PROBLEM SET 11.8

1–8 FOURIER COSINE TRANSFORM

1. Find the cosine transform $\hat{f}_c(w)$ of $f(x) = 1$ if $0 < x < 1$, $f(x) = -1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$.

2. Find f in Prob. 1 from the answer $\hat{f}_c$.

3. Find $\hat{f}_c(w)$ for $f(x) = x$ if $0 < x < 2$, $f(x) = 0$ if $x > 2$.

4. Derive formula 3 in Table I of Sec. 11.10 by integration.

5. Find $\hat{f}_c(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.

6. Continuity assumptions. Find $\hat{g}_c(w)$ for $g(x) = 2$ if $0 < x < 1$, $g(x) = 0$ if $x > 1$. Try to obtain from it $\hat{f}_c(w)$ for $f(x)$ in Prob. 5 by using (5a).

7. Existence? Does the Fourier cosine transform of $x^{-1} \sin x$ $(0 < x < \infty)$ exist? Of $x^{-1} \cos x$? Give reasons.

8. Existence? Does the Fourier cosine transform of $f(x) = k = \text{const}$ $(0 < x < \infty)$ exist? The Fourier sine transform?

9–15 FOURIER SINE TRANSFORM

9. Find $\mathcal{F}_s(e^{-ax})$, $a > 0$, by integration.

10. Obtain the answer to Prob. 9 from (5b).

11. Find $f_s(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.

12. Find $\mathcal{F}_s(xe^{-x^2/2})$ from (4b) and a suitable formula in Table I of Sec. 11.10.

13. Find $\mathcal{F}_s(e^{-x})$ from (4a) and formula 3 of Table I in Sec. 11.10.

14. Gamma function. Using formulas 2 and 4 in Table II of Sec. 11.10, prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [(30) in App. A3.1], a value needed for Bessel functions and other applications.

15. WRITING PROJECT. Finding Fourier Cosine and Sine Transforms. Write a short report on ways of obtaining these transforms, with illustrations by examples of your own.

11.9 Fourier Transform. Discrete and Fast Fourier Transforms

In Sec. 11.8 we derived two real transforms. Now we want to derive a complex transform that is called the **Fourier transform**. It will be obtained from the complex Fourier integral, which will be discussed next.

Complex Form of the Fourier Integral

The (real) Fourier integral is [see (4), (5), Sec. 11.7]

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx] \, dw$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv.$$

Substituting A and B into the integral for f, we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(v)[\cos wv \cos wx + \sin wv \sin wx] \, dv \, dw.$$

By the addition formula for the cosine [(6) in App. A3.1] the expression in the brackets $[\cdots]$ equals $\cos(wv - wx)$ or, since the cosine is even, $\cos(wx - wv)$. We thus obtain

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)dv\right]dw. \tag{1*}$$

The integral in brackets is an *even* function of w, call it $F(w)$, because $\cos(wx - wv)$ is an even function of w, the function f does not depend on w, and we integrate with respect to v (not w). Hence the integral of $F(w)$ from $w = 0$ to ∞ is $\frac{1}{2}$ times the integral of $F(w)$ from $-\infty$ to ∞. Thus (note the change of the integration limit!)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\,dv\right]dw. \tag{1}$$

We claim that the integral of the form (1) with sin instead of cos is zero:

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\,dv\right]dw = 0. \tag{2}$$

This is true since $\sin(wx - wv)$ is an odd function of w, which makes the integral in brackets an odd function of w, call it $G(w)$. Hence the integral of $G(w)$ from $-\infty$ to ∞ is zero, as claimed.

We now take the integrand of (1) plus $i\,(=\sqrt{-1})$ times the integrand of (2) and use the **Euler formula** [(11) in Sec. 2.2]

$$e^{ix} = \cos x + i\sin x. \tag{3}$$

Taking $wx - wv$ instead of x in (3) and multiplying by $f(v)$ gives

$$f(v)\cos(wx - wv) + if(v)\sin(wx - wv) = f(v)e^{i(wx-wv)}.$$

Hence the result of adding (1) plus i times (2), called the **complex Fourier integral**, is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^{\infty} f(v)e^{iw(x-v)}\,dv\,dw \qquad (i = \sqrt{-1}). \tag{4}$$

To obtain the desired Fourier transform will take only a very short step from here.

