# 1

# *Experiments, Models, and Probabilities*

## Getting Started with Probability

You have read the “Message to Students” in the Preface. Now you can begin. The title of this book is *Probability and Stochastic Processes*. We say and hear and read the word *probability* and its relatives (*possible, probable, probably*) in many contexts. Within the realm of applied mathematics, the meaning of *probability* is a question that has occupied mathematicians, philosophers, scientists, and social scientists for hundreds of years.

Everyone accepts that the probability of an event is a number between 0 and 1. Some people interpret probability as a physical property (like mass or volume or temperature) that can be measured. This is tempting when we talk about the probability that a coin flip will come up heads. This probability is closely related to the nature of the coin. Fiddling around with the coin can alter the probability of heads.

Another interpretation of probability relates to the knowledge that we have about something. We might assign a low probability to the truth of the statement, *It is raining now in Phoenix, Arizona*, because we know that Phoenix is in the desert. However, our knowledge changes if we learn that it was raining an hour ago in Phoenix. This knowledge would cause us to assign a higher probability to the truth of the statement, *It is raining now in Phoenix*.

Both views are useful when we apply probability theory to practical problems. Whichever view we take, we will rely on the abstract mathematics of probability, which consists of definitions, axioms, and inferences (theorems) that follow from the axioms. While the structure of the subject conforms to principles of pure logic, the terminology is not entirely abstract. Instead, it reflects the practical origins of probability theory, which was developed to describe phenomena that cannot be predicted with certainty. The point of view is different from the one we took when we started studying physics. There we said that if we do the same thing in the same way over and over again – send a space shuttle into orbit, for example – the result will always be the same. To predict the result, we have to take account of all relevant facts.

The mathematics of probability begins when the situation is so complex that we just can’t replicate everything important exactly – like when we fabricate and test an integrated circuit. In this case, repetitions of the same procedure yield different results. The situation is not totally chaotic, however. While each outcome may be unpredictable, there are consistent patterns to be observed when we repeat the procedure a large number of times. Understanding these patterns helps engineers establish test procedures to ensure that a factory meets quality objectives. In this repeatable procedure (making and testing a chip) with unpredictable outcomes (the quality of individual chips), the *probability* is a number between 0 and 1 that states the proportion of times we expect a certain thing to happen, such as the proportion of chips that pass a test.

As an introduction to probability and stochastic processes, this book serves three purposes:

- It introduces students to the logic of probability theory.
- It helps students develop intuition into how the theory applies to practical situations.
- It teaches students how to apply probability theory to solving engineering problems.

To exhibit the logic of the subject, we show clearly in the text three categories of theoretical material: definitions, axioms, and theorems. Definitions establish the logic of probability theory, while axioms are facts that we accept without proof. Theorems are consequences that follow logically from definitions and axioms. Each theorem has a proof that refers to definitions, axioms, and other theorems. Although there are dozens of definitions and theorems, there are only three axioms of probability theory. These three axioms are the foundation on which the entire subject rests. To meet our goal of presenting the logic of the subject, we could set out the material as dozens of definitions followed by three axioms followed by dozens of theorems. Each theorem would be accompanied by a complete proof.

While rigorous, this approach would completely fail to meet our second aim of conveying the intuition necessary to work on practical problems. To address this goal, we augment the purely mathematical material with a large number of examples of practical phenomena that can be analyzed by means of probability theory. We also interleave definitions and theorems, presenting some theorems with complete proofs, others with partial proofs, and omitting some proofs altogether. We find that most engineering students study probability with the aim of using it to solve practical problems, and we cater mostly to this goal. We also encourage students to take an interest in the logic of the subject – it is very elegant – and we feel that the material presented will be sufficient to enable these students to fill in the gaps we have left in the proofs.

Therefore, as you read this book you will find a progression of definitions, axioms, theorems, more definitions, and more theorems, all interleaved with examples and comments designed to contribute to your understanding of the theory. We also include brief quizzes that you should try to solve as you read the book. Each one will help you decide whether you have grasped the material presented just before the quiz. The problems at the end of each chapter give you more practice applying the material introduced in the chapter. They vary considerably in their level of difficulty. Some of them take you more deeply into the subject than the examples and quizzes do.

## 1.1 Set Theory

The mathematical basis of probability is the theory of sets. Most people who study probability have already encountered set theory and are familiar with such terms as *set*, *element*, *union*, *intersection*, and *complement*. For them, the following paragraphs will review material already learned and introduce the notation and terminology we use here. For people who have no prior acquaintance with sets, this material introduces basic definitions and the properties of sets that are important in the study of probability.

A *set* is a collection of things. We use capital letters to denote sets. The things that together make up the set are *elements*. When we use mathematical notation to refer to set elements, we usually use small letters. Thus we can have a set $A$ with elements $x$, $y$, and $z$. The symbol $\in$ denotes set inclusion. Thus $x \in A$ means "$x$ is an element of set $A$." The symbol $\notin$ is the opposite of $\in$. Thus $c \notin A$ means "$c$ is not an element of set $A$."

It is essential when working with sets to have a definition of each set. The definition allows someone to consider anything conceivable and determine whether that thing is an element of the set. There are many ways to define a set. One way is simply to name the elements:

$$A = \{\text{Rutgers University, Polytechnic University, the planet Mercury}\}\,. \tag{1.1}$$

Note that in stating the definition, we write the name of the set on one side of $=$ and the definition in curly brackets $\{\ \}$ on the other side of $=$.

It follows that "the planet closest to the Sun $\in A$" is a true statement. It is also true that "Bill Clinton $\notin A$." Another way of writing the set is to give a rule for testing something to determine whether it is a member of the set:

$$B = \{\text{all Rutgers juniors who weigh more than 170 pounds}\}\,. \tag{1.2}$$

In engineering, we frequently use mathematical rules for generating all of the elements of the set:

$$C = \left\{x^2 | x = 1, 2, 3, 4, 5\right\} \tag{1.3}$$

This notation tells us to form a set by performing the operation to the left of the vertical bar, $|$, on the numbers to the right of the bar. Therefore,

$$C = \{1, 4, 9, 16, 25\}\,. \tag{1.4}$$

Some sets have an infinite number of elements. For example

$$D = \left\{x^2 | x = 1, 2, 3, \ldots\right\}. \tag{1.5}$$

The dots tell us to continue the sequence to the left of the dots. Since there is no number to the right of the dots, we continue the sequence indefinitely, forming an infinite set containing all perfect squares except 0. The definition of $D$ implies that $144 \in D$ and $10 \notin D$.

In addition to set inclusion, we also have the notion of a *subset*, which describes a relationship between two sets. By definition, $A$ is a subset of $B$ if every member of $A$ is also a member of $B$. We use the symbol $\subset$ to denote subset. Thus $A \subset B$ is mathematical notation for the statement "the set $A$ is a subset of the set $B$." Using the definitions of sets $C$ and $D$ in Equations (1.3) and (1.5), we observe that $C \subset D$. If

$$I = \{\text{all positive integers, negative integers, and } 0\}\,, \tag{1.6}$$

it follows that $C \subset I$, and $D \subset I$.

The definition of set equality,

$$A = B, \tag{1.7}$$

is

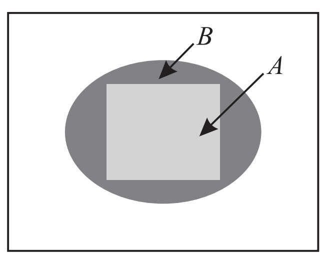
$$A = B \text{ if and only if } B \subset A \text{ and } A \subset B.$$

This is the mathematical way of stating that $A$ and $B$ are identical if and only if every element of $A$ is an element of $B$ and every element of $B$ is an element of $A$. This definition implies that a set is unaffected by the order of the elements in a definition. For example, $\{0, 17, 46\} = \{17, 0, 46\} = \{46, 0, 17\}$ are all the same set.

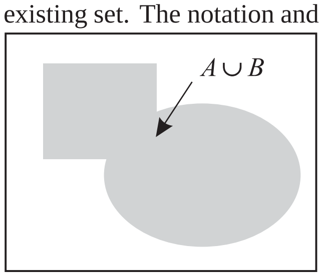
To work with sets mathematically it is necessary to define a *universal set*. This is the set of all things that we could possibly consider in a given context. In any study, all set operations relate to the universal set for that study. The members of the universal set include all of the elements of all of the sets in the study. We will use the letter $S$ to denote the universal set. For example, the universal set for $A$ could be $S = \{\text{all universities in New Jersey, all planets}\}$. The universal set for $C$ could be $S = I = \{0, 1, 2, \ldots\}$. By definition, every set is a subset of the universal set. That is, for any set $X$, $X \subset S$.

The *null set*, which is also important, may seem like it is not a set at all. By definition it has no elements. The notation for the null set is $\phi$. By definition $\phi$ is a subset of every set. For any set $A$, $\phi \subset A$.

It is customary to refer to Venn diagrams to display relationships among sets. By convention, the region enclosed by the large rectangle is the universal set $S$. Closed surfaces within this rectangle denote sets. A Venn diagram depicting the relationship $A \subset B$ is



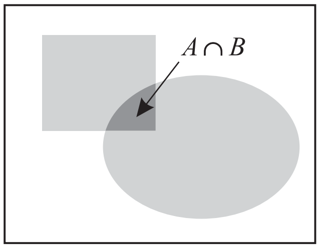

When we do set algebra, we form new sets from existing sets. There are three operations for doing this: *union*, *intersection*, and *complement*. Union and intersection combine two existing sets to produce a third set. The complement operation forms a new set from one existing set. The notation and definitions are




The *union* of sets $A$ and $B$ is the set of all elements that are either in $A$ or in $B$, or in both. The union of $A$ and $B$ is denoted by $A \cup B$. In this Venn diagram, $A \cup B$ is the complete shaded area. Formally, the definition states

$$x \in A \cup B \text{ if and only if } x \in A \text{ or } x \in B.$$

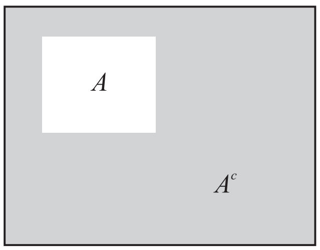
The set operation union corresponds to the logical "or" operation.




The *intersection* of two sets $A$ and $B$ is the set of all elements which are contained both in $A$ and $B$. The intersection is denoted by $A \cap B$. Another notation for intersection is $AB$. Formally, the definition is

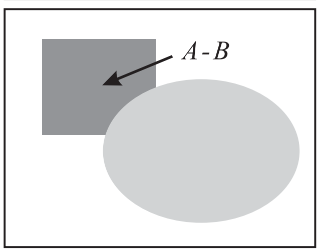
$$x \in A \cap B \text{ if and only if } x \in A \text{ and } x \in B.$$

The set operation intersection corresponds to the logical "and" function.




The *complement* of a set $A$, denoted by $A^c$, is the set of all elements in $S$ that are not in $A$. The complement of $S$ is the null set $\phi$. Formally,

$$x \in A^c \text{ if and only if } x \notin A.$$

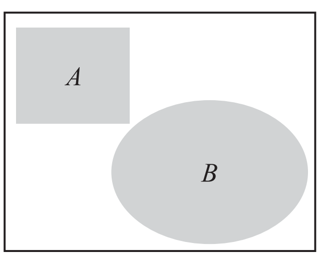



A fourth set operation is called the *difference*. It is a combination of intersection and complement. The *difference* between $A$ and $B$ is a set $A - B$ that contains all elements of $A$ that are *not* elements of $B$. Formally,

$$x \in A - B \text{ if and only if } x \in A \text{ and } x \notin B$$

Note that $A - B = A \cap B^c$ and $A^c = S - A$.

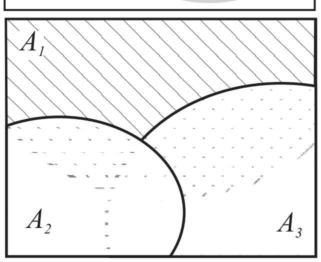
In working with probability we will frequently refer to two important properties of collections of sets. Here are the definitions.




A collection of sets $A_1, \ldots, A_n$ is *mutually exclusive* if and only if

$$A_i \cap A_j = \phi, \qquad i \neq j. \tag{1.8}$$

When there are only two sets in the collection, we say that these sets are *disjoint*. Formally, $A$ and $B$ are disjoint if and only if $A \cap B = \phi$.




A collection of sets $A_1, \ldots, A_n$ is *collectively exhaustive* if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S. \tag{1.9}$$

In the definition of *collectively exhaustive*, we used the somewhat cumbersome notation $A_1 \cup A_2 \cup \cdots \cup A_n$ for the union of $N$ sets. Just as $\sum_{i=1}^{n} x_i$ is a shorthand for $x_1 + x_2 + \cdots + x_n$, we will use a shorthand for unions and intersections of $n$ sets:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n, \tag{1.10}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n. \tag{1.11}$$

From the definition of set operations, we can derive many important relationships between sets and other sets derived from them. One example is

$$A - B \subset A. \tag{1.12}$$

To prove that this is true, it is necessary to show that if $x \in A - B$, then it is also true that $x \in A$. A proof that two sets are equal, for example, $X = Y$, requires two separate proofs: $X \subset Y$ and $Y \subset X$. As we see in the following theorem, this can be complicated to show.

***Theorem 1.1*** *De Morgan's law relates all three basic operations:*

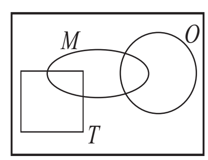
$$(A \cup B)^c = A^c \cap B^c.$$

***Proof*** There are two parts to the proof:

- To show $(A \cup B)^c \subset A^c \cap B^c$, suppose $x \in (A \cup B)^c$. That implies $x \notin A \cup B$. Hence, $x \notin A$ and $x \notin B$, which together imply $x \in A^c$ and $x \in B^c$. That is, $x \in A^c \cap B^c$.
- To show $A^c \cap B^c \subset (A \cup B)^c$, suppose $x \in A^c \cap B^c$. In this case, $x \in A^c$ and $x \in B^c$. Equivalently, $x \notin A$ and $x \notin B$ so that $x \notin A \cup B$. Hence, $x \in (A \cup B)^c$.

***Quiz 1.1***

*A pizza at Gerlanda's is either regular ($R$) or Tuscan ($T$). In addition, each slice may have mushrooms ($M$) or onions ($O$) as described by the Venn diagram at right. For the sets specified below, shade the corresponding region of the Venn diagram.*




| | |
|---|---|
| (1) $R$ | (2) $M \cup O$ |
| (3) $M \cap O$ | (4) $R \cup M$ |
| (5) $R \cap M$ | (6) $T^c - M$ |

## 1.2 Applying Set Theory to Probability

The mathematics we study is a branch of measure theory. Probability is a number that describes a set. The higher the number, the more probability there is. In this sense probability is like a quantity that measures a physical phenomenon; for example, a weight or a temperature. However, it is not necessary to think about probability in physical terms. We can do all the math abstractly, just as we defined sets and set operations in the previous paragraphs without any reference to physical phenomena.

Fortunately for engineers, the language of probability (including the word *probability* itself) makes us think of things that we experience. The basic model is a repeatable *experiment*. An experiment consists of a *procedure* and *observations*. There is uncertainty in what will be observed; otherwise, performing the experiment would be unnecessary. Some examples of experiments include

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?
4. Transmit one of a collection of waveforms over a channel. What waveform arrives at the receiver?
5. Transmit one of a collection of waveforms over a channel. Which waveform does the receiver identify as the transmitted waveform?

For the most part, we will analyze *models* of actual physical experiments. We create models because real experiments generally are too complicated to analyze. For example, to describe *all* of the factors affecting your waiting time at a bus stop, you may consider

- The time of day. (Is it rush hour?)
- The speed of each car that passed by while you waited.
- The weight, horsepower, and gear ratios of each kind of bus used by the bus company.
- The psychological profile and work schedule of each bus driver. (Some drivers drive faster than others.)
- The status of all road construction within 100 miles of the bus stop.

It should be apparent that it would be difficult to analyze the effect of each of these factors on the likelihood that you will wait less than five minutes for a bus. Consequently, it is necessary to study a *model* of the experiment that captures the important part of the actual physical experiment. Since we will focus on the model of the experiment almost exclusively, we often will use the word *experiment* to refer to the model of an experiment.

