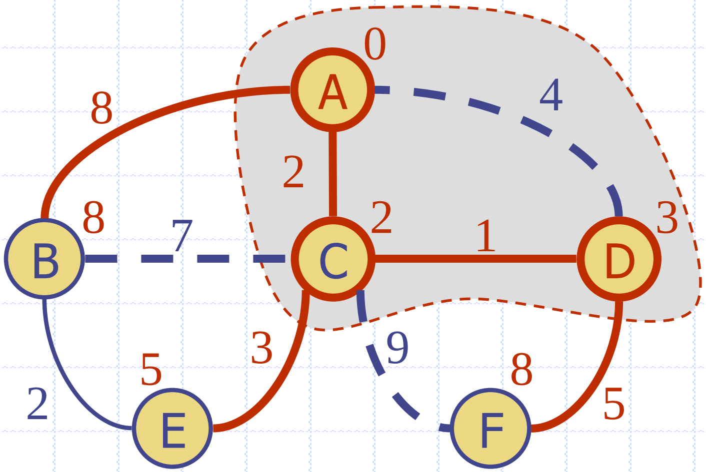
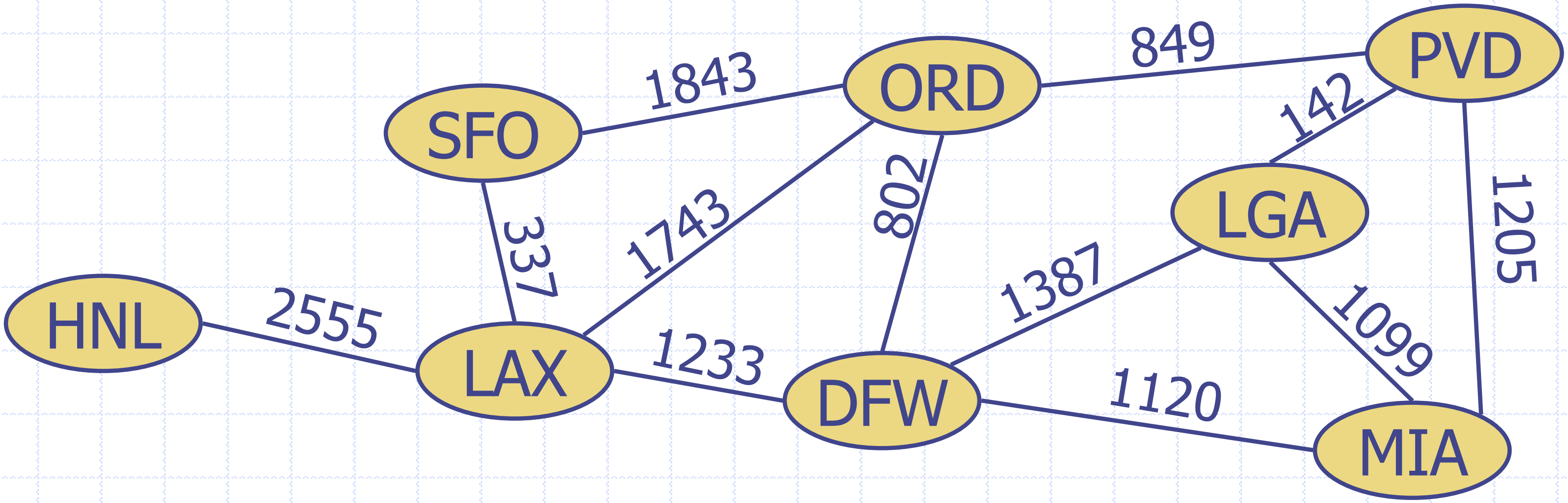
Shortest Paths



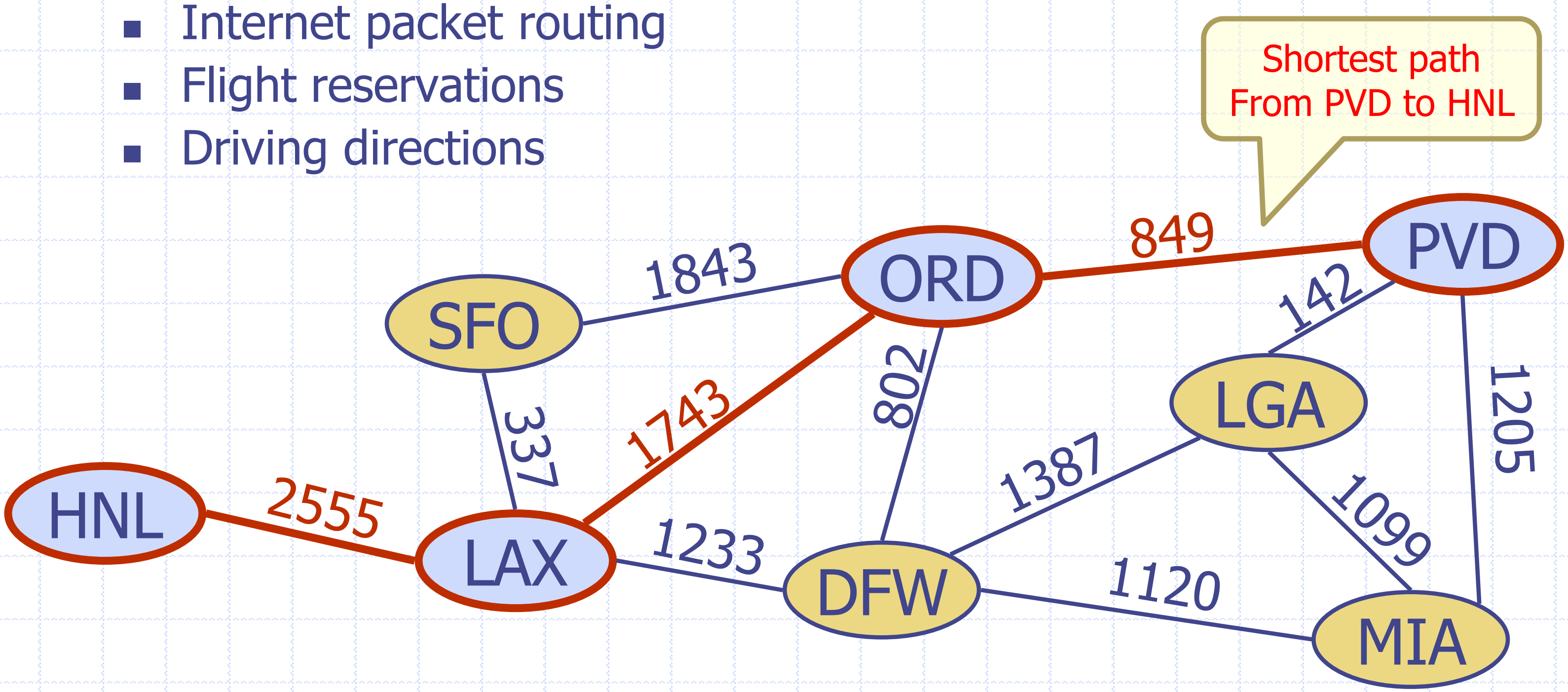
Weighted Graphs

- In a weighted graph, each edge has an associated numerical value, called the weight of the edge
- Edge weights may represent, distances, costs, etc.
- Example:
  - In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports



