Chapter 1 Ordinary Differential Equation

1.1Basic Concept and ideas

※ Ordinary differential equation:

An equation that contains one or several derivatives of an unknown function, say y(x), **which depends only on one variable**.

e.g. $$\begin{cases} y'(x) = \cos(x), \\ y''(x) + 4y = 0, \\ x^2 y''' y' + 2e^x y'' = (x^2+2) y^2 \end{cases}$$

ordinary differential equation	$\longleftrightarrow$	partial differential equation
unknown function $y(x)$		unknown function $y(u,v,\ldots)$
depend only on x		depend on u,v, or more

※ Order:

the highest derivative that appears in the equation

e.g. first order equation $$F(x,y,y') = 0 \quad or \quad y' = f(x,y) \tag{1}$$

※ Solution:

A function $y = h(x)$ satisfies (1) for all x in the defined interval, if we replace y by $h(x)$ and y' by $h'(x)$.

Implicit solution: a solution of diff. Equation in form $H(x,y)=0$

Explicit solution: a solution of diff. Equation in form $y = h(x)$

General solution: solutions can be expressed by a single formula involving some arbitrary constant c (the number of c is the same as the order of the equation)

Particular solution: the solution that a specific c is given in general solution

e.g. $y' = \cos(x)$ general solution: $y = \sin(x) + c$

particular solution: $$\begin{cases} y = \sin(x) \\ y = \sin(x) - 2 \\ y = \sin(x) + 0.75 \end{cases}$$

Singular solution: an additional solution that can not be obtained from the general solution

$$y'^2 - xy' + y = 0 --- Equation$$

$$y = cx - c^2 ---- \text{Geneal Sol.}$$

$$y = \frac{x^2}{4} ------- \text{Singular Sol}$$

Trivial solution: a solution that is identically zero, i.e. $y \equiv 0$

natural law, such as conservation of mass, energy, Newton's second law

※ Application:

Physical problem $\longrightarrow$ mathematical formulation (differential equation) $\longrightarrow$ Solve the diff. equation $\longrightarrow$ determining the particular solution $\longrightarrow$ Check the solution $\longrightarrow$ physical interpretation of the result

e.g. 1. Radioactive substance decompose at a rate proportional to the amount present $\Rightarrow \frac{dy}{dt} = ky$

$$\Rightarrow y = c \cdot e^{-kt}$$

2. the curve having the slope at each point is $-y/x$ $\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$$\Rightarrow x \cdot y = c$$

※ Initial value problem: the conditions specified at one point

e.g. $y'' = f(x, y, y')$, $\quad y(x_0) = y_0$, $\quad y'(x_0) = y_1$

※ Boundary value problem: the conditions specified at different locations

e.g. $y'' = f(x, y, y')$, $\quad y(x_a) = y_a$, $\quad y(x_b) = y_b$

$y(x_a) = y_a$, $\quad y'(x_b) = y'_b$

1-2 Geometrical meaning of $y' = f(x, y)$, $y(x_0) = y_0$ Direction field

first order differential equation $y' = f(x, y)$

y' : the slope of the unknow solution curves

solve: step 1: set $f(x,y)$ = k = constant, $\longrightarrow$ draw curves

step 2: along the curve $f(x,y)$ = k draw many linear elements of slope k

step 3: sketch the approximate solution curve

Example: $y' = xy$, $y(1) = 2$

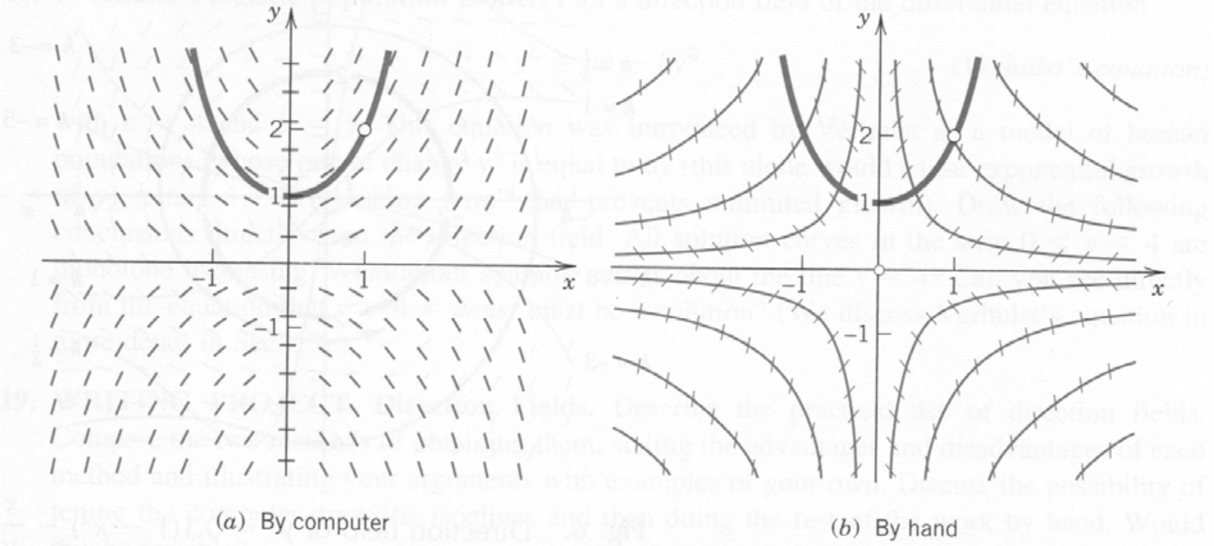


Fig. 5. Direction field of $y' = xy$

$$y = ce^{\frac{x^2}{2}}$$

1.3 Separable differential equation

※ An equation in the form

$$\left.\begin{array}{ll} g(y)dy = f(x)dx & (a) \\ \dfrac{dy}{dx} = \dfrac{f(x)}{g(y)} & (b) \\ g(y)y' = f(x) & (c) \end{array}\right\}$$