**Example 1.1** An experiment consists of the following procedure, observation, and model:

- Procedure: Flip a coin and let it land on a table.
- Observation: Observe which side (head or tail) faces you after the coin lands.
- Model: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

As we have said, an experiment consists of both a procedure and observations. It is important to understand that two experiments with the same procedure but with different observations are different experiments. For example, consider these two experiments:

**Example 1.2** Flip a coin three times. Observe the sequence of heads and tails.

**Example 1.3** Flip a coin three times. Observe the number of heads.

These two experiments have the same procedure: flip a coin three times. They are different experiments because they require different observations. We will describe models of experiments in terms of a set of possible experimental outcomes. In the context of probability, we give precise meaning to the word *outcome*.

***Definition 1.1*** ***Outcome***
*An **outcome** of an experiment is any possible observation of that experiment.*

Implicit in the definition of an outcome is the notion that each outcome is distinguishable from every other outcome. As a result, we define the universal set of all possible outcomes. In probability terms, we call this universal set the *sample space*.

***Definition 1.2*** ***Sample Space***
*The **sample space** of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes.*

The *finest-grain* property simply means that all possible distinguishable outcomes are identified separately. The requirement that outcomes be mutually exclusive says that if one outcome occurs, then no other outcome also occurs. For the set of outcomes to be collectively exhaustive, every outcome of the experiment must be in the sample space.

**Example 1.4**

- The sample space in Example 1.1 is $S = \{h, t\}$ where $h$ is the outcome "observe head," and $t$ is the outcome "observe tail."
- The sample space in Example 1.2 is

$$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\} \tag{1.13}$$

- The sample space in Example 1.3 is $S = \{0, 1, 2, 3\}$.

**Example 1.5** Manufacture an integrated circuit and test it to determine whether it meets quality objectives. The possible outcomes are "accepted" $(a)$ and "rejected" $(r)$. The sample space is $S = \{a, r\}$.

In common speech, an event is just something that occurs. In an experiment, we may say that an event occurs when a certain phenomenon is observed. To define an event mathematically, we must identify *all* outcomes for which the phenomenon is observed. That is, for each outcome, either the particular event occurs or it does not. In probability terms, we define an event in terms of the outcomes of the sample space.

| **Set Algebra** | **Probability** |
|---|---|
| Set | Event |
| Universal set | Sample space |
| Element | Outcome |

***Table 1.1*** The terminology of set theory and probability.

***Definition 1.3*** ***Event***
*An **event** is a set of outcomes of an experiment.*

Table 1.1 relates the terminology of probability to set theory. All of this may seem so simple that it is boring. While this is true of the definitions themselves, applying them is a different matter. Defining the sample space and its outcomes are key elements of the solution of any probability problem. A probability problem arises from some practical situation that can be modeled as an experiment. To work on the problem, it is necessary to define the experiment carefully and then derive the sample space. Getting this right is a big step toward solving the problem.

**Example 1.6** Suppose we roll a six-sided die and observe the number of dots on the side facing upwards. We can label these outcomes $i = 1, \ldots, 6$ where $i$ denotes the outcome that $i$ dots appear on the up face. The sample space is $S = \{1, 2, \ldots, 6\}$. Each subset of $S$ is an event. Examples of events are

- The event $E_1 = \{$Roll 4 or higher$\} = \{4, 5, 6\}$.
- The event $E_2 = \{$The roll is even$\} = \{2, 4, 6\}$.
- $E_3 = \{$The roll is the square of an integer$\} = \{1, 4\}$.

**Example 1.7** Wait for someone to make a phone call and observe the duration of the call in minutes. An outcome $x$ is a nonnegative real number. The sample space is $S = \{x | x \geq 0\}$. The event "the phone call lasts longer than five minutes" is $\{x | x > 5\}$.

**Example 1.8** A short-circuit tester has a red light to indicate that there is a short circuit and a green light to indicate that there is no short circuit. Consider an experiment consisting of a sequence of three tests. In each test the observation is the color of the light that is on at the end of a test. An outcome of the experiment is a sequence of red ($r$) and green ($g$) lights. We can denote each outcome by a three-letter word such as $rgr$, the outcome that the first and third lights were red but the second light was green. We denote the event that light $n$ was red or green by $R_n$ or $G_n$. The event $R_2 = \{grg, grr, rrg, rrr\}$. We can also denote an outcome as an intersection of events $R_i$ and $G_j$. For example, the event $R_1 G_2 R_3$ is the set containing the single outcome $\{rgr\}$.

In Example 1.8, suppose we were interested only in the status of light 2. In this case, the set of events $\{G_2, R_2\}$ describes the events of interest. Moreover, for each possible outcome of the three-light experiment, the second light was either red or green, so the set of events $\{G_2, R_2\}$ is both mutually exclusive and collectively exhaustive. However, $\{G_2, R_2\}$

is not a sample space for the experiment because the elements of the set do not completely describe the set of possible outcomes of the experiment. The set $\{G_2, R_2\}$ does not have the finest-grain property. Yet sets of this type are sufficiently useful to merit a name of their own.

***Definition 1.4*** ***Event Space***

*An **event space** is a collectively exhaustive, mutually exclusive set of events.*

An event space and a sample space have a lot in common. The members of both are mutually exclusive and collectively exhaustive. They differ in the finest-grain property that applies to a sample space but not to an event space. Because it possesses the finest-grain property, a sample space contains all the details of an experiment. The members of a sample space are *outcomes*. By contrast, the members of an event space are *events*. The event space is a set of events (sets), while the sample space is a set of outcomes (elements). Usually, a member of an event space contains many outcomes. Consider a simple example:

**Example 1.9** Flip four coins, a penny, a nickel, a dime, and a quarter. Examine the coins in order (penny, then nickel, then dime, then quarter) and observe whether each coin shows a head ($h$) or a tail ($t$). What is the sample space? How many elements are in the sample space?

................................................................................

The sample space consists of 16 four-letter words, with each letter either $h$ or $t$. For example, the outcome $tthh$ refers to the penny and the nickel showing tails and the dime and quarter showing heads. There are 16 members of the sample space.

**Example 1.10** Continuing Example 1.9, let $B_i = \{\text{outcomes with } i \text{ heads}\}$. Each $B_i$ is an event containing one or more outcomes. For example, $B_1 = \{ttth, ttht, thtt, httt\}$ contains four outcomes. The set $B = \{B_0, B_1, B_2, B_3, B_4\}$ is an event space. Its members are mutually exclusive and collectively exhaustive. It is not a sample space because it lacks the finest-grain property. Learning that an experiment produces an event $B_1$ tells you that one coin came up heads, but it doesn't tell you which coin it was.

The experiment in Example 1.9 and Example 1.10 refers to a "toy problem," one that is easily visualized but isn't something we would do in the course of our professional work. Mathematically, however, it is equivalent to many real engineering problems. For example, observe a pair of modems transmitting four bits from one computer to another. For each bit, observe whether the receiving modem detects the bit correctly ($c$), or makes an error ($e$). Or, test four integrated circuits. For each one, observe whether the circuit is acceptable ($a$), or a reject ($r$). In all of these examples, the sample space contains 16 four-letter words formed with an alphabet containing two letters. If we are interested only in the number of times one of the letters occurs, it is sufficient to refer only to the event space $B$, which does not contain all of the information about the experiment but does contain all of the information we need. The event space is simpler to deal with than the sample space because it has fewer members (there are five events in the event space and 16 outcomes in the sample space). The simplification is much more significant when the complexity of the experiment is higher. For example, in testing 20 circuits the sample space has $2^{20} = 1{,}048{,}576$ members, while the corresponding event space has only 21 members.

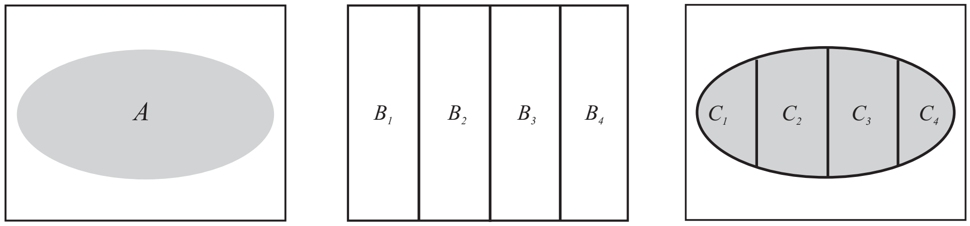



***Figure 1.1*** In this example of Theorem 1.2, the event space is $B = \{B_1, B_2, B_3, B_4\}$ and $C_i = A \cap B_i$ for $i = 1, \ldots, 4$. It should be apparent that $A = C_1 \cup C_2 \cup C_3 \cup C_4$.

The concept of an event space is useful because it allows us to express any event as a union of mutually exclusive events. We will observe in the next section that the entire theory of probability is based on unions of mutually exclusive events. The following theorem shows how to use an event space to represent an event as a union of mutually exclusive events.

***Theorem 1.2*** *For an event space $B = \{B_1, B_2, \ldots\}$ and any event $A$ in the sample space, let $C_i = A \cap B_i$. For $i \neq j$, the events $C_i$ and $C_j$ are mutually exclusive and*

$$A = C_1 \cup C_2 \cup \cdots .$$

Figure 1.1 is a picture of Theorem 1.2.

**Example 1.11** In the coin-tossing experiment of Example 1.9, let $A$ equal the set of outcomes with less than three heads:

$$A = \{tttt, httt, thtt, ttht, ttth, hhtt, htht, htth, tthh, thth, thht\}. \tag{1.14}$$

From Example 1.10, let $B_i = \{\text{outcomes with } i \text{ heads}\}$. Since $\{B_0, \ldots, B_4\}$ is an event space, Theorem 1.2 states that

$$A = (A \cap B_0) \cup (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup (A \cap B_4) \tag{1.15}$$

In this example, $B_i \subset A$, for $i = 0, 1, 2$. Therefore $A \cap B_i = B_i$ for $i = 0, 1, 2$. Also, for $i = 3$ and $i = 4$, $A \cap B_i = \phi$ so that $A = B_0 \cup B_1 \cup B_2$, a union of disjoint sets. In words, this example states that the event "less than three heads" is the union of events "zero heads," "one head," and "two heads."

We advise you to make sure you understand Theorem 1.2 and Example 1.11. Many practical problems use the mathematical technique contained in the theorem. For example, find the probability that there are three or more bad circuits in a batch that comes from a fabrication machine.

***Quiz 1.2*** *Monitor three consecutive phone calls going through a telephone switching office. Classify each one as a voice call ($v$) if someone is speaking, or a data call ($d$) if the call is carrying a modem or fax signal. Your observation is a sequence of three letters (each letter is either*

*$v$ or $d$). For example, two voice calls followed by one data call corresponds to $vvd$. Write the elements of the following sets:*

*(1) $A_1 = \{$first call is a voice call$\}$* | *(2) $B_1 = \{$first call is a data call$\}$*

*(3) $A_2 = \{$second call is a voice call$\}$* | *(4) $B_2 = \{$second call is a data call$\}$*

*(5) $A_3 = \{$all calls are the same$\}$* | *(6) $B_3 = \{$voice and data alternate$\}$*

*(7) $A_4 = \{$one or more voice calls$\}$* | *(8) $B_4 = \{$two or more data calls$\}$*

*For each pair of events $A_1$ and $B_1$, $A_2$ and $B_2$, and so on, identify whether the pair of events is either mutually exclusive or collectively exhaustive or both.*

## 1.3 Probability Axioms

Thus far our model of an experiment consists of a procedure and observations. This leads to a set-theory representation with a sample space (universal set $S$), outcomes ($s$ that are elements of $S$), and events ($A$ that are sets of elements). To complete the model, we assign a probability $P[A]$ to every event, $A$, in the sample space. With respect to our physical idea of the experiment, the probability of an event is the proportion of the time that event is observed in a large number of runs of the experiment. This is the *relative frequency* notion of probability. Mathematically, this is expressed in the following axioms.

***Definition 1.5*** ***Axioms of Probability***

*A probability measure $P[\cdot]$ is a function that maps events in the sample space to real numbers such that*

**Axiom 1** *For any event $A$, $P[A] \geq 0$.*

**Axiom 2** $P[S] = 1$.

**Axiom 3** *For any countable collection $A_1, A_2, \ldots$ of mutually exclusive events*

$$P[A_1 \cup A_2 \cup \cdots] = P[A_1] + P[A_2] + \cdots .$$

We will build our entire theory of probability on these three axioms. Axioms 1 and 2 simply establish a probability as a number between 0 and 1. Axiom 3 states that the probability of the union of mutually exclusive events is the sum of the individual probabilities. We will use this axiom over and over in developing the theory of probability and in solving problems. In fact, it is really all we have to work with. Everything else follows from Axiom 3. To use Axiom 3 to solve a practical problem, we refer to Theorem 1.2 to analyze a complicated event in order to express it as the union of mutually exclusive events whose probabilities we can calculate. Then, we add the probabilities of the mutually exclusive events to find the probability of the complicated event we are interested in.

A useful extension of Axiom 3 applies to the union of two disjoint events.

***Theorem 1.3*** *For mutually exclusive events $A_1$ and $A_2$,*

$$P[A_1 \cup A_2] = P[A_1] + P[A_2].$$

Although it may appear that Theorem 1.3 is a trivial special case of Axiom 3, this is not so. In fact, a simple proof of Theorem 1.3 may also use Axiom 2! If you are curious, Problem 1.4.8 gives the first steps toward a proof. It is a simple matter to extend Theorem 1.3 to any finite union of mutually exclusive sets.

***Theorem 1.4*** *If $A = A_1 \cup A_2 \cup \cdots \cup A_m$ and $A_i \cap A_j = \phi$ for $i \neq j$, then*

$$P[A] = \sum_{i=1}^{m} P[A_i].$$

In Chapter 7, we show that the probability measure established by the axioms corresponds to the idea of relative frequency. The correspondence refers to a sequential experiment consisting of $n$ repetitions of the basic experiment. We refer to each repetition of the experiment as a *trial*. In these $n$ trials, $N_A(n)$ is the number of times that event $A$ occurs. The relative frequency of $A$ is the fraction $N_A(n)/n$. Theorem 7.9 proves that $\lim_{n\to\infty} N_A(n)/n = P[A]$.

Another consequence of the axioms can be expressed as the following theorem:

***Theorem 1.5*** *The probability of an event $B = \{s_1, s_2, \ldots, s_m\}$ is the sum of the probabilities of the outcomes contained in the event:*

$$P[B] = \sum_{i=1}^{m} P[\{s_i\}].$$

***Proof*** Each outcome $s_i$ is an event (a set) with the single element $s_i$. Since outcomes by definition are mutually exclusive, $B$ can be expressed as the union of $m$ disjoint sets:

$$B = \{s_1\} \cup \{s_2\} \cup \cdots \cup \{s_m\} \tag{1.16}$$

with $\{s_i\} \cap \{s_j\} = \phi$ for $i \neq j$. Applying Theorem 1.4 with $B_i = \{s_i\}$ yields

$$P[B] = \sum_{i=1}^{m} P\left[\{s_i\}\right]. \tag{1.17}$$

## Comments on Notation

We use the notation $P[\cdot]$ to indicate the probability of an event. The expression in the square brackets is an event. Within the context of one experiment, $P[A]$ can be viewed as a function that transforms event $A$ to a number between 0 and 1.

Note that $\{s_i\}$ is the formal notation for a set with the single element $s_i$. For convenience, we will sometimes write $P[s_i]$ rather than the more complete $P[\{s_i\}]$ to denote the probability of this outcome.

We will also abbreviate the notation for the probability of the intersection of two events, $P[A \cap B]$. Sometimes we will write it as $P[A, B]$ and sometimes as $P[AB]$. Thus by definition, $P[A \cap B] = P[A, B] = P[AB]$.

**Example 1.12** Let $T_i$ denote the duration (in minutes) of the $i$th phone call you place today. The probability that your first phone call lasts less than five minutes and your second phone call lasts at least ten minutes is $P[T_1 < 5, T_2 \geq 10]$.

## Equally Likely Outcomes

A large number of experiments have a sample space $S = \{s_1, \ldots, s_n\}$ in which our knowledge of the practical situation leads us to believe that no one outcome is any more likely than any other. In these experiments we say that the $n$ outcomes are *equally likely*. In such a case, the axioms of probability imply that every outcome has probability $1/n$.

