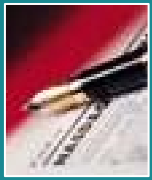
  

Chapter 6
Time and Frequency Characterization of Signals and Systems

6.0 Introduction

The frequency-domain characterization of an LTI system in terms of its frequency response represents an alternative to the time-domain characterization through convolution. In analyzing LTI systems, it is often particularly convenient to utilize the Frequency domain because differential and difference equations and convolution operations in the time domain become algebraic operations in the frequency domain.

6.0 Introduction

一個LTI系統以頻率響應表示的頻域特性，可說是它的時域特性透過迴旋運算所得的另一種表示法。在分析LTI系統時，利用頻域來處理是相當方便的。

在頻率選擇濾波器的脈衝響應中，常呈現出我們所不期望的明顯的振盪。因此，為了符合脈衝響應的需求，我們往往需要在某種程度上犧牲一些頻率選擇性。

在系統的分析和設計上，將時域和頻域特性的關係建立起來，且能尋求折衷方案是很重要的。

6.1 The Magnitude-Phase Representation of The Fourier Transform

The magnitude-phase representation of the continuous-time Fourier transform $X(j\omega)$ is

$$X(j\omega) = |X(j\omega)|e^{j\measuredangle X(j\omega)}. \tag{6.1}$$

連續時間傅立葉轉換的大小—相位表示法

Similarly the magnitude-phase representation of the discrete-time Fourier transform $X(e^{j\omega})$ is

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\measuredangle X(e^{j\omega})}. \tag{6.2}$$

離散時間傅立葉轉換的大小—相位表示法

6.1 The Magnitude-Phase Representation of The Fourier Transform

we can think of $X(j\omega)$ as providing us with a decomposition of the signal $x(t)$ into a “sum” of complex exponentials at different frequencies.

$X(j\omega)$告訴我們訊號$x(t)$可分解成不同頻率的複指數的和。

6.1 The Magnitude-Phase Representation of The Fourier Transform

The magnitude $|X(j\omega)|$ describes the basic frequency content of a signal—i.e., $|X(j\omega)|$ provides us with the information about the relative magnitudes of the complex exponentials that make up *x(t)*.

大小 $|X(j\omega)|$ 表示一訊號的頻率分量，即組成*x(t)*的各種頻率的複指數的相對大小。

6.1 The Magnitude-Phase Representation of The Fourier Transform

The phase angle $\measuredangle X(j\omega)$, on the other hand, does not affect the amplitudes of the individual frequency components, but instead provides us with information concerning the relative phases of these exponentials.

相角 $\measuredangle X(j\omega)$ 對於個別的頻率分量的大小並無影響，但它提供了不同頻率分量的相對相角的訊息。

既使各分量的大小函數不變，但因相角函數的不同，可使訊號的外觀大不同。

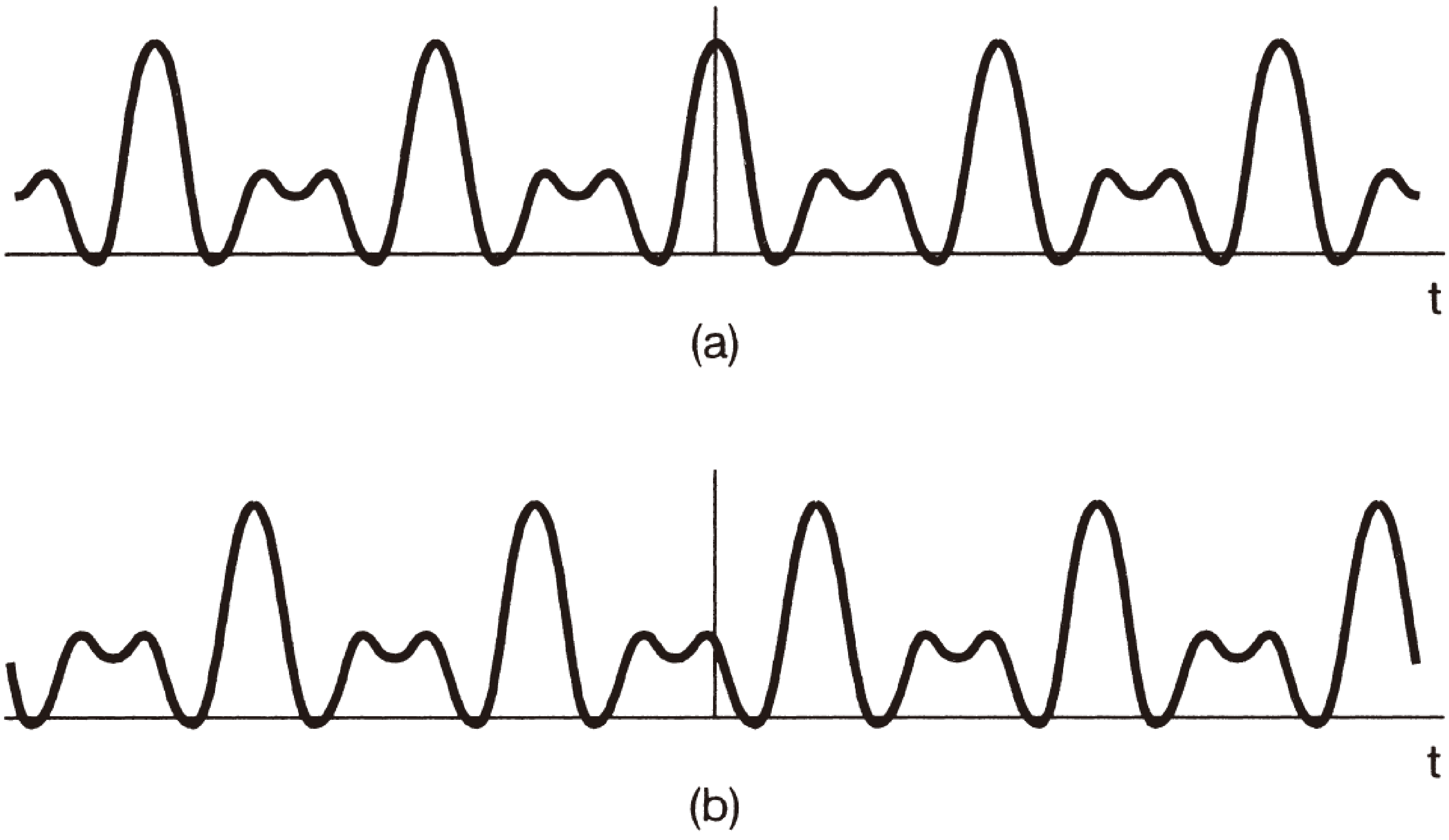
6.1 The Magnitude-Phase Representation of The Fourier Transform

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3). \quad (6.3)$$

In general, changes in the phase function of $X(j\omega)$ lead to changes in the time-domain characteristics of the signal $x(t)$. In some instances phase distortion may be important, whereas in others it is not. For example, a well-known property of the auditory system is a relative insensitivity to phase.

一般而言，$X(j\omega)$相角函數的改變將使訊號$x(t)$的時域特性也隨著改變。有些案例中，相角失真是很重要的，但有些不然。如聽覺系統，於相角是不靈敏的。

6.1 The Magnitude-Phase Representation of The Fourier Transform



6.1 The Magnitude-Phase Representation of The Fourier Transform

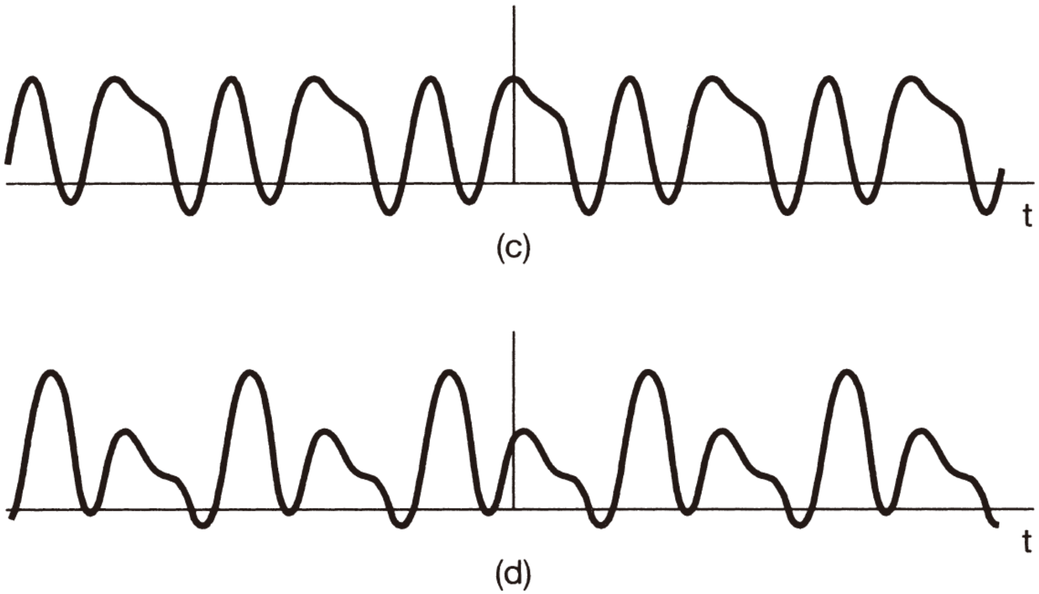


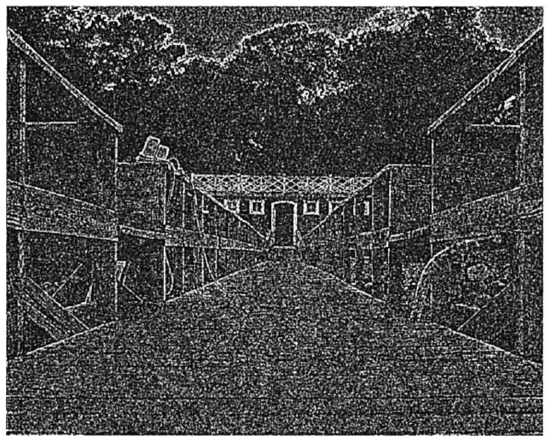
Figure 6.1 The signal $x(t)$ given in eq. (6.3) for several different choices of the phase angles ϕ_1, ϕ_2, and ϕ_3: (a) $\phi_1 = \phi_2 = \phi_3 = 0$; (b) $\phi_1 = 4$ rad, $\phi_2 = 8$ rad, $\phi_3 = 12$ rad; (c) $\phi_1 = 6$ rad, $\phi_2 = -2.7$ rad, $\phi_3 = 0.93$ rad; (d) $\phi_1 = 1.2$ rad, $\phi_2 = 4.1$ rad, $\phi_3 = -7.02$ rad.

6.1 The Magnitude-Phase Representation of The Fourier Transform

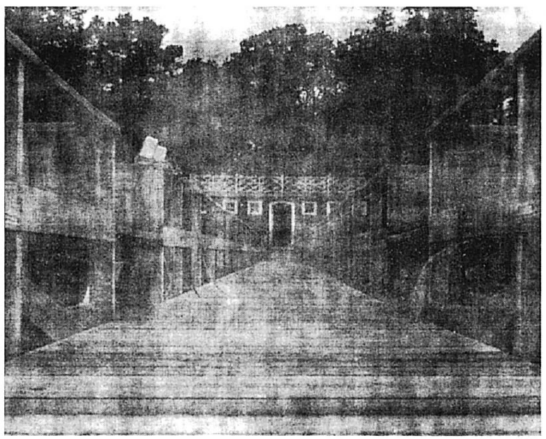
The corresponding effect in the frequency domain is to replace the Fourier transform phase by its negative:

$$F\{x(-t)\} = X(-j\omega) = |X(j\omega)| e^{-j\angle X(j\omega)}.$$

6.1 The Magnitude-Phase Representation of The Fourier Transform



(e)



(f)

(g)

Figure 6.2 (a) The image shown in Figure 1.4; (b) magnitude of the two-dimensional Fourier transform of (a); (c) phase of the Fourier transform of (a); (d) picture whose Fourier transform has magnitude as in (b) and phase equal to zero; (e) picture whose Fourier transform has magnitude equal to 1 and phase as in (c); (f) picture whose Fourier transform has phase as in (c) and magnitude equal to that of the transform of the picture shown in (g).

