Contents

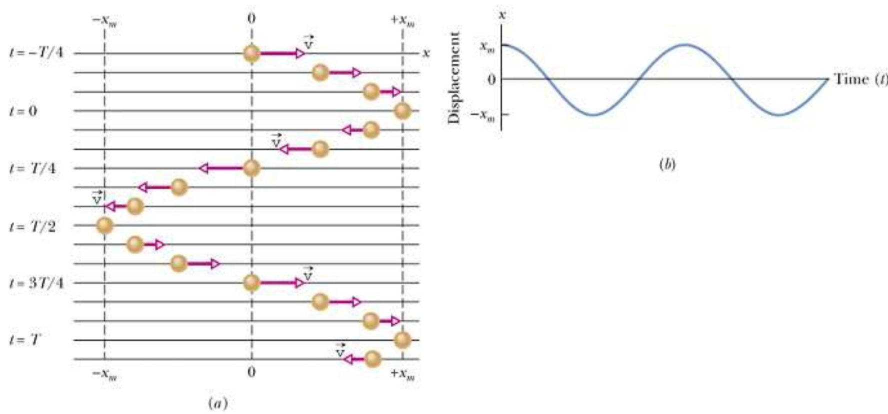
15 Oscillations

In this chapter we will cover the following topics: Displacement, velocity and acceleration of a simple harmonic oscillator, Energy of a simple harmonic oscillator.

Examples of simple harmonic oscillators: spring-mass system, simple pendulum, physical pendulum, torsion pendulum, Damped harmonic oscillator, Forced oscillations/Resonance.

15.1 Simple Harmonic Motion (SHM)

In the following, we show snapshots of a simple oscillating system.



The motion is periodic i.e. it repeats in time. The time needed to complete one repetition is known as the period (symbol T, units: s). The number of repetitions per unit time is called the frequency (symbol f , unit hertz)

$$f = \frac{1}{T}$$

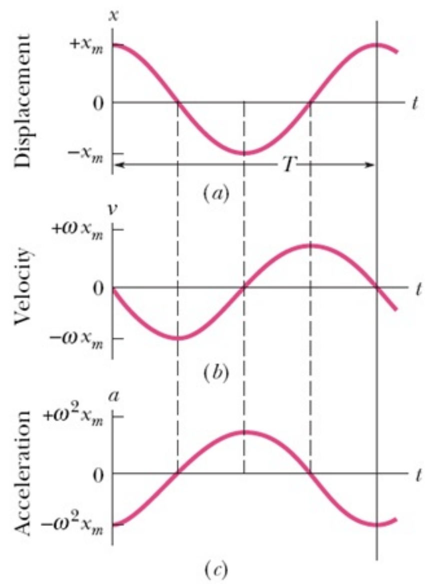
The displacement of the particle is given by the equation:

$$x(t) = x_m \cos(\omega t + \phi)$$

Fig. b is a plot of $x(t)$ versus t. The quantity x_m is called the amplitude of the motion. It gives the maximum possible displacement of the oscillating object. The quantity ω is called the angular frequency of the oscillator. It is given by the equation:

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The quantity ϕ is called the phase angle of the oscillator. The value of ϕ is determined from the displacement $x(0)$ and the velocity $v(0)$ at $t = 0$. In fig. a below, $x(t)$ is plotted versus t for $\phi = 0$. $x(t) = x_m \cos \omega t$



15.1.1 Velocity of SHM:

$$v(t) = \frac{dx(t)}{dt} = \frac{d(x_m \cos(\omega t + \phi))}{dt} = -\omega x_m \sin(\omega t + \phi)$$

The quantity ωx_m is called the velocity amplitude v_m. It expresses the maximum possible value of $v(t)$. In fig. b the velocity $v(t)$ is plotted versus t for $\phi = 0$. $v(t) = -\omega x_m \sin \omega t$.

