Contents

13 Gravitation

In this chapter we will explore the following topics:
-Newton's law of gravitation that describes the attractive force between two point masses and its application to extended objects.
-The acceleration of gravity on the surface of the earth, above it, as well as below it.
-Gravitational potential energy.
-Kepler's three laws on planetary motion.
-Satellites (orbits, energy , escape velocity)

13.1 Newton's Law of Gravitation

Newton realized that the force which holds the moon it its orbit is of the same nature with the force that makes an apple drop near the surface of the earth. Newton concluded that the earth attracts both apples as well as the moon but also that every object in the universe attracts every other object. The tendency of objects to move towards each other is known as gravitation.

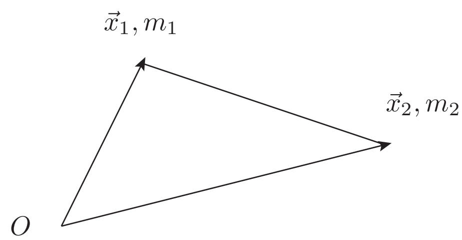
Newton formulated a force law known as Newton's law of gravitation. Every particle attracts any other particle with a gravitational force that has the following characteristics:
1. The force acts along the line that connects the two particles.
2. Its magnitude is given by the equation:

$$F = G\frac{m_1 m_2}{r^2}$$

Here m_1 and m_2 are the masses of the two particles, r is their separation and G is the gravitational constant. Its value is:

$$G = 6.67 \times 10^{-11}\frac{N.m^2}{kg^2}.$$

Let $\vec{x}_1$ be the position vector of m_1 and $\vec{x}_2$ be the position vector of m_2.



The separation between the two masses

$$r = |\vec{x}_1 - \vec{x}_2|$$

and the unit vector $\hat{r}_{12}$ along the direction from $\vec{x}_2$ to $\vec{x}_1$ is

$$\hat{r}_{12} = \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|} = \frac{\vec{x}_1 - \vec{x}_2}{r}$$

Let $\vec{F}_{12}$ denote the gravitational force exerted on m_1 due to m_2. Then

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{r}_{12} = -G\frac{m_1 m_2}{r^3}\left(\vec{x}_1 - \vec{x}_2\right)$$

Likewise

$$\vec{F}_{21} = -G\frac{m_1 m_2}{r^2}\hat{r}_{21} = -G\frac{m_1 m_2}{r^3}\left(\vec{x}_2 - \vec{x}_1\right) = -\vec{F}_{12}$$

Newton proved that a uniform shell attracts a particle that is outside the shell as if the shell's mass were concentrated at the shell center

$$F_1 = G\frac{m_1 m_2}{r^2}$$

Note: If the particle is inside the shell, the net force is zero.

Consider the force F the earth (radius R, mass M) exerts on an apple of mass m. The earth can be thought of as consisting of concentric shells. Thus from the apple's point of view the earth behaves like a point mass at the earth center. The magnitude of the force is given by the equation:

$$F = G\frac{mM}{R^2}$$

13.2 Gravitation and the Principle of Superposition

The net gravitational force exerted by a group of particles is equal to the vector sum of the contribution from each particle. For example the net force $\vec{F}_1$ exerted on m_1 by m_2 and m_3 is equal to:

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13}$$

Here $\vec{F}_{12}$ and $\vec{F}_{13}$ are the forces exerted on m_1 by m_2 and m_3, respectively. In general the force exerted on m_1 by n particles is given by the equation:

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + ...\vec{F}_{1n} = \sum_{i=1}^{n} \vec{F}_{1i}$$

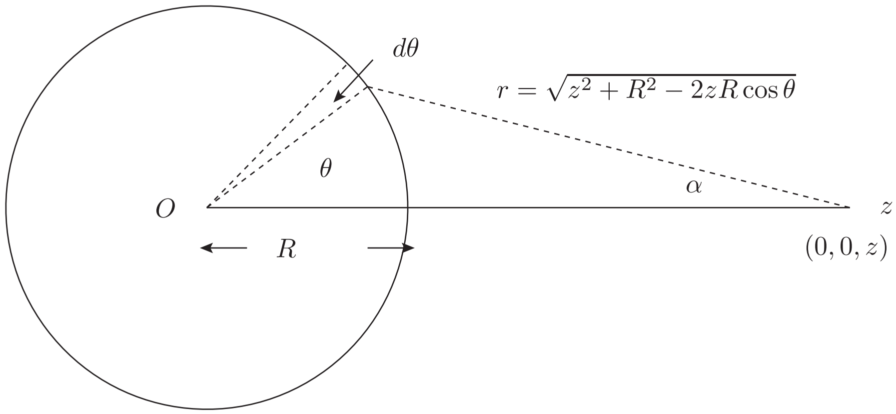
The gravitation force exerted by a continuous extended object on a particle of mass m_1 can be calculated using the principle of superposition. The object is divided into elements of mass dm the net force on m_1 is the vector sum of the forces exerted by each element. The sum takes the form of an integral

$$\vec{F}_1 = \int d\vec{F}$$

Here $d\vec{F}$ is the force exerted on m_1 by $.dm$

$$d\vec{F} = -G\frac{m_1 dm}{r^2}\hat{r}$$

where $\hat{r}$ is the unit vector pointing from dm to m_1



$$dm = \frac{M}{4\pi R^2} da = \frac{M}{4\pi R^2} 2\pi R \sin\theta \, (R d\theta) = \frac{M}{2} \sin\theta d\theta$$