Shortest Paths

- Given a weighted graph and two vertices u and v, we want to find a path of minimum total weight between u and v.
- Example:
  - Shortest path between Providence and Honolulu
- Applications
  - Internet packet routing
  - Flight reservations
  - Driving directions



Shortest Path Properties

Property 1:

A subpath of a shortest path is itself a shortest path

DP!

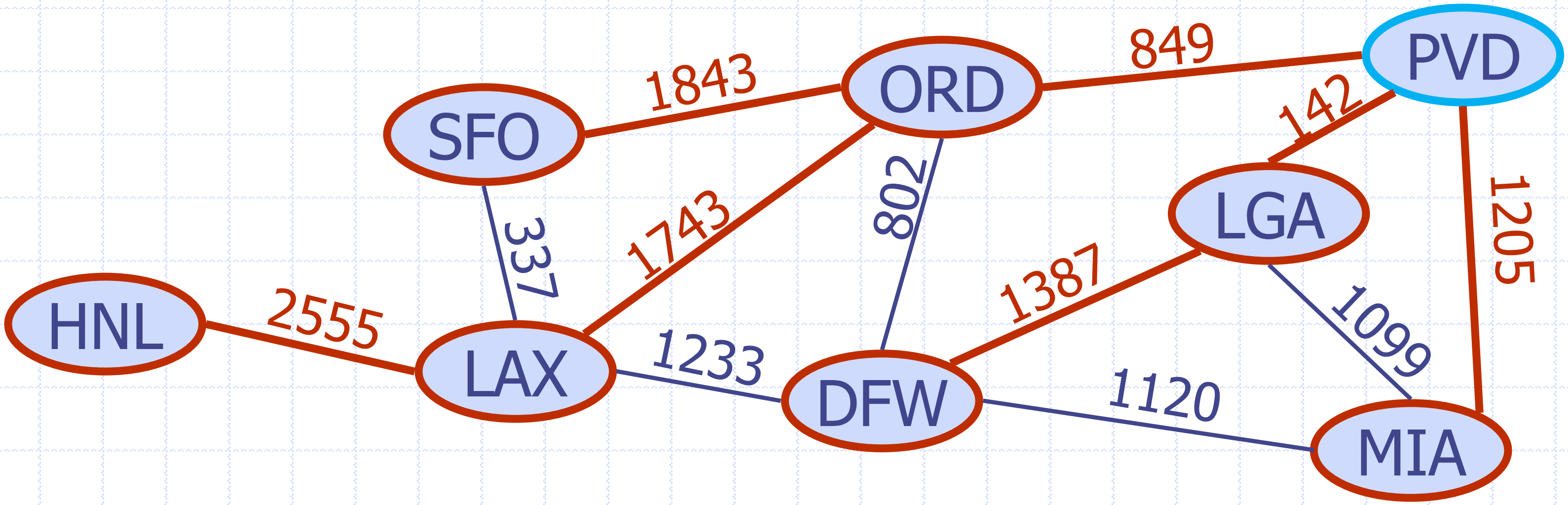
Property 2:

There is a tree of shortest paths from a start vertex to all the other vertices

Connected & no cycles

Example:

Tree of shortest paths from Providence



Dijkstra's Algorithm

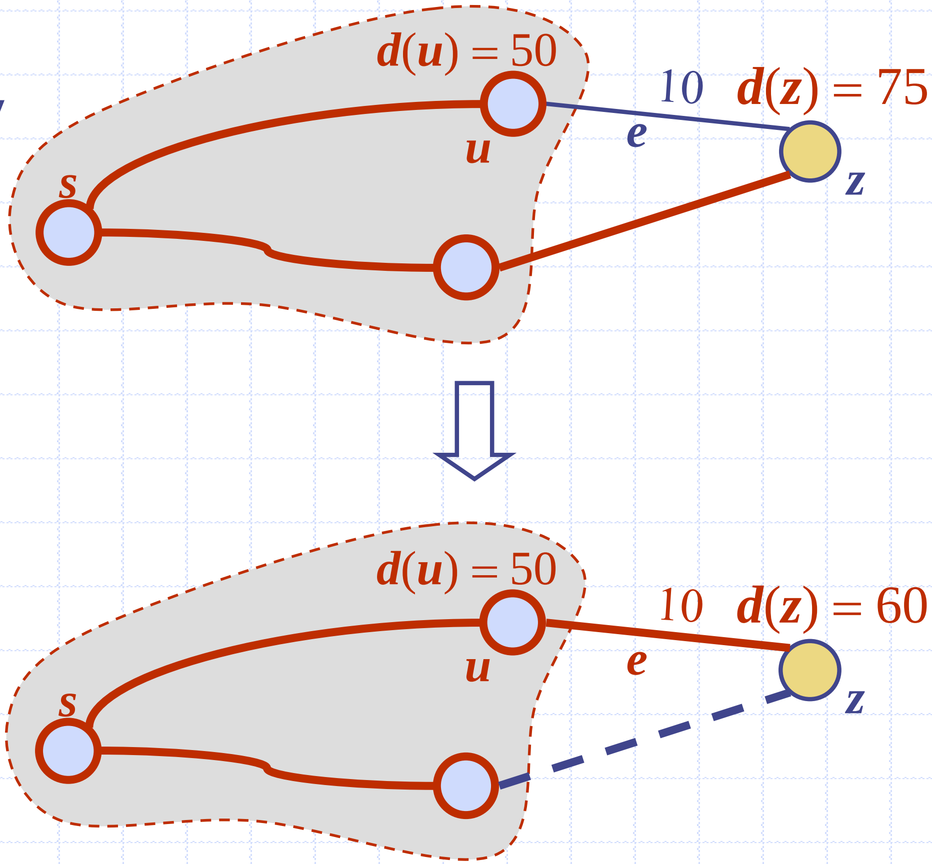
It was designed in 20 min without pencil and paper! (source)