Fourier Transform and Its Inverse

Writing the exponential function in (4) as a product of exponential functions, we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}\,dv\right]e^{iwx}\,dw. \tag{5}$$

The expression in brackets is a function of w, is denoted by $\hat{f}(w)$, and is called the **Fourier transform** of f; writing $v = x$, we have

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx. \tag{6}$$

With this, (5) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw \tag{7}$$

and is called the **inverse Fourier transform** of $\hat{f}(w)$.

Another notation for the Fourier transform is

$$\hat{f} = \mathcal{F}(f),$$

so that

$$f = \mathcal{F}^{-1}(\hat{f}).$$

The process of obtaining the Fourier transform $\mathcal{F}(f) = \hat{f}$ from a given f is also called the **Fourier transform** or the *Fourier transform method*.

Using concepts defined in Secs. 6.1 and 11.7 we now state (without proof) conditions that are sufficient for the existence of the Fourier transform.

THEOREM 1

Existence of the Fourier Transform

If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(w)$ of $f(x)$ given by (6) *exists.*

EXAMPLE 1

Fourier Transform

Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise.

Solution. Using (6) and integrating, we obtain

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \left.\frac{e^{-iwx}}{-iw}\right|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}} (e^{-iw} - e^{iw}).$$

As in (3) we have $e^{iw} = \cos w + i \sin w$, $e^{-iw} = \cos w - i \sin w$, and by subtraction

$$e^{iw} - e^{-iw} = 2i \sin w.$$

Substituting this in the previous formula on the right, we see that i drops out and we obtain the answer

$$\hat{f}(w) = \sqrt{\frac{\pi}{2}} \frac{\sin w}{w}.$$

■

EXAMPLE 2

Fourier Transform

Find the Fourier transform $\mathcal{F}(e^{-ax})$ of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$ if $x < 0$; here $a > 0$.

Solution. From the definition (6) we obtain by integration

$$\mathcal{F}(e^{-ax}) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left.\frac{e^{-(a+iw)x}}{-(a + iw)}\right|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}(a + iw)}.$$

This proves formula 5 of Table III in Sec. 11.10. ■

Physical Interpretation: Spectrum

The nature of the representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, called a **spectral representation**. This name is suggested by optics, where light is such a superposition of colors (frequencies). In (7), the "**spectral density**" $\hat{f}(w)$ measures the intensity of $f(x)$ in the frequency interval between w and $w + \Delta w$ (Δw small, fixed). We claim that, in connection with vibrations, the integral

$$\int_{-\infty}^{\infty} |\hat{f}(w)|^2 \, dw$$

can be interpreted as the **total energy** of the physical system. Hence an integral of $|\hat{f}(w)|^2$ from a to b gives the contribution of the frequencies w between a and b to the total energy.

To make this plausible, we begin with a mechanical system giving a single frequency, namely, the harmonic oscillator (mass on a spring, Sec. 2.4)

$$my'' + ky = 0.$$

Here we denote time t by x. Multiplication by y' gives $my'y'' + ky'y = 0$. By integration,

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2 = E_0 = \text{const}$$

where $v = y'$ is the velocity. The first term is the kinetic energy, the second the potential energy, and E_0 the total energy of the system. Now a general solution is (use (3) in Sec. 11.4 with $t = x$)

$$y = a_1 \cos w_0 x + b_1 \sin w_0 x = c_1 e^{iw_0 x} + c_{-1} e^{-iw_0 x}, \qquad w_0^2 = k/m$$

where $c_1 = (a_1 - ib_1)/2$, $c_{-1} = \overline{c}_1 = (a_1 + ib_1)/2$. We write simply $A = c_1 e^{iw_0 x}$, $B = c_{-1} e^{-iw_0 x}$. Then $y = A + B$. By differentiation, $v = y' = A' + B' = iw_0(A - B)$. Substitution of v and y on the left side of the equation for E_0 gives

$$E_0 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2 = \tfrac{1}{2}m(iw_0)^2(A - B)^2 + \tfrac{1}{2}k(A + B)^2.$$

Here $w_0^2 = k/m$, as just stated; hence $mw_0^2 = k$. Also $i^2 = -1$, so that

$$E_0 = \tfrac{1}{2}k[-(A - B)^2 + (A + B)^2] = 2kAB = 2kc_1 e^{iw_0 x} c_{-1} e^{-iw_0 x} = 2kc_1 c_{-1} = 2k|c_1|^2.$$

Hence *the energy is proportional to the square of the amplitude* $|c_1|$.

