Chapter 4
Shear Forces and Bending Moments

4.1 Introduction

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4.3 Shear Forces and Bending Moments

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4.1 INTRODUCTION

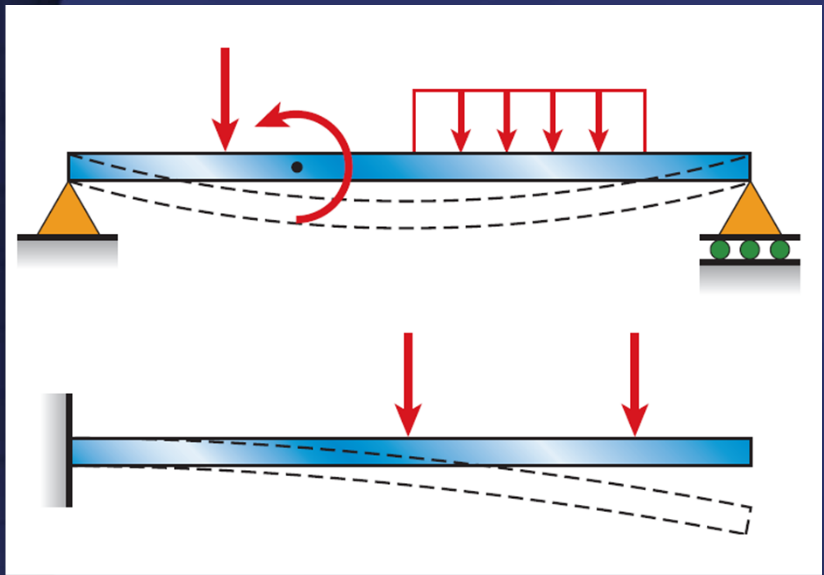


FIG. 4-1 Examples of beams subjected to lateral loads

(1) Beams
(2) Plane of bending
(3) Planar structures

4.2 TYPES OF BEAMS, LOADS, AND REACTIONS

(1) Simply supported beam
(2) Roller support
(3) Cantilever beam
(4) Fixed support
(5) Beam with an overhang
(6) Conventional symbols

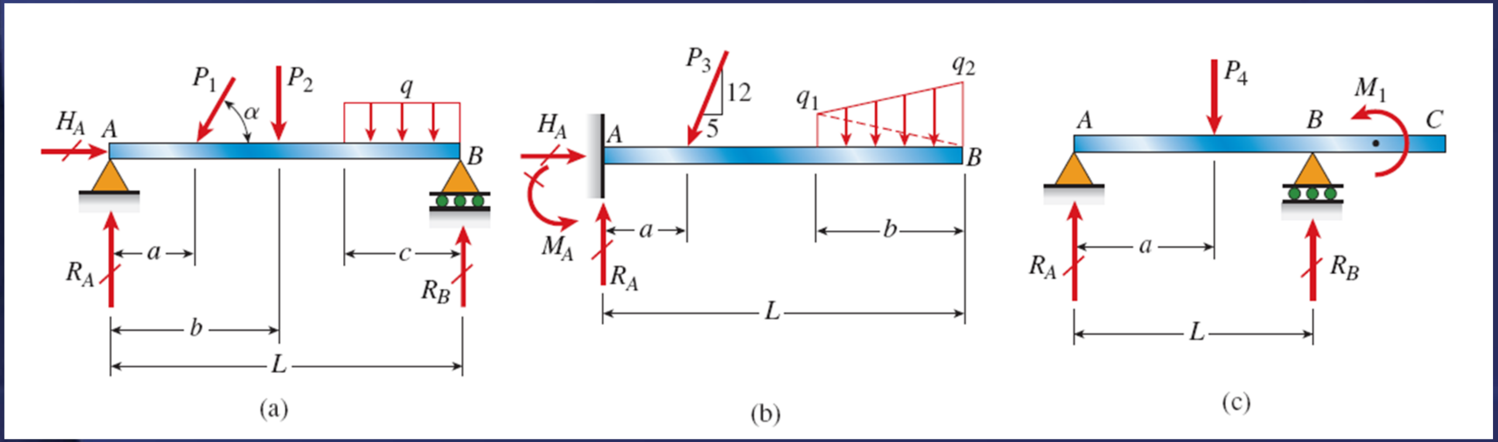
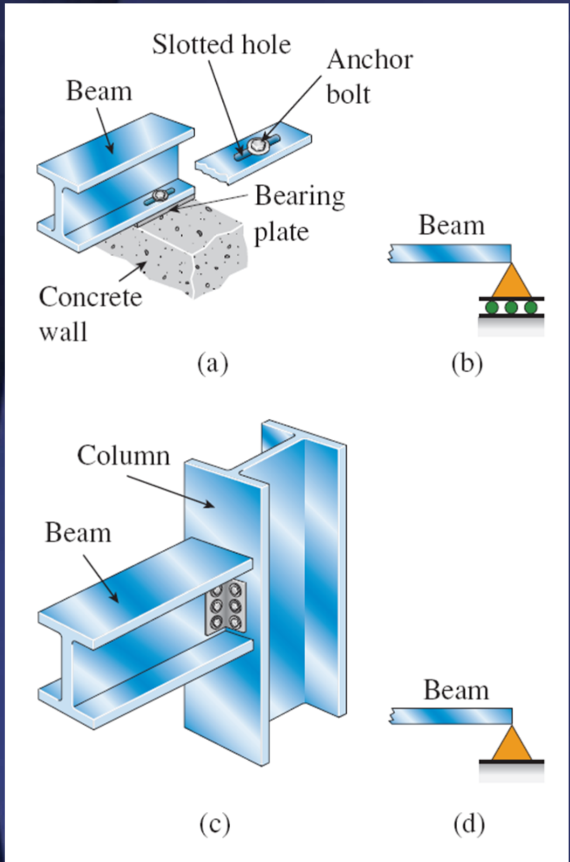


FIG. 4-2 Types of beams: (a) simple beam, (b) cantilever beam, and (c) beam with an overhang

Types of Loads

(1)Concentrated load
(2)Distributed load intensity
(3)Uniformly distributed load or uniform load
(4)Linearly varying load
(5)Couple



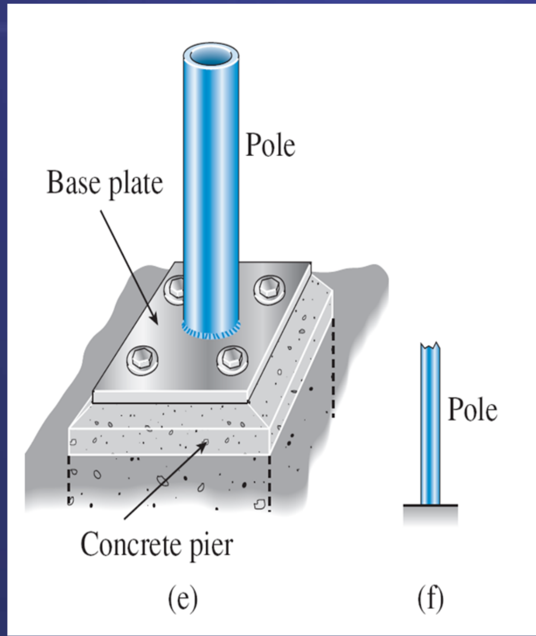


FIG. 4-3 Pole anchored to a concrete pier:
(e) actual construction, and
(f) representation as a fixed support

FIG. 4-3 Beam supported on a wall:
(a) actual construction, and
(b) representation as a roller support.
Beam-to-column connection:
(c) actual construction, and
(d) representation as a pin support.