15.1.2 Acceleration of SHM:

$$a(t) = \frac{dv(t)}{dt} = \frac{d(-\omega x_m \sin(\omega t + \phi))}{dt} = -\omega^2 x_m \cos(\omega t + \phi)$$

The quantity $\omega^2 x_m$ is called the acceleration amplitude a_m. It expresses the maximum possible value of $a(t)$. In fig. c the acceleration $a(t)$ is plotted versus t for $\phi = 0$. $a(t) = -\omega^2 x_m \cos(\omega t)$

15.2 The Force Law for Simple Harmonic Motion

We saw that the acceleration of an object undergoing SHM is:

$$a = -\omega^2 x$$

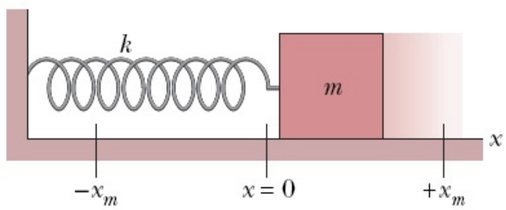
If we apply Newton's second law we get:

$$F = ma = -m\omega^2 x$$

Simple harmonic motion occurs when the force acting on an object is proportional to the displacement but opposite in sign. The force is given by Hooke's law and can be written as:

$$F = m\frac{d^2x}{dt} = -kx \tag{1}$$

where k is a constant.



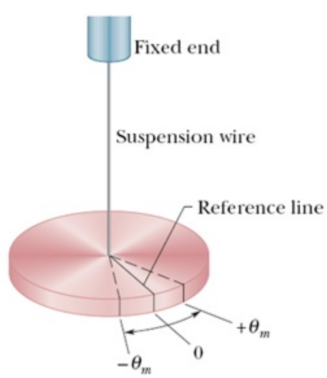
If we compare the above two expressions for F we have: $m\omega^2 = k$ and

$$\omega = \sqrt{\frac{k}{m}} \rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

15.3 An Angular Simple Harmonic Oscillator

15.3.1 Torsion Pendulum

In the figure below, we show another type of oscillating system.



It consists of a disc of rotational inertia I suspended from a wire that twists as m rotates by an angle θ . The wire exerts on the disc a restoring torque

$$\tau = -\kappa\theta$$

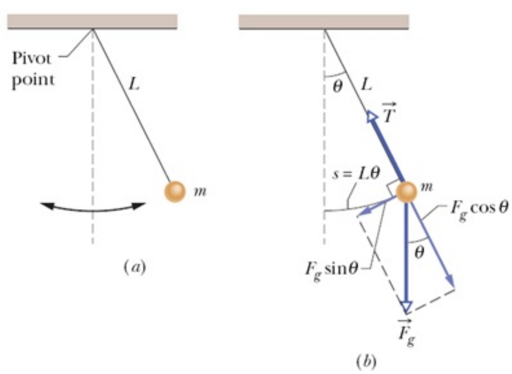
This is the angular form of Hooke's law. The constant κ is called the torsion constant of the wire. If we compare the expression $\tau = -\kappa\theta$ for the torque with the force equation $F = -kx$ we realize that we identify the constant k with the torsion constant κ . We can thus readily determine the angular frequency ω and the period T of the oscillation.

$$\omega = \sqrt{\frac{\kappa}{I}}, T = 2\pi\sqrt{\frac{I}{\kappa}}$$

We note that I is the rotational inertia of the disc about an axis that coincides with the wire. The angle θ is given by the equation:

$$\theta\left(t\right) = \theta_m \cos\left(\omega t + \phi\right)$$

15.3.2 The Simple Pendulum



A simple pendulum consists of a particle of mass m suspended by a string of length L from a pivot point. If the mass is disturbed from its equilibrium position the net force acting on it is such that the system executes simple harmonic motion. There are two forces acting on m: The gravitational force and the tension from the string. The net torque of these forces is:

$$\vec{\tau} = \vec{L} \times m\vec{g}$$
$$\vec{\tau} \cdot \hat{n} = -Lmgsin\theta$$

Here $\hat{n}$ is the unit vector pointing toward the classroom. θ is the angle that the thread makes with the vertical. If $\theta \ll 1$, then we can make the following approximation:

$$sin\theta \simeq \theta$$

where θ is expressed in radians. With this approximation,

$$\vec{\tau} \cdot \hat{n} = I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2} \simeq -Lmg\theta$$