6.2 The Magnitude-Phase Representation of The Frequency Response of LTI Systems

The transform $Y(j\omega)$ of the output of an LTI system is related to the transform $X(j\omega)$ of the input to the system by the equation

$$Y(j\omega) = H(j\omega)X(j\omega),$$

連續時間LTI系統的輸出輸入傅立葉轉換和系統頻率響應的關係

6.2 The Magnitude-Phase Representation of The Frequency Response of LTI Systems

The Fourier transforms of the input $X(e^{j\omega})$ and output $Y(e^{j\omega})$ of an LTI system with frequency response $H(e^{j\omega})$ are related by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}). \qquad (6.4)$$

離散時間LTI系統的輸出輸入傅立葉轉換和系統頻率響應的關係

故—LTI 系統對於輸入的效應，在於改變訊號的各頻率分量的複數值振幅。

6.2 The Magnitude-Phase Representation of The Frequency Response of LTI Systems

The nature of the effect in more detail. Specifically, in continuous time,

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)| \tag{6.5}$$

輸入輸出在各頻率的振幅關係

and

$$\sphericalangle Y(j\omega) = \sphericalangle H(j\omega) + \sphericalangle X(j\omega), \tag{6.6}$$

輸入輸出在各頻率的相角關係

6.2 The Magnitude-Phase Representation of The Frequency Response of LTI Systems

For this reason, $|H(j\omega)|$ (or $H(e^{j\omega})$) is commonly referred to as the *gain* of the system.

故 $|H(j\omega)|$或 $H(e^{j\omega})$)常稱為系統的「增益」。

and $\measuredangle H(j\omega)$ is typically referred to as the *phase shift* of the system.

$\measuredangle H(j\omega)$稱為系統的「相位移」。

(6.5)及(6.6)式常稱為大小和相角的「失真」。

6.2.1 Linear and Nonlinear Phase

Consider the continuous-time LTI system with frequency response

若頻率響應 $$H(j\omega) = e^{-j\omega t_0}, \qquad (6.7)$$

so that the system has unit gain and linear phase—i.e.,

$$|H(j\omega)| = 1, \qquad \measuredangle H(j\omega) = -\omega t_0. \qquad (6.8)$$

則系統具有單位增益及線性相位移。

6.2.1 Linear and Nonlinear Phase

The system with this frequency response characteristic produces an output that is simply a time shift of the input—i.e.,

$$y)(t) = x(t - t_0). \qquad (6.9)$$

連續時間的時間延遲為線性相位移的一例。

6.2.1 Linear and Nonlinear Phase

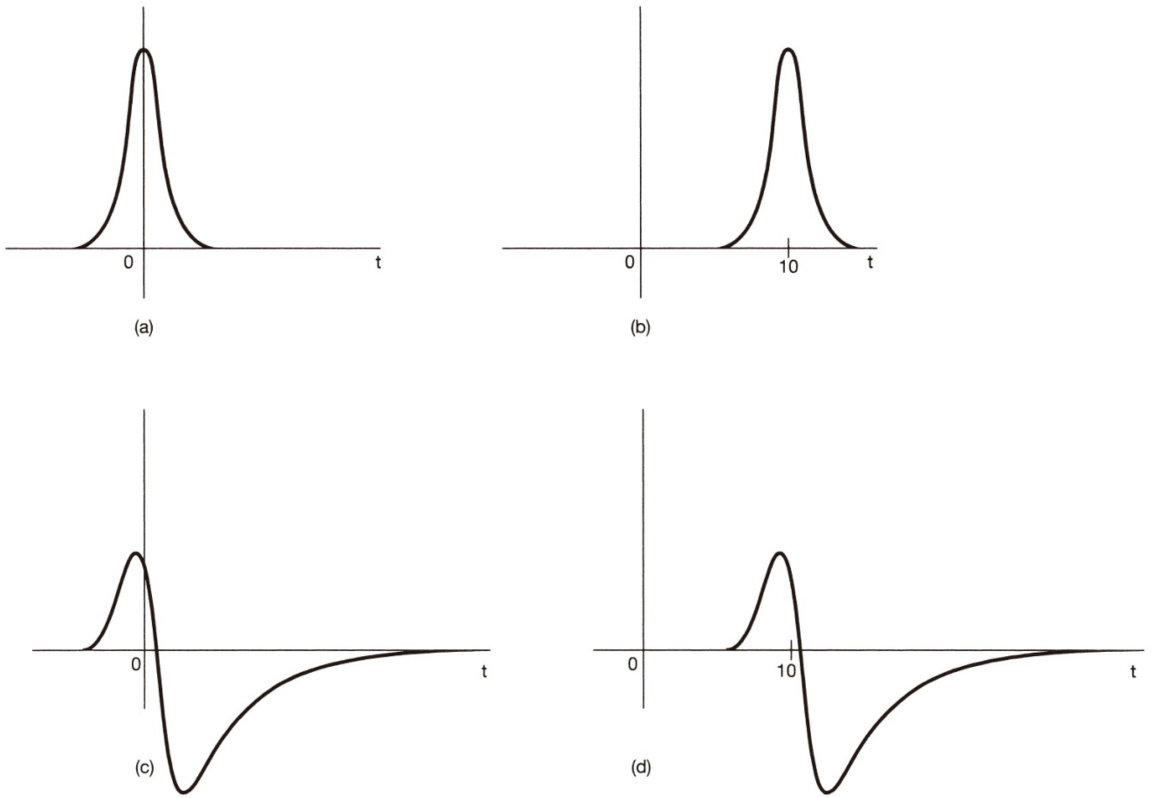


Figure 6.3 (a) Continuous-time signal that is applied as the input to several systems for which the frequency response has unity magnitude; (b) response for a system with linear phase; (c) response for a system with nonlinear phase; and (d) response for a system with phase equal to the nonlinear phase of the system in part (c) plus a linear phase term.

6.2.1 Linear and Nonlinear Phase

display the output when the signal is applied to a system with unity gain and nonlinear phase function—i.e.,

$$H_2(j\omega) = e^{j\angle H_2(j\omega)}, \tag{6.10}$$

In this case, the corresponding frequency response has a phase shift that is obtained by adding a linear phase term to $\angle H_2(j\omega)$ —i.e.,

$$H_3(j\omega) = H_2(j\omega)e^{-j\omega t_0}. \tag{6.11}$$

6.2.2 Group Delay

By taking the band to be very small, we can accurately approximate the phase of this system in the band with the linear approximation

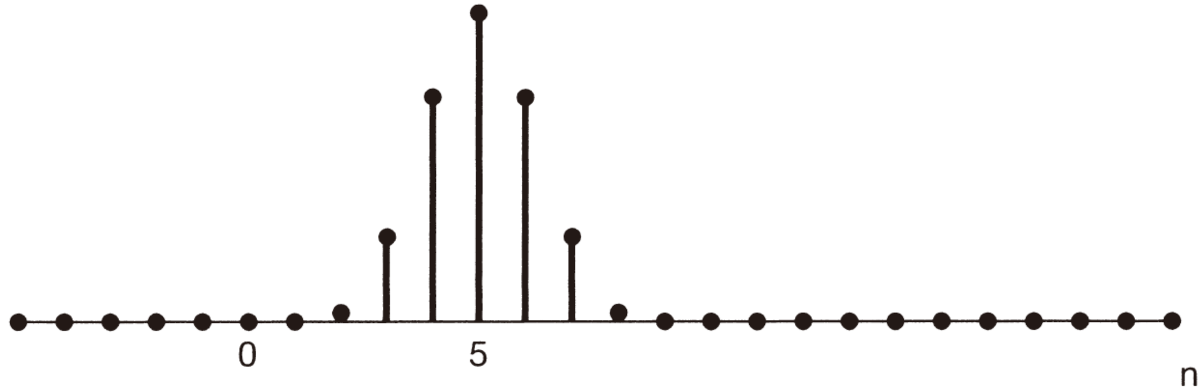
$$\measuredangle H(j\omega) \approx -\phi - \omega\alpha, \tag{6.12}$$

so that

$$Y(j\omega) \approx X(j\omega)\left|H(j\omega)\right| e^{-j\phi} e^{-j\omega\alpha}. \tag{6.13}$$

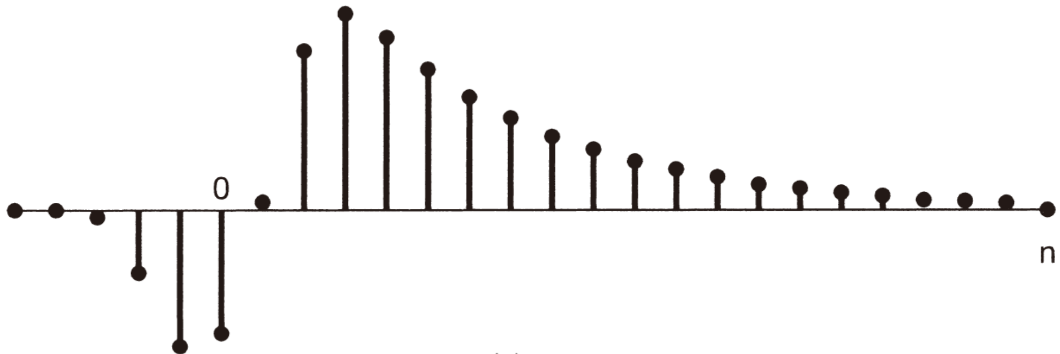
6.2.2 Group Delay

(a)

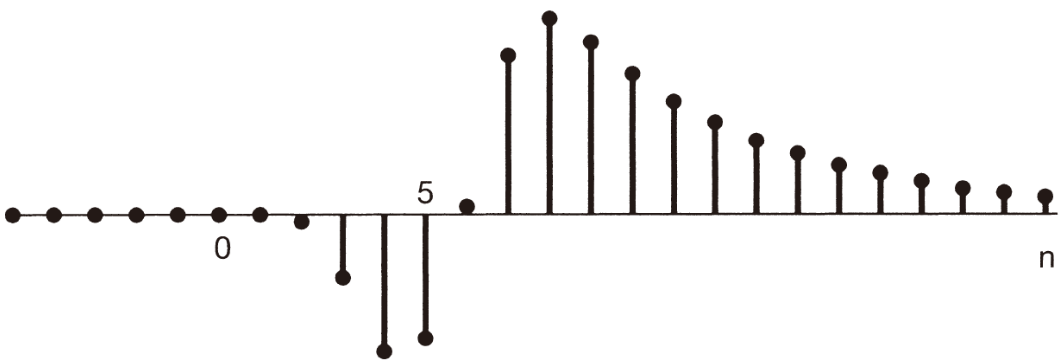


(b)

6.2.2 Group Delay



(c)



(d)

Figure 6.4 (a) Discrete-time signal that is applied as input to several systems for which the frequency response has unity magnitude; (b) response for a system with linear phase with slope of -5; (c) response for a system with nonlinear phase; and (d) response for a system whose phase characteristic is that of part (c) plus a linear phase term with integer slope.

6.2.2 Group Delay

The group delay at each frequency equals the negative of the slope of the phase at that frequency; i.e., the group delay is defined as

$$\tau(\omega) = -\frac{d}{d\omega}\left\{\angle H(j\omega)\right\}. \qquad (6.14)$$

6.2.3 Log-Magnitude and Phase Plots

In graphically displaying continuous-time or discrete-time Fourier transforms and system frequency responses in polar form, it is often convenient to use a logarithmic scale for the magnitude of the Fourier transform.

傅立葉轉換和系統頻率響應，常用對數大小刻度的圖形表示。

6.2.3 Log-Magnitude and Phase Plots

from eqs. (6.5) and (6.6), which relate the magnitude and phase of the output of an LTI system to those of the input and frequency response. Note that the phase relationship is additive, while the magnitude relationship involves the product of $\left|H(j\omega)\right|$ and $\left|X(j\omega)\right|$. Thus, if the magnitudes of the Fourier transform are displayed on a logarithmic amplitude scale, eq. (6.5) takes the form of an additive relationship, namely,

6.2.3 Log-Magnitude and Phase Plots

$$\log\left|Y(j\omega)\right| = \log\left|H(j\omega)\right| + \log\left|X(j\omega)\right|, \qquad (6.16)$$

因(6.5)及(6.6)式關係式中，相角關係是加成性的，而大小關係涉及 $\left|H(j\omega)\right|$ 及 $\left|X(j\omega)\right|$ 的乘法，故以對數大小刻度，可將它改為加成性後可方便處理。

6.2.3 Log-Magnitude and Phase Plots

For example, on a linear scale, the detailed magnitude characteristics in the stop band of a frequency-selective filter with high attenuation are typically not evident, whereas they are on a logarithmic scale.

