PROBLEM SET 5.1

1. **WRITING AND LITERATURE PROJECT. Power Series in Calculus.** **(a)** Write a review (2–3 pages) on power series in calculus. Use your own formulations and examples—do not just copy from textbooks. No proofs. **(b)** Collect and arrange Maclaurin series in a systematic list that you can use for your work.

2–5 REVIEW: RADIUS OF CONVERGENCE

Determine the radius of convergence. Show the details of your work.

2. $\sum_{m=0}^{\infty} (m+1)mx^m$

3. $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$

4. $\sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$

5. $\sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m x^{2m}$

6–9 SERIES SOLUTIONS BY HAND

Apply the power series method. Do this by hand, not by a CAS, to get a feel for the method, e.g., why a series may terminate, or has even powers only, etc. Show the details.

6. $(1 + x)y' = 2y$

7. $y' = -4xy$

8. $xy' - 4y = k$ (k a constant)

9. $y'' + y = 0$

10–14 SERIES SOLUTIONS

Find a power series solution in powers of x. Show the details.

10. $y'' - y' + xy = 0$

11. $y'' + y' + x^2y = 0$

12. $(1 - x^2)y'' - 2xy' + 2y = 0$

13. $y'' + (1 + x^2)y = 0$

14. $y'' - 4xy' + (4x^2 - 2)y = 0$

15. **Shifting summation indices** is often convenien[t] necessary in the power series method. Shift the in[dex] so that the power under the summation sign is Check by writing the first few terms explicity.

$$\sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1} x^{s-1}, \qquad \sum_{p=1}^{\infty} \frac{p^2}{(p+1)!} x^{p+4}$$

16–19 CAS PROBLEMS. IVPs

Solve the initial value problem by a power series. the partial sums of the powers up to and including x the value of the sum s (5 digits) at x_1.

16. $y' + 4y = 1, \quad y(0) = 1.25, \quad x_1 = 0.2$

17. $y'' + 4xy' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$, $x_1 = 0.25$

18. $(1 - x^2)y'' - 2xy' + 30y = 0, \quad y(0) = 0,$ $y'(0) = 1.875, \quad x_1 = 0.5$

19. $(x - 1)y' = 2xy, \quad y(0) = 4$

20. **CAS Experiment. Information from G[r] Partial Sums.** In numerics we use partial power series. To get a feel for the accuracy f x, experiment with $\sin x$. Graph partial su Maclaurin series of an increasing number describing qualitatively the "breakaway these graphs from the graph of $\sin x$. Cons Maclaurin series of your choice.

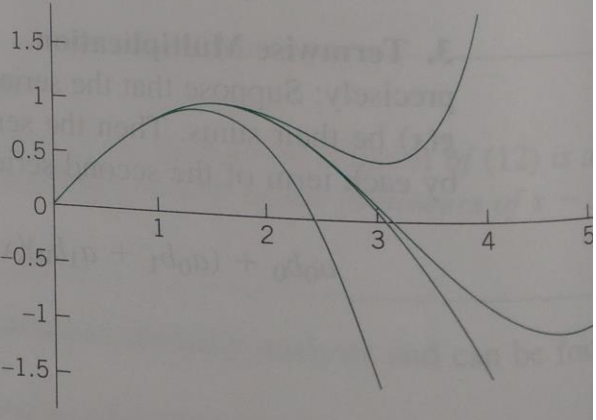

Fig. 106. CAS Experiment 20. $\sin x$ and sums s_3, s_5, s_7

PROBLEM SET 5.2

1–5 LEGENDRE POLYNOMIALS AND FUNCTIONS

1. **Legendre functions for $n = 0$.** Show that (6) with $n = 0$ gives $P_0(x) = 1$ and (7) gives (use $\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$)

$$y_2(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

Verify this by solving (1) with $n = 0$, setting $z = y'$ and separating variables.