- Dijkstra's algorithm computes the distances of all the vertices from a given start vertex s ➔ Single-source all-destination
- Assumption:
  - the edge weights are nonnegative
- We grow a "cloud" of vertices, beginning with s and eventually covering all the vertices
- We store with each vertex v a label $d(v)$ representing the distance of v from s in the subgraph consisting of the cloud and its adjacent vertices
- At each step
  - We add to the cloud the vertex u outside the cloud with the smallest distance label, $d(u)$
  - We update the labels of the vertices adjacent to u

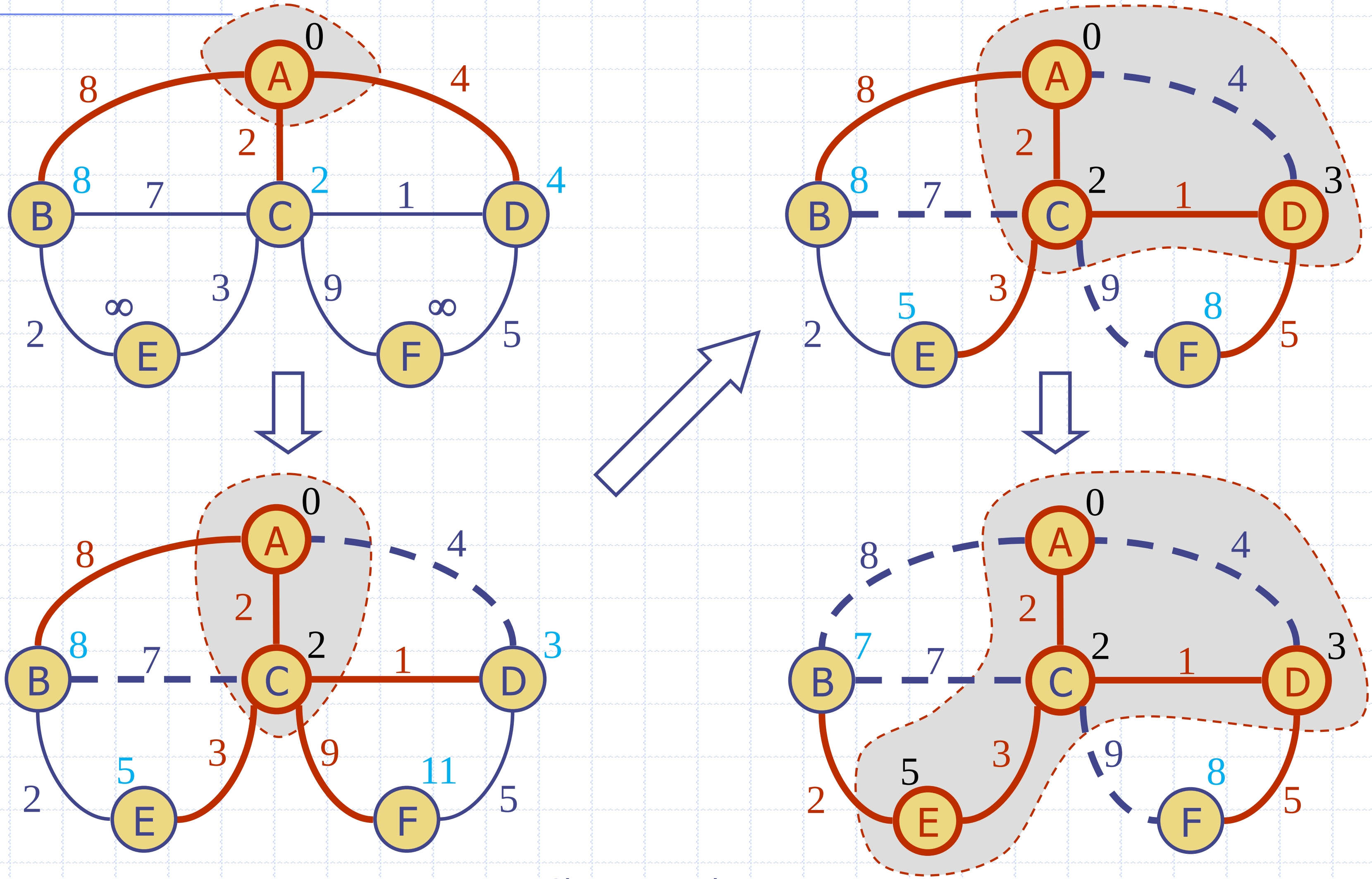
Edge Relaxation

- Consider an edge $e = (u,z)$ such that
  - u is the vertex most recently added to the cloud
  - z is not in the cloud
- The relaxation of edge e updates distance $d(z)$ as follows:

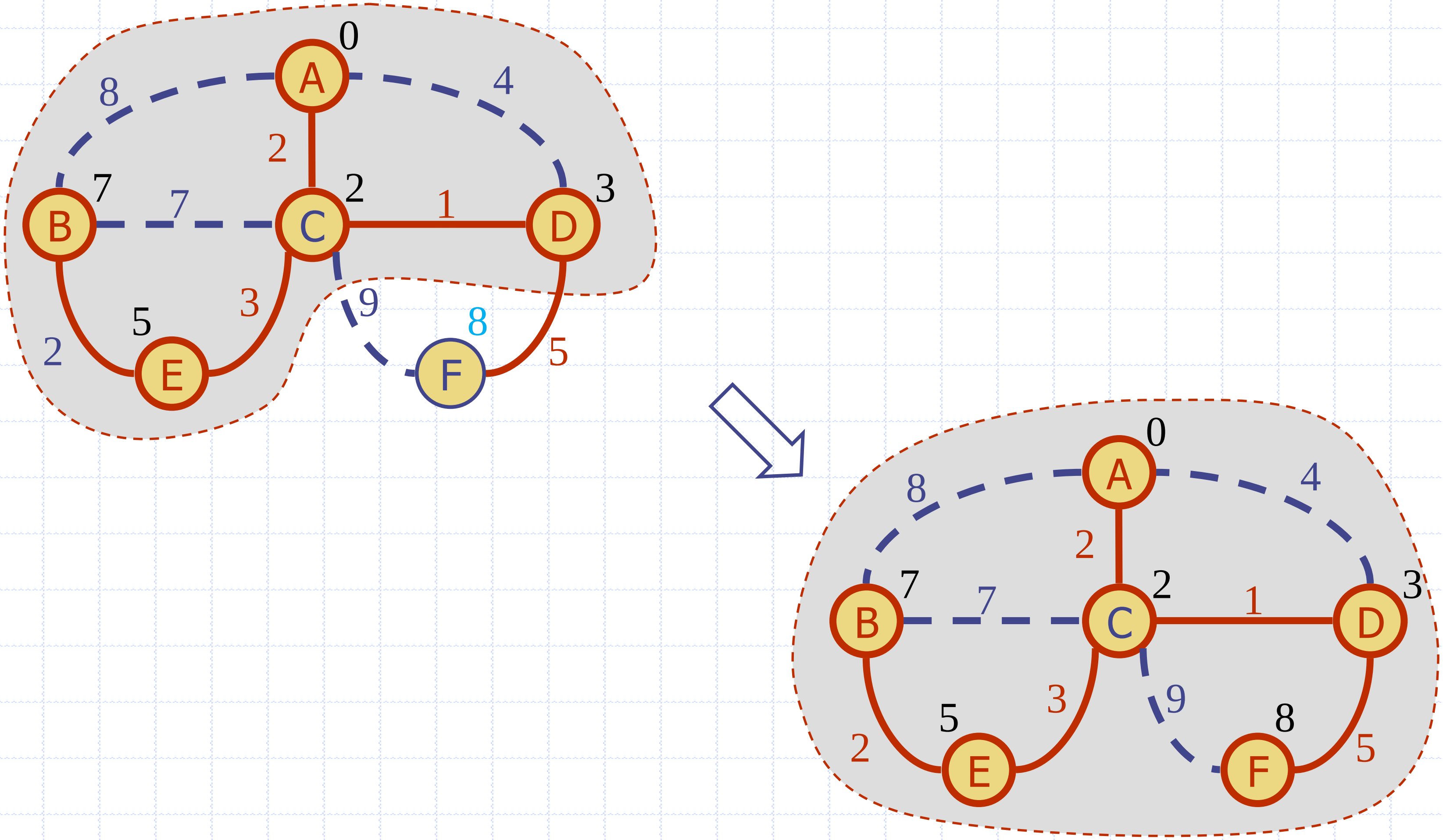
$$d(z) \leftarrow \min\{d(z), d(u) + weight(e)\}$$



Step-by-step Example

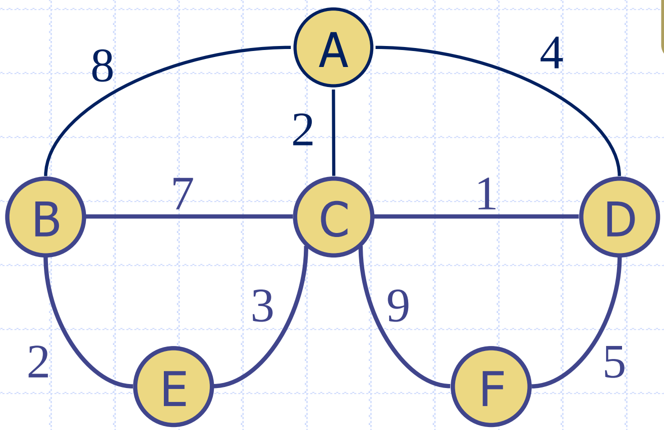


Example (cont.)



Example by Table Filling (1/2)

Quiz!

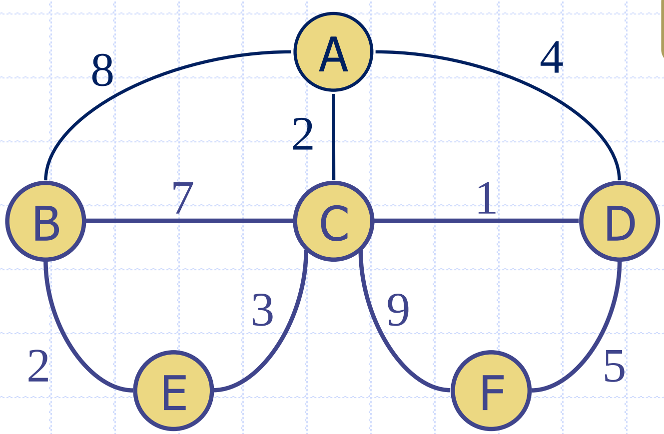


Source node

	B	C	D	E	F
A					

Example by Table Filling (1/2)

Quiz!



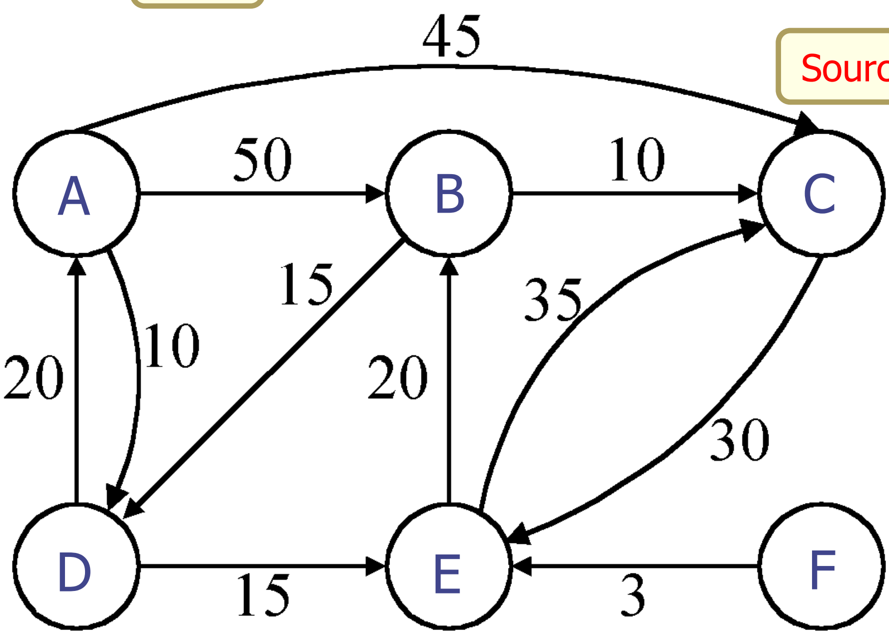
Source node

	B	C	D	E	F
A	8_A	$\underline{2_A}$	4_A	∞	∞
C			$\underline{3_C}$	5_C	11_C
D				$\underline{\quad}$	8_D
E	$\underline{7_E}$				
B					$\underline{\quad}$
F					

Selected nodes

Example by Table Filling (2/2)

Quiz!