By symmetry only the z-component of $d\vec{F}$ may survive after integration, and we thus have

$$\begin{aligned}
\vec{F} &== \hat{k} \cdot \left(- \int G \frac{m_1 dm}{r^3} \vec{r} \right) \hat{k} \\
&= -G \frac{m_1 M}{2} \int_0^\pi \frac{z - R\cos\theta}{r^3} \sin\theta d\theta \hat{k} \\
&= -G \frac{m_1 M}{2} \int_0^\pi \frac{z - R\cos\theta}{(z^2 + R^2 - 2zR\cos\theta)^{\frac{3}{2}}} \sin\theta d\theta \hat{k}
\end{aligned}$$

Change the integrating variable to $t = \cos\theta$. We get

$$\begin{aligned}
\vec{F} &= -G\frac{m_1 M}{2}\int_{-1}^{1}\frac{z-Rt}{(z^2+R^2-2zRt)^{\frac{3}{2}}}dt\hat{k} \\
&= -G\frac{m_1 M}{4z}\int_{-1}^{1}\left(\frac{z^2-R^2}{(z^2+R^2-2zRt)^{\frac{3}{2}}}+\frac{1}{(z^2+R^2-2zRt)^{\frac{1}{2}}}\right)dt\hat{k} \\
&= -G\frac{m_1 M}{4z}\int_{-1}^{1}\frac{1}{zR}\frac{d}{dt}\left(\frac{z^2-R^2}{(z^2+R^2-2zRt)^{\frac{1}{2}}}-\left(z^2+R^2-2zRt\right)^{\frac{1}{2}}\right)dt\hat{k} \\
&= -G\frac{m_1 M}{4z}\frac{1}{zR}\frac{z^2-R^2-(z^2+R^2-2zRt)}{(z^2+R^2-2zRt)^{\frac{1}{2}}}|_{t=-1}^{t=1}\hat{k} \\
&= -G\frac{m_1 M}{2z^2}\frac{zt-R}{(z^2+R^2-2zRt)^{\frac{1}{2}}}|_{t=-1}^{t=1}\hat{k} = -G\frac{m_1 M}{2z^2}\left(\frac{z-R}{|z-R|}+1\right)\hat{k} \\
&= \left\{\begin{array}{c} -G\frac{m_1 M}{z^2}\hat{k},\ z>R \\ 0,\ z<R \end{array}\right\}
\end{aligned}$$

where we have used

$$\begin{aligned}
\frac{d}{dt}\left(z^2+R^2-2zRt\right)^{\frac{1}{2}} &= \frac{-zR}{(z^2+R^2-2zRt)^{\frac{1}{2}}} \\
\frac{d}{dt}\frac{1}{(z^2+R^2-2zRt)^{\frac{1}{2}}} &= \frac{zR}{(z^2+R^2-2zRt)^{\frac{3}{2}}}
\end{aligned}$$

13.3 Gravitation near the earth's Surface

If we assume that the earth is a sphere of mass M, the magnitude F of the force exerted by the earth on an object of mass m placed at a distance r from the center of the earth is:

$$F = G\frac{mM}{r^2}$$

The gravitational force results in an acceleration $\vec{a}_g$ known as gravitational acceleration. Using Newton's second law we have that

$$a_g = G\frac{M}{r^2}$$

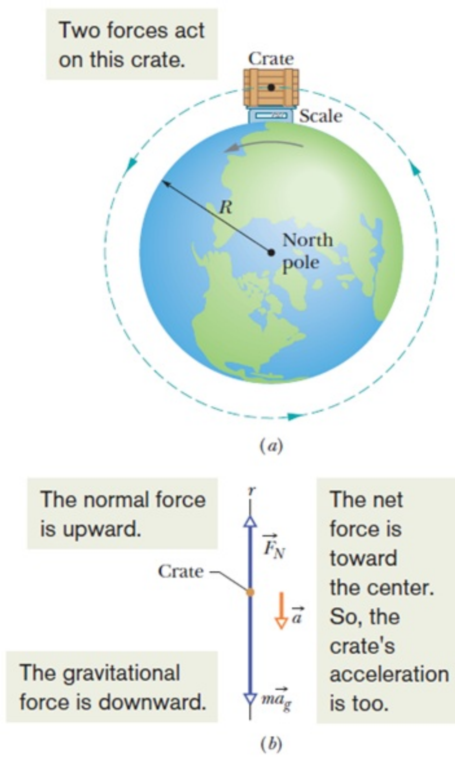
Up to this point we have assumed that the free fall acceleration g near the surface of the earth is constant. However if we measure g at various points on the surface of the earth we find that its value is not constant. This is attributed to three reasons:

13.3.1 1. The earth's mass is not uniformly distributed

The density of the earth varies radially. The density of the outer section varies from region to region over the earth's surface. Thus g varies from point to point.

