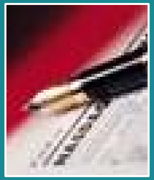


Chapter 2
Linear Time-Invariant Systems

2.1.1 The Representation of Discrete-Time Signals in Terms of Impulses

$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases},$$

$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases},$$

$$x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}.$$

將任何訊號$x[n]$以一連串不同大小的脈衝函數的合成來表示。

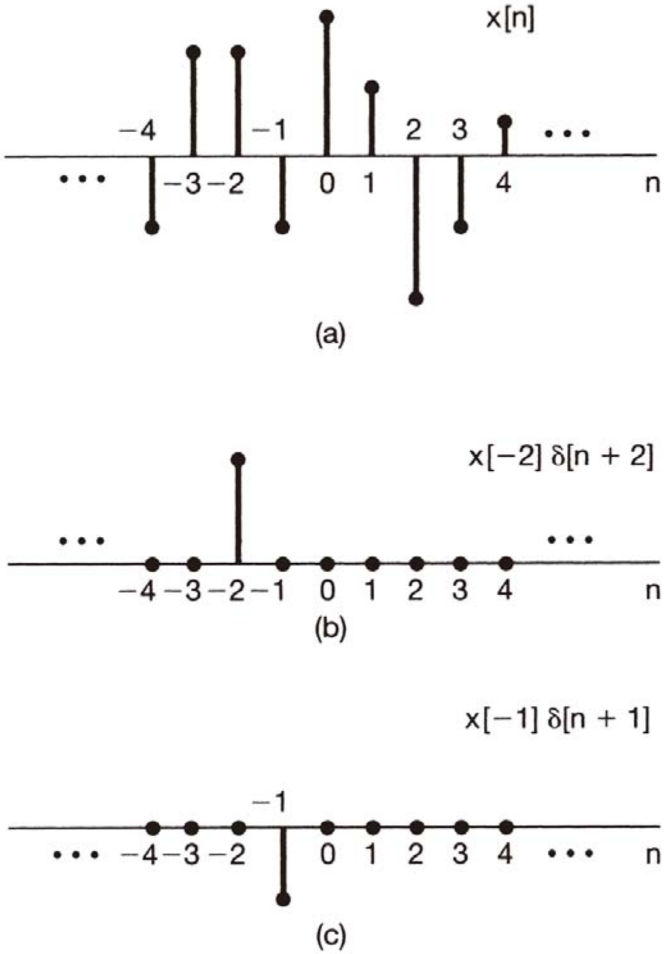
2.1.1 The Representation of Discrete-Time Signals in Terms of Impulses

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta + x[-1]\delta[n+1] + x[0]\delta[n] \\ + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots \qquad (2.1)$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]. \qquad (2.2)$$

任何序列$x[n]$均可用不同時間移位的單位脈衝$\delta[n\text{-}k]$的線性組合來表示。

2.1.1 The Representation of Discrete-Time Signals in Terms of Impulses



2.1.1 The Representation of Discrete-Time Signals in Terms of Impulses

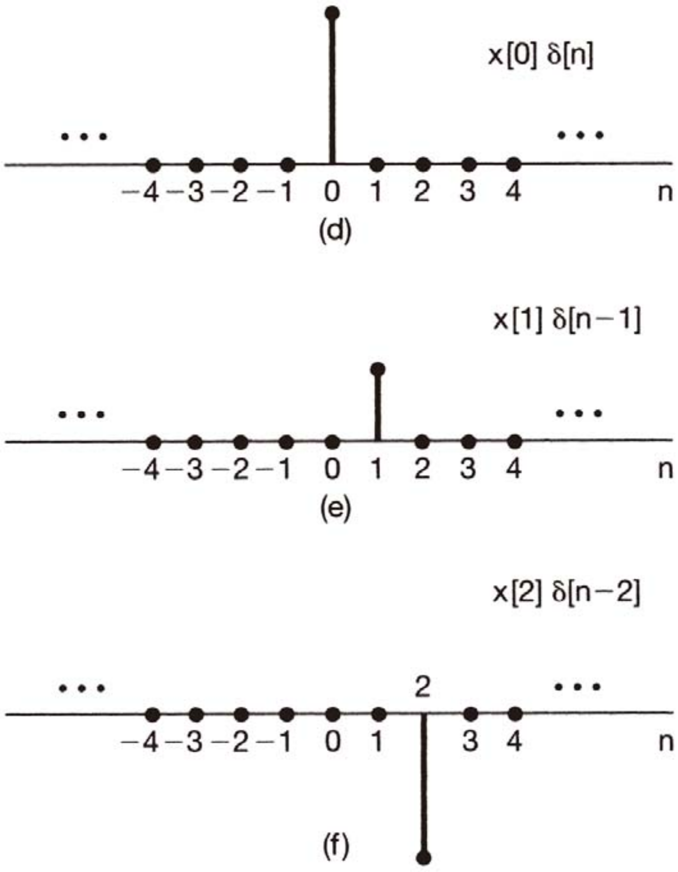


Figure 2.1 Decomposition of a discrete-time signal into a weighted sum of shifted impulses.

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]. \qquad (2.3)$$

任何輸入$x[n]$可表為時間移位的脈衝函數$\delta[n\text{-}k]$的重疊，則線性系統對於$x[n]$的響應，即為系統對每一個時間移位的脈衝響應的重疊。

$h_k[n]$代表一線性系統對於時間移位單位脈衝$\delta[n\text{-}k]$的響應。

系統輸入$y[n]$與任意輸入$x[n]$，及系統單位脈衝響應$h_k[n]$的關係式。

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

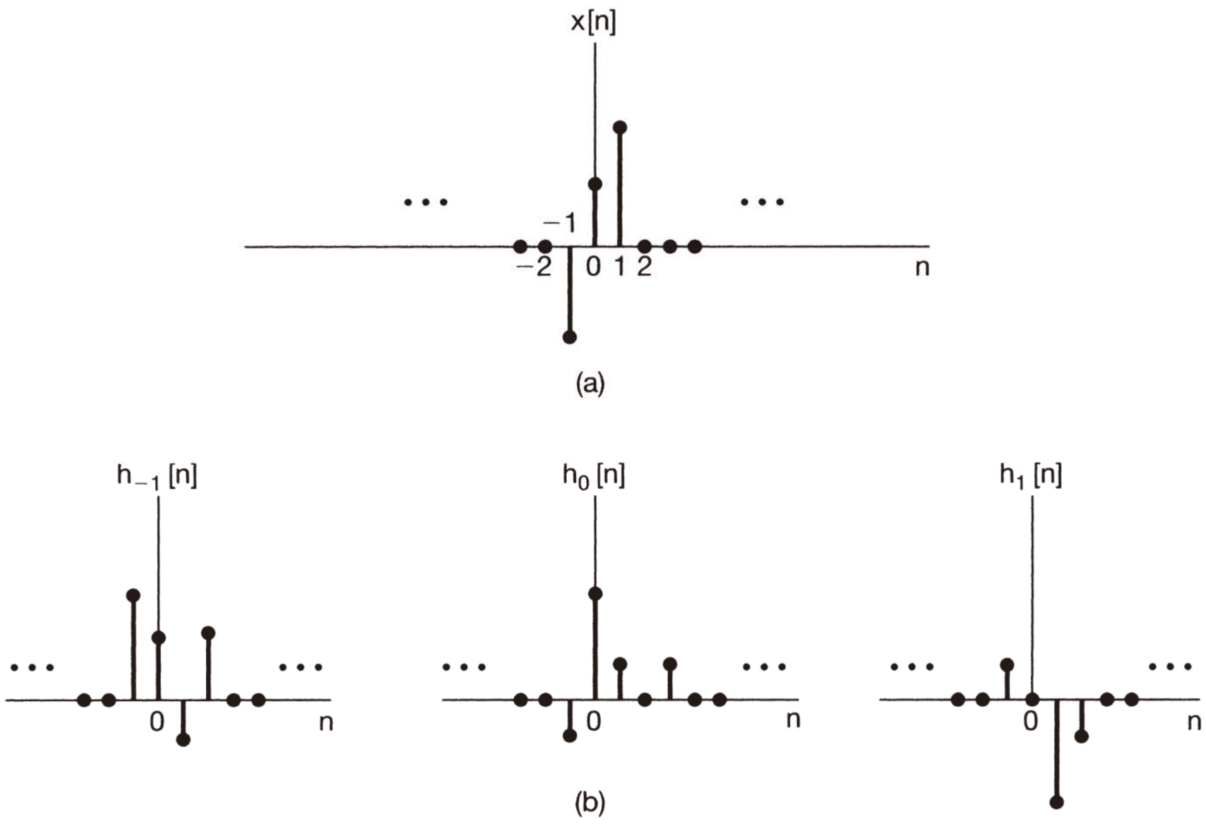


Figure 2.2 Graphical interpretation of the response of a discrete-time linear system as expressed in eq. (2.3).