As the next step, if a more complicated system leads to a periodic solution $y = f(x)$ that can be represented by a Fourier series, then instead of the single energy term $|c_1|^2$ we get a series of squares $|c_n|^2$ of Fourier coefficients c_n given by (6), Sec. 11.4. In this case we have a "**discrete spectrum**" (or "**point spectrum**") consisting of countably many isolated frequencies (infinitely many, in general), the corresponding $|c_n|^2$ being the contributions to the total energy.

Finally, a system whose solution can be represented by an integral (7) leads to the above integral for the energy, as is plausible from the cases just discussed.

Linearity. Fourier Transform of Derivatives

New transforms can be obtained from given ones by using

THEOREM 2

Linearity of the Fourier Transform

The Fourier transform is a ***linear operation****; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b, the Fourier transform of $af + bg$ exists, and*

$$\mathscr{F}(af + bg) = a\mathscr{F}(f) + b\mathscr{F}(g). \tag{8}$$

PROOF This is true because integration is a linear operation, so that (6) gives

$$\begin{aligned}\mathscr{F}\{af(x) + bg(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x) + bg(x)]e^{-iwx}\,dx \\ &= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}\,dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{-iwx}\,dx \\ &= a\mathscr{F}\{f(x)\} + b\mathscr{F}\{g(x)\}.\end{aligned}$$

In applying the Fourier transform to differential equations, the key property is that differentiation of functions corresponds to multiplication of transforms by iw:

THEOREM 3

Fourier Transform of the Derivative of $f(x)$

Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathscr{F}\{f'(x)\} = iw\mathscr{F}\{f(x)\}. \tag{9}$$

PROOF From the definition of the Fourier transform we have

$$\mathscr{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f'(x)e^{-iwx}\,dx.$$

Integrating by parts, we obtain

$$\mathscr{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-iwx}\Big|_{-\infty}^{\infty} - (-iw)\int_{-\infty}^{\infty}f(x)e^{-iwx}\,dx\right].$$

Since $f(x) \to 0$ as $|x| \to \infty$, the desired result follows, namely,

$$\mathscr{F}\{f'(x)\} = 0 + iw\mathscr{F}\{f(x)\}.$$

Two successive applications of (9) give

$$\mathcal{F}(f'') = iw\mathcal{F}(f') = (iw)^2\mathcal{F}(f).$$

Since $(iw)^2 = -w^2$, we have for the transform of the second derivative of f

$$\mathcal{F}\{f''(x)\} = -w^2\mathcal{F}\{f(x)\}. \tag{10}$$

Similarly for higher derivatives.

An application of (10) to differential equations will be given in Sec. 12.6. For the time being we show how (9) can be used to derive transforms.

EXAMPLE 3 **Application of the Operational Formula (9)**

Find the Fourier transform of xe^{-x^2} from Table III, Sec 11.10.

Solution. We use (9). By formula 9 in Table III

$$\begin{aligned}\mathcal{F}(xe^{-x^2}) &= \mathcal{F}\{-\tfrac{1}{2}(e^{-x^2})'\}\\ &= -\tfrac{1}{2}\mathcal{F}\{(e^{-x^2})'\}\\ &= -\tfrac{1}{2}iw\mathcal{F}(e^{-x^2})\\ &= -\frac{1}{2}iw\frac{1}{\sqrt{2}}e^{-w^2/4}\\ &= -\frac{iw}{2\sqrt{2}}e^{-w^2/4}.\end{aligned}$$

Convolution

The **convolution** $f * g$ of functions f and g is defined by

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\,dp = \int_{-\infty}^{\infty} f(x-p)g(p)\,dp. \tag{11}$$

The purpose is the same as in the case of Laplace transforms (Sec. 6.5): taking the convolution of two functions and then taking the transform of the convolution is the same as multiplying the transforms of these functions (and multiplying them by $\sqrt{2\pi}$):

THEOREM 4 **Convolution Theorem**

Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g). \tag{12}$$

PROOF By the definition,

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x - p)\, dp\, e^{-iwx}\, dx.$$

An interchange of the order of integration gives

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(x - p)\, e^{-iwx}\, dx\, dp.$$

Instead of x we now take $x - p = q$ as a new variable of integration. Then $x = p + q$ and

$$\mathcal{F}(f * g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)\, g(q)\, e^{-iw(p+q)}\, dq\, dp.$$

