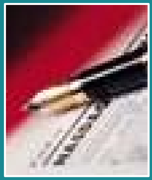


Chapter 4
The continuous-time Fourier Transform

4.0 Introduction

- 本章將利用只有週期訊號適用的傅立葉級數，擴展非週期訊號亦適用的傅立葉轉換。

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

To gain some insight into the nature of the Fourier transform representation, we begin by revisiting the Fourier series representation for the continuous-time periodic square wave examined in Example 3.5. Specifically, over one period,

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

As determined in Example 3.5, the Fourier series coefficients a_k for this square wave are

[eq.(3.44)] $$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \qquad (4.1)$$

圖4.1的方波*x(t)*的數學式，週期為*T*。

An alternative way of interpreting eq.(4.1) is as samples of an envelope function, specifically,

*x(t)*的傅立葉係數 $$Ta_k = \frac{2\sin\omega T_1}{\omega}\bigg|_{\omega=k\omega_0}. \qquad (4.2)$$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

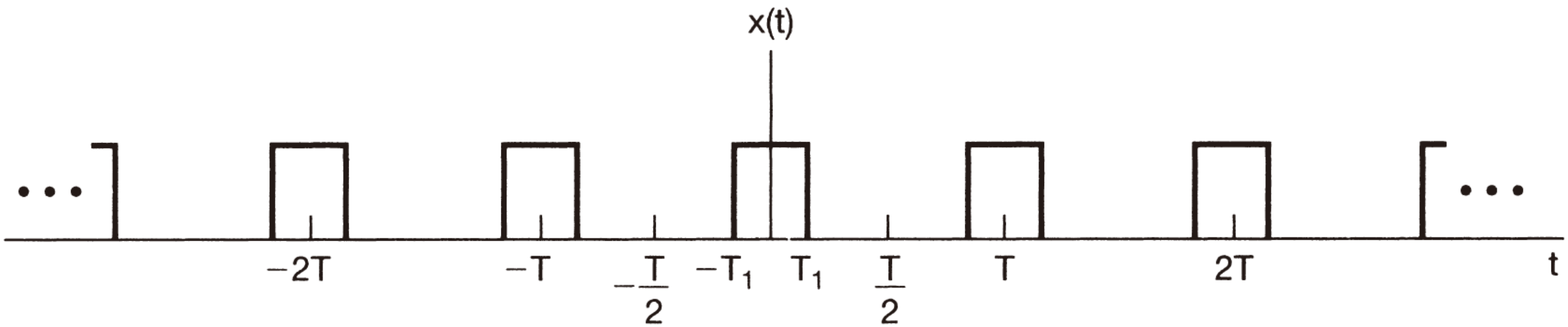


Figure 4.1 A continuous-time periodic square wave.

圖 4.1 爲週期 T 且脈波寬度 T_1 的方波 $x(t)$。

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

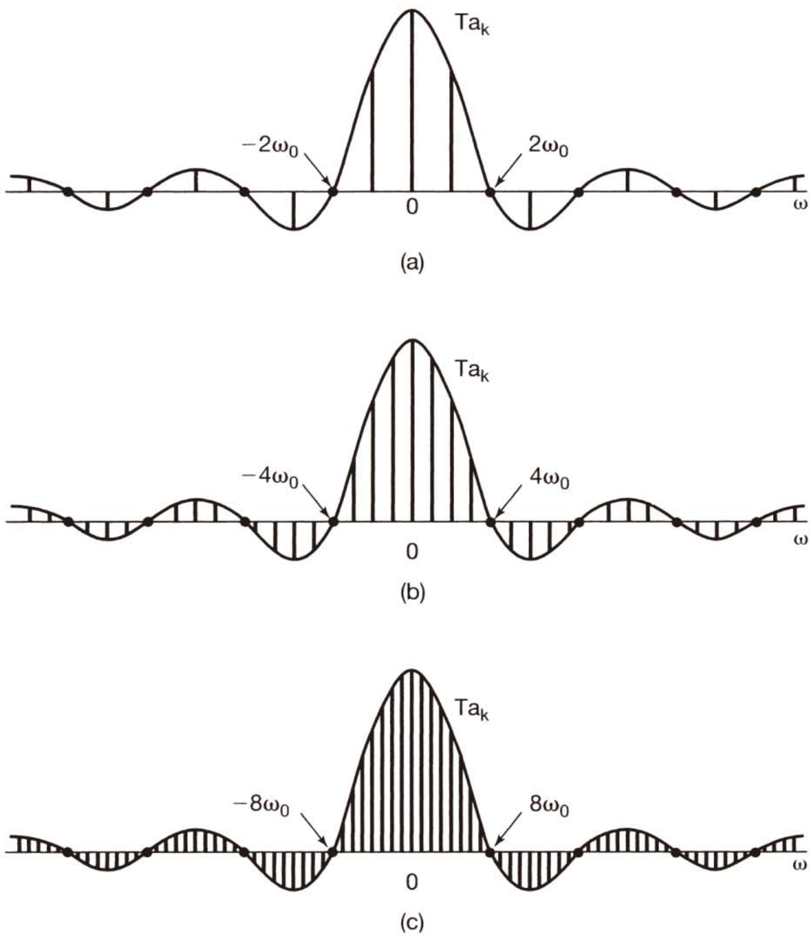


圖 4.2 所示為圖 4.1 的方波 $x(t)$ 的傅立葉係數及包絡線。

Figure 4.2 The Fourier series coefficients and their envelope for the periodic square wave in Figure 4.1 for several values of T (with T_1 fixed): (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$.

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\pi / T$ decreases, the envelope is sampled with a closer and closer spacing.

當*T*愈大，則基頻 $\omega_0 = 2\pi / T$ 愈小，且包絡線上的傅立葉係數取點愈密集。

the set of Fourier series coefficients approaches the envelope function as $T \to \infty$.

當$T \to \infty$時，傅立葉係數趨近於包絡函數。

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

In particular, consider a signal x(t) that is of finite duration. That is, for some number $T_1, x(t) = 0$ if $|t| > T_1$, as illustrated in Figure 4.3(a). From this aperiodic signal, we can construct a periodic signal $\tilde{x}(t)$ for which *x(t)* is one period, as indicated in Figure 4.3(b).

對於有限時間訊號*x(t)*，可建立另一個週期訊號 $x_1(t)$，使得 $x_1(t)$ 的一個週期正好等於*x(t)*。再將 $T \to \infty$，則 $\tilde{x}(t)$ 在任何有限時間上均等於*x(t)*。