The equation of horizontal equilibrium is

$$\sum F_{\text{horiz}} = 0 \qquad H_A - P_1 \cos\alpha = 0$$

$$H_A = P_1 \cos\alpha$$

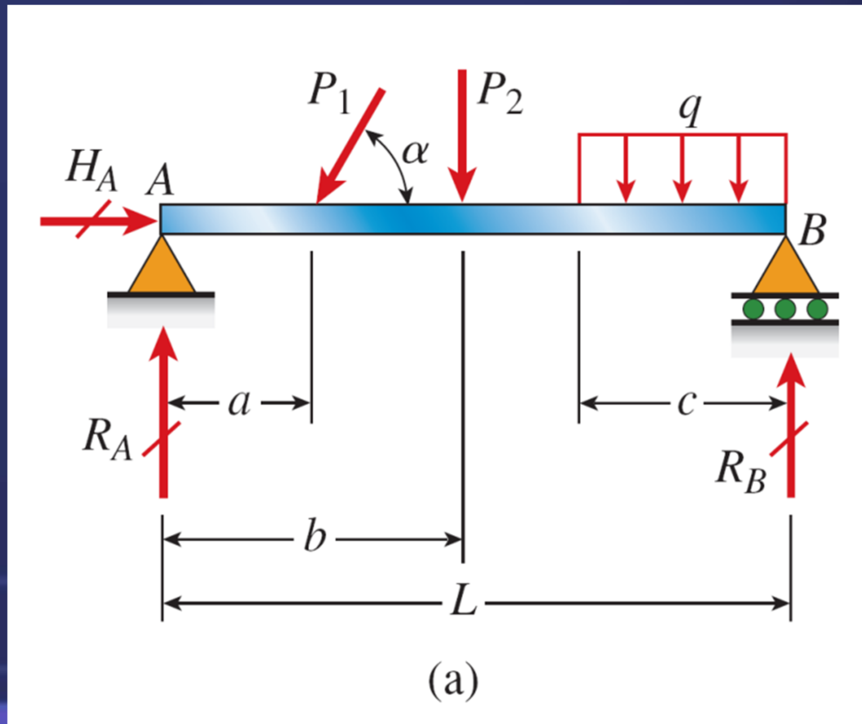


FIG. 4-2a Simple beam. (Repeated)

$$\sum M_B = 0 \qquad -R_A L + (P_1 \sin\alpha)(L-a) + P_2(L-b) + qc^2/2 = 0$$

$$\sum M_A = 0 \qquad R_B L - (P_1 \sin\alpha)(a) - P_2 b - qc(L - c/2) = 0$$

Solving for R_A and R_B, we get

$$R_A = \frac{(P_1 \sin\alpha)(L-a)}{L} + \frac{P_2(L-b)}{L} + \frac{qc^2}{2L}$$

$$R_B = \frac{(P_1 \sin\alpha)(a)}{L} + \frac{P_2 b}{L} + \frac{qc(L-c/2)}{L}$$

$$\sum F_{\text{horiz}} = 0 \qquad H_{\text{A}} = \frac{5P_3}{13} \qquad\qquad \sum F_{vert} = 0 \qquad R_A = \frac{12P_3}{13} + \left(\frac{q_1 + q_2}{2}\right)b$$

$$\sum M_A = 0 \qquad M_A - \left(\frac{12P_3}{13}\right)a - \frac{q_1 b}{2}\left(L - \frac{2b}{3}\right) - \frac{q_2 b}{2}\left(L - \frac{b}{3}\right) = 0$$

from which

$$M_A = \frac{12P_3 a}{13} + \frac{q_1 b}{2}\left(L - \frac{2b}{3}\right) + \frac{q_2 b}{2}\left(L - \frac{b}{3}\right)$$

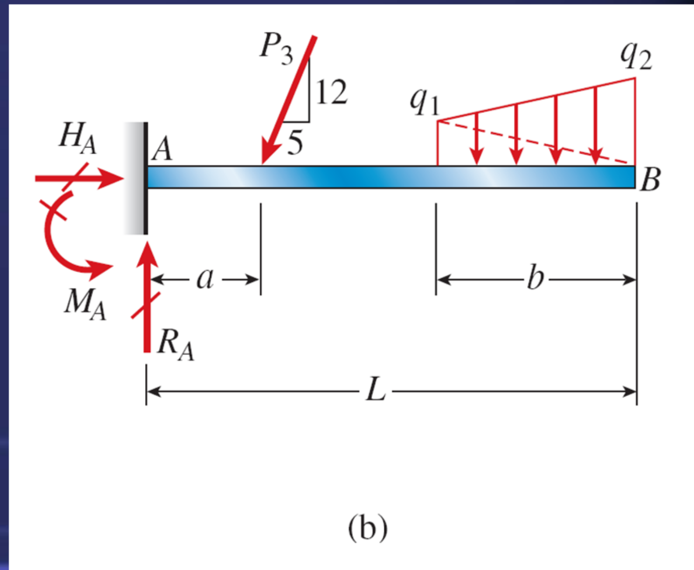


FIG. 4-2b Cantilever beam.
(Repeated)

