Chapter 7
Applications of Plane Stress (Pressure Vessels, Beams, and Combined Loadings)

7.1 Introduction
7.2 Spherical Pressure Vessels
7.3 Cylindrical Pressure Vessels
7.4 Combined Loadings

7.1 INTRODUCTION

(1)Plane stress
1. Buildings
2. Machines
3. Vehicles
4. Aircrafts

(2)Pressure vessels

(3)Stresses in beams

(4)Combined Loading

7.2 SPHERICAL PRESSURE VESSELS

(1) Shell structures
(2) Thin walled
(3) Gage pressure

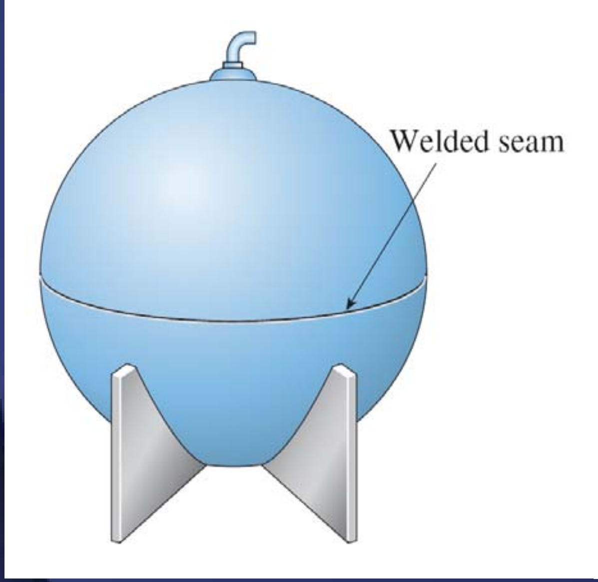


FIG. 7-1 Spherical pressure vessel

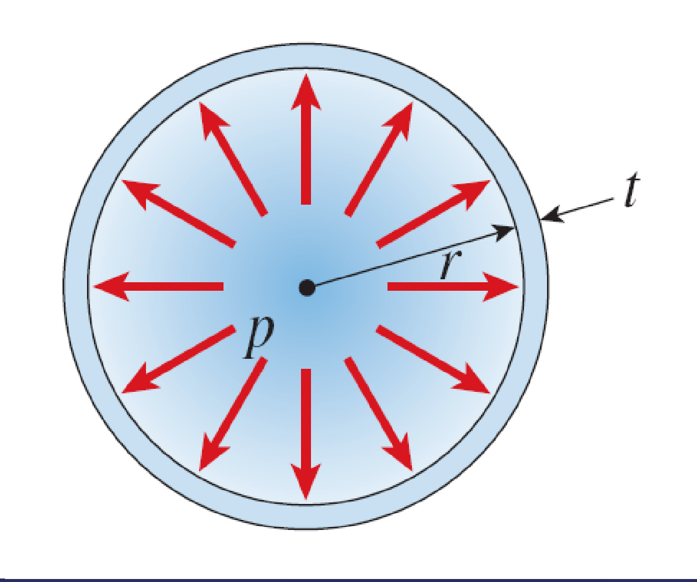


FIG. 7-2 Cross section of spherical pressure vessel showing inner radius *r*, wall thickness *t*, and internal pressure *p*

The resultant pressure force *P is*

$$P = p\left(\pi r^2\right)$$

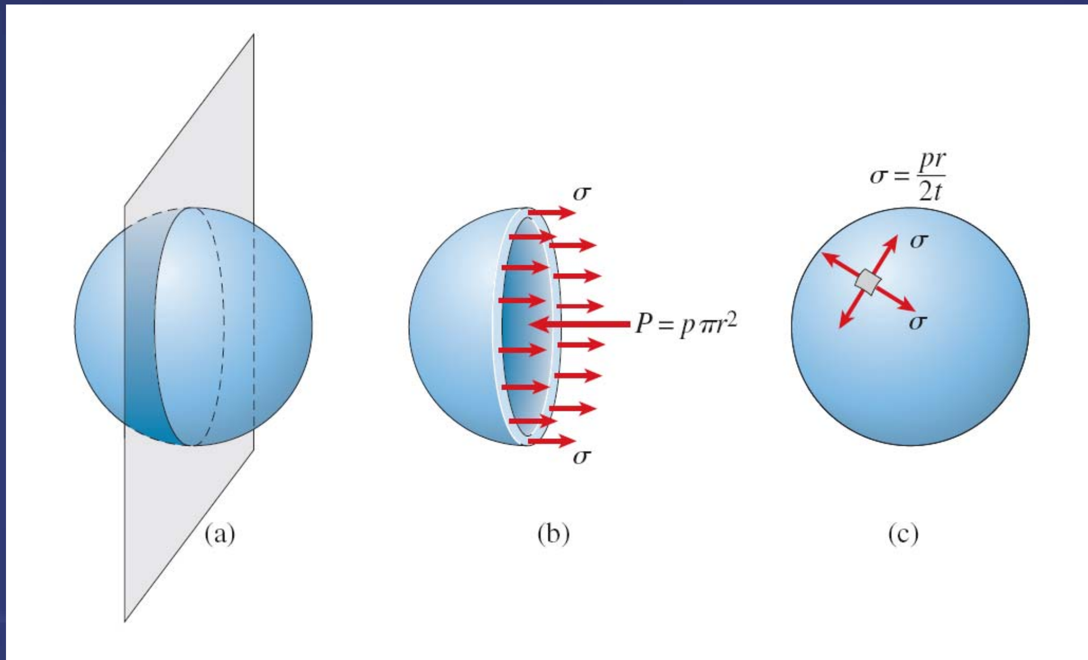


FIG. 8-3 Tensile stresses *σ* in the wall of a spherical pressure vessel

The resultant of the tensile stresses σ *in the wall is*

$$\sigma\left(2\pi r_m t\right)$$

where *t is the thickness of the wall and* r_m *is its mean radius:*

$$r_m = r + \frac{t}{2} \qquad \text{(b)}$$

Consequently,

$$\sum F_{\text{horiz}} = 0 \qquad \sigma\left(2\pi r_m t\right) - p\left(\pi r^2\right) = 0 \qquad \text{(c)}$$

The tensile stresses in the wall of the vessel is

$$\sigma = \frac{pr^2}{2r_m t} \qquad \text{(d)}$$

we can disregard the small difference between the two radii replace r_m *by r.* The **tensile stresses in the wall of a spherical shell** is

$$\sigma = \frac{pr}{2t} \qquad \text{(7-1)}$$

Conclusion: *The wall of a pressurized spherical vessel is subjected to uniform tensile stresses σ in all directions.*

General Comments

1. The wall thickness must be small in comparison to the other dimensions (the ratio *r/t should be 10 or more).*
2. The internal pressure must exceed the external pressure (to avoid inward buckling).
3. The analysis presented in this section is based only on the effects of internal pressure (the effects of external loads, reactions, the weight of the contents, and the weight of the structure are not considered).
4. The formulas derived in this section are valid throughout the wall of the vessel *except near points of stress concentrations.*

The following example illustrates how the principal stresses and maximum shear stresses are used in the analysis of a spherical shell.