利用對數大小刻度比線性刻度更易呈現出頻率選擇濾波器的衰減作用。

6.2.3 Log-Magnitude and Phase Plots

If *h(t)* is real, then $|H(j\omega)|$ is an even function of ω and $\angle H(j\omega)$ is an odd function of ω. Because of this , the plots for negative ω are superfluous and can be obtained immediately from the plots for positive ω.

若*h(t)*為實值，則$|H(j\omega)|$ 為偶函數，而$\angle H(j\omega)$ 為奇函數，故只需畫出 $\omega \geq 0$ 部份的波德圖即可。

6.2.3 Log-Magnitude and Phase Plots

The use of a logarithmic frequency scale offers a number of advantages in continuous time. For example, it often allows a much wider range of frequencies to be displayed than does a linear frequency scale. In addition, on a logarithmic frequency scale, the shape of a particular response curve doesn't change if the frequency is scaled.

在連續時間中，對數頻率刻度有多項優點：

其一為可顯示更大的頻率範圍。

6.2.3 Log-Magnitude and Phase Plots

Furthermore for continuous-time LTI systems described by differential equations, an approximate sketch of the log magnitude vs. log frequency can often be easily obtained through the use of asymptotes.

其二，利用微分方程表示的LTI系統可在對數大小對對數頻率圖中簡單地使用漸近線描繪。

6.2.3 Log-Magnitude and Phase Plots

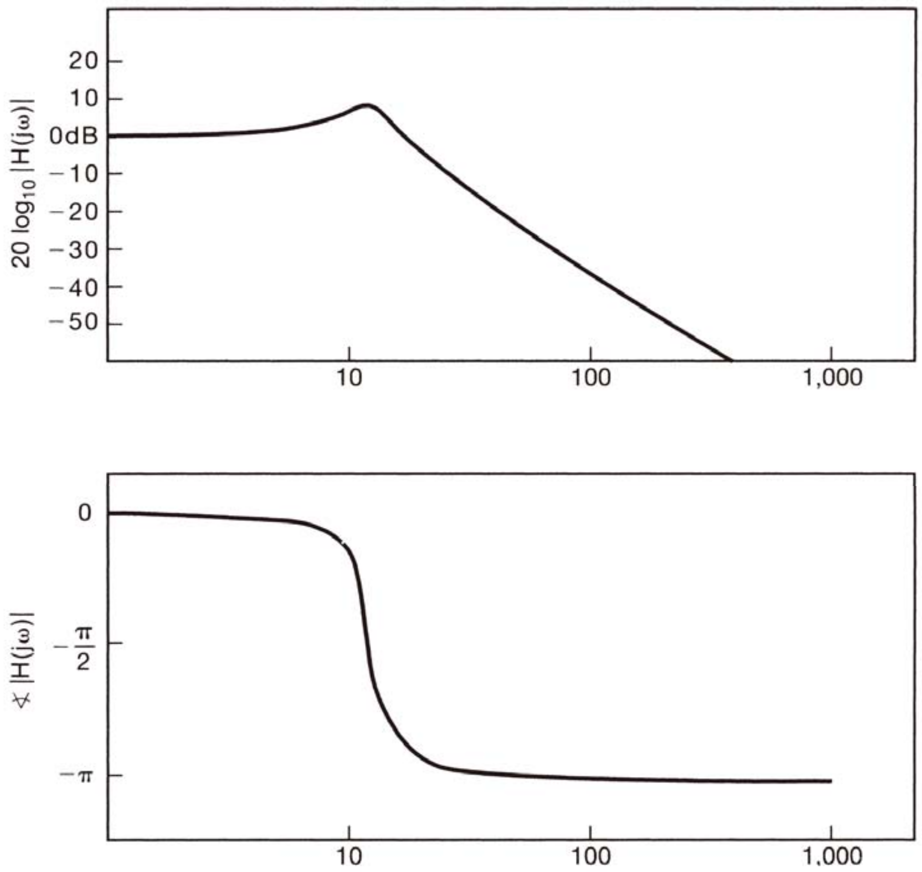


Figure 6.8 A typical Bode plot. (Note that ω is plotted using a logarithmic scale.)

6.2.3 Log-Magnitude and Phase Plots

In discrete time, the magnitudes of Fourier transforms and frequency responses are often displayed is dB for the same reasons that they are in continuous time.

離散時間中因頻率範圍有限，對數頻率刻度的波德圖不常用。

6.2.3 Log-Magnitude and Phase Plots

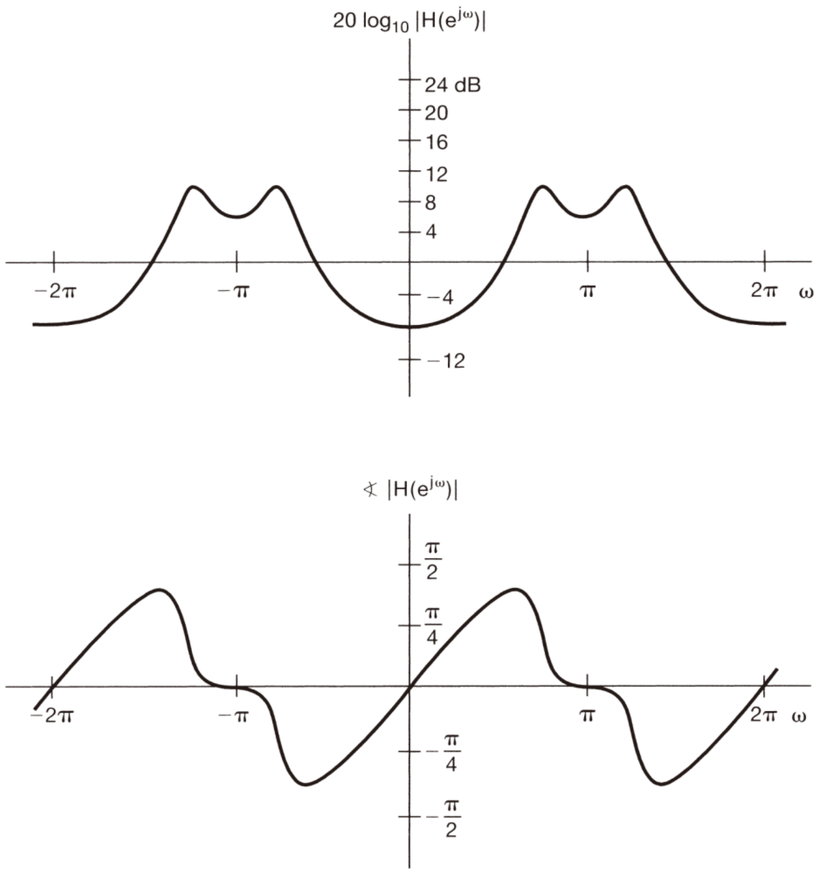


Figure 6.9 Typical graphical representations of the magnitude and phase of a discrete-time frequency response $H(e^{j\omega})$.

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

As introduced in Chapter 3, a continuous-time ideal lowpass filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & |\omega| \langle \omega_c \\ 0 & |\omega| \rangle \omega_c \end{cases}. \qquad (6.17)$$

This is illustrated in Figure 6.10(a). Similarly, a discrete-time ideal lowpass filter has a frequency response

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c \langle |\omega| \le \pi \end{cases} \qquad (6.18)$$

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

連續時間理想低通濾波器的頻率響應(配合(6.17)式)

離散時間理想低通濾波器的頻率響應(配合(6.18)式)

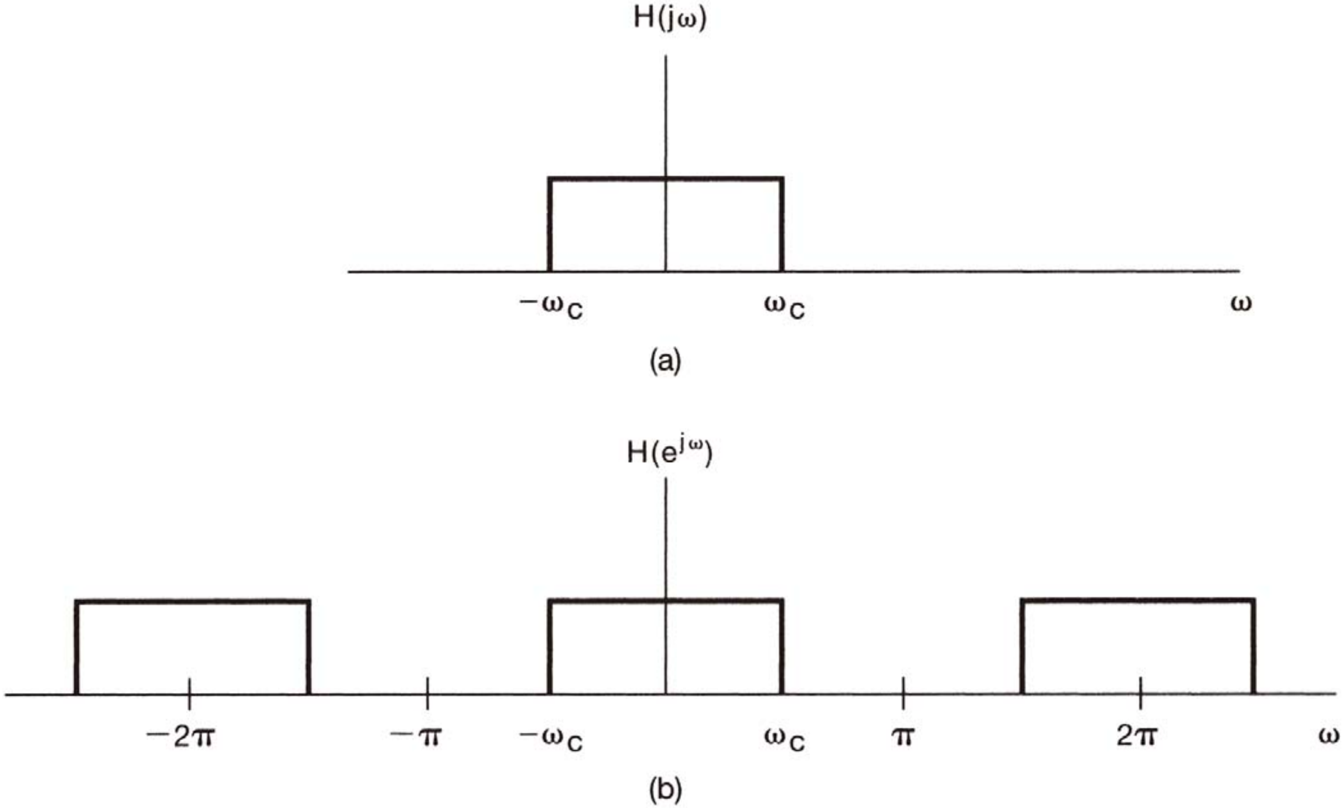


Figure 6.10 (a) The frequency response of a continuous-time ideal lowpass filter; (b) the frequency response of a discrete-time ideal lowpass filter.

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

An ideal filter with linear phase over the passband, as illustrated in Figure 6.11, introduces only a simple time shift relative to the response of the ideal lowpass filter with zero phase characteristic.