separable equation, Integrate (c) with respect to x

$$\int g(y)y'\,dx = \int f(x)dx + c \Longrightarrow \int g(y)dy = \int f(x)dx + c$$

Example 2: $y' = 1 + y^2 \Rightarrow \dfrac{dy}{1+y^2} = dx \Rightarrow \displaystyle\int \frac{dy}{1+y^2} = \int dx + c$

$$\Rightarrow \arctan y = x + c \Rightarrow \quad y = \tan(x + c)$$

Example 5:

$$y' = -2xy,\ y(0) = 1 \quad \Rightarrow \quad \frac{dy}{dx} = -2xy \quad \Rightarrow \frac{dy}{y} = -2xdx$$

$$\Rightarrow \ln|y| = -x^2 + \tilde{c} \quad \Rightarrow y = e^{-x^2+\tilde{c}} \quad \Rightarrow \quad y = ce^{-x^2}$$

$$\begin{array}{llll} e^{\tilde{c}} > 0 & when & y > 0 & \Rightarrow \quad e^{\tilde{c}} = c \\ e^{\tilde{c}} < 0 & when & y < 0 & \Rightarrow \quad e^{\tilde{c}} = -c \end{array}$$

$$\because\ y(0) = 1 \quad \Rightarrow \quad c = 1 \quad \Rightarrow y = e^{-x^2}$$

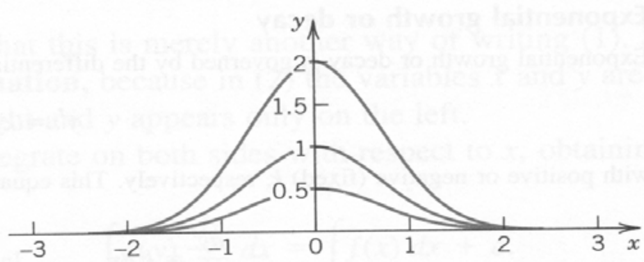


Fig. 8. Solutions of $y' = -2xy$ ("bell-shaped curves") in the upper half-plane

※ Reduction to separable form

If a differential equation (first order) is not separable, but can be expressed in form $\quad y' = g\left(\dfrac{y}{x}\right) \qquad$ (a)

then it can be made separable by let $u = \dfrac{y}{x}$ i.e. $y = u\,x$

$\Rightarrow \quad y' = xu' + u$

$$\therefore (a) \Rightarrow xu' + u = g(u) \quad \Rightarrow \frac{du}{g(u) - u} = \frac{dx}{x} \quad \leftarrow \text{separable form}$$

Example: $2xyy' - y^2 + x^2 = 0$ Divided by x^2

$$\Rightarrow 2\frac{y}{x}y' - \left(\frac{y}{x}\right)^2 + 1 = 0$$

let $u = \dfrac{y}{x} \;\Rightarrow\; 2u(u + u'x) - u^2 + 1 = 0 \;\Rightarrow\; 2xuu' + u^2 + 1 = 0$

$$\Rightarrow \quad \frac{2udu}{1+u^2} = -\frac{dx}{x} \quad \Rightarrow \quad \ln(1+u^2) = -\ln|x| + \tilde{c} \; or \ln \tilde{c}$$

$$\Rightarrow \quad u^2 + 1 = \frac{c}{x} \quad \Rightarrow x^2 + y^2 = cx \Rightarrow (x - \frac{c}{2})^2 + y^2 = \frac{c^2}{4}$$

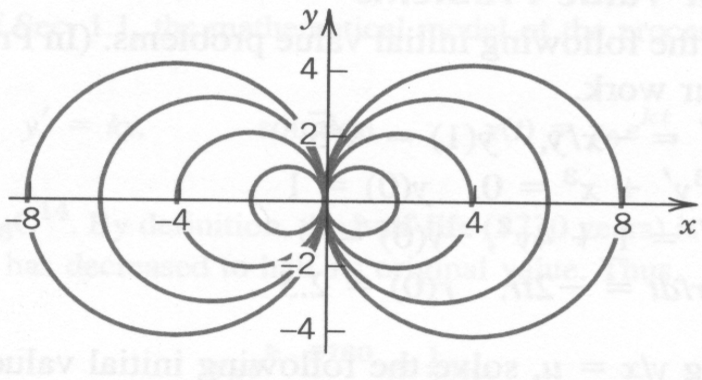


Fig. 9. General solution (family of circles) in Example 6

Modeling example:

Suppose that you turn off the heat in your home at night 2 hours before you go to bed; call this time t=0. If the temperature T at t=0 is 66^0F and at the time you go to bed(t=2) has dropped to 63^0F, what temperature can you expect in the morning, say, 8 hours later (t=10)? Suppose the outside temperature T_A is constant at 32^0F.