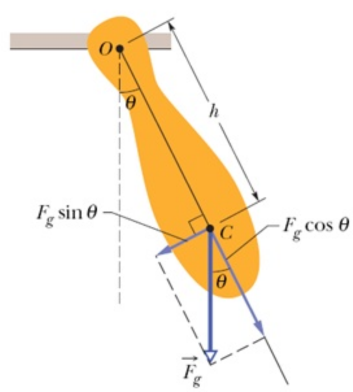
If we compare the expression for τ with the force equation $F = -kx$ we realize that we identify the constant k with the term Lmg. We can thus readily determine the angular frequency ω.

$$\omega = \sqrt{\frac{Lmg}{I}} = \sqrt{\frac{g}{L}}$$

where we have identified the rotational inertia I about the pivot point to be equal to mL^2. Thus the period T of the oscillation

$$T = 2\pi\sqrt{\frac{L}{g}}$$

15.3.3 Physical Pendulum



A physical pendulum is an extended rigid body that is suspended from a fixed point O and oscillates under the influence of gravity. The net torque

$$\vec{\tau} \cdot \hat{n} = -mgh\sin\theta$$

Here $\hat{n}$ is the unit vector pointing toward the classroom, h is the distance between point O and the center of mass C of the suspended body. If we make the small angle approximation θ, we have:

$$\vec{\tau} \cdot \hat{n} = I \frac{d^2\theta}{dt} \simeq -mgh\theta$$

If we compare the torques with the force equation $F = -kx$, we realize that we identify the constant k with the term hmg. We can thus readily determine

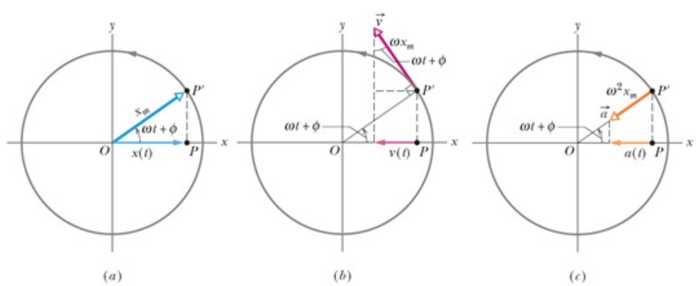
$$\omega = \sqrt{\frac{mgh}{I}}$$

Here I is the rotational inertia about the axis $\hat{n}$ through O.

$$I = I_{cm} + mh^2$$

15.3.4 Simple Harmonic Motion and Uniform Circular Motion

Consider an object moving on a circular path of radius x_m with a uniform speed v.



If we project the position of the moving particle at point P' on the x-axis, we get point P. The coordinate of P is:

$$x\left(t\right) = x_m \cos\left(\omega t + \phi\right).$$

While point P' executes uniform circular motion its projection P moves along the x-axis with simple harmonic motion. The speed v of point P' is equal to ωx_m. The direction of the velocity vector is along the tangent to the circular path. If we project the velocity $\vec{v}$ on the x-axis we get:

$$v(t) = -\omega x_m \sin\left(\omega t + \phi\right)$$

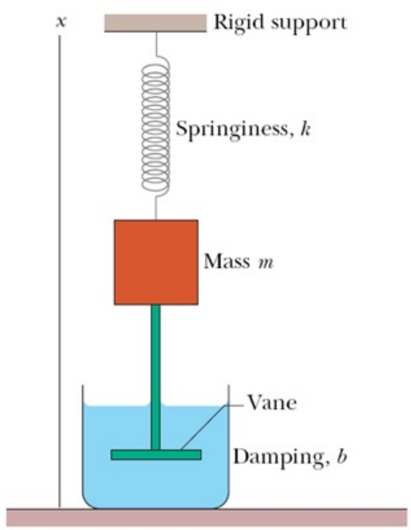
The acceleration $\vec{a}$ points along the center O. If we project $\vec{a}$ along the x-axis we get:

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi).$$

Conclusion: Whether we look at the displacement, the velocity, or the acceleration, the projection of uniform circular motion on the x-axis diameter is SHM.