This double integral can be written as a product of two integrals and gives the desired result

$$\begin{aligned}\mathcal{F}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) e^{-iwp}\, dp \int_{-\infty}^{\infty} g(q)\, e^{-iwq}\, dq \\ &= \frac{1}{\sqrt{2\pi}} [\sqrt{2\pi}\, \mathcal{F}(f)][\sqrt{2\pi}\, \mathcal{F}(g)] = \sqrt{2\pi}\, \mathcal{F}(f)\mathcal{F}(g).\end{aligned}$$ ■

By taking the inverse Fourier transform on both sides of (12), writing $\hat{f} = \mathcal{F}(f)$ and $\hat{g} = \mathcal{F}(g)$ as before, and noting that $\sqrt{2\pi}$ and $1/\sqrt{2\pi}$ in (12) and (7) cancel each other, we obtain

(13)
$$(f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)e^{iwx}\, dw,$$

a formula that will help us in solving partial differential equations (Sec. 12.6).

Discrete Fourier Transform (DFT), Fast Fourier Transform (FFT)

In using Fourier series, Fourier transforms, and trigonometric approximations (Sec. 11.6) we have to assume that a function $f(x)$, to be developed or transformed, is given on some interval, over which we integrate in the Euler formulas, etc. Now very often a function $f(x)$ is given only in terms of values at finitely many points, and one is interested in extending Fourier analysis to this case. The main application of such a "discrete Fourier analysis" concerns large amounts of equally spaced data, as they occur in telecommunication, time series analysis, and various simulation problems. In these situations, dealing with sampled values rather than with functions, we can replace the Fourier transform by the so-called **discrete Fourier transform (DFT)** as follows.

Let $f(x)$ be periodic, for simplicity of period 2π. We assume that N measurements of $f(x)$ are taken over the interval $0 \leqq x \leqq 2\pi$ at regularly spaced points

$$x_k = \frac{2\pi k}{N}, \qquad k = 0, 1, \cdots, N-1. \tag{14}$$

We also say that $f(x)$ is being **sampled** at these points. We now want to determine a **complex trigonometric polynomial**

$$q(x) = \sum_{n=0}^{N-1} c_n e^{inx_k} \tag{15}$$

that **interpolates** $f(x)$ at the nodes (14), that is, $q(x_k) = f(x_k)$, written out, with f_k denoting $f(x_k)$,

$$f_k = f(x_k) = q(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k}, \qquad k = 0, 1, \cdots, N-1. \tag{16}$$

Hence we must determine the coefficients $c_0, \cdots, c_{N-1}$ such that (16) holds. We do this by an idea similar to that in Sec. 11.1 for deriving the Fourier coefficients by using the orthogonality of the trigonometric system. Instead of integrals we now take sums. Namely, we multiply (16) by e^{-imx_k} (note the minus!) and sum over k from 0 to $N-1$. Then we interchange the order of the two summations and insert x_k from (14). This gives

$$\sum_{k=0}^{N-1} f_k e^{-imx_k} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_n e^{i(n-m)x_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N}. \tag{17}$$

Now

$$e^{i(n-m)2\pi k/N} = [e^{i(n-m)2\pi/N}]^k.$$

We donote $[\cdots]$ by r. For $n = m$ we have $r = e^0 = 1$. The sum of *these* terms over k equals N, the number of these terms. For $n \neq m$ we have $r \neq 1$ and by the formula for a geometric sum [(6) in Sec. 15.1 with $q = r$ and $n = N-1$]

$$\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} = 0$$

because $r^N = 1$; indeed, since k, m, and n are integers,

$$r^N = e^{i(n-m)2\pi k} = \cos 2\pi k(n-m) + i \sin 2\pi k(n-m) = 1 + 0 = 1.$$

This shows that the right side of (17) equals $c_m N$. Writing n for m and dividing by N, we thus obtain the desired coefficient formula

$$\boxed{c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-inx_k}} \qquad f_k = f(x_k), \quad n = 0, 1, \cdots, N-1. \tag{18*}$$

Since computation of the c_n (by the fast Fourier transform, below) involves successive halfing of the problem size N, it is practical to drop the factor $1/N$ from c_n and define the

discrete Fourier transform of the given signal $\mathbf{f} = [f_0 \quad \cdots \quad f_{N-1}]^\mathsf{T}$ to be the vector $\hat{\mathbf{f}} = [\hat{f}_0 \quad \cdots \quad \hat{f}_{N-1}]$ with components

$$\textbf{(18)} \qquad \hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}, \qquad f_k = f(x_k), \quad n = 0, \cdots, N-1.$$

This is the frequency spectrum of the signal.