2. **Legendre functions for $n = 1$.** Show that (7) with $n = 1$ gives $y_2(x) = P_1(x) = x$ and (6) gives

$$y_1 = 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \cdots$$
$$= 1 - \frac{1}{2}x\ln\frac{1+x}{1-x}.$$

3. **Special n.** Derive (11′) from (11).
4. **Legendre's ODE.** Verify that the polynomials in (11′) satisfy (1).
5. Obtain P_6 and P_7.

6–9 CAS PROBLEMS

6. Graph $P_2(x), \cdots, P_{10}(x)$ on common axes. For what x (approximately) and $n = 2, \cdots, 10$ is $|P_n(x)| < \frac{1}{2}$?
7. From what n on will your CAS no longer produce faithful graphs of $P_n(x)$? Why?
8. Graph $Q_0(x)$, $Q_1(x)$, and some further Legendre functions.
9. Substitute $a_s x^s + a_{s+1}x^{s+1} + a_{s+2}x^{s+2}$ into Legendre's equation and obtain the coefficient recursion (4).
10. **TEAM PROJECT. Generating Functions.** Generating functions play a significant role in modern applied mathematics (see [GenRef5]). The idea is simple. If we want to study a certain sequence $(f_n(x))$ and can find a function

$$G(u, x) = \sum_{n=0}^{\infty} f_n(x)u^n,$$

we may obtain properties of $(f_n(x))$ from those of G, which "generates" this sequence and is called a **generating function** of the sequence.

(a) Legendre polynomials. Show that

$$(12)\quad G(u, x) = \frac{1}{\sqrt{1 - 2xu + u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n$$

is a generating function of the Legendre polynomials. *Hint:* Start from the binomial expansion of $1/\sqrt{1-v}$, then set $v = 2xu - u^2$, multiply the powers of $2xu - u^2$ out, collect all the terms involving u^n, and verify that the sum of these terms is $P_n(x)u^n$.

(b) Potential theory. Let A_1 and A_2 be two points in space (Fig. 108, $r_2 > 0$). Using (12), show that

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}}$$
$$= \frac{1}{r_2}\sum_{m=0}^{\infty} P_m(\cos\theta)\left(\frac{r_1}{r_2}\right)^m.$$

This formula has applications in potential theory. (Q/r is the electrostatic potential at A_2 due to a charge Q located at A_1. And the series expresses $1/r$ in terms of the distances of A_1 and A_2 from any origin O and the angle θ between the segments OA_1 and OA_2.)

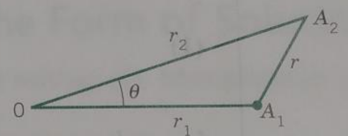

Fig. 108. Team Project 10

(c) Further applications of (12). Show that $P_n(1) = 1$, $P_n(-1) = (-1)^n$, $P_{2n+1}(0) = 0$, and $P_{2n}(0) = (-1)^n \cdot 1 \cdot 3 \cdots (2n-1)/[2 \cdot 4 \cdots (2n)]$.

11–15 FURTHER FORMULAS

11. **ODE.** Find a solution of $(a^2 - x^2)y'' - 2xy' + n(n+1)y = 0$, $a \neq 0$, by reduction to the Legendre equation.
12. **Rodrigues's formula (13)**[2] Applying the binomial theorem to $(x^2 - 1)^n$, differentiating it n times term by term, and comparing the result with (11), show that

$$(13)\qquad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2 - 1)^n].$$

[2]OLINDE RODRIGUES (1794–1851), French mathematician and economist.

PROBLEM SET 5.3

1. **WRITING PROJECT. Power Series Method and Frobenius Method.** Write a report of 2–3 pages explaining the difference between the two methods. No proofs. Give simple examples of your own.

2–13 FROBENIUS METHOD

Find a basis of solutions by the Frobenius method. Try to identify the series as expansions of known functions. Show the details of your work.

