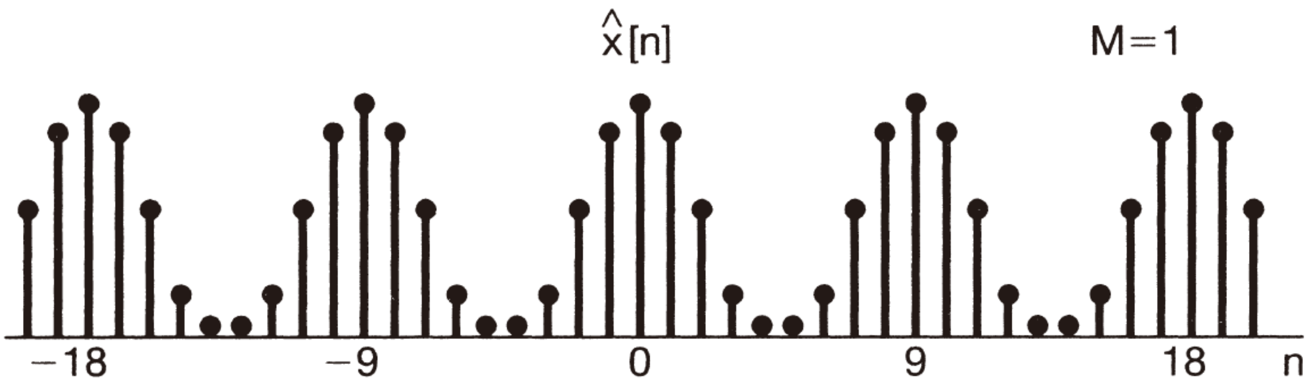
3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

Assume that the period N is odd. In Figure 3.18, we have depicted the signals

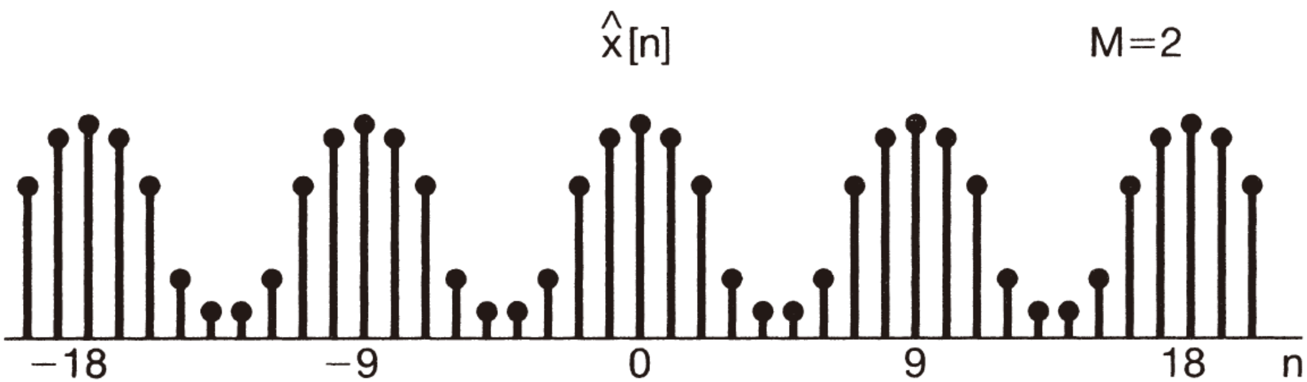
$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n} \tag{3.106}$$

for the example of Figure 3.16 with $N = 9$, $2N_1 + 1 = 5$, and for several values of M. For $M = 4$, the partial sum exactly equals $x[n]$.

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal



(a)



(b)

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

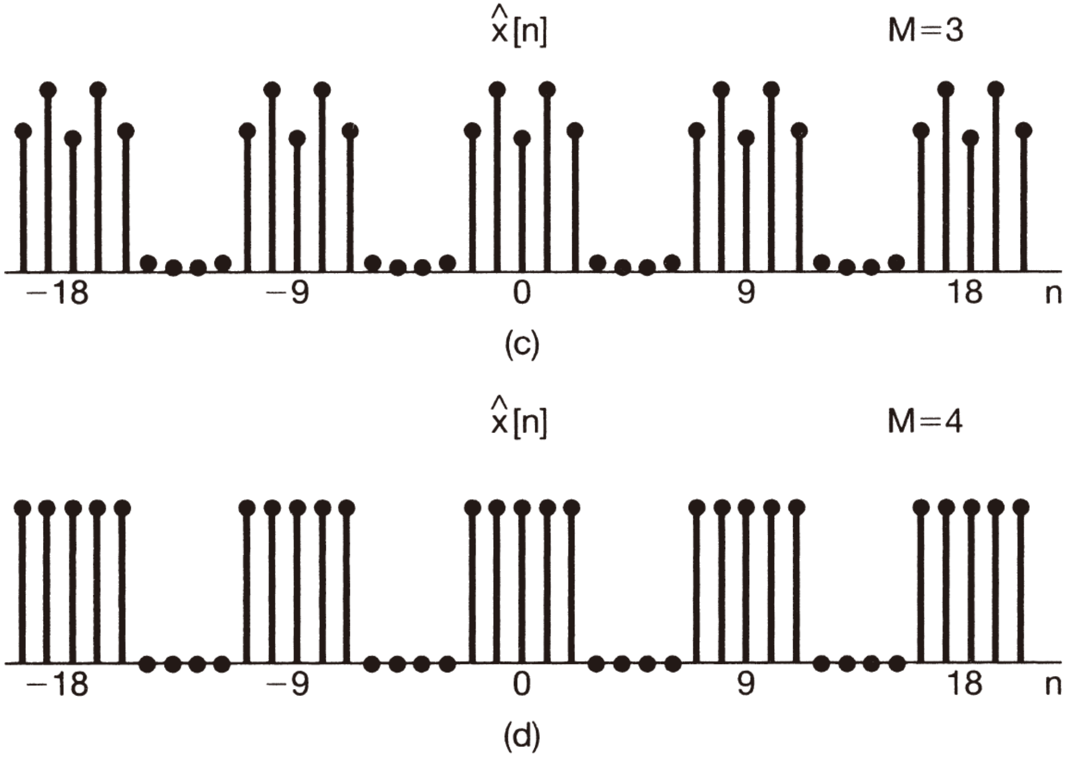


Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

If N is odd and we take $M = (N - 1)/2$ in eq. (3.106), the sum includes exactly N terms, and consequently, from the synthesis equations, we have $\tilde{x}[n] = x[n]$. Similarly, if N is even and we let