2. The earth is not a sphere. Earth is approximately an ellipsoid flattened at the poles and bulging at the equator. Its equatorial radius is larger than the polar radius by 21 km. Thus the value of g at sea level increases as one goes from the equator to the poles.

3. The earth is rotating. Consider the crate shown in the following figure.



The crate is resting on a scale at a point on the equator. The net force along the y - axis

$$\vec{F}_y = \vec{F}_g + \vec{F}_N$$

$$\left|\vec{F}_y\right| = \left|\vec{F}_g\right| - \left|\vec{F}_N\right| = ma_g - F_N$$

Here $|\vec{a}_g| = \frac{GM}{R^2}$ and $\left|\vec{F}_N\right| = mg$ is the normal force exerted on the crate by the scale. The crate has an acceleration $a = \omega^2 R$ due to the rotation of the earth about its axis every 24 hours. If we apply Newton's second law we get:

$$ma_g - mg = m\omega^2 R \to mg = ma_g - m\omega^2 R \to g = a_g - \omega^2 R.$$

Free fall acceleration = gravitational acceleration - centripetal acceleration. The term $\omega^2 R = 0.034 m/s^2$ which is much smaller than $9.8 m/s^2$.

13.4 Gravitation inside the earth.

Newton proved that the net gravitational force on a particle by a shell depends on the position of the particle with respect to the shell. If the particle is inside the shell, the net force is zero If the particle is outside the shell the force is given by:

$$F_1 = G\frac{m_1 m_2}{r^2}$$

Consider a mass m inside the earth at a distance r from the center of the earth. If we divide the earth in a series of concentric shells, only the shells with radius less than r exert a force on m. The net force on m is:

$$F = G\frac{mM_{ins}}{r^2}$$

Here M_{ins} is the mass of the part of the earth inside a sphere of radius r.

$$M_{ins} = \rho V_{ins} = \rho\frac{4\pi}{3}r^3 \to F = \frac{4\pi}{3}Gm\rho r$$

F is linear in r.

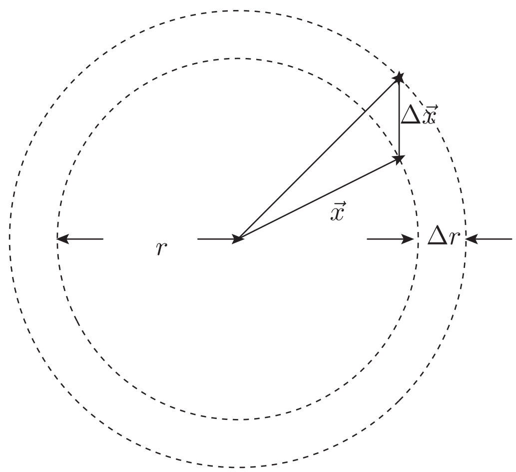
13.5 Gravitational Potential Energy

13.5.1 Spherically Symmetric Central Force

The magnitude of force $\left|\vec{F}\right|$ is constant on a spherical surface centered at the origin. In addition, the direction of $\vec{F}$ is in the radial direction.

