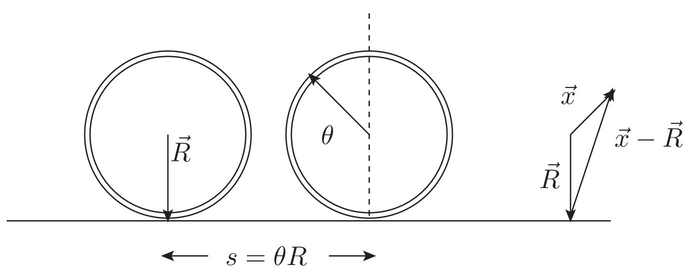
Contents

11 Rotational Motion—II

11.1 Rolling as Translation and Rotation Combined



Let $\vec{v}'$ be the velocity for a point $\vec{x}$ on the rotating body in the center of mass frame. The velocity observed in the laboratory system is

$$\vec{v} = \vec{v}_{cm} + \vec{v}' = \vec{v}_{cm} + \vec{\omega} \times \vec{x}$$

Let $\hat{n}$ be the unit vector pointing in the direction of the rotation axis. The angular velocity is related to the angle of rotation θ by

$$\vec{\omega} = \omega\hat{n} = \frac{d\theta}{dt}\hat{n}.$$

The center of mass is moving in the direction of $\frac{\vec{R}}{R} \times \hat{n}$ with the magnitude

$$|\vec{v}_{cm}| = \left| R\frac{d\theta}{dt} \right| = |R\omega|$$

Thus

$$\vec{v}_{cm} = R\omega \left(\frac{\vec{R}}{R} \times \hat{n} \right) = \vec{R} \times \vec{\omega}$$

and

$$\vec{v} = \vec{v}_{cm} + \vec{\omega} \times \vec{x} = \vec{\omega} \times \left(\vec{x} - \vec{R} \right)$$

In particular, $\vec{x} = \vec{R}$ at the bottom of the wheel, $\vec{v} = 0$. Furthermore, $\left(\vec{x} - \vec{R} \right)$ is the displacement relative to the bottom of the wheel. The rolling motion can be also treated as a rotation around the axis passing at the bottom of the wheel.

11.2 The Kinetic Energy of Rolling

First let us prove an identity that relates the total kinetic energy observed in the laboratory system to that in the center of mass frame. In the laboratory system,

$$\begin{aligned} K &= \frac{1}{2}\sum_i m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2}\sum_i m_i \left(\vec{v}_{cm} + \vec{v}_i' \right) \cdot \left(\vec{v}_{cm} + \vec{v}_i' \right) \\ &= \frac{1}{2}\sum_i m_i \vec{v}_{cm} \cdot \vec{v}_{cm} + \frac{1}{2}\sum_i m_i \vec{v}_i' \cdot \vec{v}_i' + \left(\sum_i m_i \vec{v}_i' \right) \cdot \vec{v}_{cm} \\ &= \frac{1}{2} M \left| \vec{v}_{cm} \right|^2 + K' + M\vec{v}_{cm}' \cdot \vec{v}_{cm} \end{aligned}$$

where $M = \sum_i m_i$ is the total mass of the system, K' is the total kinetic energy observed in the center of mass frame, and $\vec{v}_{cm}' = \frac{\sum_i m_i \vec{v}_i'}{M}$ the velocity of the center of mass observed in the center of mass frame. By definition, center of mass is stationary in the center of mass frame, so we must have $\vec{v}'_{cm} = 0$ and we have

$$K = K' + \frac{1}{2}M\left|\vec{v}_{cm}\right|^2$$

The kinetic energy is equal the kinetic energy calculated in the center of mass frame plus the kinetic energy of a point particle of mass M moving at the $\vec{v}_{cm}$.

For the rolling wheel, the kinetic energy observed in the center of mass frame is the kinetic energy due to the rotation described by $\vec{\omega}$.

$$K' = \frac{1}{2}I_{cm}\omega^2$$

The kinetic energy relative to the ground floor is then

$$K = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}M\left|\vec{v}_{cm}\right|^2 = \frac{1}{2}\left(I_{cm} + MR^2\right)\omega^2$$

where we have utilized $\left|\vec{v}_{cm}\right|^2 = (R\omega)^2$.