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk(2\pi/N)n},$$

3.7 Properties of Discrete-Time Fourier Series

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES 表 3.2 離散時間傅立葉級數的性質

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l}x[n]\\y[n]\end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l}a_k\\b_k\end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic}\\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only}\\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

3.7 Properties of Discrete-Time Fourier Series

If $x[n]$ is a periodic signal with period N and with Fourier series coefficients denoted by a_k , then we will write

$$x[n] \stackrel{FS}{\longleftrightarrow} a_k .$$

3.7.1 Multiplication

The product of two continuous-time signals of period T results in a periodic signal with period T whose sequence of Fourier series coefficients is the convolution of the sequences of Fourier series coefficients of the two signals being multiplied.

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$

and

$$y[n] \overset{FS}{\longleftrightarrow} b_k$$

3.7.1 Multiplication

are both periodic with period *N*. Then the product *x[n]y[n]* is also periodic with period *N*, and, as shown in Problem 3.57, its Fourier coefficients, , are given by

$$x[n]y[n] \xleftrightarrow{FS} d_k = \sum_{l=\langle N \rangle} a_l b_{k-l} \tag{3.108}$$

訊號相乘，其傅立葉係數具有類似迴旋和的結果。

3.7.1 Multiplication

Equation (3.108) is analogous to the definition of convolution, except that the summation variable is now restricted to an interval of N consecutive samples.

(3.108)式右側可稱為週期性迴旋和。

3.7.2 First Difference

If *x[n]* is periodic with period *N*, then so is *y[n]*, since shifting *x[n]* or linearly combining *x[n]* with another periodic signal whose period is N always results in a periodic signal with period N. Also, if

$$x[n] \overset{FS}{\longleftrightarrow} a_k,$$

then the Fourier coefficients corresponding to the first difference of *x[n]* may be expressed as

$$x[n] - x[n-1] \overset{FS}{\longleftrightarrow} (1 - e^{-jk(2\pi/N)})a_k \quad (3.109)$$

3.7.3 Parseval's Relation for Discrete-Time Periodic Signals

As shown in Problem 3.57, Perseval's relation for discrete-time periodic signals is given by

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2=\sum_{k=\langle N\rangle}|a_k|^2, \qquad (3.110)$$

離散時間週期訊號的巴斯瓦關係式

where the a_k are the Fourier series coefficients of $x[n]$ of N is the period.

即訊號的平均功率等於各次諧波的平均功率的總和。

3.8 Fourier Series and LTI Systems

In continuous time, if $x(t) = e^{st}$ is the input to a continuous-time LTI system, then the output is given by $y(t) = H(s)e^{st}$, where, from eq. (3.6),

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau, \qquad (3.119)$$

連續時間LTI系統脈衝響應*h(t)*與系統函數*H(s)*的關係

3.8 Fourier Series and LTI Systems

Similarly, if $x[n] = z^n$ is the input to a discrete-time LTI system, then the output is given by $y[n] = H(z)z^n$, where, from eq. (3.10),

$$H(z0 = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}, \qquad (3.120)$$

離散時間LTI系統脈衝響應*h(n)*與系統函數*H(z)*的關係

3.8 Fourier Series and LTI Systems

The system function of the form $s = j\omega$—i.e., $H(j\omega)$ viewed as a function of ω—is referred to as the *frequency response* of the system and is given by

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt. \qquad (3.121)$$

連續時間系統的頻率響應函數

3.8 Fourier Series and LTI Systems

Then the system function *H(z)* for *z* restricted to the form $z = e^{j\omega}$ is referred to as the frequency response of the system and is given by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}. \qquad (3.122)$$

離散時間系統的頻率響應函數

3.8 Fourier Series and LTI Systems

Consider first the continuous-time case, and let *x(t)* be a periodic signal with a Fourier series representation given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}. \qquad (3.123)$$

若輸入*x(t)*為週期訊號，且可表為傅立葉級數，且*k*次諧波係數為 a_k。

3.8 Fourier Series and LTI Systems

In eq. (3.13) with $s_k = jk\omega_0$, it follows that the output is

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}.$$

則輸出*y(t)*亦為週期訊號，且可表為型式相同的傅立葉級數，但*k*次諧波係數為 $a_k H(jk\omega_0)$。

3.8 Fourier Series and LTI Systems

Thus, *y(t)* is also periodic with the same fundamental frequency as *x(t).* Furthermore, if $\{a_k\}$ is the set of Fourier series coefficients for the input *x(t),* then $\{a_k H(jk\omega_0)\}$ is the set of coefficients for the output *y(t).*

3.8 Fourier Series and LTI Systems

let *x[n]* be a periodic signal with Fourier series representation given by

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

在離散時間系統中，若*x[n]*為週期訊號，且可表為傅立葉級數，其*k*次諧波係數為 a_k 。

3.8 Fourier Series and LTI Systems

If we apply this signal as the input to an LTI system with impulse response *h[n]*, then, as in eq. (3.16) with $z_k = e^{jk(2\pi/N)}$, the output is

$$y[n] = \sum_{k=\langle N\rangle} a_k H(e^{j2\pi k/N}) e^{jk(2\pi/N)n}.$$

則輸出*y[n]*亦為週期訊號且可表為同型式的傅立葉級數，但*k*次諧波係數為 $a_k H(e^{j2\pi k/N})$。

3.9 Filtering

Filtering can be conveniently accomplished through the use of LTI systems with an appropriately chose frequency response, and frequency-domain methods provide us with the ideal tools to examine this very important class of applications. In this and the following two sections, we take a first look at filtering through a few examples.

