(a) Chebyshev polynomials[6] of the first and second kind are defined by

$$T_n(x) = \cos(n \arccos x)$$
$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}$$

respectively, where $n = 0, 1, \cdots$. Show that

$$T_0 = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x,$$
$$U_0 = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x.$$

Show that the Chebyshev polynomials $T_n(x)$ are orthogonal on the interval $-1 \leqq x \leqq 1$ with respect to the weight function $r(x) = 1/\sqrt{1-x^2}$. (*Hint.* To evaluate the integral, set $\arccos x = \theta$.) Verify that $T_n(x)$, $n = 0, 1, 2, 3$, satisfy the **Chebyshev equation**

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

(b) Orthogonality on an infinite interval: Laguerre polynomials[7] are defined by $L_0 = 1$, and

$$L_n(x) = \frac{e^x}{n!}\frac{d^n(x^n e^{-x})}{dx^n}, \qquad n = 1, 2, \cdots.$$

Show that

$$L_n(x) = 1 - x, \qquad L_2(x) = 1 - 2x + x^2/2,$$
$$L_3(x) = 1 - 3x + 3x^2/2 - x^3/6.$$

Prove that the Laguerre polynomials are orthogonal on the positive axis $0 \leqq x < \infty$ with respect to the weight function $r(x) = e^{-x}$. *Hint.* Since the highest power in L_m is x^m, it suffices to show that $\int e^{-x}x^k L_n\,dx = 0$ for $k < n$. Do this by k integrations by parts.

11.6 Orthogonal Series. Generalized Fourier Series

Fourier series are made up of the trigonometric system (Sec. 11.1), which is orthogonal, and orthogonality was essential in obtaining the Euler formulas for the Fourier coefficients. Orthogonality will also give us coefficient formulas for the desired generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series. This generalization is as follows.

Let $y_0, y_1, y_2, \cdots$ be orthogonal with respect to a weight function $r(x)$ on an interval $a \leqq x \leqq b$, and let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots. \tag{1}$$

This is called an **orthogonal series**, **orthogonal expansion**, or **generalized Fourier series**. If the y_m are the eigenfunctions of a Sturm–Liouville problem, we call (1) an **eigenfunction expansion**. In (1) we use again m for summation since n will be used as a fixed order of Bessel functions.

Given $f(x)$, we have to determine the coefficients in (1), called the **Fourier constants** *of* $f(x)$ *with respect to* $y_0, y_1, \cdots$. Because of the orthogonality, this is simple. Similarly to Sec. 11.1, we multiply both sides of (1) by $r(x)y_n(x)$ (n ***fixed***) and then integrate on

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[6]PAFNUTI CHEBYSHEV (1821–1894), Russian mathematician, is known for his work in approximation theory and the theory of numbers. Another transliteration of the name is TCHEBICHEF.

[7]EDMOND LAGUERRE (1834–1886), French mathematician, who did research work in geometry and in the theory of infinite series.

both sides from a to b. We assume that term-by-term integration is permissible. (This is justified, for instance, in the case of "uniform convergence," as is shown in Sec. 15.5.) Then we obtain

$$(f, y_n) = \int_a^b r f y_n \, dx = \int_a^b r\left(\sum_{m=0}^{\infty} a_m y_m\right) y_n \, dx = \sum_{m=0}^{\infty} a_m \int_a^b r y_m y_n \, dx = \sum_{m=0}^{\infty} a_m (y_m, y_n).$$

Because of the orthogonality all the integrals on the right are zero, except when $m = n$. Hence the whole infinite series reduces to the single term

$$a_n (y_n, y_n) = a_n \|y_n\|^2. \qquad \text{Thus} \qquad (f, y_n) = a_n \|y_n\|^2.$$

Assuming that all the functions y_n have nonzero norm, we can divide by $\|y_n\|^2$; writing again m for n, to be in agreement with (1), we get the desired formula for the Fourier constants

(2)
$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x) \, dx \qquad (n = 0, 1, \cdots).$$

This formula generalizes the Euler formulas (6) in Sec. 11.1 as well as the principle of their derivation, namely, by orthogonality.