Alternatively, if we consider the rolling motion as a rotation around the bottom of the wheel, by parallel-axis theorem, the moment of inertia is

$$I = I_{cm} + MR^2$$

Therefore, we also get

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(I_{cm} + MR^2\right)\omega^2$$

11.3 Torque by Uniform gravitational Force

For a system of particles in a uniform gravitational field, the total torque due to this uniform gravity is

$$\vec{\tau} = \sum_i \vec{x}_i \times (m_i\vec{g}) = \left(\sum_i m_i\vec{x}_i\right) \times \vec{g} = M\vec{x}_{cm} \times \vec{g} = \vec{x}_{cm} \times (M\vec{g})$$

which is also to the torque due to the the total gravitational force $M\vec{g}$ exerted at center of mass $\vec{x}_{cm}$.

11.4 The Force of Rolling

$$
\begin{aligned}
dW &= \sum_i \vec{F}_i \cdot d\vec{x}_i = \sum_i \vec{F}_i \cdot (\vec{\omega}dt \times \vec{x}_i) \\
&= \sum_i \left(\vec{x}_i \times \vec{F}_i\right) \cdot \vec{\omega}dt = \vec{\tau} \cdot \vec{\omega}dt = \vec{\tau} \cdot \hat{n}\omega dt
\end{aligned}
$$

If $\frac{dI}{dt} = 0$,

$$dW = dK = I\omega d\omega$$

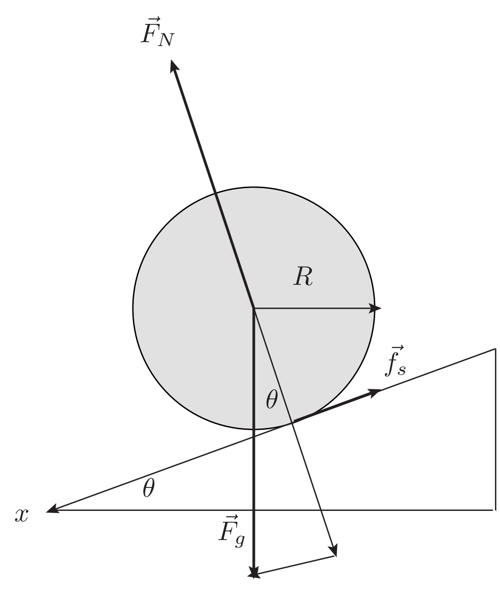
We then have

$$I\omega d\omega = \vec{\tau} \cdot \hat{n}\omega dt$$

or

$$I\frac{d\omega}{dt} = \vec{\tau} \cdot \hat{n} \tag{1}$$

11.4.1 Rolling Down a Ramp



In the center of mass frame, the total torque acting on the rolling wheel is

$$\vec{\tau}_{cm} = \vec{R} \times \vec{F}_N + \vec{R} \times \vec{f}_s = \vec{R} \times \vec{f}_s$$

where $\vec{R}$ is the vector pointing from the center of wheel to the contact point between the wheel and the ramp. Note that the gravitational force $\vec{F}_g$ does not contribute to $\vec{\tau}_{cm}$ because it is exerted at the origin and the normal force $\vec{F}_N$ does not contribute to $\vec{\tau}_{cm}$ either because $\vec{R}$ is anti-parallel to $\vec{F}_N$.

Let $\hat{n}$ be the rotation axis which points toward the classroom. Then

$$\vec{\tau}_{cm} \cdot \hat{n} = \left(\vec{R} \times \vec{f}_s\right) \cdot \hat{n} = Rf_s$$

where the friction force $\vec{f}_s = -f_s\hat{\imath}$ with the positive x-axis points toward the lower-left direction. From (1), we get

$$I_{cm}\frac{d\omega}{dt} = Rf_s \tag{2}$$

For the linear motion, we have

$$M\vec{a}_{cm} = \vec{F}_N + \vec{F}_g + \vec{f}_s$$

For the linear motion along the x direction with

$$\vec{a}_{cm} = a_{cm}\hat{\imath} = \frac{dv_{cm}}{dt}\hat{\imath},$$

we have

$$M\frac{dv_{cm}}{dt} = Ma_{cm} = \left(\vec{F}_N + \vec{F}_g + \vec{f}_s\right) \cdot \hat{\imath} = -f_s + F_g \sin\theta \tag{3}$$