***Theorem 1.6*** *For an experiment with sample space $S = \{s_1, \ldots, s_n\}$ in which each outcome $s_i$ is equally likely,*

$$P[s_i] = 1/n \qquad 1 \leq i \leq n.$$

***Proof*** Since all outcomes have equal probability, there exists $p$ such that $P[s_i] = p$ for $i = 1, \ldots, n$. Theorem 1.5 implies

$$P[S] = P[s_1] + \cdots + P[s_n] = np. \tag{1.18}$$

Since Axiom 2 says $P[S] = 1$, we must have $p = 1/n$.

**Example 1.13** As in Example 1.6, roll a six-sided die in which all faces are equally likely. What is the probability of each outcome? Find the probabilities of the events: "Roll 4 or higher," "Roll an even number," and "Roll the square of an integer."

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The probability of each outcome is

$$P[i] = 1/6 \qquad i = 1, 2, \ldots, 6. \tag{1.19}$$

The probabilities of the three events are

- $P[$Roll 4 or higher$] = P[4] + P[5] + P[6] = 1/2$.
- $P[$Roll an even number$] = P[2] + P[4] + P[6] = 1/2$.
- $P[$Roll the square of an integer$] = P[1] + P[4] = 1/3$.

***Quiz 1.3*** *A student's test score $T$ is an integer between 0 and 100 corresponding to the experimental outcomes $s_0, \ldots, s_{100}$. A score of 90 to 100 is an $A$, 80 to 89 is a $B$, 70 to 79 is a $C$, 60 to 69 is a $D$, and below 60 is a failing grade of $F$. Given that all scores between 51 and 100 are equally likely and a score of 50 or less never occurs, find the following probabilities:*

(1) $P[\{s_{79}\}]$
(2) $P[\{s_{100}\}]$
(3) $P[A]$
(4) $P[F]$
(5) $P[T \geq 80]$
(6) $P[T < 90]$
(7) $P[\text{a } C \text{ grade or better}]$
(8) $P[\text{student passes}]$

## 1.4 Some Consequences of the Axioms

Here we list some properties of probabilities that follow directly from the three axioms. While we do not supply the proofs, we suggest that students prove at least some of these theorems in order to gain experience working with the axioms.

***Theorem 1.7*** *The probability measure $P[\cdot]$ satisfies*

(a) $P[\phi] = 0$.

(b) $P[A^c] = 1 - P[A]$.

(c) *For any $A$ and $B$ (not necessarily disjoint),*

$$P[A \cup B] = P[A] + P[B] - P[A \cap B].$$

(d) *If $A \subset B$, then $P[A] \leq P[B]$.*

The following useful theorem refers to an event space $B_1, B_2, \ldots, B_m$ and any event, $A$. It states that we can find the probability of $A$ by adding the probabilities of the parts of $A$ that are in the separate components of the event space.

***Theorem 1.8*** *For any event $A$, and event space $\{B_1, B_2, \ldots, B_m\}$,*

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i].$$

***Proof*** The proof follows directly from Theorem 1.2 and Theorem 1.4. In this case, the disjoint sets are $C_i = \{A \cap B_i\}$.

Theorem 1.8 is often used when the sample space can be written in the form of a table. In this table, the rows and columns each represent an event space. This method is shown in the following example.

**Example 1.14** A company has a model of telephone usage. It classifies all calls as either long ($l$), if they last more than three minutes, or brief ($b$). It also observes whether calls carry voice ($v$), data ($d$), or fax ($f$). This model implies an experiment in which the procedure is to monitor a call and the observation consists of the type of call, $v$, $d$, or $f$, and the length, $l$ or $b$. The sample space has six outcomes $S = \{lv, bv, ld, bd, lf, bf\}$. In this problem, each call is classifed in two ways: by length and by type. Using $L$ for the event that a call is long and $B$ for the event that a call is brief, $\{L, B\}$ is an event space. Similarly, the voice ($V$), data ($D$) and fax ($F$) classification is an event space $\{V, D, F\}$. The sample space can be represented by a table in which the rows and columns are labeled by events and the intersection of each row and column event contains a single outcome. The corresponding table entry is the probability of that outcome. In this case, the table is

| | $V$ | $D$ | $F$ |
|---|---|---|---|
| $L$ | 0.3 | 0.12 | 0.15 |
| $B$ | 0.2 | 0.08 | 0.15 |

(1.20)

For example, from the table we can read that the probability of a brief data call is $P[bd] = P[BD] = 0.08$. Note that $\{V, D, F\}$ is an event space corresponding to $\{B_1, B_2, B_3\}$ in Theorem 1.8. Thus we can apply Theorem 1.8 to find the probability of a long call:

$$P[L] = P[LV] + P[LD] + P[LF] = 0.57. \tag{1.21}$$

***Quiz 1.4*** *Monitor a phone call. Classify the call as a voice call ($V$) if someone is speaking, or a data call ($D$) if the call is carrying a modem or fax signal. Classify the call as long ($L$) if the call lasts for more than three minutes; otherwise classify the call as brief ($B$). Based on data collected by the telephone company, we use the following probability model: $P[V] = 0.7$, $P[L] = 0.6$, $P[V L] = 0.35$. Find the following probabilities:*

*(1)* $P[DL]$ *(2)* $P[D \cup L]$

*(3)* $P[VB]$ *(4)* $P[V \cup L]$

*(5)* $P[V \cup D]$ *(6)* $P[LB]$

## 1.5 Conditional Probability

As we suggested earlier, it is sometimes useful to interpret $P[A]$ as our knowledge of the occurrence of event $A$ before an experiment takes place. If $P[A] \approx 1$, we have advance knowledge that $A$ will almost certainly occur. $P[A] \approx 0$ reflects strong knowledge that $A$ is unlikely to occur when the experiment takes place. With $P[A] \approx 1/2$, we have little knowledge about whether or not $A$ will occur. Thus $P[A]$ reflects our knowledge of the occurrence of $A$ *prior* to performing an experiment. Sometimes, we refer to $P[A]$ as the *a priori probability*, or the *prior probability*, of $A$.

In many practical situations, it is not possible to find out the precise outcome of an experiment. Rather than the outcome $s_i$, itself, we obtain information that the outcome is in the set $B$. That is, we learn that some event $B$ has occurred, where $B$ consists of several outcomes. Conditional probability describes our knowledge of $A$ when we know that $B$ has occurred but we still don't know the precise outcome. The notation for this new probability is $P[A|B]$. We read this as "the probability of $A$ given $B$." Before going to the mathematical definition of conditional probability, we provide an example that gives an indication of how conditional probabilities can be used.

**Example 1.15** Consider an experiment that consists of testing two integrated circuits that come from the same silicon wafer, and observing in each case whether a circuit is accepted ($a$) or rejected ($r$). The sample space of the experiment is $S = \{rr, ra, ar, aa\}$. Let $B$ denote the event that the first chip tested is rejected. Mathematically, $B = \{rr, ra\}$. Similarly, let $A = \{rr, ar\}$ denote the event that the second circuit is a failure.

The circuits come from a high-quality production line. Therefore the prior probability $P[A]$ is very low. In advance, we are pretty certain that the second circuit will be accepted. However, some wafers become contaminated by dust, and these wafers have a high proportion of defective chips. Given the knowledge of event $B$ that the first chip was rejected, our knowledge of the quality of the second chip changes. With the event $B$ that the first chip is a reject, the probability $P[A|B]$ that the second chip will also be rejected is higher than the *a priori* probability $P[A]$ because of the likelihood that dust contaminated the entire wafer.

***Definition 1.6*** ***Conditional Probability***

*The conditional probability of the event $A$ given the occurrence of the event $B$ is*

$$P[A|B] = \frac{P[AB]}{P[B]}.$$

Conditional probability is defined only when $P[B] > 0$. In most experiments, $P[B] = 0$ means that it is certain that $B$ never occurs. In this case, it is illogical to speak of the probability of $A$ given that $B$ occurs. Note that $P[A|B]$ is a respectable probability measure relative to a sample space that consists of all the outcomes in $B$. This means that $P[A|B]$ has properties corresponding to the three axioms of probability.

***Theorem 1.9*** *A conditional probability measure $P[A|B]$ has the following properties that correspond to the axioms of probability.*

*Axiom 1:* $P[A|B] \geq 0$.

*Axiom 2:* $P[B|B] = 1$.

*Axiom 3: If $A = A_1 \cup A_2 \cup \cdots$ with $A_i \cap A_j = \phi$ for $i \neq j$, then*

$$P[A|B] = P[A_1|B] + P[A_2|B] + \cdots$$

You should be able to prove these statements using Definition 1.6.

**Example 1.16** With respect to Example 1.15, consider the a priori probability model

$$P[rr] = 0.01, \quad P[ra] = 0.01, \quad P[ar] = 0.01, \quad P[aa] = 0.97. \tag{1.22}$$

Find the probability of $A =$ "second chip rejected" and $B =$ "first chip rejected." Also find the conditional probability that the second chip is a reject given that the first chip is a reject.

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We saw in Example 1.15 that $A$ is the union of two disjoint events (outcomes) $rr$ and $ar$. Therefore, the a priori probability that the second chip is rejected is

$$P[A] = P[rr] + P[ar] = 0.02 \tag{1.23}$$

This is also the a priori probability that the first chip is rejected:

$$P[B] = P[rr] + P[ra] = 0.02. \tag{1.24}$$

The conditional probability of the second chip being rejected given that the first chip is rejected is, by definition, the ratio of $P[AB]$ to $P[B]$, where, in this example,

$$P[AB] = P\left[\text{both rejected}\right] = P[rr] = 0.01 \tag{1.25}$$

Thus

$$P[A|B] = \frac{P[AB]}{P[B]} = 0.01/0.02 = 0.5. \tag{1.26}$$

The information that the first chip is a reject drastically changes our state of knowledge about the second chip. We started with near certainty, $P[A] = 0.02$, that the second chip would not fail and ended with complete uncertainty about the quality of the second chip, $P[A|B] = 0.5$.

**Example 1.17** Shuffle a deck of cards and observe the bottom card. What is the conditional probability that the bottom card is the ace of clubs given that the bottom card is a black card?

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The sample space consists of the 52 cards that can appear on the bottom of the deck. Let $A$ denote the event that the bottom card is the ace of clubs. Since all cards are equally likely to be at the bottom, the probability that a particular card, such as the ace of clubs, is at the bottom is $P[A] = 1/52$. Let $B$ be the event that the bottom card is a black card. The event $B$ occurs if the bottom card is one of the 26 clubs or spades, so that $P[B] = 26/52$. Given $B$, the conditional probability of $A$ is

$$P[A|B] = \frac{P[AB]}{P[B]} = \frac{P[A]}{P[B]} = \frac{1/52}{26/52} = \frac{1}{26}. \tag{1.27}$$

The key step was observing that $AB = A$, because if the bottom card is the ace of clubs, then the bottom card must be a black card. Mathematically, this is an example of the fact that $A \subset B$ implies that $AB = A$.

**Example 1.18** Roll two fair four-sided dice. Let $X_1$ and $X_2$ denote the number of dots that appear on die 1 and die 2, respectively. Let $A$ be the event $X_1 \geq 2$. What is $P[A]$? Let $B$ denote the event $X_2 > X_1$. What is $P[B]$? What is $P[A|B]$?

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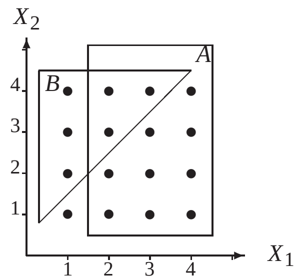


We begin by observing that the sample space has 16 elements corresponding to the four possible values of $X_1$ and the same four values of $X_2$. Since the dice are fair, the outcomes are equally likely, each with probability 1/16. We draw the sample space as a set of black circles in a two-dimensional diagram, in which the axes represent the events $X_1$ and $X_2$. Each outcome is a pair of values $(X_1, X_2)$. The rectangle represents $A$. It contains 12 outcomes, each with probability $1/16$.

To find $P[A]$, we add up the probabilities of outcomes in $A$, so $P[A] = 12/16 = 3/4$. The triangle represents $B$. It contains six outcomes. Therefore $P[B] = 6/16 = 3/8$. The event $AB$ has three outcomes, $(2, 3), (2, 4), (3, 4)$, so $P[AB] = 3/16$. From the definition of conditional probability, we write

$$P[A|B] = \frac{P[AB]}{P[B]} = \frac{1}{2}. \tag{1.28}$$

We can also derive this fact from the diagram by restricting our attention to the six outcomes in $B$ (the conditioning event), and noting that three of the six outcomes in $B$ (one-half of the total) are also in $A$.

## Law of Total Probability

In many situations, we begin with information about conditional probabilities. Using these conditional probabilities, we would like to calculate unconditional probabilities. The law of total probability shows us how to do this.

***Theorem 1.10*** ***Law of Total Probability***
*For an event space* $\{B_1, B_2, \ldots, B_m\}$ *with* $P[B_i] > 0$ *for all* $i$,

$$P[A] = \sum_{i=1}^{m} P[A|B_i]\, P[B_i].$$

***Proof*** This follows from Theorem 1.8 and the identity $P[AB_i] = P[A|B_i]P[B_i]$, which is a direct consequence of the definition of conditional probability.

The usefulness of the result can be seen in the next example.

**Example 1.19** A company has three machines $B_1$, $B_2$, and $B_3$ for making 1 kΩ resistors. It has been observed that 80% of resistors produced by $B_1$ are within 50 Ω of the nominal value. Machine $B_2$ produces 90% of resistors within 50 Ω of the nominal value. The percentage for machine $B_3$ is 60%. Each hour, machine $B_1$ produces 3000 resistors, $B_2$ produces 4000 resistors, and $B_3$ produces 3000 resistors. All of the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships a resistor that is within 50 Ω of the nominal value?

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Let $A = \{$resistor is within 50 $\Omega$ of the nominal value$\}$. Using the resistor accuracy information to formulate a probability model, we write

$$P\left[A|B_1\right] = 0.8, \quad P\left[A|B_2\right] = 0.9, \quad P\left[A|B_3\right] = 0.6 \tag{1.29}$$

The production figures state that $3000 + 4000 + 3000 = 10{,}000$ resistors per hour are produced. The fraction from machine $B_1$ is $P[B_1] = 3000/10{,}000 = 0.3$. Similarly, $P[B_2] = 0.4$ and $P[B_3] = 0.3$. Now it is a simple matter to apply the law of total probability to find the accuracy probability for all resistors shipped by the company:

$$P[A] = P\left[A|B_1\right] P\left[B_1\right] + P\left[A|B_2\right] P\left[B_2\right] + P\left[A|B_3\right] P\left[B_3\right] \tag{1.30}$$

$$= (0.8)(0.3) + (0.9)(0.4) + (0.6)(0.3) = 0.78. \tag{1.31}$$

For the whole factory, 78% of resistors are within 50 $\Omega$ of the nominal value.

## Bayes' Theorem

In many situations, we have advance information about $P[A|B]$ and need to calculate $P[B|A]$. To do so we have the following formula:

***Theorem 1.11*** ***Bayes' theorem***

$$P[B|A] = \frac{P[A|B]\,P[B]}{P[A]}.$$

***Proof***

$$P[B|A] = \frac{P[AB]}{P[A]} = \frac{P[A|B]\,P[B]}{P[A]}. \tag{1.32}$$

Bayes' theorem is a simple consequence of the definition of conditional probability. It has a name because it is extremely useful for making inferences about phenomena that cannot be observed directly. Sometimes these inferences are described as "reasoning about causes when we observe effects." For example, let $\{B_1, \ldots, B_m\}$ be an event space that includes all possible states of something that interests us but which we cannot observe directly (for example, the machine that made a particular resistor). For each possible state, $B_i$, we know the prior probability $P[B_i]$ and $P[A|B_i]$, the probability that an event $A$ occurs (the resistor meets a quality criterion) if $B_i$ is the actual state. Now we observe the actual event (either the resistor passes or fails a test), and we ask about the thing we are interested in (the machines that might have produced the resistor). That is, we use Bayes' theorem to find $P[B_1|A], P[B_2|A], \ldots, P[B_m|A]$. In performing the calculations, we use the law of total probability to calculate the denominator in Theorem 1.11. Thus for state $B_i$,

$$P[B_i|A] = \frac{P[A|B_i]\,P[B_i]}{\sum_{i=1}^{m} P[A|B_i]\,P[B_i]}. \tag{1.33}$$

**Example 1.20** In Example 1.19 about a shipment of resistors from the factory, we learned that:

- The probability that a resistor is from machine $B_3$ is $P[B_3] = 0.3$.
- The probability that a resistor is *acceptable*, i.e., within 50 Ω of the nominal value, is $P[A] = 0.78$.
- Given that a resistor is from machine $B_3$, the conditional probability that it is acceptable is $P[A|B_3] = 0.6$.