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

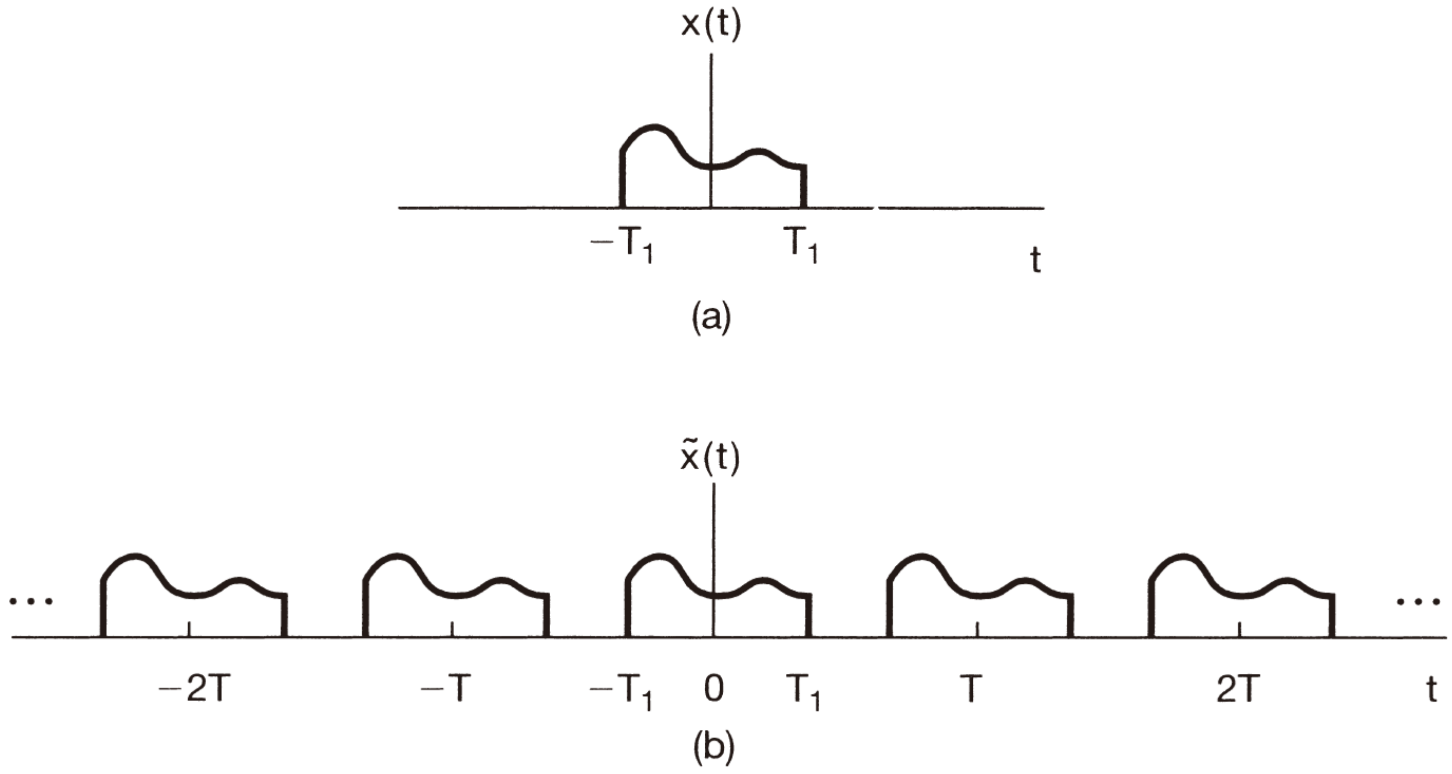


Figure 4.3 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

in eq. (3.39) carried out over the interval $-T/2 \le t \le T/2$, we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \qquad (4.3)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt, \qquad (4.4)$$

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

where $\omega_0 = 2\pi / T$. Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since *x(t)*=0 outside this interval, eq. (4.4) can be rewritten as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Therefore, defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt, \tag{4.5}$$

we have, for the coefficients a_k,

$$a_k = \frac{1}{T} X(jk\omega_0). \tag{4.6}$$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

Combining eqs. (4.6) and (4.3), we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t},$$

or equivalently, since $2\pi / T = \omega_0$,

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0. \qquad (4.7)$$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

As *T*→∞, $\tilde{x}(t)$ approaches *x(t),* and consequently, in the limit eq. (4.7) becomes a representation of *x(t).*

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega \qquad (4.8)$$

and

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt. \qquad (4.9)$$

在*T*→∞之下，$\tilde{x}(t)\rightarrow x(t)$則在此極限下可得*x(t)*的傅立葉轉換為(4.9)式，而傅立葉反轉換為(4.8)式。

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

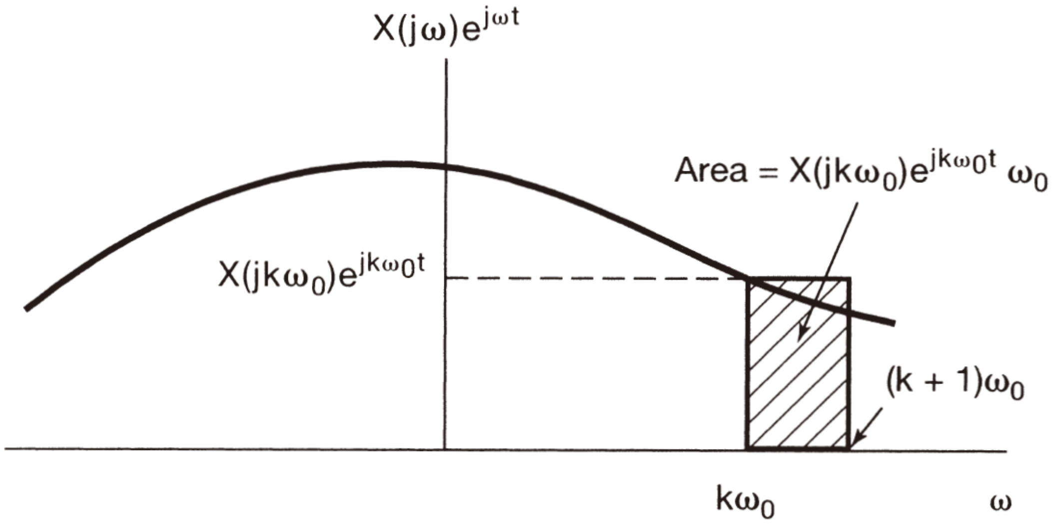


Figure 4.4 Graphical interpretation of eq. (4.7).

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

(4.9)式的$X(j\omega)$稱為「傅立葉轉換」或「傅立葉積分」。

(4.8)式稱為「反傅立葉轉換」。

上述兩式合稱「傅立葉轉換對」。

$X(j\omega)$通常稱為$x(t)$的「頻譜」。

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

since eq.(3.39) allows us to compute the Fourier coefficients of $\tilde{x}(t)$ by integrating over any period, we can write

$$a_k = \frac{1}{T}\int_{s}^{s+T} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{s}^{s+T} x(t)e^{jk\omega_0 t}dt.$$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

Since *x(t)* is zero outside the range

$s \leq t \leq s+T$ we can equivalently write

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt.$$

Comparing with eq. (4.9) we conclude that

$$a_k = \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0}, \qquad (4.10)$$

4.1.2 Convergence of Fourier Transforms

Consider $X(j\omega)$ evaluated according to eq. (4.9), and let $\hat{x}(t)$ denote the signal obtained by using in the right-hand side of eq. (4.8). That is,

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega.$$

4.1.2 Convergence of Fourier Transforms

If *x(t)* has finite energy, i.e., if it is square integrable, so that

$$\int_{-\infty}^{+\infty} \left| x(t) \right|^2 dt < \infty, \qquad (4.11)$$

若*x(t)*為有限能量(即平方可積分)，則$X(j\omega)$存在。

then we are guaranteed that $X(j\omega)$ is finite [i.e., eq. (4.9) converges] and that ,with $e(t)$ denoting the error between $\tilde{x}(t)$ and *x(t)* [i.e.,$e(t) = \hat{x}(t) - x(t)$]

$$\int_{-\infty}^{+\infty} \left| e(t) \right|^2 dt = 0. \qquad (4.12)$$

4.1.2 Convergence of Fourier Transforms

These conditions, again referred to as the Dirichlet conditions, require that:

1. *x(t)* be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} \left| x(t) \right| dt < \infty. \qquad (4.13)$$

2. *x(t)* have a finite number of maxima and minima within any finite interval.
3. *x(t)* have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

4.1.2 Convergence of Fourier Transforms

傅立葉轉換存在的充分條件(迪利斯雷條件)：

1. $x(t)$為絕對可積分。
2. $x(t)$在任何有限時間內含有有限個極大極小值。
3. $x(t)$在任何有限時間內含有有限個不連續點，且每個不連續點必須是有限值。