15.4 Damped Simple Harmonic Motion

When the amplitude of an oscillating object is reduced due to the presence of an external force the motion is said to be damped. An example is given in the figure below.



A mass m attached to a spring of spring constant k oscillates vertically. The oscillating mass is attached to a vane submerged in a liquid. The liquid exerts a damping force $\vec{F_d}$ whose magnitude is given by the equation:

$$F_d = -bv$$

The negative sign indicates that $\vec{F_d}$ opposes the motion of the oscillating mass. The parameter b is called the damping constant. The net force on m is:

$$F_{net} = -kx - bv$$

From Newton's second law we have:

$$-kx - bv = ma$$

We substitute v with $\frac{dx}{dt}$ and a with $\frac{d^2x}{dt^2}$ and get the following differential equation for the damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{2}$$

This is called a linear 2nd order homogeneous differential equation with constant coefficients. In general, it has two independent solutions (Two solutions are independent if one solution is not the multiple of the other.). Let us consider instead the following equation:

$$m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + kz = 0 \tag{3}$$

where $z\left(t\right)$ may be complex. Assume

$$z\left(t\right) = e^{\alpha t}$$

where α may be complex. Then

$$\frac{dz}{dt} = \alpha z, \frac{d^2z}{dt^2} = \alpha^2 z$$

(3) becomes

$$\left(m\alpha^2 + b\alpha + k\right) z = 0$$

and we have

$$\alpha = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \tag{4}$$

15.4.1 Over Damping: $\left(\frac{b}{2m}\right)^2 - \frac{k}{m} > 0$

Let $\alpha_\pm = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$. Both $\alpha_\pm$ are less than zero. The general solution is

$$x\left(t\right) = Ae^{\alpha_+ t} + Be^{\alpha_- t} = e^{-\frac{b}{2m}t}\left(Ae^{\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}t} + Be^{-\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}t}\right)$$

Note that the above solution vanishes in the limit $t \to \infty$.

15.4.2 Critical Damping: $\left(\frac{b}{2m}\right)^2 - \frac{k}{m} = 0$

Under this circumstance, $e^{-\frac{b}{2m}t}$ is a solution. There is another solution $te^{-\frac{b}{2m}t}$ as can be verified by noting that

$$
\begin{aligned}
& m\frac{d^2\left(te^{-\frac{b}{2m}t}\right)}{dt^2} + b\frac{d\left(te^{-\frac{b}{2m}t}\right)}{dt} + k\left(te^{-\frac{b}{2m}t}\right) \\
&= t\left[m\frac{d^2\left(e^{-\frac{b}{2m}t}\right)}{dt^2} + b\frac{d\left(e^{-\frac{b}{2m}t}\right)}{dt} + k\left(e^{-\frac{b}{2m}t}\right)\right] + 2m\frac{dt}{dt}\frac{de^{-\frac{b}{2m}t}}{dt} + b\frac{dt}{dt}e^{-\frac{b}{2m}t} \\
&= 2m\frac{de^{-\frac{b}{2m}t}}{dt} + be^{-\frac{b}{2m}t} = 0
\end{aligned}
$$

The general solution is

$$x(t) = e^{-\frac{b}{2m}t}(A + Bt)$$

which also vanishes as $t \to \infty$.

15.4.3 Under Damping: $\left(\frac{b}{2m}\right)^2 - \frac{k}{m} < 0$

Let

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}.$$

Note that $\omega_0 = \sqrt{\frac{k}{m}}$ is the angular frequency of the SHM if there is no damping ($b = 0$). Then from (4), we get

$$\alpha = -\frac{b}{2m} \pm i\omega'$$

and the solution

$$z(t) = e^{\alpha t} = e^{-\frac{b}{2m}t}e^{\pm i\omega' t} = e^{-\frac{b}{2m}t}(\cos\omega' t \pm i\sin\omega' t) \tag{5}$$

Express $z(t) = R(t) + iI(t)$ where R and I are all real. (3) then becomes

$$
\begin{aligned}
& m\frac{d^2(R(t) + iI(t))}{dt^2} + b\frac{d(R(t) + iI(t))}{dt} + k(R(t) + iI(t)) \\
&= m\frac{d^2R(t)}{dt^2} + b\frac{dR(t)}{dt} + kR(t) + i\left(m\frac{d^2I(t)}{dt^2} + b\frac{dI(t)}{dt} + kI(t)\right) \\
&= 0
\end{aligned}
$$