「濾波」即在對於訊號中某些頻率分量改變振幅或消除。用以改變訊號的頻譜形狀的LTI系統稱為「頻率整形濾波器」。用以在不失真之外通過某些頻率，或大大地縮減或消除其它頻率成分的，稱為「頻率選擇濾波器」。

3.9.1 Frequency-Shaping Filters

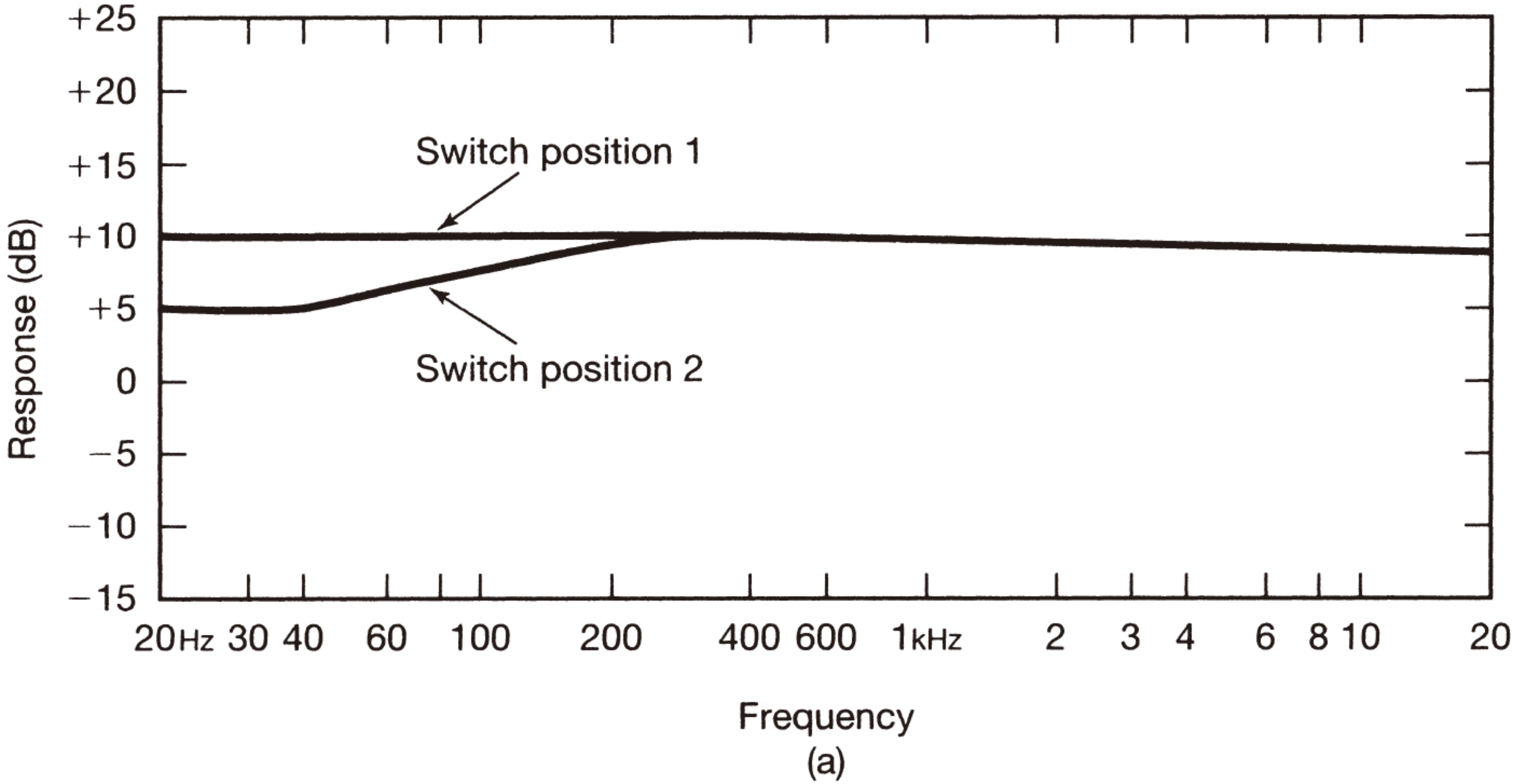
One application in which frequency-shaping filters are often encountered is audio systems.