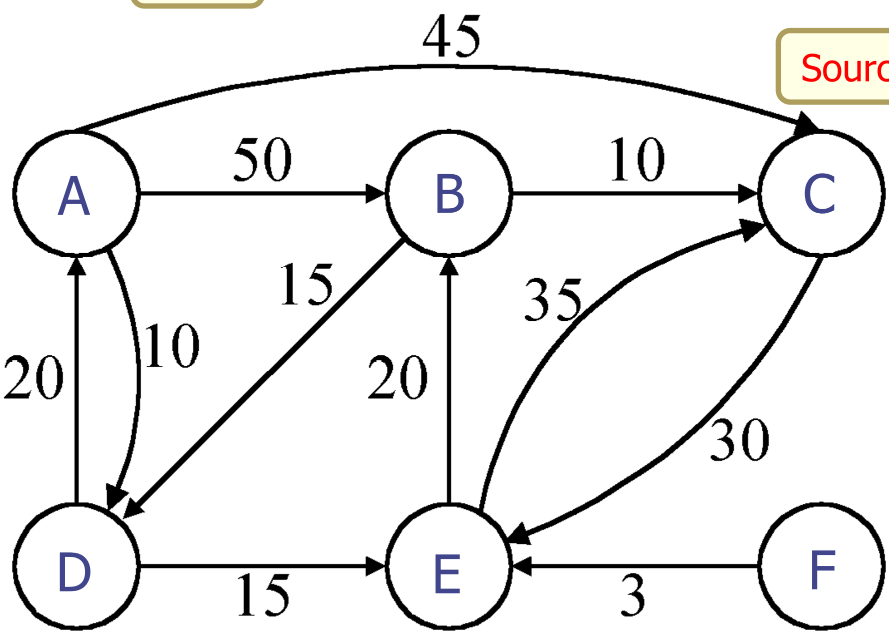
Source node

	B	C	D	E	F
A					

Animation

Example by Table Filling (2/2)

Quiz!



Source node

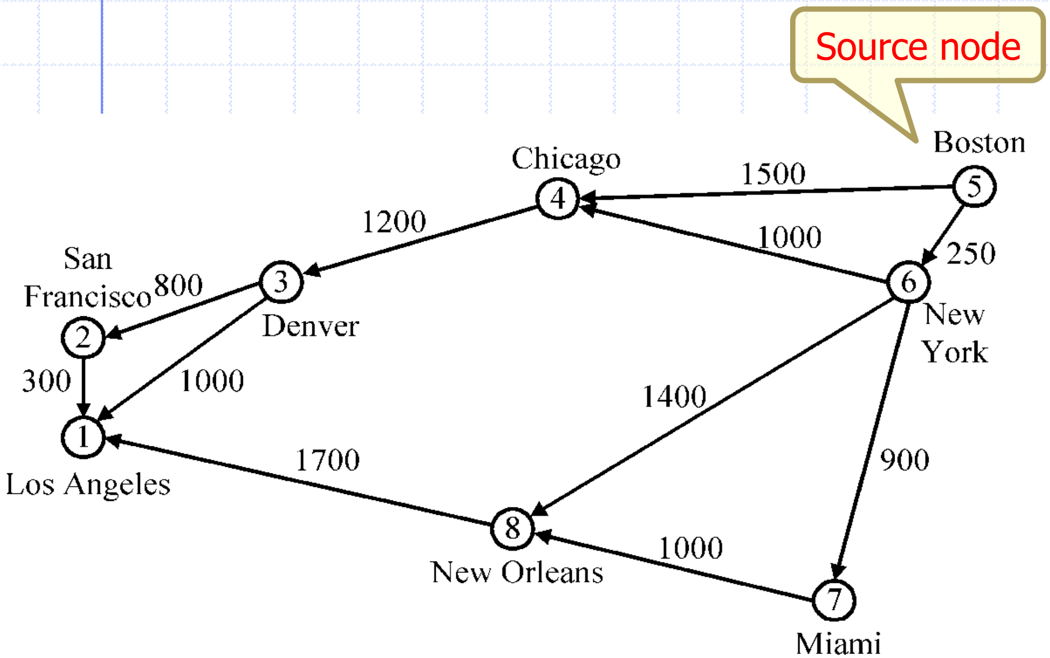
	B	C	D	E	F
A	50_A	45_A	$\underline{10_A}$	∞	∞
D				$\underline{25_D}$	
E	$\underline{45_E}$				
B		__			
C					__
F					

Selected nodes

Animation

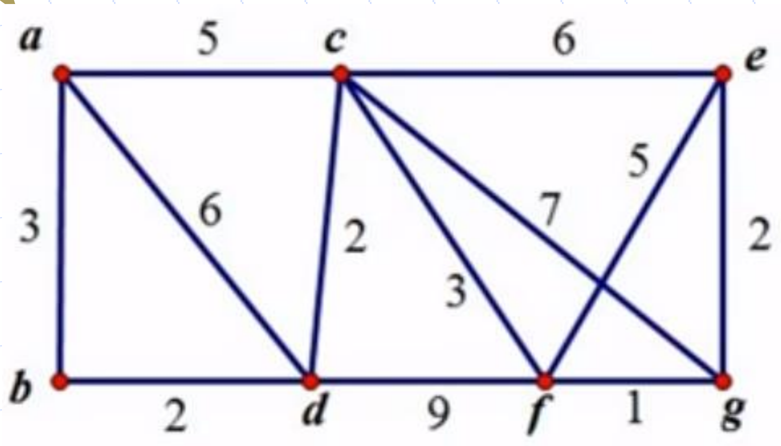
Exercise 1

Quiz!



Exercise 2

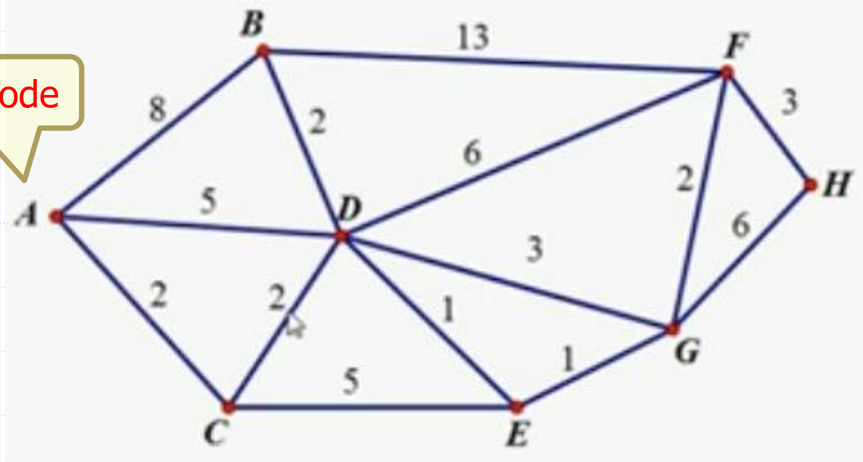
Source node



Solution!