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

Since $\delta[n-k]$ is a time-shifted version of $\delta[n]$, the response $h_k[n]$ is a time-shifted version of $h_0[n]$

$$h_k[n] = h_0[n-k] \qquad (2.4)$$

若系統為非時變則$h_k[n]$將與$\delta[n]$為輸入在時間n-k處的響應相同，即：$h_k[n] = h_0[n-k]$

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

we ill drop the subscript on $h_0[n]$ and define the unit impulse response

將$h_0[n]$簡寫成$h[n]$

$$h[n] = h_0[n] \qquad (2.5)$$

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]. \qquad (2.6)$$

LTI 系統的輸出與任意輸入及單位脈衝響應的關係式

$$y[n] = x[n] * h[n]. \qquad (2.7)$$

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases} \tag{2.24}$$

then, since $\Delta\delta_\Delta(t)$ has unit amplitude, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta. \tag{2.25}$$

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

$$x[t] = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta. \qquad (2.26)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(1 - \tau)\, dt. \qquad (2.27)$$

訊號$x(t)$的脈衝函數表示法

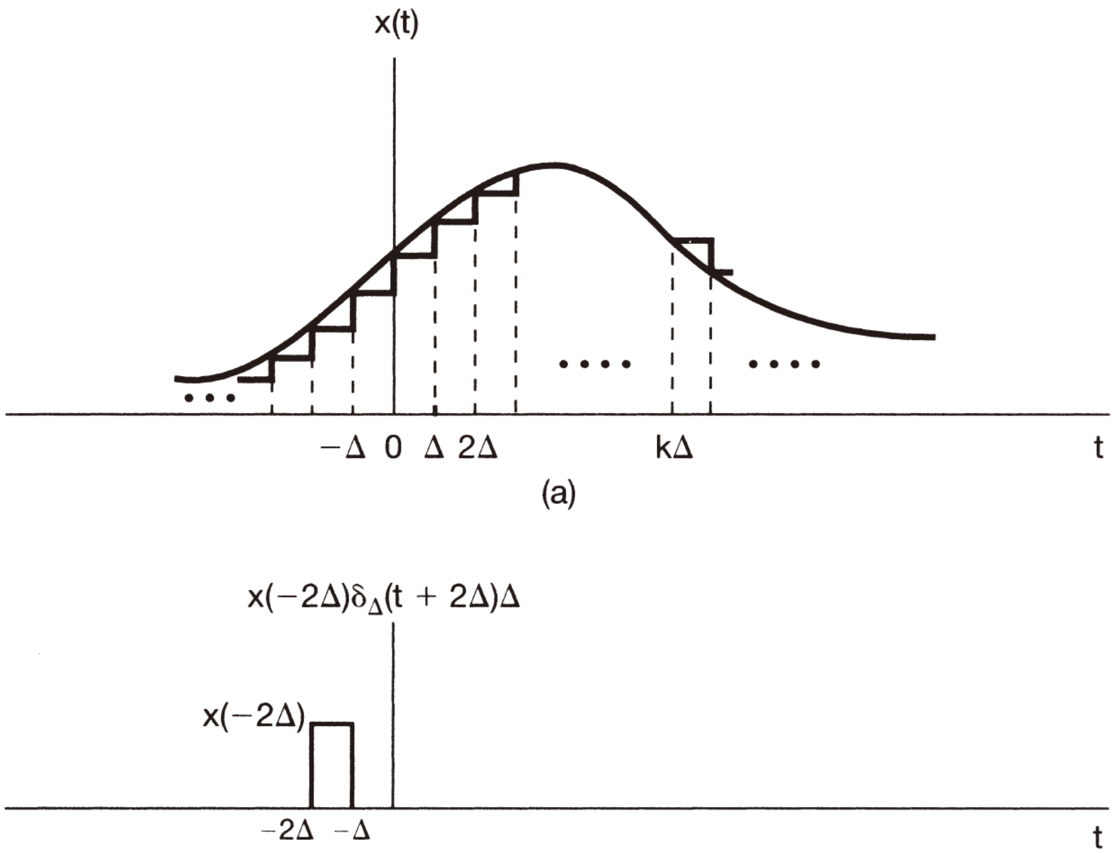
2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

- As in discrete time, we refer to eq.(2.27) as the sifting property of the continuous-time impulse. We note that, for the specific example of $x(t) = u(t)$, eq. (2.27) becomes

$$u(t) = \int_{-\infty}^{+\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau, \qquad (2.28)$$

since $u(\tau) = 0$ for $\tau < 0$ and $u(\tau) = 1$ for $\tau > 0$. Equation (2.28) is identical to eq.(1.75), derived in Section 1.42

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses



(a)

(b)

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

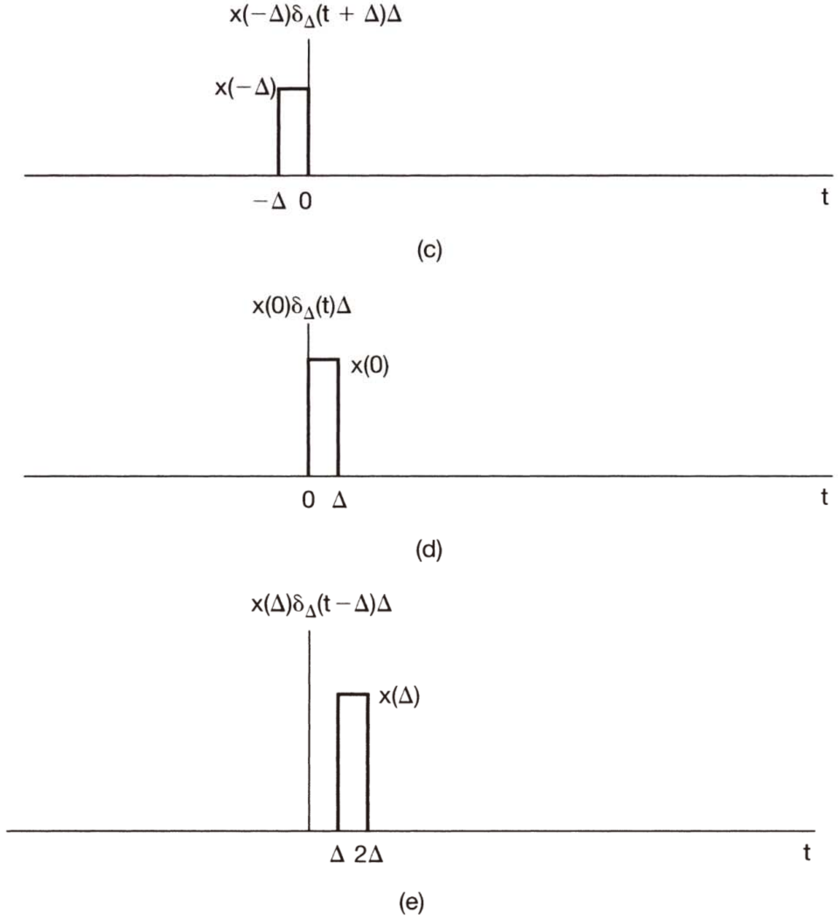


Figure 2.12 Staircase approximation to a continuous-time signal.

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

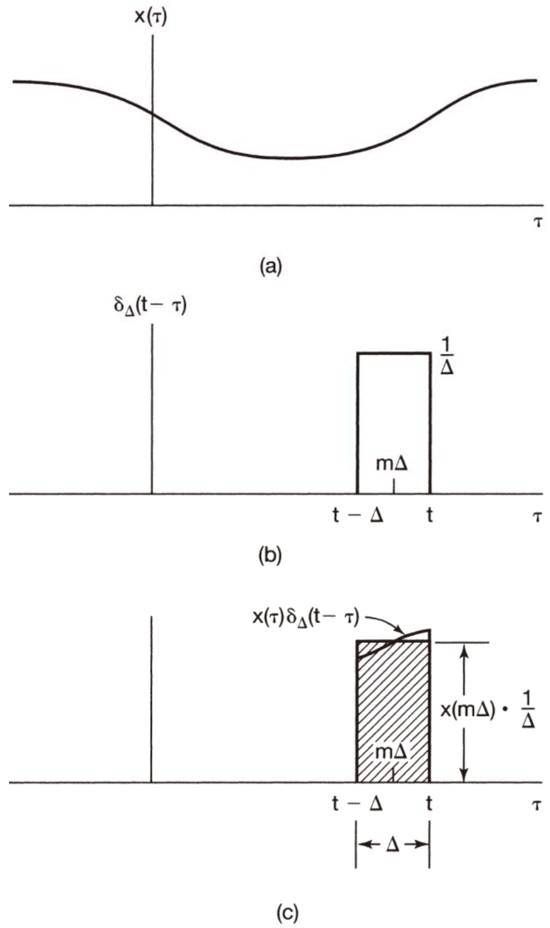
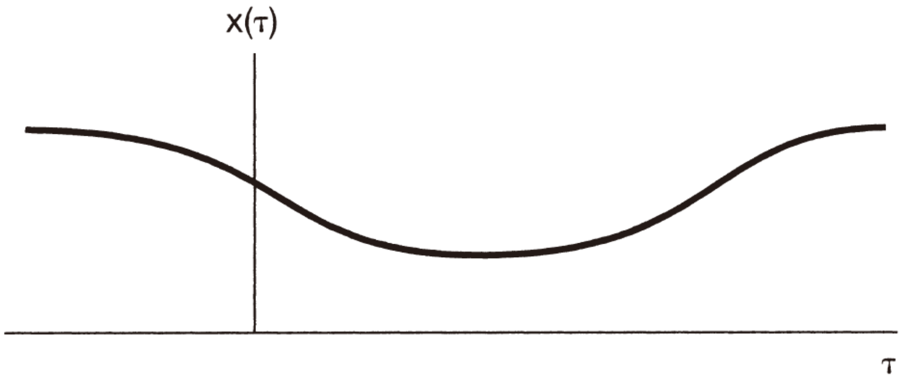
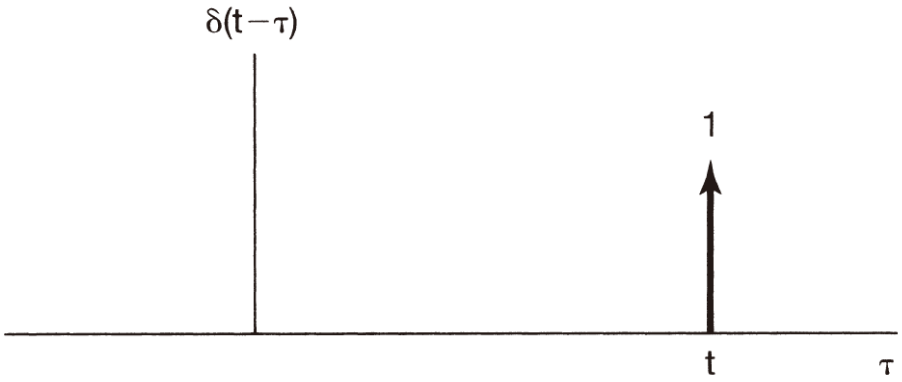


Figure 2.13 Graphical interpretation of eq. (2.26).