Since the coefficients m, b and k are all real, the above equation yields

$$m\frac{d^2R(t)}{dt^2} + b\frac{dR(t)}{dt} + kR(t) = 0$$
$$m\frac{d^2I(t)}{dt^2} + b\frac{dI(t)}{dt} + kI(t) = 0$$

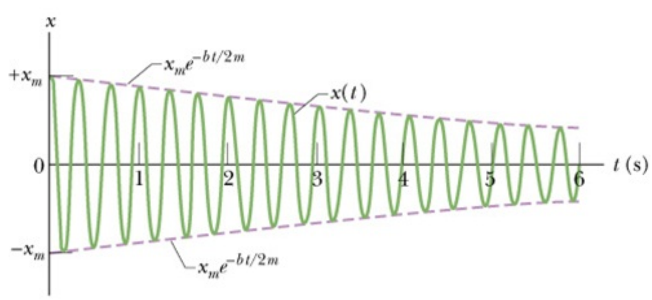
i.e., both the real part and the imaginary part of $z(t)$ also satisfy the same equation (3). (5) thus gives us two independent real solutions

$$x_1(t) = e^{-\frac{b}{2m}t}\cos\omega' t$$
$$x_2(t) == e^{-\frac{b}{2m}t}\sin\omega' t$$

In particular, the liner combinations of the above two equations is the most general solution

$$\begin{aligned} x(t) = Ax_1(t) + Bx_2(t) &= e^{-\frac{b}{2m}t}(A\cos\omega' t + B\sin\omega' t) \\ &= x_m e^{-\frac{b}{2m}t}\cos(\omega' t + \phi) \end{aligned} \tag{6}$$

where $A = x_m\cos\phi, B = -x_m\sin\phi$ or $x_m = \sqrt{A^2+B^2}$, $\tan\phi = -\frac{B}{A}$. In the picture below, we plot $x(t)$ versus t.



Note that the solution $x(t)$ vanishes in the limit $t\to\infty$. For an undamped harmonic oscillator the energy

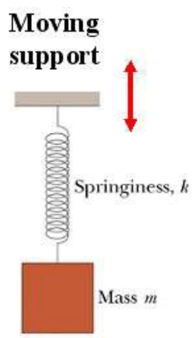
$$E = \frac{1}{2}kx_m^2$$

If the oscillator is damped its energy is not constant but decreases with time. If the damping is small we can replace x_m with $x_m e^{-\frac{b}{2m}t}$. By doing so we find that:

$$E(t) \approx \frac{1}{2} k x_m^2 e^{-\frac{b}{m}t}$$

The mechanical energy decreases exponentially with time.

15.5 Forced Oscillations and Resonance



If an oscillating system is disturbed and then allowed to oscillate freely the corresponding angular frequency ω is called the natural frequency. The same system can also be driven as shown in the figure by a moving support that oscillates at an arbitrary angular frequency ω_d. The displacement x is determined from an inhomogeneous second order linear differential equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t) = f_0 \cos(\omega_d t + \theta) \tag{7}$$

Let us defined the operator $\hat{O}$:

$$\hat{O} = m\frac{d^2}{dt^2} + b\frac{d}{dt} + k$$

and the above equation can be written as

$$\hat{O}x(t) = f(t) \tag{8}$$

The homogeneous solution x_h is the solution for the equation

$$\hat{O}x_h(t) = m\frac{d^2x_h}{dt^2} + b\frac{dx_h}{dt} + kx_h = 0$$

which is also the aforementioned equation (2) with the solution (6)

$$x_h(t) = Ce^{-\frac{b}{2m}t}\cos(\omega' t + \phi')$$

If $x_1(t)$ and $x_2(t)$ are two particular solutions of (8):