Eliminate f_s in (3) by (2), we get

$$MR\frac{dv_{cm}}{dt} = -Rf_s + RF_g \sin\theta = -I_{cm}\frac{d\omega}{dt} + RF_g \sin\theta$$

or

$$\frac{d}{dt}\left(Mv_{cm} + \frac{I_{cm}}{R}\omega\right) = F_g \sin\theta \tag{4}$$

11.4.2 Rolling without Slipping

If the wheel rolls down the ramp without slipping, we have

$$v_{cm} = R\omega$$

and (4) can be written as

$$\left(M + \frac{I_{cm}}{R^2}\right) \frac{dv_{cm}}{dt} = F_g \sin\theta = Mg\sin\theta$$

which shows that

$$a_{cm} = \frac{F_g}{M + \frac{I_{cm}}{R^2}} \sin\theta = \frac{M}{M + \frac{I_{cm}}{R^2}} g\sin\theta$$

For the wheel to travel a distance s during a time interval of T along the ramp starting from the position at which the wheel is at rest $v_{cm}(0) = 0$, we have

$$v_{cm}(T) = a_{cm}T$$

and

$$s = \frac{1}{2}a_{cm}T^2 = \frac{v_{cm}^2(T)}{2a_{cm}}$$

Let

$$h = s\sin\theta$$

be the height that the wheel has descended. We then have

$$\frac{1}{2}Mv_{cm}^2 = Ma_{cm}s = \frac{1}{1 + \frac{I_{cm}}{MR^2}} Mgs\sin\theta = \frac{1}{1 + \frac{I_{cm}}{MR^2}} Mgh$$

or

$$Mgh = \frac{1}{2}M\left(1 + \frac{I_{cm}}{MR^2}\right)v_{cm}^2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\left(\frac{v_{cm}}{R}\right)^2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \tag{5}$$

which is consistent with the conservation of mechanical energy. Note that the friction force does no work because the point of contact where the friction force is exerted is always stationary.

There is another derivation, by using the axis at the bottom of the wheel as the rotation axis. Then the kinetic energy is all due to the rotation around this axis with

$$K = \frac{1}{2}\left(I_{cm} + MR^2\right)\omega^2 = Mgh$$

Since $R\omega = v_{cm}$, the above identity is identical to (5).

11.4.3 Rolling with Slipping

(4) can be integrated to yield

$$Mv_{cm}(t) + \frac{I_{cm}}{R}\omega(t) = Mv_{cm,0} + \frac{I_{cm}}{R}\omega_0 + F_g \sin\theta t \tag{6}$$

where $v_{cm,0} = v_{cm}(0)$ and $\omega_0 = \omega(0)$. If $R\omega \neq v_{cm}$, the contact point on the wheel is in relative motion to the contact point of the ramp or floor. Let us consider the case when $\theta = 0$. The above identity becomes

$$Mv_{cm}(t) + \frac{I_{cm}}{R}\omega(t) = Mv_{cm,0} + \frac{I_{cm}}{R}\omega_0$$

Suppose at the instant T, $R\omega(T) = v_{cm}(T)$ and the wheel begins to roll without slipping.

$$\begin{aligned} Mv_{cm,0} + \frac{I_{cm}}{R}\omega_0 &= Mv_{cm}(T) + \frac{I_{cm}}{R}\omega(T) \\ &= \left(1 + \frac{I_{cm}}{MR^2}\right) Mv_{cm}(T) = \frac{I_{cm} + MR^2}{R}\omega(T) \end{aligned}$$

In particular, if $\omega_0 = 0$ and the wheel was not rotating initially, then

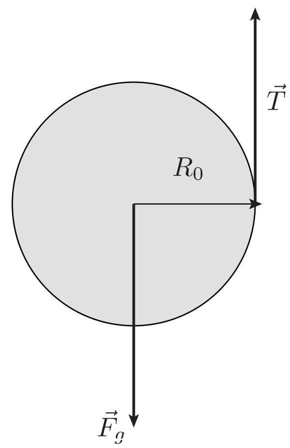
$$v_{cm}(T) = \frac{v_{cm,0}}{1 + \frac{I_{cm}}{MR^2}} < v_{cm,0}$$

On the other hand, if $v_{cm,0} = 0$ and the wheel was not moving initially, then

$$\omega(T) = \frac{I_{cm}}{I_{cm} + MR^2}\omega_0 < \omega_0$$

11.5 The Yo-Yo

Instead rolling down a ramp at an angle θ, the yo-yo rolls down a string at angle $\theta = 90°$ with the horizontal. Instead of being slowed by friction force $\vec{f}_s$, the yo-yo is slowed by the tension $\vec{T}$ from the string. Instead of rolling on its outer surface at radius R, the yo-yo rolls on an axle of radius R_0. Let $\hat{\imath}$ be the direction that points vertically downward.