EXAMPLE 1 Fourier–Legendre Series

A **Fourier–Legendre series** is an eigenfunction expansion

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + a_2 P_2(x) + \cdots = a_0 + a_1 x + a_2 (\tfrac{3}{2} x^2 - \tfrac{1}{2}) + \cdots$$

in terms of Legendre polynomials (Sec. 5.3). The latter are the eigenfunctions of the Sturm–Liouville problem in Example 4 of Sec. 11.5 on the interval $-1 \leqq x \leqq 1$. We have $r(x) = 1$ for Legendre's equation, and (2) gives

(3)
$$a_m = \frac{2m + 1}{2} \int_{-1}^{1} f(x) P_m(x) \, dx, \qquad m = 0, 1, \cdots$$

because the norm is

(4)
$$\|P_m\| = \sqrt{\int_{-1}^{1} P_m(x)^2 \, dx} = \sqrt{\frac{2}{2m + 1}} \qquad (m = 0, 1, \cdots)$$

as we state without proof. The proof of (4) is tricky; it uses Rodrigues's formula in Problem Set 5.2 and a reduction of the resulting integral to a quotient of gamma functions.

For instance, let $f(x) = \sin \pi x$. Then we obtain the coefficients

$$a_m = \frac{2m + 1}{2} \int_{-1}^{1} (\sin \pi x) P_m(x) \, dx, \qquad \text{thus} \qquad a_1 = \frac{3}{2} \int_{-1}^{1} x \sin \pi x \, dx = \frac{3}{\pi} = 0.95493, \qquad \text{etc.}$$

Hence the Fourier–Legendre series of $\sin \pi x$ is

$$\sin \pi x = 0.95493P_1(x) - 1.15824P_3(x) + 0.21929P_5(x) - 0.01664P_7(x) + 0.00068P_9(x) - 0.00002P_{11}(x) + \cdots .$$

The coefficient of P_{13} is about $3 \cdot 10^{-7}$. The sum of the first three nonzero terms gives a curve that practically coincides with the sine curve. Can you see why the even-numbered coefficients are zero? Why a_3 is the absolutely biggest coefficient? ■

EXAMPLE 2 Fourier–Bessel Series

These series model vibrating membranes (Sec. 12.9) and other physical systems of circular symmetry. We derive these series in three steps.

***Step 1.* Bessel's equation as a Sturm–Liouville equation.** The Bessel function $J_n(x)$ with fixed integer $n \geqq 0$ satisfies Bessel's equation (Sec. 5.5)

$$\tilde{x}^2 \ddot{J}_n(\tilde{x}) + \tilde{x}\dot{J}_n(\tilde{x}) + (\tilde{x}^2 - n^2)J_n(\tilde{x}) = 0$$

where $\dot{J}_n = dJ_n/d\tilde{x}$ and $\ddot{J}_n = d^2J_n/d\tilde{x}^2$. We set $\tilde{x} = kx$. Then $x = \tilde{x}/k$ and by the chain rule, $\dot{J}_n = dJ_n/d\tilde{x} = (dJ_n/dx)/k$ and $\ddot{J}_n = J_n''/k^2$. In the first two terms of Bessel's equation, k^2 and k drop out and we obtain

$$x^2J_n''(kx) + xJ_n'(kx) + (k^2x^2 - n^2)J_n(kx) = 0.$$

Dividing by x and using $(xJ_n'(kx))' = xJ_n''(kx) + J_n'(kx)$ gives the Sturm–Liouville equation

$$(5) \qquad [xJ_n'(kx)]' + \left(-\frac{n^2}{x} + \lambda x\right)J_n(kx) = 0 \qquad \lambda = k^2$$

with $p(x) = x$, $q(x) = -n^2/x$, $r(x) = x$, and parameter $\lambda = k^2$. Since $p(0) = 0$, Theorem 1 in Sec. 11.5 implies orthogonality on an interval $0 \leqq x \leqq R$ (R given, fixed) of those solutions $J_n(kx)$ that are zero at $x = R$, that is,

$$(6) \qquad J_n(kR) = 0 \qquad (n \text{ fixed}).$$

Note that $q(x) = -n^2/x$ is discontinuous at 0, but this does not affect the proof of Theorem 1.