In vector notation, $\hat{\mathbf{f}} = \mathbf{F}_N\mathbf{f}$, where the $N \times N$ **Fourier matrix** $\mathbf{F}_N = [e_{nk}]$ has the entries [given in (18)]

$$(19) \qquad e_{nk} = e^{-inx_k} = e^{-2\pi ink/N} = w^{nk}, \qquad w = w_N = e^{-2\pi i/N},$$

where $n, k = 0, \cdots, N-1$.

EXAMPLE 4 Discrete Fourier Transform (DFT). Sample of $N = 4$ Values

Let $N = 4$ measurements (sample values) be given. Then $w = e^{-2\pi i/N} = e^{-\pi i/2} = -i$ and thus $w^{nk} = (-i)^{nk}$. Let the sample values be, say $\mathbf{f} = [0 \quad 1 \quad 4 \quad 9]^\mathsf{T}$. Then by (18) and (19),

$$(20) \qquad \hat{\mathbf{f}} = \mathbf{F}_4\mathbf{f} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}.$$

From the first matrix in (20) it is easy to infer what $\mathbf{F}_N$ looks like for arbitrary N, which in practice may be 1000 or more, for reasons given below. ■

From the DFT (the frequency spectrum) $\hat{\mathbf{f}} = \mathbf{F}_N\mathbf{f}$ we can recreate the given signal $\hat{\mathbf{f}} = \mathbf{F}_N^{-1}\mathbf{f}$, as we shall now prove. Here $\mathbf{F}_N$ and its complex conjugate $\overline{\mathbf{F}}_N = \dfrac{1}{N}[\overline{w}^{nk}]$ satisfy

$$(21\text{a}) \qquad \overline{\mathbf{F}}_N\mathbf{F}_N = \mathbf{F}_N\overline{\mathbf{F}}_N = N\mathbf{I}$$

where $\mathbf{I}$ is the $N \times N$ unit matrix; hence $\mathbf{F}_N$ has the inverse

$$(21\text{b}) \qquad \mathbf{F}_N^{-1} = \frac{1}{N}\overline{\mathbf{F}}_N.$$

PROOF We prove (21). By the multiplication rule (row times column) the product matrix $\mathbf{G}_N = \overline{\mathbf{F}}_N\mathbf{F}_N = [g_{jk}]$ in (21a) has the entries $g_{jk} = $ *Row* j *of* $\overline{\mathbf{F}}_N$ *times Column* k *of* $\mathbf{F}_N$. That is, writing $W = \overline{w}^j w^k$, we prove that

$$\begin{aligned} g_{jk} &= (\overline{w}^j w^k)^0 + (\overline{w}^j w^k)^1 + \cdots + (\overline{w}^j w^k)^{N-1} \\ &= W^0 + W^1 + \cdots + W^{N-1} = \begin{cases} 0 & \text{if} \quad j \neq k \\ N & \text{if} \quad j = k. \end{cases} \end{aligned}$$

Indeed, when $j = k$, then $\overline{w}^k w^k = (\overline{w}w)^k = (e^{2\pi i/N} e^{-2\pi i/N})^k = 1^k = 1$, so that the sum of *these* N terms equals N; these are the diagonal entries of $\mathbf{G}_N$. Also, when $j \neq k$, then $W \neq 1$ and we have a geometric sum (whose value is given by (6) in Sec. 15.1 with $q = W$ and $n = N - 1$)

$$W^0 + W^1 + \cdots + W^{N-1} = \frac{1 - W^N}{1 - W} = 0$$

because $W^N = (\overline{w}^j w^k)^N = (e^{2\pi i})^j (e^{-2\pi i})^k = 1^j \cdot 1^k = 1$. ∎