What is the probability that an acceptable resistor comes from machine $B_3$?

Now we are given the event $A$ that a resistor is within 50 Ω of the nominal value, and we need to find $P[B_3|A]$. Using Bayes' theorem, we have

$$P\left[B_3|A\right] = \frac{P\left[A|B_3\right] P\left[B_3\right]}{P\left[A\right]}. \tag{1.34}$$

Since all of the quantities we need are given in the problem description, our answer is

$$P\left[B_3|A\right] = (0.6)(0.3)/(0.78) = 0.23. \tag{1.35}$$

Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$. Of all resistors within 50 Ω of the nominal value, only 23% come from machine $B_3$ (even though this machine produces 30% of all resistors). Machine $B_1$ produces 31% of the resistors that meet the 50 Ω criterion and machine $B_2$ produces 46% of them.

***Quiz 1.5*** *Monitor three consecutive phone calls going through a telephone switching office. Classify each one as a voice call* $(v)$ *if someone is speaking, or a data call* $(d)$ *if the call is carrying a modem or fax signal. Your observation is a sequence of three letters (each one is either* $v$ *or* $d$*). For example, three voice calls corresponds to* $vvv$. *The outcomes* $vvv$ *and* $ddd$ *have probability* $0.2$ *whereas each of the other outcomes* $vvd$, $vdv$, $vdd$, $dvv$, $dvd$, *and* $ddv$ *has probability* $0.1$. *Count the number of voice calls* $N_V$ *in the three calls you have observed. Consider the four events* $N_V = 0$, $N_V = 1$, $N_V = 2$, $N_V = 3$. *Describe in words and also calculate the following probabilities:*

*(1)* $P[N_V = 2]$ *(2)* $P[N_V \geq 1]$

*(3)* $P[\{vvd\}|N_V = 2]$ *(4)* $P[\{ddv\}|N_V = 2]$

*(5)* $P[N_V = 2|N_V \geq 1]$ *(6)* $P[N_V \geq 1|N_V = 2]$

## 1.6 Independence

***Definition 1.7*** ***Two Independent Events***
*Events* $A$ *and* $B$ *are* **independent** *if and only if*

$$P\left[AB\right] = P\left[A\right] P\left[B\right].$$

When events $A$ and $B$ have nonzero probabilities, the following formulas are equivalent to the definition of independent events:

$$P[A|B] = P[A], \qquad P[B|A] = P[B]. \tag{1.36}$$

To interpret independence, consider probability as a description of our knowledge of the result of the experiment. $P[A]$ describes our prior knowledge (before the experiment is performed) that the outcome is included in event $A$. The fact that the outcome is in $B$ is partial information about the experiment. $P[A|B]$ reflects our knowledge of $A$ when we learn that $B$ occurs. $P[A|B] = P[A]$ states that learning that $B$ occurs does not change our information about $A$. It is in this sense that the events are independent.

Problem 1.6.7 at the end of the chapter asks the reader to prove that if $A$ and $B$ are independent, then $A$ and $B^c$ are also independent. The logic behind this conclusion is that if learning that event $B$ occurs does not alter the probability of event $A$, then learning that $B$ does not occur also should not alter the probability of $A$.

Keep in mind that **independent and disjoint are *not* synonyms**. In some contexts these words can have similar meanings, but this is not the case in probability. Disjoint events have no outcomes in common and therefore $P[AB] = 0$. In most situations independent events are not disjoint! Exceptions occur only when $P[A] = 0$ or $P[B] = 0$. When we have to calculate probabilities, knowledge that events $A$ and $B$ are *disjoint* is very helpful. Axiom 3 enables us to *add* their probabilities to obtain the probability of the *union*. Knowledge that events $C$ and $D$ are *independent* is also very useful. Definition 1.7 enables us to *multiply* their probabilities to obtain the probability of the *intersection*.

**Example 1.21** Suppose that for the three lights of Example 1.8, each outcome (a sequence of three lights, each either red or green) is equally likely. Are the events $R_2$ that the second light was red and $G_2$ that the second light was green independent? Are the events $R_1$ and $R_2$ independent?

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Each element of the sample space

$$S = \{rrr, rrg, rgr, rgg, grr, grg, ggr, ggg\} \tag{1.37}$$

has probability $1/8$. Each of the events

$$R_2 = \{rrr, rrg, grr, grg\} \quad \text{and} \quad G_2 = \{rgr, rgg, ggr, ggg\} \tag{1.38}$$

contains four outcomes so $P[R_2] = P[G_2] = 4/8$. However, $R_2 \cap G_2 = \phi$ and $P[R_2G_2] = 0$. That is, $R_2$ and $G_2$ must be disjoint because the second light cannot be both red and green. Since $P[R_2G_2] \neq P[R_2]P[G_2]$, $R_2$ and $G_2$ are not independent. Learning whether or not the event $G_2$ (second light green) occurs drastically affects our knowledge of whether or not the event $R_2$ occurs. Each of the events $R_1 = \{rgg, rgr, rrg, rrr\}$ and $R_2 = \{rrg, rrr, grg, grr\}$ has four outcomes so $P[R_1] = P[R_2] = 4/8$. In this case, the intersection $R_1 \cap R_2 = \{rrg, rrr\}$ has probability $P[R_1R_2] = 2/8$. Since $P[R_1R_2] = P[R_1]P[R_2]$, events $R_1$ and $R_2$ are independent. Learning whether or not the event $R_2$ (second light red) occurs does not affect our knowledge of whether or not the event $R_1$ (first light red) occurs.

In this example we have analyzed a probability model to determine whether two events are independent. In many practical applications we reason in the opposite direction. Our knowledge of an experiment leads us to *assume* that certain pairs of events are independent. We then use this knowledge to build a probability model for the experiment.

**Example 1.22** Integrated circuits undergo two tests. A mechanical test determines whether pins have the correct spacing, and an electrical test checks the relationship of outputs to inputs. We *assume* that electrical failures and mechanical failures occur independently. Our information about circuit production tells us that mechanical failures occur with probability 0.05 and electrical failures occur with probability 0.2. What is the probability model of an experiment that consists of testing an integrated circuit and observing the results of the mechanical and electrical tests?

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To build the probability model, we note that the sample space contains four outcomes:

$$S = \{(ma, ea), (ma, er), (mr, ea), (mr, er)\} \tag{1.39}$$

where $m$ denotes mechanical, $e$ denotes electrical, $a$ denotes accept, and $r$ denotes reject. Let $M$ and $E$ denote the events that the mechanical and electrical tests are acceptable. Our prior information tells us that $P[M^c] = 0.05$, and $P[E^c] = 0.2$. This implies $P[M] = 0.95$ and $P[E] = 0.8$. Using the independence assumption and Definition 1.7, we obtain the probabilities of the four outcomes in the sample space as

$$P[(ma, ea)] = P[ME] = P[M]\,P[E] = 0.95 \times 0.8 = 0.76, \tag{1.40}$$

$$P[(ma, er)] = P\left[ME^c\right] = P[M]\,P\left[E^c\right] = 0.95 \times 0.2 = 0.19, \tag{1.41}$$

$$P[(mr, ea)] = P\left[M^cE\right] = P\left[M^c\right]P[E] = 0.05 \times 0.8 = 0.04, \tag{1.42}$$

$$P[(mr, er)] = P\left[M^cE^c\right] = P\left[M^c\right]P\left[E^c\right] = 0.05 \times 0.2 = 0.01. \tag{1.43}$$

Thus far, we have considered independence as a property of a pair of events. Often we consider larger sets of independent events. For more than two events to be *independent*, the probability model has to meet a set of conditions. To define mutual independence, we begin with three sets.

***Definition 1.8*** ***3 Independent Events***

*$A_1$, $A_2$, and $A_3$ are **independent** if and only if*

*(a) $A_1$ and $A_2$ are independent,*

*(b) $A_2$ and $A_3$ are independent,*

*(c) $A_1$ and $A_3$ are independent,*

*(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.*

The final condition is a simple extension of Definition 1.7. The following example shows why this condition is insufficient to guarantee that “everything is independent of everything else,” the idea at the heart of independence.

**Example 1.23** In an experiment with equiprobable outcomes, the event space is $S = \{1, 2, 3, 4\}$. $P[s] = 1/4$ for all $s \in S$. Are the events $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \phi$ independent?

These three sets satisfy the final condition of Definition 1.8 because $A_1 \cap A_2 \cap A_3 = \phi$, and

$$P\left[A_1 \cap A_2 \cap A_3\right] = P\left[A_1\right] P\left[A_2\right] P\left[A_3\right] = 0. \tag{1.44}$$

However, $A_1$ and $A_2$ are not independent because, with all outcomes equiprobable,

$$P\left[A_1 \cap A_2\right] = P\left[\{3, 4\}\right] = 1/2 \neq P\left[A_1\right] P\left[A_2\right] = 3/4 \times 3/4. \tag{1.45}$$

Hence the three events are dependent.

The definition of an arbitrary number of independent events is an extension of Definition 1.8.

***Definition 1.9*** ***More than Two Independent Events***

*If $n \geq 3$, the sets $A_1, A_2, \ldots, A_n$ are independent if and only if*

*(a) every set of $n-1$ sets taken from $A_1, A_2, \ldots A_n$ is independent,*

*(b) $P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2]\cdots P[A_n]$.*

This definition and Example 1.23 show us that when $n > 2$ it is a complex matter to determine whether or not a set of $n$ events is independent. On the other hand, if we know that a set is independent, it is a simple matter to determine the probability of the intersection of any subset of the events. Just multiply the probabilities of the events in the subset.

***Quiz 1.6***

*Monitor two consecutive phone calls going through a telephone switching office. Classify each one as a voice call ($v$) if someone is speaking, or a data call ($d$) if the call is carrying a modem or fax signal. Your observation is a sequence of two letters (either $v$ or $d$). For example, two voice calls corresponds to $vv$. The two calls are independent and the probability that any one of them is a voice call is 0.8. Denote the identity of call $i$ by $C_i$. If call $i$ is a voice call, then $C_i = v$; otherwise, $C_i = d$. Count the number of voice calls in the two calls you have observed. $N_V$ is the number of voice calls. Consider the three events $N_V = 0$, $N_V = 1$, $N_V = 2$. Determine whether the following pairs of events are independent:*

*(1) $\{N_V = 2\}$ and $\{N_V \geq 1\}$*

*(2) $\{N_V \geq 1\}$ and $\{C_1 = v\}$*

*(3) $\{C_2 = v\}$ and $\{C_1 = d\}$*

*(4) $\{C_2 = v\}$ and $\{N_V$ is even$\}$*

## 1.7 Sequential Experiments and Tree Diagrams

Many experiments consist of a sequence of *subexperiments*. The procedure followed for each subexperiment may depend on the results of the previous subexperiments. We often find it useful to use a type of graph referred to as a *tree diagram* to represent the sequence of subexperiments. To do so, we assemble the outcomes of each subexperiment into sets in an

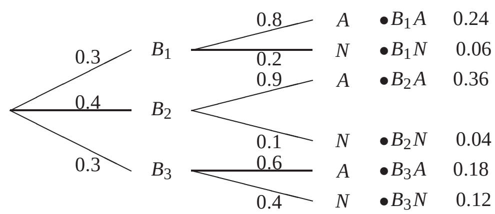



***Figure 1.2*** The sequential tree for Example 1.24.

event space. Starting at the root of the tree, [1] we represent each event in the event space of the first subexperiment as a branch and we label the branch with the probability of the event. Each branch leads to a node. The events in the event space of the second subexperiment appear as branches growing from every node at the end of the first subexperiment. The labels of the branches of the second subexperiment are the *conditional* probabilities of the events in the second subexperiment. We continue the procedure taking the remaining subexperiments in order. The nodes at the end of the final subexperiment are the leaves of the tree. Each leaf corresponds to an outcome of the entire sequential experiment. The probability of each outcome is the product of the probabilities and conditional probabilities on the path from the root to the leaf. We usually label each leaf with a name for the event and the probability of the event.

This is a complicated description of a simple procedure as we see in the following five examples.

**Example 1.24** For the resistors of Example 1.19, we have used $A$ to denote the event that a randomly chosen resistor is "within 50 Ω of the nominal value." This could mean "acceptable." We use the notation $N$ ("not acceptable") for the complement of $A$. The experiment of testing a resistor can be viewed as a two-step procedure. First we identify which machine ($B_1$, $B_2$, or $B_3$) produced the resistor. Second, we find out if the resistor is acceptable. Sketch a sequential tree for this experiment. What is the probability of choosing a resistor from machine $B_2$ that is not acceptable?

This two-step procedure corresponds to the tree shown in Figure 1.2. To use the tree to find the probability of the event $B_2N$, a nonacceptable resistor from machine $B_2$, we start at the left and find that the probability of reaching $B_2$ is $P[B_2] = 0.4$. We then move to the right to $B_2N$ and multiply $P[B_2]$ by $P[N|B_2] = 0.1$ to obtain $P[B_2N] = (0.4)(0.1) = 0.04$.

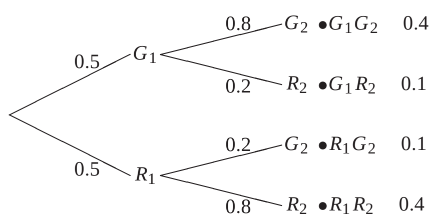
We observe in this example a general property of all tree diagrams that represent sequential experiments. The probabilities on the branches leaving any node add up to 1. This is a consequence of the law of total probability and the property of conditional probabilities that

[1] Unlike biological trees, which grow from the ground up, probabilities usually grow from left to right. Some of them have their roots on top and leaves on the bottom.

corresponds to Axiom 3 (Theorem 1.9). Moreover, Axiom 2 implies that the probabilities of all of the leaves add up to 1.

**Example 1.25** Suppose traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights. In particular, the timing was designed so that with probability $0.8$ a driver will find the second light to have the same color as the first. Assuming the first light is equally likely to be red or green, what is the probability $P[G_2]$ that the second light is green? Also, what is $P[W]$, the probability that you wait for at least one light? Lastly, what is $P[G_1|R_2]$, the conditional probability of a green first light given a red second light?

In the case of the two-light experiment, the complete tree is




The probability that the second light is green is

$$P\left[G_2\right] = P\left[G_1G_2\right] + P\left[R_1G_2\right] = 0.4 + 0.1 = 0.5. \tag{1.46}$$

The event $W$ that you wait for at least one light is

$$W = \{R_1G_2 \cup G_1R_2 \cup R_1R_2\}. \tag{1.47}$$

The probability that you wait for at least one light is

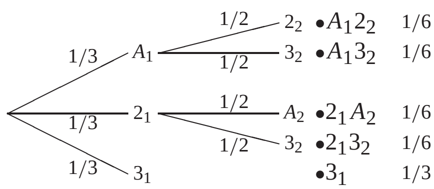
$$P[W] = P\left[R_1G_2\right] + P\left[G_1R_2\right] + P\left[R_1R_2\right] = 0.1 + 0.1 + 0.4 = 0.6. \tag{1.48}$$

To find $P[G_1|R_2]$, we need $P[R_2] = 1 - P[G_2] = 0.5$. Since $P[G_1R_2] = 0.1$, the conditional probability that you have a green first light given a red second light is

$$P\left[G_1|R_2\right] = \frac{P\left[G_1R_2\right]}{P\left[R_2\right]} = \frac{0.1}{0.5} = 0.2. \tag{1.49}$$

**Example 1.26** Consider the game of Three. You shuffle a deck of three cards: ace, 2, 3. With the ace worth 1 point, you draw cards until your total is 3 or more. You win if your total is 3. What is $P[W]$, the probability that you win?