在通帶內具有線性相位移的理想濾波器，與零相位移的理想低通濾波器相較，只是增加了一個單純的時間移位。

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

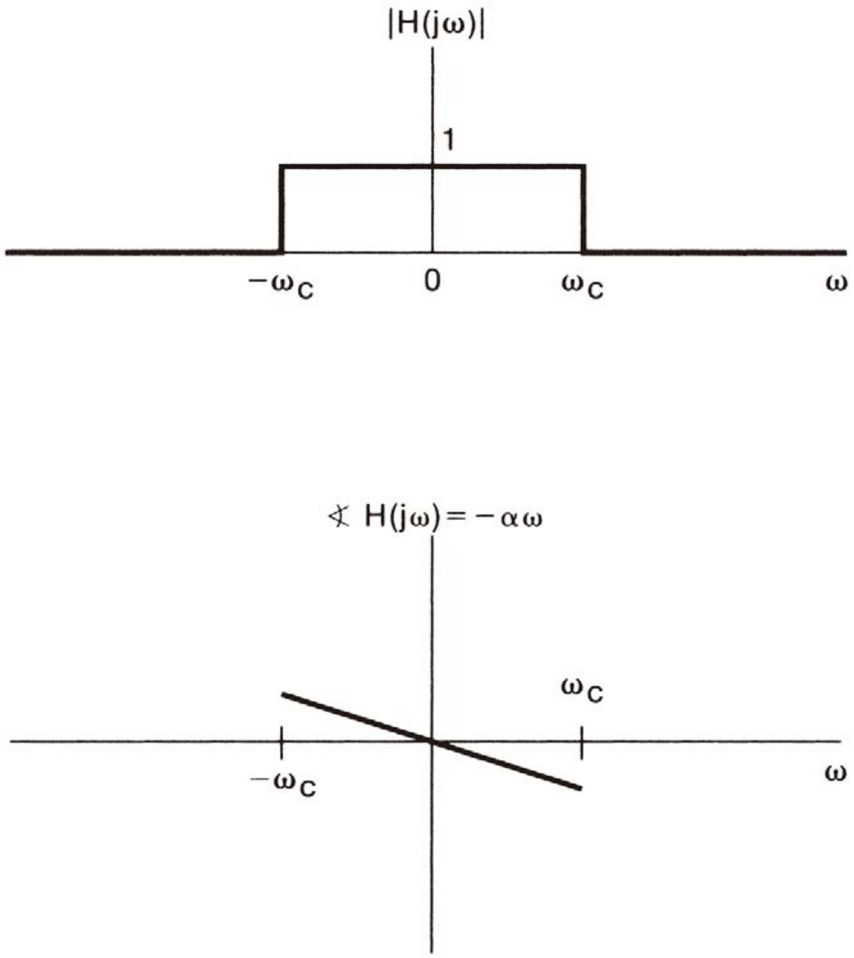


Figure 6.11 Continuous-time ideal lowpass filter with linear phase characteristic.

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

In particular, the impulse response corresponding to the filter in eq. (6.17) is

$$h(t) = \frac{\sin \omega_c t}{\pi t}, \qquad (6.19)$$

連續時間理想低通濾波器的脈衝響應

Similarly, the impulse response of the discrete-time ideal filter in eq. (6.18) is

$$h[n] = \frac{\sin \omega_c n}{\pi n}, \qquad (6.20)$$

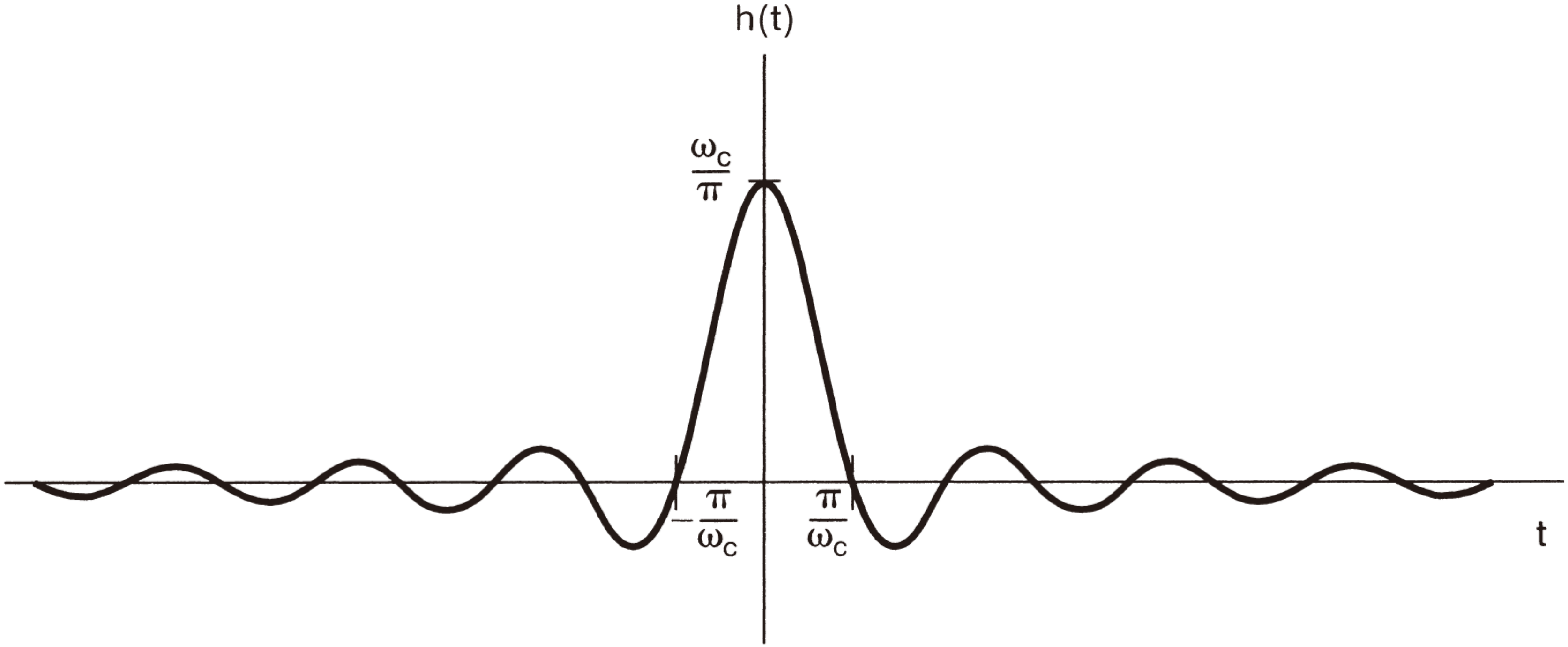
離散時間理想低通濾波器的脈衝響應

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

Note that in both contiuous and discrete time, the width of the filter passband is proportional to ω_c ,while the width of the main lobe of the impulse is proportional to $1/\omega_c$.

上述理想濾波器的通帶寬度與ω_c 成正比，而脈衝響應的主葉寬度與$1/\omega_c$ 成正比。當ω_c 增大，則脈衝響應愈窄(編者按：因頻寬增大，高頻成分增加使響應變化變快之故)。

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters



(a)

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

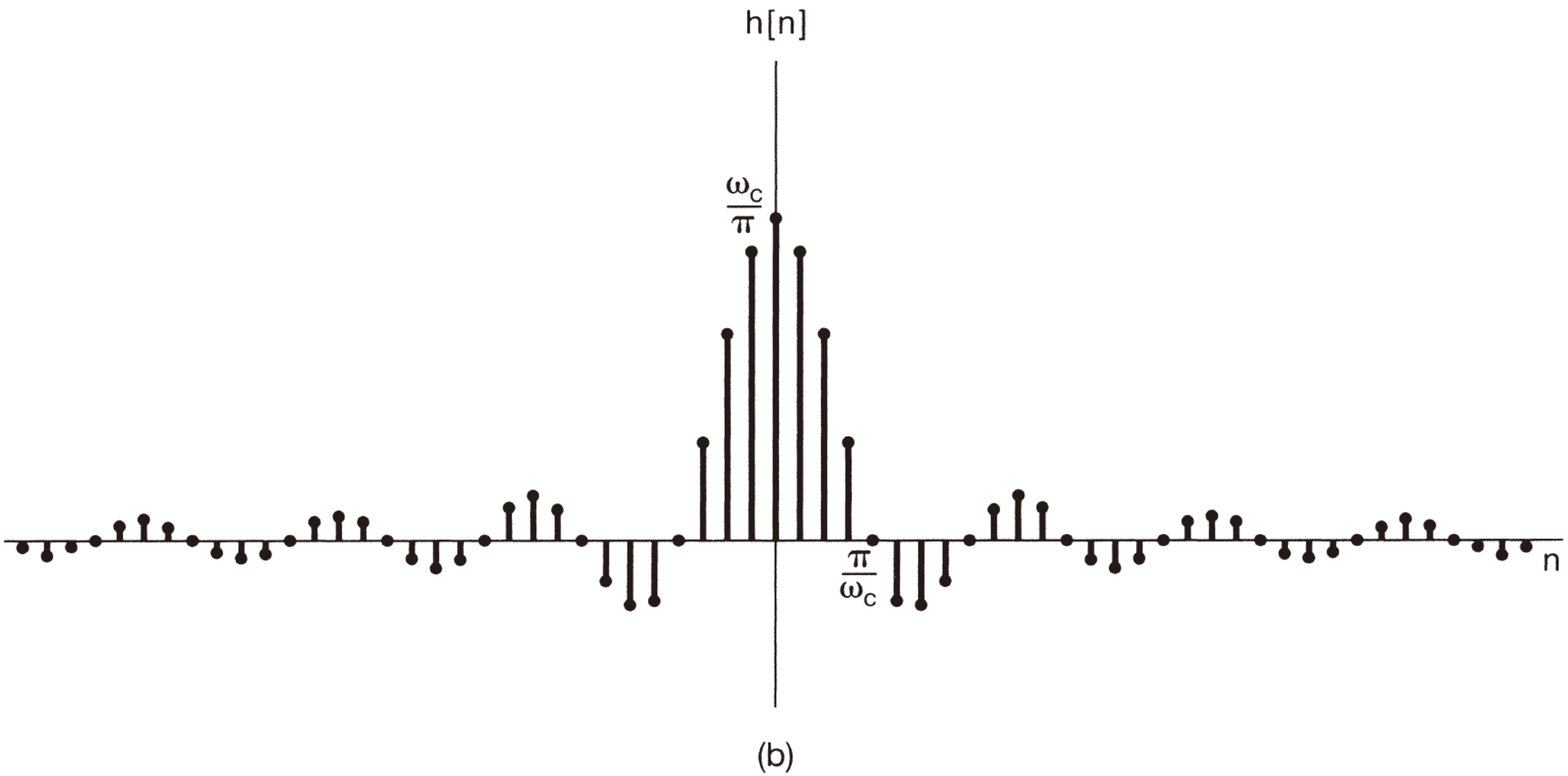


Figure 6.12 (a) The impulse response of the continuous-time ideal lowpass filter of Figure 6.10(a); (b) the impulse response of the discrete-time ideal lowpass filter of Figure 6.10(b) with $\omega_c = \pi/4$.