7.3 CYLINDRICAL PRESSURE VESSELS

(1) **Circumferential stress** or the **hoop stress**
(2) **Longitudinal stress** or the **axial stress**

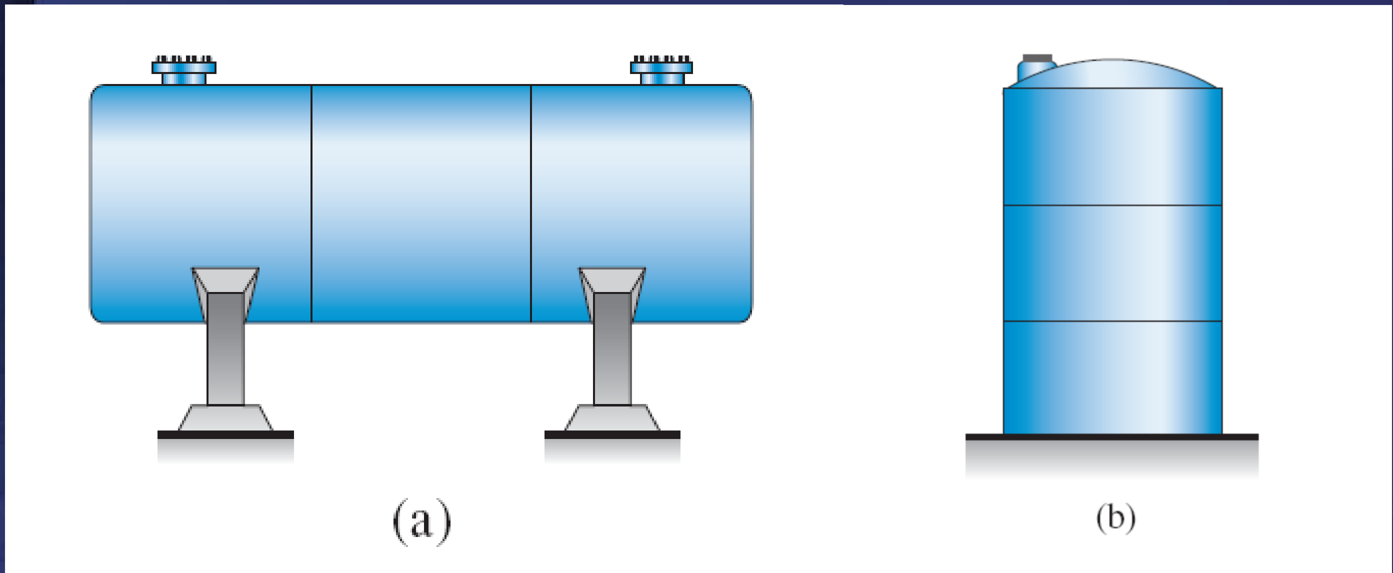
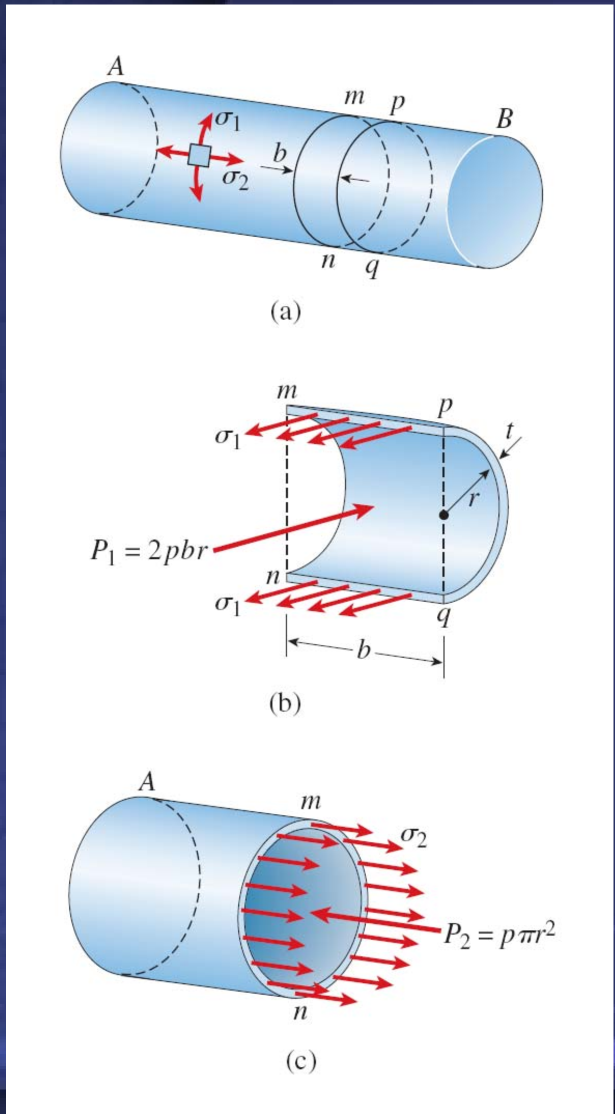


FIG. 7-6 Cylindrical pressure vessels with circular cross sections



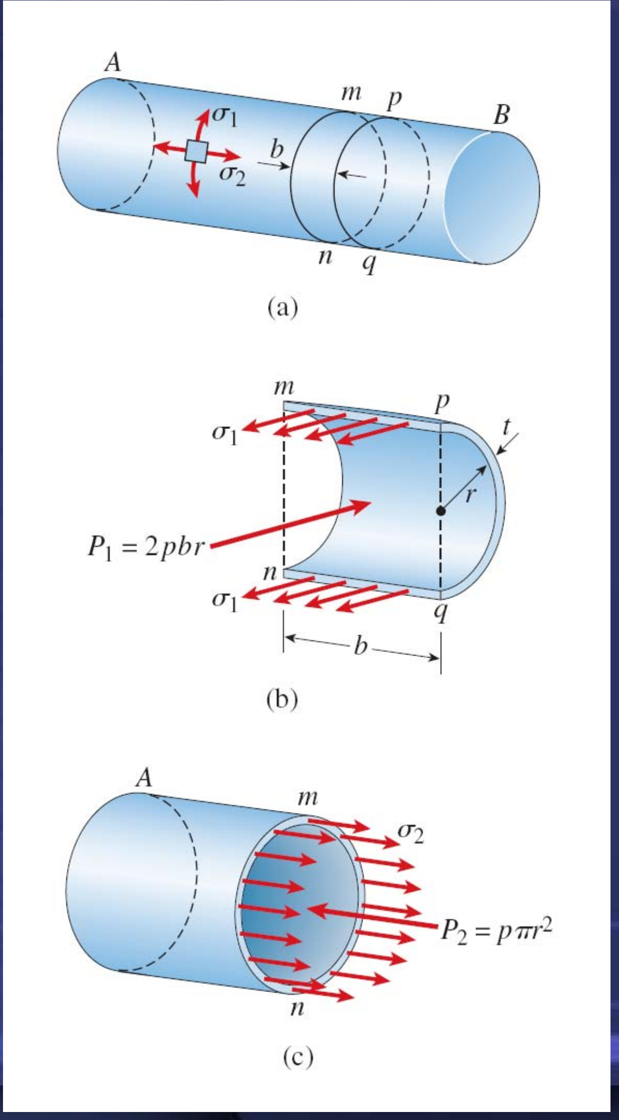
The circumferential stresses *σ_1 acting in the wall of the vessel have* a resultant equal to *$\sigma_1(2bt)$, where t is the thickness of the wall. Also,* the resultant force *P_1 of the internal pressure is equal to 2pbr. Therefore:*

$$\sigma_1 \left(2bt\right) - 2pbr = 0$$

we obtain the following formula for the *circumferential stress in a pressurized cylinder*