Example 4.1

Consider the signal

$$x(t) = e^{-at}u(t) \qquad a > 0.$$

From eq. (4.9),

$$X(j\omega) = \int_0^\infty e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^\infty.$$

That is,

$$X(j\omega) = \frac{1}{a+j\omega}, \qquad a > 0.$$

4.1.3 Examples of Continuous-Time Fourier Transforms

in fact, the case. Specifically, consider the inverse Fourier transform for the rectangular pulse signal:

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\frac{\sin \omega T_1}{\omega} e^{j\omega t} d\omega.$$

Then, since *x(t)* in square integrable,

$$\int_{-\infty}^{+\infty} \left| x(t) - \hat{x}(t) \right|^2 dt = 0.$$

4.1.3 Examples of Continuous-Time Fourier Transforms

in analogy with the finite Fourier series approximation, eq. (3.47), consider the following integral over a finite-length interval of frequencies:

$$\frac{1}{2\pi}\int_{-W}^{W} 2\frac{\sin \omega T_1}{\omega} e^{j\omega t} d\omega.$$

4.1.3 Examples of Continuous-Time Fourier Transforms

Comparing Figures 4.8 and 4.9 or, equivalently, eqs. (4.16) and (4.17) with eqs. (4.18) and (4.19), we see an interesting relationship. In each case, the Fourier transform pair consists of a function of the form $(\sin a\theta)/b\theta$ and a rectangular pulse.

比較圖4.8及4.9，範例4.4的訊號為脈波，而傅立葉轉換為sinc函數，對於範例4.5則正好相。這正是傅立葉轉換的對偶性質。

4.1.3 Examples of Continuous-Time Fourier Transforms

A commonly used precise form for the sinc function is

$$\sin c(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{4.20}$$

The sinc function is plotted in figure 4.10. Both of the signals in eqs.(4.17) and (4.19) can be expressed in terms of the sinc function:

$$\frac{2\sin \omega T_1}{\omega} = 2T_1 \sin c\left(\frac{\omega T_1}{\pi}\right)$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi} \sin c\left(\frac{Wt}{\pi}\right).$$

4.1.3 Examples of Continuous-Time Fourier Transforms

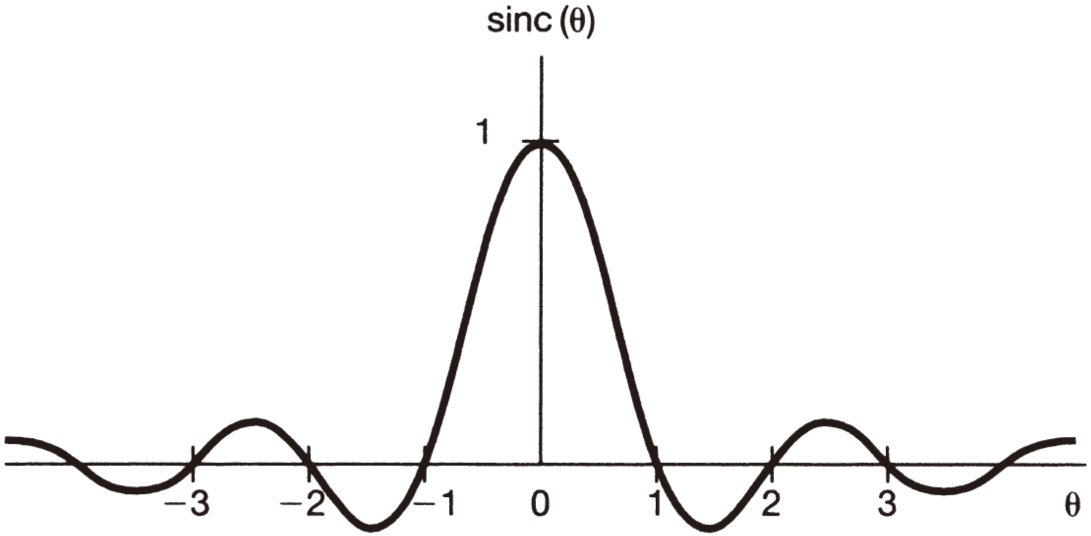
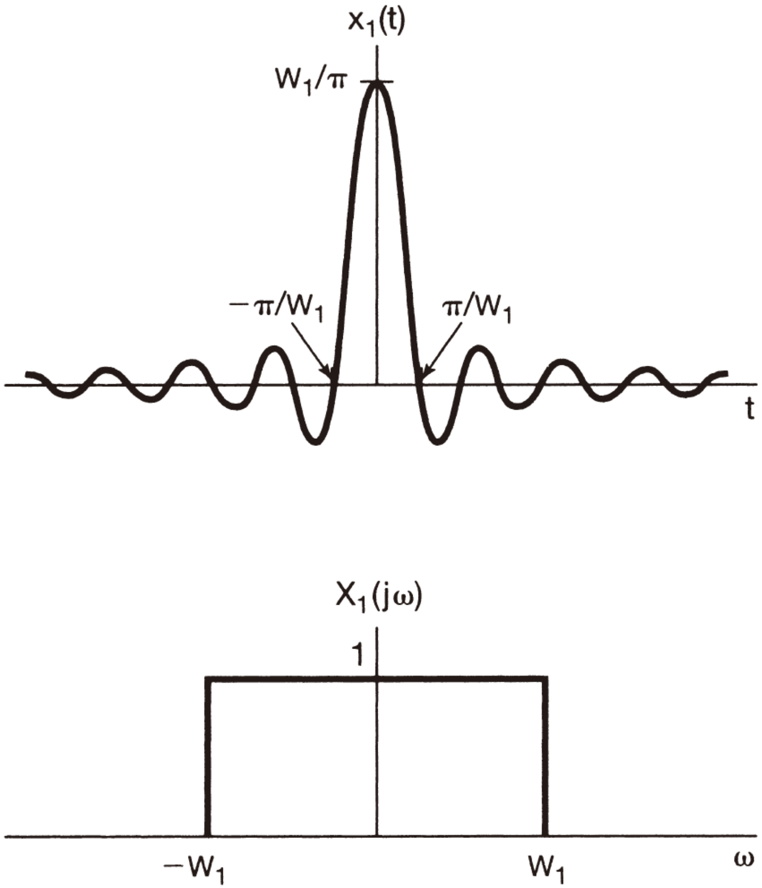


