Lecture 2
Introduction to Logic Circuits

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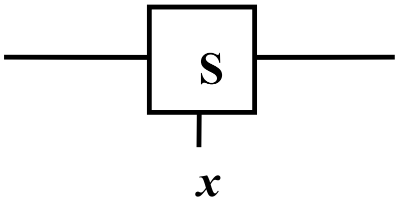
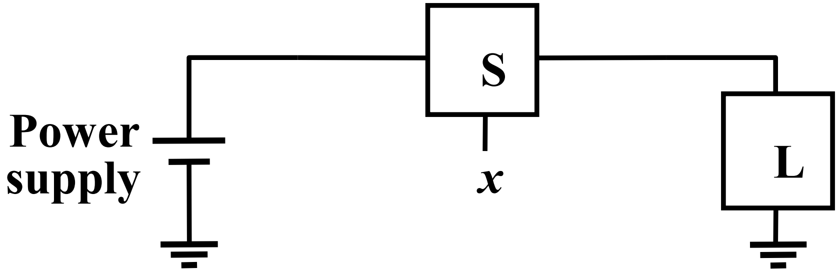
One Variable Logic Function

- Simplest binary logic, a switch with two states.

- Symbol of a switch.



- Light control switch, $L(x) = x$, a one variable logic function.
  - $L = 1$ if $x = 1$.
  - $L = 0$ if $x = 0$.

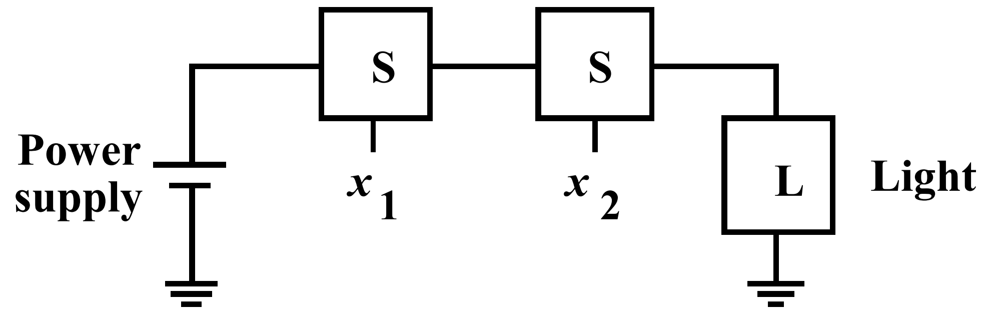


Two Variable Logic Function

- Two switches to control a lamp.
- And logic(series):

  $L(x_1) = x_1 \cdot x_2$

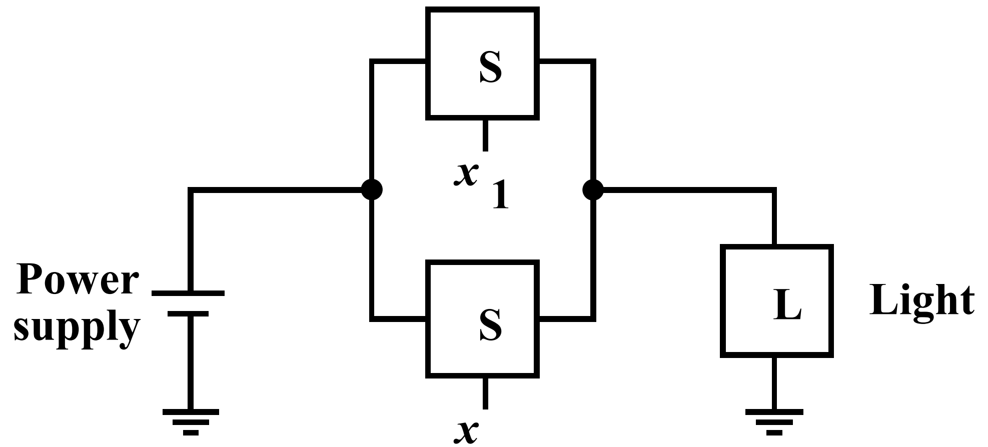
  - $L = 1$
    if $x_1 = 1$ and $x_2 = 1$
  - $L = 0$ otherwise.