Let $C_i$ denote the event that card $C$ is the $i$th card drawn. For example, $3_2$ is the event that the $3$ was the second card drawn. The tree is




You win if $A_1 2_2$, $2_1 A_2$, or $3_1$ occurs. Hence, the probability that you win is

$$P[W] = P\left[A_1 2_2\right] + P\left[2_1 A_2\right] + P\left[3_1\right] \tag{1.50}$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \frac{1}{3} = \frac{2}{3}. \tag{1.51}$$

**Example 1.27** Suppose you have two coins, one biased, one fair, but you don't know which coin is which. Coin 1 is biased. It comes up heads with probability $3/4$, while coin 2 will flip heads with probability $1/2$. Suppose you pick a coin at random and flip it. Let $C_i$ denote the event that coin $i$ is picked. Let $H$ and $T$ denote the possible outcomes of the flip. Given that the outcome of the flip is a head, what is $P[C_1|H]$, the probability that you picked the biased coin? Given that the outcome is a tail, what is the probability $P[C_1|T]$ that you picked the biased coin?

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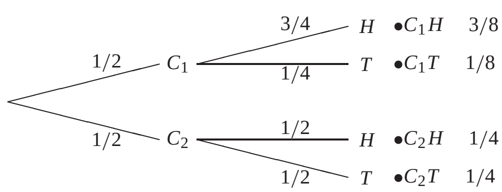
First, we construct the sample tree.




To find the conditional probabilities, we see

$$P\left[C_1|H\right] = \frac{P\left[C_1 H\right]}{P[H]} = \frac{P\left[C_1 H\right]}{P\left[C_1 H\right] + P\left[C_2 H\right]} = \frac{3/8}{3/8 + 1/4} = \frac{3}{5}. \tag{1.52}$$

Similarly,

$$P\left[C_1|T\right] = \frac{P\left[C_1 T\right]}{P[T]} = \frac{P\left[C_1 T\right]}{P\left[C_1 T\right] + P\left[C_2 T\right]} = \frac{1/8}{1/8 + 1/4} = \frac{1}{3}. \tag{1.53}$$

As we would expect, we are more likely to have chosen coin 1 when the first flip is heads, but we are more likely to have chosen coin 2 when the first flip is tails.

***Quiz 1.7*** *In a cellular phone system, a mobile phone must be paged to receive a phone call. However, paging attempts don't always succeed because the mobile phone may not receive the paging* *signal clearly. Consequently, the system will page a phone up to three times before giving up. If a single paging attempt succeeds with probability* 0.8, *sketch a probability tree for this experiment and find the probability* $P[F]$ *that the phone is found.*

## 1.8 Counting Methods

Suppose we have a shuffled full deck and we deal seven cards. What is the probability that we draw no queens? In theory, we can draw a sample space tree for the seven cards drawn. However, the resulting tree is so large that this is impractical. In short, it is too difficult to enumerate all 133 million combinations of seven cards. (In fact, you may wonder if 133 million is even approximately the number of such combinations.) To solve this problem, we need to develop procedures that permit us to count how many seven-card combinations there are and how many of them do not have a queen.

The results we will derive all follow from the fundamental principle of counting.

***Definition 1.10*** ***Fundamental Principle of Counting***

*If subexperiment A has n possible outcomes, and subexperiment B has k possible outcomes, then there are nk possible outcomes when you perform both subexperiments.*

This principle is easily demonstrated by a few examples.

**Example 1.28** There are two subexperiments. The first subexperiment is "Flip a coin." It has two outcomes, $H$ and $T$. The second subexperiment is "Roll a die." It has six outcomes, $1, 2, \ldots, 6$. The experiment, "Flip a coin and roll a die," has $2 \times 6 = 12$ outcomes:

$$\begin{array}{cccccc} (H,1), & (H,2), & (H,3), & (H,4), & (H,5), & (H,6), \\ (T,1), & (T,2), & (T,3), & (T,4), & (T,5), & (T,6). \end{array}$$

Generally, if an experiment $E$ has $k$ subexperiments $E_1, \ldots, E_k$ where $E_i$ has $n_i$ outcomes, then $E$ has $\prod_{i=1}^{k} n_i$ outcomes.

**Example 1.29** Shuffle a deck and observe each card starting from the top. The outcome of the experiment is an ordered sequence of the 52 cards of the deck. How many possible outcomes are there?

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The procedure consists of 52 subexperiments. In each one the observation is the identity of one card. The first subexperiment has 52 possible outcomes corresponding to the 52 cards that could be drawn. After the first card is drawn, the second subexperiment has 51 possible outcomes corresponding to the 51 remaining cards. The total number of outcomes is

$$52 \times 51 \times \cdots \times 1 = 52!\,. \tag{1.54}$$

**Example 1.30** Shuffle the deck and choose three cards in order. How many outcomes are there?

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In this experiment, there are 52 possible outcomes for the first card, 51 for the second card, and 50 for the third card. The total number of outcomes is $52 \times 51 \times 50$.

In Example 1.30, we chose an ordered sequence of three objects out of a set of 52 *distinguishable objects*. In general, an ordered sequence of $k$ distinguishable objects is called a *$k$-permutation*. We will use the notation $(n)_k$ to denote the number of possible $k$-permutations of $n$ distinguishable objects. To find $(n)_k$, suppose we have $n$ distinguishable objects, and the experiment is to choose a sequence of $k$ of these objects. There are $n$ choices for the first object, $n-1$ choices for the second object, etc. Therefore, the total number of possibilities is

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1). \tag{1.55}$$

Multiplying the right side by $(n-k)!/(n-k)!$ yields our next theorem.

***Theorem 1.12*** *The number of $k$-permutations of $n$ distinguishable objects is*

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

### Sampling without Replacement

Choosing objects from a collection is also called *sampling*, and the chosen objects are known as a *sample*. A $k$-permutation is a type of sample obtained by specific rules for selecting objects from the collection. In particular, once we choose an object for a $k$-permutation, we remove the object from the collection and we cannot choose it again. Consequently, this is also called *sampling without replacement*. When an object can be chosen repeatedly, we have *sampling with replacement*, which we examine in the next subsection.

When we choose a $k$-permutation, different outcomes are distinguished by the order in which we choose objects. However, in many practical problems, the order in which the objects are chosen makes no difference. For example, in many card games, only the set of cards received by a player is of interest. The order in which they arrive is irrelevant. Suppose there are four objects, $A$, $B$, $C$, and $D$, and we define an experiment in which the procedure is to choose two objects, arrange them in alphabetical order, and observe the result. In this case, to observe $AD$ we could choose $A$ first or $D$ first or both $A$ and $D$ simultaneously. What we are doing is picking a subset of the collection of objects. Each subset is called a *$k$-combination*. We want to find the number of $k$-combinations.

We will use $\binom{n}{k}$, which is read as "$n$ choose $k$," to denote the number of $k$-combinations of $n$ objects. To find $\binom{n}{k}$, we perform the following two subexperiments to assemble a $k$-permutation of $n$ distinguishable objects:

1. Choose a $k$-combination out of the $n$ objects.
2. Choose a $k$-permutation of the $k$ objects in the $k$-combination.

Theorem 1.12 tells us that the number of outcomes of the combined experiment is $(n)_k$. The first subexperiment has $\binom{n}{k}$ possible outcomes, the number we have to derive. By Theorem 1.12, the second experiment has $(k)_k = k!$ possible outcomes. Since there are $(n)_k$ possible outcomes of the combined experiment,

$$(n)_k = \binom{n}{k} \cdot k! \tag{1.56}$$

Rearranging the terms yields our next result.

***Theorem 1.13*** *The number of ways to choose $k$ objects out of $n$ distinguishable objects is*

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$

We encounter $\binom{n}{k}$ in other mathematical studies. Sometimes it is called a *binomial coefficient* because it appears (as the coefficient of $x^k y^{n-k}$) in the expansion of the binomial $(x+y)^n$. In addition, we observe that

$$\binom{n}{k} = \binom{n}{n-k}. \tag{1.57}$$

The logic behind this identity is that choosing $k$ out of $n$ elements to be part of a subset is equivalent to choosing $n-k$ elements to be excluded from the subset.

In many (perhaps all) other books, $\binom{n}{k}$ is undefined except for integers $n$ and $k$ with $0 \le k \le n$. Instead, we adopt the following extended definition:

***Definition 1.11*** *$n$* ***choose*** *$k$*

*For an integer $n \ge 0$, we define*

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & k = 0, 1, \ldots, n, \\ 0 & \textit{otherwise}. \end{cases}$$

This definition captures the intuition that given, say, $n = 33$ objects, there are zero ways of choosing $k = -5$ objects, zero ways of choosing $k = 8.7$ objects, and zero ways of choosing $k = 87$ objects. Although this extended definition may seem unnecessary, and perhaps even silly, it will make many formulas in later chapters more concise and easier for students to grasp.

**Example 1.31**

- The number of five-card poker hands is

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{2 \cdot 3 \cdot 4 \cdot 5} = 2{,}598{,}960. \tag{1.58}$$

- The number of ways of picking 60 out of 120 students is $\binom{120}{60}$.

- The number of ways of choosing 5 starters for a basketball team with 11 players is $\binom{11}{5} = 462$.
- A baseball team has 15 field players and 10 pitchers. Each field player can take any of the 8 nonpitching positions. Therefore, the number of possible starting lineups is $N = \binom{10}{1}\binom{15}{8} = 64{,}350$ since you must choose 1 of the 10 pitchers and you must choose 8 out of the 15 field players. For each choice of starting lineup, the manager must submit to the umpire a batting order for the 9 starters. The number of possible batting orders is $N \times 9! = 23{,}351{,}328{,}000$ since there are $N$ ways to choose the 9 starters, and for each choice of 9 starters, there are $9! = 362{,}880$ possible batting orders.

**Example 1.32** To return to our original question of this section, suppose we draw seven cards. What is the probability of getting a hand without any queens?

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There are $H = \binom{52}{7}$ possible hands. All $H$ hands have probability $1/H$. There are $H_{NQ} = \binom{48}{7}$ hands that have no queens since we must choose 7 cards from a deck of 48 cards that has no queens. Since all hands are equally likely, the probability of drawing no queens is $H_{NQ}/H = 0.5504$.

## Sampling with Replacement

Now we address sampling with replacement. In this case, each object can be chosen repeatedly because a selected object is replaced by a duplicate.

**Example 1.33** A laptop computer has PCMCIA expansion card slots $A$ and $B$. Each slot can be filled with either a modem card $(m)$, a SCSI interface $(i)$, or a GPS card $(g)$. From the set $\{m, i, g\}$ of possible cards, what is the set of possible ways to fill the two slots when we sample with replacement? In other words, how many ways can we fill the two card slots when we allow both slots to hold the same type of card?

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Let $xy$ denote the outcome that card type $x$ is used in slot $A$ and card type $y$ is used in slot $B$. The possible outcomes are

$$S = \{mm, mi, mg, im, ii, ig, gm, gi, gg\}. \tag{1.59}$$

As we see from $S$, the number of possible outcomes is nine.

The fact that Example 1.33 had nine possible outcomes should not be surprising. Since we were sampling with replacement, there were always three possible outcomes for each of the subexperiments to choose a PCMCIA card. Hence, by the fundamental theorem of counting, Example 1.33 must have $3 \times 3 = 9$ possible outcomes.

In Example 1.33, $mi$ and $im$ are distinct outcomes. This result generalizes naturally when we want to choose with replacement a sample of $n$ objects out of a collection of $m$ distinguishable objects. The experiment consists of a sequence of $n$ identical subexperiments. Sampling with replacement ensures that in each subexperiment, there are $m$ possible outcomes. Hence there are $m^n$ ways to choose with replacement a sample of $n$ objects.

**Theorem 1.14** *Given m distinguishable objects, there are $m^n$ ways to choose with replacement an ordered sample of n objects.*

**Example 1.34** There are $2^{10} = 1024$ binary sequences of length 10.

**Example 1.35** The letters $A$ through $Z$ can produce $26^4 = 456{,}976$ four-letter words.

Sampling with replacement also arises when we perform $n$ repetitions of an identical subexperiment. Each subexperiment has the same sample space $S$. Using $x_i$ to denote the outcome of the $i$th subexperiment, the result for $n$ repetitions of the subexperiment is a sequence $x_1, \ldots, x_n$. Note that each observation $x_i$ is some element $s$ in t“‘he sample space $S$.

**Example 1.36** A chip fabrication facility produces microprocessors. Each microprocessor is tested to determine whether it runs reliably at an acceptable clock speed. A subexperiment to test a microprocessor has sample space $S = \{0, 1\}$ to indicate whether the test was a failure $(0)$ or a success $(1)$. For test $i$, we record $x_i = 0$ or $x_i = 1$ to indicate the result. In testing four microprocessors, the observation sequence $x_1x_2x_3x_4$ is one of 16 possible outcomes:

$$\begin{array}{cccccccc} 0000, & 0001, & 0010, & 0011, & 0100, & 0101, & 0110, & 0111, \\ 1000, & 1001, & 1010, & 1011, & 1100, & 1101, & 1110, & 1111. \end{array}$$

Note that we can think of the observation sequence $x_1, \ldots, x_n$ as having been generated by sampling with replacement $n$ times from a collection $S$. For sequences of identical subexperiments, we can formulate the following restatement of Theorem 1.14.

**Theorem 1.15** *For n repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, there are $m^n$ possible observation sequences.*

**Example 1.37** A chip fabrication facility produces microprocessors. Each microprocessor is tested and assigned a grade $s \in S = \{s_0, \ldots, s_3\}$. A grade of $s_j$ indicates that the microprocessor will function reliably at a maximum clock rate of $s_j$ megahertz (MHz). In testing 10 microprocessors, we use $x_i$ to denote the grade of the $i$th microprocessor tested. Testing 10 microprocessors, for example, may produce an observation sequence

$$x_1x_2 \cdots x_{10} = s_3s_0s_3s_1s_2s_3s_0s_2s_2s_1. \tag{1.60}$$

The entire set of possible sequences contains $4^{10} = 1{,}048{,}576$ elements.

In the preceding examples, repeating a subexperiment $n$ times and recording the observation can be viewed as constructing a word with $n$ symbols from the alphabet $\{s_0, \ldots, s_{m-1}\}$.

For example, for $m = 2$, we have a binary alphabet with symbols $s_0$ and $s_1$ and it is common to simply define $s_0 = 0$ and $s_1 = 1$.

A more challenging problem is to calculate the number of observation sequences such that each subexperiment outcome appears a certain number of times. We start with the case in which each subexperiment is a trial with sample space $S = \{0, 1\}$ indicating failure or success.

**Example 1.38** For five subexperiments with sample space $S = \{0, 1\}$, how many observation sequences are there in which 0 appears $n_0 = 2$ times and 1 appears $n_1 = 3$ times?

The set of five-letter words with 0 appearing twice and 1 appearing three times is

$$\{00111, 01011, 01101, 01110, 10011, 10101, 10110, 11001, 11010, 11100\}.$$

There are exactly 10 such words.

Writing down all 10 sequences of Example 1.38 and making sure that no sequences are overlooked is surprisingly difficult. However, with some additional effort, we will see that it is not so difficult to count the number of such sequences. Each sequence is uniquely determined by the placement of the ones. That is, given five slots for the five subexperiment observations, a possible observation sequence is completely specified by choosing three of the slots to hold a 1. There are exactly $\binom{5}{3} = 10$ such ways to choose those three slots. More generally, for length $n$ binary words with $n_1$ 1's, we must choose $\binom{n}{n_1}$ slots to hold a 1.