$$\sigma_1 = \frac{pr}{t} \qquad (7\text{-}5)$$

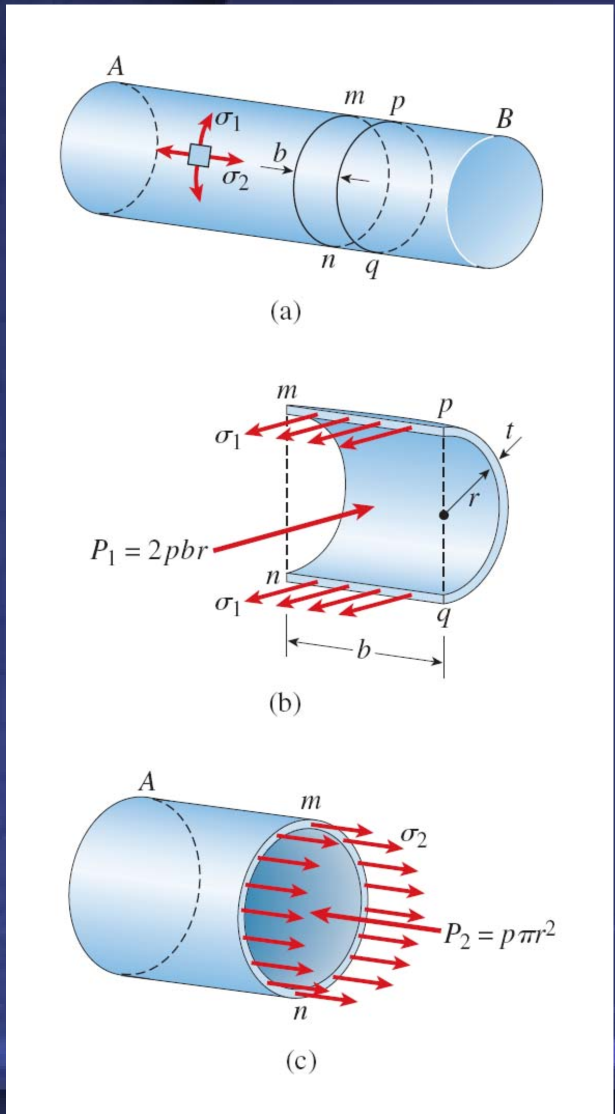
Longitudinal Stress



The stresses σ_2 *act longitudinally and have a resultant force equal to* $\sigma_2 (2\pi rt)$*.*

The resultant force P_2 *of the internal pressure is a force equal to* $p\pi r^2$*. Thus, the equation of equilibrium for the free body is*

$$\sigma_2 \left(2\pi rt\right) - p\pi r^2 = 0$$



The *longitudinal stress in a cylindrical pressure vessel is*

$$\sigma_2 = \frac{pr}{2t} \qquad (7\text{-}6)$$

Comparing Eqs. (7-5) and (7-6), we see that

$$\sigma_1 = 2\sigma_2 \qquad (7\text{-}7)$$

Stresses at the Outer Surface

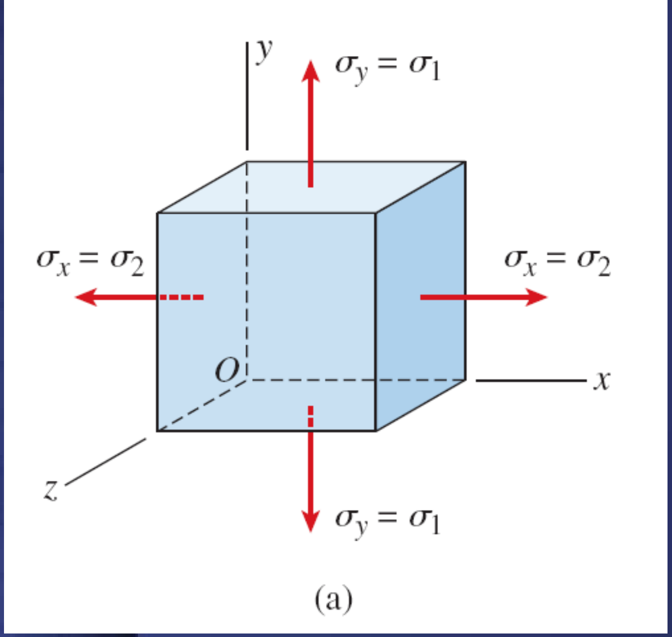
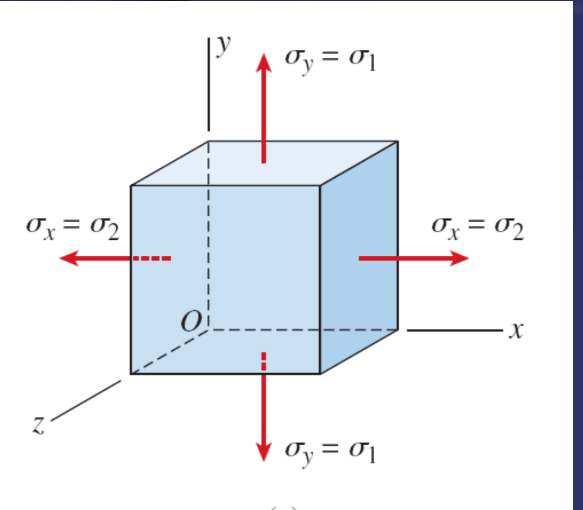


FIG. 8-8 Stresses in a circular *z* cylindrical pressure vessel at (a) the outer surface and (b) the inner surface

The maximum *in-plane shear stresses occur on planes that are* rotated 45° about the *z axis; these stresses are*

$$\left(\tau_{\max}\right)_z = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_1}{4} = \frac{pr}{4t}$$

(7-8)



(a)

The maximum *out-of-plane shear stresses are obtained by 45° rotations* about the *x and y axes, respectively; thus,*

$$\left(\tau_{\max}\right)_x = \frac{\sigma_1}{2} = \frac{pr}{2t}$$

$$\left(\tau_{\max}\right)_y = \frac{\sigma_2}{2} = \frac{pr}{4t} \qquad \text{(7-9a,b)}$$

Comparing the preceding results, we see that the *absolute maximum shear stress is*

$$\tau_{\max} = \frac{\sigma_1}{2} = \frac{pr}{2t} \qquad \text{(7-10)}$$

Stresses at the Inner Surface

The principal stresses are

$$\sigma_1 = \frac{pr}{t} \qquad \sigma_2 = \frac{pr}{2t} \qquad \sigma_3 = -p \qquad \text{(a,b,c)}$$

The three maximum shear stresses, obtained by 45° rotations about the *x, y, and z axes, are*

$$\left(\tau_{\max}\right)_x = \frac{\sigma_1 - \sigma_3}{2} = \frac{pr}{2t} + \frac{p}{2}$$

$$\left(\tau_{\max}\right)_y = \frac{\sigma_2 - \sigma_3}{2} = \frac{pr}{4t} + \frac{p}{2} \qquad \text{(d,e)}$$

$$\left(\tau_{\max}\right)_z = \frac{\sigma_1 - \sigma_2}{2} = \frac{pr}{4t} \qquad \text{(f)}$$

Example 7-2

A cylindrical pressure vessel is constructed from a long, narrow steel plate by wrapping the plate around a mandrel and then welding along the edges of the plate to make a helical joint (Fig. 8-9). The helical weld makes an angle $\alpha = 55^\circ$ with the longitudinal axis. The vessel has inner radius $r = 1.8$ m and wall thickness $t = 20$ mm. The material is steel with modulus $E = 200$ GPa and Poisson's ratio $v = 0.30$. The internal pressure p is 800 kPa.