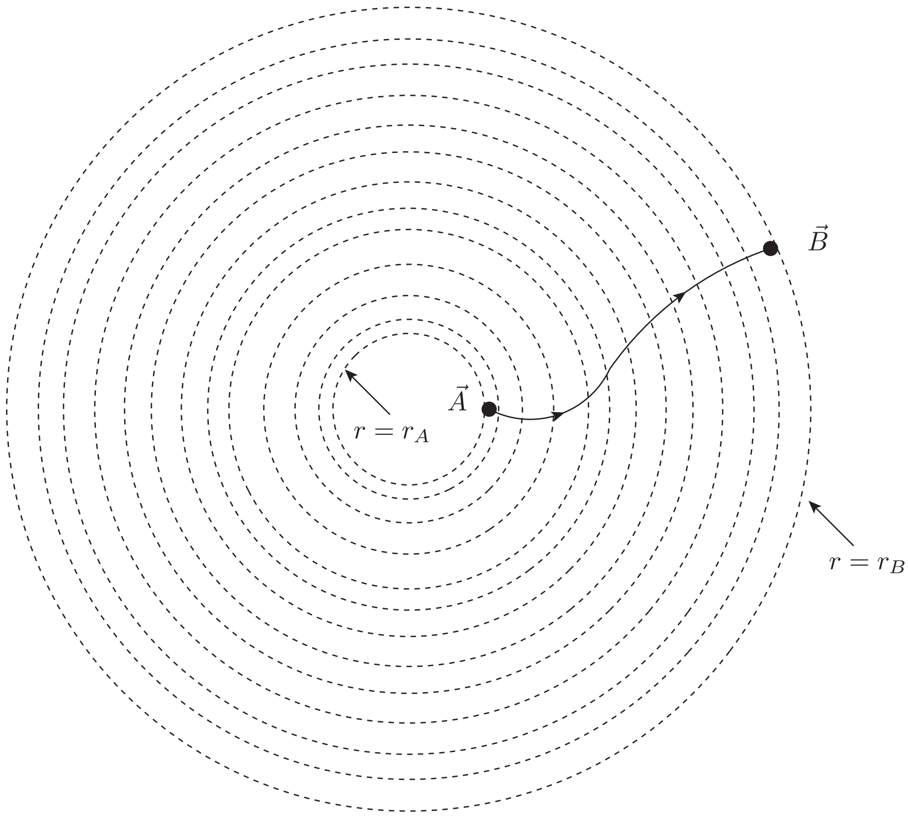
$$\vec{F}\left(\vec{x}\right) = f\left(r\right)\hat{r}$$

where $r = |\vec{x}|$ is the radial distance of the point $\vec{x}$ from the origin and $\hat{r} = \frac{\vec{x}}{r}$ is the unit vector in the radial direction. We shall show immediately that this $\vec{F}$ is conservative. To begin with, consider the work ΔW done for an infinitesimal displacement from $\vec{x}$ to $\vec{x} + \Delta\vec{x}$



$$
\begin{aligned}
\Delta W = \vec{F} \cdot \Delta\vec{x} = f(r)\,\hat{r} \cdot \Delta\vec{x} &= \frac{f(r)}{r}\vec{x} \cdot \Delta\vec{x} \\
&\simeq \frac{f(r)}{2r}\left((\vec{x} + \Delta\vec{x}) \cdot (\vec{x} + \Delta\vec{x}) - \vec{x} \cdot \vec{x}\right) \\
&= \frac{f(r)}{2r}\Delta\left(\vec{x} \cdot \vec{x}\right) = \frac{f(r)}{2r}\Delta\left(|\vec{x}|^2\right) \\
&= \frac{f(r)}{2r}\Delta\left(r^2\right) = f(r)\,\Delta r
\end{aligned}
$$

Thus for a path from $\vec{A}$ to $\vec{B}$ as shown below:



$$\Delta W\left(\vec{A} \to \vec{B}\right) = \sum_i \Delta W_i = \int_{r_A=|\vec{x}_A|}^{r_B=|\vec{x}_B|} f(r)\, dr$$

The work done depends only on the radial distances $r_A = |\vec{x}_A|$ and $r_B = |\vec{x}_B|$. Thus $\vec{F}$ is conservative and the potential energy is

$$U(x, y, z) = U(r) = -\int f(r)\, dr$$

$$f(r) = -\frac{dU(r)}{dr}$$

13.5.2 Gravitational Force

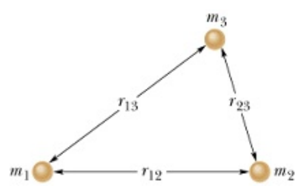
Assume that the mass m can move away from the surface of the earth, at a distance r from the center of the earth to a distance of $r + dr$. In this case the change gravitational potential energy is:

$$\begin{aligned} dU &= U(r+dr) - U(r) = -dW \\ &= -\left(-\frac{GmM}{r^2}\right) dr = d\left(-\frac{GmM}{r}\right) \end{aligned}$$

A possible solution satisfying $U(\infty) = 0$ is

$$U = -\frac{GmM}{r}$$

The negative sign of U expresses the fact that the corresponding gravitational force is attractive. Note: The gravitational potential energy is not only associated with the mass m but with M as well i.e. with both objects. If we have three masses m_1, m_2, and m_3 positioned as shown in the following figure:

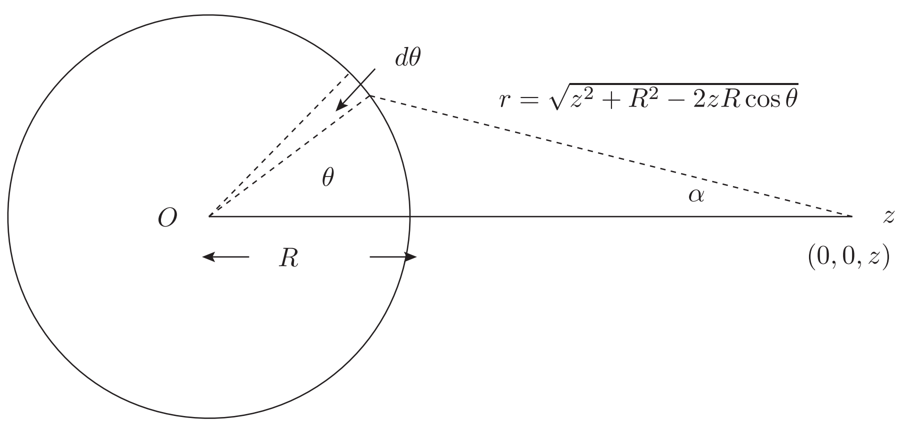


The potential energy U due to the gravitational forces among the objects is:

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right)$$

We take into account each pair once.

13.5.3 Potential Due to a Spherical Shell



$$dm = \frac{M}{4\pi R^2} da = \frac{M}{4\pi R^2} 2\pi R \sin\theta \left(R d\theta\right) = \frac{M}{2} \sin\theta d\theta$$

$$\begin{aligned} U &= -\int G \frac{m_1 dm}{r} \\ &= -G \frac{m_1 M}{2} \int_0^\pi \frac{1}{\left(z^2 + R^2 - 2zR\cos\theta\right)^{\frac{1}{2}}} \sin\theta d\theta \end{aligned}$$

Change the integrating variable to $t = \cos\theta$. We get

$$\begin{aligned} U &= -G \frac{m_1 M}{2} \int_{-1}^1 \frac{1}{\left(z^2 + R^2 - 2zR\cos\theta\right)^{\frac{1}{2}}} dt \\ &= G \frac{m_1 M}{2zR} \left(z^2 + R^2 - 2zRt\right)^{\frac{1}{2}} |_{t=-1}^{t=1} \\ &= G \frac{m_1 M}{2zR} \left(|z - R| - |z + R|\right) = \left\{ \begin{array}{c} -G \frac{m_1 M}{z}, \ z > R \\ -G \frac{m_1 M}{R}, \ z < R \end{array} \right\} \end{aligned}$$

where we have used

$$-\frac{1}{zR} \frac{d}{dt} \left(z^2 + R^2 - 2zRt\right)^{\frac{1}{2}} = \frac{1}{\left(z^2 + R^2 - 2zRt\right)^{\frac{1}{2}}}$$

13.6 Escape Speed

If a projectile of mass m is fired upward from the earth surface, the projectile will stop momentarily and return to the earth. There is however a minimum initial speed for which the projectile will escape from the gravitational pull of the earth and will stop at infinity. This minimum speed is known as escape velocity. We can determine the escape velocity using energy conservation.

$$K + U = \frac{1}{2} m v^2 - G \frac{Mm}{R} = K(\infty) + U(\infty) = K(\infty).$$

The escape speed form the earth occurs when $K(\infty) = 0$ or

$$\frac{1}{2} m v^2 = G \frac{Mm}{R}$$

i.e.,

$$v = \sqrt{\frac{2GM}{R}} = \sqrt{2Rg}$$

Note: The escape speed does not depend on m.

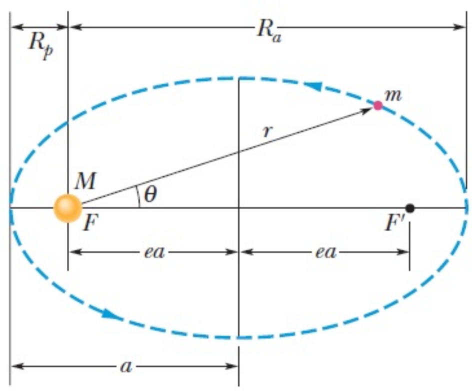
13.7 Planets and Satellites: Kepler's Laws

Stars follow regular paths in the evening sky. They rotate once every 24 hours about an axis that passes through the star Polaris. Polaris is the only star that does not move in the sky. The stars have fixed spatial relationships among them. Humans have classified them in groups known as constellations.

In contrast, planets follow complicated paths in the sky. Tycho Brahe made very careful measurements planetary motions but he died before he had the chance to analyze his data. This task was carried out by his assistant Johannes Kepler who summarized the results into three empirical laws known by his name. Later, Newton used his second law of motion with his gravitational law and the newly developed methods of calculus and derived Kepler's laws.

13.7.1 Kepler's First Law.

All planets move on elliptical orbits with the sun at one focus.