***Theorem 1.16*** *The number of observation sequences for $n$ subexperiments with sample space $S = \{0, 1\}$ with 0 appearing $n_0$ times and 1 appearing $n_1 = n - n_0$ times is $\binom{n}{n_1}$.*

Theorem 1.16 can be generalized to subexperiments with $m > 2$ elements in the sample space. For $n$ trials of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, we want to find the number of observation sequences in which $s_0$ appears $n_0$ times, $s_1$ appears $n_1$ times, and so on. Of course, there are no such sequences unless $n_0 + \cdots + n_{m-1} = n$. The number of such words is known as the *multinomial coefficient* and is denoted by

$$\binom{n}{n_0, \ldots, n_{m-1}}.$$

To find the multinomial coefficient, we generalize the logic used in the binary case. Representing the observation sequence by $n$ slots, we first choose $n_0$ slots to hold $s_0$, then $n_1$ slots to hold $s_1$, and so on. The details can be found in the proof of the following theorem:

***Theorem 1.17*** *For $n$ repetitions of a subexperiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, the number of length $n = n_0 + \cdots + n_{m-1}$ observation sequences with $s_i$ appearing $n_i$ times is*

$$\binom{n}{n_0, \ldots, n_{m-1}} = \frac{n!}{n_0! n_1! \cdots n_{m-1}!}.$$

***Proof*** Let $M = \binom{n}{n_0, \ldots, n_{m-1}}$. Start with $n$ empty slots and perform the following sequence of subexperiments:

| Subexperiment | Procedure |
|---|---|
| 0 | Label $n_0$ slots as $s_0$. |
| 1 | Label $n_1$ slots as $s_1$. |
| $\vdots$ | $\vdots$ |
| $m-1$ | Label the remaining $n_{m-1}$ slots as $s_{m-1}$. |

There are $\binom{n}{n_0}$ ways to perform subexperiment 0. After $n_0$ slots have been labeled, there are $\binom{n-n_0}{n_1}$ ways to perform subexperiment 1. After subexperiment $j-1$, $n_0+\cdots+n_{j-1}$ slots have already been filled, leaving $\binom{n-(n_0+\cdots+n_{j-1})}{n_j}$ ways to perform subexperiment $j$. From the fundamental counting principle,

$$M = \binom{n}{n_0}\binom{n-n_0}{n_1}\binom{n-n_0-n_1}{n_2}\cdots\binom{n-n_0-\cdots-n_{m-2}}{n_{m-1}} \tag{1.61}$$

$$= \frac{n!}{(n-n_0)!n_0!}\frac{(n-n_0)!}{(n-n_0-n_1)!n_1!}\cdots\frac{(n-n_0-\cdots-n_{m-2})!}{(n-n_0-\cdots-n_{m-1})!n_{m-1}!}. \tag{1.62}$$

Canceling the common factors, we obtain the formula of the theorem.

Note that a binomial coefficient is the special case of the multinomial coefficient for an alphabet with $m = 2$ symbols. In particular, for $n = n_0 + n_1$,

$$\binom{n}{n_0, n_1} = \binom{n}{n_0} = \binom{n}{n_1}. \tag{1.63}$$

Lastly, in the same way that we extended the definition of the binomial coefficient, we will employ the following extended definition for the multinomial coefficient.

**Definition 1.12** ***Multinomial Coefficient***

*For an integer $n \geq 0$, we define*

$$\binom{n}{n_0,\ldots,n_{m-1}} = \begin{cases} \dfrac{n!}{n_0!n_1!\cdots n_{m-1}!} & n_0+\cdots+n_{m-1}=n;\\ & n_i \in \{0,1,\ldots,n\},\ i=0,1,\ldots,m-1,\\ 0 & \textit{otherwise}.\end{cases}$$

**Quiz 1.8** *Consider a binary code with 4 bits (0 or 1) in each code word. An example of a code word is* 0110.

*(1) How many different code words are there?*

*(2) How many code words have exactly two zeroes?*

*(3) How many code words begin with a zero?*

*(4) In a constant-ratio binary code, each code word has $N$ bits. In every word, $M$ of the $N$ bits are 1 and the other $N - M$ bits are 0. How many different code words are in the code with $N = 8$ and $M = 3$?*

## 1.9 Independent Trials

We now apply the counting methods of Section 1.8 to derive probability models for experiments consisting of independent repetitions of a subexperiment. We start with a simple subexperiment in which there are two outcomes: a success occurs with probability $p$; otherwise, a failure occurs with probability $1-p$. The results of all trials of the subexperiment are mutually independent. An outcome of the complete experiment is a sequence of successes and failures denoted by a sequence of ones and zeroes. For example, $10101\ldots$ is an alternating sequence of successes and failures. Let $S_{n_0,n_1}$ denote the event $n_0$ failures and $n_1$ successes in $n = n_0 + n_1$ trials. To find $P[S_{n_0,n_1}]$, we consider an example.

**Example 1.39** What is the probability $P[S_{2,3}]$ of two failures and three successes in five independent trials with success probability $p$.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

To find $P[S_{2,3}]$, we observe that the outcomes with three successes in five trials are 11100, 11010, 11001, 10110, 10101, 10011, 01110, 01101, 01011, and 00111. We note that the probability of each outcome is a product of five probabilities, each related to one subexperiment. In outcomes with three successes, three of the probabilities are $p$ and the other two are $1-p$. Therefore each outcome with three successes has probability $(1-p)^2p^3$.

From Theorem 1.16, we know that the number of such sequences is $\binom{5}{3}$. To find $P[S_{2,3}]$, we add up the probabilities associated with the 10 outcomes with 3 successes, yielding

$$P\left[S_{2,3}\right] = \binom{5}{3}(1-p)^2p^3. \tag{1.64}$$

In general, for $n = n_0 + n_1$ independent trials we observe that

- Each outcome with $n_0$ failures and $n_1$ successes has probability $(1-p)^{n_0}p^{n_1}$.
- There are $\binom{n}{n_0} = \binom{n}{n_1}$ outcomes that have $n_0$ failures and $n_1$ successes.

Therefore the probability of $n_1$ successes in $n$ independent trials is the sum of $\binom{n}{n_1}$ terms, each with probability $(1-p)^{n_0}p^{n_1} = (1-p)^{n-n_1}p^{n_1}$.

***Theorem 1.18*** *The probability of $n_0$ failures and $n_1$ successes in $n = n_0 + n_1$ independent trials is*

$$P\left[S_{n_0,n_1}\right] = \binom{n}{n_1}(1-p)^{n-n_1}p^{n_1} = \binom{n}{n_0}(1-p)^{n_0}p^{n-n_0}.$$

The second formula in this theorem is the result of multiplying the probability of $n_0$ failures in $n$ trials by the number of outcomes with $n_0$ failures.

**Example 1.40** In Example 1.19, we found that a randomly tested resistor was acceptable with probability $P[A] = 0.78$. If we randomly test 100 resistors, what is the probability of $T_i$, the event that $i$ resistors test acceptable?

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Testing each resistor is an independent trial with a success occurring when a resistor is acceptable. Thus for $0 \le i \le 100$,

$$P\left[T_i\right] = \binom{100}{i}(0.78)^i(1-0.78)^{100-i} \tag{1.65}$$

We note that our intuition says that since 78% of the resistors are acceptable, then in testing 100 resistors, the number acceptable should be near 78. However, $P[T_{78}] \approx 0.096$, which is fairly small. This shows that although we might expect the number acceptable to be close to 78, that does not mean that the probability of exactly 78 acceptable is high.

**Example 1.41** To communicate one bit of information reliably, cellular phones transmit the same binary symbol five times. Thus the information "zero" is transmitted as 00000 and "one" is 11111. The receiver detects the correct information if three or more binary symbols are received correctly. What is the information error probability $P[E]$, if the binary symbol error probability is $q = 0.1$?

In this case, we have five trials corresponding to the five times the binary symbol is sent. On each trial, a success occurs when a binary symbol is received correctly. The probability of a success is $p = 1-q = 0.9$. The error event $E$ occurs when the number of successes is strictly less than three:

$$P[E] = P\left[S_{0,5}\right] + P\left[S_{1,4}\right] + P\left[S_{2,3}\right] \tag{1.66}$$

$$= \binom{5}{0}q^5 + \binom{5}{1}pq^4 + \binom{5}{2}p^2q^3 = 0.00856. \tag{1.67}$$

By increasing the number of binary symbols per information bit from 1 to 5, the cellular phone reduces the probability of error by more than one order of magnitude, from 0.1 to 0.0081.

Now suppose we perform $n$ independent repetitions of a subexperiment for which there are $m$ possible outcomes for any subexperiment. That is, the sample space for each subexperiment is $(s_0, \ldots, s_{m-1})$ and every event in one subexperiment is independent of the events in all the other subexperiments. Therefore, in every subexperiment the probabilities of corresponding events are the same and we can use the notation $P[s_k] = p_k$ for all of the subexperiments.

An outcome of the experiment consists of a sequence of $n$ subexperiment outcomes. In the probability tree of the experiment, each node has $m$ branches and branch $i$ has probability $p_i$. The probability of an experimental outcome is just the product of the branch probabilities encountered on a path from the root of the tree to the leaf representing the outcome. For example, the experimental outcome $s_2s_0s_3s_2s_4$ occurs with probability $p_2p_0p_3p_2p_4$. We want to find the probability of the event

$$S_{n_0,\ldots,n_{m-1}} = \{s_0 \text{ occurs } n_0 \text{ times}, \ldots, s_{m-1} \text{ occurs } n_{m-1} \text{ times}\} \tag{1.68}$$

Note that buried in the notation $S_{n_0,\ldots,n_{m-1}}$ is the implicit fact that there is a sequence of $n = n_0 + \cdots + n_{m-1}$ trials.

To calculate $P[S_{n_0,\ldots,n_{m-1}}]$, we observe that the probability of the outcome

$$\underbrace{s_0 \cdots s_0}_{n_0 \text{ times}}\underbrace{s_1 \cdots s_1}_{n_1 \text{ times}} \cdots \underbrace{s_{m-1} \cdots s_{m-1}}_{n_{m-1} \text{ times}} \tag{1.69}$$

is

$$p_0^{n_0} p_1^{n_1} \cdots p_{m-1}^{n_{m-1}}. \tag{1.70}$$

Next, we observe that any other experimental outcome that is a reordering of the preceding sequence has the same probability because on each path through the tree to such an outcome there are $n_i$ occurrences of $s_i$. As a result,

$$P\left[S_{n_0,\ldots,n_{m-1}}\right] = M p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} \tag{1.71}$$

where $M$, the number of such outcomes, is the multinomial coefficient $\binom{n}{n_0,\ldots,n_{m-1}}$ of Definition 1.12. Applying Theorem 1.17, we have the following theorem:

**Theorem 1.19** *A subexperiment has sample space $S = \{s_0, \ldots, s_{m-1}\}$ with $P[s_i] = p_i$. For $n = n_0 + \cdots + n_{m-1}$ independent trials, the probability of $n_i$ occurences of $s_i$, $i = 0, 1, \ldots, m-1$, is*

$$P\left[S_{n_0,\ldots,n_{m-1}}\right] = \binom{n}{n_0, \ldots, n_{m-1}} p_0^{n_0} \cdots p_{m-1}^{n_{m-1}}.$$

**Example 1.42** Each call arriving at a telephone switch is independently either a voice call with probability $7/10$, a fax call with probability $2/10$, or a modem call with probability $1/10$. Let $S_{v,f,m}$ denote the event that we observe $v$ voice calls, $f$ fax calls, and $m$ modem calls out of 100 observed calls. In this case,

$$P\left[S_{v,f,m}\right] = \binom{100}{v,\ f,\ m} \left(\frac{7}{10}\right)^v \left(\frac{2}{10}\right)^f \left(\frac{1}{10}\right)^m \tag{1.72}$$

Keep in mind that by the extended definition of the multinomial coefficient, $P[S_{v,f,m}]$ is nonzero only if $v$, $f$, and $m$ are nonnegative integers such that $v + f + m = 100$.

**Example 1.43** Continuing with Example 1.37, suppose in testing a microprocessor that all four grades have probability $0.25$, independent of any other microprocessor. In testing $n = 100$ microprocessors, what is the probability of exactly $25$ microprocessors of each grade?

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Let $S_{25,25,25,25}$ denote the probability of exactly 25 microprocessors of each grade. From Theorem 1.19,

$$P\left[S_{25,25,25,25}\right] = \binom{100}{25, 25, 25, 25} (0.25)^{100} = 0.0010. \tag{1.73}$$

**Quiz 1.9** *Data packets containing $100$ bits are transmitted over a communication link. A transmitted bit is received in error (either a 0 sent is mistaken for a 1, or a 1 sent is mistaken for a 0) with probability $\epsilon = 0.01$, independent of the correctness of any other bit. The packet has been coded in such a way that if three or fewer bits are received in error, then those bits can be corrected. If more than three bits are received in error, then the packet is decoded with errors.*

*(1) Let $S_{k,100-k}$ denote the event that a received packet has $k$ bits in error and $100 - k$ correctly decoded bits. What is $P[S_{k,100-k}]$ for $k = 0, 1, 2, 3$?*

*(2) Let $C$ denote the event that a packet is decoded correctly. What is $P[C]$?*

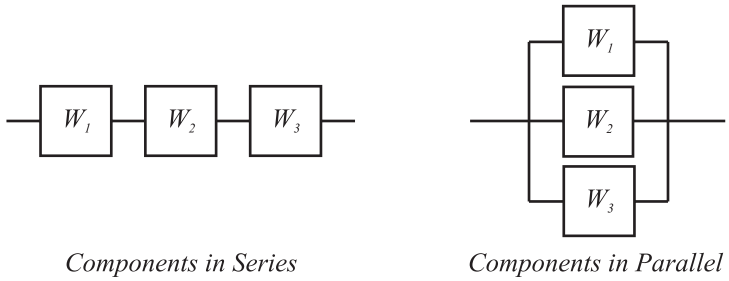


**Figure 1.3** Serial and parallel devices.

## 1.10 Reliability Problems

Independent trials can also be used to describe reliability problems in which we would like to calculate the probability that a particular operation succeeds. The operation consists of $n$ components and each component succeeds with probability $p$, independent of any other component. Let $W_i$ denote the event that component $i$ succeeds. As depicted in Figure 1.3, there are two basic types of operations.

- *Components in series*. The operation succeeds if *all* of its components succeed.

  One example of such an operation is a sequence of computer programs in which each program after the first one uses the result of the previous program. Therefore, the complete operation fails if any component program fails. Whenever the operation consists of $k$ components in series, we need all $k$ components to succeed in order to have a successful operation. The probability that the operation succeeds is

$$P[W] = P[W_1 W_2 \cdots W_n] = p \times p \times \cdots \times p = p^n \tag{1.74}$$

- *Components in parallel*. The operation succeeds if *any* component works.

  This operation occurs when we introduce redundancy to promote reliability. In a redundant system, such as a space shuttle, there are $n$ computers on board so that the shuttle can continue to function as long as at least one computer operates successfully. If the components are in parallel, the operation fails when all elements fail, so we have

$$P\left[W^c\right] = P\left[W_1^c W_2^c \cdots W_n^c\right] = (1-p)^n. \tag{1.75}$$

  The probability that the parallel operation succeeds is

$$P[W] = 1 - P\left[W^c\right] = 1 - (1-p)^n. \tag{1.76}$$

We can analyze complicated combinations of components in series and in parallel by reducing several components in parallel or components in series to a single equivalent component.

**Example 1.44** An operation consists of two redundant parts. The first part has two components in series ($W_1$ and $W_2$) and the second part has two components in series ($W_3$ and $W_4$). All components succeed with probability $p = 0.9$. Draw a diagram of the operation

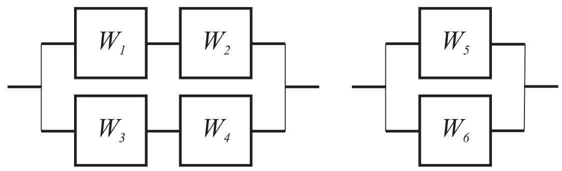


***Figure 1.4*** The operation described in Example 1.44. On the left is the original operation. On the right is the equivalent operation with each pair of series components replaced with an equivalent component.

and calculate the probability that the operation succeeds.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

A diagram of the operation is shown in Figure 1.4. We can create an equivalent component, $W_5$, with probability of success $p_5$ by observing that for the combination of $W_1$ and $W_2$,

$$P\left[W_5\right] = p_5 = P[W_1 W_2] = p^2 = 0.81. \tag{1.77}$$

Similarly, the combination of $W_3$ and $W_4$ in series produces an equivalent component, $W_6$, with probability of success $p_6 = p_5 = 0.81$. The entire operation then consists of $W_5$ and $W_6$ in parallel which is also shown in Figure 1.4. The success probability of the operation is

$$P\left[W\right] = 1 - (1 - p_5)^2 = 0.964 \tag{1.78}$$

We could consider the combination of $W_5$ and $W_6$ to be an equivalent component $W_7$ with success probability $p_7 = 0.964$ and then analyze a more complex operation that contains $W_7$ as a component.

Working on these reliability problems leads us to the observation that in calculating probabilities of events involving independent trials, it is easy to find the probability of an intersection and difficult to find directly the probability of a union. Specifically, for a device with components in series, it is difficult to calculate directly the probability that device fails. Similarly, when the components are in parallel, calculating the probability that the device succeeds is hard. However, De Morgan's law (Theorem 1.1) allows us to express a union as the complement of an intersection and vice versa. Therefore when it is difficult to calculate directly the probability we need, we can often calculate the probability of the complementary event first and then subtract this probability from one to find the answer. This is how we calculated the probability that the parallel device works.