4.3 SHEAR FORCES AND BENDING MOMENTS

(1)Shear force
(2)Bending moment
(3)Stress resultants

$$\sum F_{vert} = 0 \qquad P - V = 0 \quad \text{or} \quad V = P$$

$$\sum M = 0 \qquad M - Px = 0 \quad \text{or} \quad M = Px$$

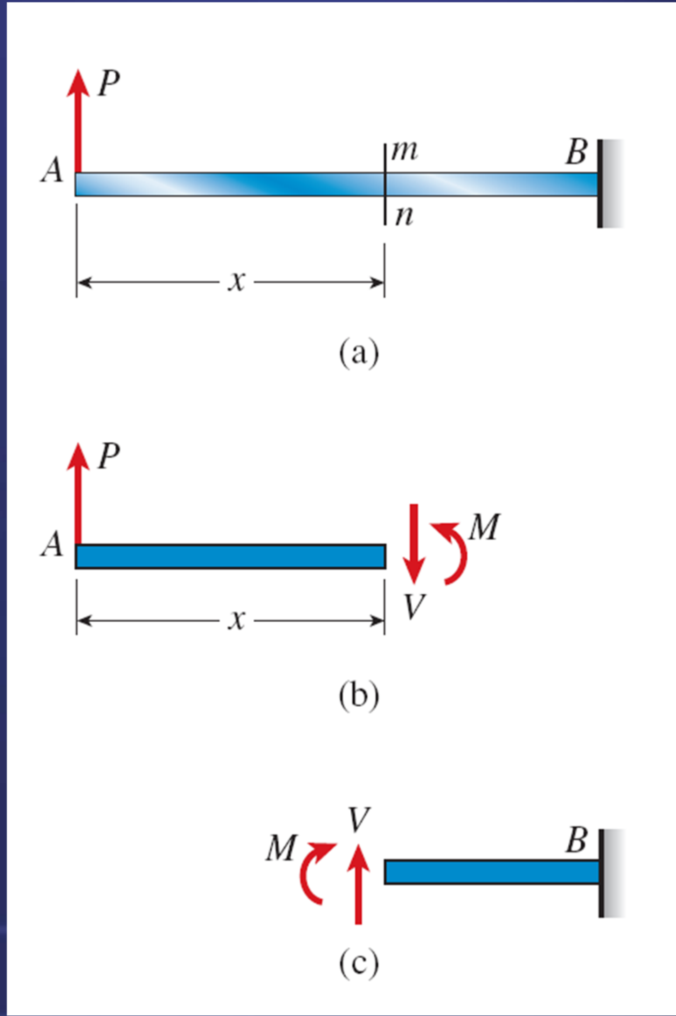


FIG. 4-8 Shear force *V and bending moment M in a beam*

Sign Conventions

(1) Deformation
(2) Static sign conventions

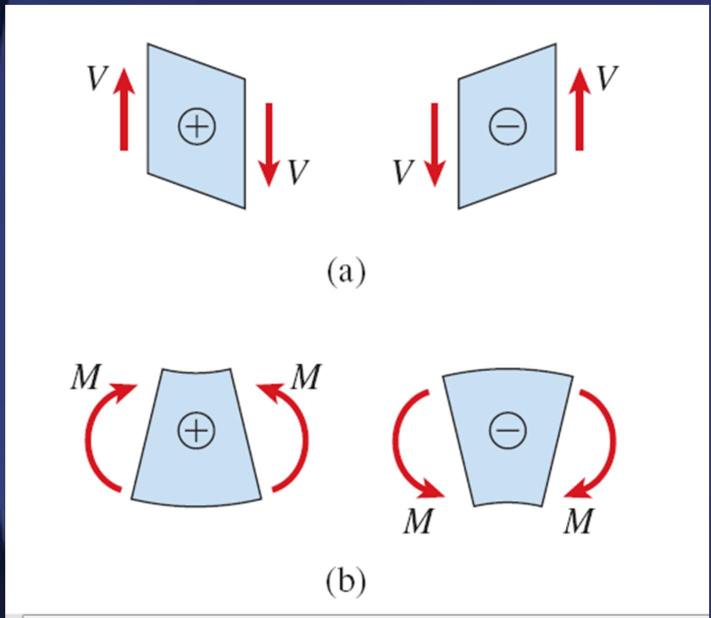


FIG. 4-10 Deformations (highly exaggerated) of a beam element caused by
(a) shear forces, and
(b) bending moments

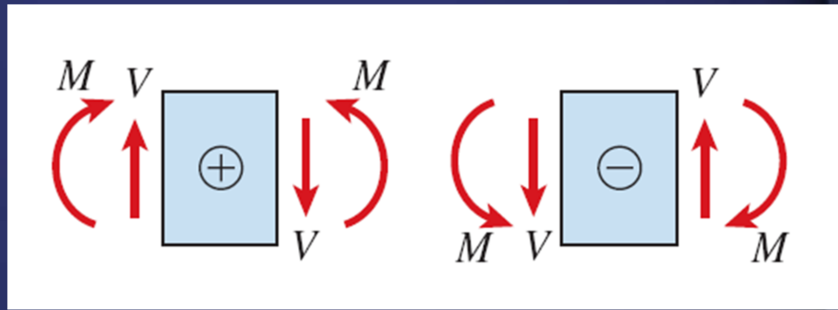


FIG. 4-9 Sign conventions for shear force *V* and bending moment *M*

Example 4-1

A simple beam AB supports two loads, a force P and a couple M_0, acting as shown in Fig. 4-11a.

Find the shear force V and bending moment M in the beam at cross sections located as follows:

(a) a small distance to the left of the midpoint of the beam, and

(b) a small distance to the right of the midpoint of the beam.

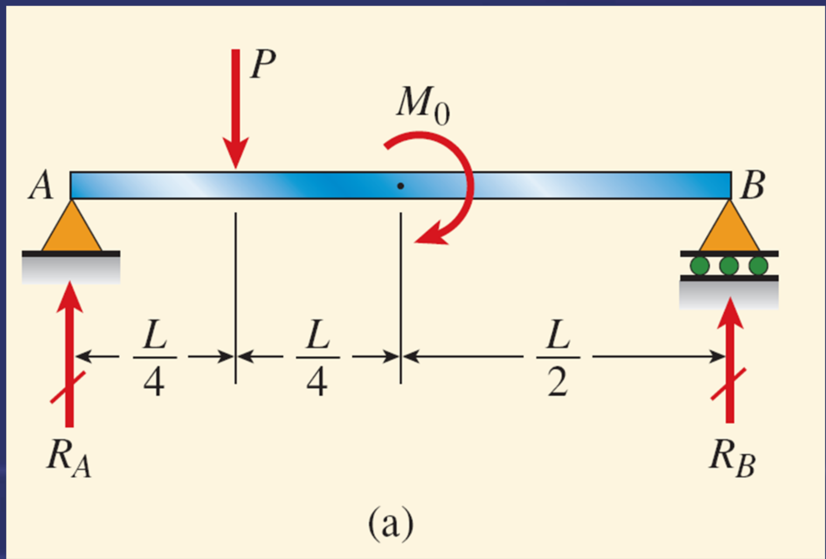


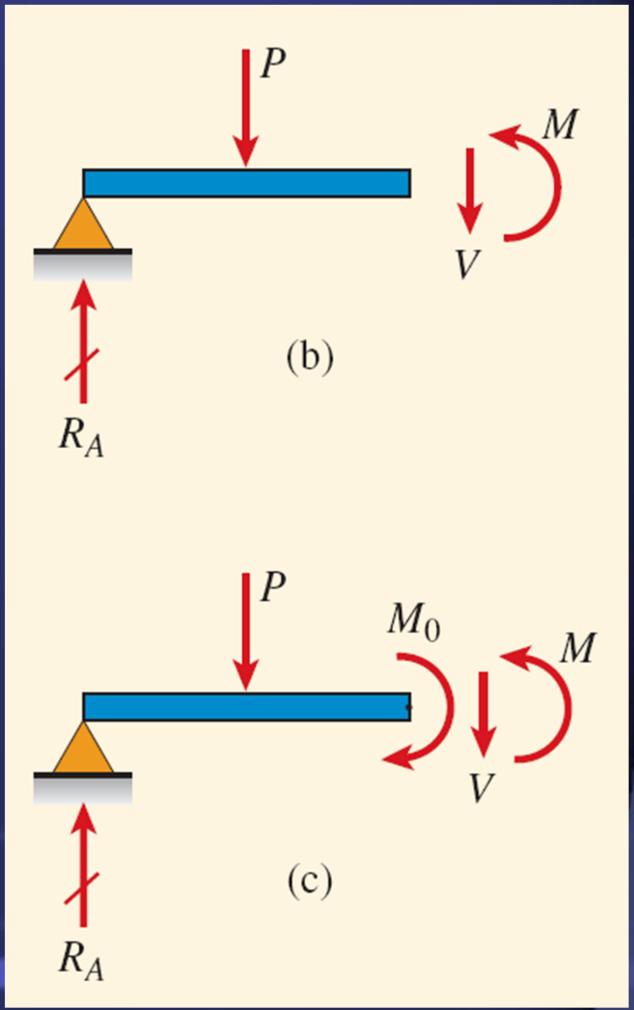
FIG. 4-11 Example 4-1. Shear forces and bending moment in a simple beam

Solution

Reactions.

$$\sum M_B = 0 \qquad R_A = \frac{3P}{4} - \frac{M_0}{L}$$

$$\sum M_A = 0 \qquad R_B = \frac{P}{4} + \frac{M_0}{L} \qquad \text{(a)}$$



(a) Shear force and bending moment to the left of the midpoint.