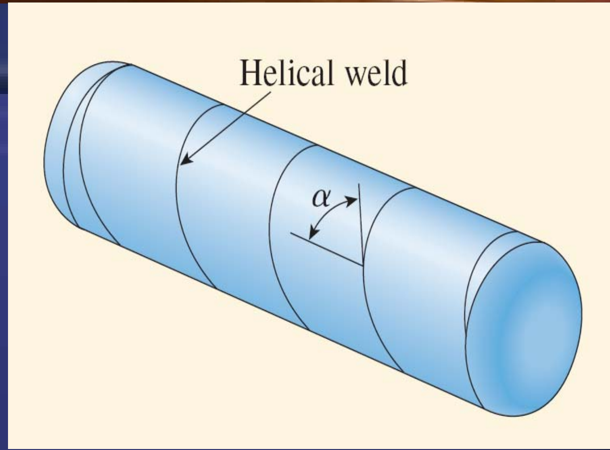


FIG. 8-9 Example 8-2. Cylindrical pressure vessel with a helical weld

Calculate the following quantities for the cylindrical part of the vessel: (a) the circumferential and longitudinal stresses σ_1 and σ_2, respectively; (b) the maximum in-plane and out-of-plane shear stresses; (c) the circumferential and longitudinal strains ϵ_1 and ϵ_2, respectively; and (d) the normal stress σ_w and shear stress τ_w acting perpendicular and parallel, respectively, to the welded seam.

Solution

(a) *Circumferential and longitudinal stresses.*

$$\sigma_1 = \frac{pr}{t} = \frac{(800\text{ kPa})(1.8\text{ m})}{20\text{ mm}} = 72\text{ MPa} \qquad \sigma_2 = \frac{pr}{2t} = \frac{\sigma_1}{2} = 36\text{ MPa}$$

(b) *Maximum shear stresses. The largest in-plane shear stress is obtained* from Eq. (7-8):

$$(\tau_{\max})_z = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_1}{4} = \frac{pr}{4t} = 18\text{ MPa}$$

the largest out-of-plane shear stress is obtained from Eq. (7-9a):

$$\tau_{\max} = \frac{\sigma_1}{2} = \frac{pr}{2t} = 36 \text{ MPa}$$

(c) *Circumferential and longitudinal strains.* The strains in the *x and y directions are*

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \epsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \text{(g,h)}$$

Which can be written in the following forms:

$$\in_2 = \frac{\sigma_2}{E}(1-2\nu) = \frac{pr}{2tE}(1-2\nu) \qquad \text{(7-11a)}$$

$$\in_1 = \frac{\sigma_1}{2E}(2-\nu) = \frac{pr}{2tE}(2-2\nu) \qquad \text{(7-11b)}$$

Substituting numerical values, we find

$$\in_2 = \frac{\sigma_2}{E}(1-2\nu) = \frac{(36\text{ MPa})\left[1-2(0.30)\right]}{200\text{ GPa}} = 72\times10^{-6}$$

$$\in_1 = \frac{\sigma_1}{2E}(2-\nu) = \frac{(72\text{ MPa})(2-0.30)}{2(200\text{ GPa})} = 306\times10^{-6}$$

(d) *Normal and shear stresses acting on the welded seam.* The angle θ *for the element is*

$$\theta = 90^\circ - \alpha = 35^\circ$$

Stress-transformation equations. The normal stress σ_{x_1} *and the shear stress* $\tau_{x_1y_1}$ *acting on the* x_1 *face of the element (Fig. 7-10c) are*

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (7\text{-}12a)$$

$$\tau_{x_1y_1} = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (7\text{-}12b)$$

Substituting *$\sigma_x = \sigma_2 = pr/2t$, $\sigma_y = \sigma_1 = pr/t$, and $\tau_{xy} = 0$, we obtain*

$$\sigma_{x_1} = \frac{pr}{4t}\left(3 - \cos 2\theta\right) \qquad \tau_{x_1 y_1} = \frac{pr}{4t}\sin 2\theta \qquad \text{(7-13a,b)}$$

Substituting *$pr/4t = 18$ MPa and $\theta = 35^\circ$ into Eqs. (7-13a) and (7-13b),* we obtain

$$\sigma_{x_1} = 47.8 \text{ MPa} \qquad \tau_{x_1 y_1} = 16.9 \text{ MPa}$$

The normal stress σ_{y_1} *is*

$$\sigma_{y_1} = \sigma_1 + \sigma_2 - \sigma_{x_1}$$

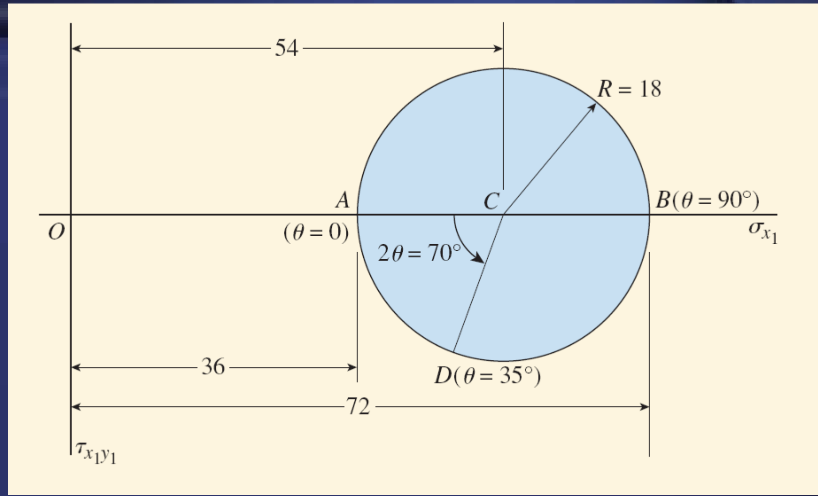
$$= 72\ \text{MPa} + 36\ \text{MPa} - 47.8\ \text{MPa} = 60.2\ \text{MPa}$$

The normal and shear stresses are:

$$\sigma_w = 47.8\ \text{MPa} \qquad \tau_w = 16.9\ \text{MPa}$$

Mohr's circle.

$$R = \frac{72\ \text{MPa} - 36\ \text{MPa}}{2} = 18\ \text{MPa}$$



A counterclockwise angle $2\theta = 70^\circ$ *(measured on the circle from point A)* locates point *D, which corresponds to the stresses on the* x_1 *face (θ = 35°) of the* element. The coordinates of point *D (from the geometry of the circle) are*

$$\sigma_{x_1} = 54\ \text{MPa} - R\cos 70^\circ = 54\ \text{MPa} - (18\ \text{MPa})(\cos 70^\circ) = 47.8\ \text{MPa}$$

$$\tau_{x_1y_1} = R\sin 70^\circ = (18\ \text{MPa})(\sin 70^\circ) = 16.9\ \text{MPa}$$