頻率整形濾波器常應用於音訊系統。

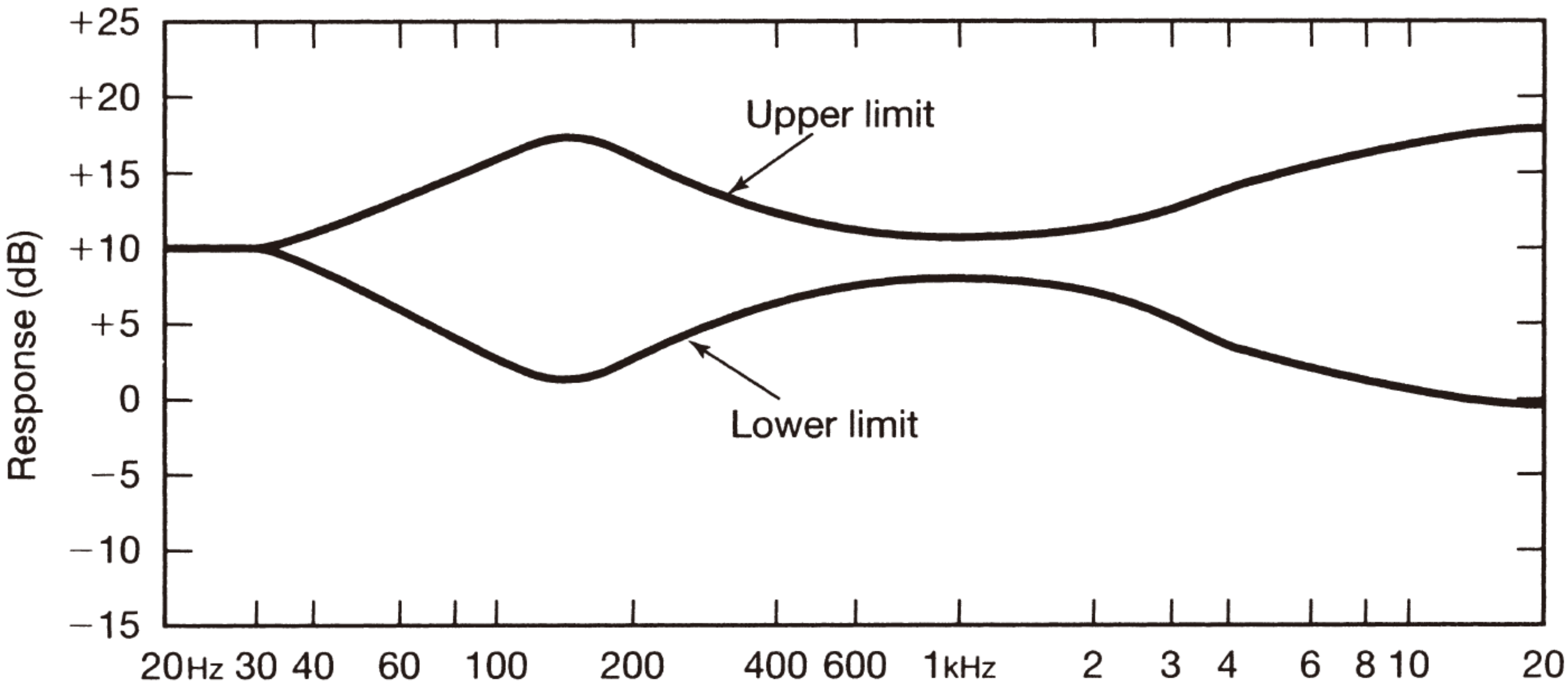
Another class of frequency-shaping filters often encountered is that for which the filter output is the derivative of the filter input, i.e., $y(t) = dx(t)/dt$. With $x(t)$ of the form $x(t) = e^{j\omega t}$, $y(t)$ will be $y(t) = j\omega e^{j\omega t}$, from which it follows that the frequency response is

$$H(j\omega) = j\omega. \tag{3.137}$$

3.9.1 Frequency-Shaping Filters



3.9.1 Frequency-Shaping Filters



(b)

3.9.1 Frequency-Shaping Filters

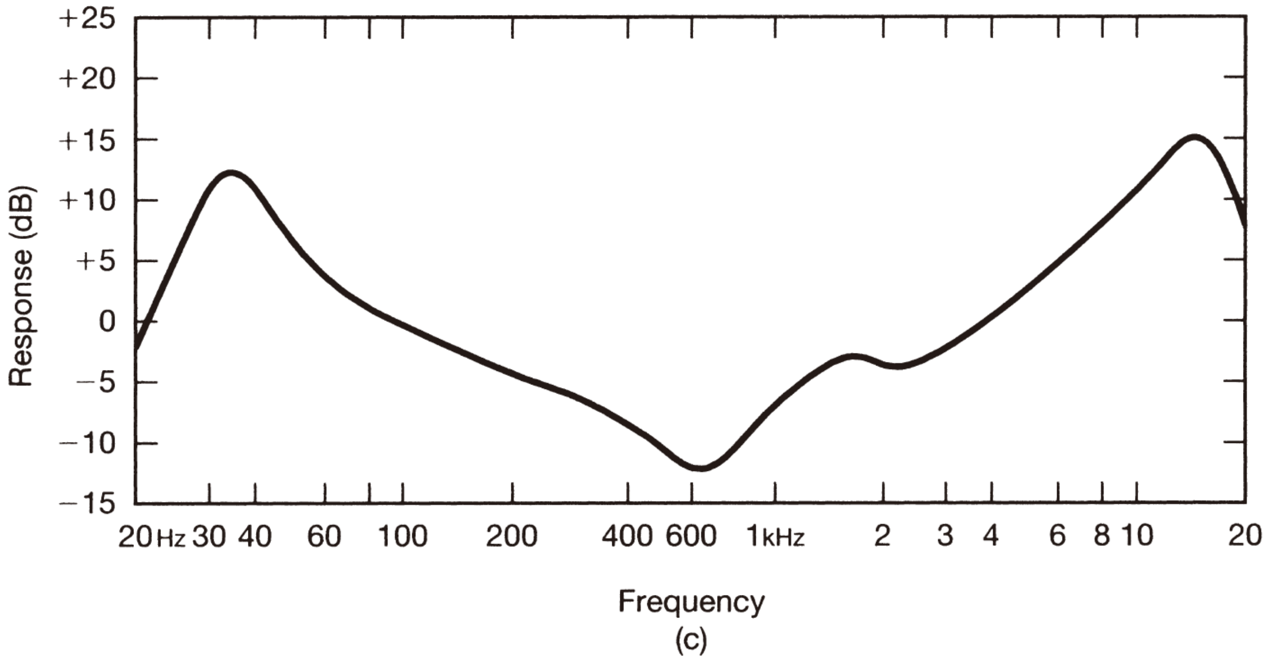


Figure 3.22 Magnitudes of the frequency responses of the equalizer circuits for one particular series of audio speakers, shown on a scale of $20\log_{10}|H(j\omega)|$, which is referred to as a decibel (or dB) scale. (a) Low-frequency filter controlled by a two-position switch; (b) upper and lower frequency limits on a continuously adjustable shaping filter; (c) fixed frequency response of the equalizer stage.

3.9.2 Frequency-Selective Filters

Frequency-selective filters are a class of filters specifically intended to accurately or approximately select some bands of frequencies and reject others.

頻率選擇濾波器常用於雜訊消除、通訊系統(如AM等)。

3.9.2 Frequency-Selective Filters

a *lowpass filter* is a filter that passes low frequencies—i.e., frequencies around $\omega = 0$—and attenuates or rejects higher frequencies. A *highpass filter* is a filter that passes high frequencies and attentuates or rejects low ones, and a *bandpass filter* is a filter that passes a band of frequencies and attenuates frequencies both higher and lower than those in the band that is passed.