$$\sum F_{vert} = 0 \qquad R_A - P - V = 0$$

$$V = R_A - P = -\frac{P}{4} - \frac{M_0}{L} \qquad \text{(b)}$$

$$\sum M = 0 \qquad -R_A\left(\frac{L}{2}\right) + P\left(\frac{L}{4}\right) + M = 0$$

$$M = R_A\left(\frac{L}{2}\right) - P\left(\frac{L}{4}\right) = \frac{PL}{8} - \frac{M_0}{2} \qquad \text{(c)}$$

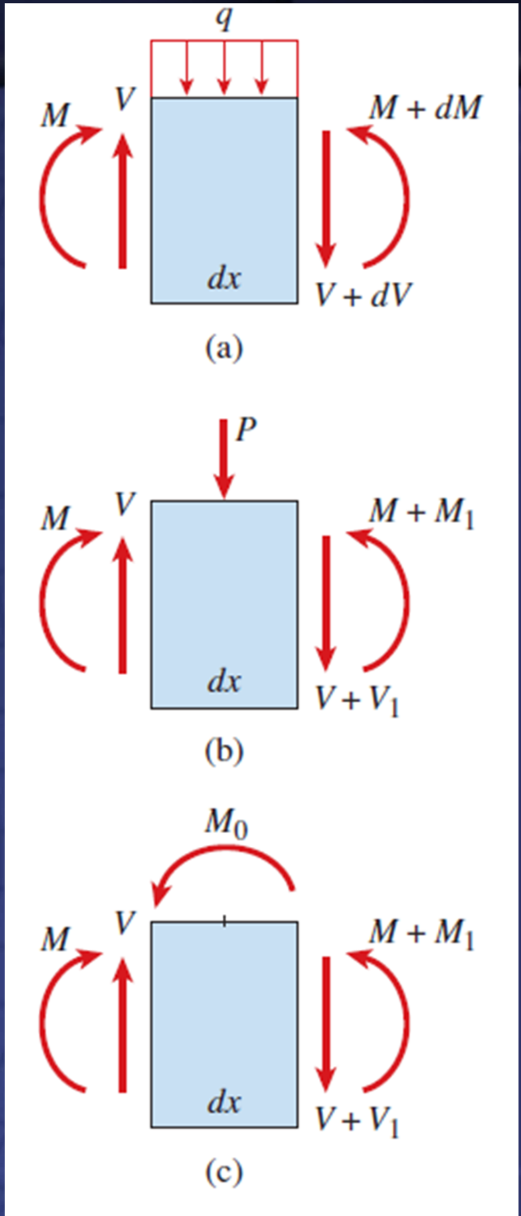
(b) Shear force and bending moment to the right of the midpoint.

$$V = -\frac{P}{4} - \frac{M_0}{L} \qquad M = \frac{PL}{8} + \frac{M_0}{2} \qquad \text{(d,e)}$$

4.4 RELATIONSHIPS BETWEEN LOADS, SHEAR FORCES, AND BENDING MOMENTS

Sign conventions

(1)Distributed loads
(2)Concentrated loads
(3)Couple



Distributed Loads (Fig. 4-14a)

$$\sum F_{vert} = 0 \qquad V - q\,dx - (V + dV) = 0$$

$$\frac{dV}{dx} = -q \qquad \text{(4-4)}$$

dV/dx = 0 and the shear force is constant in that part of the beam.

FIG. 4-14 Element of a beam used in deriving the relationships between loads, shear forces, and bending moments. (All loads and stress resultants are shown in their positive directions.)

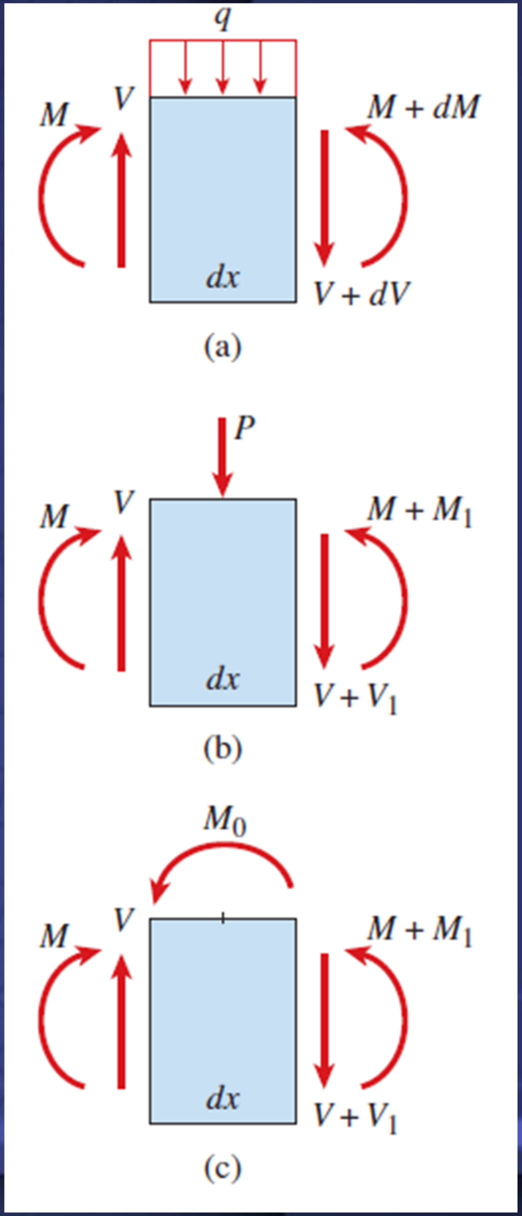
The shear forces at two different cross sections of a beam is

$$\int_A^B dV = -\int_A^B q\ dx \qquad \text{(a)}$$

Thus

$$V_B - V_A = -\int_A^B q\ dx$$

$$= -\left(\text{area of the loading diagram between A and B}\right) \qquad \text{(4-5)}$$



$$\sum M = 0$$

$$-M - q\,dx\left(\frac{dx}{2}\right) - (V + dV)\,dx + M + dM = 0$$

$$\frac{dM}{dx} = V \qquad (4\text{-}6)$$

Integrating Eq. (4-6) between two points *A and B on the beam axis* gives

$$\int_A^B dM = \int_A^B V\ dx \qquad \text{(b)}$$

Therefore, we can express Eq. (b) in the following manner:

$$M_B - M_A = \int_A^B V\ dx \qquad \text{(4-7)}$$

= (area of the shear-force diagram between *A and B)*

Concentrated Loads (Fig. 4-14b)

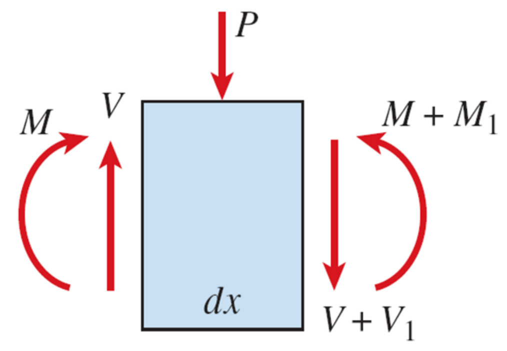
$$V - P - (V + V_1) = 0 \qquad \text{or} \qquad V_1 = -P \qquad \text{(4-8)}$$

From equilibrium of forces in the vertical direction, we get

$$-M - P\left(\frac{dx}{2}\right) - (V + V_1)\,dx + M + M_1 = 0$$

or

$$M_1 = P\left(\frac{dx}{2}\right) + V\,dx + V_1\,dx \qquad \text{(c)}$$



(b)

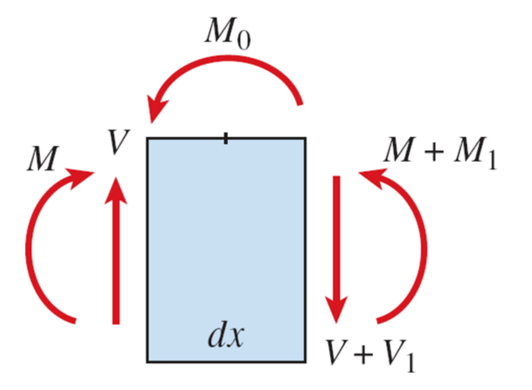
Therefore, at the point of application of a concentrated load P, the rate of change dM/dx of the bending moment decreases abruptly by an amount equal to P.