We then have

$$v_{cm} = R_0\omega$$
$$I_{cm}\frac{d\omega}{dt} = \vec{\tau}\cdot\hat{n} = R_0 T$$
$$m\vec{a}_{cm} = m\frac{dv_{cm}}{dt}\hat{\imath} = \vec{F}_g + \vec{T} = (mg - T)\,\hat{\imath}$$

Combining the above 3 identities we get

$$g = a_{cm} + \frac{T}{m} = a_{cm} + \frac{I_{cm}}{mR_0}\frac{d\omega}{dt} = \left(1 + \frac{I_{cm}}{mR_0^2}\right)a_{cm}$$

11.6 Torque and Angular Momentum

The Newton's second law can be written as

$$\vec{f}_i = m_i\vec{a}_i = \frac{d}{dt}\left(m_i\vec{v}_i\right) = \frac{d\vec{p}_i}{dt}$$

where $\vec{p}_i = m_i\vec{v}_i$ is the linear momentum of the particle with mass m_i and velocity $\vec{v}_i$. For a system of particles,

$$\vec{F}_{total} = \sum_i \vec{f}_i = \sum_i \frac{d\vec{p}_i}{dt} = \frac{d\vec{P}_{toal}}{dt}$$

The rate of change for the total linear momentum is equal to the net total force acting on the system. Note in evaluating $\vec{F}_{total}$ only the external forces need to be taken into account because the internal forces cancel in pairs by Newton's 3rd law.

$$\vec{F}_{total} = \vec{F}^{(ext)}_{total}$$

For a particle located at $\vec{x}_i$ with mass m_i, define the angular momentum

$$\vec{L}_i \equiv \vec{x}_i \times \vec{p}_i$$

and it is subjected to the torque

$$\begin{aligned}\vec{\tau}_i &= \vec{x}_i \times \vec{f}_i = \vec{x}_i \times \frac{d\vec{p}_i}{dt} = \frac{d}{dt}\left(\vec{x}_i \times \vec{p}_i\right) - \frac{d\vec{x}_i}{dt} \times \vec{p}_i \\ &= \frac{d\vec{L}_i}{dt} - \vec{v}_i \times \vec{p}_i = \frac{d\vec{L}_i}{dt}\end{aligned}$$

Note we have set $\vec{v}_i \times \vec{p}_i = 0$ in the last step since $\vec{v}_i \parallel \vec{p}_i$. Summing over

$$\vec{\tau}_i = \frac{d\vec{L}_i}{dt}$$

for all particles in a system, we get

$$\vec{\tau}_{total} = \sum_i \vec{\tau}_i = \frac{d\sum_i \vec{L}_i}{dt} = \frac{d\vec{L}_{total}}{dt}$$

Similar to the linear case, the rate of change for the total angular momentum is equal to the net total torque acting on the system. We have shown in Sec. (10.4) that only external forces contribute to the net total torque.

$$\vec{\tau}_{total} = \sum_i \vec{x}_i \times \vec{f}_i = \sum_i \vec{x}_i \times \vec{f}^{(external)}_i$$

The total angular momentum will be conserved (a constant vector) if the net torque $\vec{\tau}_{total}$ is zero.

As a simple example, the net torque on a particle moving in a spherically symmetric central force

$$\vec{f} = f\left(r\right)\frac{\vec{x}}{r}$$

where $r = |\vec{x}|$, is

$$\vec{\tau} = \vec{x} \times \vec{f} = \frac{f\left(r\right)}{r}\vec{x} \times \vec{x} = 0$$

Therefore, $\frac{d\vec{L}}{dt}=0$ and $\vec{L}=m\vec{x}\times\vec{v}$ is constant. Since $\vec{x}\cdot\vec{L}=\vec{v}\cdot\vec{L}=0$, the position vector and velocity vector of the particle are always perpendicular to the constant angular momentum $\vec{L}$. This shows that the orbit of the particle is always on a plane (passing through the origin) with its normal direction along $\vec{L}$.

