11.8 Fourier Cosine and Sine Transforms

An **integral transform** is a transformation in the form of an integral that produces from given functions new functions depending on a different variable. One is mainly interested in these transforms because they can be used as tools in solving ODEs, PDEs, and integral equations and can often be of help in handling and applying special functions. The Laplace transform of Chap. 6 serves as an example and is by far the most important integral transform in engineering.

Next in order of importance are Fourier transforms. They can be obtained from the Fourier integral in Sec. 11.7 in a straightforward way. In this section we derive two such transforms that are real, and in Sec. 11.9 a complex one.

Fourier Cosine Transform

The Fourier cosine transform concerns ***even functions*** $f(x)$. We obtain it from the Fourier cosine integral [(10) in Sec. 10.7]

$$f(x) = \int_0^\infty A(w) \cos wx \, dw, \qquad \text{where} \qquad A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos wv \, dv.$$

Namely, we set $A(w) = \sqrt{2/\pi}\,\hat{f}_c(w)$, where c suggests "cosine." Then, writing $v = x$ in the formula for $A(w)$, we have

(1a)
$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \cos wx \, dx$$

and

(1b)
$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w) \cos wx \, dw.$$

Formula (1a) gives from $f(x)$ a new function $\hat{f}_c(w)$, called the **Fourier cosine transform** of $f(x)$. Formula (1b) gives us back $f(x)$ from $\hat{f}_c(w)$, and we therefore call $f(x)$ the **inverse Fourier cosine transform** of $\hat{f}_c(w)$.

The process of obtaining the transform $\hat{f}_c$ from a given f is also called the **Fourier cosine transform** or the *Fourier cosine transform method.*

Fourier Sine Transform

Similarly, in (11), Sec. 11.7, we set $B(w) = \sqrt{2/\pi}\,\hat{f}_s(w)$, where s suggests "sine." Then, writing $v = x$, we have from (11), Sec. 11.7, the **Fourier sine transform**, of $f(x)$ given by

(2a)
$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \sin wx \, dx,$$

and the **inverse Fourier sine transform** of $\hat{f}_s(w)$, given by

(2b)
$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw.$$

The process of obtaining $f_s(w)$ from $f(x)$ is also called the **Fourier sine transform** or the *Fourier sine transform method.*

Other notations are

$$\mathcal{F}_c(f) = \hat{f}_c, \qquad \mathcal{F}_s(f) = \hat{f}_s$$

and $\mathcal{F}_c^{-1}$ and $\mathcal{F}_s^{-1}$ for the inverses of $\mathcal{F}_c$ and $\mathcal{F}_s$, respectively.

EXAMPLE 1 Fourier Cosine and Fourier Sine Transforms

Find the Fourier cosine and Fourier sine transforms of the function

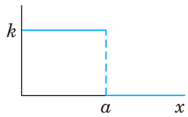

Fig. 285. $f(x)$ in Example 1

$$f(x) = \begin{cases} k & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases} \qquad \text{(Fig. 285).}$$

Solution. From the definitions (1a) and (2a) we obtain by integration

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \cos wx \, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{\sin aw}{w}\right)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \sin wx \, dx = \sqrt{\frac{2}{\pi}}\, k \left(\frac{1 - \cos aw}{w}\right).$$

This agrees with formulas 1 in the first two tables in Sec. 11.10 (where $k = 1$).

Note that for $f(x) = k = \text{const}$ $(0 < x < \infty)$, these transforms do not exist. (Why?) ■

EXAMPLE 2 Fourier Cosine Transform of the Exponential Function

Find $\mathcal{F}_c(e^{-x})$.

Solution. By integration by parts and recursion,

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx \, dx = \sqrt{\frac{2}{\pi}} \frac{e^{-x}}{1 + w^2}(-\cos wx + w \sin wx)\Big|_0^\infty = \frac{\sqrt{2/\pi}}{1 + w^2}.$$

This agrees with formula 3 in Table I, Sec. 11.10, with $a = 1$. See also the next example. ■

What did we do to introduce the two integral transforms under consideration? Actually not much: We changed the notations A and B to get a "symmetric" distribution of the constant $2/\pi$ in the original formulas (1) and (2). This redistribution is a standard convenience, but it is not essential. One could do without it.

What have we gained? We show next that these transforms have operational properties that permit them to convert differentiations into algebraic operations (just as the Laplace transform does). This is the key to their application in solving differential equations.

Linearity, Transforms of Derivatives

If $f(x)$ is absolutely integrable (see Sec. 11.7) on the positive x-axis and piecewise continuous (see Sec. 6.1) on every finite interval, then the Fourier cosine and sine transforms of f exist.