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses



(a)



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2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

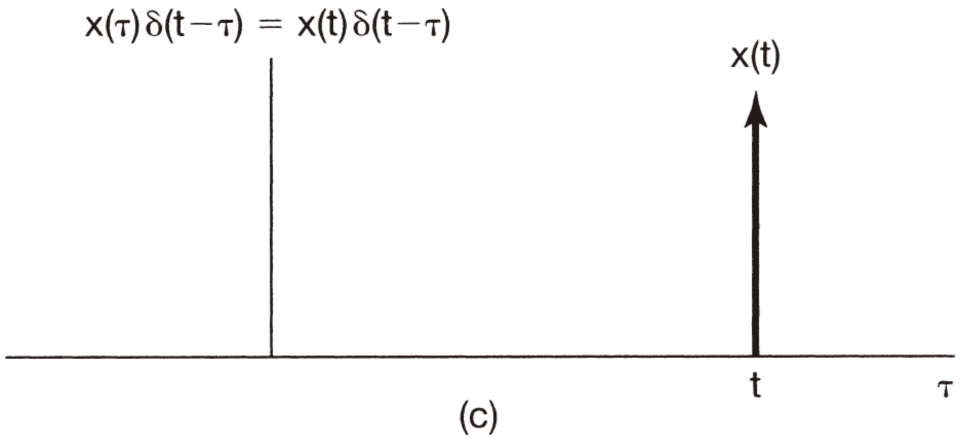


Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a function of τ with t fixed; (c) product of these two signals.

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

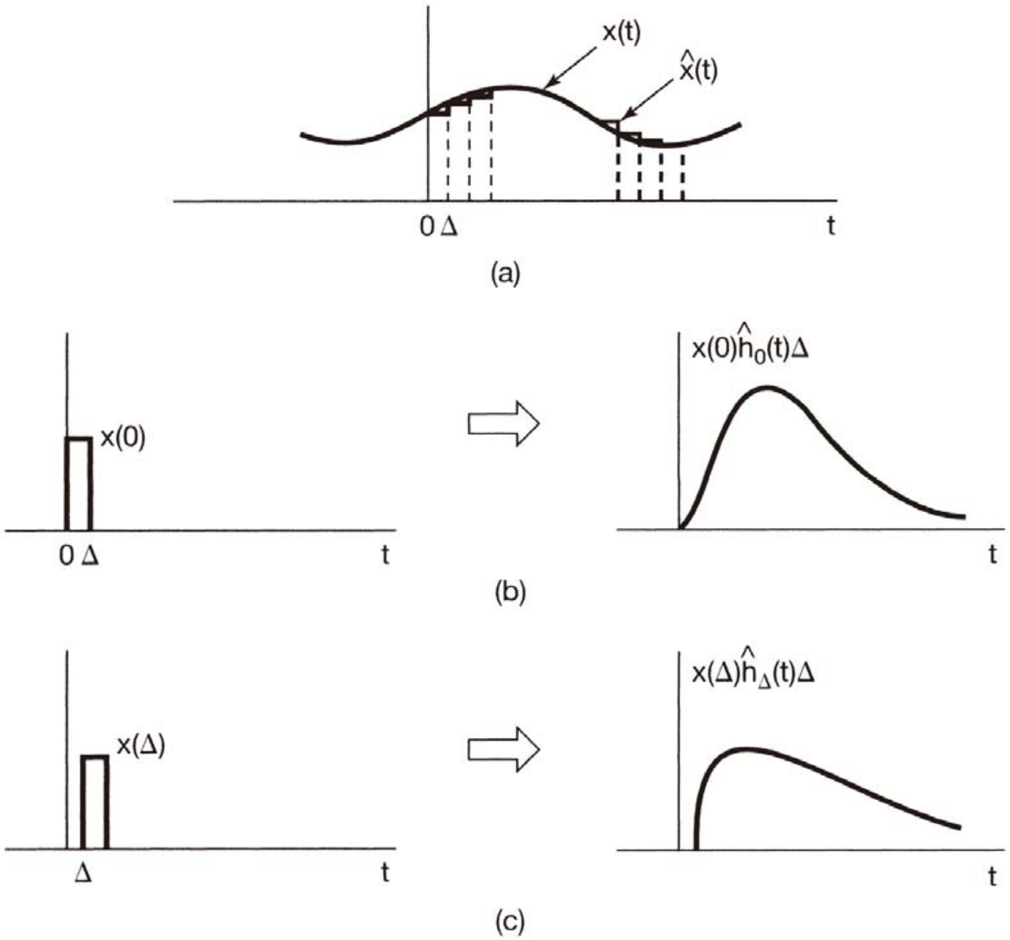
as the response of an LTI system to the input $\delta_\Delta(t-k\Delta)$. Then from eq.(2.25) and the superposition property, for continuous-time linear systems, we see that

$$\hat{y}(t)=\sum_{k=-\infty}^{+\infty}x(k\Delta)\hat{h}_{k\Delta}(t)\Delta. \qquad (2.29)$$

Therefore, if we let $h_\tau(t)$ denote the response at time t to a unit impulse $\delta(t-\tau)$ located at time τ , then

$$y(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{+\infty}x(k\Delta)\hat{h}_{k\Delta}(t)\Delta. \qquad (2.30)$$

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems



2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

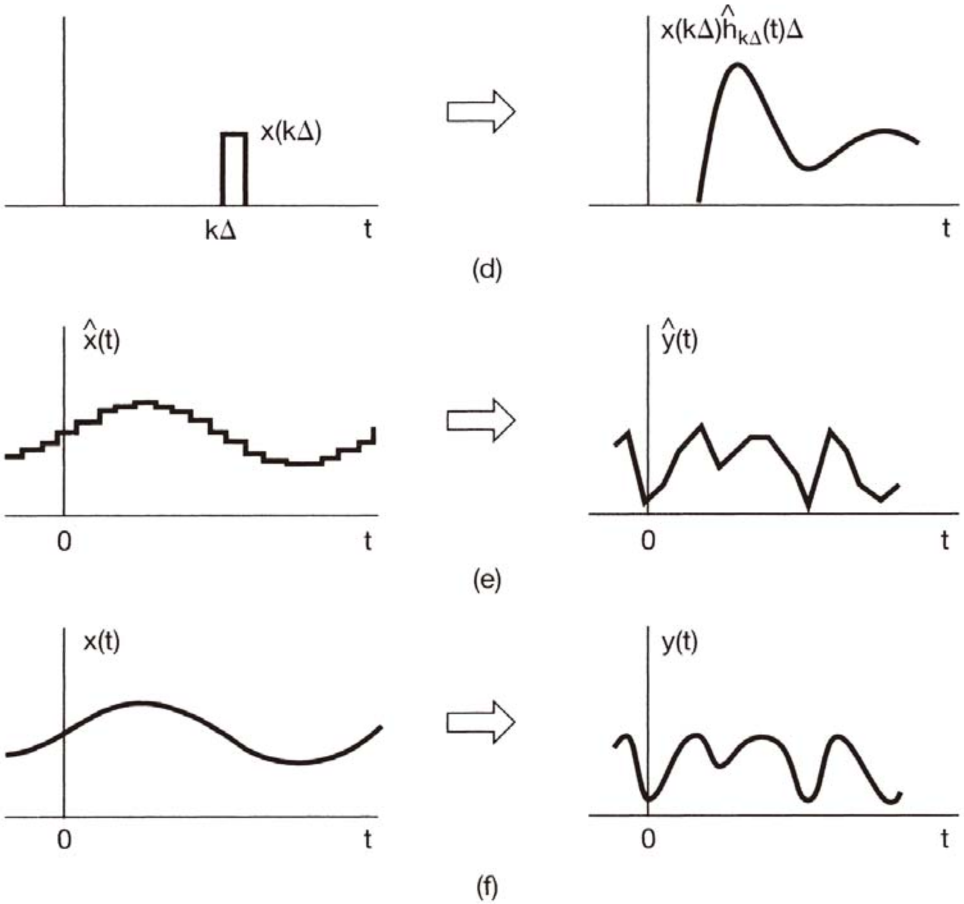


Figure 2.15 Graphical interpretation of the response of a continuous-time linear system as expressed in eqs. (2.29) and (2.30).