The orbits are described by two parameters: The semimajor axis a and the eccentricity e. The orbit in the figure has $e = 0.74$. The actual eccentricity of the earth's orbit is only 0.0167. Recall that the equation for ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is the semimajor axis and b is the semiminor axis. The foci are at $(c, 0)$ and $(-c, 0)$ with $c = \sqrt{a^2 - b^2}$. The eccentricity e is defined as

$$e = \frac{c}{a}$$

***** The following discussion for the derivation of Kepler's first law is optional *****

The Kepler's 1st Law can be derived from the equation of motion

$$\frac{d^2\vec{x}}{dt^2} = -\frac{GM}{r^2}\frac{\vec{x}}{r} \tag{1}$$

where $r = |\vec{x}|$. Suppose the orbit is on the xy plane which is perpendicular to the angular momentum $\vec{L}$. Let

$$\hat{e}_r = \frac{\vec{x}}{r} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

and

$$\hat{e}_\theta = \frac{d\hat{e}_r}{d\theta} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

Then

$$\frac{d\hat{e}_\theta}{d\theta} = -\cos\theta\hat{\imath} - \sin\theta\hat{\jmath} = -\hat{e}_r$$

Thus

$$\frac{d\hat{e}_r}{dt} = \frac{d\theta}{dt}\frac{d\hat{e}_r}{d\theta} = \frac{d\theta}{dt}\hat{e}_\theta$$
$$\frac{d\hat{e}_\theta}{dt} = \frac{d\theta}{dt}\frac{d\hat{e}_\theta}{d\theta} = -\frac{d\theta}{dt}\hat{e}_r$$

$$\begin{aligned}\frac{d^2\vec{x}}{dt^2} &= \frac{d}{dt}\frac{d\left(r\hat{e}_r\right)}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}\right) = \frac{d}{dt}\left(\frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta\right) \\ &= \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{e}_\theta + r\frac{d\theta}{dt}\frac{d\hat{e}_\theta}{dt} \\ &= \frac{d^2r}{dt^2}\hat{e}_r + 2\frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d^2\theta}{dt^2}\hat{e}_\theta - r\left(\frac{d\theta}{dt}\right)^2\hat{e}_r\end{aligned} \tag{2}$$

From (1) and (2), we get

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \tag{3}$$

and

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0 \tag{4}$$

The above identity multiplied by r becomes

$$2r\frac{dr}{dt}\frac{d\theta}{dt}+r^2\frac{d^2\theta}{dt^2}=\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0$$

Thus

$$r^2\frac{d\theta}{dt}=\frac{\ell}{m}\text{ is a constant.}\tag{5}$$

This can also be seen from

$$\begin{aligned}\vec{L}=\vec{x}\times\vec{p}=m\vec{x}\times\frac{d\vec{x}}{dt}&=mr\hat{e}_r\times\left(\frac{dr}{dt}\hat{e}_r+r\frac{d\theta}{dt}\hat{e}_\theta\right)\\ =mr^2\frac{d\theta}{dt}\hat{e}_r\times\hat{e}_\theta&=mr^2\frac{d\theta}{dt}\hat{k}\end{aligned}$$

As a result, ℓ is identified as

$$\ell=\left|\vec{L}\right|$$

and

$$\frac{d}{dt}=\frac{d\theta}{dt}\frac{d}{d\theta}=\frac{\ell}{mr^2}\frac{d}{d\theta}\tag{6}$$

Combining (2) and (5), we get

$$\frac{d^2r}{dt^2}=\frac{\ell^2}{m^2r^3}-\frac{GM}{r^2}\tag{7}$$

(6) enables us to write

$$\begin{aligned}\frac{d^2r}{dt^2}&=\frac{\ell}{mr^2}\frac{d}{d\theta}\left(\frac{\ell}{mr^2}\frac{dr}{d\theta}\right)=-\frac{\ell^2}{m^2r^2}\frac{d}{d\theta}\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)\\ &=-\frac{\ell^2}{m^2r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\end{aligned}$$

So (7) becomes

$$-\frac{\ell^2}{m^2r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)=\frac{\ell^2}{m^2r^3}-\frac{GM}{r^2}$$

or

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}-\frac{GMm^2}{\ell^2}\right)=-\left(\frac{1}{r}-\frac{GMm^2}{\ell^2}\right)$$

The above equation has the general solution

$$\frac{1}{r}-\frac{GMm^2}{\ell^2}=-\frac{GMm^2}{\ell^2}e\cos\left(\theta-\theta_0\right) \tag{8}$$

that depends on two parameters e and θ_0. For simplicity, let us consider the orbital equation

$$\frac{1}{r}-\frac{GMm^2}{\ell^2}=-\frac{GMm^2}{\ell^2}e\cos\theta \tag{9}$$

The orbit for (8) can be obtained from(9) by substituting $\theta\to\theta-\theta_0$ or by rotating θ_0 counterclockwise.