Loads in the Form of Couples (Fig. 4-14c)

From equilibrium of the element in the vertical direction we obtain $V_1 = 0$, *which shows that the shear force does not change at the point of application of a couple.*

$$-M + M_0 - (V + V_1)\,dx + M + M_1 = 0$$

$$M_1 = -M_0 \qquad \text{(4-9)}$$



(c)

Thus, the bending moment changes abruptly at the point of application of a couple.

distributed loads

$$\frac{dV}{dx} = -q \qquad \text{(4-1)}$$

$$V_B - V_A = -\int_A^B q\,dx$$
$$= -(\text{area of the loading diagram between } A \text{ and } B) \qquad \text{(4-3)}$$

$$\frac{dM}{dx} = V \qquad \text{(4-4)}$$

$$M_B - M_A = \int_A^B V\,dx$$
$$= (\text{area of the shear-force diagram between } A \text{ and } B) \qquad \text{(4-6)}$$

concentrated loads

$$V_1 = -P$$

couple

$$M_1 = -M_0 \qquad \text{(4-9)}$$

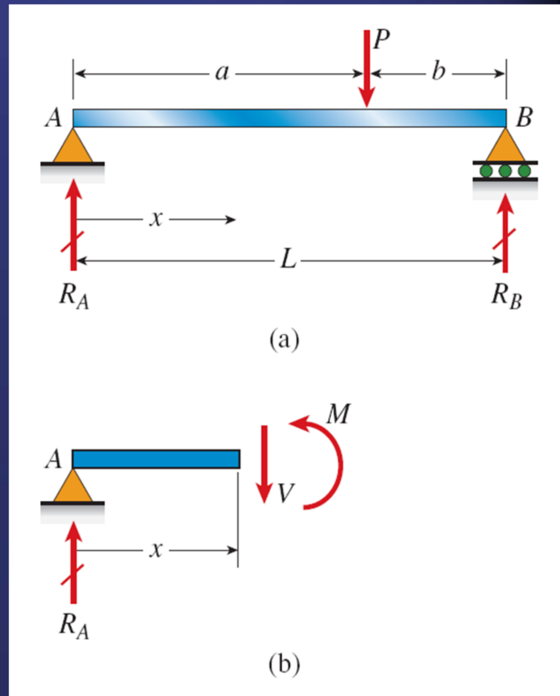
4.5 SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS

(1) Shear-force diagrams
(2) Bending-moment diagrams

Concentrated Load

The reactions of the beam at A, B are

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L} \qquad \text{(4-10a,b)}$$



We now cut through the beam at a cross section to the left of the load *P and at distance x from the support at A.*

$$V = R_A = \frac{Pb}{L} \qquad M = R_A x = \frac{Pbx}{L} \qquad (0 < x < a) \qquad \text{(4-11a,b)}$$

Next, we cut through the beam to the right of the load *P (that is, in the* region *a < x < L)*

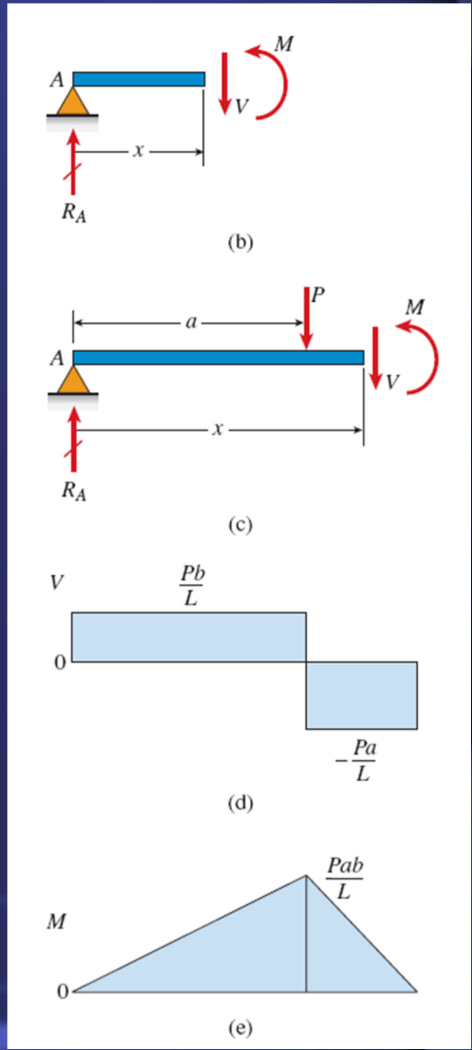
$$V = R_A - P = \frac{Pb}{L} - P = -\frac{Pa}{L} \qquad (a < x < L) \qquad \text{(4-12a)}$$