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

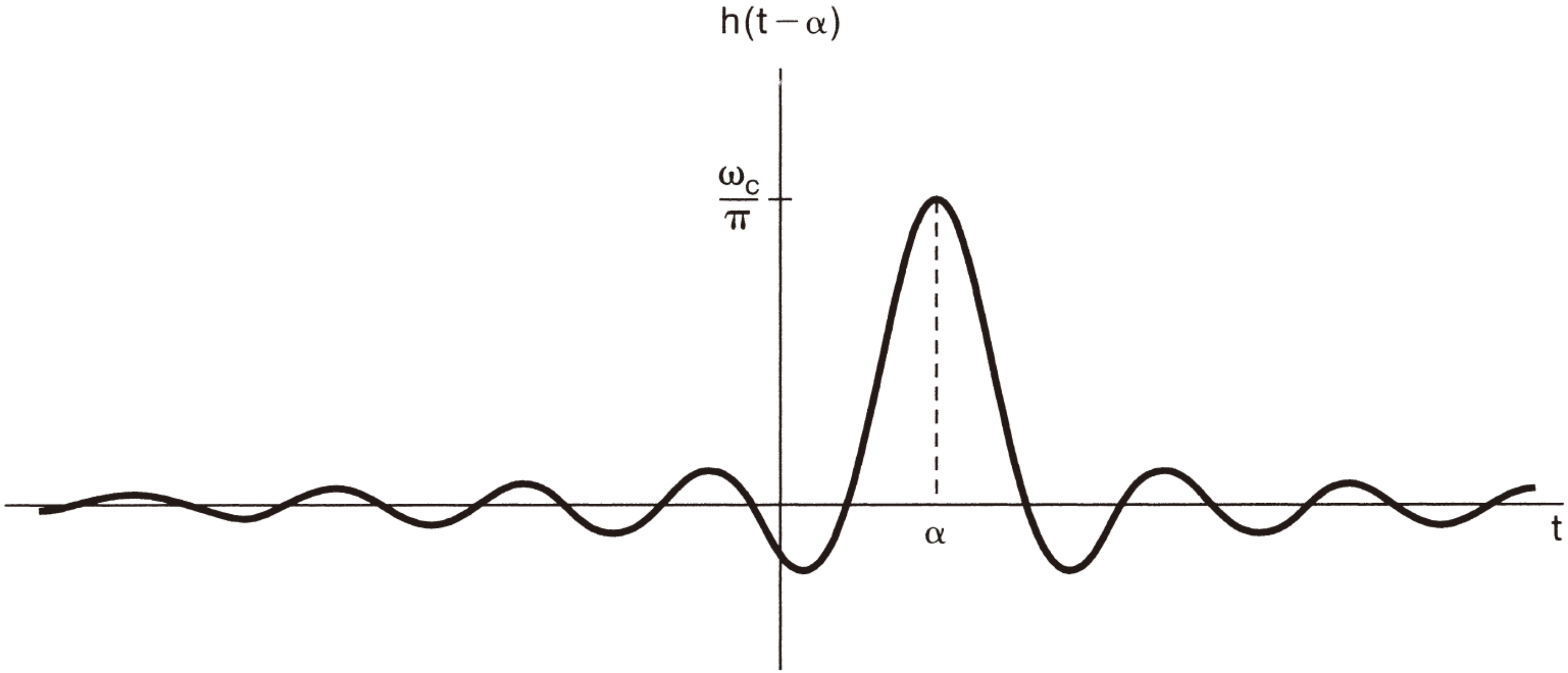


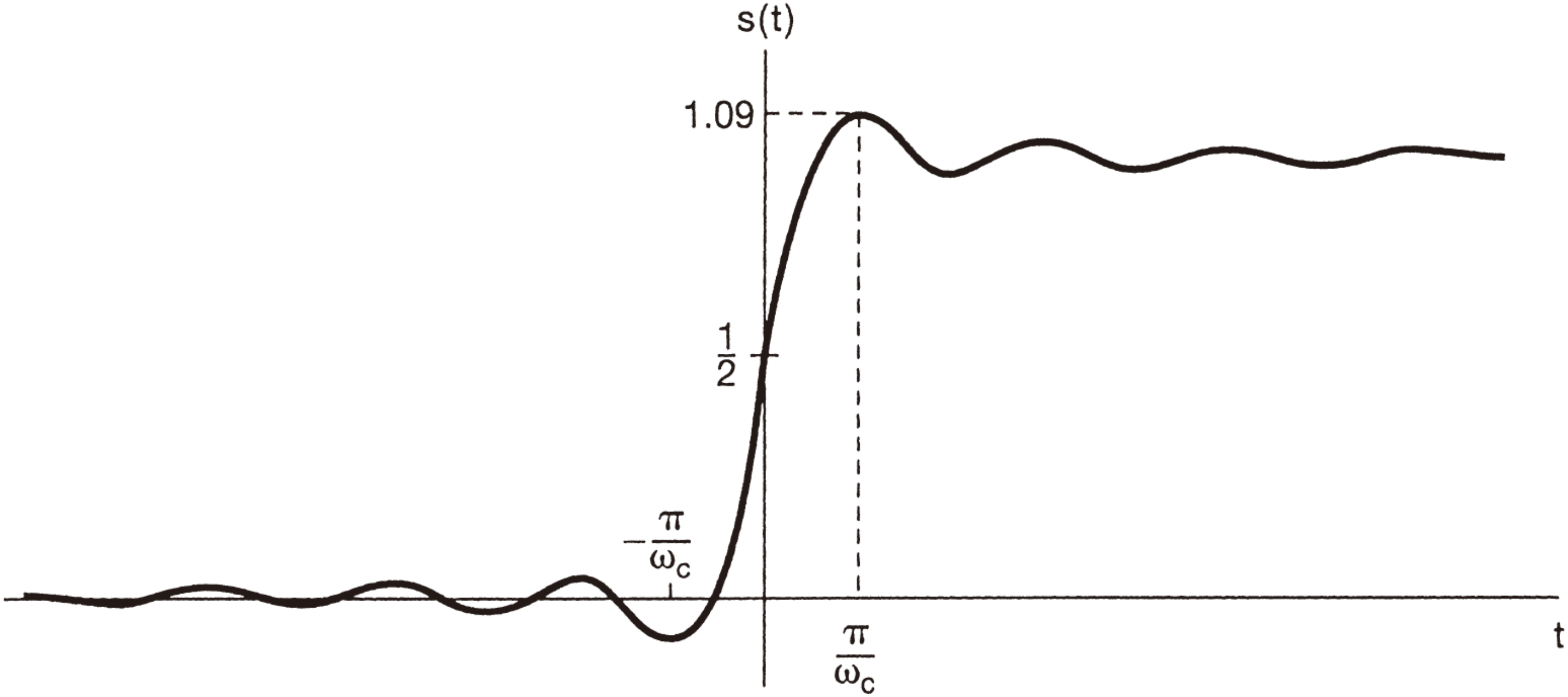
Figure 6.13 Impulse response of an ideal lowpass filter with magnitude and phase shown in Figure 6.11.

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

The step responses $s(t)$ and $s[n]$ of the ideal lowpass filters in continuous time and discrete time are displayed in Figure 6.14.

理想低通濾波器具有一些不好的步級響應特性：超過終值的超越量呈現出振盪現象，常稱為「振鈴」。

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters



(a)

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

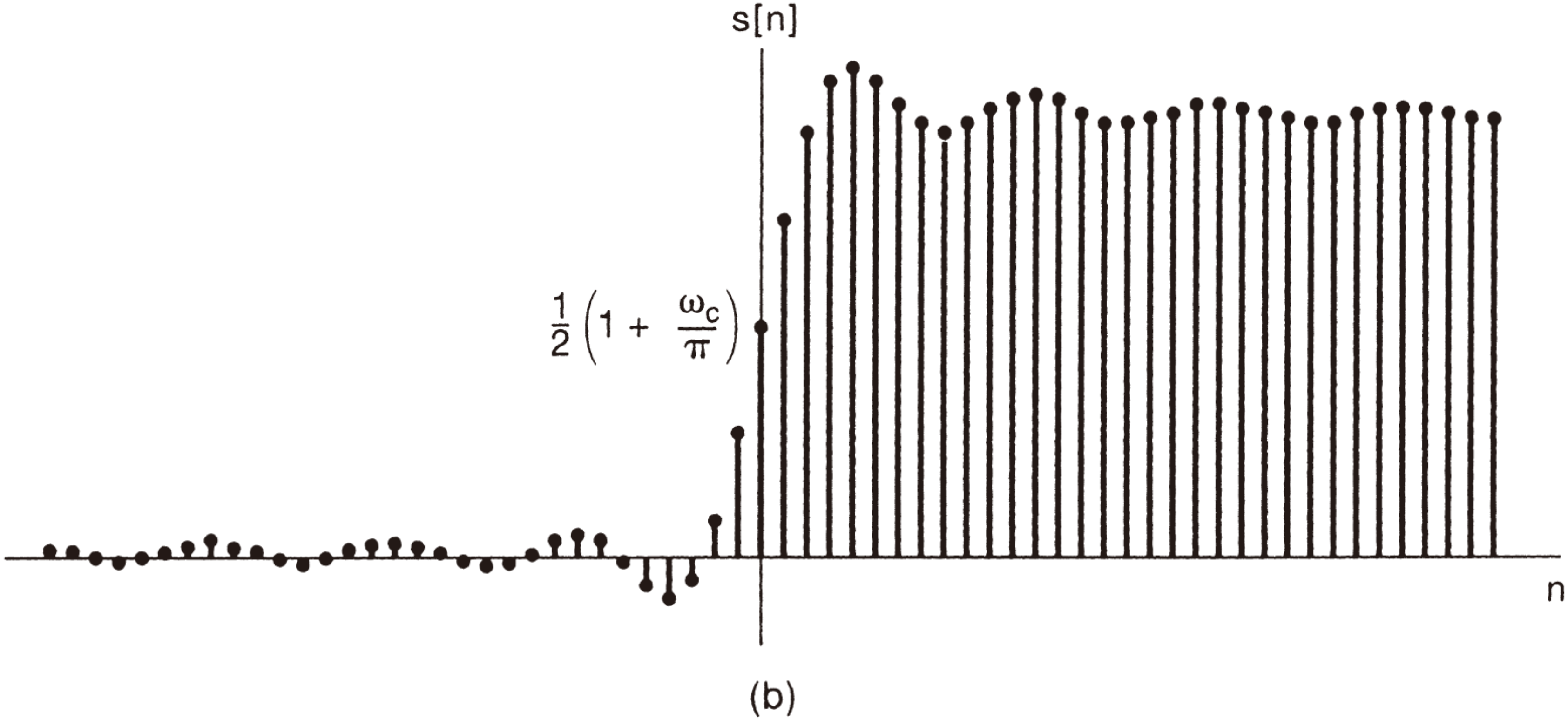


Figure 6.14 (a) Step response of a continuous-time ideal lowpass filter; (b) step response of a discrete-time ideal lowpass filter.

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

步級響應的「上升時間」，為濾波器響應時間的一種量測，與濾波器的頻寬有反比關係。

The characteristics of ideal filter are not always desirable in practice.

對於頻譜重疊的訊號的濾波，一般常用的是從通帶到止帶的逐漸過渡的濾波器。

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

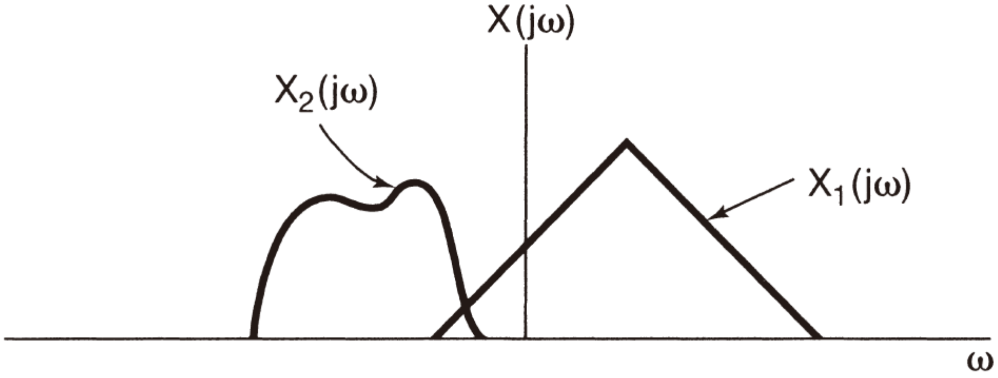


Figure 6.15 Two spectra that are slightly overlapping.

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

Moreover, even in cases where the ideal frequency-selective characteristics are desirable, they may not be attainable.

對於理想低通濾波器在輸入的不連續點處，將產生超越量而呈現振鈴現象，此種時域表現可能不符需求。

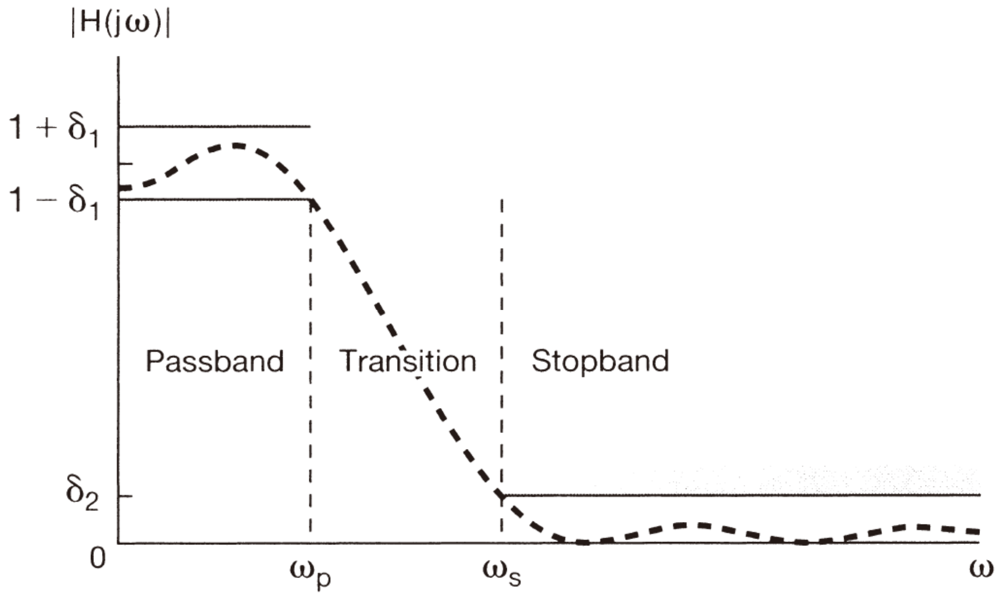
理想低通濾波器是非因果的，而因果性是實現的必要條件，故應尋求理想濾波器的因果近似系統。

愈精確近似理想的濾波，往往結構較複雜，成本較高，故常使用較單純的濾波器。

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

For all of these reasons, nonideal filters are of considerable practical importance, and the characteristics of such filters are frequently specified or quantified in terms of several parameters in both the frequency and time domain.