***Step 2.* Orthogonality.** It can be shown (see Ref. [A13]) that $J_n(\tilde{x})$ has infinitely many zeros, say, $\tilde{x} = a_{n,1} < a_{n,2} < \cdots$ (see Fig. 110 in Sec. 5.4 for $n = 0$ and 1). Hence we must have

$$(7) \qquad kR = \alpha_{n,m} \qquad \text{thus} \qquad k_{n,m} = \alpha_{n,m}/R \qquad (m = 1, 2, \cdots).$$

This proves the following orthogonality property.

THEOREM 1

Orthogonality of Bessel Functions

For each fixed nonnegative integer n the sequence of Bessel functions of the first kind $J_n(k_{n,1}x)$, $J_n(k_{n,2}x)$, $\cdots$ with $k_{n,m}$ as in (7) forms an orthogonal set on the interval $0 \leqq x \leqq R$ with respect to the weight function $r(x) = x$, that is,

$$(8) \qquad \int_0^R xJ_n(k_{n,m}x)J_n(k_{n,j}x)\, dx = 0 \qquad (j \neq m,\ n \text{ fixed}).$$

Hence we have obtained ***infinitely many orthogonal sets*** of Bessel functions, one for each of $J_0, J_1, J_2, \cdots$. Each set is orthogonal on an interval $0 \leqq x \leqq R$ with a fixed positive R of our choice and with respect to the weight x. The orthogonal set for J_n is $J_n(k_{n,1}x), J_n(k_{n,2}x), J_n(k_{n,3}x), \cdots$, where n is *fixed* and $k_{n,m}$ is given by (7).

Step 3. **Fourier–Bessel series.** The Fourier–Bessel series corresponding to J_n (n fixed) is

$$f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) = a_1 J_n(k_{n,1}x) + a_2 J_n(k_{n,2}x) + a_3 J_n(k_{n,3}x) + \cdots \qquad (n \text{ fixed}). \tag{9}$$

The coefficients are (with $\alpha_{n,m} = k_{n,m}R$)

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_0^R x\, f(x)\, J_n(k_{n,m}x)\, dx, \qquad m = 1, 2, \cdots \tag{10}$$

because the square of the norm is

$$\|J_n(k_{n,m}x)\|^2 = \int_0^R x J_n^2(k_{n,m}x)\, dx = \frac{R^2}{2} J_{n+1}^2(k_{n,m}R) \tag{11}$$

as we state without proof (which is tricky; see the discussion beginning on p. 576 of [A13]). ■

EXAMPLE 3 Special Fourier–Bessel Series

For instance, let us consider $f(x) = 1 - x^2$ and take $R = 1$ and $n = 0$ in the series (9), simply writing λ for $\alpha_{0,m}$. Then $k_{n,m} = \alpha_{0,m} = \lambda = 2.405, 5.520, 8.654, 11.792$, etc. (use a CAS or Table A1 in App. 5). Next we calculate the coefficients a_m by (10)

$$a_m = \frac{2}{J_1^2(\lambda)} \int_0^1 x(1 - x^2) J_0(\lambda x)\, dx.$$

This can be integrated by a CAS or by formulas as follows. First use $[xJ_1(\lambda x)]' = \lambda x J_0(\lambda x)$ from Theorem 1 in Sec. 5.4 and then integration by parts,

$$a_m = \frac{2}{J_1^2(\lambda)} \int_0^1 x(1 - x^2) J_0(\lambda x)\, dx = \frac{2}{J_1^2(\lambda)} \left[\frac{1}{\lambda}(1 - x^2) x J_1(\lambda x) \Big|_0^1 - \frac{1}{\lambda} \int_0^1 x J_1(\lambda x)(-2x)\, dx \right].$$