2. $(x+1)^2y'' + (x+1)y' - y = 0$
3. $xy'' + 2y' + xy = 0$
4. $xy'' - y = 0$
5. $x^2y'' + x(2x-1)y' + (x+1)y = 0$
6. $xy'' + 2x^3y' + (x^2-2)y = 0$
7. $y'' + (x - \frac{1}{2})y = 0$
8. $xy'' + y' - xy = 0$
9. $2x(x-1)y'' - (x+1)y' + y = 0$
10. $xy'' + 2y' + 16xy = 0$
11. $xy'' + (2-2x)y' + (x-2)y = 0$
12. $x^2y'' + 6xy' + (4x^2+6)y = 0$
13. $xy'' + (2x+1)y' + (x+1)y = 0$
14. **TEAM PROJECT. Hypergeometric Equation, Series, and Function. Gauss's hypergeometric ODE**[5] is

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0. \tag{15}$$

Here, a, b, c are constants. This ODE is of the form $p_2y'' + p_1y' + p_0y = 0$, where p_2, p_1, p_0 are polynomials of degree 2, 1, 0, respectively. These polynomials are written so that the series solution takes a most practical form, namely,

$$\begin{aligned} y_1(x) = 1 &+ \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 \\ &+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\,c(c+1)(c+2)}x^3 + \cdots . \end{aligned} \tag{16}$$

This series is called the **hypergeometric series.** Its sum $y_1(x)$ is called the **hypergeometric function** and is denoted by $F(a, b, c; x)$. Here, $c \neq 0, -1, -2, \cdots$. By choosing specific values of a, b, c we can obtain an incredibly large number of special functions as solutions of (15) [see the small sample of elementary functions in part (c)]. This accounts for the importance of (15).

(a) Hypergeometric series and function. Show that the indicial equation of (15) has the roots $r_1 = 0$ and $r_2 = 1 - c$. Show that for $r_1 = 0$ the Frobenius method gives (16). Motivate the name for (16) by showing that

$$F(1, 1, 1; x) = F(1, b, b; x) = F(a, 1, a; x) = \frac{1}{1-x}$$

(b) Convergence. For what a or b will (16) reduce to a polynomial? Show that for any other a, b, c $(c \neq 0, -1, -2, \cdots)$ the series (16) converges when $|x| < 1$.

(c) Special cases. Show that

$$\begin{aligned} (1+x)^n &= F(-n, b, b; -x), \\ (1-x)^n &= 1 - nxF(1-n, 1, 2; x), \\ \arctan x &= xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; -x^2) \\ \arcsin x &= xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}; x^2), \\ \ln(1+x) &= xF(1, 1, 2; -x), \\ \ln\frac{1+x}{1-x} &= 2xF(\tfrac{1}{2}, 1, \tfrac{3}{2}; x^2). \end{aligned}$$

Find more such relations from the literature on special functions, for instance, from [GenRef1] in App. 1.

(d) Second solution. Show that for $r_2 = 1 - c$ the Frobenius method yields the following solution (where $c \neq 2, 3, 4, \cdots$):

$$\begin{aligned} y_2(x) = x^{1-c}\Big(1 &+ \frac{(a-c+1)(b-c+1)}{1!\,(-c+2)} \\ &+ \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2!\,(-c+2)(-c+3)} + \cdots \end{aligned} \tag{17}$$

Show that

$$y_2(x) = x^{1-c}F(a-c+1, b-c+1, 2-c; x).$$

(e) On the generality of the hypergeometric equation. Show that

$$(t^2 + At + B)\ddot{y} + (Ct + D)\dot{y} + Ky = 0 \tag{18}$$

PROBLEM SET 5.4

1. **Convergence.** Show that the series (11) converges for all x. Why is the convergence very rapid?

2–10 ODEs REDUCIBLE TO BESSEL'S ODE

This is just a sample of such ODEs; some more follow in the next problem set. Find a general solution in terms of J_ν and $J_{-\nu}$ or indicate when this is not possible. Use the indicated substitutions. Show the details of your work.

2. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$
3. $xy'' + y' + y = 0 \quad (2\sqrt{x} = z)$
4. $y'' + (e^{-2x} - \frac{1}{16})y = 0 \quad (e^{-x} = z)$
5. **Two-parameter ODE**
   $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \quad (\lambda x = z)$
6. $x^2y'' + (\frac{3}{16} + x)\, y = 0 \quad (y = 2u\sqrt{x},\ \sqrt{x} = z)$
7. $x^2y'' + xy' + \frac{1}{16}(x^2 - 1)y = 0 \quad (x = 4z)$
8. $(x-1)^2y'' - (1-x)y' + x(x-2)y = 0,$
   $(x - 1 = z)$
9. $xy'' + (2\nu + 1)y' + xy = 0 \quad (y = x^{-\nu}u)$
10. $x^2y'' + (1 - 2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0$
    $(y = x^\nu u,\ x^\nu = z)$
11. **CAS EXPERIMENT. Change of Coefficient.** Find and graph (on common axes) the solutions of

$$y'' + kx^{-1}y' + y = 0,\ y(0) = 1,\ y'(0) = 0,$$

for $k = 0, 1, 2, \cdots, 10$ (or as far as you get useful graphs). For what k do you get elementary functions? Why? Try for noninteger k, particularly between 0 and 2, to see the continuous change of the curve. Describe the change of the location of the zeros and of the extrema as k increases from 0. Can you interpret the ODE as a model in mechanics, thereby explaining your observations?

12. **CAS EXPERIMENT. Bessel Functions for Large x.**
    (a) Graph $J_n(x)$ for $n = 0, \cdots, 5$ on common axes.

    (b) Experiment with (14) for integer n. Using graphs find out from which $x = x_n$ on the curves of (11 and (14) practically coincide. How does x_n chang with n?

    (c) What happens in (b) if $n = \pm\frac{1}{2}$? (Our usual notatio in this case would be ν.)

    (d) How does the error of (14) behave as a func tion of x for fixed n? [Error = exact value minu approximation (14).]

    (e) Show from the graphs that $J_0(x)$ has extrema whe $J_1(x) = 0$. Which formula proves this? Find furthe relations between zeros and extrema.

13–15 ZEROS of Bessel functions play a key role i modeling (e.g. of vibrations; see Sec. 12.9).

13. **Interlacing of zeros.** Using (21) and Rolle's theorer show that between any two consecutive positive zer of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$.
14. **Zeros.** Compute the first four positive zeros of $J_0($ and $J_1(x)$ from (14). Determine the error and commer
15. **Interlacing of zeros.** Using (21) and Rolle's theorer show that between any two consecutive zeros of $J_0($ there is precisely one zero of $J_1(x)$.

16–18 HALF-INTEGER PARAMETER: APPROAC BY THE ODE

16. **Elimination of first derivative.** Show that $y =$ with $v(x) = \exp(-\frac{1}{2}\int p(x)\,dx)$ gives from the OD $y'' + p(x)y' + q(x)y = 0$ the ODE

$$u'' + \left[q(x) - \tfrac{1}{4}p(x)^2 - \tfrac{1}{2}p'(x)\right]u = 0,$$

not containing the first derivative of u.

17. Bessel's equation. Show that for (1) the substitution in Prob. 16 is $y = ux^{-1/2}$ and gives

$$x^2u'' + (x^2 + \tfrac{1}{4} - \nu^2)u = 0. \tag{27}$$

18. Elementary Bessel functions. Derive (22) in Example 3 from (27).

19–25 APPLICATION OF (21): DERIVATIVES, INTEGRALS

Use the powerful formulas (21) to do Probs. 19–25. Show the details of your work.

19. Derivatives. Show that $J_0'(x) = -J_1(x)$, $J_1'(x) = J_0(x) - J_1(x)/x$, $J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$.