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters



由圖 6.16 可知，通帶和止帶之間應存在逐漸過渡的地帶，且通帶和止帶在大小上亦應保留允許「通帶漣波」及「止帶漣波」的存在空間。

Figure 6.16 Tolerances for the magnitude characteristic of a lowpass filter. The allowable passband ripple is δ_1 and stopband ripple is δ_2. The dashed curve illustrates one possible frequency response that stays within the tolerable limits.

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

In addition to the specification of magnitude characteristics in the frequency domain, in some cases the specification of phase characteristics is also important.

在相位特性上，濾波器常常期望具有線性或接近線性的相位移特性。

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

用來衡量振鈴特性的三個量值為：超越量△、振鈴頻率 ω_r 及安定時間 t_s (請配合圖6.17)。

If such ringing is present, then there are three other quantities that are often used to characterize the nature of these oscillations: the overshoot △ of the final value of the step response, the ringing frequency ω_r , and the settling time t_s —i.e., the time required for the step response to settle to within a specified tolerance of its final value.

非理想低通濾波器常在過渡帶寬度(頻域特性)及安定時間(時域特性)上尋求折衷。

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

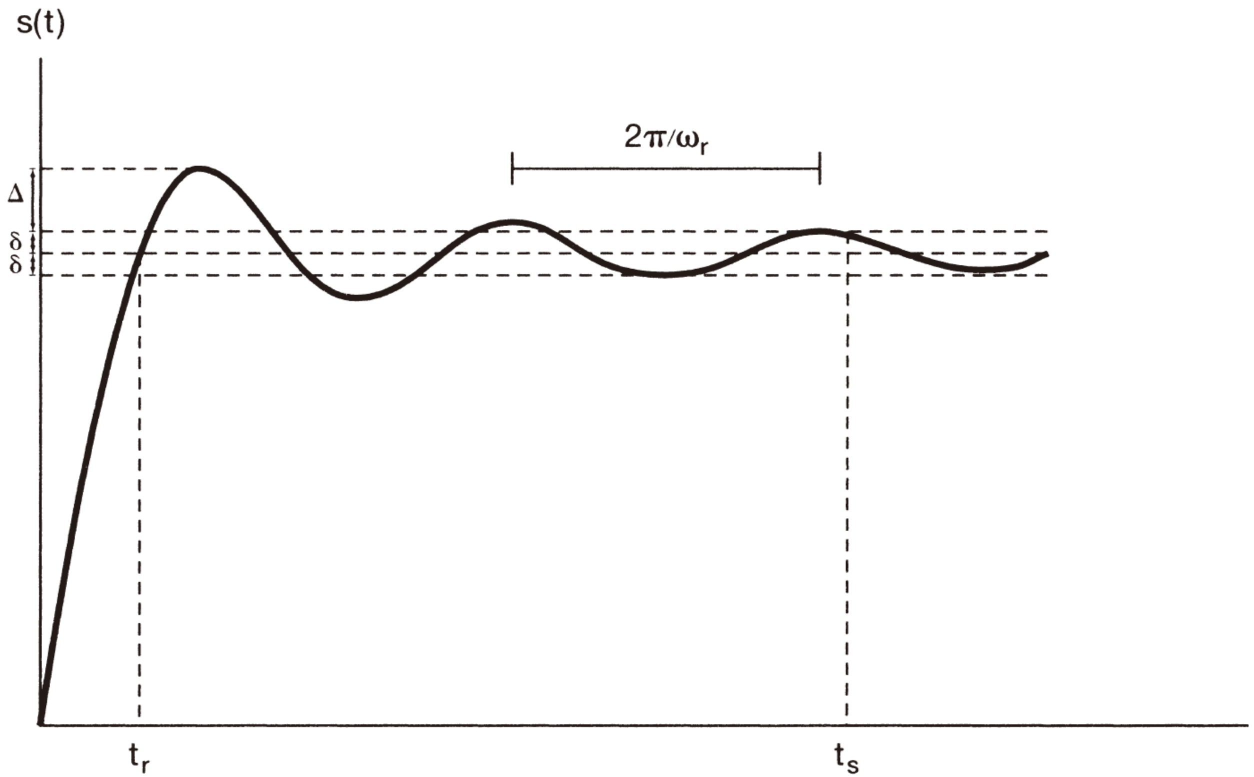


Figure 6.17 Step response of a continuous-time lowpass filter, indicating the rise time t_r, overshoot Δ, ringing frequency ω_r, and settling time t_s—i.e., the time at which the step response settles to within $\pm\delta$ of its final value.

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

巴特沃斯濾波器的過渡帶比橢圓濾波器寬

橢圓濾波器的步級響應，比巴特沃斯濾波器呈現出較為明顯的振鈴現象及較長的安定時間。

6.5 First-Order and Second-Order Continuous-Time Systems

高階系統常可用一階或二階系統串接或並接表示，故一階或二階系統對於高階系統在時域及頻域的分析、設計和瞭解上扮演重要的角色。

6.5.1 First-Order Continuous-Time Systems

The differential equation for a first-order system is often expressed in the form

$$\tau \frac{dy(t)}{dt} + y(t) = x(t), \tag{6.21}$$

一階系統的微分方程

6.5.1 First-Order Continuous-Time Systems

The corresponding frequency response for the first-order system is

$$H(j\omega) = \frac{1}{j\omega\tau + 1}, \qquad (6.22)$$

一階系統的頻率響應

and the impulse response is

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t), \qquad (6.23)$$

一階系統的脈衝響應

6.5.1 First-Order Continuous-Time Systems

which is sketched in Figure 6.19(a). The step response of the system is

$$s(t) = h(t) * u(t) = \left[1 - e^{-t/\tau}\right] u(t). \qquad (6.24)$$

一階系統的步級響應

τ 為系統的時間常數。

6.5.1 First-Order Continuous-Time Systems

Specifically, from eq. (6.22), we obtain

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left[(\omega\tau)^2+1\right] \qquad (6.25)$$

From this we see that for $\omega\tau << 1$,the log magnitude is approximately zero, while for $\omega\tau >> 1$,the log magnitude is approximately a linear function of $\log_{10}(\omega)$. That is,

$$20\log_{10}|H(j\omega)| \approx 0 \qquad for \qquad \omega << 1/\tau, \qquad (6.26)$$

當 $\omega\tau << 1/\tau$, 則系統的對數大小值(分貝值)為0；
當 $\omega\tau >> 1/\tau$，則系統的對數小大值近似 $\log_{10}(\omega)$ 的線性函數。

6.5.1 First-Order Continuous-Time Systems

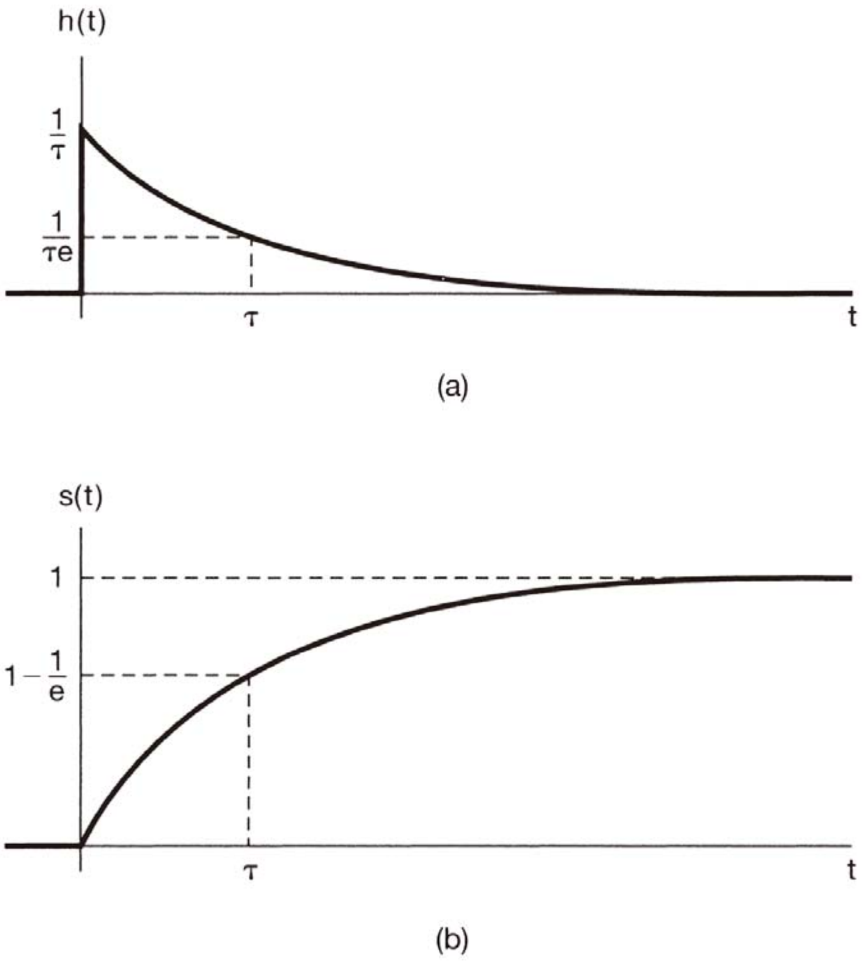


Figure 6.19 Continuous-time first-order system: (a) impulse response; (b) step response.

6.5.1 First-Order Continuous-Time Systems

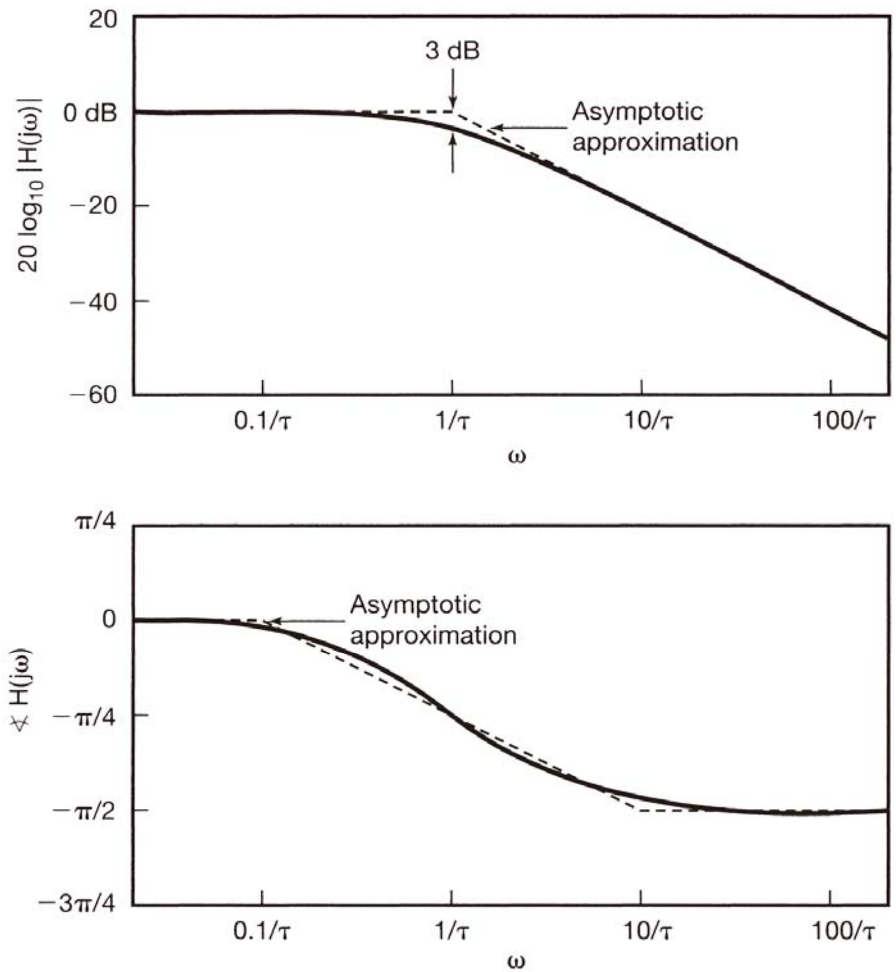


圖 6.20 顯示一階系統的波德圖及漸近線近似圖。

Figure 6.20 Bode plot for a continuous-time first-order system.