1) $e=1:$ (9) becomes

$$\frac{1}{r}=\frac{GMm^2}{\ell^2}\left(1-\cos\theta\right)\to r^2=\left(\frac{\ell^2}{GMm^2}+r\cos\theta\right)^2$$

$$\begin{aligned} x^2+y^2&=\left(\frac{\ell^2}{GMm^2}+x\right)^2=x^2+\left(\frac{\ell^2}{GMm^2}\right)^2+2x\frac{\ell^2}{GMm^2}\\ \to y^2&=\left(\frac{\ell^2}{GMm^2}\right)^2+2x\frac{\ell^2}{GMm^2}\end{aligned}$$

which an equation for parabola.

When $e\neq 1$, we may identify

$$\frac{\ell^2}{GMm^2}=a\left|1-e^2\right| \tag{10}$$

and

$$r=\frac{a\left|1-e^2\right|}{1-e\cos\theta} \tag{11}$$

2) $0\leq e<1$**:** (11) becomes

$$r=\frac{a\left(1-e^2\right)}{1-e\cos\theta} \tag{12}$$

Since

$$\frac{a\left(1-e^2\right)}{1-e\cos\theta}\leq\frac{a\left(1-e^2\right)}{1-e}=a\left(1+e\right)$$

we have

$$a(1-e) \leq r \leq a(1+e) < 2a$$

For (12), the distance r' from the position $(r\cos\theta, r\sin\theta)$ to $(2ae, 0)$ is

$$\begin{aligned} r' &= \sqrt{(r\cos\theta - 2ae)^2 + r^2\sin^2\theta} \\ &= \sqrt{-4are\cos\theta + 4a^2e^2 + r^2} \\ &= \sqrt{4ar(1 - e\cos\theta) + 4a^2e^2 - 4ar + r^2} \\ &= \sqrt{4a^2(1-e^2) + 4a^2e^2 - 4ar + r^2} \\ &= \sqrt{(r-2a)^2} = 2a - r \end{aligned}$$

Thus

$$r + r' = 2a$$

and (12) is the equation for an elliptical orbit with two foci at $(0,0)$ and $(2ae, 0)$, e being the eccentricity and a being the length of semimajor axis.

3) $e > 1$**:** (11) becomes

$$r = \frac{a(e^2-1)}{1 - e\cos\theta} \tag{13}$$

θ in the above must be in the range of

$$\cos^{-1}\left(\frac{1}{e}\right) < \theta < 2\pi - \cos^{-1}\left(\frac{1}{e}\right)$$

to have a positive value of r. The distance from the position $(r\cos\theta, r\sin\theta)$ to $(-2ae, 0)$ is

$$\begin{aligned} r' &= \sqrt{(r\cos\theta + 2ae)^2 + r^2\sin^2\theta} \\ &= \sqrt{4are\cos\theta + 4a^2e^2 + r^2} \\ &= \sqrt{-4ar(1 - e\cos\theta) + 4a^2e^2 + 4ar + r^2} \\ &= \sqrt{4a^2(1-e^2) + 4a^2e^2 + 4ar + r^2} \\ &= \sqrt{(r+2a)^2} = 2a + r \end{aligned}$$

We have

$$r' - r = 2a$$

This is the equation for the right branch of a hyperbola with foci at $(0,0)$ and $(-2ae, 0)$

Total Energy

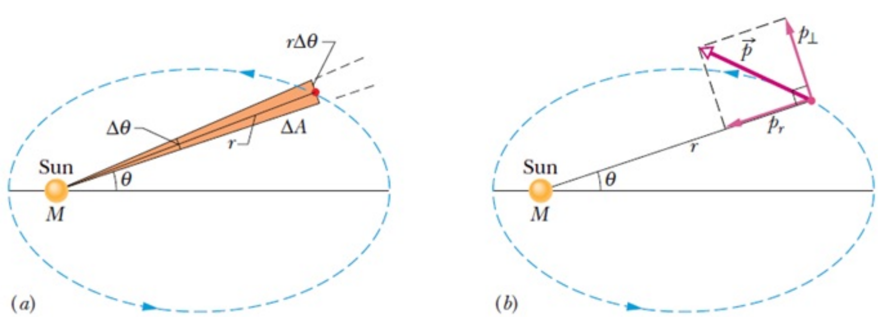
$$
\begin{aligned}
E &= \frac{1}{2}m\left(\frac{d\vec{x}}{dt}\right)^2 - \frac{GMm}{r} = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) - \frac{GMm}{r} \\
&= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2mr^2} - \frac{GMm}{r} = \frac{1}{2}\frac{\ell^2}{m}\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2 + \frac{\ell^2}{2mr^2} - \frac{GMm}{r} \\
&= \frac{1}{2}\frac{\ell^2}{m}\frac{e^2\sin^2(\theta-\theta_0)}{a^2(1-e^2)^2} + \frac{\ell^2}{2m}\frac{(1+e\cos(\theta-\theta_0))^2}{a^2(1-e^2)^2} - GMm\frac{1+e\cos(\theta-\theta_0)}{a\,|1-e^2|} \\
&= \frac{GMm}{2}\left(\frac{e^2\sin^2(\theta-\theta_0)}{a\,|1-e^2|} + \frac{(1+e\cos(\theta-\theta_0))^2}{a\,|1-e^2|} - 2\frac{1+e\cos(\theta-\theta_0)}{a\,|1-e^2|}\right) \\
&= -\frac{GMm}{2a}\frac{1-e^2}{|1-e^2|}
\end{aligned}
$$

where we have used (8) and (10).