Newton's law of cooling: the time rate change of temperature T of a body is proportional to the difference between T and the temperature T_A of the surrounding medium.

$$\frac{dT}{dt} = k(T - T_A) = k(T - 32), \qquad T(0) = 66 \text{ and } T(2) = 63$$

solve:

$$\frac{dT}{T-32} = kdt, \quad \Rightarrow \quad \ln|T - 32| = kt + \tilde{c}$$

$$\Rightarrow T(t) = 32 + ce^{kt} \qquad (e^{\tilde{c}} = c)$$

I.C. $T(0) = 32 + c = 66, \quad \rightarrow \quad c = 34$

$$T(t) = 32 + 34e^{kt}$$

Determine k ,

for $T(2)$=63, $\quad \Rightarrow 63 = 32 + 34e^{2k}, \quad \Rightarrow \quad k = -0.046187$

Thus, $\quad T(10) = 32 + 34e^{-0.046187 \cdot 10} = 53.4$

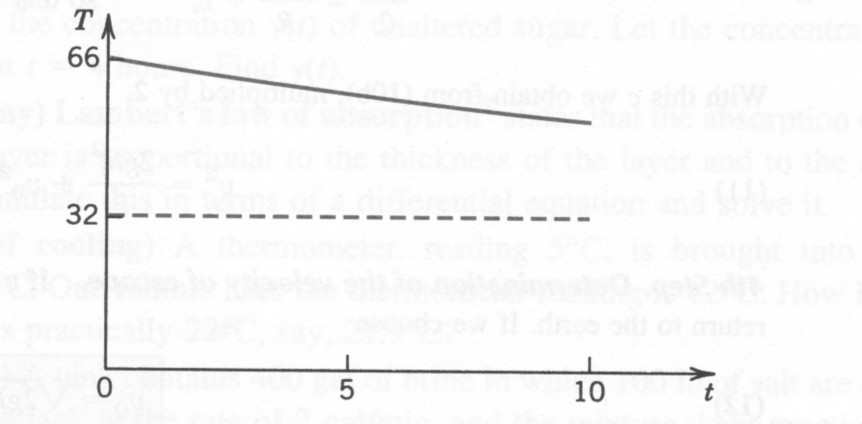


Fig. 11. Temperature in Example 3, Step 3

1.4 Exact Differential equations

From calculus if $u(x,y)$has continuous partial derivatives, then the total or exact differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

if $\quad u(x,y) = c \quad (const) \quad \Rightarrow du = 0$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \Rightarrow \text{exact}$$

A diff. equation of the form $M(x,y)dx + N(x,y)dy = 0$ is called exact differential equation if $M(x,y)dx + N(x,y)dy$ is the total differential or exact of some function $u(x,y)$.

i.e. $M(x,y) = \dfrac{\partial u}{\partial x} \quad \& \quad N(x,y) = \dfrac{\partial u}{\partial y}$

e.g. $2xydx + x^2dy \Rightarrow u(x,y) = x^2y$

If M, N defined and have continuous first partial derivative

Then $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial^2 u}{\partial x \partial y}, \quad \dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \quad \Leftarrow$ necessary and sufficient condition for

$M(x,y)dx + N(x,y)dy$ exact differential.

If $M(x,y)dx + N(x,y)dy = 0$ is exact, then $u(x,y)$ can be found by

$$u(x,y) = \int M(x,y)dx + k(y)$$

or

$$u(x,y) = \int N(x,y)dy + \ell(x)$$

To determine $\quad k(y), \quad \Rightarrow \dfrac{\partial u}{\partial y} = N(x,y) \Rightarrow \quad k(y)$

$$\ell(x), \quad \Rightarrow \frac{\partial u}{\partial x} = M(x,y) \Rightarrow \quad \ell(x)$$

Total solution is $u(x,y) = c$

this equation can also be solved by reduction to separable form

Example: $\quad (x^3 + 3xy^2)dx + (3x^2 y + y^3)dy = 0$

$$M(x,y) = x^3 + 3xy^2 \,, \quad N(x,y) = 3x^2 y + y^3$$

$$\Rightarrow \frac{\partial M}{\partial y} = 6xy, \qquad \frac{\partial N}{\partial x} = 6xy, \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \text{exact}$$

$$u(x,y) = \int M(x,y)dx + k(y) = \int (x^3 + 3xy^2)dx + k(y)$$
$$= \quad \frac{1}{4}x^4 + \frac{3}{2}x^2 y^2 + k(y)$$

To find $k(y)$ $\quad \Rightarrow \dfrac{\partial u}{\partial y} = 3x^2 y + k'(y) = N(x,y) = 3x^2 y + y^3$

$$\Rightarrow k'(y) = y^3 \quad \Rightarrow k(y) = \frac{y^4}{4} + \tilde{c}$$

The solution is $u(x,y) = \dfrac{1}{4}x^4 + \dfrac{3}{2}x^2 y^2 + \dfrac{1}{4}y^4 + \tilde{c} = c_1$

i.e. $\dfrac{1}{4}x^4 + \dfrac{3}{2}x^2 y^2 + \dfrac{1}{4}y^4 = c$

※ Integration factors

Consider a differential equation

$$P(x,y)dx + Q(x,y)dy = 0 \quad \Rightarrow \quad \text{if it is not exact} \qquad \text{(1-0)}$$

if multiple it by a function $F(x,y)$

$$F(x,y)P(x,y)dx + F(x,y)Q(x,y)dy = 0 \quad \text{become exact} \quad (1)$$

then it can be solved by that mentioned above.