We have seen that $\hat{\mathbf{f}}$ is the frequency spectrum of the signal $f(x)$. Thus the components $\hat{f}_n$ of $\hat{\mathbf{f}}$ give a resolution of the 2π-periodic function $f(x)$ into simple (complex) harmonics. Here one should use only n's that are much smaller than $N/2$, to avoid **aliasing**. By this we mean the effect caused by sampling at too few (equally spaced) points, so that, for instance, in a motion picture, rotating wheels appear as rotating too slowly or even in the wrong sense. Hence in applications, N is usually large. But this poses a problem. Eq. (18) requires $O(N)$ operations for any particular n, hence $O(N^2)$ operations for, say, all $n < N/2$. Thus, already for 1000 sample points the straightforward calculation would involve millions of operations. However, this difficulty can be overcome by the so-called **fast Fourier transform (FFT)**, for which codes are readily available (e.g., in Maple). The FFT is a computational method for the DFT that needs only $O(N)\log_2 N$ operations instead of $O(N^2)$. It makes the DFT a practical tool for large N. Here one chooses $N = 2^p$ (p integer) and uses the special form of the Fourier matrix to break down the given problem into smaller problems. For instance, when $N = 1000$, those operations are reduced by a factor $1000/\log_2 1000 \approx 100$.

The breakdown produces two problems of size $M = N/2$. This breakdown is possible because for $N = 2M$ we have in (19)

$$w_N^2 = w_{2M}^2 = (e^{-2\pi i/N})^2 = e^{-4\pi i/(2M)} = e^{-2\pi i/(M)} = w_M.$$

The given vector $\mathbf{f} = [f_0 \quad \cdots \quad f_{N-1}]^\mathsf{T}$ is split into two vectors with M components each, namely, $\mathbf{f}_{\text{ev}} = [f_0 \quad f_2 \quad \cdots \quad f_{N-2}]^\mathsf{T}$ containing the even components of $\mathbf{f}$, and $\mathbf{f}_{\text{od}} = [f_1 \quad f_3 \quad \cdots \quad f_{N-1}]^\mathsf{T}$ containing the odd components of $\mathbf{f}$. For $\mathbf{f}_{\text{ev}}$ and $\mathbf{f}_{\text{od}}$ we determine the DFTs

$$\hat{\mathbf{f}}_{\text{ev}} = [\hat{f}_{\text{ev},0} \quad \hat{f}_{\text{ev},2} \quad \cdots \quad \hat{f}_{\text{ev},N-2}]^\mathsf{T} = \mathbf{F}_M \mathbf{f}_{\text{ev}}$$

and

$$\hat{\mathbf{f}}_{\text{od}} = [\hat{f}_{\text{od},1} \quad \hat{f}_{\text{od},3} \quad \cdots \quad \hat{f}_{\text{od},N-1}]^\mathsf{T} = \mathbf{F}_M \mathbf{f}_{\text{od}}$$

involving the same $M \times M$ matrix $\mathbf{F}_M$. From these vectors we obtain the components of the DFT of the given vector f by the formulas

$$\text{(22)} \qquad \begin{aligned} &\text{(a)} \qquad \hat{f}_n = \hat{f}_{\text{ev},n} + w_N^n \hat{f}_{\text{od},n} && n = 0, \cdots, M-1 \\ &\text{(b)} \quad \hat{f}_{n+M} = \hat{f}_{\text{ev},n} - w_N^n \hat{f}_{\text{od},n} && n = 0, \cdots, M-1. \end{aligned}$$

For $N = 2^p$ this breakdown can be repeated $p - 1$ times in order to finally arrive at $N/2$ problems of size 2 each, so that the number of multiplications is reduced as indicated above.

We show the reduction from $N = 4$ to $M = N/2 = 2$ and then prove (22).

EXAMPLE 5 Fast Fourier Transform (FFT). Sample of $N = 4$ Values

When $N = 4$, then $w = w_N = -i$ as in Example 4 and $M = N/2 = 2$, hence $w = w_M = e^{-2\pi i/2} = e^{-\pi i} = -1$. Consequently,

$$\hat{\mathbf{f}}_{\text{ev}} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_2 \end{bmatrix} = \mathbf{F}_2 \mathbf{f}_{\text{ev}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix}$$

$$\hat{\mathbf{f}}_{\text{od}} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_3 \end{bmatrix} = \mathbf{F}_2 \mathbf{f}_{\text{od}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix}.$$

From this and (22a) we obtain

$$\hat{f}_0 = \hat{f}_{\text{ev},0} + w_N^0 \hat{f}_{\text{od},0} = (f_0 + f_2) + (f_1 + f_3) = f_0 + f_1 + f_2 + f_3$$
$$\hat{f}_1 = \hat{f}_{\text{ev},1} + w_N^1 \hat{f}_{\text{od},1} = (f_0 - f_2) - i(f_1 + f_3) = f_0 - if_1 - f_2 + if_3.$$