11.7 Angular Momentum for a Rigid Body about a Fixed Axis

The angular momentum $\vec{L}$ is not necessarily parallel to the angular velocity $\vec{\omega}$. For example, a system consisting of a rotating particle of mass m located at $\vec{x}$ with angular velocity $\vec{\omega}$ will have

$$\vec{L}=m\vec{x}\times\vec{v}=m\vec{x}\times(\vec{\omega}\times\vec{x})=m\left(|\vec{x}|^2\,\vec{\omega}-(\vec{\omega}\cdot\vec{x})\,\vec{x}\right)$$

$\vec{L}$ is not parallel to $\vec{\omega}$ unless $\vec{\omega}$ is perpendicular to $\vec{x}$. In general,

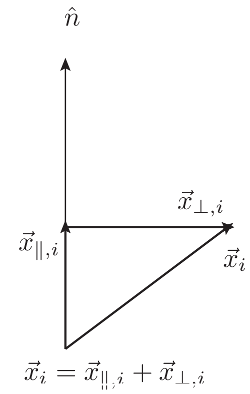
$$\begin{aligned}
\vec{L}&=\sum_i m_i\vec{x}_i\times\vec{v}_i=\sum_i m_i\vec{x}_i\times(\vec{\omega}\times\vec{x}_i)\\
&=\sum_i m_i\left(|\vec{x}_i|^2\,\vec{\omega}-(\vec{\omega}\cdot\vec{x}_i)\,\vec{x}_i\right)\\
&=\sum_i m_i\omega\left(|\vec{x}_i|^2\,\hat{n}-(\hat{n}\cdot\vec{x}_i)\left(\vec{x}_{\parallel,i}+\vec{x}_{\perp,i}\right)\right)\\
&=\sum_i m_i\omega\left(|\vec{x}_i|^2\,\hat{n}-\left(\hat{n}\cdot\vec{x}_{\parallel,i}\right)^2\hat{n}-\left(\hat{n}\cdot\vec{x}_{\parallel,i}\right)\vec{x}_{\perp,i}\right)\\
&=\sum_i \omega\left(m_i\left|\vec{x}_{\perp,i}\right|^2\hat{n}-m_i\left(\hat{n}\cdot\vec{x}_{\parallel,i}\right)\vec{x}_{\perp,i}\right)
\end{aligned}$$

where we have used

$$\vec{x}_i=(\vec{x}_i\cdot\hat{n})\,\hat{n}+\hat{n}\times(\vec{x}_i\times\hat{n})=\vec{x}_{\parallel,i}+\vec{x}_{\perp,i}$$

and

$$\vec{x}_{\parallel,i}=(\vec{x}_i\cdot\hat{n})\,\hat{n},\vec{x}_{\perp,i}=\hat{n}\times(\vec{x}_i\times\hat{n})=\vec{x}_i-(\vec{x}_i\cdot\hat{n})\,\hat{n}$$



If $\hat{n}$ is a line of symmetry, then a particle of mass m located at $\vec{x} = \vec{x}_\parallel + \vec{x}_\perp$ is symmetric with respect to another particle of the same mass m located $\vec{x} = \vec{x}_\parallel - \vec{x}_\perp$, therefore $\sum_i m_i \left(\hat{n} \cdot \vec{x}_{\parallel,i}\right) \vec{x}_{\perp,i} = 0$, and as a result,

$$\vec{L} = \sum_i m_i \left|\vec{x}_{\perp,i}\right|^2 \omega \hat{n} = I_{\hat{n}} \vec{\omega}$$

and $\vec{L}$ is parallel to $\vec{\omega}$.

The total kinetic energy of a rigid body under rotation can be written as

$$\begin{aligned} K &= \frac{1}{2} \sum_i m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2} \sum_i m_i \left(\vec{\omega} \times \vec{x}_i\right) \cdot \vec{v}_i \\ &= \frac{1}{2} \vec{\omega} \cdot \sum_i m_i \left(\vec{x}_i \times \vec{v}_i\right) = \frac{1}{2} \vec{\omega} \cdot \sum_i \vec{L}_i = \frac{1}{2} \vec{\omega} \cdot \vec{L}_{total} \end{aligned}$$