- Or logic(parallel):

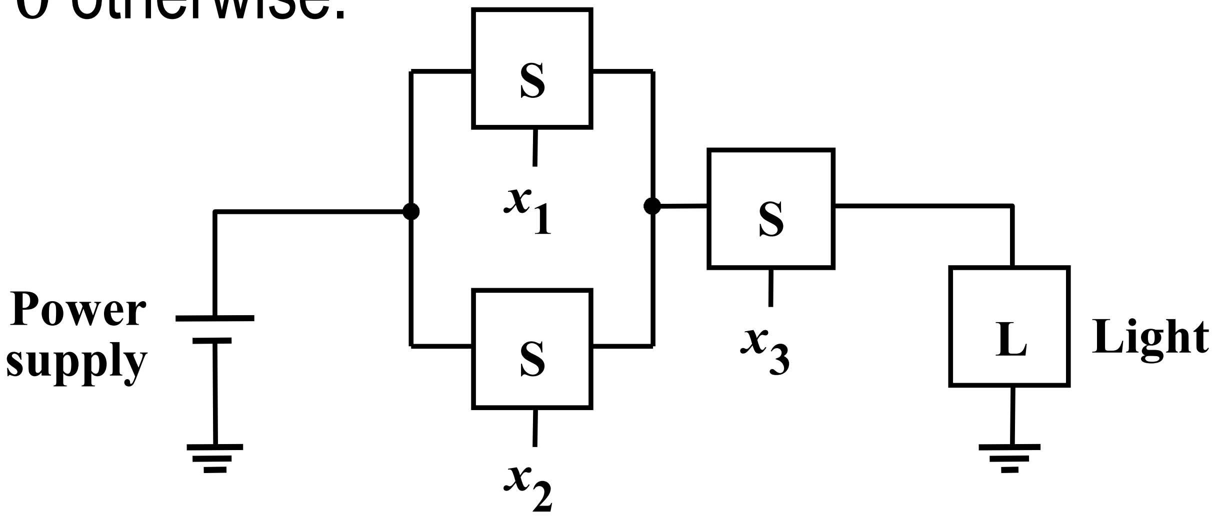
  $L(x_1) = x_1 + x_2$

  - $L = 1$
    if $x_1 = 1$ or $x_2 = 1$
  - $L = 0$ otherwise



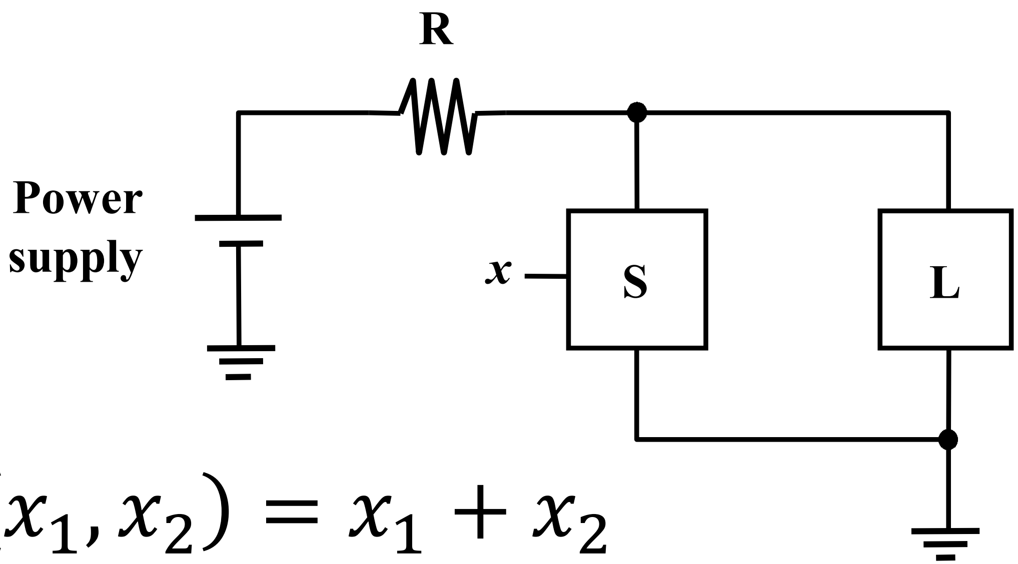
Three Variable Logic Function

- $L(x_1) = (x_1 + x_2) \cdot x_3$
  - $L = 1$ if $x_3 = 1$ and, at the same time either $x_1 = 1$ or $x_2 = 1$
  - $L = 0$ otherwise.