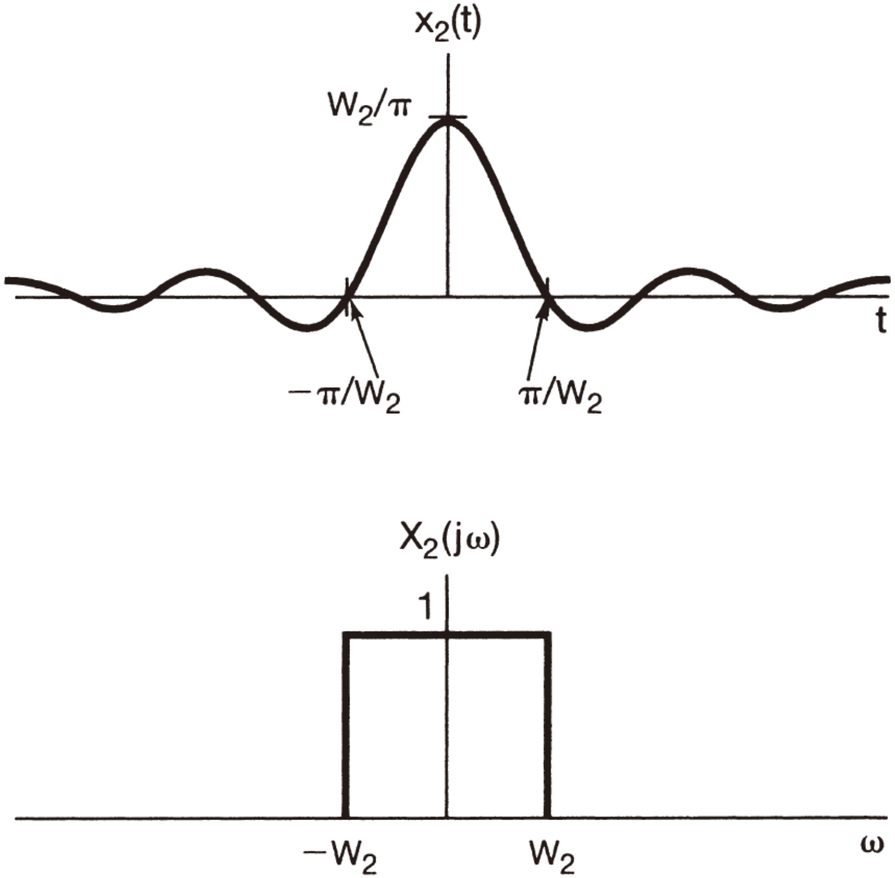
Figure 4.10 The sinc function.

4.1.3 Examples of Continuous-Time Fourier Transforms



(a)

4.1.3 Examples of Continuous-Time Fourier Transforms



(b)

4.1.3 Examples of Continuous-Time Fourier Transforms

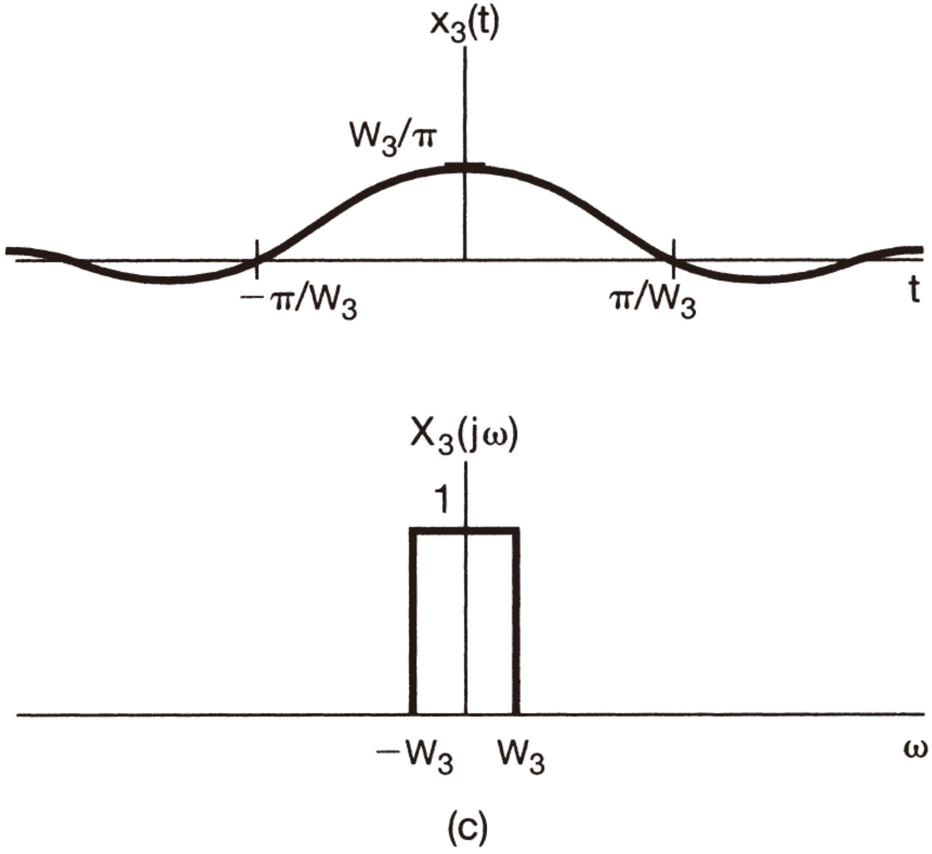


Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of *W*.

4.2 The Fourier Transform for periodic signals

To suggest the general result, let us consider a signal *x(t)* with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega=\omega_0$; that is,

$$X(j\omega)=2\pi\delta(\omega-\omega_0). \qquad (4.21)$$

To determine the signal *x(t)* for which this is the Fourier transform, we can apply the inverse transform relation, eq. (4.8), to obtain

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega$$

$$e^{j\omega_0 t}\leftrightarrow 2\pi\delta(\omega-\omega_0)$$

$$=e^{j\omega_0 t}.$$

4.2 The Fourier Transform for periodic signals

More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency, that is,

$$X(j\omega)=\sum_{k=\infty}^{+\infty} 2\pi a_k \delta(\omega-k\omega_0), \qquad (4.22)$$

then the application of eq. (4.8) yields

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow$$

$$\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-\omega_0)$$

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}. \qquad (4.23)$$

上式左側為一週期訊號的傅立葉級數，右側為其傅立葉轉換。故週期訊號的傅之葉轉換，呈現出以各傅立葉級數為大小且間隔 ω_0的脈衝串。

4.3 Properties of the Continuous-time Fourier Transform

As developed in Section 4.1, a signal *x(t)* and its Fourier transform $X(j\omega)$ are related by the Fourier transform synthesis and analysis equations,

[eq. (4.8)]

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \qquad (4.24)$$

and

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt. \qquad (4.25)$$

[eq. (4.9)]

4.3 Properties of the Continuous-time Fourier Transform

refer to *x(t)* and $X(j\omega)$ as a Fourier transform pair with the notation

$$x(t) \xleftrightarrow{F} X(j\omega).$$

Thus, with reference to Example 4.1,

$$\frac{1}{a+j\omega} = F\left\{e^{-at}u(t)\right\},$$

$$e^{-at}u(t) = F^{-1}\left\{\frac{1}{a+j\omega}\right\},$$

and

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}.$$

4.3.1 Linearity

If

$$x(t) \xleftrightarrow{F} X(j\omega)$$

and

$$y(t) \xleftrightarrow{F} Y(j\omega),$$

Then

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega). \quad (4.26)$$

線性性質(重疊性質適用)

4.3.2 Time Shifting

If

$$x(t) \xleftrightarrow{F} X(j\omega),$$

Then

時間移位性質 $$x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega). \quad (4.27)$$

To establish this property, consider eq. (4.24);

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega.$$

4.3.2 Time Shifting

Replacing *t* by $t - t_0$ in this equation, we obtain

$$x(t - t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(e^{-j\omega t_0} X(j\omega)\right)e^{j\omega t}d\omega.$$

Recognizing this as the synthesis equation for $x(t - t_0)$, we conclude that

$$F\{x(t - t_0)\} = e^{-j\omega t_0} X(j\omega).$$

4.3.2 Time Shifting

if we express *X(jω)* in polar form as

$$F\{x(t)\} = X(j\omega) = |X(j\omega)| e^{j\measuredangle X(j\omega)},$$

then

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega) = |X(j\omega)| e^{j[\measuredangle X(j\omega) - \omega t_0]}.$$

4.3.3 Conjugation and Conjugate Symmetry

The conjugation property states that if

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega),$$

then

共軛性質 $$x^*(t) \overset{F}{\longleftrightarrow} X^*(-j\omega). \qquad (4.28)$$

4.3.3 Conjugation and Conjugate Symmetry

This property follows from the evaluation of the complex conjugate eq. (4.25):

$$X^*(j\omega) = \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]^*$$

$$= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}dt.$$

Replacing ω by $-\omega$, we see that

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}dt. \qquad (4.29)$$

4.3.3 Conjugation and Conjugate Symmetry

if *x(t)* is real, then $X(j\omega)$ has *conjugate symmetry*; that is,

$$X(-j\omega) = X^*(j\omega) \qquad [x(t) \quad real].$$

(4.30)

Specifically, if *x(t)* is real so that $x^*(t) = x(t)$, we have, from eq. (4.29),

$$X^*(-j\omega) = \int_{-\infty}^{+\infty} x(t)e^{j\omega t}dt = X(j\omega),$$

and eq. (4.30) follows by replacing ω and $-\omega$.