We also know that, with $\vec{\omega} = \omega \hat{n}$,

$$K = \frac{1}{2} I_{\hat{n}} \omega^2$$

Thus

$$I_{\hat{n}} \omega^2 = \vec{\omega} \cdot \vec{L}_{total} = \hat{n} \cdot \vec{L}_{total} \omega$$

or

$$\hat{n} \cdot \vec{L}_{total} = I_{\hat{n}} \omega$$

The projection of the net total angular momentum along the axis of rotation is $I_{\hat{n}} \omega$. Taking the time derivative of the above, we get

$$\frac{d}{dt} \left(\hat{n} \cdot \vec{L}_{total}\right) = \frac{d}{dt} \left(I_{\hat{n}} \omega\right) \tag{7}$$

If the rotation axis is fixed $\frac{d\hat{n}}{dt} = 0$, then

$$\hat{n} \cdot \frac{d\vec{L}_{total}}{dt} = \hat{n} \cdot \vec{\tau}_{total} = \frac{d}{dt}\left(I_{\hat{n}}\omega\right) \tag{8}$$

If, in addition, the body is rigid so that $\frac{dI_{\hat{n}}}{dt} = 0$. We then have

$$\hat{n} \cdot \vec{\tau}_{total} = I_{\hat{n}} \frac{d\omega}{dt} = I_{\hat{n}}\alpha \tag{9}$$

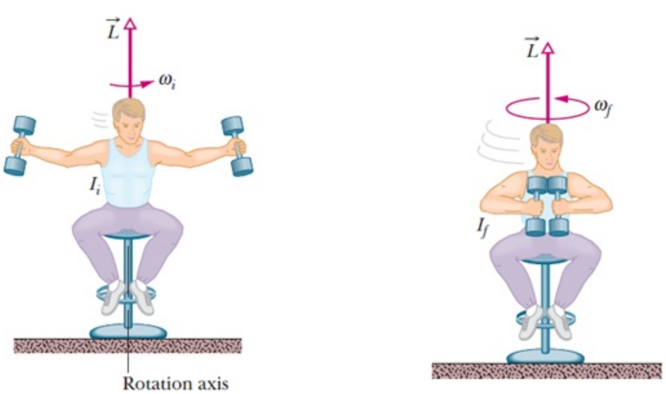
11.8 Conservation of Angular Momentum

From (8), for a fixed direction $\hat{n}$, if $\hat{n} \cdot \vec{\tau}_{total} = 0$, $I_{\hat{n}}\omega$ is a constant and we have

$$I_{\hat{n},i}\omega_i = I_{\hat{n},f}\omega_f$$

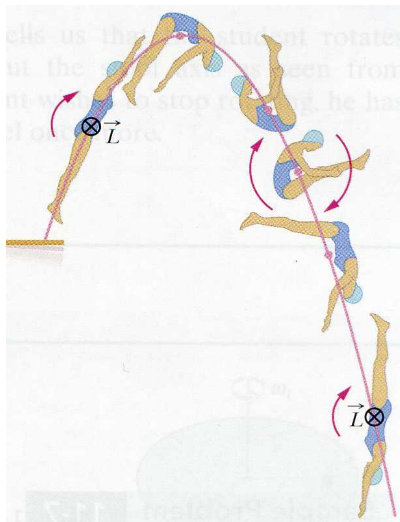
The above is applicable for a body which somehow redistributes its mass relative to the rotation axis, changing its rotational inertia about that axis. Here, the subscripts refer to the values of the rotational inertia and the angular velocity before and after the redistribution of mass.

- The spinning Volunteer



No net external torque along the rotation axis acts on the system ($\vec{\tau} \cdot \hat{n} = 0$) consisting of the student, the stool, and dumbbells. The angular momentum of the system about the rotation axis must remain constant, no matter how the student maneuvers the dumbbells. The student's angular speed ω_i in (Fig a) is relatively low and his rotational inertia I_i is relatively high. In (Fig b), his angular speed ω_f must be greater to compensate for the decreased I_f since $I_i\omega_i = I_f\omega_f$.