Inversion Logic (One Variable)

- The light with be turned on when the switch is opened.
  - $L(x) = \bar{x}$
  - $L = 1$ if $x = 0$
  - $L = 0$ if $x = 1$

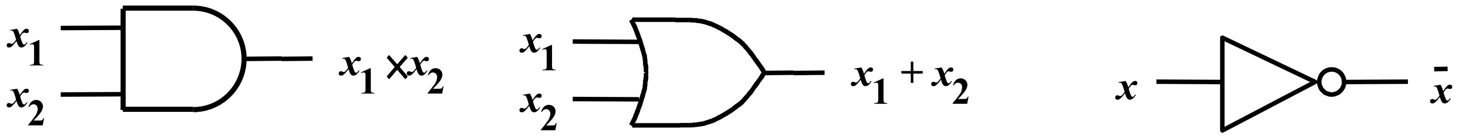


- $\bar{x} = x' = \,!\,x = \sim x$
- Complex operator: $f(x_1, x_2) = x_1 + x_2$
- Complement function: $\bar{f}(x_1, x_2) = \overline{x_1 + x_2}$
- $\overline{x_1 + x_2} = (x_1 + x_2)' = \,!\,(x_1 + x_2) = \sim(x_1 + x_2)$

Truth Table

- A truth table for AND and OR operations.

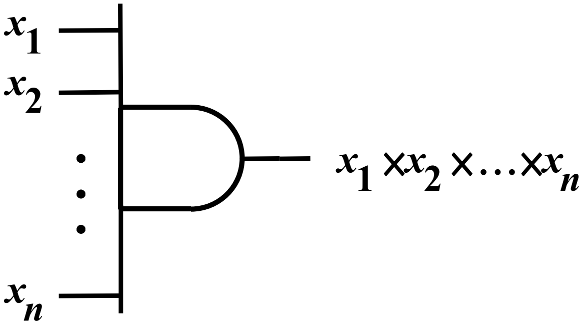
x_1	x_2	$x_1 \cdot x_2$	$x_1 + x_2$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1
		AND	OR

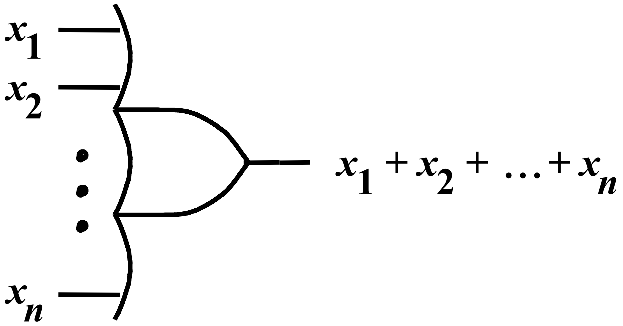


Three-input to Multiple input AND, OR

- Truth table for 3 input AND, OR

x_1	x_2	x_3	$x_1 \cdot x_2 \cdot x_3$	$x_1 + x_2 + x_3$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1





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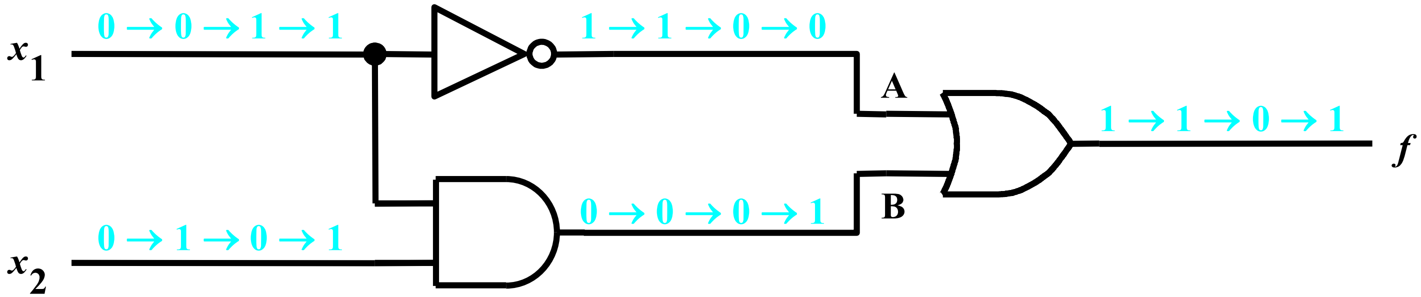
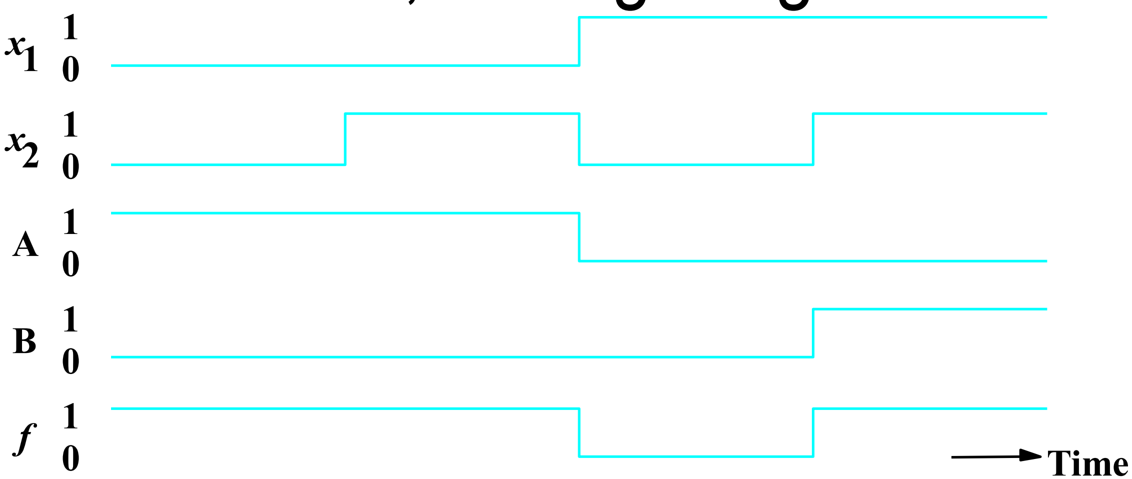
Logic Network Analysis