4.3.3 Conjugation and Conjugate Symmetry

From Example 4.1, with $x(t)=e^{-at}u(t)$,

$$X(j\omega)=\frac{1}{a+j\omega}$$

and

$$X(-j\omega)=\frac{1}{a-j\omega}=X^{*}(j\omega).$$

4.3.3 Conjugation and Conjugate Symmetry

As one consequence of eq. (4.30), if we express *X(jω)* in rectangular form as

$$X(j\omega) = \Re e\{X(j\omega)\} + j\Im m\{X(j\omega)\},$$

Then if *x(t)* is real,

$$\Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\}$$

and

$$\Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\}.$$

4.3.3 Conjugation and Conjugate Symmetry

if we express $X(j\omega)$ in polar form as

$$X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)},$$

As a further consequence of eq. (4.30), if $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even. To see this, we write

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t) e^{j\omega t} dt,$$

4.3.3 Conjugation and Conjugate Symmetry

or, with the substitution $\tau = -t$,

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(-\tau) e^{-j\omega\tau} d\tau.$$

since $x(-\tau) = x(\tau)$, we have

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega t} d\tau$$

$$= X(j\omega).$$

4.3.3 Conjugation and Conjugate Symmetry

Finally, as was discussed in Chapter 1, a real function *x(t)* can always be expressed in terms of the sum of an even function $x_e(t)$ = $\mathcal{E}v\{x(t)\}$ and an odd function $x_0(t)$= $\mathcal{O}d\{x(t)\}$; that is,

$$x(t) = x_e(t) + x_0(t).$$

From the linearity of the Fourier transform,

$$F\{x(t)\} = F\{x_e(t)\} + F\{x_0(t)\},$$

4.3.3 Conjugation and Conjugate Symmetry

and from the preceding discussion, $F\{x_e(t)\}$ is a real function and $F\{x_0(t)\}$ is purely imaginary. Thus, we can conclude that, with *x(t)* real,

$$x(t) \xleftrightarrow{F} X(j\omega),$$

$$\mathcal{E}v\{x(t)\} \xleftrightarrow{F} \Re e\{X(j\omega)\},$$

$$\mathcal{O}d\{x(t)\} \xleftrightarrow{F} j\Im m\{X(j\omega)\}.$$

訊號的偶函數部份的傅立葉轉換等於原訊號傅立葉轉換的實部。訊號的奇函數部份的傅立葉轉換等於原訊號傅立葉轉換的虛部。

4.3.4 Differentiation and Integration

By differentiation both sides of the Fourier transform synthesis equation (4.24), we obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega X(j\omega)e^{j\omega t}\,d\omega.$$

Therefore,

$$\frac{dx(t)}{dt} \stackrel{F}{\longleftrightarrow} j\omega X(j\omega). \qquad (4.31)$$

訊號微分性質

時域的微分相對於頻域乘以$j\omega$。

4.3.4 Differentiation and Integration

This is indeed the case, but it is only one part of the picture. The precise relationship is

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega). \tag{4.32}$$

訊號積分性質

4.3.5 Time and Frequency Scaling

If

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega),$$

Then

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right), \tag{4.34}$$

時間刻度變換

where a is a nonzero real number. This property follows directly from the definition of the Fourier transform—specifically,

$$F\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt.$$

4.3.5 Time and Frequency Scaling

Using the substitution τ =at , we obtain

$$F\{x(at)\} = \begin{cases} \dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a\rangle 0 \\ -\dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a\langle 0 \end{cases}$$

Thus, aside from the amplitude factor 1/lal, a linear scaling in time by a factor of corresponds to a linear scaling in frequency by a factor of 1/a,and vice versa. Also, letting a = -1

$$x(-t) \xleftrightarrow{F} X(-j\omega). \qquad (4.35)$$

訊號時間倒轉對應的傅立葉轉換為頻率倒轉。

4.3.5 Time and Frequency Scaling

we decrease its frequency. Also, as we saw in Example 4.5 (see Figure 4.11), if we consider the transform

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases},$$

4.3.6 Duality

In the former example we derived the Fourier transform pair

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \xleftrightarrow{F} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}, \quad (4.36)$$

脈波型式(時域) $\longleftrightarrow$ sin函數型式(頻域)

while in the latter we considered the pair

$$x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{F} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}. \quad (4.37)$$

sin函數型式(時域) $\longleftrightarrow$ 脈波型式(頻域)

