PROBLEM SET 2.1

REDUCTION OF ORDER is important because it gives a simpler ODE. A general second-order ODE $F(x, y, y', y'') = 0$, linear or not, can be reduced to first order if y does not occur explicitly (Prob. 1) or if x does not occur explicitly (Prob. 2) or if the ODE is homogeneous linear and we know a solution (see the text).

1. **Reduction.** Show that $F(x, y', y'') = 0$ can be reduced to first order in $z = y'$ (from which y follows by integration). Give two examples of your own.
2. **Reduction.** Show that $F(y, y', y'') = 0$ can be reduced to a first-order ODE with y as the independent variable and $y'' = (dz/dy)z$, where $z = y'$; derive this by the chain rule. Give two examples.

3–10 REDUCTION OF ORDER

Reduce to first order and solve, showing each step in detail.

3. $y'' - y' = 0$
4. $2xy'' = 3y'$
5. $yy'' = 3y'^2$
6. $xy'' + 2y' + xy = 0, \quad y_1 = (\cos x)/x$
7. $y'' + y'^3 \cos y = 0$
8. $y'' = 1 + y'^2$
9. $x^2y'' + xy' - 4y = 0, \quad y_1 = x^2$
10. $y'' + (1 + 1/y)y'^2 = 0$

11–14 APPLICATIONS OF REDUCIBLE ODEs

11. **Curve.** Find the curve through the origin in the xy-plane which satisfies $y'' = 2y'$ and whose tangent at the origin has slope 1.
12. **Hanging cable.** It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving $y'' = k\sqrt{1 + y'^2}$, where the constant k depends on the weight. This curve is called *catenary* (from Latin *catena* = the chain). Find and graph $y(x)$, assuming that $k = 1$ and those fixed points are $(-1, 0)$ and $(1, 0)$ in a vertical xy-plane.
13. **Motion.** If, in the motion of a small body on a straight line, the sum of velocity and acceleration equals a positive constant, how will the distance $y(t)$ depend on the initial velocity and position?
14. **Motion.** In a straight-line motion, let the velocity be the reciprocal of the acceleration. Find the distance $y(t)$ for arbitrary initial position and velocity.

15–19 GENERAL SOLUTION. INITIAL VALUE PROBLEM (IVP)

(More in the next set.) **(a)** Verify that the given functions are linearly independent and form a basis of solutions of the given ODE. **(b)** Solve the IVP. Graph or sketch the solution.

15. $y'' + 9y = 0, \quad y(0) = 2; y'(0) = -1, \cos 3x, \sin 3x$
16. $y'' + 2y' + y = 0, \quad y(0) = 2, y'(0) = -1,$ e^{-x}, xe^{-x}
17. $4x^2y'' - 3y = 0, \quad y(1) = -3, \quad y'(1) = 0,$ $x^{3/2}, x^{-1/2}$
18. $x^2y'' - xy' + y = 0, \quad y(1) = 1; y'(1) = 2,$ $x, x \ln(x)$
19. $y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 15,$ $e^{-x}\cos x, e^{-x}\sin x$
20. **CAS PROJECT. Linear Independence.** Write a program for testing linear independence and dependence. Try it out on some of the problems in this and the next problem set and on examples of your own.

PROBLEM SET 2.2

1–15 GENERAL SOLUTION

Find a general solution. Check your answer by substitution. ODEs of this kind have important applications to be discussed in Secs. 2.4, 2.7, and 2.9.

1. $y'' - 0.25y = 0$
2. $y'' + 36y = 0$
3. $y'' + 4y' + 2.5y = 0$
4. $y'' + 4y' + (\pi^2 + 4)y = 0$
5. $y'' + 2\pi y' + \pi^2 y = 0$
6. $10y'' - 32y' + 25.6y = 0$
7. $y'' + 1.25y' = 0$
8. $y'' + y' + 3.25y = 0$
9. $y'' + 1.75y' - 0.5y = 0$
10. $100y'' + 240y' + (196\pi^2 + 144)y = 0$
11. $4y'' - 4y' - 3y = 0$
12. $y'' + 8y' + 15y = 0$
13. $9y'' - 30y' + 25y = 0$
14. $y'' + 2k^2y' + k^4y = 0$
15. $y'' + 0.54y' + (0.0729 + \pi)y = 0$

16–20 FIND AN ODE

$y'' + ay' + by = 0$ for the given basis.