F(*x*,*y*) is called " integrating factor" of Eq.(1-0)

e.g.

$$2ydx + xdy = 0 - - - - (A) \to \text{not exact}$$

$$x \cdot (A) \Rightarrow 2xydx + x^2dy = 0 - - - - (B) \to \text{exact}$$

$$\Rightarrow u = x^2 y$$

$$\Rightarrow x^2 y = c(\text{constant}) \ \text{ is a solution}$$

How to get *F*(*x*,*y*) (integrating factor)

(1) exact $\Rightarrow \ M = FP, \quad N = FQ$ and $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\Rightarrow \quad \frac{\partial (FP)}{\partial y} = \frac{\partial (FQ)}{\partial x} \Rightarrow \quad F_y P + FP_y = F_x Q + FQ_x \qquad (2)$$

not easy to solve

Alternately, look for *F*(*x*,*y*) depend only on one variable *x* or *y*

If $\ F = F(x) \ \Rightarrow \quad F_y = 0, \quad (2) \Rightarrow \quad FP_y = F_x Q + FQ_x$

$$\Rightarrow \quad \frac{1}{F}\frac{dF}{dx} = \underbrace{\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}_{\text{if depend only on } x} = R(x) \qquad (3)$$

Then $\qquad F(x) = e^{\int R(x)dx} = e^{\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx}$

If $\ F = F(y) \ \Rightarrow \quad F_x = 0, \quad (2) \Rightarrow \quad F_y P + FP_y = FQ_x$

$$\Rightarrow \quad \frac{1}{F}\frac{dF}{dy} = \underbrace{\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{if depend only on } y} = \tilde{R}(y) \qquad (4)$$

Then $$F(y) = e^{\int \tilde{R}(y)dy} = e^{\int \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy}$$

Example: $2\sin(y^2)dx + xy\cos(y^2)dy = 0$ (5)

$$\Rightarrow \quad \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 3y\cos(y^2) \quad \Rightarrow \quad \text{not exact}$$

in(3) $$\Rightarrow R(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xy\cos(y^2)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{3}{x}$$

integrating factor is $F(x) = e^{\int R(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = x^3$

(5) $\times x^3$ $\Rightarrow 2x^3\sin(y^2)dx + x^4y\cos(y^2)dy = 0$ exact

$$\Rightarrow \quad u(x,y) = \int 2x^3\sin(y^2)dx + k(y) = \frac{1}{2}x^4\sin(y^2) + k(y)$$

$$\Rightarrow \quad u_y(x,y) = x^4y\cos(y^2) + k'(y) = N(x,y) = x^4y\cos(y^2)$$

Hence $k'(y) = 0$ and k = const.

The general solution is $u(x,y) = const.$ i.e.

$$\Rightarrow \quad u(x,y) = \frac{1}{2}x^4\sin(y^2) = c$$

1.5 Linear Differential Equations

An equation of the form is said to be first order linear differential equation

$$y' + p(x)y = r(x) \qquad (1)$$

if $r(x) = 0 \quad \Rightarrow \quad$ (1) said to be homogeneous equation

$$(1) \rightarrow \quad y' + p(x)y = 0$$

$$\rightarrow \frac{dy}{y} = -p(x)dx \rightarrow \ln y = -\int p(x)dx + c$$

$$\rightarrow y = ce^{-\int p(x)dx} \text{ ------General Sol. of (1)}$$

If $r(x) \neq 0 \quad \Rightarrow \quad$ (1) said to be non-homogeneous equation

rearrange (1) $\quad \Rightarrow \quad [p(x)y - r(x)]dx + dy = 0 \quad$ not exact

$$\Rightarrow P(x, y)dx + Q(x, y)dy = 0,$$

$$\Rightarrow P(x, y) = p(x)y - r(x) \quad \& \quad Q(x, y) = 1$$

eq.(3) in previous section

$$\Rightarrow R(x) = \frac{1}{Q}\left(\frac{\partial P(x, y)}{\partial y} - \frac{\partial Q(x, y)}{\partial x}\right) = p(x)$$

Thus the integrating factor $\quad F(x) = e^{\int p(x)dx}$

$$(1) \times F(x) \quad \Rightarrow \quad e^{\int p(x)dx}\left(y' + p(x)y\right) = r(x)\, e^{\int p(x)dx}$$

$$\Rightarrow \quad \frac{d}{dx}\left[y\, e^{\int p(x)dx}\right] = r(x)\, e^{\int p(x)dx} \quad \text{integrate with respect to } x$$

$$\Rightarrow y\, e^{\int p(x)dx} = \int r(x)\, e^{\int p(x)dx} dx + c$$

$$y = e^{-\int p(x)dx} \int r(x)\, e^{\int p(x)dx} dx + c\, e^{-\int p(x)dx}$$

Example: $\quad y' - y = e^{2x} \quad \Rightarrow p(x) = -1, \quad r(x) = e^{2x}$

$$F(x) = e^{\int p(x)dx} = e^{-\int dx} = e^{-x}$$

$$\therefore\ e^{-x}y' - e^{-x}y = e^{x} \quad \Rightarrow \frac{d}{dx}\left[ye^{-x}\right] = e^{x}$$

$$\therefore\ ye^{-x} = e^{x} + c \quad \Rightarrow \quad y = e^{2x} + c\, e^{x}$$