4.3.6 Duality

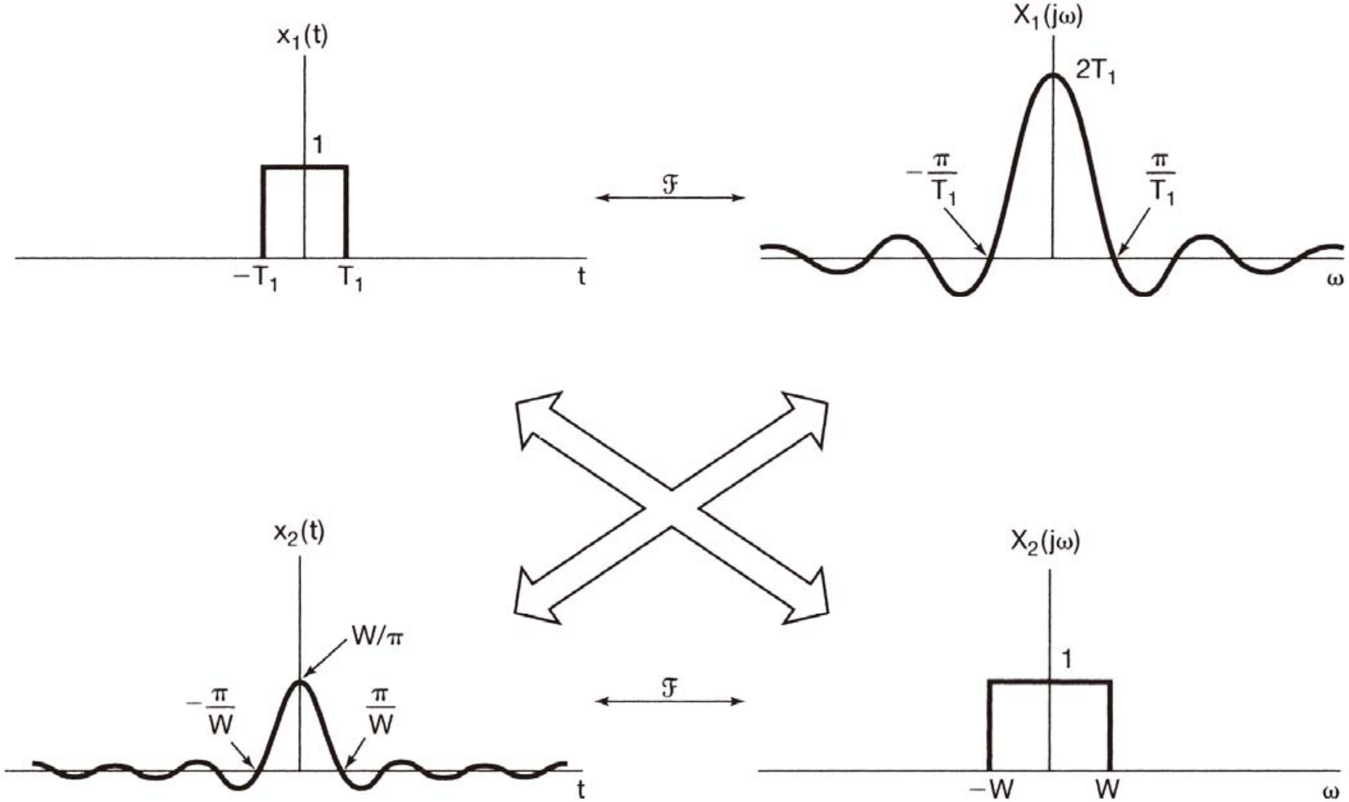


Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

4.3.6 Duality

To determine the precise form of this dual property, we can proceed in a fashion exactly analogous to that used in Section 4.3.4. Thus, if we differentiate the analysis equation (4.25) with respect to ω, we obtain

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} -jtx(t)e^{-j\omega t}dt. \qquad (4.39)$$

That is,

$$-jtx(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega}. \qquad (4.40)$$

由微分性質及對偶性質而得。

4.3.6 Duality

Similarly, we can derive the dual properties of eqs. (4.27) and (4.32):

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X\left(j(\omega - \omega_0)\right) \qquad (4.41)$$

由時間移位性質及對偶性質而得。

and

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{F} \int_{-\omega}^{\omega} x(\eta)\,d\eta. \qquad (4.42)$$

由積分性質及對偶性質而得。

4.3.7 Parseval’s Relation

If *x(t)* and *X(jω)* are a Fourier transform pair, then

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega. \qquad (4.43)$$

巴斯瓦關係式(定理)

This expression, referred to as Parseval’s relation, follows from direct application of the Fourier transform. Specifically,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x9t)x^{*}9t)dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^{*}(j\omega)e^{-j\omega t} d\omega\right]dt.$$

4.3.7 Parseval's Relation

Reversing the order of integration gives

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega) \left[\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \right] d\omega.$$

The bracketed term is simply the Fourier transform of *x(t);* thus,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.$$

此定理意指訊號*x(t)*的總能量等於 $|X(j\omega)|^2/2\pi$ 對整個頻率軸積分。故 $|X(j\omega)|^2$ 常稱為「能量密度頻譜」。

4.4 The Convolution Property

referring back to eq. (4.7), *x(t)* is expressed as the limit of a sum; that is,

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}X(jk\omega_0)e^{jk\omega_0 t}\omega_0. \tag{4.47}$$

As developed in Sections 3.2 and 3.8, the response of a linear system with impulse response *h(t)* to a complex exponential $e^{jk\omega_0 t}$ is $H(jk\omega_0)e^{jk\omega_0 t}$, where

$$H(jk\omega_0)=\int_{-\infty}^{+\infty}h(t)e^{-jk\omega_0 t}dt. \tag{4.48}$$

4.4 The Convolution Property

From superposition [see eq.(3.124)], we then have

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \rightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0,$$

and thus, from eq. (4.47), the response of the linear system to *x(t)* is

$$y(t) = \lim_{\omega_0 \to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)H(j\omega)e^{j\omega t}d\omega. \qquad (4.49)$$

4.4 The Convolution Property

Since *y(t)* and its Fourier transform *Y(jω)* are related by

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\omega)e^{j\omega t}d\omega, \qquad (4.50)$$

we can identify *Y(jω)* from eq. (4.49), yielding

$$Y(j\omega) = X(j\omega)H(j\omega). \qquad (4.51)$$

As a more formal derivation, we consider the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \qquad (4.52)$$

4.4 The Convolution Property

We desire *Y(jω)*, which is

$$Y(j\omega) = F\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt. \qquad (4.53)$$

Interchanging the order of integration and noting that x(τ) does not depend on t, we have

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau. \qquad (4.54)$$

4.4 The Convolution Property

By the time-shift property, eq. (4.27), the bracketed term is $e^{-j\omega\tau}H(j\omega)$. Substituting this into eq. (4.54) yields

$$Y(j\omega)=\int_{-\infty}^{+\infty}x(\tau)e^{-j\omega t}H(j\omega)d\tau=H(j\omega)\int_{-\infty}^{+\infty}x(\tau)e^{-j\omega t}d\tau.$$

The integral is $X(j\omega)$, and hence,

$$Y(j\omega)=H(j\omega)X(j\omega).$$

That is,

$$y(t)=h(t)*x(t)\xleftrightarrow{F}Y(j\omega)=H(j\omega)X(j\omega). \quad (4.56)$$

連續時間傅立葉轉換的迴旋運算性質

上式在訊號與系統分析上是極為重要的，它將時域中較複雜的迴旋運算轉換至頻域中較簡單的乘法。

4.4 The Convolution Property

As illustrated in Figure 4.19, since the impulse response of the cascade of two LTI systems is the convolution of the individual impulse responses, the convolution property then implies that the overall frequency response of the cascade of two systems is simply the product of the individual frequency responses.

兩個串接的LTI系統的脈衝響應為個別的脈衝響應的迴旋積分，故其串接系統頻率響應即為個別的頻率響應的乘積。

4.4 The Convolution Property

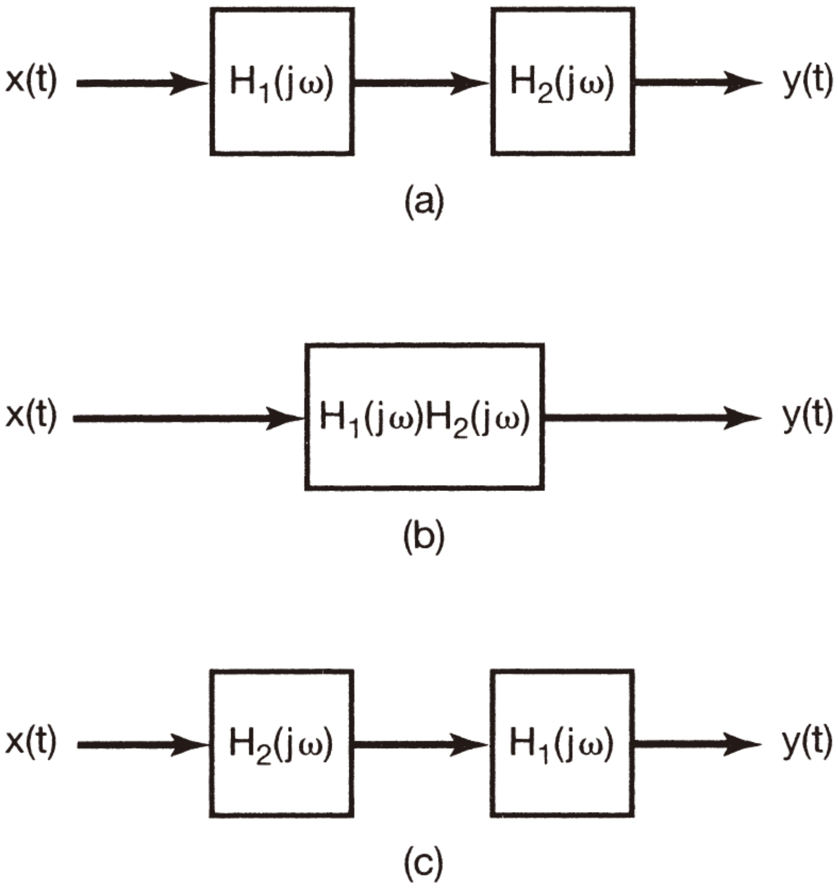


Figure 4.19 Three equivalent LTI systems. Here, each block represents an LTI system with the indicated frequency response.