$$M = R_A x - P(x-a) = \frac{Pbx}{L} - P(x-a)$$

$$= \frac{Pa}{L}(L-x) \qquad (a < x < L) \qquad \text{(4-12b)}$$

$$M_{\max} = \frac{Pab}{L} \qquad \text{(4-13)}$$

Certain characteristics of the shear-force and bending moment diagrams

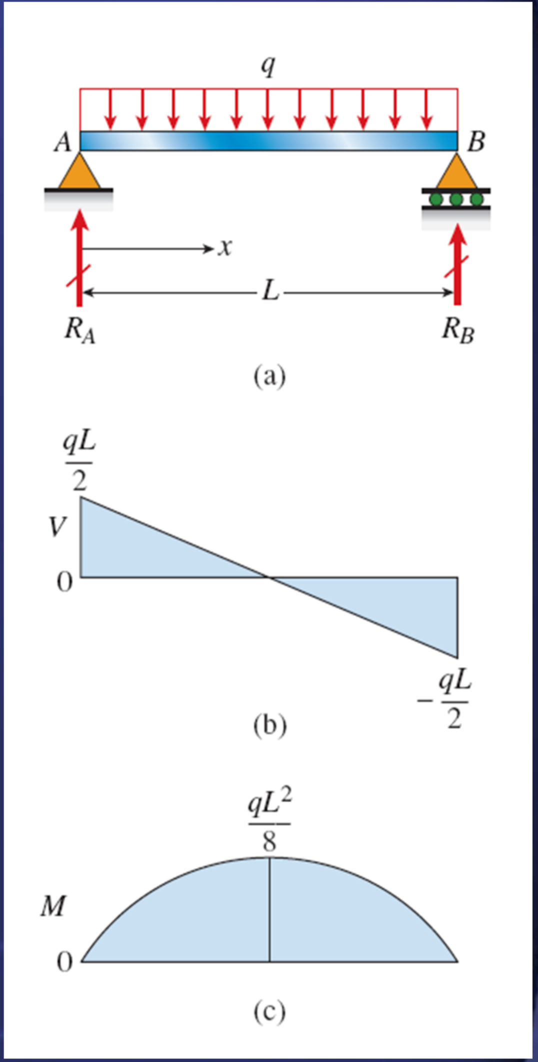


Uniform Load

The shear force and bending moment at distance *x from the left-hand end are*

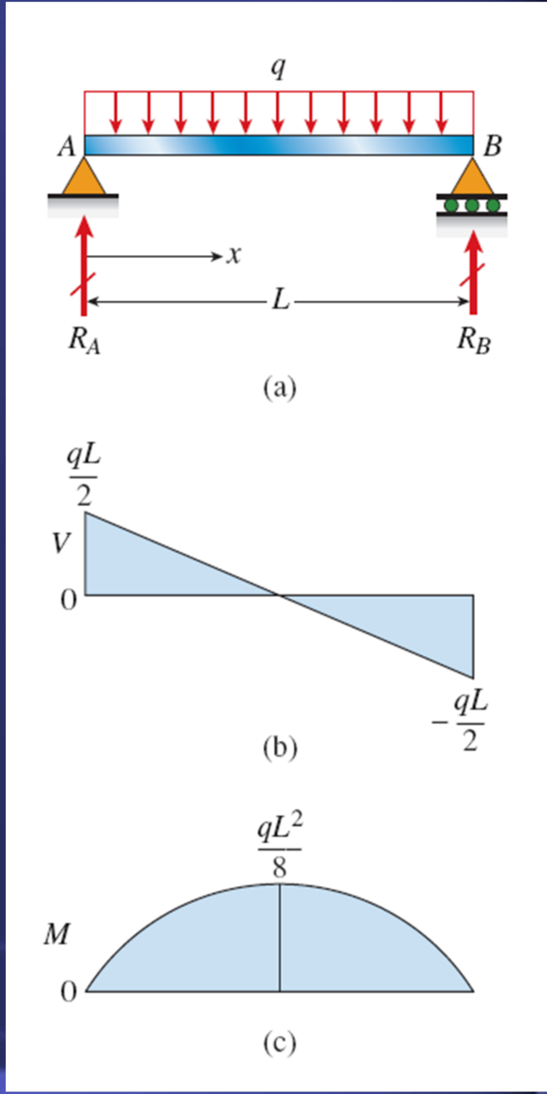
$$V = R_A - qx = \frac{qL}{2} - qx \qquad (4\text{-}14a)$$

$$M = R_A x - qx\left(\frac{x}{2}\right) = \frac{qLx}{2} - \frac{qx^2}{2} \qquad (4\text{-}14b)$$



At each cross section the slope of the bending-moment diagram is equal to the shear force (see Eq. 4-6):

$$\frac{dM}{dx}=\frac{d}{dx}\left(\frac{qLx}{2}-\frac{qx^2}{2}\right)=\frac{qL}{2}-qx=V$$



The maximum value of the bending moment occurs at the midpoint of the beam

$$M_{\max}=\frac{qL^2}{8} \qquad \text{(4-15)}$$

Several Concentrated Loads

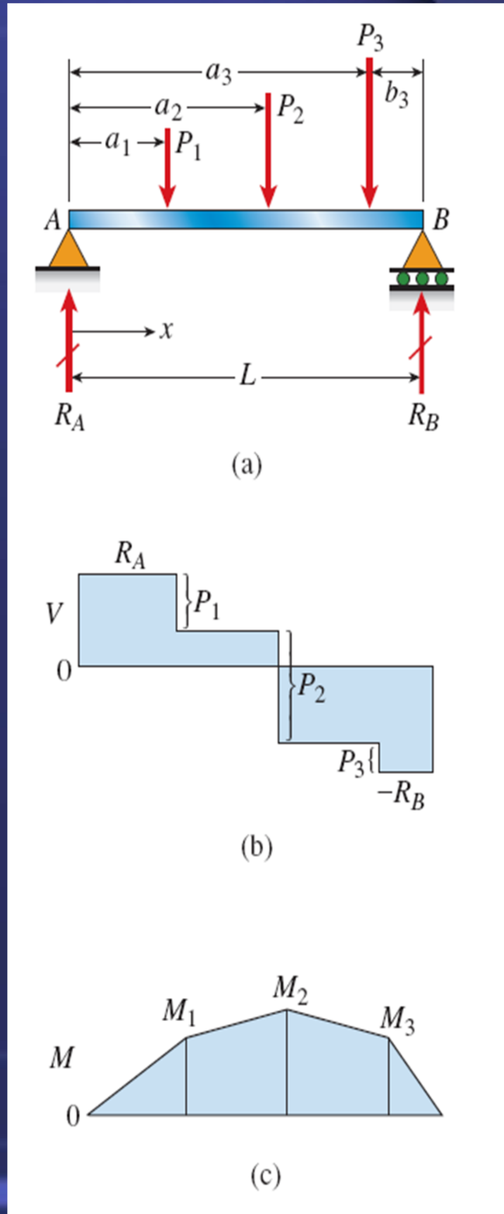
for the first segment of the beam

$$V = R_A \qquad M = R_A x \quad \left(0 < x < a_1\right) \qquad \text{(4-16a,b)}$$

For the second segment, we get

$$V = R_A - P_1$$

$$M = R_A x - P_1\left(x - a_1\right) \quad \left(a_1 < x < a_2\right) \qquad \text{(4-17a,b)}$$



For the third segment of the beam,

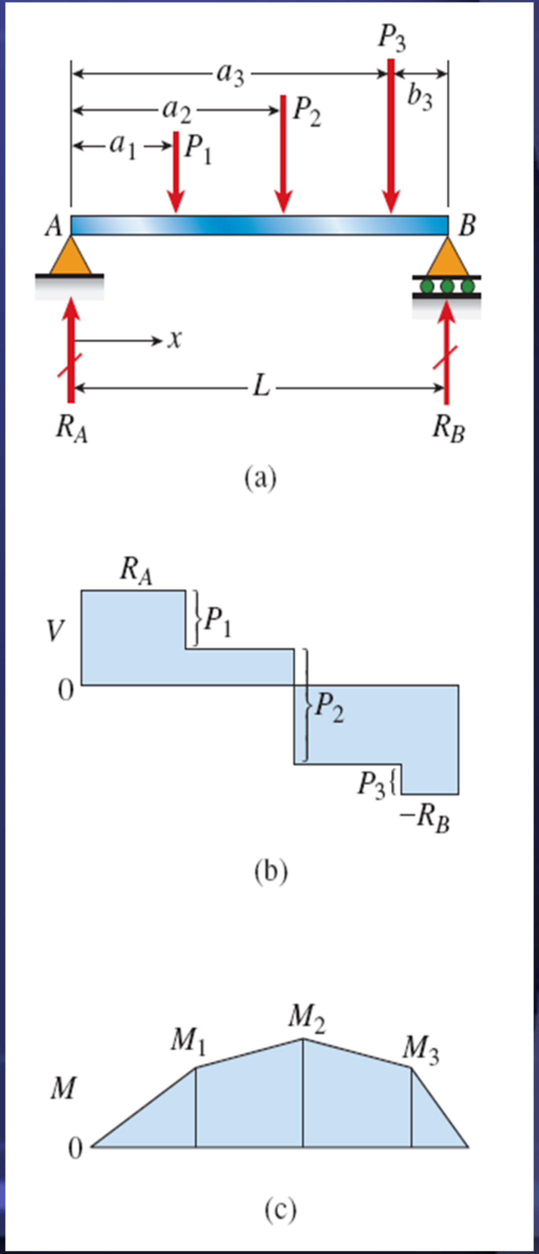
$$V = -R_B + P_3 \qquad \text{(4-18a)}$$