Similarly, by (22b),

$$\hat{f}_2 = \hat{f}_{\text{ev},0} - w_N^0 \hat{f}_{\text{od},0} = (f_0 + f_2) - (f_1 + f_3) = f_0 - f_1 + f_2 - f_3$$
$$\hat{f}_3 = \hat{f}_{\text{ev},1} - w_N^1 \hat{f}_{\text{od},1} = (f_0 - f_2) - (-i)(f_1 - f_3) = f_0 + if_1 - f_2 - if_3.$$

This agrees with Example 4, as can be seen by replacing 0, 1, 4, 9 with f_0, f_1, f_2, f_3. ■

We prove (22). From (18) and (19) we have for the components of the DFT

$$\hat{f}_n = \sum_{k=0}^{N-1} w_N^{kn} f_k.$$

Splitting into two sums of $M = N/2$ terms each gives

$$\hat{f}_n = \sum_{k=0}^{M-1} w_N^{2kn} f_{2k} + \sum_{k=0}^{M-1} w_N^{(2k+1)n} f_{2k+1}.$$

We now use $w_N^2 = w_M$ and pull out w_N^n from under the second sum, obtaining

$$\hat{f}_n = \sum_{k=0}^{M-1} w_M^{kn} f_{\text{ev},k} + w_N^n \sum_{k=0}^{M-1} w_M^{kn} f_{\text{od},k}. \tag{23}$$

The two sums are $f_{\text{ev},n}$ and $f_{\text{od},n}$, the components of the "half-size" transforms $\mathbf{Ff}_{\text{ev}}$ and $\mathbf{Ff}_{\text{od}}$.

Formula (22a) is the same as (23). In (22b) we have $n + M$ instead of n. This causes a sign changes in (23), namely $-w_N^n$ before the second sum because

$$w_N^M = e^{-2\pi iM/N} = e^{-2\pi i/2} = e^{-\pi i} = -1.$$

This gives the minus in (22b) and completes the proof. ■

PROBLEM SET 11.9

1. **Review in complex.** Show that $1/i = -i$, $e^{-ix} = \cos x - i \sin x$, $e^{ix} + e^{-ix} = 2 \cos x$, $e^{ix} - e^{-ix} = 2i \sin x$, $e^{ikx} = \cos kx + i \sin kx$.

2–11 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show details.

2. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

5. $f(x) = \begin{cases} e^{x} & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = e^{-|x|} \quad (-\infty < x < \infty)$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

11. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

12–17 USE OF TABLE III IN SEC. 11.10. OTHER METHODS

12. Find $\mathcal{F}(f(x))$ for $f(x) = xe^{-x}$ if $x > 0$, $f(x) = 0$ if $x < 0$, by (9) in the text and formula 5 in Table III (with $a = 1$). *Hint.* Consider xe^{-x} and e^{-x}.

13. Obtain $\mathcal{F}(e^{-x^2/2})$ from Table III.

14. In Table III obtain formula 7 from formula 8.

15. In Table III obtain formula 1 from formula 2.

16. **TEAM PROJECT. Shifting (a)** Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathcal{F}\{f(x - a)\} = e^{-iwa}\mathcal{F}\{f(x)\}$.

    (b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.

    (c) Shifting on the *w*-Axis. Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.

    (d) Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.

17. What could give you the idea to solve Prob. 11 by using the solution of Prob. 9 and formula (9) in the text? Would this work?

18–25 DISCRETE FOURIER TRANSFORM

18. Verify the calculations in Example 4 of the text.

19. Find the transform of a general signal $f = [f_1 \quad f_2 \quad f_3 \quad f_4]^{\mathsf{T}}$ of four values.

20. Find the inverse matrix in Example 4 of the text and use it to recover the given signal.

21. Find the transform (the frequency spectrum) of a general signal of two values $[f_1 \quad f_2]^{\mathsf{T}}$.

22. Recreate the given signal in Prob. 21 from the frequency spectrum obtained.

23. Show that for a signal of eight sample values, $w = e^{-i/4} = (1 - i)/\sqrt{2}$. Check by squaring.

24. Write the Fourier matrix **F** for a sample of eight values explicitly.

25. **CAS Problem.** Calculate the inverse of the 8×8 Fourier matrix. Transform a general sample of eight values and transform it back to the given data.