13.7.2 Kepler's Second Law

The line that connects a planet to the sun sweeps out equal areas ΔA in the plane of the orbit in equal time intervals Δt.

$$\frac{dA}{dt} = constant$$



The area ΔA swept out by the planet (fig. a) is given by the equation: $\Delta A \approx \frac{1}{2}r^2\Delta\theta$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega$$

whereω is the planet's angular speed. The planet's angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} = mrv_{\perp}\hat{L} = 2m\frac{dA}{dt}\hat{L}$$

So

$$\frac{dA}{dt} = \frac{\left|\vec{L}\right|}{2m}$$

Kepler's second law is equivalent to the law of conservation of angular momentum.

13.7.3 Kepler's Third law

The square of the period T of any planet is proportional to the cube of the semimajor axis of its orbit a.

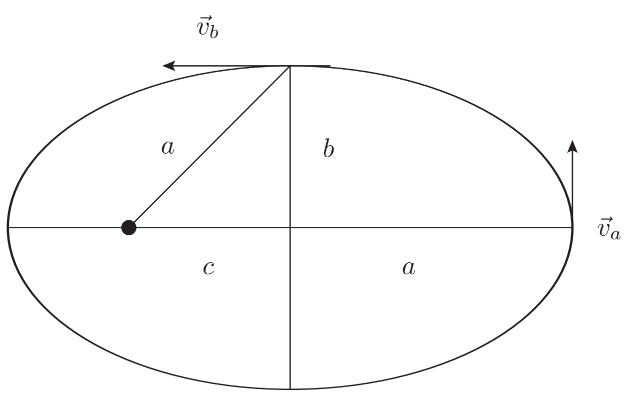
$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

Let us first consider the simpler case of circular orbit. A planet of mass m moves on a circular orbit of radius a around a star of mass M. We apply Newton's second law to the motion:

$$F_g = G\frac{Mm}{a^2} = ma\omega^2 = ma\left(\frac{2\pi}{T}\right)^2 \rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

Note that the ratio $\frac{T^2}{a^3}$ does not depend on the mass m of the planet but only on the mass M of the central star.

In general, for elliptical orbits,



Conservation of angular momentum:

$$\left|\vec{L}\right| = m\left(a+c\right)v_a = mbv_b$$

gives us

$$v_a = \frac{b}{a+c}v_b \tag{14}$$

Energy conservation:

$$\begin{aligned} E &= K_a + U_a = K_b + U_b \\ &= \frac{1}{2}mv_a^2 - \frac{GMm}{a+c} = \frac{1}{2}mv_b^2 - \frac{GMm}{a} \end{aligned} \tag{15}$$

(14) and (15) may be used to solve for v_b.

$$\left(\frac{b}{a+c}v_b\right)^2 - \frac{2GM}{a+c} = v_b^2 - \frac{2GM}{a}$$

$$v_b^2\frac{\left(a+c\right)^2 - b^2}{\left(a+c\right)^2} = \frac{2GM}{a} - \frac{2GM}{a+c}$$

$$v_b^2 = 2GM\left(\frac{1}{a} - \frac{1}{a+c}\right)\frac{\left(a+c\right)^2}{\left(a+c\right)^2 - b^2} = 2GM\frac{c}{a\left(a+c\right)}\frac{\left(a+c\right)^2}{2c^2+2ac} = \frac{GM}{a} \tag{16}$$

Now

$$T = \frac{\pi ab}{\frac{dA}{dt}} = \frac{\pi ab}{\frac{\left|\vec{L}\right|}{2m}} = 2m\frac{\pi ab}{\left|\vec{L}\right|} = \frac{2\pi a}{v_b}$$

$$T^2 = \frac{4\pi^2 a^2}{v_b^2} = \frac{4\pi^2}{GM}a^3$$

13.8 Satellites: Orbits and Energy

Consider a satellite that follows a circular orbit of radius r around a planet of mass M. We apply Newton's second law and have:

$$\frac{GMm}{r^2} = ma = m\frac{v^2}{r} \to v^2 = \frac{GM}{r}$$

The kinetic energy

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}$$

The potential energy

$$U = -\frac{GMm}{r} = -2K$$

The total energy

$$E = K + U = -\frac{1}{2}\frac{GMm}{r} = -K$$

For elliptical orbits

$$E = K_b + U_b = \frac{1}{2}mv_b^2 - \frac{GMm}{a} = -\frac{1}{2}\frac{GMm}{a} = -K_b$$