20. Bessel's equation. Derive (1) from (21).

21. Basic integral formula. Show that

$$\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) + c.$$

22. Basic integral formulas. Show that

$$\int x^{-\nu} J_{\nu+1}(x)\,dx = -x^{-\nu} J_\nu(x) + c,$$

$$\int J_{\nu+1}(x)\,dx = \int J_{\nu-1}(x)\,dx - 2J_\nu(x).$$

23. Integration. Show that $\int x^2 J_0(x)\,dx = x^2 J_1(x) + xJ_0(x) - \int J_0(x)\,dx$. (The last integral is nonelementary; tables exist, e.g., in Ref. [A13] in App. 1.)

24. Integration. Evaluate $\int x^{-1} J_2(x)\,dx$.

25. Integration. Evaluate $\int x^{-1} J_3(x)\,dx$.

5.5 Bessel Functions $Y_\nu(x)$. General Solution

PROBLEM SET 5.5

1–9 FURTHER ODE's REDUCIBLE TO BESSEL'S ODE

Find a general solution in terms of J_ν and Y_ν. Indicate whether you could also use $J_{-\nu}$ instead of Y_ν. Use the indicated substitution. Show the details of your work.

1. $x^2y'' + xy' + (x^2 - 9)y = 0$
2. $xy'' + 3y' + xy = 0 \quad (y = u/x)$
3. $9x^2y'' + 9xy' + (9x^4 - 4)y = 0 \quad (z = \frac{x^2}{2})$
4. $y'' + xy = 0 \quad (y = u\sqrt{x}, \frac{2}{3}x^{3/2} = z)$
5. $4xy'' + 4y' + y = 0 \quad (\sqrt{x} = z)$
6. $xy'' + y' + 4y = 0 \quad (z = 4\sqrt{x})$
7. $y'' + k^2x^2y = 0 \quad (y = u\sqrt{x}, \frac{1}{2}kx^2 = z)$
8. $y'' + k^2x^4y = 0 \quad (y = u\sqrt{x}, \frac{1}{3}kx^3 = z)$
9. $xy'' - 3y' + xy = 0 \quad (y = x^2u)$

10. CAS EXPERIMENT. Bessel Functions for Large x. It can be shown that for large x,

$$Y_n(x) \sim \sqrt{2/(\pi x)} \sin\left(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right) \tag{11}$$

with $\sim$ defined as in (14) of Sec. 5.4.

(a) Graph $Y_n(x)$ for $n = 0, \cdots, 5$ on common axes. Are there relations between zeros of one function and extrema of another? For what functions?

(b) Find out from graphs from which $x = x_n$ on the curves of (8) and (11) (both obtained from your CAS) practically coincide. How does x_n change with n?

(c) Calculate the first ten zeros x_m, $m = 1, \cdots, 10$, of $Y_0(x)$ from your CAS and from (11). How does the error behave as m increases?

(d) Do (c) for $Y_1(x)$ and $Y_2(x)$. How do the errors compare to those in (c)?

11–15 HANKEL AND MODIFIED BESSEL FUNCTIONS

11. Hankel functions. Show that the Hankel functions (10) form a basis of solutions of Bessel's equation for any ν.

12. Modified Bessel functions of the first kind of order ν are defined by $I_\nu(x) = i^{-\nu}J_\nu(ix)$, $i = \sqrt{-1}$. Show that I_ν satisfies the ODE

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0. \tag{12}$$

13. Modified Bessel functions. Show that $I_\nu(x)$ has the representation

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{x^{2m+\nu}}{2^{2m+\nu} m!\, \Gamma(m + \nu + 1)}. \tag{13}$$

14. Reality of I_ν. Show that $I_\nu(x)$ is real for all real x (and real ν), $I_\nu(x) \neq 0$ for all real $x \neq 0$, and $I_{-n}(x) = I_n(x)$ where n is any integer.

15. Modified Bessel functions of the third kind (sometime called *of the second kind*) are defined by the formula (14) below. Show that they satisfy the ODE (12).

$$K_\nu(x) = \frac{\pi}{2 \sin \nu\pi}[I_{-\nu}(x) - I_\nu(x)]. \tag{14}$$