Exercise 3

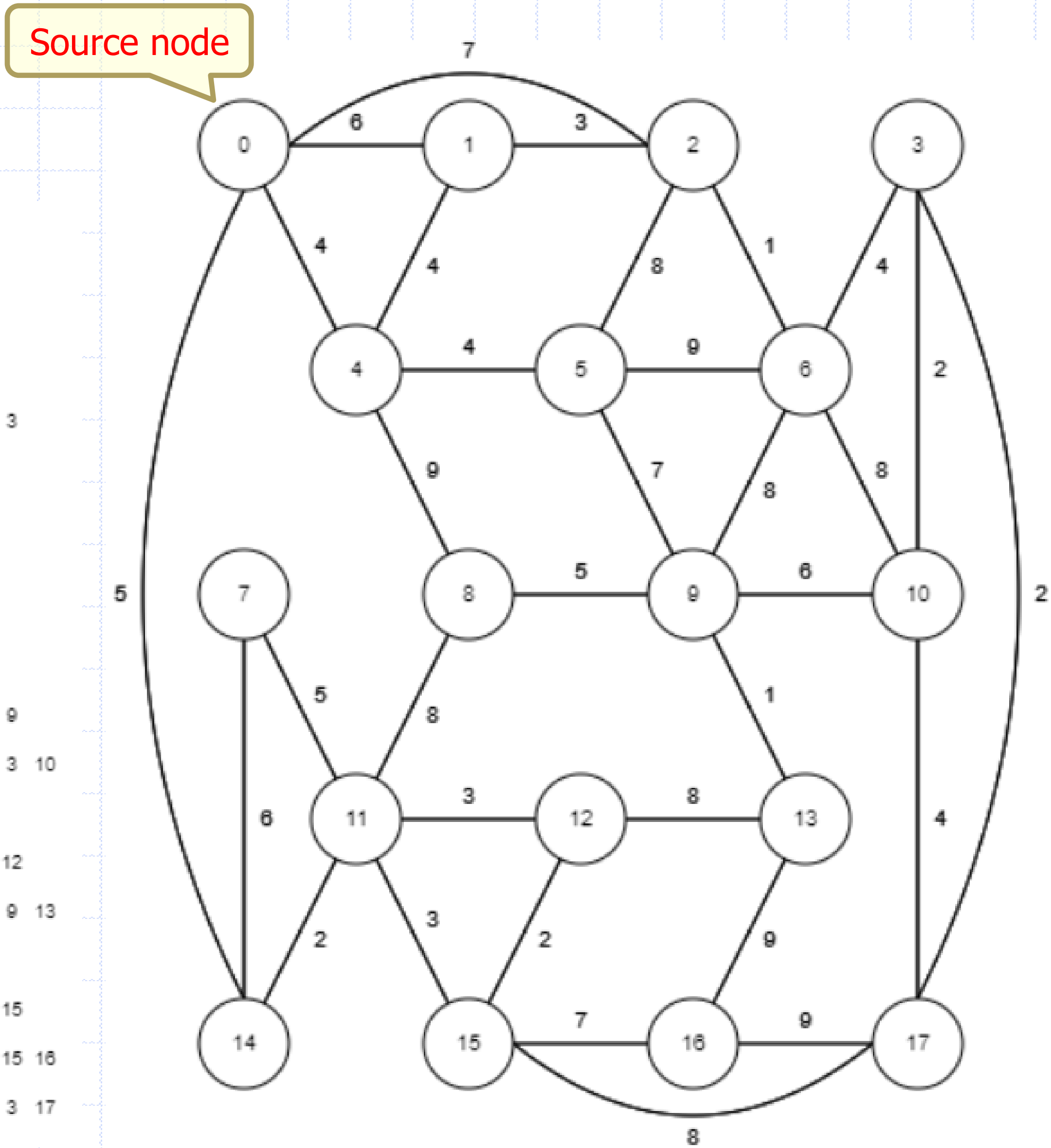
Source node



Solution!

Exercise 4

Vertex	Known	Cost	Path	
0	T	0	-1	0
1	T	6	0	0 1
2	T	7	0	0 2
3	T	12	6	0 2 6 3
4	T	4	0	0 4
5	T	8	4	0 4 5
6	T	8	2	0 2 6
7	T	11	14	0 14 7
8	T	13	4	0 4 8
9	T	15	5	0 4 5 9
10	T	14	3	0 2 6 3 10
11	T	7	14	0 14 11
12	T	10	11	0 14 11 12
13	T	16	9	0 4 5 9 13
14	T	5	0	0 14
15	T	10	11	0 14 11 15
16	T	17	15	0 14 11 15 16
17	T	14	3	0 2 6 3 17



Dijkstra's Algorithm

- A heap-based adaptable priority queue with location-aware entries stores the vertices outside the cloud
  - Key: distance
  - Value: vertex
  - Recall that method *replaceKey*(*l*,*k*) changes the key of entry *l*
- We store two labels with each vertex:
  - Distance
  - Entry in priority queue

```
Algorithm DijkstraDistances(G, s)
    Q ← new heap-based priority queue
    for all v ∈ G.vertices()
        if v = s
            v.setDistance(0)
        else
            v.setDistance(∞)
        l ← Q.insert(v.getDistance(), v)
        v.setEntry(l)
    while ¬Q.empty()
        l ← Q.removeMin()
        u ← l.getValue()
        for all e ∈ u.incidentEdges() { relax e }
            z ← e.opposite(u)
            r ← u.getDistance() + e.weight()
            if r < z.getDistance()
                z.setDistance(r)
                Q.replaceKey(z.getEntry(), r)
```

Analysis of Dijkstra's Algorithm

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Dijkstra's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- The running time can also be expressed as $O(m \log n)$ since the graph is connected

Shortest Paths Tree

- Using the template method pattern, we can extend Dijkstra's algorithm to return a tree of shortest paths from the start vertex to all other vertices
- We store with each vertex a third label:
  - parent edge in the shortest path tree
- In the edge relaxation step, we update the parent label