低通濾波器用來通過低頻訊號，而較高頻的訊號則被縮減或拒絕。

3.9.2 Frequency-Selective Filters

高通濾波器用來通過高頻訊號，而較低頻的訊號則被縮減或拒絕。

帶通濾波器用來通過某一頻帶的訊號，而較高和較低的頻段的訊號則被縮減。

截止頻率為通帶和足帶的邊界頻率。

the frequency response of a continuous-time ideal lowpass filter is

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \qquad (3.140)$$

3.9.2 Frequency-Selective Filters

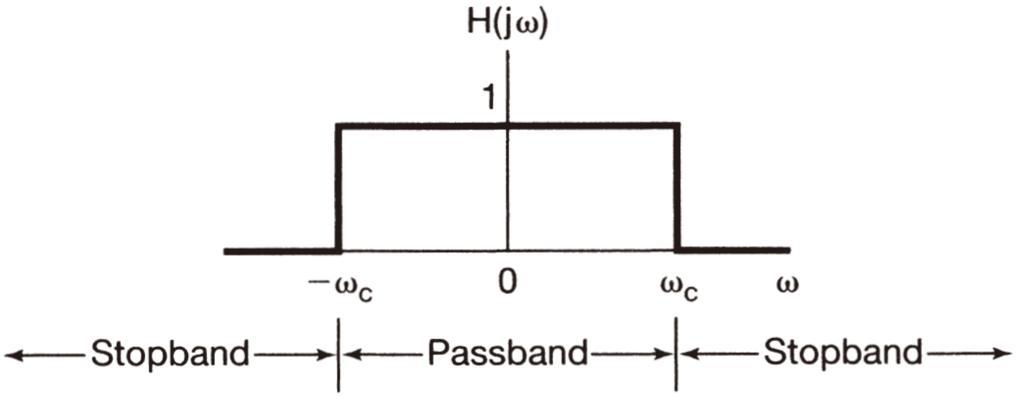


Figure 3.26 Frequency response of an ideal lowpass filter.

3.9.2 Frequency-Selective Filters

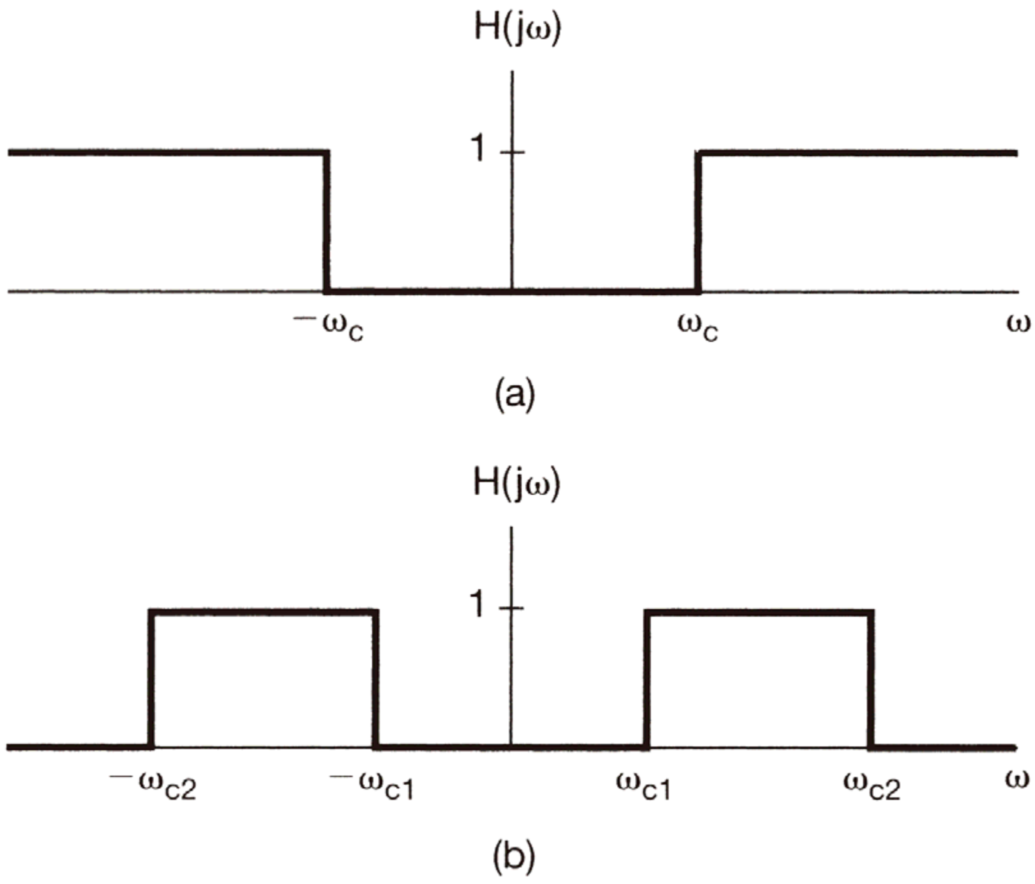


Figure 3.27 (a) Frequency response of an ideal highpass filter; (b) frequency response of an ideal bandpass filter.