7.4 COMBINED LOADINGS

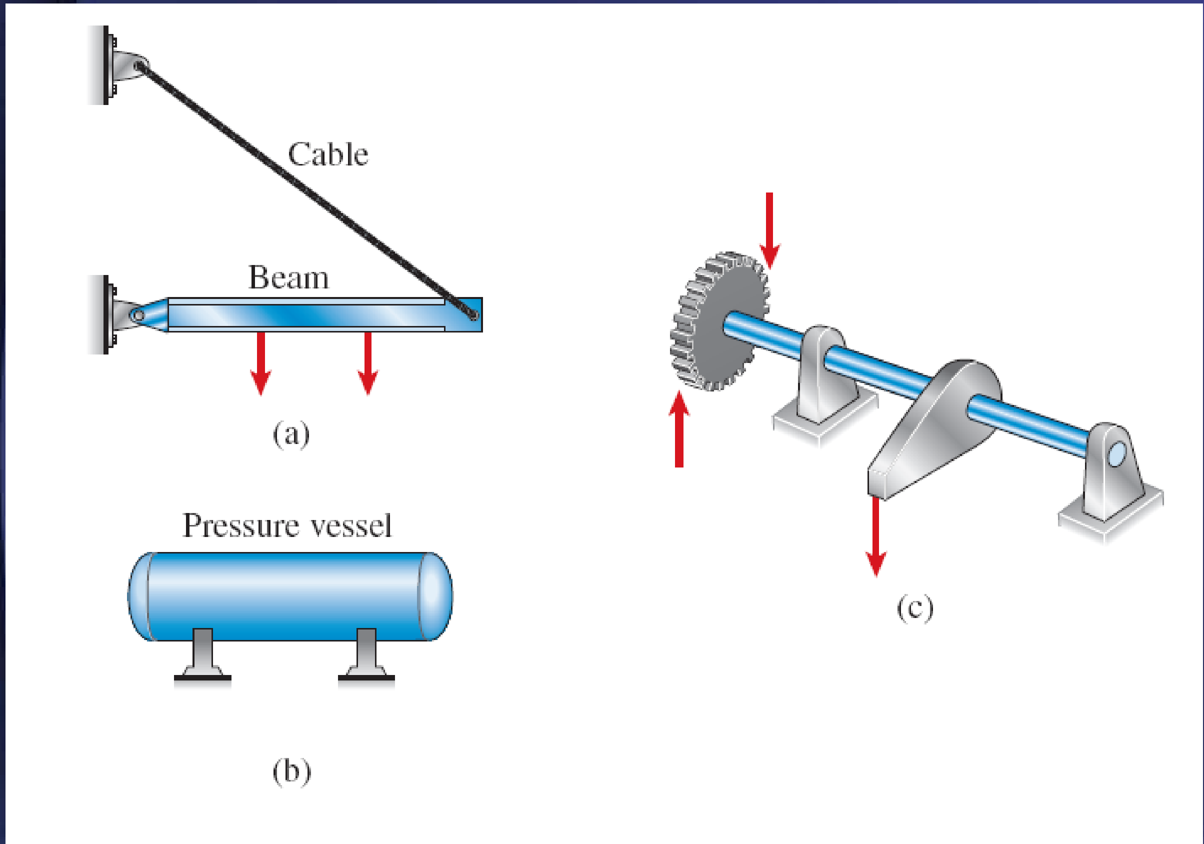


FIG. 8-20 Examples of structures subjected to combined loadings:
(a) wide-flange beam supported by a cable (combined bending and axial load), (b) cylindrical pressure vessel supported as a beam, and (c) shaft in combined torsion and bending

Method of Analysis

1. Select a point in the structure where the stresses and strains are to be determined. (The point is usually selected at a cross section where the stresses are large, such as at a cross section where the bending moment has its maximum value.)

2. For each load on the structure, determine the stress resultants at the cross section containing the selected point. (The possible stress resultants are an axial force, a twisting moment, a bending moment, and a shear force.)

3. Calculate the normal and shear stresses at the selected point due to each of the stress resultants. Also, if the structure is a pressure vessel, determine the stresses due to the internal pressure. (The stresses are found from the stress formulas derived previously; for instance, $\sigma = P/A$, $\tau = T\rho/I_P$, $\sigma = My/I$, $\tau = VQ/Ib$, and $\sigma = pr/t$.)

4. Combine the individual stresses to obtain the resultant stresses at the selected point. In other words, obtain the stresses *sx, sy, and txy* acting on a stress element at the point. (Note that in this chapter we are dealing only with elements in plane stress.)

5. Determine the principal stresses and maximum shear stresses at the selected point, using either the stress-transformation equations or Mohr's circle. If required, determine the stresses acting on other inclined planes.

6. Determine the strains at the point with the aid of Hooke's law for plane stress.

7. Select additional points and repeat the process. Continue until enough stress and strain information is available to satisfy the purposes of the analysis.

Illustration of the Method

The stress resultants acting at the cross section (Fig. 7-14b) are a twisting moment equal to the torque *T*, a bending moment M equal to the load *P* times the distance b from the free end of the bar to the cross section, and a shear force *V* equal to the load *P*. The stresses acting at points A and B are shown in Fig. 7-14c. The twisting moment *T* produces torsional shear stresses