The integral-free part is zero. The remaining integral can be evaluated by $[x^2 J_2(\lambda x)]' = \lambda x^2 J_1(\lambda x)$ from Theorem 1 in Sec. 5.4. This gives

$$a_m = \frac{4 J_2(\lambda)}{\lambda^2 J_1^2(\lambda)} \qquad (\lambda = \alpha_{0,m}).$$

Numeric values can be obtained from a CAS (or from the table on p. 409 of Ref. [GenRef1] in App. 1, together with the formula $J_2 = 2x^{-1}J_1 - J_0$ in Theorem 1 of Sec. 5.4). This gives the eigenfunction expansion of $1 - x^2$ in terms of Bessel functions J_0, that is,

$$1 - x^2 = 1.1081 J_0(2.405x) - 0.1398 J_0(5.520x) + 0.0455 J_0(8.654x) - 0.0210 J_0(11.792x) + \cdots.$$

A graph would show that the curve of $1 - x^2$ and that of the sum of first three terms practically coincide. ■

Mean Square Convergence. Completeness

Ideas on approximation in the last section generalize from Fourier series to orthogonal series (1) that are made up of an orthonormal set that is "complete," that is, consists of "sufficiently many" functions so that (1) can represent large classes of other functions (definition below).

In this connection, convergence is **convergence in the norm**, also called **mean-square convergence**; that is, a sequence of functions f_k is called **convergent** *with the limit f* if

$$\lim_{k \to \infty} \|f_k - f\| = 0; \tag{12*}$$

written out by (5) in Sec. 11.5 (where we can drop the square root, as this does not affect the limit)

$$\lim_{k\to\infty} \int_a^b r(x)[f_k(x) - f(x)]^2\, dx = 0. \tag{12}$$

Accordingly, the series (1) converges and represents f if

$$\lim_{k\to\infty} \int_a^b r(x)[s_k(x) - f(x)]^2\, dx = 0 \tag{13}$$

where s_k is the kth partial sum of (1).

$$s_k(x) = \sum_{m=0}^{k} a_m y_m(x). \tag{14}$$

Note that the integral in (13) generalizes (3) in Sec. 11.4.

We now define completeness. An **orthonormal** set $y_0, y_1, \cdots$ on an interval $a \leqq x \leqq b$ is **complete** *in a set of functions* S defined on $a \leqq x \leqq b$ if we can approximate every f belonging to S arbitrarily closely in the norm by a linear combination $a_0y_0 + a_1y_1 + \cdots + a_ky_k$, that is, technically, if for every $\epsilon > 0$ we can find constants $a_0, \cdots, a_k$ (with k large enough) such that

$$\|f - (a_0y_0 + \cdots + a_ky_k)\| < \epsilon. \tag{15}$$

Ref. [GenRef7] in App. 1 uses the more modern term **total** for *complete*.

We can now extend the ideas in Sec. 11.4 that guided us from (3) in Sec. 11.4 to Bessel's and Parseval's formulas (7) and (8) in that section. Performing the square in (13) and using (14), we first have (analog of (4) in Sec. 11.4)

$$\begin{aligned}\int_a^b r(x)[s_k(x) - f(x)]^2\, dx &= \int_a^b rs_k^2\, dx - 2\int_a^b rfs_k\, dx + \int_a^b rf^2\, dx \\ &= \int_a^b r\left[\sum_{m=0}^{k} a_m y_m\right]^2 dx - 2\sum_{m=0}^{k} a_m \int_a^b rfy_m\, dx + \int_a^b rf^2\, dx.\end{aligned}$$