- The springboard diver

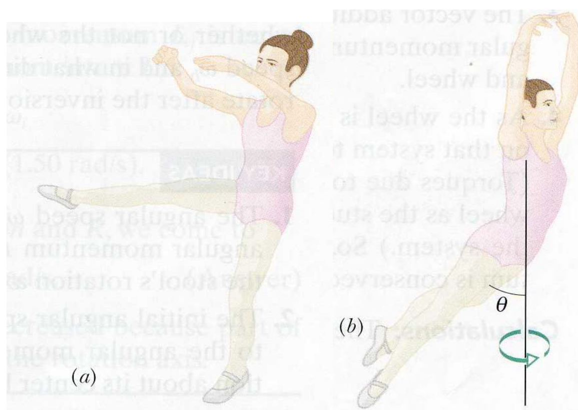


The above figure shows a diver doing a forward one-and-a-half somersault dive. The center of mass of the diver follows a parabolic path. She leaves the springboard with a definite angular momentum $\vec{L}$ about an axis through her center of mass, represented by a vector pointing into the plane of the figure, perpendicular to the page. When she is in the air, no net external torque acts on her about her center of mass, so her angular momentum about her center of mass cannot change. By pulling her arms and legs into the closed tuck position, she can considerably reduce her rotational inertia about the same axis and thus, considerably increase her angular speed. Pulling out of the tuck position at the end of the dive increases her rotational inertia and thus slows her rotation rate so that she can enter the water with little splash.

- Long jump

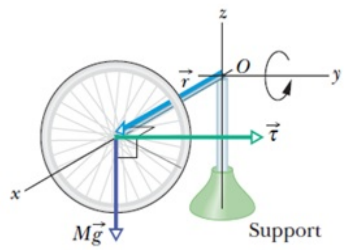
When an athlete takes off from the ground in a running long jump, the force on the launching foot gives the athlete an angular momentum with a forward rotation around a horizontal axis. Such rotation would not allow the jumper to land properly: In the landing, the legs should be together and extended forward together at an angle so that the heels mark the sand at the greatest distance. Once airborne, the angular momentum cannot change because no external torque acts to change. However, the jumper can shift most of the angular momentum to the arms by rotating them in the windmill fashion. The the body remains upright and in the proper orientation for landing.

- A ballet performer

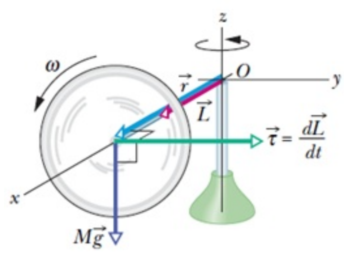


A ballet performer leaps with a small twisting motion on the floor with one foot while holding the other leg perpendicular to the body (Fig a). The angular speed is so small that it may not be perceptible to the audience. As the performer ascends, the outstretched leg is brought down and the other leg is brought up, with both ending up at an angle θ to the body (Fig b). The motion is graceful, but it also serves to increase the rotation because bringing in the initially outstretched leg decreases the performer's rotational inertia. Since no external torque acts on the airborne performer, the angular momentum can not change. Thus, with a decrease in rotational inertia, the angular speed must increase.

11.9 Precession of Gyroscope



If one end of the shaft of a non-spinning gyroscope is placed on a support as in the above figure and the gyroscope is released, the gyroscope falls by rotating downward about the tip of the support. A rapid spinning gyroscope behaves differently.



$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times (M\vec{g}) = Mr\hat{n} \times \vec{g}$$

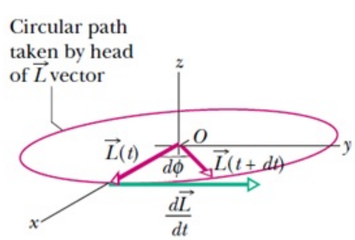
For fast spinning gyroscope, the angular momentum is almost parallel to its angular velocity $\vec{\omega} = \omega\hat{n}$ with

$$\vec{L} \simeq I\vec{\omega} = I\omega\hat{n}$$

Thus

$$\frac{d\vec{L}}{dt} \simeq I\omega\frac{d\hat{n}}{dt} = \vec{\tau} = Mr\hat{n} \times \vec{g}$$

$$\frac{d\hat{n}}{dt} = -\frac{Mr}{I\omega}\vec{g} \times \hat{n} = \vec{\Omega} \times \hat{n}$$



The axis $\hat{n}$ rotates with the angular velocity $\vec{\Omega} = -\frac{Mr}{I\omega}\vec{g}$ or $\vec{L}$ precess around the vertical z axis with a precession rate

$$\Omega = \left|\vec{\Omega}\right| = \frac{d\phi}{dt} = \frac{Mgr}{I\omega}$$