16. $e^{2.6x}, e^{-4.3x}$
17. $e^{-\sqrt{2}x}, xe^{-\sqrt{2}x}$
18. $\cos 2\pi x, \sin 2\pi x$
19. $e^{(-1+i\sqrt{2})x}, e^{(-1-i\sqrt{2}x)}$
20. $e^{-3.1x}\cos 2.1x, e^{-3.1x}\sin 2.1x$

21–30 INITIAL VALUES PROBLEMS

Solve the IVP. Check that your answer satisfies the ODE as well as the initial conditions. Show the details of your work.

21. $y'' + 9y = 0, \quad y(0) = 0.2, \quad y'(0) = -1.5$
22. The ODE in Prob. 4, $y(\frac{1}{2}) = 1, \quad y'(\frac{1}{2}) = -2$
23. $y'' - 3y' - 4y = 0, \quad y(0) = 2, \quad y'(0) = 1$
24. $y'' - 2y' - 3y = 0, \quad y(-1) = e, \quad y'(-1) = -e/4$
25. $y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -2$
26. $y'' - k^2y = 0 \ (k \neq 0), \quad y(0) = 1, \quad y'(0) = 1$

PROBLEM SET 2.3

1–5 APPLICATION OF DIFFERENTIAL OPERATORS

Apply the given operator to the given functions. Show all steps in detail.

1. $D^2 + 2D; \quad \sinh 2x, e^x + e^{-2x}, \sin x$
2. $D - 3I; \quad 3x^2 + 3x, 3e^{3x}, \cos 4x - \sin 4x$
3. $(D - 3I)^2; \quad e^x, xe^x, e^{-x}$
4. $(D + 6I)^2; \quad 6x + \sin 6x, xe^{-6x}$
5. $(D + I)(D - 2I); \quad e^{4x}, xe^{4x}, e^{-2x}$

6–12 GENERAL SOLUTION

Factor as in the text and solve.

6. $(D^2 + 4.00D + 3.36I)y = 0$
7. $(9D^2 - I)y = 0$
8. $(D^2 + 3I)y = 0$
9. $(D^2 - 4.20D + 4.41I)y = 0$
10. $(D^2 + 4.80D + 5.76I)y = 0$
11. $(D^2 - 6D + 6.75D) = 0$
12. $(D^2 + 3.0D + 2.5I)y = 0$

2-4

5. **Springs in parallel.** What are the frequencies of vibration of a body of mass $m = 5$ kg (i) on a spring of modulus $k_1 = 20$ nt/m, (ii) on a spring of modulus $k_2 = 45$ nt/m, (iii) on the two springs in parallel? See Fig. 41.

Fig. 41. Parallel springs (Problem 5)

6. **Spring in series.** If a body hangs on a spring s_1 of modulus $k_1 = 6$, which in turn hangs on a spring s_2 of modulus $k_2 = 8$, what is the modulus k of this combination of springs?

7. **Pendulum.** Find the frequency of oscillation of a pendulum of length L (Fig. 42), neglecting air resistance and the weight of the rod, and assuming θ to be so small that $\sin\theta$ practically equals θ.

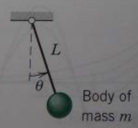

Fig. 42. Pendulum (Problem 7)

8. **Archimedian principle.** This principle states that the buoyancy force equals the weight of the water displaced by the body (partly or totally submerged).

First guess.

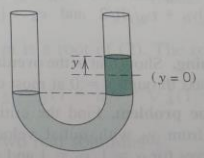

Fig. 44. Tube (Problem 9)

10. **TEAM PROJECT. Harmonic Motions of Similar Models.** The *unifying power of mathematical methods* results to a large extent from the fact that different physical (or other) systems may have the same or very similar models. Illustrate this for the following three systems

(a) **Pendulum clock.** A clock has a 1-meter pendulum. The clock ticks once for each time the pendulum completes a full swing, returning to its original position. How many times a minute does the clock tick?

(b) **Flat spring** (Fig. 45). The harmonic oscillations of a flat spring with a body attached at one end and horizontally clamped at the other are also governed by (3). Find its motions, assuming that the body weighs 8 nt (about 1.8 lb), the system has its static equilibrium 1 cm below the horizontal line, and we let it start from this position with initial velocity 10 cm/sec.