***Quiz 1.10*** *A memory module consists of nine chips. The device is designed with redundancy so that it works even if one of its chips is defective. Each chip contains n transistors and functions properly if all of its transistors work. A transistor works with probability p independent of any other transistor. What is the probability $P[C]$ that a chip works? What is the probability $P[M]$ that the memory module works?*

## 1.11 MATLAB

Engineers have studied and applied probability theory long before the invention of MATLAB. If you don't have access to MATLAB or if you're not interested in MATLAB, feel free to skip this section. You can use this text to learn probability without MATLAB. Nevertheless, MATLAB provides a convenient programming environment for solving probability problems and for building models of probabilistic systems. Versions of MATLAB, including a low cost student edition, are available for most computer systems.

At the end of each chapter, we include a MATLAB section (like this one) that introduces ways that MATLAB can be applied to the concepts and problems of the chapter. We assume you already have some familiarity with the basics of running MATLAB. If you do not, we encourage you to investigate the built-in tutorial, books dedicated to MATLAB, and various Web resources.

MATLAB can be used two ways to study and apply probability theory. Like a sophisticated scientific calculator, it can perform complex numerical calculations and draw graphs. It can also simulate experiments with random outcomes. To simulate experiments, we need a source of randomness. MATLAB uses a computer algorithm, referred to as a *pseudo-random number generator*, to produce a sequence of numbers between 0 and 1. Unless someone knows the algorithm, it is impossible to examine some of the numbers in the sequence and thereby calculate others. The calculation of each random number is similar to an experiment in which all outcomes are equally likely and the sample space is all binary numbers of a certain length. (The length depends on the machine running MATLAB.) Each number is interpreted as a fraction, with a binary point preceding the bits in the binary number. To use the pseudo-random number generator to simulate an experiment that contains an event with probability $p$, we examine one number, $r$, produced by the MATLAB algorithm and say that the event occurs if $r < p$; otherwise it does not occur.

A MATLAB simulation of an experiment starts with the `rand` operator: `rand(m,n)` produces an $m \times n$ array of pseudo-random numbers. Similarly, `rand(n)` produces an $n \times n$ array and `rand(1)` is just a scalar random number. Each number produced by `rand(1)` is in the interval $(0, 1)$. Each time we use `rand`, we get new, unpredictable numbers. Suppose $p$ is a number between 0 and 1. The comparison `rand(1) < p` produces a 1 if the random number is less than $p$; otherwise it produces a zero. Roughly speaking, the function `rand(1) < p` simulates a coin flip with $P[\text{tail}] = p$.

**Example 1.45**

```
» X=rand(1,4)
X =
  0.0879  0.9626  0.6627  0.2023
» X<0.5
ans =
     1     0     0     1
```

Since `rand(1,4)<0.5` compares four random numbers against $0.5$, the result is a random sequence of zeros and ones that simulates a sequence of four flips of a fair coin. We associate the outcome $1$ with $\{\text{head}\}$ and $0$ with $\{\text{tail}\}$.

Because MATLAB can simulate these coin flips much faster than we can actually flip coins, a few lines of MATLAB code can yield quick simulations of many experiments.

**Example 1.46** Using MATLAB, perform 75 experiments. In each experiment, flip a coin 100 times and record the number of heads in a vector $\mathbf{Y}$ such that the $i$th element $Y_i$ is the number of heads in subexperiment $i$.

```
X=rand(75,100)<0.5;
Y=sum(X,2);
```

The MATLAB code for this task appears on the left. The $75 \times 100$ matrix $\mathbf{X}$ has $i, j$th element $X_{ij} = 0$ (tails) or $X_{ij} = 1$ (heads) to indicate the result of flip $j$ of subexperiment $i$.

Since $\mathbf{Y}$ sums $\mathbf{X}$ across the second dimension, $Y_i$ is the number of heads in the $i$th subexperiment. Each $Y_i$ is between 0 and 100 and generally in the neighborhood of 50.

**Example 1.47** Simulate the testing of 100 microprocessors as described in Example 1.43. Your output should be a $4 \times 1$ vector $\mathbf{X}$ such that $X_i$ is the number of grade $i$ microprocessors.

```
%chiptest.m
G=ceil(4*rand(1,100));
T=1:4;
X=hist(G,T);
```

The first line generates a row vector `G` of random grades for 100 microprocessors. The possible test scores are in the vector `T`. Lastly, `X=hist(G,T)` returns a histogram vector `X` such that `X(j)` counts the number of elements `G(i)` that equal `T(j)`.

Note that "`help hist`" will show the variety of ways that the hist function can be called. Morever, `X=hist(G,T)` does more than just count the number of elements of `G` that equal each element of `T`. In particular, `hist(G,T)` creates bins centered around each `T(j)` and counts the number of elements of `G` that fall into each bin.

Note that in MATLAB all variables are assumed to be matrices. In writing MATLAB code, `X` may be an $n \times m$ matrix, an $n \times 1$ column vector, a $1 \times m$ row vector, or a $1 \times 1$ scalar. In MATLAB, we write `X(i,j)` to index the $i, j$th element. By contrast, in this text, we vary the notation depending on whether we have a scalar $X$, or a vector or matrix $\mathbf{X}$. In addition, we use $X_{i,j}$ to denote the $i, j$th element. Thus, $\mathbf{X}$ and `X` (in a MATLAB code fragment) may both refer to the same variable.

***Quiz 1.11*** *The flip of a thick coin yields heads with probability* 0.4*, tails with probability* 0.5*, or lands on its edge with probability* 0.1*. Simulate* 100 *thick coin flips. Your output should be a* $3 \times 1$ *vector* $\mathbf{X}$ *such that* $X_1$, $X_2$, *and* $X_3$ *are the number of occurrences of heads, tails, and edge.*

## Chapter Summary

An experiment consists of a procedure and observations. Outcomes of the experiment are elements of a sample space. A probability model assigns a number to every set in the sample space. Three axioms contain the fundamental properties of probability. The rest of this book uses these axioms to develop methods of working on practical problems.

- *Sample space*, *event*, and *outcome* are probability terms for the set theory concepts of universal set, set, and element.

- A *probability measure* $P[A]$ is a function that assigns a number between 0 and 1 to every event $A$ in a sample space. The assigned probabilities conform to the three axioms presented in Section 1.3.
- A *conditional probability* $P[A|B]$ describes the likelihood of $A$ given that $B$ has occurred. If we consider $B$ to be the sample space, the conditional probability $P[A|B]$ also satisfies the three axioms of probability.
- $A$ and $B$ are *independent events* if and only if $P[AB] = P[A]P[B]$.
- *Tree diagrams* illustrate experiments that consist of a sequence of steps. The labels on the tree branches can be used to calculate the probabilities of outcomes of the combined experiment.
- *Counting methods* determine the number of outcomes of complicated experiments. These methods are particularly useful for sequences of independent trials in which the probability tree is too large to draw.

## Problems

Difficulty: ● Easy ■ Moderate ◆ Difficult ◆◆ Experts Only

**1.1.1** ● For Gerlanda's pizza in Quiz 1.1, answer these questions:

(a) Are $T$ and $M$ mutually exclusive?

(b) Are $R$, $T$, and $M$ collectively exhaustive?

(c) Are $T$ and $O$ mutually exclusive? State this condition in words.

(d) Does Gerlanda's make Tuscan pizzas with mushrooms and onions?

(e) Does Gerlanda's make regular pizzas that have neither mushrooms nor onions?

**1.1.2** ● Continuing Quiz 1.1, write Gerlanda's entire menu in words (supply prices if you wish).

**1.2.1** ● A fax transmission can take place at any of three speeds depending on the condition of the phone connection between the two fax machines. The speeds are high ($h$) at $14,400$ b/s, medium ($m$) at 9600 b/s, and low ($l$) at 4800 b/s. In response to requests for information, a company sends either short faxes of two ($t$) pages, or long faxes of four ($f$) pages. Consider the experiment of monitoring a fax transmission and observing the transmission speed and length. An observation is a two-letter word, for example, a high-speed, two-page fax is $ht$.

(a) What is the sample space of the experiment?

(b) Let $A_1$ be the event "medium-speed fax." What are the outcomes in $A_1$?

(c) Let $A_2$ be the event "short (two-page) fax." What are the outcomes in $A_2$?

(d) Let $A_3$ be the event "high-speed fax or low-speed fax." What are the outcomes in $A_3$?

(e) Are $A_1$, $A_2$, and $A_3$ mutually exclusive?

(f) Are $A_1$, $A_2$, and $A_3$ collectively exhaustive?

**1.2.2** ● An integrated circuit factory has three machines $X$, $Y$, and $Z$. Test one integrated circuit produced by each machine. Either a circuit is acceptable ($a$) or it fails ($f$). An observation is a sequence of three test results corresponding to the circuits from machines $X$, $Y$, and $Z$, respectively. For example, $aaf$ is the observation that the circuits from $X$ and $Y$ pass the test and the circuit from $Z$ fails the test.

(a) What are the elements of the sample space of this experiment?

(b) What are the elements of the sets

$$Z_F = \{\text{circuit from } Z \text{ fails}\},$$
$$X_A = \{\text{circuit from } X \text{ is acceptable}\}.$$

(c) Are $Z_F$ and $X_A$ mutually exclusive?

(d) Are $Z_F$ and $X_A$ collectively exhaustive?

(e) What are the elements of the sets

$$C = \{\text{more than one circuit acceptable}\},$$
$$D = \{\text{at least two circuits fail}\}.$$

(f) Are $C$ and $D$ mutually exclusive?

(g) Are $C$ and $D$ collectively exhaustive?

**1.2.3** Shuffle a deck of cards and turn over the first card. What is the sample space of this experiment? How many outcomes are in the event that the first card is a heart?

**1.2.4** Find out the birthday (month and day but not year) of a randomly chosen person. What is the sample space of the experiment. How many outcomes are in the event that the person is born in July?

**1.2.5** Let the sample space of an experiment consist of all the undergraduates at a university. Give four examples of event spaces.

**1.2.6** Let the sample space of the experiment consist of the measured resistances of two resistors. Give four examples of event spaces.

**1.3.1** Computer programs are classified by the length of the source code and by the execution time. Programs with more than 150 lines in the source code are big ($B$). Programs with $\leq$ 150 lines are little ($L$). Fast programs ($F$) run in less than 0.1 seconds. Slow programs ($W$) require at least 0.1 seconds. Monitor a program executed by a computer. Observe the length of the source code and the run time. The probability model for this experiment contains the following information: $P[LF] = 0.5$, $P[BF] = 0.2$, and $P[BW] = 0.2$. What is the sample space of the experiment? Calculate the following probabilities:

(a) $P[W]$

(b) $P[B]$

(c) $P[W \cup B]$

**1.3.2** There are two types of cellular phones, handheld phones ($H$) that you carry and mobile phones ($M$) that are mounted in vehicles. Phone calls can be classified by the traveling speed of the user as fast ($F$) or slow ($W$). Monitor a cellular phone call and observe the type of telephone and the speed of the user. The probability model for this experiment has the following information: $P[F] = 0.5$, $P[HF] = 0.2$, $P[MW] = 0.1$. What is the sample space of the experiment? Calculate the following probabilities:

(a) $P[W]$

(b) $P[MF]$

(c) $P[H]$

**1.3.3** Shuffle a deck of cards and turn over the first card. What is the probability that the first card is a heart?

**1.3.4** You have a six-sided die that you roll once and observe the number of dots facing upwards. What is the sample space? What is the probability of each sample outcome? What is the probability of $E$, the event that the roll is even?

**1.3.5** A student's score on a 10-point quiz is equally likely to be any integer between 0 and 10. What is the probability of an $A$, which requires the student to get a score of 9 or more? What is the probability the student gets an $F$ by getting less than 4?

**1.4.1** Mobile telephones perform *handoffs* as they move from cell to cell. During a call, a telephone either performs zero handoffs ($H_0$), one handoff ($H_1$), or more than one handoff ($H_2$). In addition, each call is either long ($L$), if it lasts more than three minutes, or brief ($B$). The following table describes the probabilities of the possible types of calls.

| | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $L$ | 0.1 | 0.1 | 0.2 |
| $B$ | 0.4 | 0.1 | 0.1 |

What is the probability $P[H_0]$ that a phone makes no handoffs? What is the probability a call is brief? What is the probability a call is long or there are at least two handoffs?

**1.4.2** For the telephone usage model of Example 1.14, let $B_m$ denote the event that a call is billed for $m$ minutes. To generate a phone bill, observe the duration of the call in integer minutes (rounding up). Charge for $M$ minutes $M = 1, 2, 3, \ldots$ if the exact duration $T$ is $M-1 < t \leq M$. A more complete probability model shows that for $m = 1, 2, \ldots$ the probability of each event $B_m$ is

$$P[B_m] = \alpha(1-\alpha)^{m-1}$$

where $\alpha = 1 - (0.57)^{1/3} = 0.171$.

(a) Classify a call as long, $L$, if the call lasts more than three minutes. What is $P[L]$?

(b) What is the probability that a call will be billed for nine minutes or less?

**1.4.3** The basic rules of genetics were discovered in mid-1800s by Mendel, who found that each characteristic of a pea plant, such as whether the seeds were green or yellow, is determined by two genes, one from each parent. Each gene is either dominant $d$ or recessive $r$. Mendel's experiment is to select a plant and observe whether the genes are both dominant $d$, both recessive $r$, or one of each (hybrid) $h$. In his pea plants, Mendel found that yellow seeds were a dominant trait over green seeds. A $yy$ pea with two yellow genes has yellow seeds; a $gg$ pea with two recessive genes has green seeds; a hybrid $gy$ or $yg$ pea has yellow seeds. In one of Mendel's experiments, he started with a parental generation in which half the pea plants were $yy$ and half the plants were $gg$. The two groups were crossbred so that each pea plant in the first generation was $gy$. In the second generation, each pea plant was equally likely to inherit a $y$ or a $g$ gene from each first generation parent. What is the probability $P[Y]$ that a randomly chosen pea plant in the second generation has yellow seeds?

**1.4.4** Use Theorem 1.7 to prove the following facts:

(a) $P[A \cup B] \geq P[A]$

(b) $P[A \cup B] \geq P[B]$

(c) $P[A \cap B] \leq P[A]$

(d) $P[A \cap B] \leq P[B]$

**1.4.5** Use Theorem 1.7 to prove by induction the *union bound*: For any collection of events $A_1, \ldots, A_n$,

$$P\left[A_1 \cup A_2 \cup \cdots \cup A_n\right] \leq \sum_{i=1}^{n} P\left[A_i\right].$$

**1.4.6** Suppose a cellular telephone is equally likely to make zero handoffs ($H_0$), one handoff ($H_1$), or more than one handoff ($H_2$). Also, a caller is either on foot ($F$) with probability $5/12$ or in a vehicle ($V$).

(a) Given the preceding information, find three ways to fill in the following probability table:

| | $H_0$ | $H_1$ | $H_2$ |
|---|---|---|---|
| $F$ | | | |
| $V$ | | | |

(b) Suppose we also learn that $1/4$ of all callers are on foot making calls with no handoffs and that $1/6$ of all callers are vehicle users making calls with a single handoff. Given these additional facts, find all possible ways to fill in the table of probabilities.

**1.4.7** Using *only* the three axioms of probability, prove $P[\phi] = 0$.

**1.4.8** Using the three axioms of probability and the fact that $P[\phi] = 0$, prove Theorem 1.4. Hint: Define $A_i = B_i$ for $i = 1, \ldots, m$ and $A_i = \phi$ for $i > m$.

**1.4.9** For each fact stated in Theorem 1.7, determine which of the three axioms of probability are needed to prove the fact.

**1.5.1** Given the model of handoffs and call lengths in Problem 1.4.1,

(a) What is the probability that a brief call will have no handoffs?

(b) What is the probability that a call with one handoff will be long?

(c) What is the probability that a long call will have one or more handoffs?

**1.5.2** You have a six-sided die that you roll once. Let $R_i$ denote the event that the roll is $i$. Let $G_j$ denote the event that the roll is greater than $j$. Let $E$ denote the event that the roll of the die is even-numbered.

(a) What is $P[R_3|G_1]$, the conditional probability that 3 is rolled given that the roll is greater than 1?

(b) What is the conditional probability that 6 is rolled given that the roll is greater than 3?

(c) What is $P[G_3|E]$, the conditional probability that the roll is greater than 3 given that the roll is even?