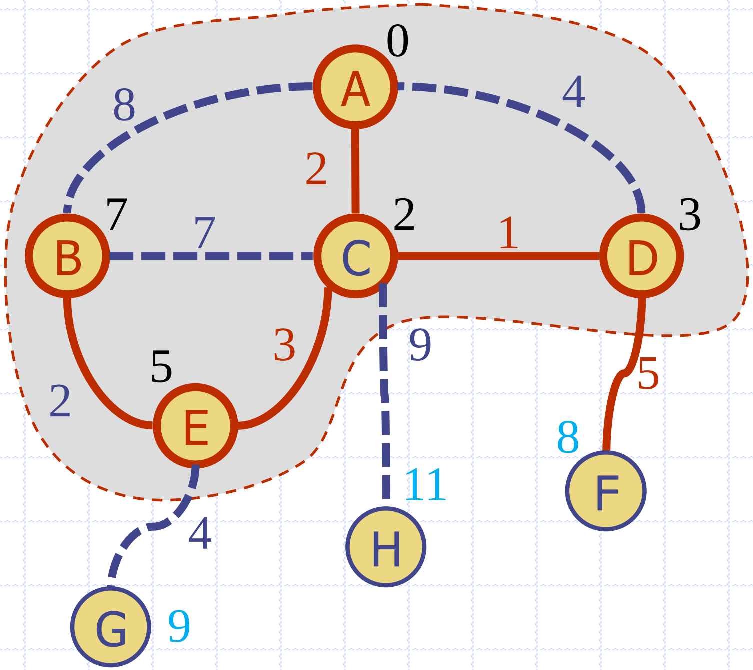
Easy for backtracking

Algorithm *DijkstraShortestPathsTree*(G, s)

```
    …
  for all v ∈ G.vertices()
    …
    v.setParent(∅)
    …
    for all e ∈ u.incidentEdges()
      { relax edge e }
      z ← e.opposite(u)
      r ← u.getDistance() + e.weight()
      if r < z.getDistance()
        z.setDistance(r)
        z.setParent(e)
        Q.replaceKey(z.getEntry(),r)
```

Why Dijkstra's Algorithm Works

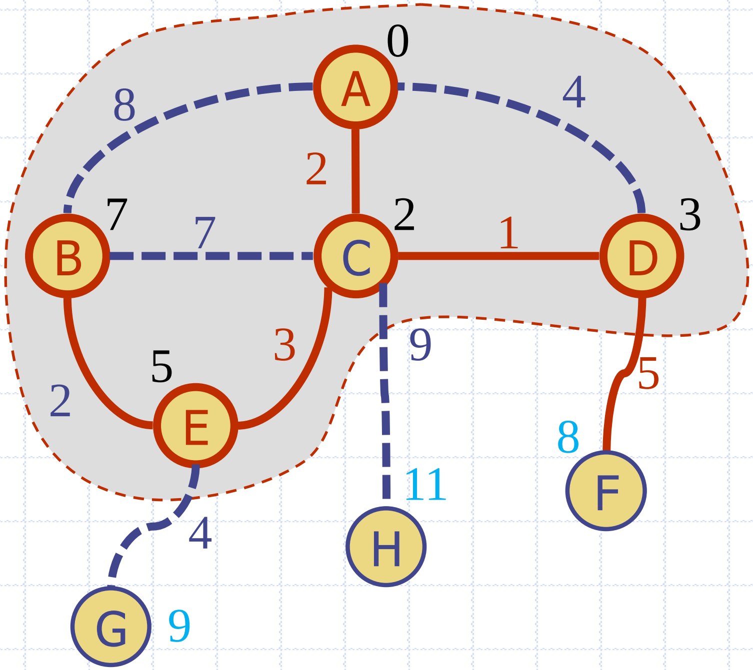
- Dijkstra's algorithm is based on DP. It adds vertices by increasing distance.
  - Suppose it didn't find all shortest distances. Let F be the first wrong vertex the algorithm processed.
  - When the previous node, D, on the true shortest path was considered, its distance was correct
  - But the edge (D,F) was relaxed at that time!
  - Thus, so long as d(F)$\geq$d(D), F's distance cannot be wrong. That is, there is no wrong vertex



Why It Doesn't Work for Negative-Weight Edges

- Dijkstra's algorithm is based on the DP. It adds vertices by increasing distance, which does not hold if we have negative-weight edges.

Quiz!



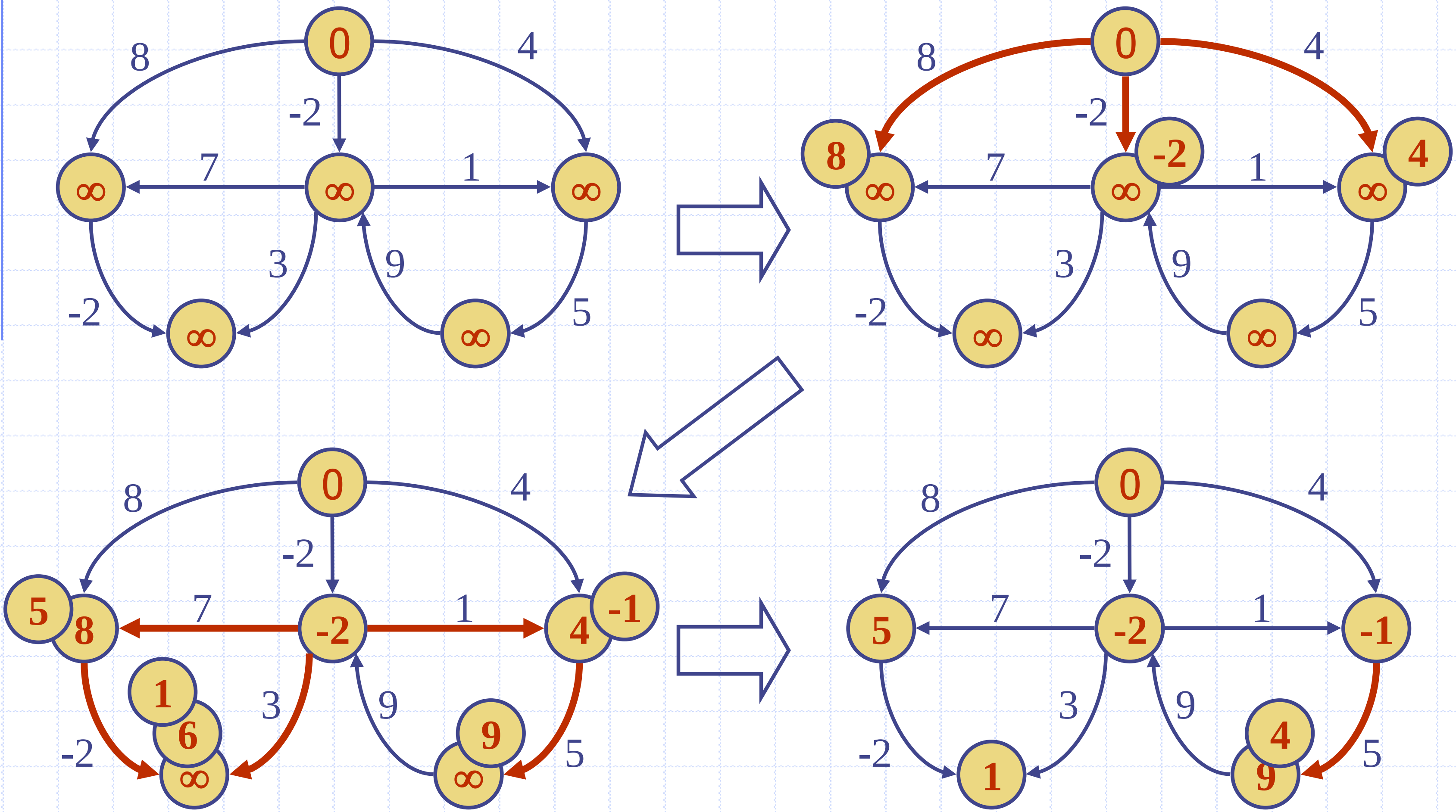
Bellman-Ford Algorithm (not in book)