Example: $\quad y' + y\tan x = \sin(2x), \quad y(0) = 1$

$$p(x) = \tan x, \quad r(x) = \sin(2x)$$

$$F(x) = e^{\int p(x)dx} = e^{\int \tan(x)dx} = e^{\ln|\sec(x)|} = \sec(x)$$

$$\boxed{\begin{aligned} &\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx, \quad \text{let } u = \cos(x), \quad du = -\sin(x)dx \\ &= -\int \frac{du}{u} = -\ln|\cos(x)| = \ln\left[\frac{1}{\cos(x)}\right] = \ln|\sec(x)| \end{aligned}}$$

$$\Rightarrow \quad \sec x\ y' + y\tan x \sec x = \sin(2x)\sec x$$

$$\Rightarrow \quad \sec x\ y' + y\tan x \sec x = 2\sin x$$

$$\Rightarrow \quad \frac{d}{dx}\left[\sec x\ y\right] = 2\sin x \quad \Rightarrow y\sec x = \int 2\sin x dx + c$$

$$\Rightarrow \quad y = \cos x \int 2\sin x dx + c\cos x = -2\cos^2 x + c\cos x$$

$$y(0) = 1 \quad \Rightarrow -2 + c = 1 \quad \Rightarrow c = 3$$

$$\therefore y = 3\cos x - 2\cos^2 x$$

※ Bernoulli Equation:

Equation form: $\quad y' + p(x)y = g(x)y^{a} \qquad$ (A)

a: any real number and $a \neq 0 \quad \& \quad a \neq 1 \quad \Rightarrow$ nonlinear equation

How to reduce it to a linear equation ?

Set $\quad u(x) = \left[y(x)\right]^{1-a}, \quad \Rightarrow u' = (1-a)y^{-a}y'$

$$\Rightarrow \quad (A)/y^a \quad \Rightarrow \quad y^{-a}y' + p(x)y^{1-a} = g(x)$$

$$\Rightarrow \quad \frac{u(x)'}{1-a} + p(x)u(x) = g(x)$$

$$\Rightarrow \quad u(x)' + (1-a)p(x)u(x) = (1-a)g(x) \quad \rightarrow \text{linear equation}$$

example: $y' - Ay = -By^2$, i.e. $a = 2$

let $u(x) = y^{-1}$, $\Rightarrow u' = -y^{-2}y'$

original equation$/y^2 \Rightarrow y^{-2}y' - Ay^{-1} = -B \quad \Rightarrow -u' - Au = -B$

$$\Rightarrow u' + Au = B \qquad \therefore F(x) = e^{\int A dx} = e^{Ax}$$

$$\Rightarrow e^{Ax}u' + e^{Ax}Au = e^{Ax}B$$

$$\Rightarrow \frac{d}{dx}(u\,e^{Ax}) = B\,e^{Ax} \Rightarrow u\,e^{Ax} = B\int e^{Ax}dx + c = \frac{B}{A}e^{Ax} + c$$

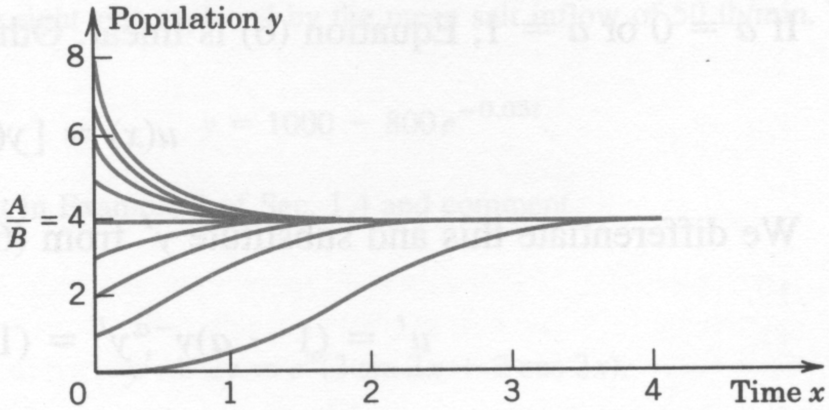


Fig. 18. Logistic population model. Curves (9) in Example 5 with $A/B = 4$

1.6 Modeling: Electric Circuits

※ voltage drop E_R across a resistor is proportional to the instantaneous current

$$\Rightarrow \quad E_R = R\,I$$

※ voltage drop E_L across a inductor is proportional to the instantaneous time rate of change of the current

$$\Rightarrow \quad E_L = L\frac{dI}{dt}$$

※ voltage drop E_C across a capacitor is proportional to the instantaneous electric charge Q on the capacitor

$$\Rightarrow \quad E_C = \frac{Q}{C}$$

I: current, Q: charge, C: capacitance, L: inductance, R: resistance and $I = \dfrac{dQ}{dt}$

※ Kirchhoff's voltage law:
The algebraic sum of all the instantaneous voltage drops around any closed loop is zero, or the voltage impressed on a closed loop is equal to the sum of the voltage drops in the rest of the loop.