6.5.1 First-Order Continuous-Time Systems

And

$$20\log_{10}|H(j\omega)| \approx -20\log_{10}(\omega\tau) \qquad (6.27)$$

$$= -20\log_{10}(\omega) - 20\log_{10}(\tau) \qquad \textit{for} \qquad \omega >> 1/\tau.$$

In other words, for the first-order system, the low- and high-frequency asymptotes of the log magnitude are straight lines.

一階系統的低頻和高頻漸近線為直線。低頻漸近線為0dB水平線。高頻漸近線為ω每十倍頻減少20dB的斜直線。

6.5.1 First-Order Continuous-Time Systems

The point at which the slope of the approximation changes is precisely $\omega = 1/\tau$.

$\omega = 1/\tau$ 為「折點頻率」。

note that at $\omega = 1/\tau$ the two terms [$(\omega\tau)^2$ and 1] in the argument of the logarithm in eq. (6.25) are equal.

$$20\log_{10}\left|H\left(j\frac{1}{\tau}\right)\right| = -10\log_{10}(2) \approx -3dB. \quad (6.28)$$

此頻率處的分貝值之差為-3dB，又稱「3dB頻率」。

6.5.1 First-Order Continuous-Time Systems

It is also possible to obtain a useful straight-line approximation to

$$\sphericalangle H(j\omega) = -\tan^{-1}(\omega\tau) \approx \begin{cases} 0, & \omega \le 0.1/\tau \\ -(\pi/4)\left[\log_{10}(\omega\tau)+1\right], & 0.1/\tau \le \omega \le 10/\tau. \\ -\pi/2, & \omega \ge 10/\tau \end{cases} \quad (6.29)$$

6.5.1 First-Order Continuous-Time Systems

Note that this approximation decreases linearly (from 0 to $-\pi/2$) as a function of $\log_{10}(\omega)$ in the range

$$\frac{0.1}{\tau} \le \omega \le \frac{10}{\tau},$$

在 $0.1/\tau \le \omega \le 10/\tau$ 範圍內，相位圖以 $\log_{10}(\omega)$ 的線性函數減少。

the approximation agrees with the actual value of $\sphericalangle H(j\omega)$ at the break frequency $\omega = 1/\tau$, at which point

$$\sphericalangle H\left(j\frac{1}{\tau}\right) = -\frac{\pi}{4}. \qquad (6.30)$$

折點頻率處，相位移之值為$-\pi/4$(即 -45°)。

6.5.1 First-Order Continuous-Time Systems

As we make τ smaller, we speed up the time response of the system [i.e., *h(t)* becomes more compressed toward the origin, and the rise time of the step response is reduced] and we simultaneously make the break frequency large [i.e., *H(j* ω*)* becomes broader, since $|H(j\omega)| \approx 1$ for a larger range of frequencies].

當 τ 減少時，系統的時間響應速率增加(上升時間減少)，且折點頻率增加。

$$\tau h(t) = e^{-t/\tau}u(t), \qquad H(j\omega) = \frac{1}{j\omega\tau + 1}.$$

6.5.2 Second-Order Continuous-Time Systems

The linear constant-coefficient differential equation for a second-order system is

$$\tau h(t) = e^{-t/\tau} u(t), \qquad H(j\omega) = \frac{1}{j\omega\tau + 1}. \qquad (6.31)$$

二階系統的線性常係數微分方程

6.5.2 Second-Order Continuous-Time Systems

In the figure, the input is the applied force *x(t)* and the output is the displacement of the mass *y(t)* from some equilibrium position at which the spring exerts no restoring force.

$$m\frac{d^2y(t)}{dt^2} = x(t) - ky(t) - b\frac{dy(t)}{dt},$$

or

$$\frac{d^2y(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \frac{1}{m}x(t).$$

6.5.2 Second-Order Continuous-Time Systems

Comparing this to eq. (6.31), we see that if we identify

$$\omega_n = \sqrt{\frac{k}{m}} \tag{6.32}$$

and

$$\zeta = \frac{b}{2\sqrt{km}},$$

6.5.2 Second-Order Continuous-Time Systems

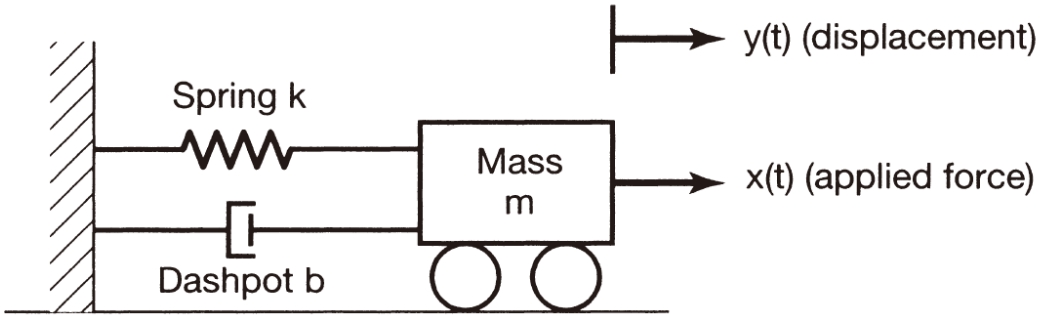


Figure 6.21 Second-order system consisting of a spring and dashpot attached to a moveable mass and a fixed support.

6.5.2 Second-Order Continuous-Time Systems

The frequency response for the second-order system of eq. (6.31) is

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}. \qquad (6.33)$$

二階系統的頻率響應

6.5.2 Second-Order Continuous-Time Systems

The denominator of $H(j\omega)$ can be factored to yield

$$h(j\omega)=\frac{\omega_n^2}{(j\omega-c_1)(j\omega-c_2)},$$

where

$$\begin{aligned} c_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}, \\ c_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}. \end{aligned} \qquad (6.34)$$

c_1 及 c_2 為二階系統的極點

6.5.2 Second-Order Continuous-Time Systems

For $\zeta \neq 1$, c_1 and c_2 are unequal, and we can perform a partial-fraction expansion of the form

$$H(j\omega) = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2}, \qquad (6.35)$$

where

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}. \qquad (6.36)$$

6.5.2 Second-Order Continuous-Time Systems

From eq. (6.35), the corresponding impulse response for the system is

$$h(t) = M\left[e^{c_1 t} - e^{c_2 t}\right]u(t). \qquad (6.37)$$

If $\zeta = 1$, then $c_1 = c_2 = -\omega_n$, and

$$H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}. \qquad (6.38)$$

From Table 4.2, we find that in this case the impulse response is

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t). \qquad (6.39)$$

6.5.2 Second-Order Continuous-Time Systems

Note from eqs. (6.37) and (6.39), that h(t)/ ω_n is a function of $\omega_n t$. Furthermore, eq. (6.33) can be rewritten as

$$H(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1},$$

二階系統的頻率響應(以 ω/ω_n 表示)

ζ 稱為「阻尼比」：ω_n 稱為「無阻尼自然頻率」。

6.5.2 Second-Order Continuous-Time Systems

from eq. (6.35), we see that for $0 < \zeta < 1$, c_1 and c_2 are complex, and we can rewrite the impulse response in eq. (6.37) in the form

$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{2j\sqrt{1-\zeta^2}}\left\{\exp\left[(\omega_n\sqrt{1-\zeta^2})t\right] - \exp\left[-j(\omega_n\sqrt{1-\zeta^2})t\right]\right\}u(t) \tag{6.40}$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_n\sqrt{1-\zeta^2})t\right]u(t).$$

二階系統的脈衝響應

6.5.2 Second-Order Continuous-Time Systems

Thus, for $0 < \zeta < 1$, the second-order system has in impulse response that has damped oscillatory behavior, and this case the system is referred to as being under-damped. If $\zeta > 1$, both c_1 and c_2 are real and negative, and the impulse response is the difference between two decaying exponentials. In this case, the system is overdamped. The case of ζ = 1, when $c_1 = c_2$, is called the critically damped case.

6.5.2 Second-Order Continuous-Time Systems

當$0 < \zeta < 1$時，系統為「欠阻尼」，呈現阻尼振盪現象。

當$\zeta > 1$時，系統為「過阻尼」，呈現沒有振盪的衰減指數。

當$\zeta = 1$時，系統為「臨界阻尼」，沒有振盪。

6.5.2 Second-Order Continuous-Time Systems

不同 ζ 值的二階系統脈衝響應圖

不同 ζ 值的二階系統步級響應圖

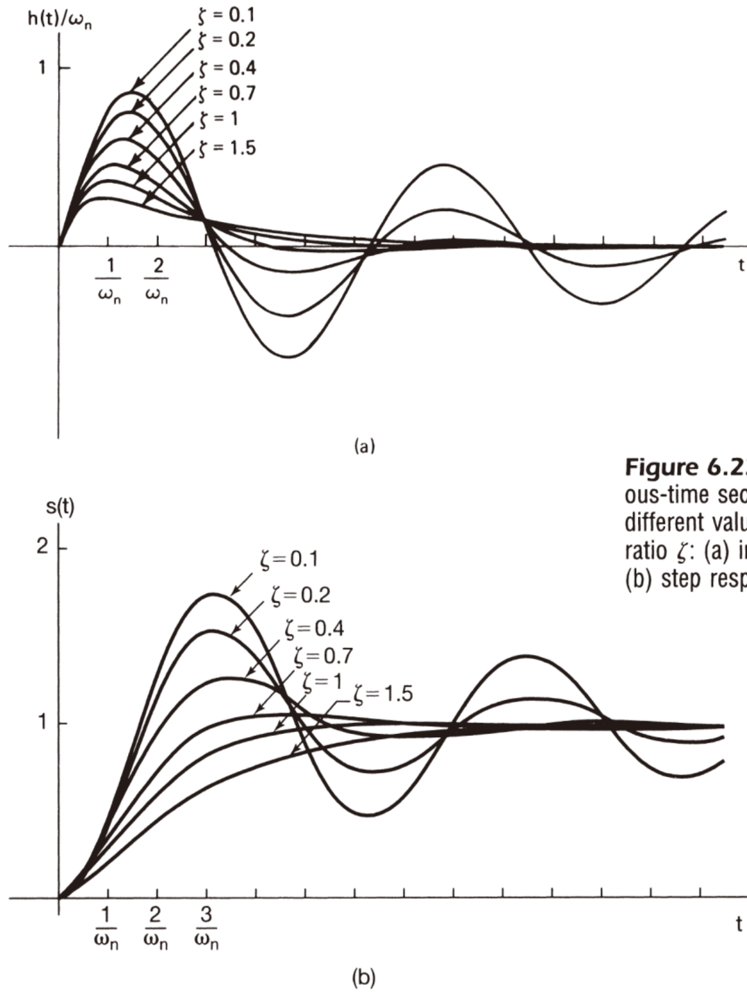


Figure 6.22 Response of continuous-time second-order systems with different values of the damping ratio ζ: (a) impulse response; (b) step response.

6.5.2 Second-Order Continuous-Time Systems

The step response of a second-order system can be calculated from eq. (6.37) for $\zeta \neq 1$. This yields the expression

$$s(t) = h(t) * u(t) = \left\{ 1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2} \right] \right\} u(t). \qquad (6.41)$$

For $\zeta = 1$, we can use eq. (6.39) to obtain

$$s(t) = \left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \right] u(t). \qquad (6.42)$$

6.5.2 Second-Order Continuous-Time Systems

The step response of a second-order system is plotted in Figure 6.22(b) for several values of ζ.

For $\zeta = 1$, the step response has the fastest response (i.e., the shortest rise time) that is possible without overshoot and thus has the shortest settling time.