3.9.2 Frequency-Selective Filters

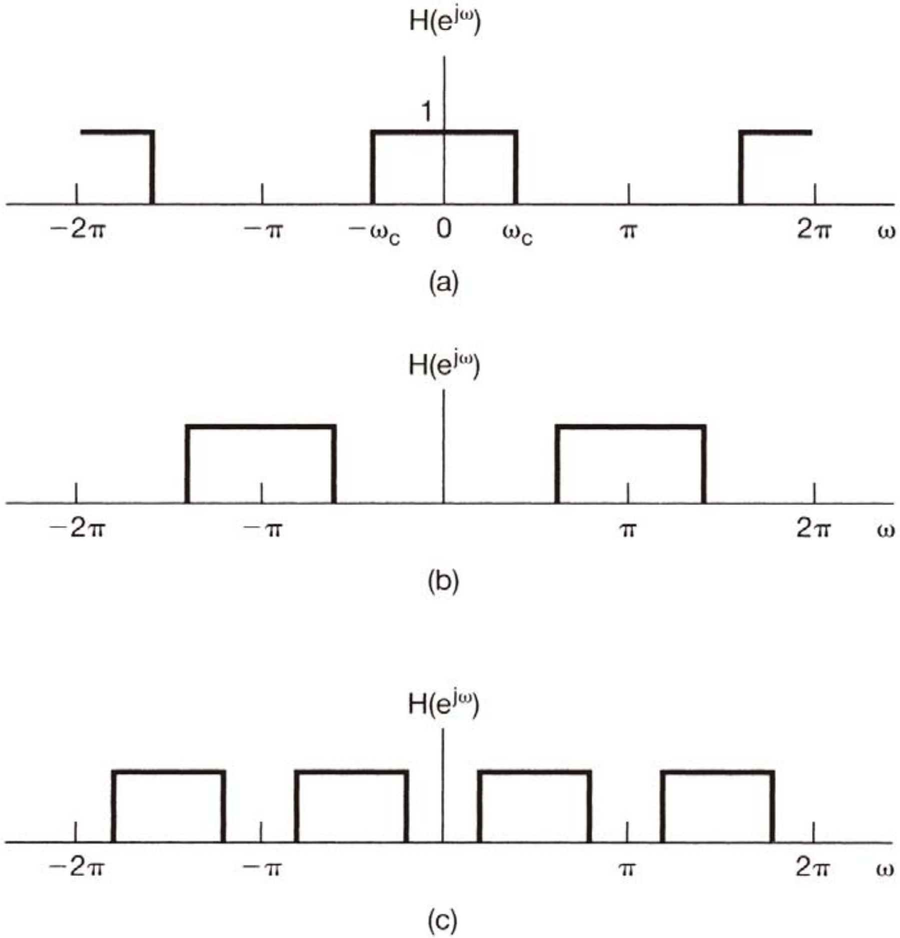


Figure 3.28 Discrete-time ideal frequency-selective filters: (a) lowpass; (b) highpass; (c) bandpass.

3.9.2 Frequency-Selective Filters

Idea filter are quite useful in describing idealized system configurations for s variety of applications.

理想濾波器實際上是無法實現的，必須用近似的可實現濾波器來代替。

3.10 Examples of Continuous-Time Filters Described by Differentail Equations

In many applications, frequency-selective filtering is accomplished through the use of LTI systems described by linear constant-coefficient differential or difference equations.

頻率選擇濾波器常用由線性常係數微分或差分方程述的LTI系統來完成。主要原因有三：

其一為許多實際系統可用微分或差分方程表示。

其二為利用微分或差分方程表示的系統可以很容易實現。

其三為以微分或差分方程表示的系統，具有極為寬廣而彈性的設計範圍。

3.10.1 A Simple RC Lowpass Filter

In this case, the output voltage is related to the input voltage through the linear constant-coefficient differential equation

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t). \qquad (3.141)$$

圖3.29的RC濾波器之微分方程(輸入為v_s ，輸出為電容電壓 v_c)

3.10.1 A Simple RC Lowpass Filter

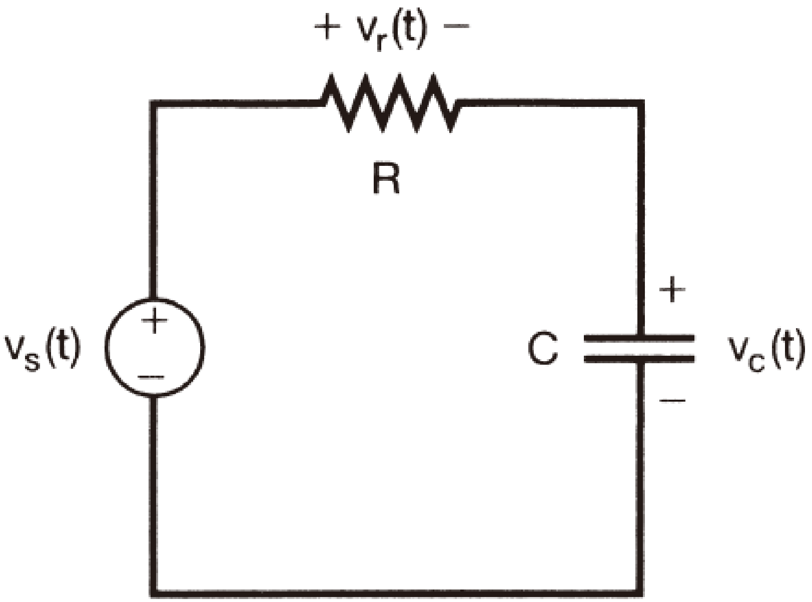


Figure 3.29 First-order *RC* filter.

3.10.1 A Simple RC Lowpass Filter

In order to determine its frequency response $H(j\omega)$, we note that, by definition, with input voltage $v_s(t) = e^{j\omega t}$, we must have the output voltage $v_c(t) = H(j\omega)e^{j\omega t}$.

$$RC\frac{d}{dt}\left[H(j\omega)e^{j\omega t}\right] + H(j\omega)e^{j\omega t} = e^{j\omega t}, \quad (3.142)$$

or

$$RCj\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}, \quad (3.143)$$

3.10.1 A Simple RC Lowpass Filter

From which it follows directly that

$$H(j\omega)e^{j\omega t} = \frac{1}{1+RCj\omega}e^{j\omega t}, \tag{3.144}$$

or

$$H(j\omega) = \frac{1}{1+RCj\omega}. \tag{3.145}$$

圖3.29的RC濾波器在ω = 0附近時，$|H(j\omega)| \approx 1$，而當ω極大時$|H(j\omega)|$極小。配合圖3.30可知為一個非理想的低通濾波器。

3.10.1 A Simple RC Lowpass Filter

To provide a first glimpse at the trade-offs involved in filter design, let us briefly consider the time-domain behavior of the circuit. In particular, the impulse response of the system described by eq. (3.141) is