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h_{\tau}(t) d_{\tau}. \qquad (2.31)$$

線性系統的輸出與任意輸入及脈衝響應關係式

$$h(t) = h_0(t) \qquad (2.32)$$

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \qquad (2.33)$$

LTI 系統的輸出與任意輸入及單位脈衝響應的關係

$$y(t) = x(t) * h(t). \qquad (2.34)$$

上式稱為「迴旋積分」或「重疊積分」，可寫成 $y(t)=x(t)*h(t)$。

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

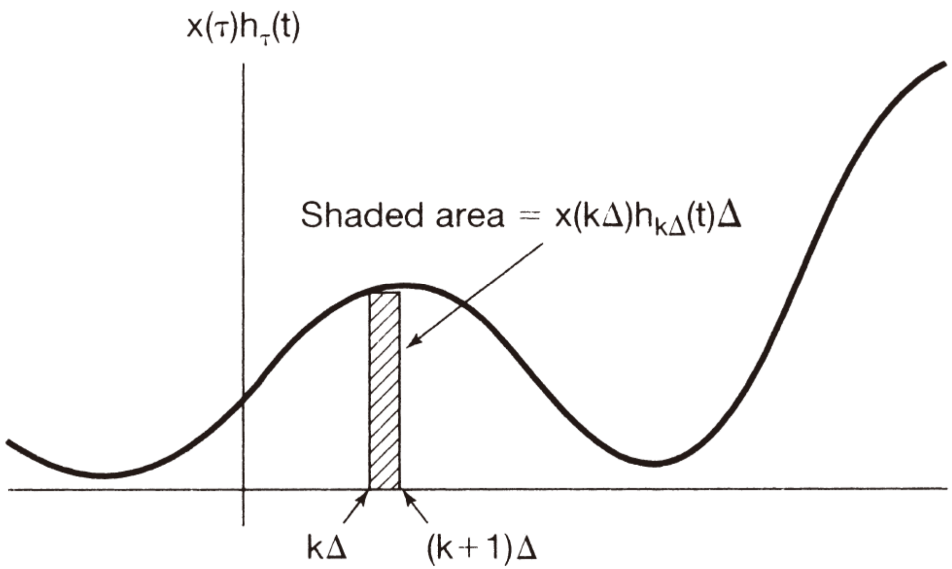


Figure 2.16 Graphical illustration of eqs. (2.30) and (2.31).

2.3 Properties Of Linear Time-Invariant Systems

$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]=x[n]*h[n] \qquad (2.39)$$

$$y(t0=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau=x(t)*h(t) \qquad (2.40)$$

連續時間與離散時間 LTI 系統的輸出為輸入與單位脈衝響應的迴旋運算式

上述性質只適用於 LTI 系統。

Example 29

- Consider a discrete-time system with unit impulse response

$$h[n] = \begin{cases} 1, & n=0,1 \\ 0, & \text{otherwise} \end{cases}. \qquad (2.41)$$

If the system is LTI, then eq. (2.41) completely determines its input-output behavior.

Example 29

$$y[n] = x[n] + x[n-1] \qquad (2.42)$$

For example, both of the following systems have this property:

$$y[n] = (x[n] + x[n-1])^2,$$
$$y[n] = \max(x[n], x[n-1])$$

2.3.1 The Commutative Property

$$x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty}^{+\infty}h[k]x[n-k], \qquad (2.43)$$

連續與離散時間迴旋運算適合交換律。即兩個運算元位置，其結果不變。

$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{+\infty}h(\tau)x(t-\tau)d\tau. \qquad (2.44)$$

2.3.1 The Commutative Property

- These expressions can be verified in a straightforward manner by means of a substitution of variables in eq. (2.39) and (2.40). For example, in the discrete-time case, if we let $r = n - k$ or , equivalently, $k = n - r$, eq. (2.39) becomes

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=-\infty}^{+\infty} x[n-r]h[r] = h[n] * x[n]. \qquad (2.45)$$

2.3.2 The Distributive Property

in discrete time

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n], \quad (2.46)$$

連續與離散時間迴旋運算適合分配律。

and in continuous time

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t). \quad (2.47)$$

2.3.2 The Distributive Property

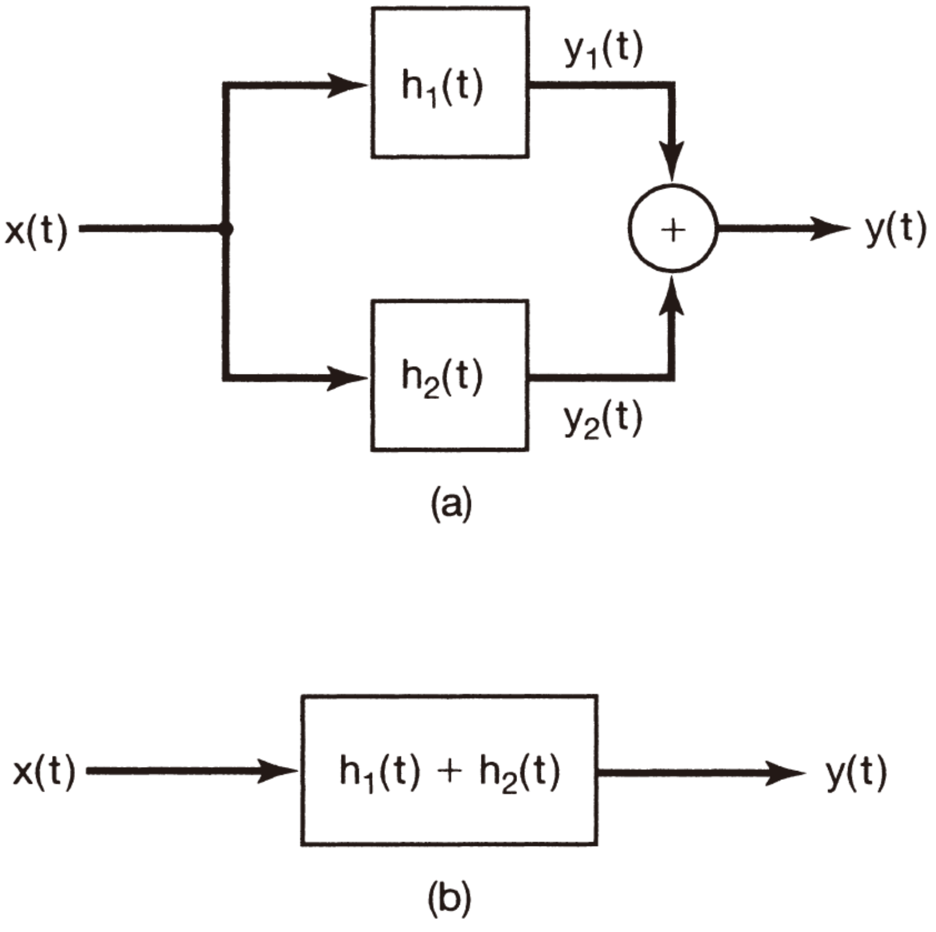


Figure 2.23 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

2.3.2 The Distributive Property

The two systems, with impulse response $h_1(t)$ and $h_2(t)$, have identical inputs, and their outputs are added. Since

$$y_1(t) = x(t) * h_1(t)$$

and

$$y_2(t) = x(t) * h_2(t)$$

2.3.2 The Distributive Property

the system of Figure 2.23(a) has output

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t), \tag{2.48}$$

corresponding to the right-hand side of eq.(2.47). The system of Figure 2.33(b) has output

$$y(t) = x(t) * [h_1(t) + h_2(t)], \tag{2.49}$$

2.3.2 The Distributive Property

Also, as a consequence of both the commutative and distributive properties, we have

$$\left[x_1[n]+x_2[n]\right]*h[n]=x_1[n]*h[n]+x_2[n]*h[n] \qquad (2.50)$$

and

$$\left[x_1(t)+x_2(t)\right]*h(t)=x_1(t)*h(t)+x_2(t)*h(t), \qquad (2.51)$$

which simply state that the response of an LTI system to the sum of two inputs must equal the sum of the response to these signals individually.