對於二階系統的步級響應：

欠阻尼系統具有超越量及振鈴現象；臨界阻尼系統具有在沒有超越量之下的最快響應及最短的安定時間。

若$\zeta > 1$，則響應著ζ值增大而減緩。

對於較大的ζ值，系統安定時間較長。

6.5.2 Second-Order Continuous-Time Systems

In Figure 6.23, we have depicted the Bode plot of the frequency response given in eq. (6.33) for several values of ζ.

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}. \qquad (6.41)$$

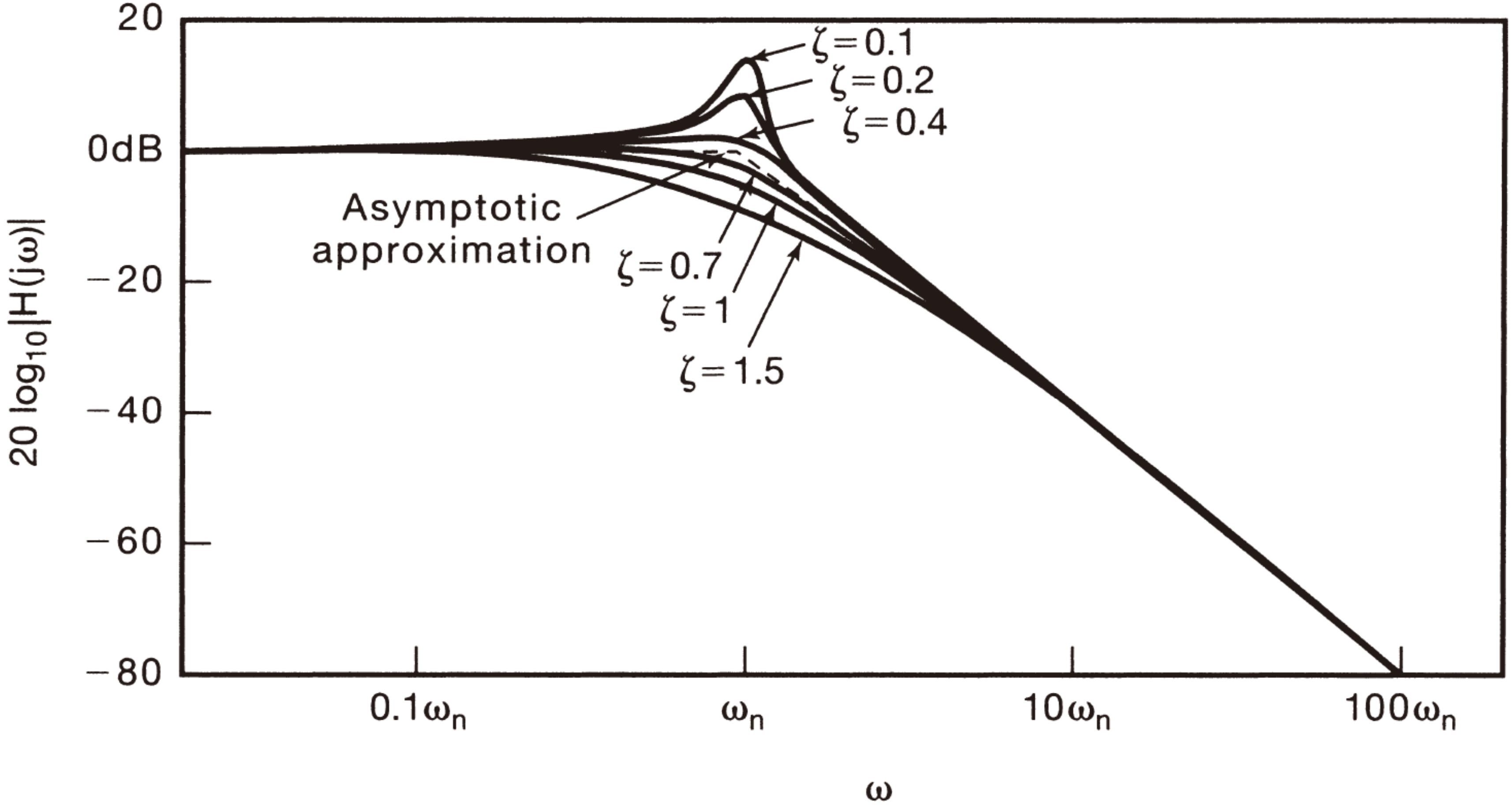
二階系統的對數大小函數

From this expression, it follows that

$$20\log_{10}|H(j\omega)| \cong \begin{cases} 0, & for\,\omega \langle\langle \omega_n \\ -40\log_{10}\omega + 40\log_{10}\omega_n, & for\,\omega \rangle\rangle \omega_n \end{cases}. \qquad (6.42)$$

對數大小函數的漸近線

6.5.2 Second-Order Continuous-Time Systems



注意 ξ 值較小時，大小圖將存在一個突出的最高點(共振點)。

6.5.2 Second-Order Continuous-Time Systems

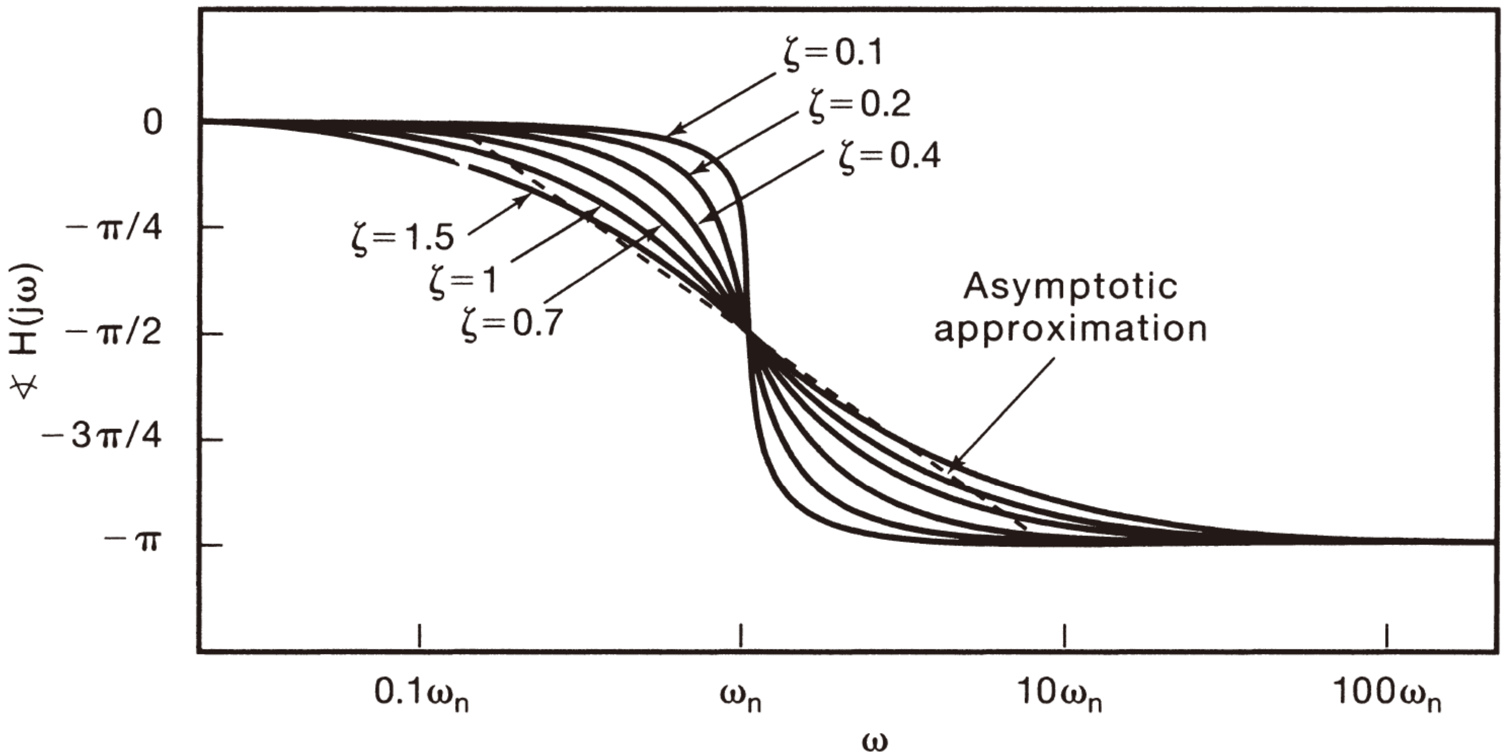


Figure 6.23 Bode plots for second-order systems with several different values of damping ratio ζ.

圖 6.23 爲在不同 ζ 值之下的波德圖。

6.5.2 Second-Order Continuous-Time Systems

Therefore, the low-frequency asymptote of the log magnitude is the 0-dB line, while the high-frequency asymptote [given by eq. (6.44)] has a slope of -40 dB per decade; i.e., $|H(j\omega)|$ decreases by 40 dB for every increase in ω of a factor of 10.

低頻漸近線為0dB水平線；高頻漸近線為斜率為每十倍頻-40dB的直線。

6.5.2 Second-Order Continuous-Time Systems

We can, in addition, obtain a straight-line approximation to $\measuredangle H(j\omega)$, whose exact expression can be obtained from eq. (6.33):

$$\measuredangle H(j\omega) = -\tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right). \tag{6.45}$$

二階系統的相位移函數

6.5.2 Second-Order Continuous-Time Systems

The approximation is

$$\measuredangle H(j\omega) \cong \begin{cases} 0, & \omega \le 0.1\omega_n \\ -\dfrac{\pi}{2}\left[\log_{10}\left(\dfrac{\omega}{\omega_n}\right)+1\right], & 0.1\omega_n \le \omega \le 10\omega_n, \\ -\pi, & \omega \ge 10\omega_n \end{cases} \qquad (6.46)$$

相位移函數的漸近線

which is also plotted in Figure 6.23. Note that the approximation and the actual value again are equal at the break frequency $\omega = \omega_n$, where

$$\measuredangle H(j\omega_n) = -\frac{\pi}{2}.$$

6.5.2 Second-Order Continuous-Time Systems

In fact, straightforward calculations using eq. (6.43) show that, for $\zeta < \sqrt{2}/2 \cong 0.707, |H(j\omega)|$ has a maximum value at

$$\omega_{\max} = \omega_n \sqrt{1 - 2\zeta^2}, \tag{6.47}$$

and the value at this maximum point is

$$|H(j\omega_{\max})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}. \tag{6.48}$$

6.5.2 Second-Order Continuous-Time Systems

若 $\zeta < 1/\sqrt{2}$，則當 ω 值為：

$\omega_{\max} = \omega_n\sqrt{1-2\zeta^2}$ 時，$|H(j\omega)|$ 將有最大值(尖峰)，此最大值為

$$|H(j\omega_{\max})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

若 $\zeta > 1/\sqrt{2}$，$|H(j\omega)|$ 為單調漸減函數。

6.5.2 Second-Order Continuous-Time Systems

For a second-order circuit described by an equation of the form of eq. (6.31), the quality is usually taken to be

$$Q = \frac{1}{2\zeta},$$

品質因數*Q*為對於尖峰的尖銳程度的衡量值。

二階電路的品質因數$Q = 1/2\zeta$。

系統阻尼愈小，則$|H(j\omega)|$的尖峰愈尖銳。

6.5.3 Bode Plots for Rational Frequency Responses

In addition, we can readily obtain the Bode plots for frequency responses of the forms

$$H(j\omega) = 1 + j\omega\tau \tag{6.49}$$

and

$$H(j\omega) = 1 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2. \tag{6.50}$$

6.5.3 Bode Plots for Rational Frequency Responses

The Bode plots for eqs. (6.49) and (6.50) follow directly from Figures 6.20 and 6.23 and from the fact that

$$20\log_{10}|H(j\omega)| = -20\log_{10}\left|\frac{1}{H(j\omega)}\right|$$

and

$$\sphericalangle\left(H(j\omega)\right) = -\sphericalangle\left(\frac{1}{H(j\omega)}\right).$$

6.5.3 Bode Plots for Rational Frequency Responses

also, consider a system function that is a constant gain

$$H(j\omega) = K.$$

Since $K = |K|e^{j\cdot 0}$ if K > 0 and $K = |K|e^{j\pi}$ if K < 0, we see that

$$20\log_{10}|H(j\omega)| = 20\log_{10}|K|$$

$$\not< H(j\omega) = \begin{cases} 0, & if \quad K\rangle 0 \\ \pi, & if \quad K\langle 0 \end{cases}$$