4.4 The Convolution Property

If, however, an LTI system is stable, then, as we saw in Section 2.3.7 and Problem 2.49, its impulse response is absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |h(t)| dt < \infty.$$

4.4 The Convolution Property

In using Fourier analysis to study LTI systems, we will be restricting ourselves to systems whose impulse responses possess Fourier transforms.

為了利用轉換法來檢視不穩定的LTI系統，必須藉助一種傅立葉轉換的一般化型式，即「拉氏轉換」。

4.5 The multiplication Property

Because of duality between the time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$r(t) = s(t)p(t) \leftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P\big(j(\omega-\theta)\big)d\theta \qquad (4.70)$$

乘法性質：

時域中訊號相乘對應至頻域中為個別的傅立葉轉換的迴旋積分。

4.5 The multiplication Property

Multiplication of one signal by another can be thought of as using one signal to scale or *modulate* the amplitude of the other, and consequently, the multiplication of two signals is often referred to as *amplitude modulation*. For this reason, eq. (4.70) is sometimes referred to as the *modulation property*.

兩訊號相乘常稱之為「振幅調變」。故(4.70)式常稱為「調變性質」。

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

In a frequency-selective bandpass filter built with elements such as resistors, operational amplifiers, and capacitors, the center frequency depends on a number of element values, all of which must be varied simultaneously in the correct way if the center frequency is to be adjusted directly.

直接利用元件值的改變，來調整帶通濾波器可變的中心頻率是很困難的。實際上可利用固定的濾波，及適當的地利用弦波振幅調變來平移訊號的頻譜達成。如圖4.26及4.27所示。

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

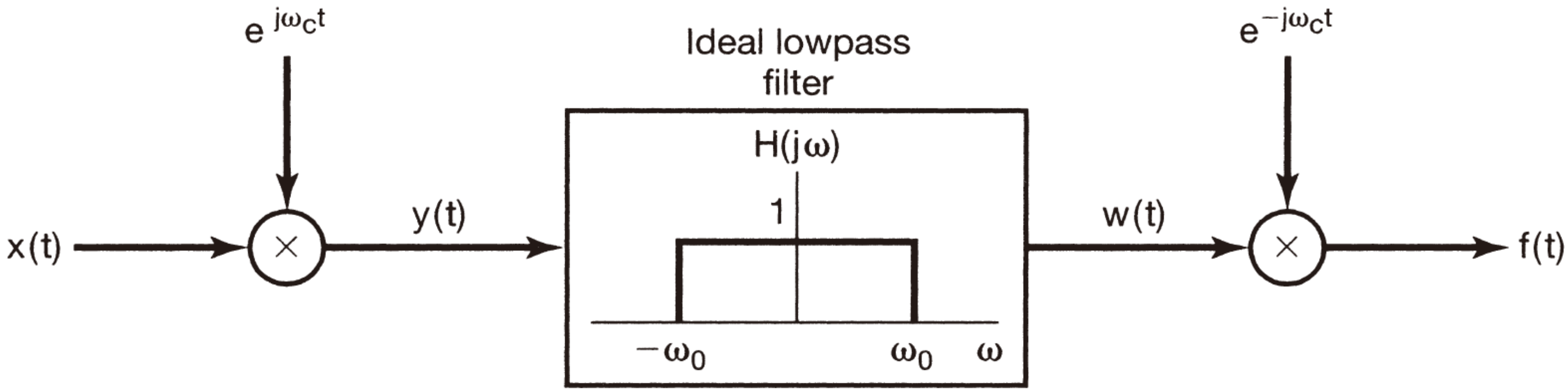
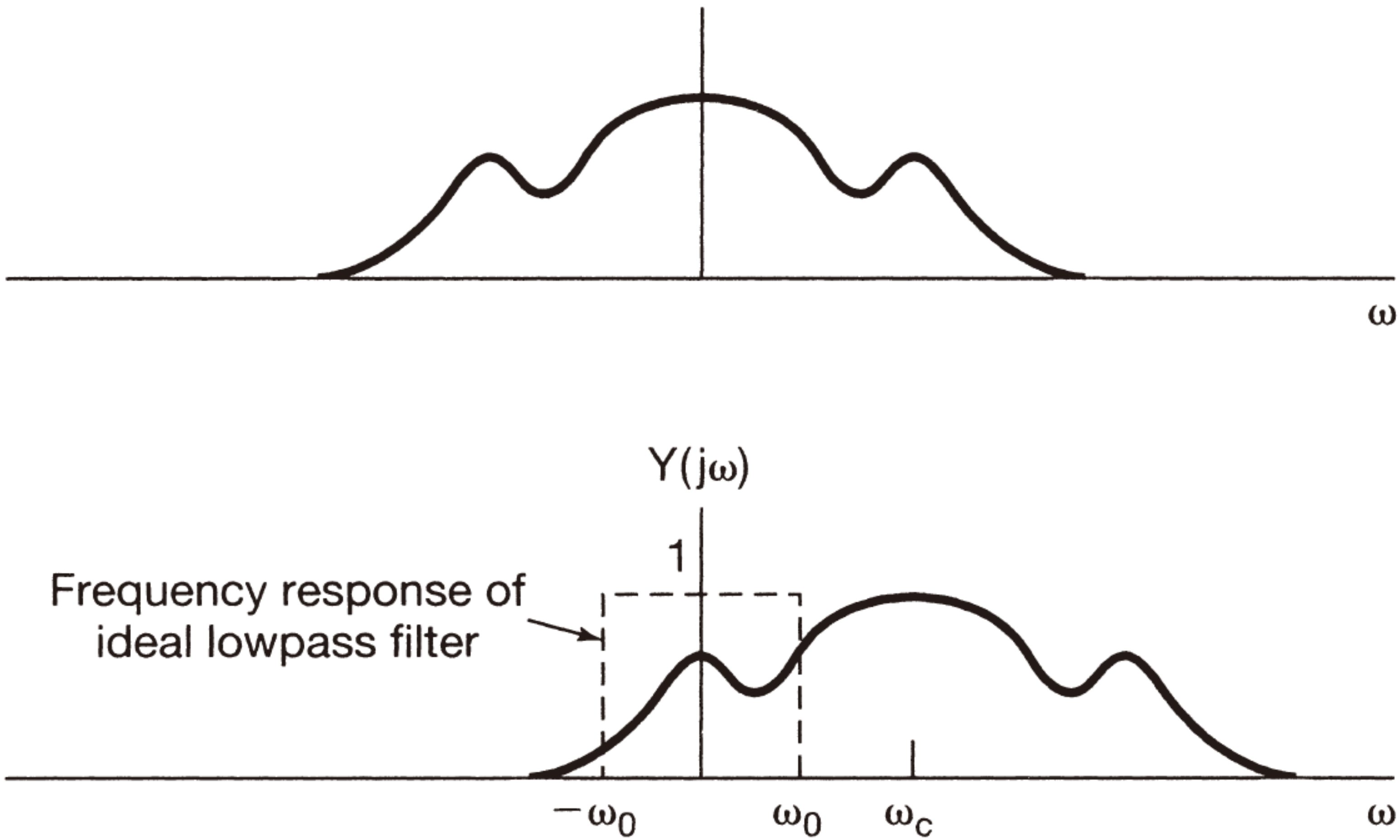