(d) Given that the roll is greater than 3, what is the conditional probability that the roll is even?

**1.5.3** You have a shuffled deck of three cards: 2, 3, and 4. You draw one card. Let $C_i$ denote the event that card $i$ is picked. Let $E$ denote the event that card chosen is a even-numbered card.

(a) What is $P[C_2|E]$, the probability that the 2 is picked given that an even-numbered card is chosen?

(b) What is the conditional probability that an even-numbered card is picked given that the 2 is picked?

**1.5.4** From Problem 1.4.3, what is the conditional probability of $yy$, that a pea plant has two dominant genes given the event $Y$ that it has yellow seeds?

**1.5.5** You have a shuffled deck of three cards: 2, 3, and 4 and you deal out the three cards. Let $E_i$ denote the event that $i$th card dealt is even numbered.

(a) What is $P[E_2|E_1]$, the probability the second card is even given that the first card is even?

(b) What is the conditional probability that the first two cards are even given that the third card is even?

(c) Let $O_i$ represent the event that the $i$th card dealt is odd numbered. What is $P[E_2|O_1]$, the conditional probability that the second card is even given that the first card is odd?

(d) What is the conditional probability that the second card is odd given that the first card is odd?

**1.5.6** Deer ticks can carry both Lyme disease and human granulocytic ehrlichiosis (HGE). In a study of ticks in the Midwest, it was found that 16% carried Lyme disease, 10% had HGE, and that 10% of the ticks that had either Lyme disease or HGE carried both diseases.

(a) What is the probability $P[LH]$ that a tick carries both Lyme disease ($L$) and HGE ($H$)?

(b) What is the conditional probability that a tick has HGE given that it has Lyme disease?

**1.6.1** Is it possible for $A$ and $B$ to be independent events yet satisfy $A = B$?

**1.6.2** Use a Venn diagram in which the event areas are proportional to their probabilities to illustrate two events $A$ and $B$ that are independent.

**1.6.3** In an experiment, $A$, $B$, $C$, and $D$ are events with probabilities $P[A] = 1/4$, $P[B] = 1/8$, $P[C] = 5/8$, and $P[D] = 3/8$. Furthermore, $A$ and $B$ are disjoint, while $C$ and $D$ are independent.

(a) Find $P[A \cap B]$, $P[A \cup B]$, $P[A \cap B^c]$, and $P[A \cup B^c]$.

(b) Are $A$ and $B$ independent?

(c) Find $P[C \cap D]$, $P[C \cap D^c]$, and $P[C^c \cap D^c]$.

(d) Are $C^c$ and $D^c$ independent?

**1.6.4** In an experiment, $A$, $B$, $C$, and $D$ are events with probabilities $P[A \cup B] = 5/8$, $P[A] = 3/8$, $P[C \cap D] = 1/3$, and $P[C] = 1/2$. Furthermore, $A$ and $B$ are disjoint, while $C$ and $D$ are independent.

(a) Find $P[A \cap B]$, $P[B]$, $P[A \cap B^c]$, and $P[A \cup B^c]$.

(b) Are $A$ and $B$ independent?

(c) Find $P[D]$, $P[C \cap D^c]$, $P[C^c \cap D^c]$, and $P[C|D]$.

(d) Find $P[C \cup D]$ and $P[C \cup D^c]$.

(e) Are $C$ and $D^c$ independent?

**1.6.5** In an experiment with equiprobable outcomes, the event space is $S = \{1, 2, 3, 4\}$ and $P[s] = 1/4$ for all $s \in S$. Find three events in $S$ that are pairwise independent but are not independent. (Note: Pairwise independent events meet the first three conditions of Definition 1.8).

**1.6.6** (Continuation of Problem 1.4.3) One of Mendel's most significant results was the conclusion that genes determining different characteristics are transmitted independently. In pea plants, Mendel found that round peas are a dominant trait over wrinkled peas. Mendel crossbred a group of $(rr, yy)$ peas with a group of $(ww, gg)$ peas. In this notation, $rr$ denotes a pea with two "round" genes and $ww$ denotes a pea with two "wrinkled" genes. The first generation were either $(rw, yg)$, $(rw, gy)$, $(wr, yg)$, or $(wr, gy)$ plants with both hybrid shape and hybrid color. Breeding among the first generation yielded second-generation plants in which genes for each characteristic were equally likely to be either dominant or recessive. What is the probability $P[Y]$ that a second-generation pea plant has yellow seeds? What is the probability $P[R]$ that a second-generation plant has round peas? Are $R$ and $Y$ independent events? How many visibly different kinds of pea plants would Mendel observe in the second generation? What are the probabilities of each of these kinds?

**1.6.7** For independent events $A$ and $B$, prove that

(a) $A$ and $B^c$ are independent.

(b) $A^c$ and $B$ are independent.

(c) $A^c$ and $B^c$ are independent.

**1.6.8** Use a Venn diagram in which the event areas are proportional to their probabilities to illustrate three events $A$, $B$, and $C$ that are independent.

**1.6.9** Use a Venn diagram in which the event areas are proportional to their probabilities to illustrate three events $A$, $B$, and $C$ that are pairwise independent but not independent.

**1.7.1** Suppose you flip a coin twice. On any flip, the coin comes up heads with probability $1/4$. Use $H_i$ and $T_i$ to denote the result of flip $i$.

(a) What is the probability, $P[H_1|H_2]$, that the first flip is heads given that the second flip is heads?

(b) What is the probability that the first flip is heads and the second flip is tails?

**1.7.2** For Example 1.25, suppose $P[G_1] = 1/2$, $P[G_2|G_1] = 3/4$, and $P[G_2|R_1] = 1/4$. Find $P[G_2]$, $P[G_2|G_1]$, and $P[G_1|G_2]$.

**1.7.3** At the end of regulation time, a basketball team is trailing by one point and a player goes to the line for two free throws. If the player makes exactly one free throw, the game goes into overtime. The probability that the first free throw is good is $1/2$. However, if the first attempt is good, the player relaxes and the second attempt is good with probability $3/4$. However, if the player misses the first attempt, the added pressure reduces the success probability to $1/4$. What is the probability that the game goes into overtime?

**1.7.4** You have two biased coins. Coin $A$ comes up heads with probability $1/4$. Coin $B$ comes up heads with probability $3/4$. However, you are not sure which is which so you choose a coin randomly and you flip it. If the flip is heads, you guess that the flipped coin is $B$; otherwise, you guess that the flipped coin is $A$. Let events $A$ and $B$ designate which coin was picked. What is the probability $P[C]$ that your guess is correct?

**1.7.5** Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (−) response. Suppose the test gives the correct answer 99% of the time. What is $P[-|H]$, the conditional probability that a person tests negative given that the person does have the HIV virus? What is $P[H|+]$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

**1.7.6** A machine produces photo detectors in pairs. Tests show that the first photo detector is acceptable with probability $3/5$. When the first photo detector is acceptable, the second photo detector is acceptable with probability $4/5$. If the first photo detector is defective, the second photo detector is acceptable with probability $2/5$.

(a) What is the probability that exactly one photo detector of a pair is acceptable?

(b) What is the probability that both photo detectors in a pair are defective?

**1.7.7** You have two biased coins. Coin $A$ comes up heads with probability $1/4$. Coin $B$ comes up heads with probability $3/4$. However, you are not sure which is which so you flip each coin once, choosing the first coin randomly. Use $H_i$ and $T_i$ to denote the result of flip $i$. Let $A_1$ be the event that coin $A$ was flipped first. Let $B_1$ be the event that coin $B$ was flipped first. What is $P[H_1H_2]$? Are $H_1$ and $H_2$ independent? Explain your answer.

**1.7.8** Suppose Dagwood (Blondie's husband) wants to eat a sandwich but needs to go on a diet. So, Dagwood decides to let the flip of a coin determine whether he eats. Using an unbiased coin, Dagwood will postpone the diet (and go directly to the refrigerator) if either (a) he flips heads on his first flip or (b) he flips tails on the first flip but then proceeds to get two heads out of the next three flips. Note that the first flip is *not* counted in the attempt to win two of three and that Dagwood never performs any unnecessary flips. Let $H_i$ be the event that Dagwood flips heads on try $i$. Let $T_i$ be the event that tails occurs on flip $i$.

(a) Sketch the tree for this experiment. Label the probabilities of all outcomes carefully.

(b) What are $P[H_3]$ and $P[T_3]$?

(c) Let $D$ be the event that Dagwood must diet. What is $P[D]$? What is $P[H_1|D]$?

(d) Are $H_3$ and $H_2$ independent events?

**1.7.9** The quality of each pair of photodiodes produced by the machine in Problem 1.7.6 is independent of the quality of every other pair of diodes.

(a) What is the probability of finding no good diodes in a collection of $n$ pairs produced by the machine?

(b) How many pairs of diodes must the machine produce to reach a probability of 0.99 that there will be at least one acceptable diode?

**1.7.10** Each time a fisherman casts his line, a fish is caught with probability $p$, independent of whether a fish is caught on any other cast of the line. The fisherman will fish all day until a fish is caught and then he will quit and go home. Let $C_i$ denote the event that on cast $i$ the fisherman catches a fish. Draw the tree for this experiment and find $P[C_1]$, $P[C_2]$, and $P[C_n]$.

**1.8.1** Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. How many different code words are there? How many code words have exactly three 0's?

**1.8.2** ● Consider a language containing four letters: $A$, $B$, $C$, $D$. How many three-letter words can you form in this language? How many four-letter words can you form if each letter appears only once in each word?

**1.8.3** ■ Shuffle a deck of cards and pick two cards at random. Observe the sequence of the two cards in the order in which they were chosen.

(a) How many outcomes are in the sample space?

(b) How many outcomes are in the event that the two cards are the same type but different suits?

(c) What is the probability that the two cards are the same type but different suits?

(d) Suppose the experiment specifies observing the set of two cards without considering the order in which they are selected, and redo parts (a)–(c).

**1.8.4** ● On an American League baseball team with 15 field players and 10 pitchers, the manager must select for the starting lineup, 8 field players, 1 pitcher, and 1 designated hitter. A starting lineup specifies the players for these positions and the positions in a batting order for the 8 field players and designated hitter. If the designated hitter must be chosen among all the field players, how many possible starting lineups are there?

**1.8.5** ■ Suppose that in Problem 1.8.4, the designated hitter can be chosen from among all the players. How many possible starting lineups are there?

**1.8.6** ■ A basketball team has three pure centers, four pure forwards, four pure guards, and one swingman who can play either guard or forward. A pure position player can play only the designated position. If the coach must start a lineup with one center, two forwards, and two guards, how many possible lineups can the coach choose?

**1.8.7** ♦ An instant lottery ticket consists of a collection of boxes covered with gray wax. For a subset of the boxes, the gray wax hides a special mark. If a player scratches off the correct number of the marked boxes (and no boxes without the mark), then that ticket is a winner. Design an instant lottery game in which a player scratches five boxes and the probability that a ticket is a winner is approximately 0.01.

**1.9.1** ● Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. In each code word, a bit is a zero with probability 0.8, independent of any other bit.

(a) What is the probability of the code word 00111?

(b) What is the probability that a code word contains exactly three ones?

**1.9.2** ● The Boston Celtics have won 16 NBA championships over approximately 50 years. Thus it may seem reasonable to assume that in a given year the Celtics win the title with probability $p = 0.32$, independent of any other year. Given such a model, what would be the probability of the Celtics winning eight straight championships beginning in 1959? Also, what would be the probability of the Celtics winning the title in 10 out of 11 years, starting in 1959? Given your answers, do you trust this simple probability model?

**1.9.3** ● Suppose each day that you drive to work a traffic light that you encounter is either green with probability $7/16$, red with probability $7/16$, or yellow with probability $1/8$, independent of the status of the light on any other day. If over the course of five days, $G$, $Y$, and $R$ denote the number of times the light is found to be green, yellow, or red, respectively, what is the probability that $P[G = 2, Y = 1, R = 2]$? Also, what is the probability $P[G = R]$?

**1.9.4** ■ In a game between two equal teams, the home team wins any game with probability $p > 1/2$. In a best of three playoff series, a team with the home advantage has a game at home, followed by a game away, followed by a home game if necessary. The series is over as soon as one team wins two games. What is $P[H]$, the probability that the team with the home advantage wins the series? Is the home advantage increased by playing a three-game series rather than one-game playoff? That is, is it true that $P[H] \geq p$ for all $p \geq 1/2$?

**1.9.5** ♦ There is a collection of field goal kickers, which can be divided into two groups 1 and 2. Group $i$ has $3i$ kickers. On any kick, a kicker from group $i$ will kick a field goal with probability $1/(i + 1)$, independent of the outcome of any other kicks by that kicker or any other kicker.

(a) A kicker is selected at random from among all the kickers and attempts one field goal. Let $K$ be the event that a field goal is kicked. Find $P[K]$.

(b) Two kickers are selected at random. For $j = 1, 2$, let $K_j$ be the event that kicker $j$ kicks a field goal. Find $P[K_1 \cap K_2]$. Are $K_1$ and $K_2$ independent events?

(c) A kicker is selected at random and attempts 10 field goals. Let $M$ be the number of misses. Find $P[M = 5]$.

**1.10.1** A particular operation has six components. Each component has a failure probability $q$, independent of any other component. The operation is successful if both

- Components 1, 2, and 3 all work, or component 4 works.
- Either component 5 or component 6 works.

Sketch a block diagram for this operation similar to those of Figure 1.3 on page 38. What is the probability $P[W]$ that the operation is successful?

**1.10.2** We wish to modify the cellular telephone coding system in Example 1.41 in order to reduce the number of errors. In particular, if there are two or three zeroes in the received sequence of 5 bits, we will say that a deletion (event $D$) occurs. Otherwise, if at least 4 zeroes are received, then the receiver decides a zero was sent. Similarly, if at least 4 ones are received, then the receiver decides a one was sent. We say that an error occurs if either a one was sent and the receiver decides zero was sent or if a zero was sent and the receiver decides a one was sent. For this modified protocol, what is the probability $P[E]$ of an error? What is the probability $P[D]$ of a deletion?

**1.10.3** Suppose a 10-digit phone number is transmitted by a cellular phone using four binary symbols for each digit, using the model of binary symbol errors and deletions given in Problem 1.10.2. If $C$ denotes the number of bits sent correctly, $D$ the number of deletions, and $E$ the number of errors, what is $P[C = c, D = d, E = e]$? Your answer should be correct for any choice of $c$, $d$, and $e$.

**1.10.4** Consider the device described in Problem 1.10.1. Suppose we can replace any one of the components with an ultrareliable component that has a failure probability of $q/2$. Which component should we replace?

**1.11.1** Following Quiz 1.3, use MATLAB to generate a vector **T** of 200 independent test scores such that all scores between 51 and 100 are equally likely.

**1.11.2** Build a MATLAB simulation of 50 trials of the experiment of Example 1.27. Your output should be a pair of $50 \times 1$ vectors **C** and **H**. For the $i$th trial, $H_i$ will record whether it was heads ($H_i = 1$) or tails ($H_i = 0$), and $C_i \in \{1, 2\}$ will record which coin was picked.

**1.11.3** Following Quiz 1.9, suppose the communication link has different error probabilities for transmitting 0 and 1. When a 1 is sent, it is received as a 0 with probability 0.01. When a 0 is sent, it is received as a 1 with probability 0.03. Each bit in a packet is still equally likely to be a 0 or 1. Packets have been coded such that if five or fewer bits are received in error, then the packet can be decoded. Simulate the transmission of 100 packets, each containing 100 bits. Count the number of packets decoded correctly.

**1.11.4** For a failure probability $q = 0.2$, simulate 100 trials of the six-component test of Problem 1.10.1. How many devices were found to work? Perform 10 repetitions of the 100 trials. Are your results fairly consistent?

**1.11.5** Write a function `N=countequal(G,T)` that duplicates the action of `hist(G,T)` in Example 1.47. Hint: Use the `ndgrid` function.

**1.11.6** In this problem, we use a MATLAB simulation to "solve" Problem 1.10.4. Recall that a particular operation has six components. Each component has a failure probability $q$, independent of any other component. The operation is successful if both

- Components 1, 2, and 3 all work, or component 4 works.
- Either component 5 or component 6 works.

With $q = 0.2$, simulate the replacement of a component with an ultrareliable component. For each replacement of a regular component, perform 100 trials. Are 100 trials sufficient to conclude which component should be replaced?