$$\hat{O}x_1(t) = \hat{O}x_2(t) = f(t),$$

then

$$0 = \hat{O}x_1(t) - \hat{O}x_2(t) = \hat{O}(x_1(t) - x_2(t))$$

and the difference $x_1(t) - x_2(t)$ is a homogeneous solution. Suppose we have found a particular solution for (6) such that

$$\hat{O}x_p(t) = f(t).$$

Then, the difference between x_p and any solution of (8):$x(t)$ must be a homogeneous solution x_h with $\hat{O}x_h(t) = 0$. This is to say that $x(t)$ always be expressed as

$$x(t) = x_p(t) + x_h(t)$$

i.e., the solution of (8) can always be written as the sum of a homogeneous solution x_h and a particular solution x_p. Since we have learned how to solve the homogeneous equation, the remaining task is to find a particular solution for the inhomogeneous equation (8). To do this, observe that the function $x(t)$ of (7) is also the real part of $z(t)$ which satisfies

$$m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + kz = f_0 e^{i(\omega_d t + \theta)} \tag{9}$$

We may assume $z(t)$ is in a form that is proportional to $e^{i\omega_d t}$

$$z(t) = Z_0 e^{i\omega_d t}$$

Then

$$\frac{dz}{dt} = i\omega_d z, \frac{d^2z}{dt^2} = (i\omega_d)^2 z$$

and

$$m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + kz = \left(m(i\omega_d)^2 + b(i\omega_d) + k\right) Z_0 e^{i\omega_d t} = f_0 e^{i(\omega_d t + \theta)}$$

We may choose to let

$$Z_0 = \frac{f_0 e^{i\theta}}{k - m\omega_d^2 + ib\omega_d} = \frac{f_0 e^{i\theta}}{m\left(\left(\omega_0^2 - \omega_d^2\right) + i\frac{b}{m}\omega_d\right)}$$

and $Z_0 e^{i\omega_d t}$ will be a solution for (9) with the real part

$$x_p = \mathrm{Re}\left[\frac{f_0 e^{i\theta}}{m\left(\left(\omega_0^2 - \omega_d^2\right) + i\frac{b}{m}\omega_d\right)} e^{i\omega_d t}\right] = \mathrm{Re}\left[x_m e^{i(\omega_d t + \theta + \Delta)}\right] = x_m(\omega_d)\cos(\omega_d t + \theta + \Delta)$$

being a particular solution for (7). Here

$$\frac{f_0}{m}\frac{1}{\left(\omega_0^2 - \omega_d^2\right) + i\frac{b}{m}\omega_d} = x_m(\omega_d)\, e^{i\Delta}$$

with

$$x_m = \frac{f_0}{m}\left|\frac{1}{\left(\omega_0^2 - \omega_d^2\right) + i\frac{b}{m}\omega_d}\right| = \frac{f_0}{m}\frac{1}{\sqrt{\left(\omega_0^2 - \omega_d^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega_d^2}}$$

and

$$\tan\Delta = \frac{\frac{b}{m}\omega_d}{\omega_d^2 - \omega_0^2}$$

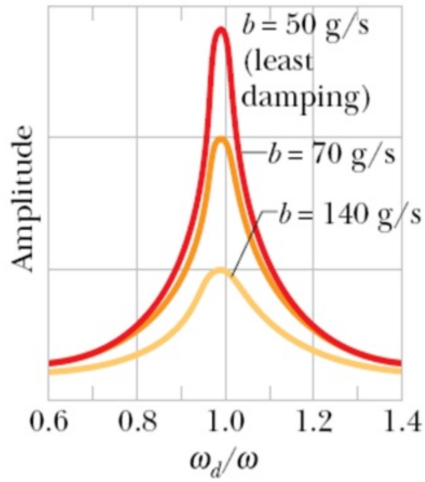
The displacement $x(t)$ in general is given by:

$$x(t) = x_m(\omega_d)\cos(\omega_d t + \theta + \Delta) + Ce^{-\frac{b}{2m}t}\cos(\omega' t + \phi')$$

Note when $\frac{b}{2m}t \gg 1$ or $t \gg \frac{2m}{b}$, $Ce^{-\frac{b}{2m}t}\cos(\omega' t + \phi') \to 0$ and any solution of $x(t)$ approaches the steady-state solution

$$x(t) \to x_m(\omega_d)\cos(\omega_d t + \theta + \Delta)$$

Such a forced oscillator oscillates at the angular frequency ω_d of the driving force. The oscillation amplitude x_m varies with the driving frequency as shown below.