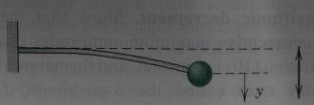

Fig. 45. Flat spring

2-4

(c) **Torsional vibrations** (Fig. 46). Undamped torsional vibrations (rotations back and forth) of a wheel attached to an elastic thin rod or wire are governed by the equation $I_0\theta'' + K\theta = 0$, where θ is the angle measured from the state of equilibrium. Solve this equation for $K/I_0 = 13.69\ \mathrm{sec}^{-2}$, initial angle $30°(= 0.5235$ rad) and initial angular velocity $20°\ \mathrm{sec}^{-1}$ $(= 0.349\ \mathrm{rad} \cdot \mathrm{sec}^{-1})$.

Fig. 46. Torsional vibrations

11–20 DAMPED MOTION

11. **Overdamping.** Show that for (7) to satisfy initial conditions $y(0) = y_0$ and $v(0) = v_0$ we must have $c_1 = [(1 + \alpha/\beta)y_0 + v_0/\beta]/2$ and $c_2 = [(1 - \alpha/\beta)y_0 - v_0/\beta]/2$.

12. **Overdamping.** Show that in the overdamped case, the body can pass through $y = 0$ at most once (Fig. 37).

13. **Initial value problem.** Find the critical motion (8) that starts from y_0 with initial velocity v_0. Graph

equals $\Delta = 2\pi\alpha/\omega^*$. Find Δ for the solutions of $y'' + 4y' + 13y = 0$.

19. **Damping constant.** Consider an underdamped motion of a body of mass $m = 1.5$ kg. If the time between two consecutive maxima is 3 sec and the maximum amplitude decreases to $\frac{1}{2}$ its initial value after 15 cycles, what is the damping constant of the system?

20. **CAS PROJECT. Transition Between Cases I, II, III.** Study this transition in terms of graphs of typical solutions. (Cf. Fig. 47.)

(a) ***Avoiding unnecessary generality is part of good modeling.*** Show that the initial value problems (A) and (B),

(A) $y'' + cy' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$

(B) the same with different c and $y'(0) = -2$ (instead of 0), will give practically as much information as a problem with other m, k, $y(0)$, $y'(0)$.

(b) *Consider* (A). Choose suitable values of c, perhaps better ones than in Fig. 47, for the transition from Case III to II and I. Guess c for the curves in the figure.

(c) *Time to go to rest.* Theoretically, this time is infinite (why?). Practically, the system is at rest when its motion has become very small, say, less than 0.1% of the initial displacement (this choice being up to us), that is in our case,

2-5

12–19 INITIAL VALUE PROBLEM

Solve and graph the solution. Show the details of your work.

12. $x^2y'' - 4xy' + 6y = 0, \quad y(1) = 0.4, \quad y'(1) = 0$

13. $x^2y'' + 3xy' + 0.75y = 0, \quad y(1) = 1, \quad y'(1) = -2.5$

14. $x^2y'' + xy' + 9y = 0, \quad y(1) = 0, \quad y'(1) = 2.5$

15. $x^2y'' - xy' + y = 0, \quad y(1) = 1.5, \quad y'(1) = 0.25$

16. $(x^2D^2 - 3xD + 4I)y = 0, \quad y(1) = -\pi, \quad y'(1) = 2\pi$

17. $(x^2D^2 + xD + I)y = 0, \quad y(1) = 1, \quad y'(1) = 1$

18. $(9x^2D^2 + 3xD + I)y = 0, \quad y(1) = 1, \quad y'(1) = 0$

19. $x^2y'' + 2xy' - 6y = 0, \quad y(1) = 0.5, \ y'(1) = 1.5$

20. **TEAM PROJECT. Double Root**

(a) Derive a second linearly independent solution of (1) by reduction of order; but instead of using (9), Sec. 2.1, perform all steps directly for the present ODE (1).

(b) Obtain $x^m \ln x$ by considering the solutions x^m and x^{m+s} of a suitable Euler–Cauchy equation and letting $s \to 0$.

(c) Verify by substitution that $x^m \ln x$, $m = (1-a)/2$, is a solution in the critical case.

(d) Transform the Euler–Cauchy equation (1) into an ODE with constant coefficients by setting $x = e^t$ $(x > 0)$.

(e) Obtain a second linearly independent solution of the Euler–Cauchy equation in the "critical case" from that of a constant-coefficient ODE.