$$\tau_1 = \frac{Tr}{I_P} = \frac{2T}{\pi r^3} \qquad \text{(a)}$$

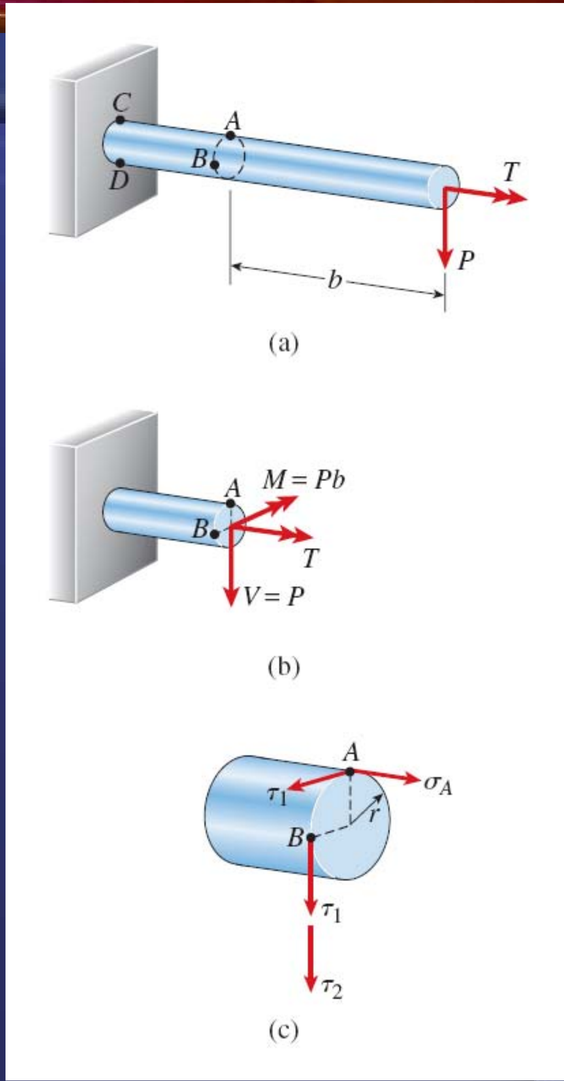


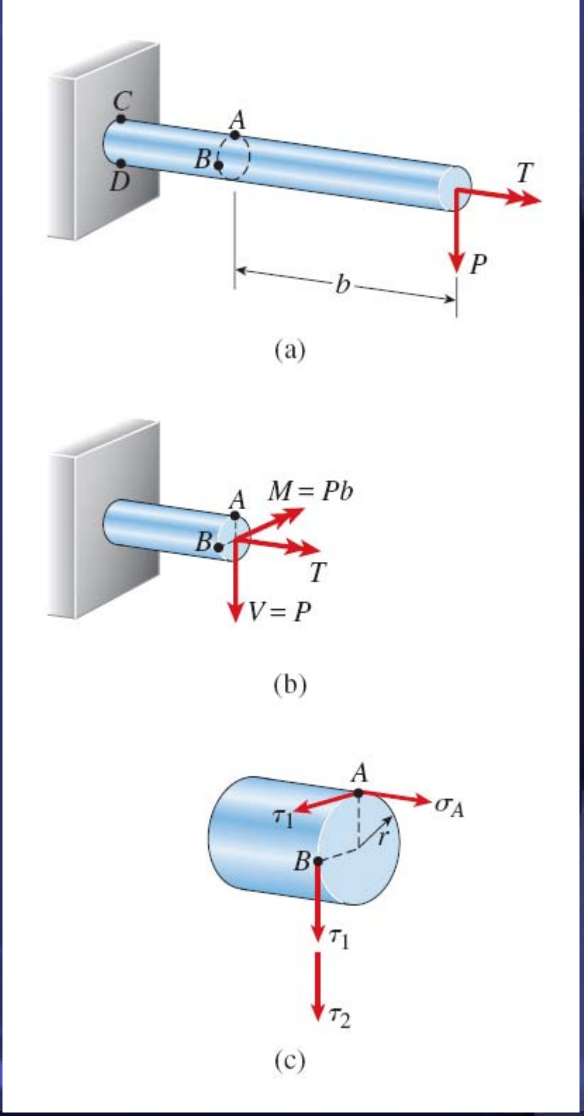
FIG. 7-14 Cantilever bar subjected to combined torsion and bending: (a) loads acting on the bar, (b) stress resultants at a cross section, and (c) stresses at points *A* and *B*

The bending moment *M produces a tensile stress at point A:*

$$\sigma_A = \frac{Mr}{I} = \frac{4M}{\pi r^3} \qquad \text{(b)}$$

The shear force *V produces no shear stress at the top of the bar* (point *A), but at point B the shear stress is as follows (see Eq. 5-39 in* Chapter 5):

$$\tau_2 = \frac{4V}{3A} = \frac{4V}{3\pi r^2} \qquad \text{(c)}$$



The stresses σ_A *and* τ_1 *acting at point A (Fig. 7-14c) are shown* acting on a stress element in Fig. 7-15a. A two-dimensional view of the element, obtained by looking vertically downward on the element, is shown in Fig. 7-15b.

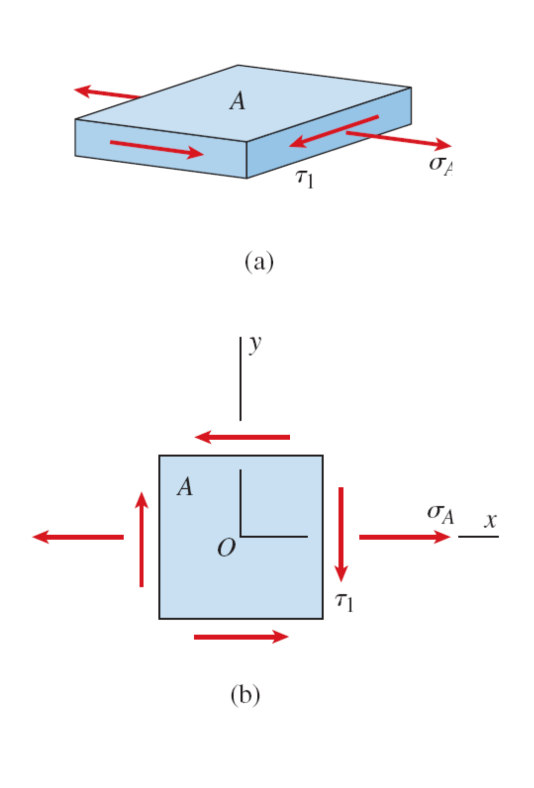


FIG. 7-15 Stress element at point A

A stress element at point *B (also in plane stress) is shown in Fig. 7-16a.* A two-dimensional view of the stress element is shown in Fig. 7-16b, selection of critical points

As a final step, the principal stresses and maximum shear stresses at the critical points can be compared with one another in order to determine the absolute maximum normal and shear stresses in the bar.

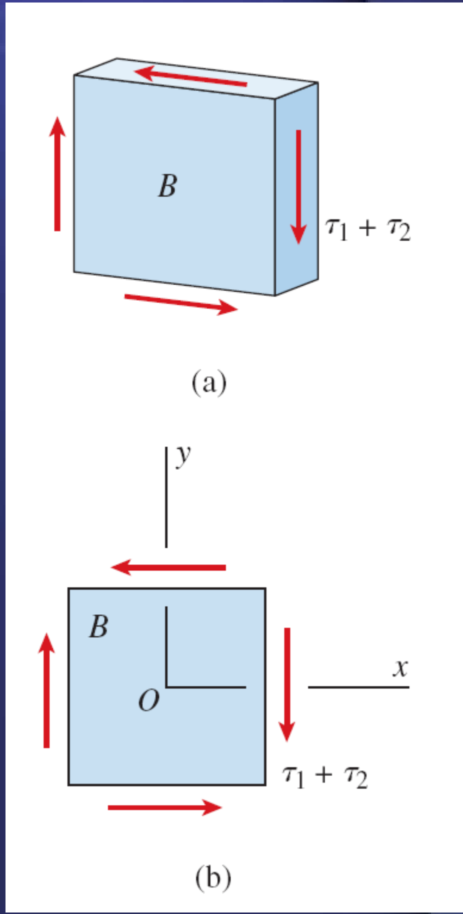


FIG. 7-16 Stress element at point *B*

Example 7-3

The rotor shaft of a helicopter drives the rotor blades that provide the lifting force to support the helicopter in the air (Fig. 8-24a). As a consequence, the shaft is subjected to a combination of torsion and axial loading (Fig. 8-24b).

For a 50-mm diameter shaft transmitting a torque *T = 2.4 kN-m and a* tensile force *P = 125 kN, determine the maximum tensile stress, maximum* compressive stress, and maximum shear stress in the shaft.