2.3.3 The Association Property

in discrete time

$$x[n]*(h_1[n]*h_2[n])=(x[n]*h_1[n])*h_2[n], \qquad (2.58)$$

and in continuous time

$$x(t)*[h_1(t)*h_2(t)]=[x(t)*h_1(t)]*h_2(t). \qquad (2.59)$$

連續與離散時間迴旋運算適合結合律。

2.3.3 The Association Property

As a consequence to the associative property, the expressions

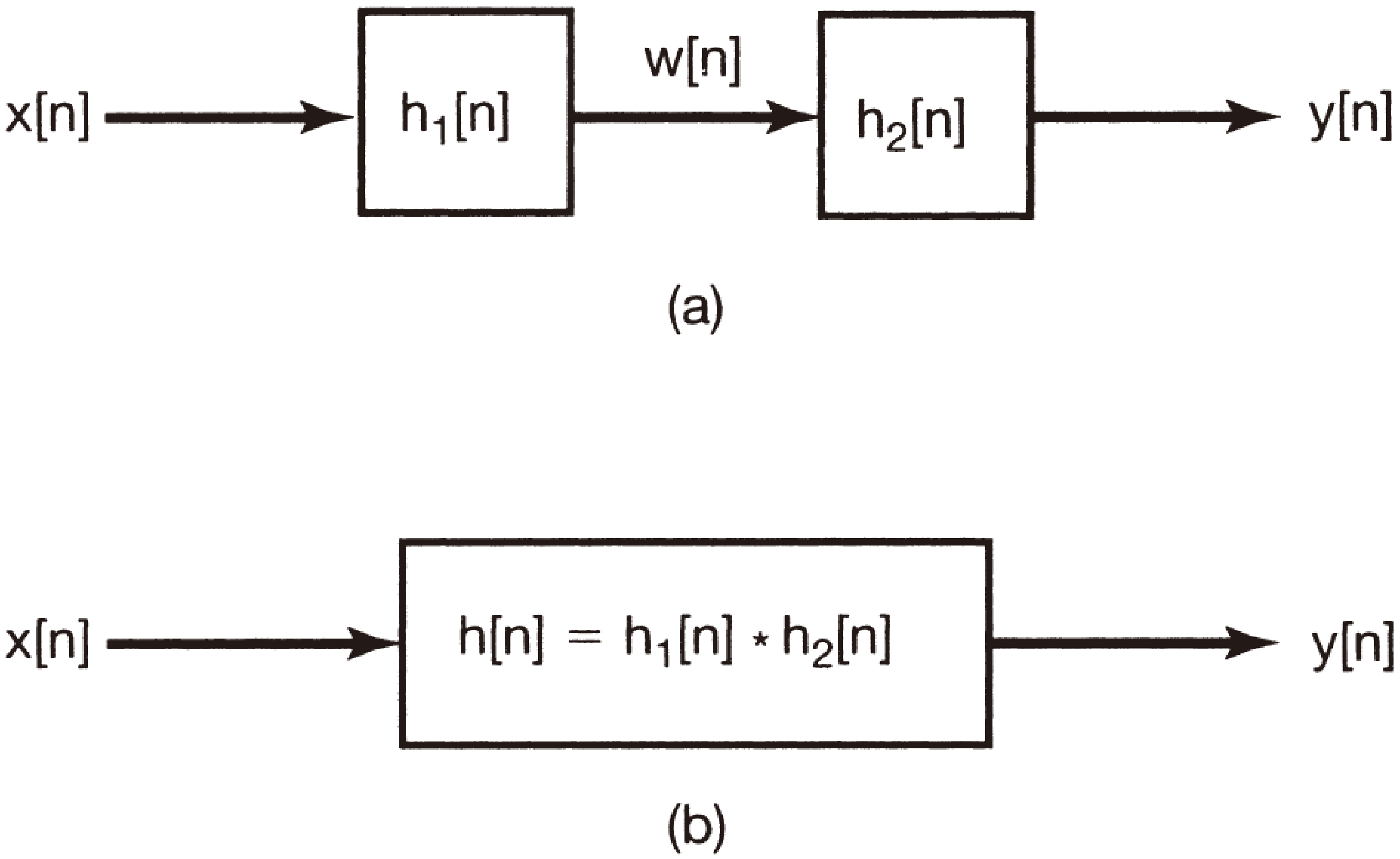
$$y[n] = x[n] * h_1[n] * h_2[n] \qquad (2.60)$$

and

$$y(t) = x(t) * h_1(t) * h_2(t) \qquad (2.61)$$

are unambiguous.

2.3.3 The Association Property



2.3.3 The Association Property

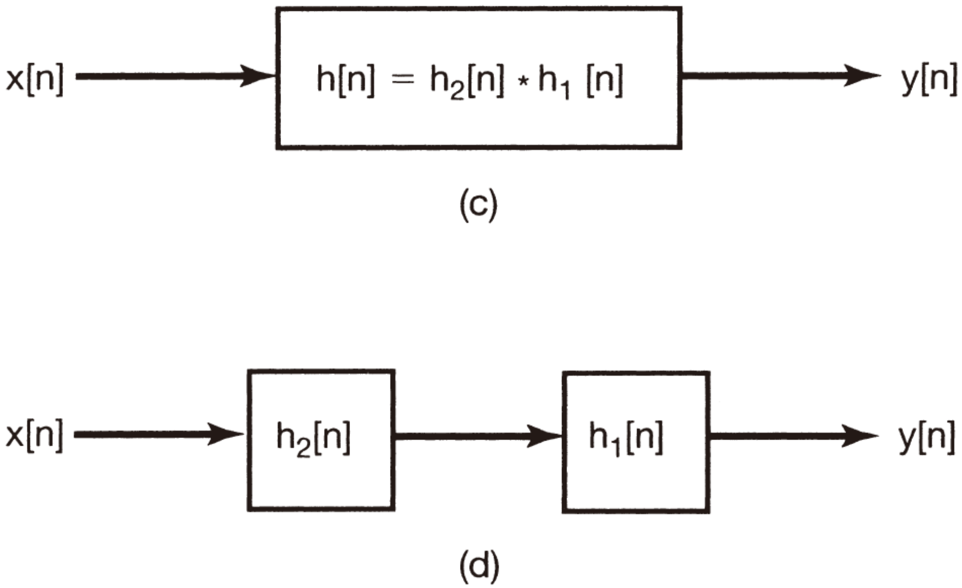


Figure 2.25 Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

2.3.4 LTI Systems with and without Memory

若且為若 $t \neq 0$ 時， $h[t]=0$ ，則連續時間 LTI 系統為無記憶性。

2.3.4 LTI Systems with and without Memory

the impulse response has the form

$$h[n] = k\delta[n], \qquad (2.62)$$

where $k = h[0]$ is a constant, and the convolution sum reduces to the relation

$$y[n] = kx[n]. \qquad (2.63)$$

2.3.4 LTI Systems with and without Memory

In particular, a continuous-time LTI system is memoryless if $h(t) = 0$ for $t \neq 0$, and such a memoryless LTI system has the form

若$t \neq 0$時，$h(t) = 0$則連續時間LTI系統為無記憶性

$$y(t) = Kx(t) \qquad (2.64)$$

$$h(t) = K\delta(t) \qquad (2.65)$$

2.3.5 Invertibility of LTI Systems

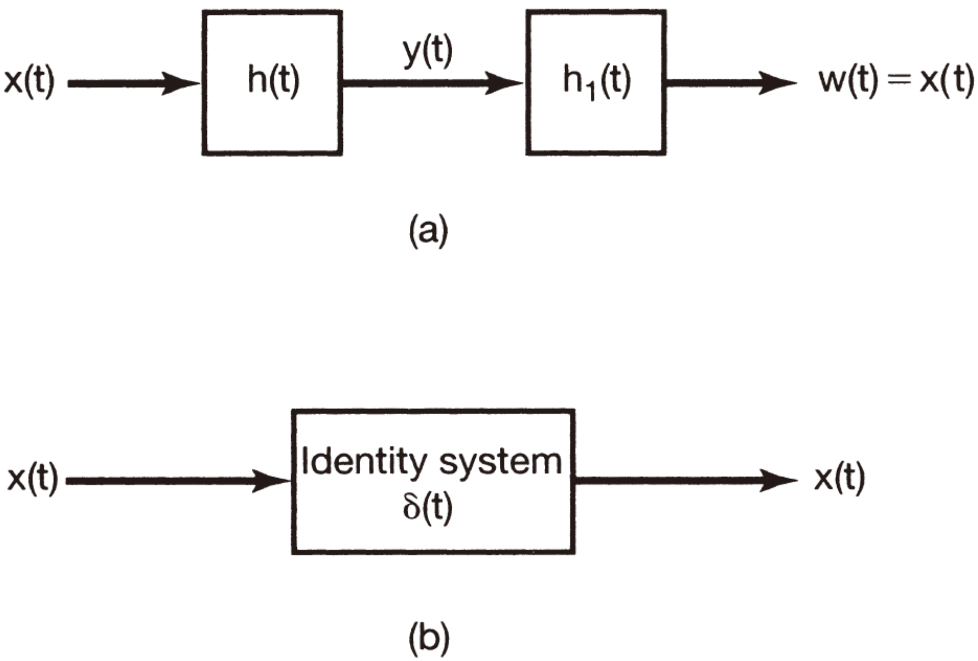


Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

若一 LTI 系統的脈衝應為 h[t]，其逆系統的脈衝響應為$h_1(t)$，則$h(t)*h_1(t)=\delta(t)$。

同理，對於離散時間LTI系統滿足$h[n]*h_1[n]=\delta[n]$。

2.3.5 Invertibility of LTI Systems

we have the condition that $h_1(t)$ must satisfy for to be the impulse response of the inverse system, namely,

$$h(t) * h_1(t) = \delta(t). \qquad (2.66)$$

the impulse response $h_1[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy

$$h[n] * h_1[n] = \delta[n]. \qquad (2.67)$$

2.3.6Causality for LTI Systems

- 一個因果系統的輸出只取決於現在和過去的系統輸入。
- 因 $h[n]$ 為一離散時間 LTI 系統在 $n=0$ 時加入單位脈衝輸入所引起的響應。故對因果 LTI 系統而言：

  對所有 $n<0, h[n]=0$
- 同理可得一個因果的連續時間 LTI 系統必滿足：

  對所有 $t<0, h(t)=0$

2.3.6Causality for LTI Systems

- The impulse response of a causal discrete-time LTI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0 \qquad (2.77)$$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k], \qquad (2.78)$$

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]. \qquad (2.79)$$

2.3.6Causality for LTI Systems

Similarly, a continuous-time LTI system is causal if

$$h(t) = 0 \qquad \text{for } \ t < 0 \qquad (2.80)$$

同理可得一個因果的連續時間LTI系統必滿足對所有$t < 0$, $h(t)$=0

and the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau. \qquad (2.81)$$

2.3.7 Stability for LTI Systems

- 若對每一個有界輸入均可產生有界輸出，則系統為穩定。
- 若一個離散時間 LTI 系統的脈衝響應為絕對可加的，即： $\sum_{k=-\infty}^{\infty}|h[k]|<\infty$ 則此系統穩定。
- 同理可得若一個連續時間 LTI 系統的脈衝響應為絕對可積分，即：$\int_{-\infty}^{+\infty}|h(\tau)|d\tau<\infty$則此系統為穩定。