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t), \qquad (3.146)$$

and the step response is

$$s(t) = \left[1 - e^{-t/RC}\right] u(t), \qquad (3.147)$$

3.10.1 A Simple RC Lowpass Filter

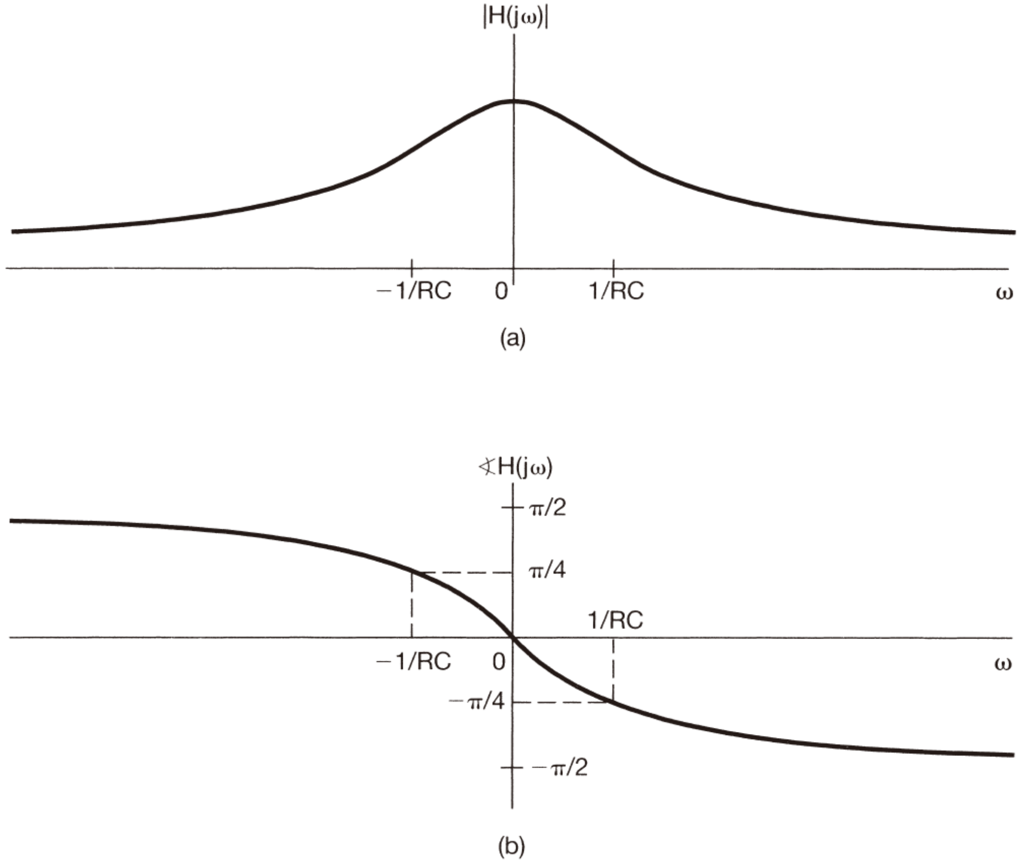


Figure 3.30 (a) Magnitude and (b) phase plots for the frequency response for the *RC* circuit of Figure 3.29 with output $v_c(t)$.

3.10.2 A Simple RC Highpass Filter

In this case, the differential equation relating input and output is

$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}. \qquad (3.148)$$

圖3.29的RC濾波器的微分方程(輸入為 v_s ，輸出改為電阻電壓 v_r)

3.10.2 A Simple RC Highpass Filter

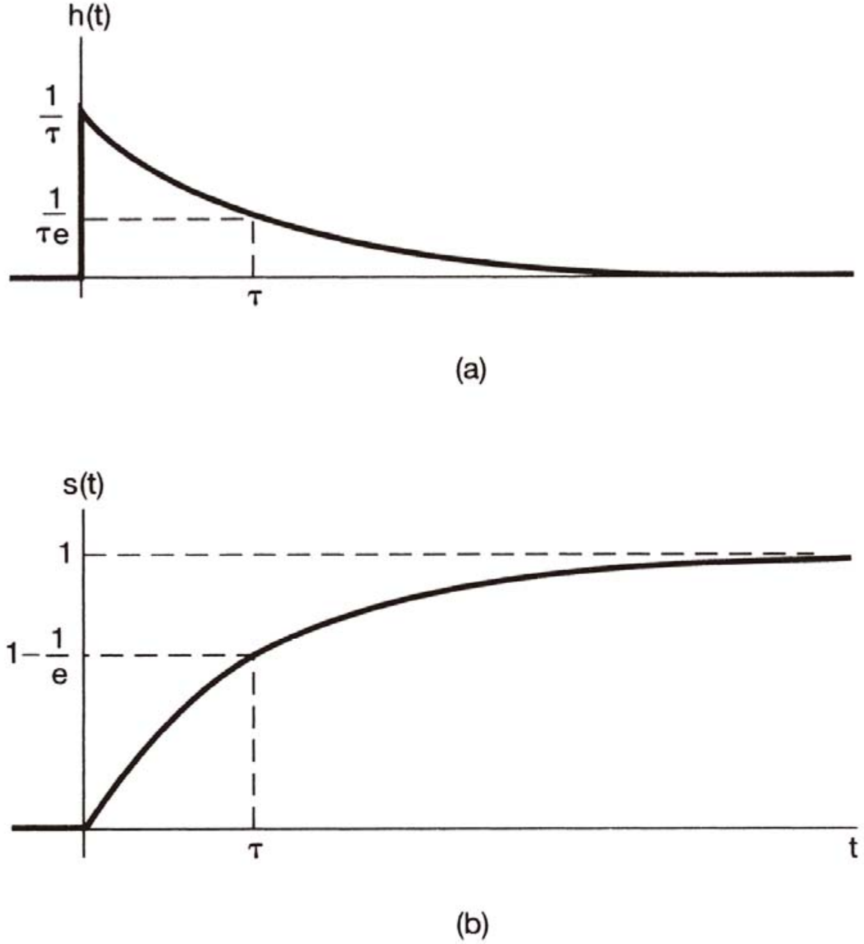


Figure 3.31 (a) Impulse response of the first-order *RC* lowpass filter with $\tau = RC$; (b) step response of *RC* lowpass filter with $\tau = RC$.