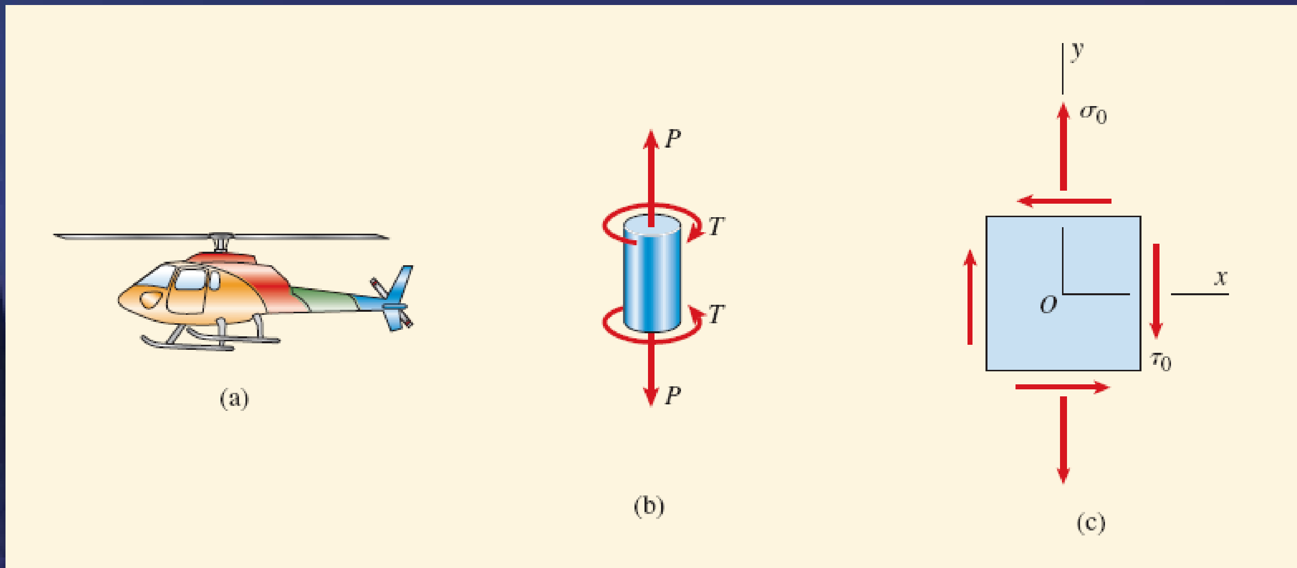


FIG. 7-17 Example 7-3. Rotor shaft of a helicopter (combined torsion and axial force)

Solution

The stresses in the rotor shaft are produced by the combined action of the axial force *P and the torque T (Fig. 7-17b). Therefore, the stresses at any point* on the surface of the shaft consist of a tensile stress σ_0 *and shear stresses* τ_0, *as* shown on the stress element of Fig. 7-17c.

The tensile stress σ_0 *is*

$$\sigma_O = \frac{P}{A} = \frac{4P}{\pi d^2} = \frac{4(125 \text{ kN})}{\pi(50 \text{ mm})^2} = 63.66 \text{ MPa}$$

The shear stress τ_0 *is*

$$\tau_O = \frac{Tr}{I_P} = \frac{16T}{\pi d^3} = \frac{16(2.4\ \text{kN}\cdot\text{m})}{\pi(50\ \text{mm})^3} = 97.78\ \text{MPa}$$

Substituting $\sigma_x = 0$, $\sigma_y = \sigma_0 = 63.66$ *MPa, and* $\tau_{xy} = \tau_0 = 97.78$ *MPa, we* get

$$\sigma_{1,2} = 32\ \text{MPa}\ \pm 103\ \text{MPa}$$

or $\quad \sigma_1 = 135\ \text{MPa} \quad \sigma_2 = -71\ \text{MPa}$

The maximum in-plane shear stresses are

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 103\ \text{MPa}$$

Example 7-5

A sign of dimensions 2.0 m x 1.2 m is supported by a hollow circular pole having outer diameter 220 mm and inner diameter 180 mm (Fig. 8-26). The sign is offset 0.5 m from the centerline of the pole and its lower edge is 6.0 m above the ground.

Determine the principal stresses and maximum shear stresses at points *A and B* at the base of the pole due to a wind pressure of 2.0 kPa against the sign.

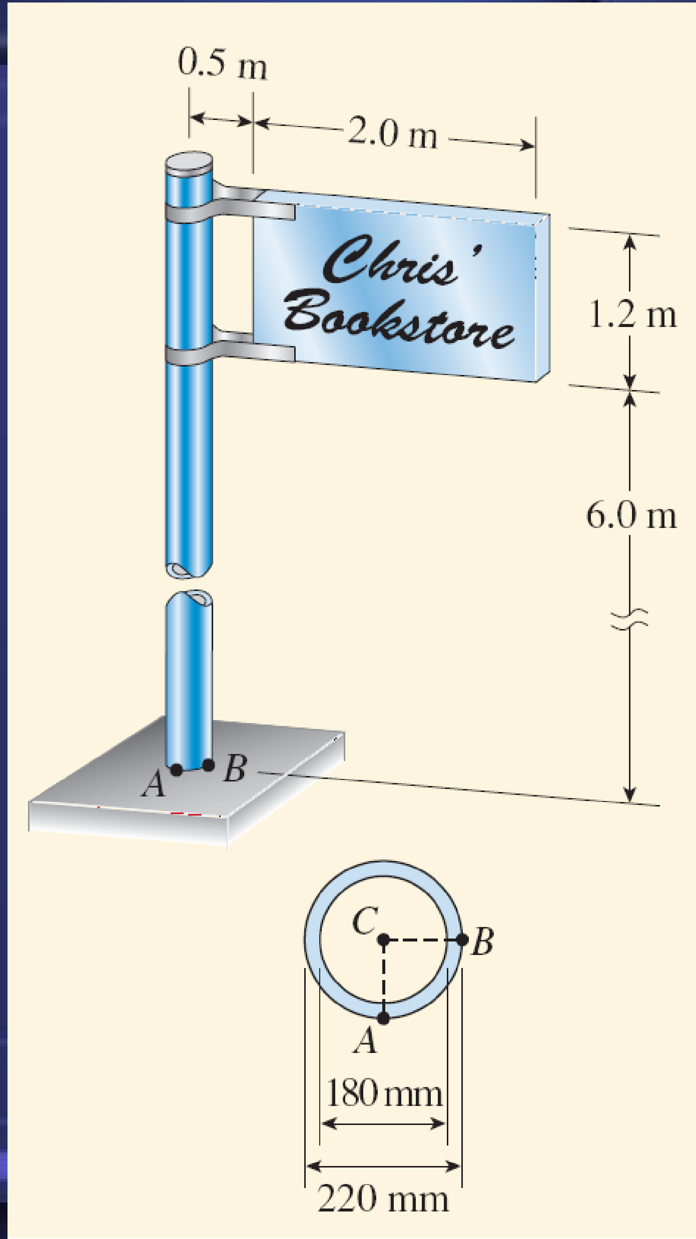


FIG. 7-19 Example 8-6. Wind pressure against a sign (combined bending, torsion, and shear of the pole)

Solution

Stress resultants. The wind pressure against the sign produces a resultant force *W that acts at the midpoint of the sign (Fig. 7-20a) is*

$$W = pA = (2.0 \text{ kPa})(2.0 \text{ m} \times 1.2 \text{ m}) = 4.8 \text{ kN}$$

The wind force acting on the sign is statically equivalent to a lateral force *W* and a torque *T acting on the pole (Fig. 7-20b). The torque is*

$$T = Wb = (4.8 \text{ kN})(1.5 \text{ m}) = 7.2 \text{ kN} \cdot \text{m}$$

The stress resultants at the base of the pole (Fig. 7-20c) consist of a bending moment *M, a torque T, and a shear force V. Their magnitudes are*

$$M = Wh = (4.8\ \mathrm{kN})(6.6\ \mathrm{m}) = 31.68\ \mathrm{kN \cdot m}$$

$$T = 7.2\ \mathrm{kN \cdot m} \qquad \mathrm{V} = W = 4.8\ \mathrm{kN}$$

Stresses at points A and B. The bending moment M produces a tensile stress σ_A at point A (Fig. 7-20d) but no stress at point B (which is located on the neutral axis). The stress σ_A *is obtained from the flexure formula:*

$$\sigma_A = \frac{M(d_2/2)}{I}$$

The moment of inertia is

$$I = \frac{\pi}{64}\left(d_2^4 - d_1^4\right) = \frac{\pi}{64}\left[(220\text{ mm})^4 - (180\text{ mm})^4\right] = 63.46\times10^{-6}\text{ m}^4$$