$$M = R_B(L-x) - P_3(L-b_3-x) \quad (a_2 < x < a_3) \qquad \text{(4-18b)}$$

Finally, for the fourth segment of the beam, we obtain

$$V = -R_B$$

$$M = R_B(L-x) \qquad (a_3 < x < L) \qquad \text{(4-19a,b)}$$

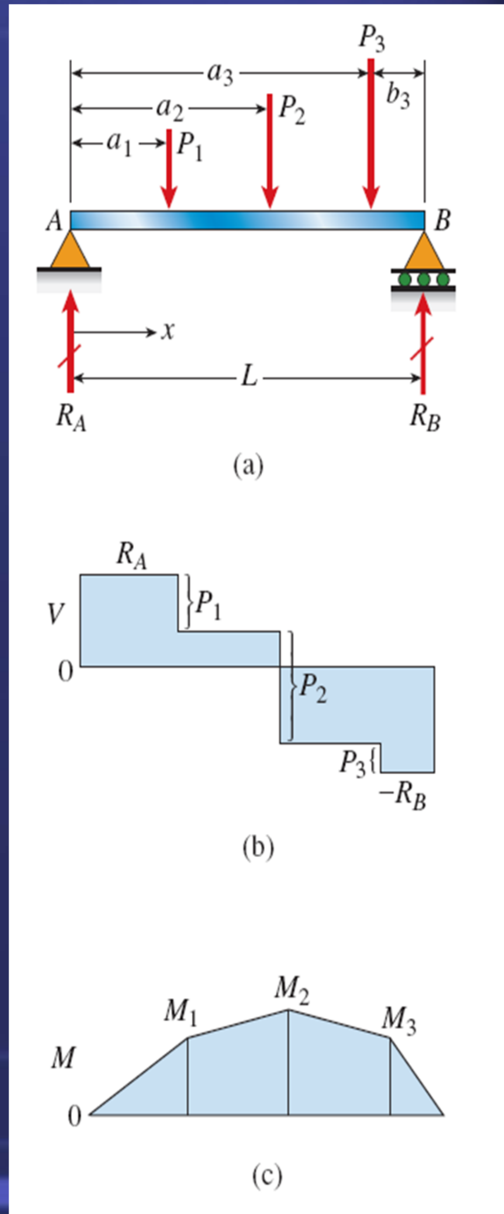


To assist in drawing these lines, we obtain the bending moments under the concentrated loads by substituting $x = a_1$, $x = a_2$, and $x = a_3$ into Eqs. (4-16b), (4-17b), and (4-18b), we obtain the following bending moments:

$$M_1 = R_A a_1$$

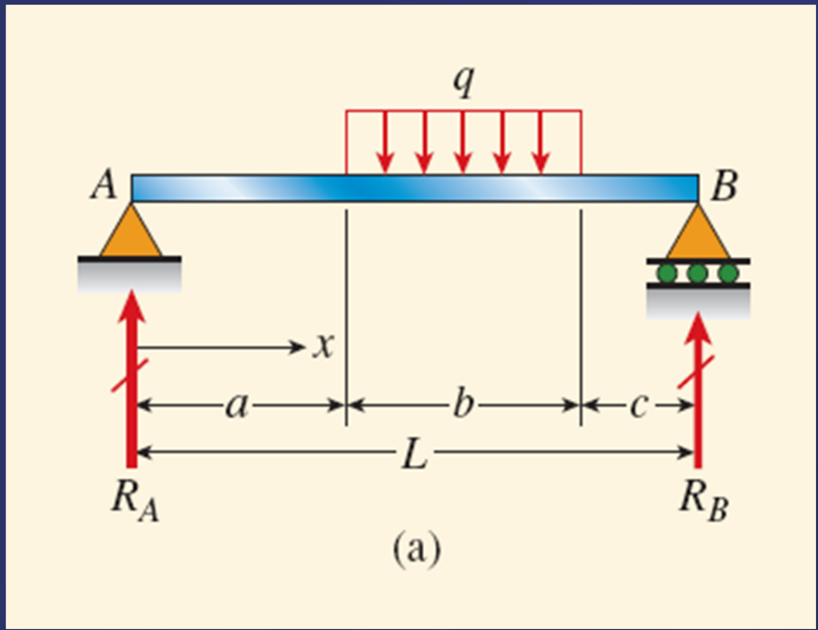
$$M_2 = R_A a_2 - P\left(a_2 - a_1\right)$$

$$M_3 = R_B b_3 \qquad \text{(4-20a,b,c)}$$



Example 4-4

Draw the shear-force and bending-moment diagrams for a simple beam with a uniform load of intensity q *acting over part of the span (Fig. 4-18a).*

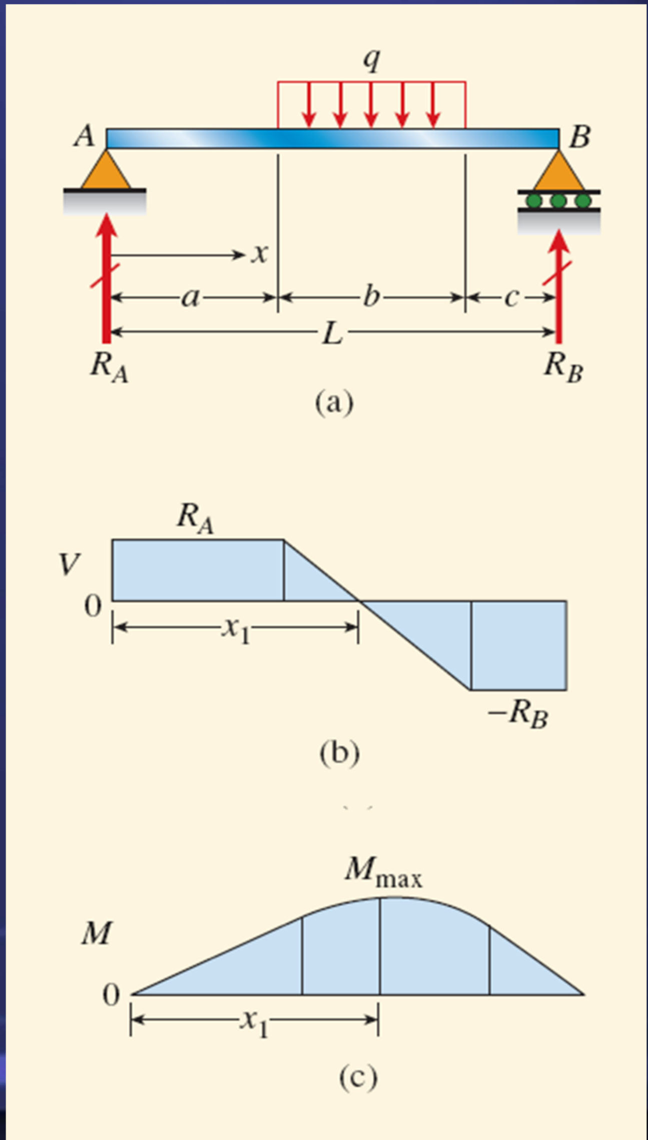


Solution

Reactions.