The first integral on the right equals $\sum a_m^2$ because $\int ry_my_l\, dx = 0$ for $m \neq l$, and $\int ry_m^2\, dx = 1$. In the second sum on the right, the integral equals a_m, by (2) with $\|y_m\|^2 = 1$. Hence the first term on the right cancels half of the second term, so that the right side reduces to (analog of (6) in Sec. 11.4)

$$-\sum_{m=0}^{k} a_m^2 + \int_a^b rf^2\, dx.$$

This is nonnegative because in the previous formula the integrand on the left is nonnegative (recall that the weight $r(x)$ is positive!) and so is the integral on the left. This proves the important **Bessel's inequality** (analog of (7) in Sec. 11.4)

$$\boxed{\sum_{m=0}^{k} a_m^2 \leqq \|f\|^2 = \int_a^b r(x)f(x)^2\, dx} \qquad (k = 1, 2, \cdots), \tag{16}$$

Here we can let $k \to \infty$, because the left sides form a monotone increasing sequence that is bounded by the right side, so that we have convergence by the familiar Theorem 1 in App. A.3.3 Hence

$$\sum_{m=0}^{\infty} a_m^2 \leqq \|f\|^2. \tag{17}$$

Furthermore, if $y_0, y_1, \cdots$ is complete in a set of functions S, then (13) holds for every f belonging to S. By (13) this implies equality in (16) with $k \to \infty$. Hence in the case of completeness every f in S saisfies the so-called **Parseval equality** (analog of (8) in Sec. 11.4)

$$\sum_{m=0}^{\infty} a_m^2 = \|f\|^2 = \int_a^b r(x) f(x)^2 \, dx. \tag{18}$$

As a consequence of (18) we prove that in the case of *completeness* there is no function orthogonal to *every* function of the orthonormal set, with the trivial exception of a function of zero norm:

THEOREM 2

Completeness

Let $y_0, y_1, \cdots$ be a complete orthonormal set on $a \leqq x \leqq b$ in a set of functions S. Then if a function f belongs to S and is orthogonal to every y_m, it must have norm zero. In particular, if f is continuous, then f must be identically zero.

PROOF Since f is orthogonal to every y_m, the left side of (18) must be zero. If f is continuous, then $\|f\| = 0$ implies $f(x) \equiv 0$, as can be seen directly from (5) in Sec. 11.5 with f instead of y_m because $r(x) > 0$ by assumption. ■

PROBLEM SET 11.6

1–7 FOURIER–LEGENDRE SERIES

Showing the details, develop

1. $63x^5 - 90x^3 + 35x$
2. $(x + 1)^2$
3. $1 - x^4$
4. $1, \quad x, \quad x^2, \quad x^3, \quad x^4$
5. Prove that if $f(x)$ is even (is odd, respectively), its Fourier–Legendre series contains only $P_m(x)$ with even m (only $P_m(x)$ with odd m, respectively). Give examples.
6. What can you say about the coefficients of the Fourier–Legendre series of $f(x)$ if the Maclaurin series of $f(x)$ contains only powers x^{4m} $(m = 0, 1, 2, \cdots)$?
7. What happens to the Fourier–Legendre series of a *polynomial* $f(x)$ if you change a coefficient of $f(x)$? Experiment. Try to prove your answer.

8–13 CAS EXPERIMENT

FOURIER–LEGENDRE SERIES. Find and graph (on common axes) the partial sums up to S_{m_0} whose graph practically coincides with that of $f(x)$ within graphical accuracy. State m_0. On what does the size of m_0 seem to depend?