Therefore

$$\sigma_A = \frac{Md_2}{2I} = \frac{(31.68\text{ kN}\cdot\text{m})(220\text{ mm})}{2\left(63.46\times10^{-6}\text{ m}^4\right)} = 54.91\text{ MPa}$$

The torque *T produces shear stresses τ_1 at points A and B (Fig. 7-20d).*

$$\tau_1 = \frac{T\left(d_2/2\right)}{I_P}$$

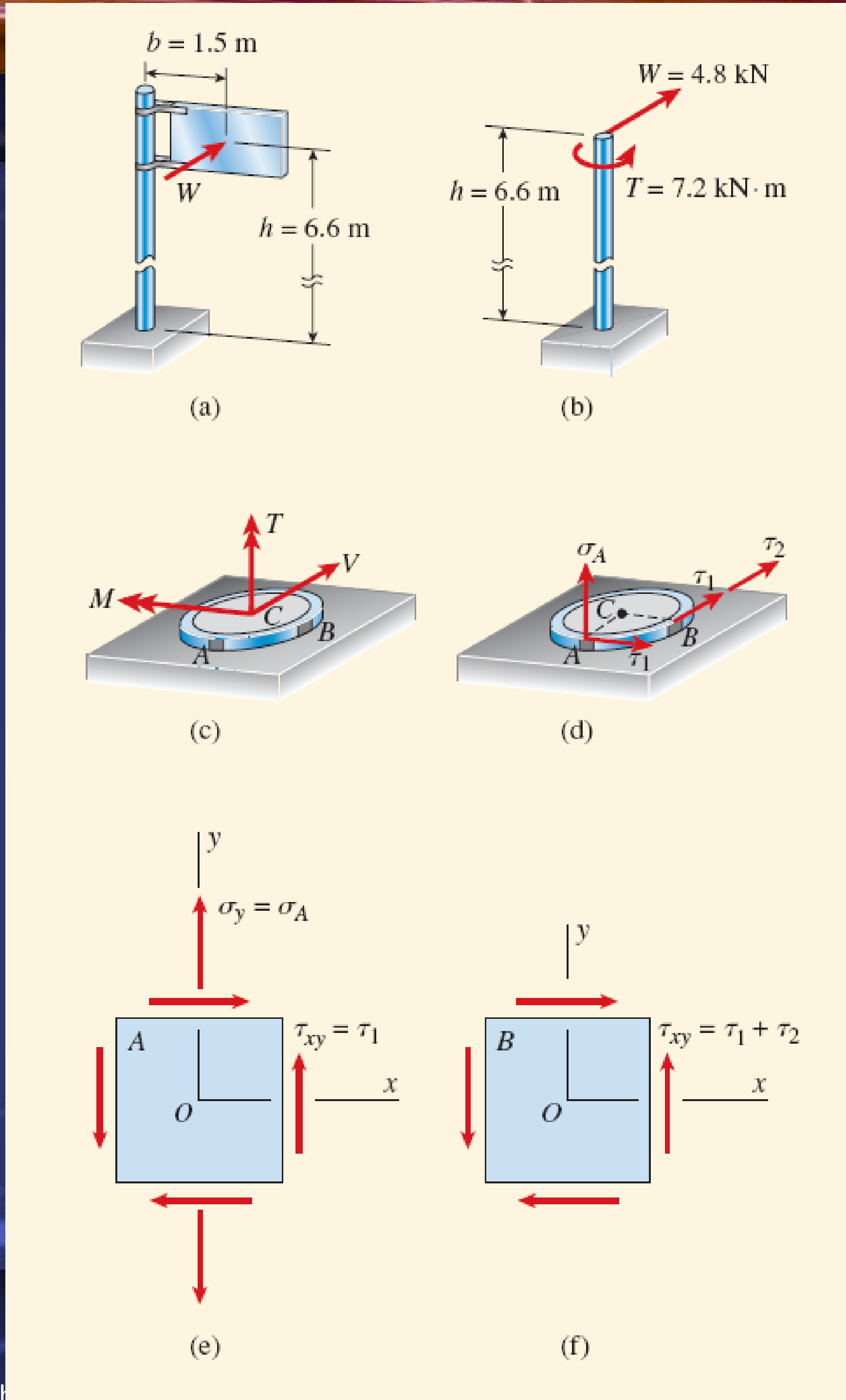


FIG. 7-20 Solution to Example 7-5

In which I_P *is the polar moment of inertia*

$$I_P = \frac{\pi}{32}\left(d_2^4 - d_1^4\right) = 2I = 126.92 \times 10^{-6}\ \text{m}^4$$

Thus,

$$\tau_1 = \frac{Td_2}{2I_P} = \frac{(7.2\ \text{kN}\cdot\text{m})(220\ \text{mm})}{2\left(126.92\times10^{-6}\ \text{m}^4\right)} = 6.24\ \text{MPa}$$

Due to the shear force *V. The shear stress at point A is zero, and the shear stress at point B is*

$$\tau_2 = \frac{4V}{3A}\left(\frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2}\right) \qquad \text{(j)}$$

in which r_2 *and* r_1 *are the outer and inner radii, respectively, and A is the crosssectional* area

$$r_2 = \frac{d_2}{2} = 110 \text{ mm} \qquad r_1 = \frac{d_1}{2} = 90 \text{ mm}$$

$$A = \pi\left(r_2^2 - r_1^2\right) = 12{,}570 \text{ mm}^2$$

Substituting numerical values into Eq. (j), we obtain

$$\tau_2 = 0.76 \text{ MPa}$$

Stress elements. At point *A the stresses acting on the* element are

$$\sigma_x = 0 \qquad \sigma_y = \sigma_A = 54.91 \text{ MPa} \qquad \tau_{xy} = \tau_1 = 6.24 \text{ MPa}$$

At point *B the stresses are*

$$\sigma_x = \sigma_y = 0$$

$$\tau_{xy} = \tau_1 + \tau_2 = 6.24 \text{ MPa} + 0.76 \text{ MPa} = 7.00 \text{ MPa}$$

Since there are no normal stresses acting on the element, point *B is in pure* shear.

Principal stresses and maximum shear stresses at point A. The principal stresses are

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{(k)}$$

Substituting *σ_x = 0, σ_y = 54.91 MPa, and τ_{xy} = 6.24 MPa, we get*

$$\sigma_{1,2} = 27.5\ \text{MPa} \pm 28.2\ \text{MPa}$$

or

$$\sigma_1 = 55.7\ \text{MPa} \qquad \sigma_2 = -0.7\ \text{MPa}$$

The maximum in-plane shear stresses is obtained

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 28.2\ \text{MPa} \qquad \text{(I)}$$

Principal stresses and maximum shear stresses at point B. The stresses at this point are *σ_x = 0, σ_y = 0, and τ_{xy} = 7.0 MPa. Since the element is in pure* shear, the principal stresses are

$$\sigma_1 = 7.0\ \text{MPa} \qquad \sigma_2 = -7.0\ \text{MPa}$$

The maximum in-plane shear stress is

$$\tau_{\max} = 7.0\ \text{MPa}$$

The End of Chap. 7