$$R_A = \frac{qb\left(b+2c\right)}{2L}$$

$$R_B = \frac{qb\left(b+2a\right)}{2L} \qquad \text{(4-21a,b)}$$

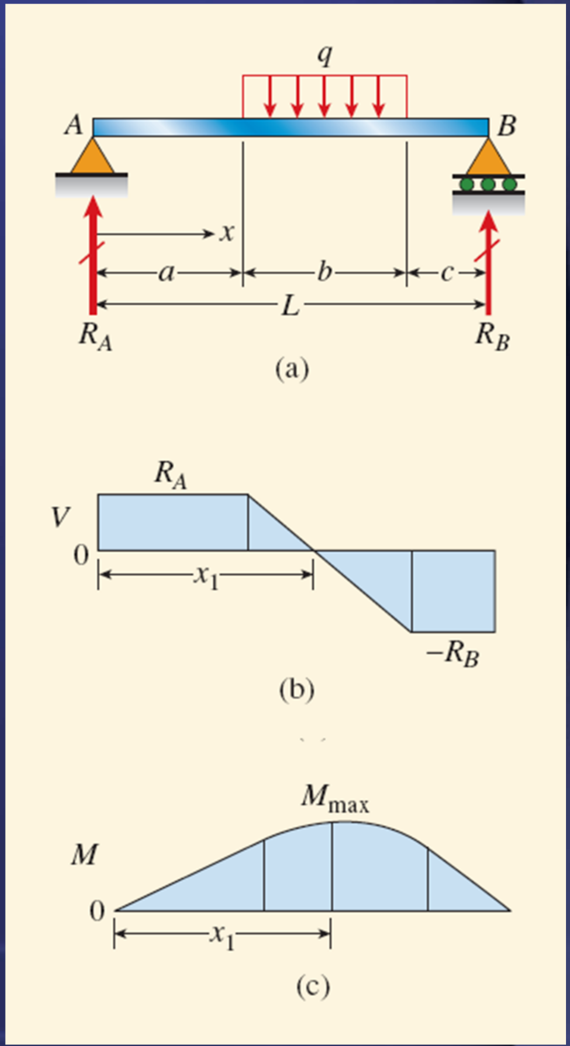


Shear forces and bending moments

$$V = R_A$$
$$M = R_A x \qquad (0 < x < a) \qquad \text{(4-22a,b)}$$

$$V = R_A - q(x-a)$$
$$M = R_A x - \frac{q(x-a)^2}{2} \qquad (a < x < a+b) \qquad \text{(4-23a,b)}$$

$$V = -R_B$$
$$M = R_B(L-x) \qquad (a+b < x < L) \qquad \text{(4-24a,b)}$$



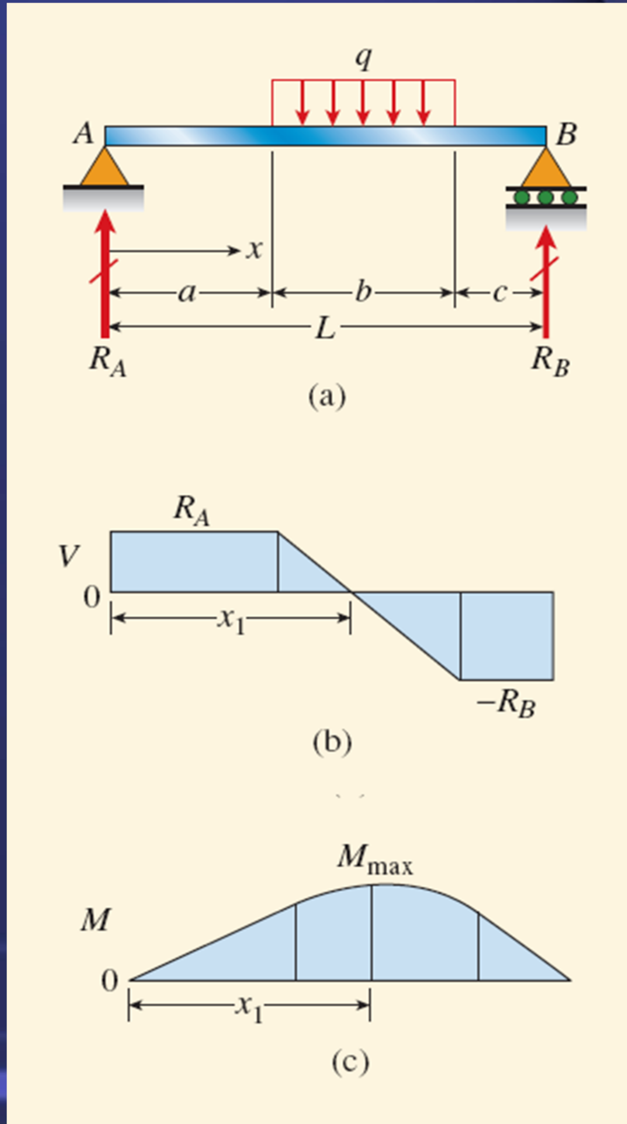
Maximum bending moment. The maximum moment occurs where the shear force equals zero.

For $r_s = 0$

$$x_1 = a + \frac{b}{2L}\left(b + 2c\right) \qquad (4\text{-}25)$$

The maximum moment.

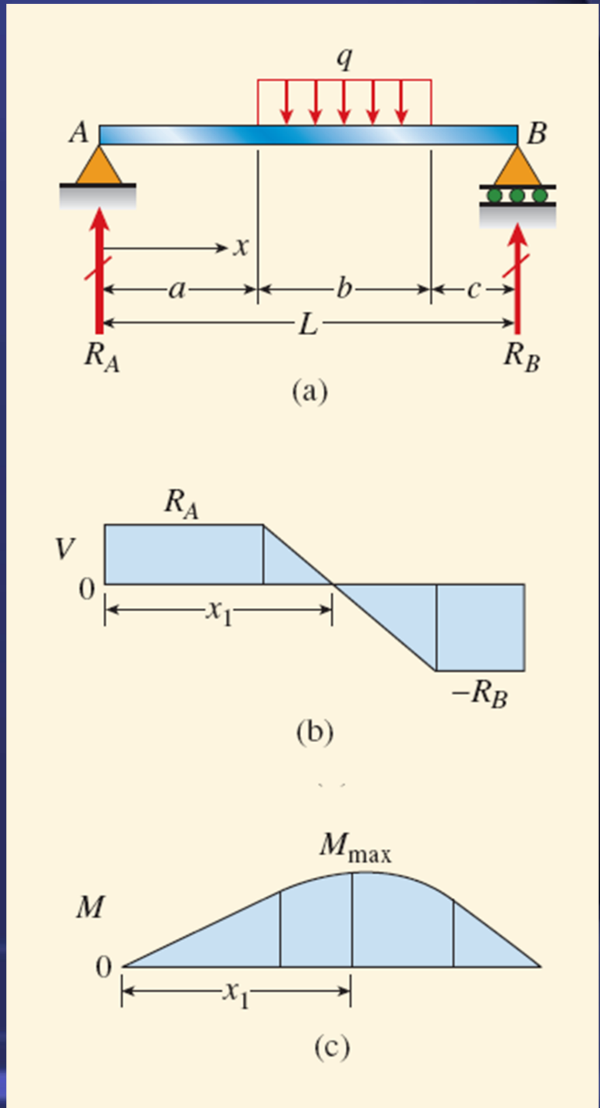
$$M_{\max} = \frac{qb}{8L^2}\left(b + 2c\right)\left(4aL + 2bc + b^2\right) \qquad (4\text{-}26)$$



Special cases.

For $a = c$

$$x_1 = \frac{L}{2} \qquad M_{\max} = \frac{qb(2L-b)}{8} \qquad \text{(4-27a,b)}$$



The End of Chap. 4