Example: RL circuit

$$E(t) = R\,I + L\frac{dI}{dt}$$

$$\Rightarrow \frac{dI}{dt} + \frac{R}{L}I = \frac{E(t)}{L}$$

$$F(t) = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

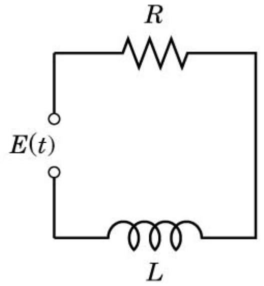


Fig. 21. *RL*-circuit

$$\Rightarrow \frac{d}{dt}\left[I\,e^{\frac{R}{L}t}\right] = \frac{E(t)}{L}e^{\frac{R}{L}t} \quad \Rightarrow I\,e^{\frac{R}{L}t} = \int \frac{E(t)}{L}e^{\frac{R}{L}t}dt + c$$

$$\Rightarrow I(t) = e^{-\frac{R}{L}t} \int \frac{E(t)}{L} e^{\frac{R}{L}t} dt + ce^{-\frac{R}{L}t}$$

If $\quad E(t) = E_0 \quad$ (D C) $\qquad \Rightarrow I(t) = \dfrac{E_0}{R} + ce^{-\frac{R}{L}t}$

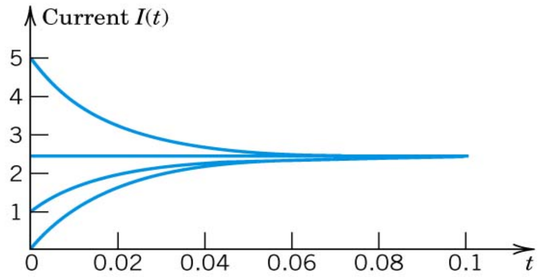


Fig. 22. Current in an *RL*-circuit due to a constant electromotive force

If $\quad I(0) = 0 \quad \Rightarrow \quad c = -\dfrac{E_0}{R}$

$$\therefore I(t) = \frac{E_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\frac{R}{L} = \alpha = \frac{1}{\tau_L},$$

τ_L : inductive time constant

If $\quad \Rightarrow E(t) = E_0 \sin \omega t \quad$ (A C)

$$\Rightarrow I(t) = e^{-\frac{R}{L}t} \frac{E_0}{L} \int e^{\frac{R}{L}t} \sin \omega t \, d\, t + ce^{-\frac{R}{L}t}$$

$$= ce^{-\frac{R}{L}t} + \frac{E_0}{R^2 + \omega^2 L^2}(R \sin \omega t - \omega L \cos \omega t)$$

$$= ce^{-\frac{R}{L}t} + \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \delta)$$

$\boxed{\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha}$

$$\cos\delta = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}, \quad \sin\delta = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\delta = \tan^{-1}\left(\frac{\omega L}{R}\right) \quad \text{phase angle with respect to } \sin\omega t$$

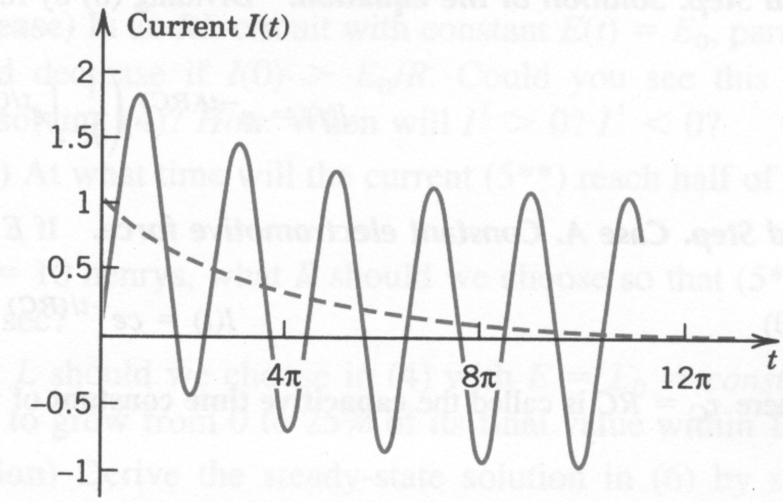


Fig. 23. Current (6) in an *RL*-circuit due to a sinusoidal electromotive force. (For simplicity, $I(t) = \exp(-0.1t) + \sin(t - \pi/4)$.) Dashed: the exponential term