3.10.2 A Simple RC Highpass Filter

Find the frequency response $G(j\omega)$ of this system in exactly the same way we did in the previous case: If $v_s(t) = e^{j\omega t}$, then we must have $v_r(t) = G(j\omega)e^{j\omega t}$

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}. \qquad (3.149)$$

3.10.2 A Simple RC Highpass Filter

當 $\omega\langle\langle 1/RC$ 時，$|G(j\omega)| \approx 0$ ；

當 $|\omega|\rangle\rangle 1/RC$ 時，則 $|G(j\omega)| \approx 1$ 。

故為一非理想高通濾波器。

From Figure 3.29, we see that $v_r(t) = v_s(t) - v_c(t)$. Thus, if $v_s(t) = u(t), v_c(t)$ must be given by eq. (3.147).

$$v_r(t) = e^{-t/RC}u(t), \qquad (3.150)$$

3.10.2 A Simple RC Highpass Filter

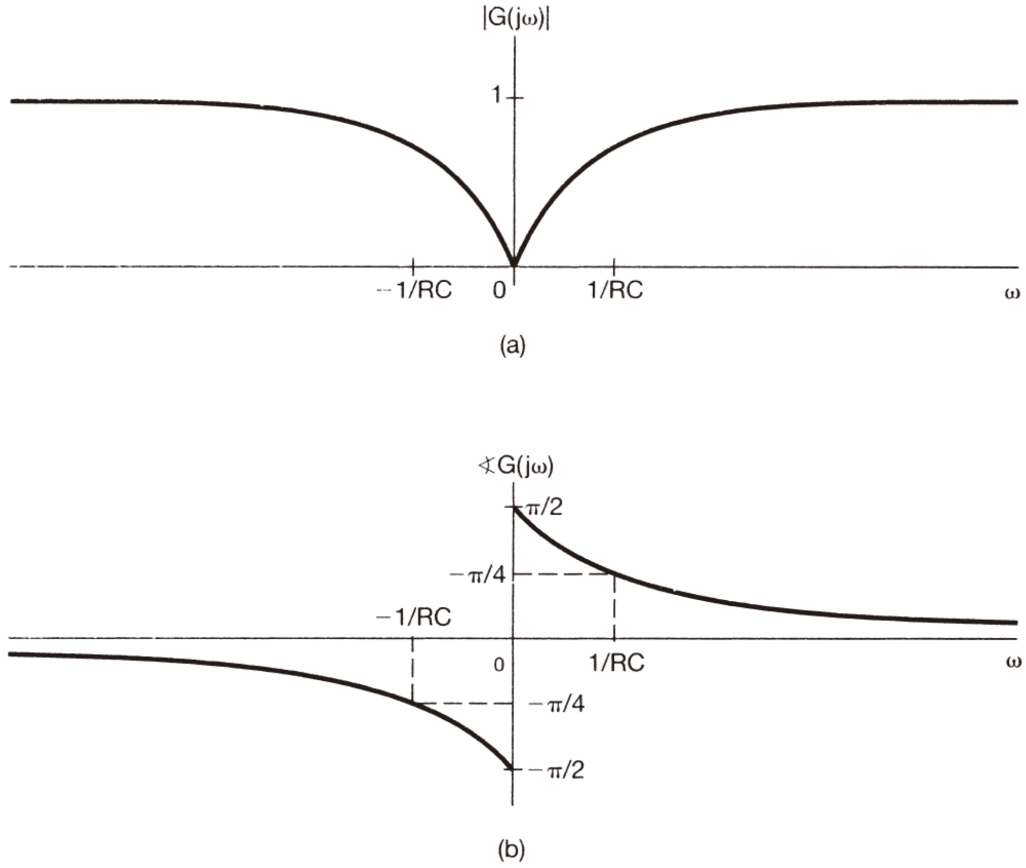


Figure 3.32 (a) Magnitude and (b) phase plots for the frequency response of the *RC* circuit of Figure 3.29 with output $v_r(t)$.

3.10.2 A Simple RC Highpass Filter

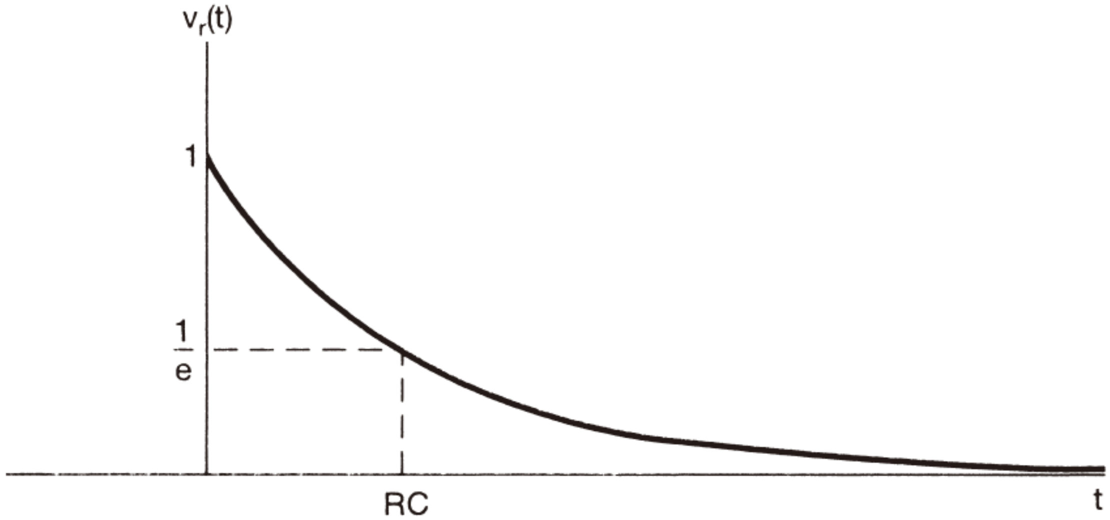


Figure 3.33 Step response of the first-order *RC* highpass filter with $\tau = RC$.