The amplitude is approximately greatest when $\omega_d = \omega_0$. This condition is called resonance. To be more precise, the energy for the oscillation is proportional to the square of the amplitude,

$$E \propto x_m^2 \propto \frac{1}{\left(\omega_0^2 - \omega_d^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega_d^2}$$

The resonance occurs at

$$\begin{aligned} 0 = \frac{dE}{d\omega_d} &= 2\omega_d \frac{dE\left(\omega_d^2\right)}{d\omega_d^2} = -2\omega_d E^2 \frac{d}{d\omega_d^2} \frac{1}{E\left(\omega_d^2\right)} \\ &= -2\omega_d E^2 \frac{d}{d\omega_d^2} \left(\left(\omega_0^2 - \omega_d^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega_d^2 \right) = -2\omega_d E^2 \left(2\left(\omega_d^2 - \omega_0^2\right) + \left(\frac{b}{m}\right)^2 \right) \end{aligned}$$

or at the frequency

$$\omega_r = \sqrt{\omega_0^2 - \frac{1}{2}\frac{b^2}{m^2}}$$

We will assume the damping is relatively small, $\omega_0^2 \gg \frac{b^2}{m^2}$ such that $\omega_r \approx \omega_0$. The frequency ω_b is defined to be the frequency at which the energy is half the peak value at resonance frequency ω_r

$$\begin{aligned} \frac{1}{\left(\omega_0^2 - \omega_b^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega_b^2} &= \frac{1}{2} \frac{1}{\left(\omega_0^2 - \omega_r^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega_r^2} = \frac{1}{2} \frac{1}{\left(\frac{1}{2}\frac{b^2}{m^2}\right)^2 + \left(\frac{b}{m}\right)^2 \omega_r^2} \\ &= \frac{1}{2\frac{b^2}{m^2}} \frac{1}{\frac{1}{4}\frac{b^2}{m^2} + \omega_r^2} \approx \frac{1}{2\frac{b^2}{m^2}} \frac{1}{\omega_0^2} \end{aligned}$$

Thus

$$\left(\omega_b^2-\omega_0^2\right)^2+\left(\frac{b}{m}\right)^2\omega_b^2\approx\left(\omega_b^2-\omega_0^2\right)^2+\left(\frac{b}{m}\right)^2\omega_0^2\approx 2\frac{b^2}{m^2}\omega_0^2$$

$$\left(\omega_b^2-\omega_0^2\right)^2\approx\frac{b^2}{m^2}\omega_0^2$$

and

$$\omega_b\approx\sqrt{\omega_0^2\pm\frac{b}{m}\omega_0}=\omega_0\sqrt{1\pm\frac{b}{m\omega_0}}\approx\omega_0\left(1\pm\frac{b}{2m\omega_0}\right)$$

The bandwidth $\Delta\omega$ is defined to be the frequency range in which the energy is more than half the peak value at resonance.

$$\Delta\omega=\left(\omega_0+\frac{b}{2m}\right)-\left(\omega_0-\frac{b}{2m}\right)=\frac{b}{m}$$

The ration $\frac{\omega_0}{\Delta\omega}$ is called the Q value

$$Q=\frac{\omega_0}{\Delta\omega}=\frac{m\omega_0}{b}$$

Note at resonance

$$x_m\left(\omega_0\right)\approx\frac{f_0}{m}\frac{1}{\frac{b}{m}\omega_0}=\frac{f_0}{b\omega_0}$$

and therefore

$$x_m\left(\omega_0\right)\Delta\omega\approx\frac{f_0}{b\omega_0}\frac{b}{m}=\frac{f_0}{m\omega_0}$$

is independent of the damping coefficient b. All mechanical structures have one or more natural frequencies and if a structure is subjected to a strong external driving force whose frequency matches one of the natural frequencies, the resulting oscillations may damage the structure.