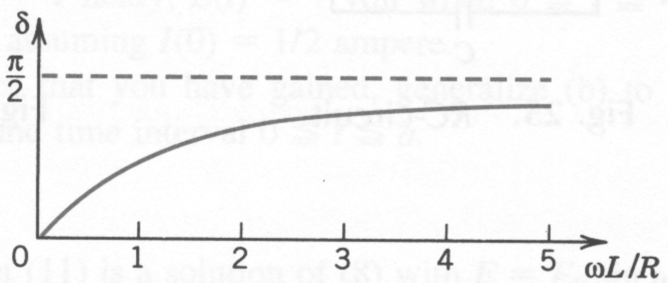


Fig. 24. Phase angle δ in (6) as a function of $\omega L/R$

1.7 Orthogonal Trajectories of Curves

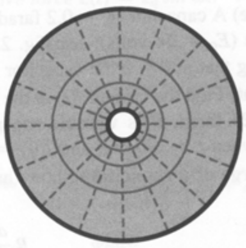
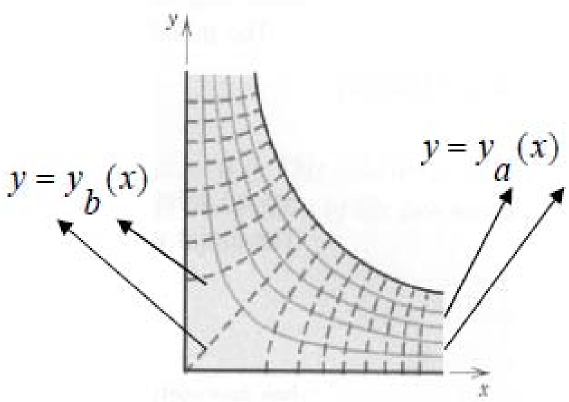


Fig. 27. Equipotential lines and lines of electric force (dashed) between two concentric cylinders



Orthongal : $y_a' \cdot y_b' = -1 \quad \Rightarrow \quad y_b' = -\dfrac{1}{y_a'}$

If a family of curves y are given, then the slope, say, $y' = f(x, y)$ is known

$\Longrightarrow$ The orthogonal trajectories can be obtained by

$$y'_{orthog} = \frac{-1}{y'_{initial}} = \frac{-1}{f(x,y)} \quad \Rightarrow \quad \text{solve } y_{orthog}$$

Example: Find the orthogonal trajectories of the parabolas $y = c\,x^2$

First find the slope of the parabolas $\Rightarrow \quad y' = 2c\,x$

And $y = c\,x^2 \Rightarrow \quad c = \dfrac{y}{x^2} \quad \Rightarrow \quad y' = 2x\dfrac{y}{x^2} = \dfrac{2y}{x} = f(x,y)$

The slope of the orthogonal trajectories is

$$y'_{orthog} == \frac{-1}{f(x,y)} = -\frac{x}{2y} \quad \Rightarrow \quad y' = -\frac{x}{2y}$$

$$\Rightarrow \quad 2y\,dy = -x\,dx \quad \Rightarrow \quad y^2 = -\frac{x^2}{2} + \tilde{C} \quad \Rightarrow y^2 + \frac{x^2}{2} = \tilde{C}$$

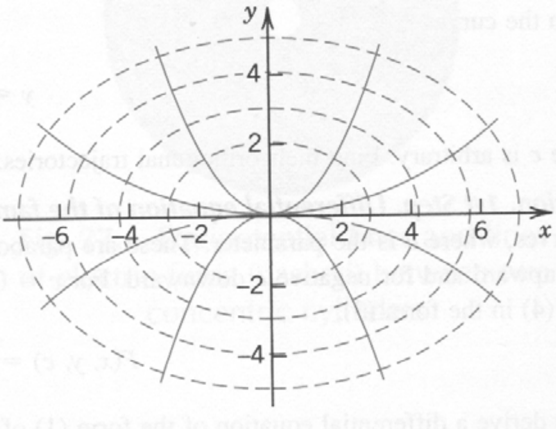


Fig. 28. Curves and orthogonal trajectories in Example 1

1.8 Existence and uniqueness solutions

Consider initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

Existence theorem: If $f(x,y)$ is continuous at all points (x,y) in some rectangle $R: |x - x_0| < a, \quad |y - y_0 < b|$ and bounded in R, say $|f(x,y)| \le K$ for all (x,y) in R. Then the initial value problem has at least one solution $y(x)$. This solution is defined at least for all x in the interval $|x - x_0| \le \alpha$ where α is the smaller of the two numbers a and b/K.

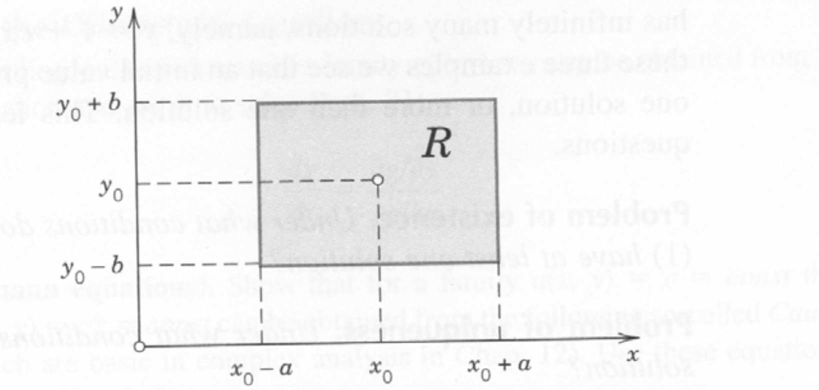


Fig. 31. Rectangle R in the existence and uniqueness theorems

Unique theorem: If $f(x,y)$ and $\partial f / \partial y$ are continuous for all (x,y) in that rectangle and bounded, say, $|f(x,y)| \le K, \quad \left|\frac{\partial f}{\partial y}\right| \le M$, for all (x,y) in R, then the initial value problem has at most one solution. Hence by existence theorem, it has precisely one solution. This solution is defined at least for all x in that interval $|x - x_0| \le \alpha$.