8. $f(x) = \sin \pi x$
9. $f(x) = \sin 2\pi x$
10. $f(x) = e^{-x^2}$
11. $f(x) = (1 + x^2)^{-1}$
12. $f(x) = J_0(\alpha_{0,1} x), \quad \alpha_{0,1} =$ the first positive zero of $J_0(x)$
13. $f(x) = J_0(\alpha_{0,2} x), \quad \alpha_{0,2} =$ the second positive zero of $J_0(x)$

14. TEAM PROJECT. Orthogonality on the Entire Real Axis. Hermite Polynomials.[8] These orthogonal polynomials are defined by $He_0(1) = 1$ and

$$(19) \qquad He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}), \qquad n = 1, 2, \cdots.$$

REMARK. As is true for many special functions, the literature contains more than one notation, and one sometimes defines as Hermite polynomials the functions

$$H_0^* = 1, \qquad H_n^*(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

This differs from our definition, which is preferred in applications.

(a) Small Values of *n*. Show that

$$He_1(x) = x, \qquad He_2(x) = x^2 - 1,$$
$$He_3(x) = x^3 - 3x, \qquad He_4(x) = x^4 - 6x^2 + 3.$$

(b) Generating Function. A generating function of the Hermite polynomials is

$$(20) \qquad e^{tx - t^2/2} = \sum_{n=0}^{\infty} a_n(x) t^n$$

because $He_n(x) = n!\, a_n(x)$. Prove this. *Hint*: Use the formula for the coefficients of a Maclaurin series and note that $tx - \frac{1}{2}t^2 = \frac{1}{2}x^2 - \frac{1}{2}(x - t)^2$.

(c) Derivative. Differentiating the generating function with respect to x, show that

$$(21) \qquad He_n'(x) = nHe_{n-1}(x).$$

(d) Orthogonality on the *x*-Axis needs a weight function that goes to zero sufficiently fast as $x \to \pm\infty$, (Why?)

Show that the Hermite polynomials are orthogonal on $-\infty < x < \infty$ with respect to the weight function $r(x) = e^{-x^2/2}$. *Hint*. Use integration by parts and (21).

(e) ODEs. Show that

$$(22) \qquad He_n'(x) = xHe_n(x) - He_{n+1}(x).$$

Using this with $n - 1$ instead of n and (21), show that $y = He_n(x)$ satisfies the ODE

$$(23) \qquad y'' = xy' + ny = 0.$$

Show that $w = e^{-x^2/4}y$ is a solution of **Weber's equation**

$$(24) \quad w'' + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)w = 0 \qquad (n = 0, 1, \cdots).$$

15. CAS EXPERIMENT. Fourier–Bessel Series. Use Example 2 and $R = 1$, so that you get the series

$$(25) \qquad f(x) = a_1 J_0(\alpha_{0,1}x) + a_2 J_0(\alpha_{0,2}x) + a_3 J_0(\alpha_{0,3}x) + \cdots$$

With the zeros $\alpha_{0,1}\alpha_{0,2}, \cdots$ from your CAS (see also Table A1 in App. 5).

(a) Graph the terms $J_0(\alpha_{0,1}x), \cdots, J_0(\alpha_{0,10}x)$ for $0 \leqq x \leqq 1$ on common axes.

(b) Write a program for calculating partial sums of (25). Find out for what $f(x)$ your CAS can evaluate the integrals. Take two such $f(x)$ and comment empirically on the speed of convergence by observing the decrease of the coefficients.

(c) Take $f(x) = 1$ in (25) and evaluate the integrals for the coefficients analytically by (21a), Sec. 5.4, with $\nu = 1$. Graph the first few partial sums on common axes.

11.7 Fourier Integral

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Sections 11.2 and 11.3 first illustrated this, and various further applications follow in Chap. 12. Since, of course, many problems involve functions that are ***nonperiodic*** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

In Example 1 we start from a special function f_L of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$. Then we do the same for an *arbitrary* function f_L of period $2L$. This will motivate and suggest the main result of this section, which is an integral representation given in Theorem 1 below.

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[8]CHARLES HERMITE (1822–1901), French mathematician, is known for his work in algebra and number theory. The great HENRI POINCARÉ (1854–1912) was one of his students.