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency



4.5.1 Frequency-Selective Filtering with Variable Center Frequency

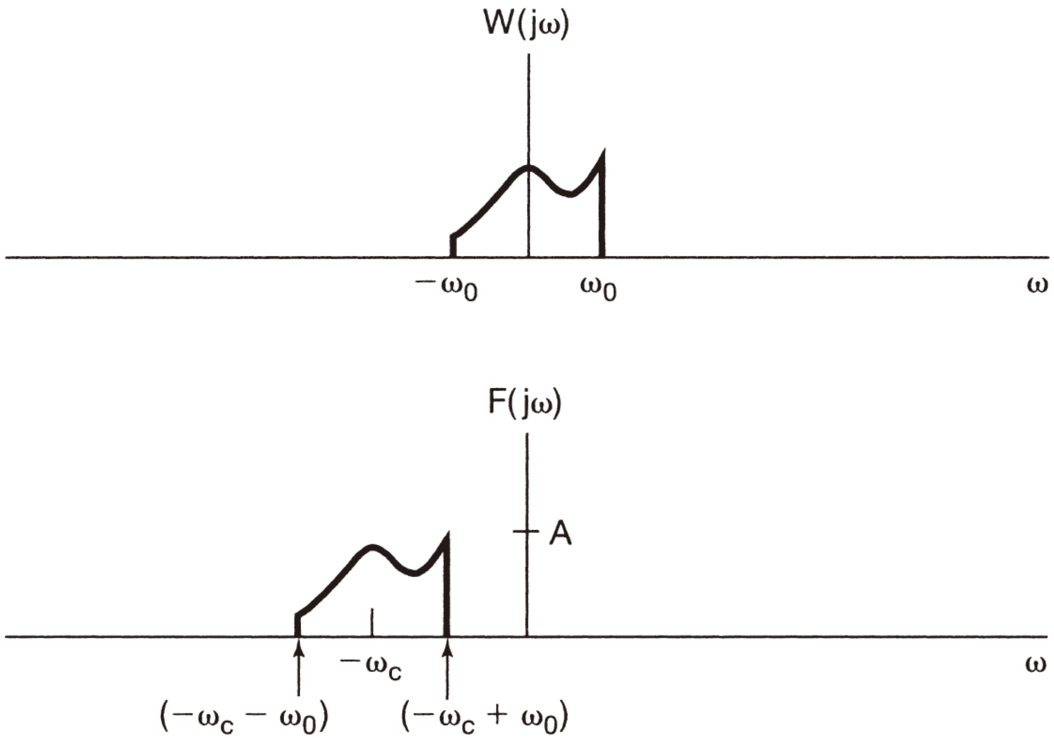


Figure 4.27 Spectra of the signals in the system of Figure 4.26.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

The Fourier transform of $y(t) = e^{j\omega_c t} x(t)$ is

$$Y(j\omega) = \int_{-\infty}^{+\infty} \delta(\theta - \omega_c) X(\omega - \theta) d\theta$$

the Fourier transform of $f(t) = e^{-jw_c t} w(t)$ is

$$F(j\omega) = W\left(j(\omega + \omega_c)\right),$$

訊號*W(t)*乘以弦波 $e^{-j\omega_0 t}$，其頻譜為原頻譜*W(jω)*平移成 $W\left(j(\omega+\omega_c)\right)$。

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

So that the Fourier transform of $F(j\omega)$ is $W(j\omega)$ shifted to left by ω_c . From Figure 4.27, we observe that the overall system of Figure 4.26 is equivalent to an ideal bandpass filter with center frequency $-\omega_c$ and bandwidth $2\omega_0$, as illustrated in Figure 4.28.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

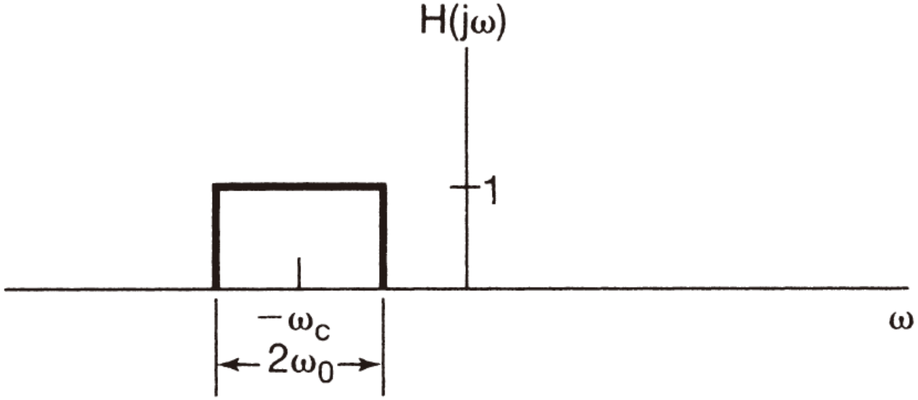


Figure 4.28 Bandpass filter equivalent of Figure 4.26.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

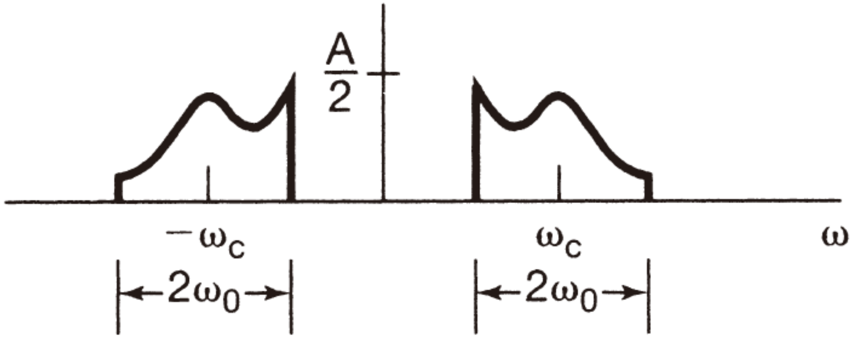


Figure 4.29 Spectrum of $\mathcal{R}e\{f(t)\}$ associated with Figure 4.26.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

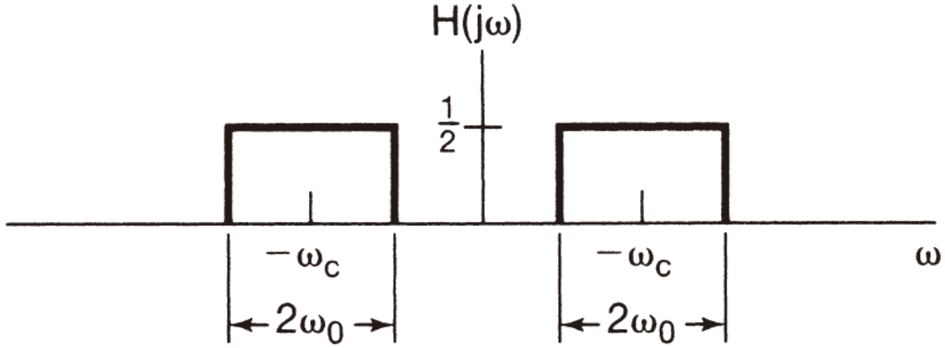


Figure 4.30 Equivalent bandpass filter for $\mathcal{R}e\{f(t)\}$ in Figure 4.29.

4.6 Tables of Fourier Proerties and of Basic Fourier Transform Pairs

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM 表 4.1 傅立葉轉換的重要性質

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\}=\mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\}=-\mathcal{Im}\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{Ev}\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{Od}\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\lvert X(j\omega)\rvert^2d\omega$$

4.6 Tables of Fourier Proerties and of Basic Fourier Transform Pairs

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS 表 4.2 基本的傅立葉轉換對

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1,\ a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\,\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\,\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t),\ \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t),\ \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—