Furthermore, if f and g have Fourier cosine and sine transforms, so does $af + bg$ for any constants a and b, and by (1a)

$$\begin{aligned}\mathcal{F}_c(af + bg) &= \sqrt{\frac{2}{\pi}} \int_0^\infty [af(x) + bg(x)] \cos wx \, dx \\ &= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos wx \, dx.\end{aligned}$$

The right side is $a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$. Similarly for $\mathcal{F}_s$, by (2). This shows that the Fourier cosine and sine transforms are **linear operations**,

$$\text{(3)} \qquad \begin{aligned}&\text{(a)} \quad \mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g), \\ &\text{(b)} \quad \mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g).\end{aligned}$$

THEOREM 1

Cosine and Sine Transforms of Derivatives

Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on every finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then

$$\textbf{(4)} \qquad \begin{aligned}&\textbf{(a)} \quad \mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0), \\ &\textbf{(b)} \quad \mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}.\end{aligned}$$

PROOF This follows from the definitions and by using integration by parts, namely,

$$\begin{aligned}\mathcal{F}_c\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[f(x) \cos wx \Big|_0^\infty + w \int_0^\infty f(x) \sin wx \, dx \right] \\ &= -\sqrt{\frac{2}{\pi}} f(0) + w\mathcal{F}_s\{f(x)\};\end{aligned}$$

and similarly,

$$\begin{aligned}\mathcal{F}_s\{f'(x)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[f(x) \sin wx \Big|_0^\infty - w \int_0^\infty f(x) \cos wx \, dx \right] \\ &= 0 - w\mathcal{F}_c\{f(x)\}. \qquad \blacksquare\end{aligned}$$

Formula (4a) with f' instead of f gives (when f', f'' satisfy the respective assumptions for f, f' in Theorem 1)

$$\mathcal{F}_c\{f''(x)\} = w\mathcal{F}_s\{f'(x)\} - \sqrt{\frac{2}{\pi}}\,f'(0);$$

hence by (4b)

(5a)
$$\boxed{\mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\,f'(0).}$$

Similarly,

(5b)
$$\boxed{\mathcal{F}_s\{f''(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\,wf(0).}$$

A basic application of (5) to PDEs will be given in Sec. 12.7. For the time being we show how (5) can be used for deriving transforms.

EXAMPLE 3 An Application of the Operational Formula (5)

Find the Fourier cosine transform $\mathcal{F}_c(e^{-ax})$ of $f(x) = e^{-ax}$, where $a > 0$.

Solution. By differentiation, $(e^{-ax})'' = a^2e^{-ax}$; thus

$$a^2 f(x) = f''(x).$$

From this, (5a), and the linearity (3a),

$$\begin{aligned} a^2\mathcal{F}_c(f) &= \mathcal{F}_c(f'') \\ &= -w^2\mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}}\,f'(0) \\ &= -w^2\mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}. \end{aligned}$$

Hence

$$(a^2 + w^2)\mathcal{F}_c(f) = a\sqrt{2/\pi}.$$

The *answer* is (see Table I, Sec. 11.10)

$$\mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + w^2}\right) \qquad (a > 0).$$ ■

Tables of Fourier cosine and sine transforms are included in Sec. 11.10.

PROBLEM SET 11.8

1–8 FOURIER COSINE TRANSFORM

1. Find the cosine transform $\hat{f}_c(w)$ of $f(x) = 1$ if $0 < x < 1$, $f(x) = -1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$.
2. Find f in Prob. 1 from the answer $\hat{f}_c$.
3. Find $\hat{f}_c(w)$ for $f(x) = x$ if $0 < x < 2$, $f(x) = 0$ if $x > 2$.
4. Derive formula 3 in Table I of Sec. 11.10 by integration.
5. Find $\hat{f}_c(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.
6. **Continuity assumptions.** Find $\hat{g}_c(w)$ for $g(x) = 2$ if $0 < x < 1$, $g(x) = 0$ if $x > 1$. Try to obtain from it $\hat{f}_c(w)$ for $f(x)$ in Prob. 5 by using (5a).
7. **Existence?** Does the Fourier cosine transform of $x^{-1} \sin x$ $(0 < x < \infty)$ exist? Of $x^{-1} \cos x$? Give reasons.
8. **Existence?** Does the Fourier cosine transform of $f(x) = k = \text{const}$ $(0 < x < \infty)$ exist? The Fourier sine transform?

9–15 FOURIER SINE TRANSFORM

9. Find $\mathscr{F}_s(e^{-ax})$, $a > 0$, by integration.
10. Obtain the answer to Prob. 9 from (5b).
11. Find $f_s(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.
12. Find $\mathscr{F}_s(xe^{-x^2/2})$ from (4b) and a suitable formula in Table I of Sec. 11.10.
13. Find $\mathscr{F}_s(e^{-x})$ from (4a) and formula 3 of Table I in Sec. 11.10.
14. **Gamma function.** Using formulas 2 and 4 in Table II of Sec. 11.10, prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [(30) in App. A3.1], a value needed for Bessel functions and other applications.
15. **WRITING PROJECT. Finding Fourier Cosine and Sine Transforms.** Write a short report on ways of obtaining these transforms, with illustrations by examples of your own.

11.9 Fourier Transform. Discrete and Fast Fourier Transforms

In Sec. 11.8 we derived two real transforms. Now we want to derive a complex transform that is called the **Fourier transform**. It will be obtained from the complex Fourier integral, which will be discussed next.

Complex Form of the Fourier Integral

The (real) Fourier integral is [see (4), (5), Sec. 11.7]

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx]\, dw$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv\, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv\, dv.$$

Substituting A and B into the integral for f, we have

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(v)[\cos wv \cos wx + \sin wv \sin wx]\, dv\, dw.$$