2.3.7 Stability for LTI Systems

- A system is stable if every bounded input produces a bounded output.
  若對每一個有界輸入均可產生有界輸出，則系統為穩定

$$|x[n]| < B \qquad \text{for all } n \qquad (2.82)$$

we obtain an expression for the magnitude of the output:

$$|y[n]| = \left|\sum_{k-\infty}^{+\infty} h[k]x[n-k]\right|. \qquad (2.83)$$

2.3.7 Stability for LTI Systems

- Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|. \tag{2.84}$$

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| \qquad \text{for all } n. \tag{2.85}$$

2.3.7 Stability for LTI Systems

we can conclude that if the impulse response is absolutely summable. That is, if

$$\sum_{k=-\infty}^{+\infty} \left|h[k]\right| < \infty \qquad (2.86)$$

若一個離散時間LTI系統的脈衝響應為絕對可加的，即：

$$\sum_{k=-\infty}^{+\infty} \left|h[k]\right| < \infty$$

則此系統穩定

2.3.7 Stability for LTI Systems

the system is stable if the impulse response is absolutely integrable

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty \qquad (2.87)$$

同理可得若一個連續時間LTI系統的脈衝響應為絕對可積分，即

$$\int_{-\infty}^{+\infty} |h(\tau)| dt < \infty$$

則此系統為穩定

2.3.8The Unit Step Response of an LTI System

設步級函數為u[n]或 u(t)，LTI 系統的單位脈衝響應為 h[n]或 h(t)，而單位步級響應為 s[n]或 s(t)，

則： $$s[n]=u[n]*h[n]=\sum_{k=-\infty}^{n}h[k]$$

反之可得 $$h[n]=s[n]-s[n-1]$$

同理，在連續時間中： $$s(t)=u(t)*h(t)=\int_{-\infty}^{t}h(\tau)d\tau$$

反之可得： $$h(t)=\frac{ds(t)}{dt}=s'(t)$$

2.3.8 The Unit Step Response of an LTI System

$$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad (2.91)$$

$$h[n] = s[n] - s[n-1]. \qquad (2.92)$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \qquad (2.93)$$

$$h(t) = \frac{ds(t)}{dt} = s'(t). \qquad (2.94)$$

2.4.1 Linear Constant-Coefficient Differential Equations

- To introduce some of the important ideas concerning systems specified by linear constant-coefficient differential equations, let us consider a first-order differential equation as in eq. (1.85), viz.,

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad (2.95)$$

2.4.1 Linear Constant-Coefficient Differential Equations

- It is important to emphasize that the condition of initial rest does not specify a zero initial condition at a fixed point in time, but rather adjusts this point in time so that the response is zero until the input becomes nonzero.

  「初始靜止」的條件並非意指在某個固定時間點上的初始條件為零，而是指在輸入開始不為零之前，系統的響應維持為零

2.4.1 Linear Constant-Coefficient Differential Equations

A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (2.109)$$

N階線性常係數(非齊次)微分方程一般式

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (2.110)$$

2.4.1 Linear Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0 \qquad (2.111)$$

The solutions to this equation referred to as the natural responses of the system.

其解常稱為系統的「自然響應」

$$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1} y(t_0)}{dt^{N-1}} = 0 \qquad (2.112)$$

2.4.2 Linear Constant-Coefficient Difference Equations

The discrete-time counterpart of eq.(2.109) is the Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (2.113)$$

N階線性常係數(非齊次)差分方程一般式

$$\sum_{k=0}^{N} a_k y[n-k] = 0 \qquad (2.114)$$

N階線性常係數齊次差分方程一般式

2.4.2 Linear Constant-Coefficient Difference Equations

The stems from the observation that eq.(2.113) can be rearranged in the form

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\} \qquad (2.115)$$

2.4.2 Linear Constant-Coefficient Difference Equations

An equation of the form eq.(2.113) or eq.(2.115) is called a recursive equation, since it specifies a recursive procedure for determining the output in terms of the input and previous outputs. In the special case when $N = 0$, eq.(2.115) reduces to

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0} \right) x[n-k] \qquad (2.116)$$

(2.113)式及(2.115)式的型是因輸出的現在值必須由輸出的過去值必須由輸出的過去值推算，故為一種「遞迴方程式」

2.4.2 Linear Constant-Coefficient Difference Equations

Furthermore, eq.(2.116) describes and LTI system, and by direct computation, the impulse response of this system is found to be

$$h[n] = \begin{cases} \dfrac{b_n}{a_0}, & 0 \le n \le M. \\ 0, & \text{otherwise} \end{cases} \qquad (2.117)$$

2.4.2 Linear Constant-Coefficient Difference Equations

Note that the impulse response for it has finite duration; that is , it is nonzero only over a finite time interval. Because of this property, the system specified by eq.(2.116) is often called a finite impulse response system.

可由非遞迴方程式描述的離散時間系統，因其脈衝響應只在有限的時間內不為零，故常稱為「有限時間脈衝響應系統」

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

We begin with the discrete-time case and, in particular, the causal system described by the first-order difference equation

$$y[n] + ay[n-1] = bx[n] \qquad (2.126)$$

rewrite this equation in the form that directly suggests a recursive algorithm for computing successive values of the output $y[n]$

$$y[n] = -ay[n-1] + bx[n] \qquad (2.127)$$

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

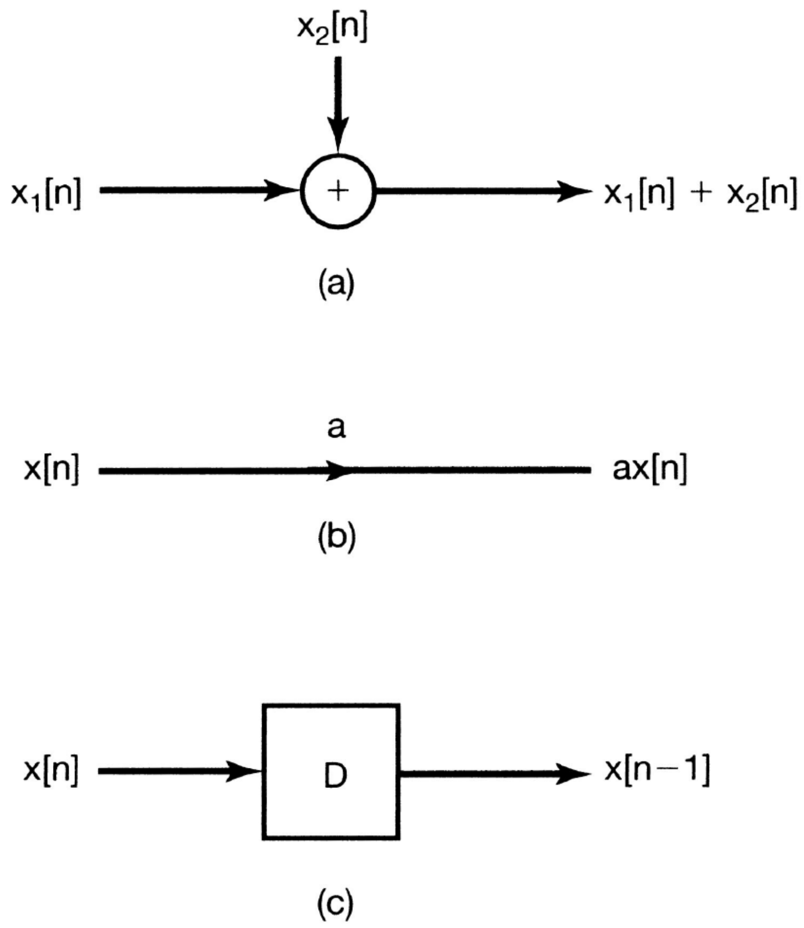


Figure 2.27 Basic elements for the block diagram representation of the causal system described by eq. (2.126): (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

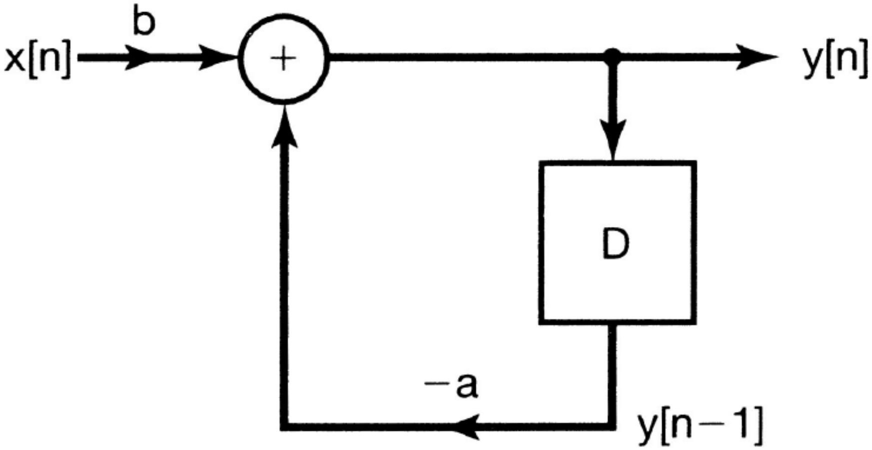


Figure 2.28 Block diagram representation for the causal discrete-time system described by eq. (2.126).