※ understanding these theorem

$\because |f(x,y)| \le K \Rightarrow |y'| \le K \Rightarrow$ the slope of the solution is between $-K$ and K. $\Rightarrow$ the solution curve pass through (x_0, y_0) must lie in region shown as follows.

Case (a) $b/K \geq a$ case (b) $b/K < a$

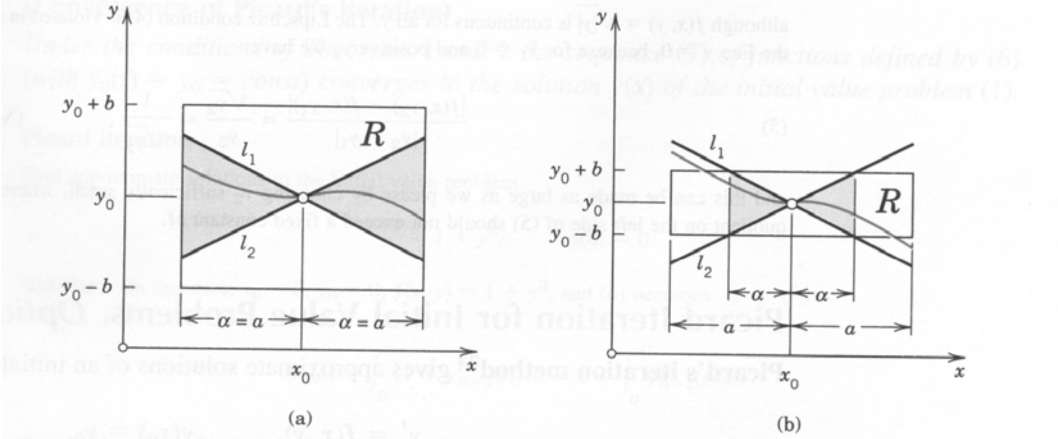


Fig. 32. The condition (2) of the existence theorem.
(a) First case. (b) Second case

※ Picard's iteration method

Consider $y' = f(x,y), \quad y(x_0) = y_0$

Integration with respect to x $\Rightarrow$ $y(x) = c + \int_{x_0}^{x} f(x,y)dx$

$\because y(x_0) = y_0 \quad \Rightarrow \; y_0 = c + \int_{x_0}^{x_0} f(x,y)dx \Rightarrow y_0 = c$

$\therefore \quad y(x) = y_0 + \int_{x_0}^{x} f(x,y)dx$

Now solve this equation by successive approximations

Suppose $y_0(x)$ is a known function that approximate to the solution, then $f(x, y_0(x))$ is a known function with $y(x)$ replaced by $y_0(x)$

$$\Rightarrow \quad y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

repeating the procedure, get

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx \qquad \text{(A)}$$

(A) is known as Picard's method of iteration.

Example: $y' = 1 + y^2, \quad y(0) = 0$

$$\therefore \quad x_0 = 0, \quad y_0 = 0, \quad f(x,y) = 1 + y^2$$

$$\Rightarrow \quad y_1(x) = 0 + \int_0^x (1 + y_0^2)dx = x$$

$$\Rightarrow \quad y_2(x) = 0 + \int_0^x (1 + x^2)dx = x + \frac{x^3}{3}$$

$$\Rightarrow \quad y_3(x) = 0 + \int_0^x \left[1 + (x + \frac{x^3}{3})^2\right]dx = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{1}{63}x^7$$

Exact: $y(x) = \tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \dfrac{17}{315}x^7 + ...$

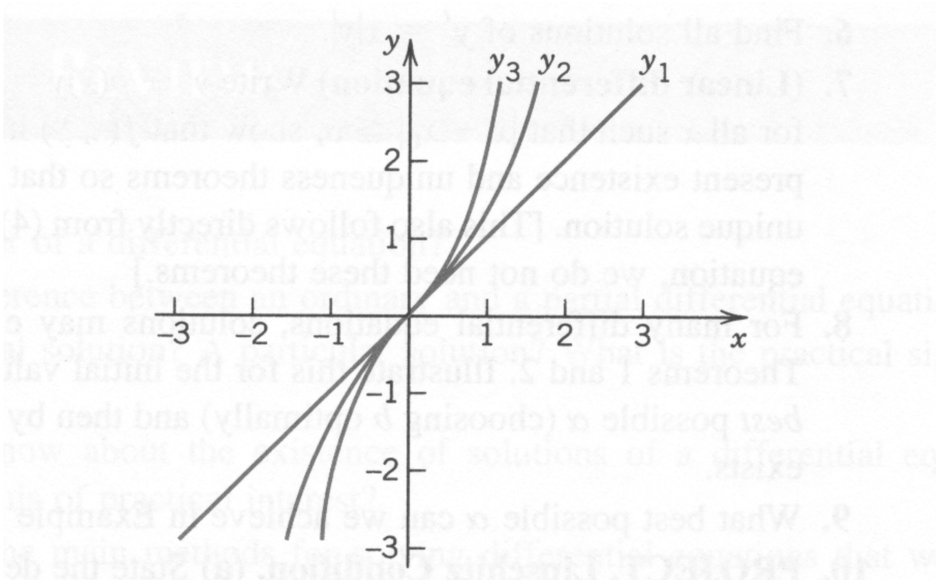


Fig. 33. Approximate solutions in Example 3