- Network implements $f = \bar{x}_1 + x_1 \cdot x_2$



- Truth table, Timing Diagram



x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

(b) Truth table for f

Switching Algebra

- In 1854, George Boole invented a 2-value algebraic system, now called Boolean algebra.
- In 1904, E. V. Huntington formulate the postulates as the formal definitions
- In 1938, Bell Labs researcher Claude E. Shannon showed how to adapt Boolean algebra to analyze and describe the behavior of circuits built from relays with his *switching algebra*.

Basic Definition

- Boolean Algebra:
  - A deductive mathematical system
  - Defined with
    - A set of elements
      - ex: B = {0, 1}
    - A set of operators
      - ex: +, •,
  - A number of unproved axioms or postulates

Two-Valued Boolean Algebra

- A two-valued Boolean algebra is
- Defined on a set of two elements B = {0, 1}
- With rules for the binary operators + and •

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

x	x'
0	1
1	0

- Called “switching algebra”, “binary logic”

Axioms of Boolean Algebra

- 1a. $0 \cdot 0 = 0$
- 1b. $1 + 1 = 1$
- 2a. $1 \cdot 1 = 1$
- 2b. $0 + 0 = 0$
- 3a. $0 \cdot 1 = 1 \cdot 0 = 0$
- 3b. $1 + 0 = 0 + 1 = 1$
- 4a. If $x = 0$, then $\bar{x} = 1$ (inverse)
- 4b. If $x = 1$, then $\bar{x} = 0$

Single-Variable Theorems

- 5a. $x \cdot 0 = 0$
- 5b. $x + 1 = 1$
- 6a. $x \cdot 1 = x$ (identity)
- 6b. $x + 0 = x$
- 7a. $x \cdot x = x$
- 7b. $x + x = x$
- 8a. $x \cdot \bar{x} = 0$
- 8b. $x + \bar{x} = 1$
- 9. $\bar{\bar{x}} = x$

Duality Principle

- Every Boolean algebraic expression remains valid if the operators and identity elements are interchanged
- Part (a) and part (b) are dual
- For one-variable Boolean algebra:
  - Interchange OR and AND operators and replace 1’s by 0’s and 0’s by 1’s
  - Ex: $X + 1 = 1 \rightarrow X \cdot 0 = 0$

Two- and Three- Variable Properties

- 10a. $x \cdot y = y \cdot x$ (Commutative 交換率)
- 10b. $x + y = y + x$
- 11a. $x \cdot (y \cdot z) = (x \cdot y) \cdot \mathrm{z}$ (Associative 結合率)
- 11b. $x + (y + z) = (x + y) + \mathrm{z}$
- 12a. $x \cdot (y + z) = x \cdot y + x \cdot z$ (Distributive 分配率)
- 12b. $x + y \cdot z = (x + y) \cdot (x + z)$
- 13a. $x + x \cdot y = x$ (Absorption)
- 13b. $x \cdot (x + y) = x$

Two- and Three- Variable Properties

- 14a. $x \cdot y + x \cdot \bar{y} = x$ (combining)
- 14b. $(x + y) \cdot (x + \bar{y}) = x$
- 15a. $\overline{x \cdot y} = \bar{x} + \bar{y}$ (DeMorgan's Theorem)
- 15b. $\overline{x + y} = \bar{x} \cdot \bar{y}$