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

Consider next the causal continuous-time system described by a first-order differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \qquad (2.128)$$

rewrite it as

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t) \qquad (2.129)$$

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

訊號合成

訊號乘以係數（放大）

訊號的微分

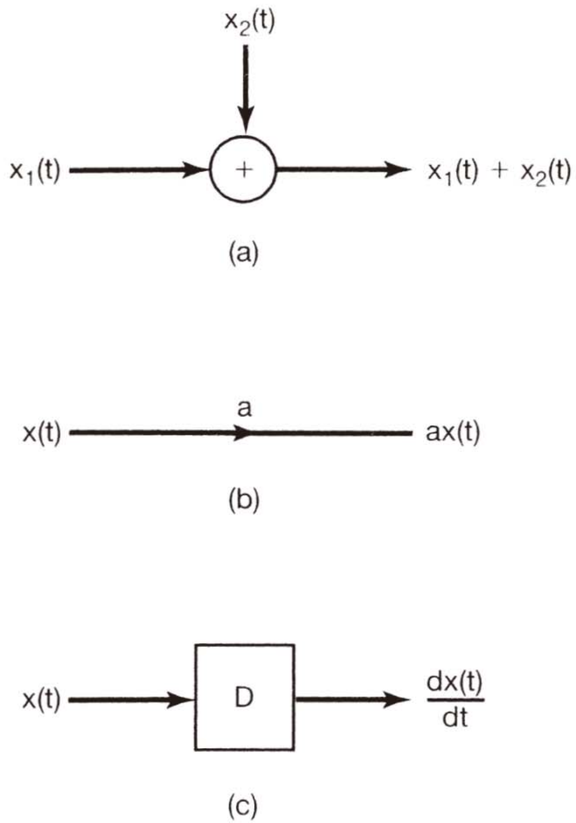


Figure 2.29 One possible set of basic elements for the block diagram representation of the continuous-time system described by eq. (2.128): (a) an adder; (b) multiplication by a coefficient; (c) a differentiator.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

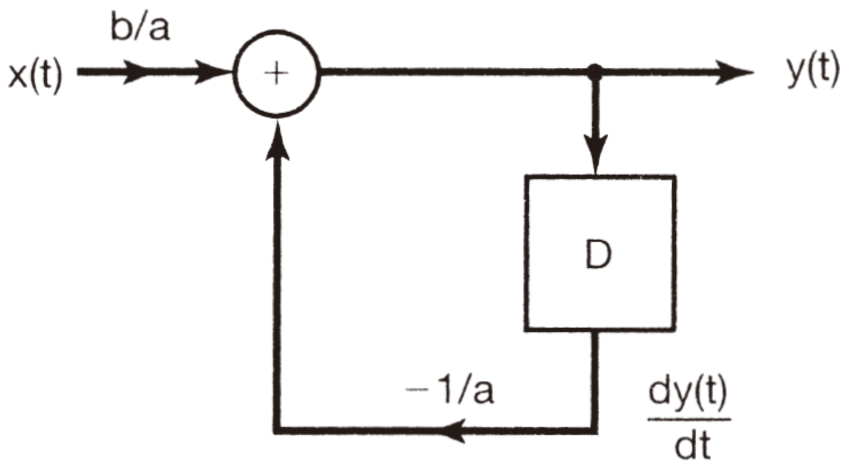


Figure 2.30 Block diagram representation for the system in eqs. (2.128) and (2.129), using adders, multiplications by coefficients, and differentiators.

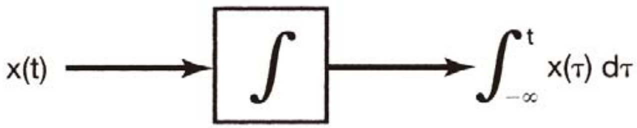


Figure 2.31 Pictorial representation of an integrator.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

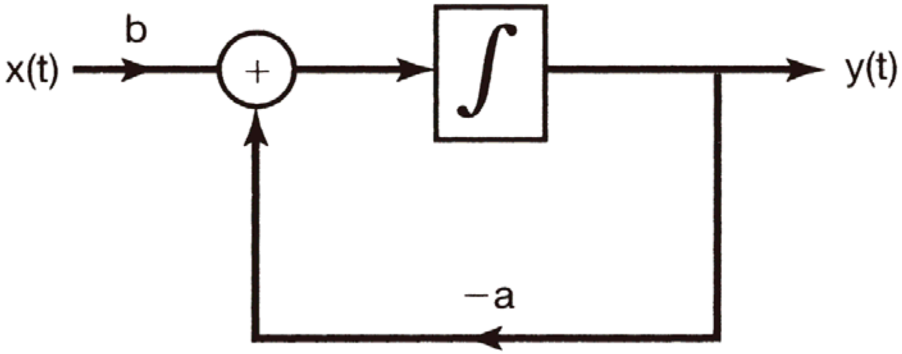


Figure 2.32 Block diagram representation for the system in eqs. (2.128) and (2.131), using adders, multiplications by coefficients, and integrators.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

$$\frac{dy(t)}{dt} = bx(t) - ay(t) \qquad (2.130)$$

Consequently, we obtain the equation

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)] d\tau \qquad (2.131)$$

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

This is perhaps more readily seen if we consider integrating eq.(2.130) from a finite point in time t_0, resulting in the expression

$$y(t) = y(t_0) + \int_{t_0}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau \qquad (2.132)$$

2.5.1 The Unit Impulse as an Idealized Short Pulse

From the sifting property, eq.(2.27), the unit impulse $\delta(t)$is the impulse response of the identity system. That is,

$$x(t) = x(t) * \delta(t) \qquad (2.133)$$

if we take $x(t) = \delta(t)$, we have

$$\delta(t) = \delta(t) * \delta(t) \qquad (2.134)$$

任何訊號$x(t)$與單位脈衝函數的迴旋積分，即為原訊號$x(t)$

$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t) \qquad (2.135)$$

2.5.1 The Unit Impulse as an Idealized Short Pulse

單位脈衝函數可視為圖 2.33 的三角脈波（面積為1）在△ → 0的極限情況。

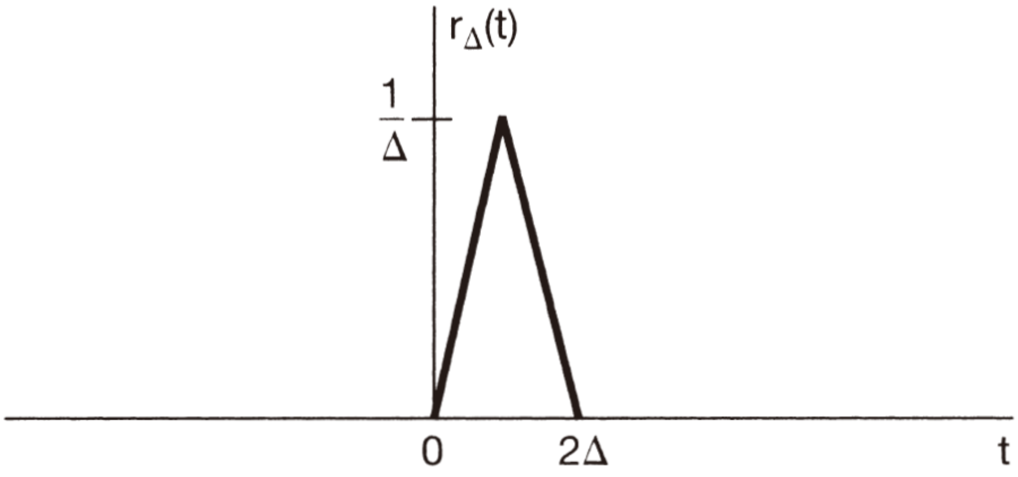


Figure 2.33 The signal $r_\Delta(t)$ defined in eq. (2.135).

2.5.2 Defining the Unit Impulse through Convolution

- We define as the signal for which

$$x(t) = x(t) * \delta(t) \qquad (2.138)$$

- We obtain

$$g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{+\infty} \delta(\tau) x(t-\tau) d\tau$$

which, for $t = 0$, yields

$$\mathrm{g}(0) = \int_{-\infty}^{+\infty} \mathrm{g}(\tau)\delta(\tau)\mathrm{d}\tau. \qquad (2.139)$$

2.5.2 Defining the Unit Impulse through Convolution

$$\int_{-\infty}^{+\infty} g(\tau) f(\tau) \delta(\tau) d\tau = g(0) f(0) \tag{2.140}$$

On the other hand, if we consider the signal $f(0)\delta(t)$, we see that

$$\int_{-\infty}^{+\infty} g(\tau) f(0) \delta(\tau) d\tau = g(0) f(0) \tag{2.141}$$

$$f(t)\delta(t) = f(0)\delta(t) \tag{2.142}$$

2.5.3 Unit Doublets and Other Singularity Functions

- Consider the LTI system for which the output is the derivative of the input

$$y(t) = \frac{dx(t)}{dt} \qquad (2.143)$$

The unit impulse response of this system is the derivative of the unit impulse, which is called the unit doublet $u_1(t)$.