- Works even with negative-weight edges
- Must assume directed edges (for otherwise we would have negative-weight cycles)
- Iteration i finds all shortest paths that use i edges.
- Running time: O(nm).
- Can be extended to detect a negative-weight cycle if it exists
  - How?

```
Algorithm BellmanFord(G, s)
  for all v ∈ G.vertices()
    if v = s
      v.setDistance(0)
    else
      v.setDistance(∞)
  for i ← 1 to n – 1 do
    for each e ∈ G.edges()
      { relax edge e }
      u ← e.origin()
      z ← e.opposite(u)
      r ← u.getDistance() + e.weight()
      if r < z.getDistance()
        z.setDistance(r)
```

Bellman-Ford Example

Nodes are labeled with their d(v) values



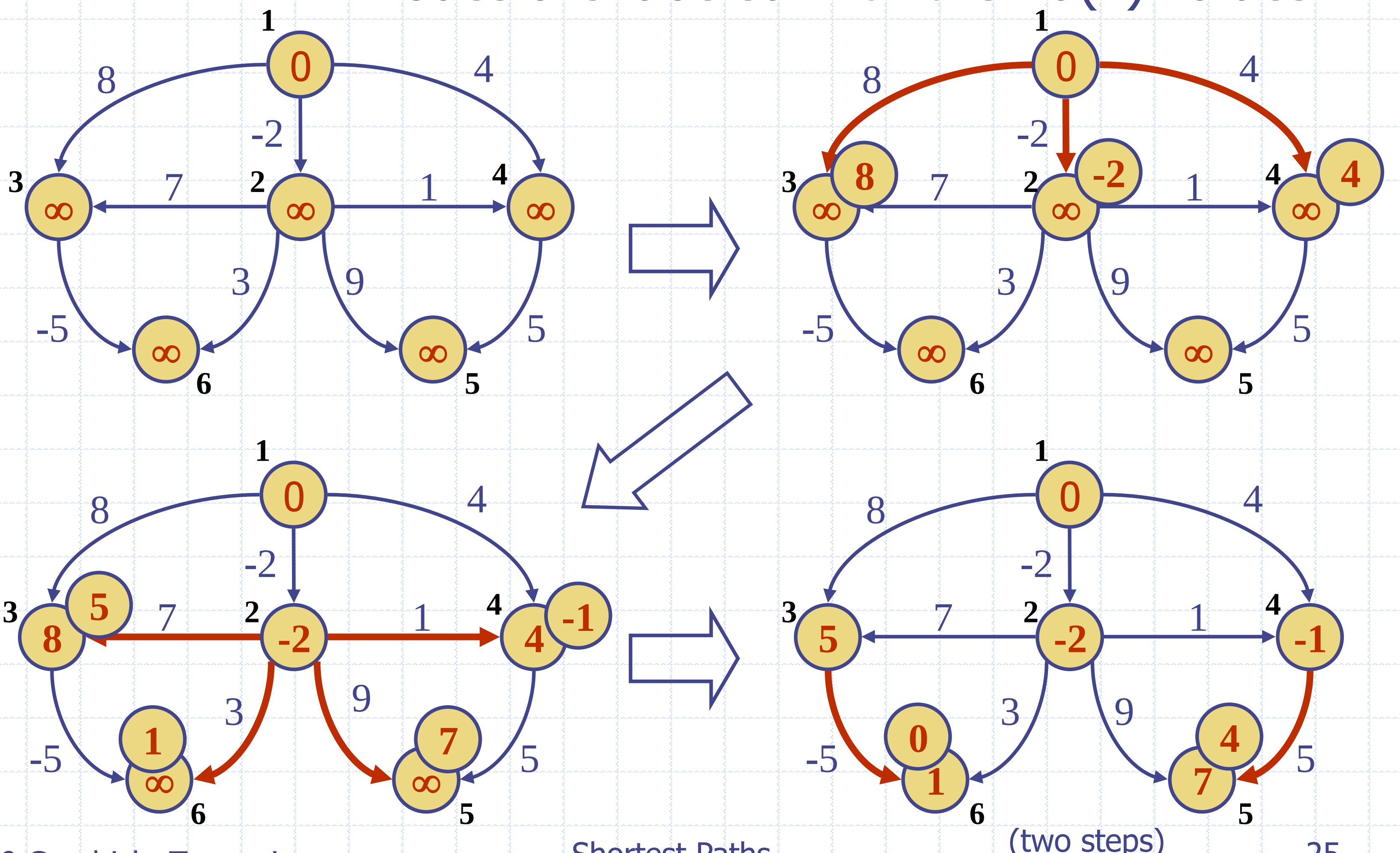
DAG-based Algorithm (not in book)

- Works even with negative-weight edges
- Uses topological order
- Doesn't use any fancy data structures
- Is much faster than Dijkstra's algorithm
- Running time: O(n+m).

```
Algorithm DagDistances(G, s)
  for all v ∈ G.vertices()
    if v = s
      v.setDistance(0)
    else
      v.setDistance(∞)
  { Perform a topological sort of the vertices }
  for u ← 1 to n do    {in topological order}
    for each e ∈ u.outEdges()
      { relax edge e }
      z ← e.opposite(u)
      r ← u.getDistance() + e.weight()
      if r < z.getDistance()
      z.setDistance(r)
```

DAG Example

Nodes are labeled with their d(v) values



Resources

- Youtube tutorials
  - Dijkstra's algorithm: Single source all destination
  - Bellman-Ford algorithm: Single source all destination
  - Floyd-Warshall algorithm: All pairs shortest path
- Animation
  - Step-by-step animation