單位脈衝函數 $\delta(t)$的微分稱為「單位脈衝偶」$u_1(t)$

2.5.3 Unit Doublets and Other Singularity Functions

From the convolution representation for LTI systems, we have

$$\frac{dx(t)}{dt} = x(t) * u_1(t) \qquad (2.144)$$

We can define $u_2(t)$, the second derivative of $\delta(t)$, as the impulse response of an LTI system that takes the second derivative of the input

$$\frac{d^2x(t)}{dt^2} = x(t) * u_2(t). \qquad (2.145)$$

令$u_2(t)$為 $\delta(t)$的二階為分，則$\frac{d^2x(t)}{dt^2} = x(t) * u_2(t)$

2.5.3 Unit Doublets and Other Singularity Functions

we see that

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = x(t) * u_1(t) * u_1(t) \quad (2.146)$$

$$u_2(t) = u_1(t) * u_1(t) \quad (2.147)$$

$$u_k(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{k \text{ times}}. \quad (2.148)$$

2.5.3 Unit Doublets and Other Singularity Functions

if we convolve the signal $g(-t)$ with $u_1(t)$, we obtain

$$\int_{-\infty}^{+\infty} g(\tau - t)u_1(\tau)d\tau = g(-t) * u_1(t) = \frac{dg(-t)}{dt} = -g'(-t)$$

which, for $t = 0$, yields

$$-g'(0) = \int_{-\infty}^{+\infty} g(\tau)u_1(\tau)d\tau \qquad (2.149)$$

2.5.3 Unit Doublets and Other Singularity Functions

$$\frac{d\delta_\Delta(t)}{dt} = \frac{1}{\Delta}\left[\delta(t) - \delta(t-\Delta)\right] \qquad (2.150)$$

Consequently, using the fact that $x(t) * \delta(t-t_0) = x(t-t_0)$, we find that

$$x(t) * \frac{d\delta_\Delta(t)}{dt} = \frac{x(t) - x(t-\Delta)}{\Delta} \cong \frac{dx(t)}{dt} \qquad (2.151)$$

2.5.3 Unit Doublets and Other Singularity Functions

the unit step is the impulse response of an integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Therefore,

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad (2.152)$$

and we also have the following operational definition of $u(t)$

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau \qquad (2.153)$$

2.5.3 Unit Doublets and Other Singularity Functions

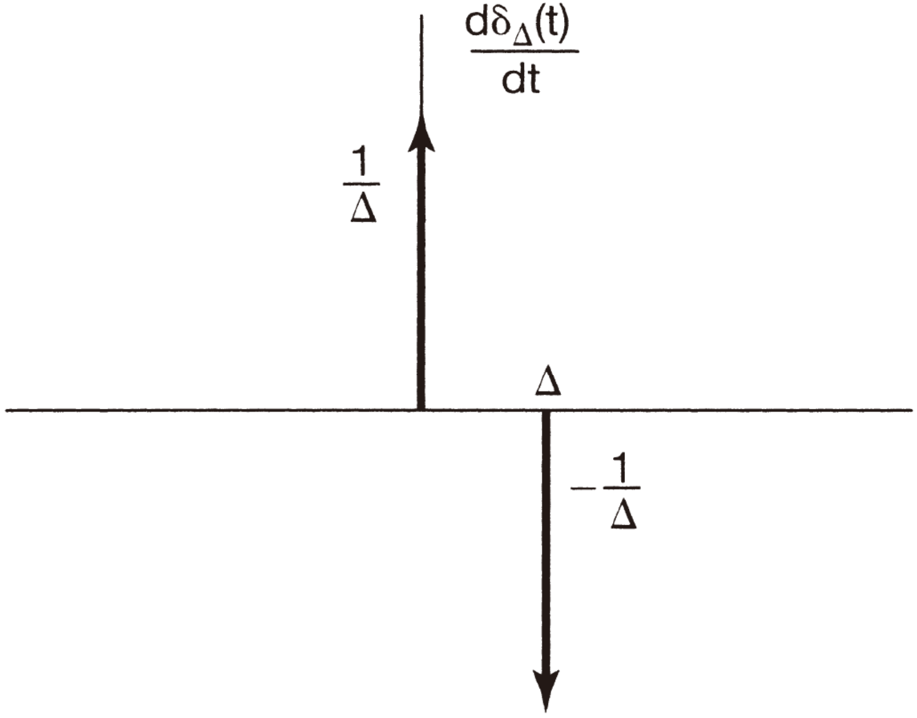


Figure 2.36 The derivative $d\delta_\Delta(t)/dt$ of the short rectangular pulse $\delta_\Delta(t)$ of Figure 1.34.

2.5.3 Unit Doublets and Other Singularity Functions

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau \qquad (2.154)$$

Since $u(t)$ equals 0 for $t < 0$ and equals 1 for $t > 0$, it follows that

$$u_{-2}(t) = tu(t) \qquad (2.155)$$

令 $u_{-2}(t) = tu(t)$，稱為「單位斜坡函數」

2.5.3 Unit Doublets and Other Singularity Functions

$$\begin{aligned} x(t) * u_{-2}(t) &= x(t) * u(t) * u(t) \\ &= \left(\int_{-\infty}^{t} x(\sigma) d\sigma \right) * u(t) \qquad (2.156) \\ &= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau \end{aligned}$$

we can define higher order integrals of $\delta(t)$ as the impulse responses of cascades of integrators:

$$u_{-k}(t) = \underbrace{u(t) * \cdots * u(t)}_{k \text{ times}} = \int_{-\infty}^{t} u_{-(k-1)}(\tau)\, d\tau. \qquad (2.157)$$

2.5.3 Unit Doublets and Other Singularity Functions

單位脈衝函數$\delta(t)$ 的微分稱為「單位脈衝偶」$u_1(t)$。

則：

$$\frac{dx(t)}{dt} = x(t) * u_1(t)$$

對任何 $x(t)$。

再令$u_2(t)$為$\delta(t)$的二階微分，則：

$$\frac{d^2x(t)}{dt^2} = x(t) * u_2(t)$$ 對任何 $x(t)$。

2.5.3 Unit Doublets and Other Singularity Functions

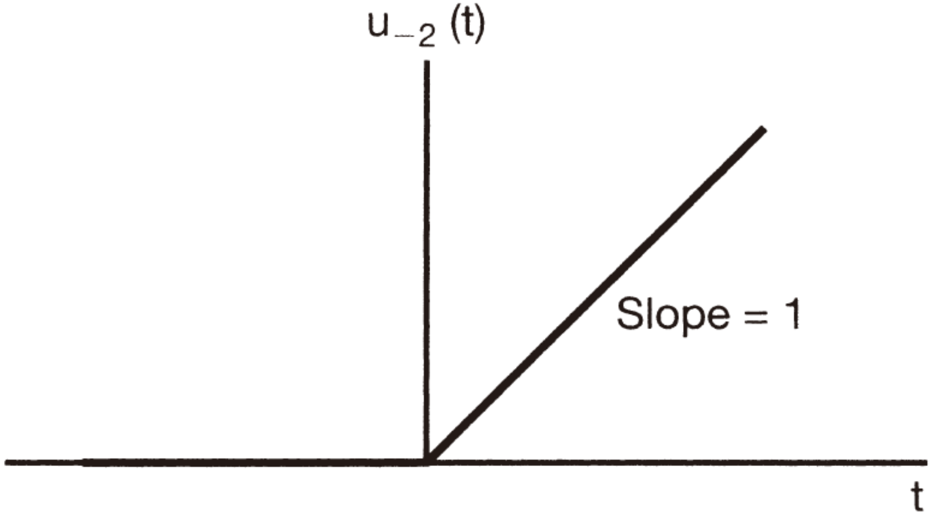


Figure 2.37 Unit ramp function.

2.5.3 Unit Doublets and Other Singularity Functions

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t) \qquad (2.158)$$

At times it will be worthwhile to use an alternative notation for $\delta(t)$ and $u(t)$, namely,

$$\delta(t) = u_0(t) \qquad (2.159)$$

$$u(t) = u_{-1}(t) \qquad (2.160)$$

$$u_{-1}(t) * u_1(t) = u_0(t) \qquad (2.161)$$

2.5.3 Unit Doublets and Other Singularity Functions

More generally, from eqs. (2.148), (2.157), and (2.161), we see that for any integers *k* and *r*,

$$u_k(t) * u_r(t) = u_{k+r}(t) \qquad (2.162)$$

2.6 Summary

- 我們推導了離散時間與連續時間兩種LTI系統重要的表示法。在離散時間中，我們推導出一種將訊號表為有時間移位的單位脈衝的加權總合的表示法，然後利用它來推導離散時間LTI系統的響應的迴旋和表示法。在連續時間中，我們導出一種類似的表示法，可以將連續時間訊號表為有時間移位的單位脈衝的加權積分，並利用它來推導連續時間LTI系統的迴旋積分表示法。

2.6 Summary

這些表示法非常重要，因為它門可允許我們利用系統的單位脈衝響應來計算對於任意輸入之下的LTI系統的響應。此外，在2.3節哩，迴旋與迴旋積分提供了一種工具，用來分析LTI系統的性質與單位脈衝響應的關係，特別是LTI系統性質的結合，包括因果性及穩定性。並且，在第2.5節裡，我們發展出連續時間單位脈衝和其他相關的奇性函數在其迴旋運算之下所代表的意義。這個意義在分析LTI系統時特別有用。

2.6 Summary

- 由線性常數係數微方程式描述的連續時間系統是一種很重要的類型。相似地，在離散時間中線性常係數差分方程式扮演了相同重要的角色。在2.4節中，我們檢視了簡單的微分及差分方程的例子，而且也討論了一些利用這些型式的方程式所描述的系統的性質。特別地是，以線性常係數微分及差分方程式描述的系統，在初始為靜止的狀況下是因果且LTI的。