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\bar{x}$	$\bar{y}$	$\bar{x} + \bar{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

LHS RHS

Two- and Three- Variable Properties

- 16a. $x + \bar{x} \cdot y = x + y$
- 16b. $x \cdot (\bar{x} + y) = x \cdot y$
- 17a. $x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$ (consensus)
- 17b. $(x + y) \cdot (y + z) \cdot (\bar{x} + z) =$
$$(x + y) \cdot (\bar{x} + z)$$

Example 2.1

- Prove $(x_1 + x_3) \cdot (\overline{x_1} + \overline{x_3}) = x_1 \cdot \overline{x_3} + \overline{x_1} \cdot x_3$
- LHS = $(x_1 + x_3) \cdot \overline{x_1} + (x_1 + x_3) \cdot \overline{x_3}$ (12a.)

  = $x_1 \cdot \overline{x_1} + x_3 \cdot \overline{x_1} + x_1 \cdot \overline{x_3} + x_3 \cdot \overline{x_3}$ (12a.)

  = $0 + x_3 \cdot \overline{x_1} + x_1 \cdot \overline{x_3} + 0$ (8a.)

  = $x_3 \cdot \overline{x_1} + x_1 \cdot \overline{x_3}$ (6a.)

  = $x_1 \cdot \overline{x_3} + \overline{x_1} \cdot x_3$ (10a. and 10b.)

  = RHS
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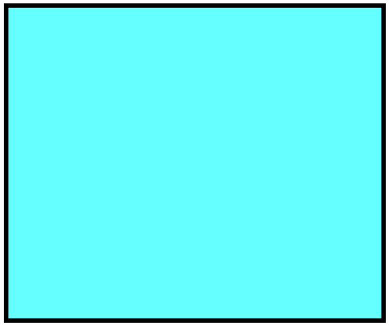
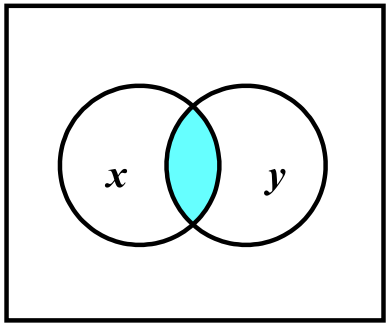
The Venn Diagram

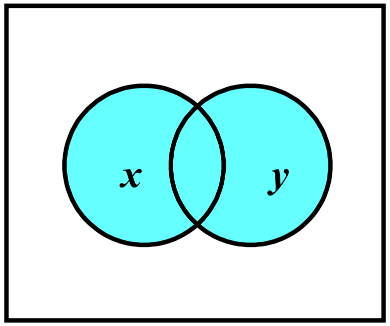


(a) Constant 1

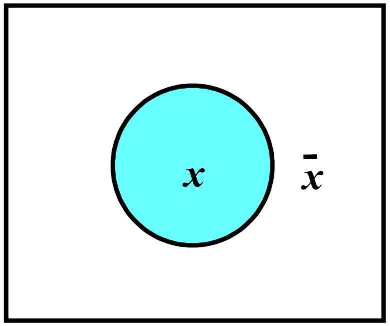
(b) Constant 0



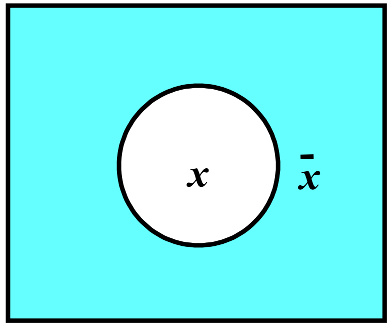
(e) $x \times y$



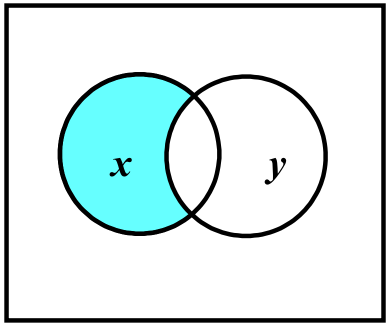
(f) $x + y$



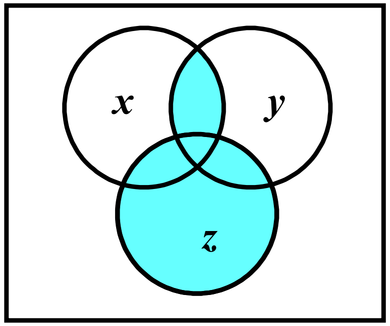
(c) Variable x



(d) $\bar{x}$



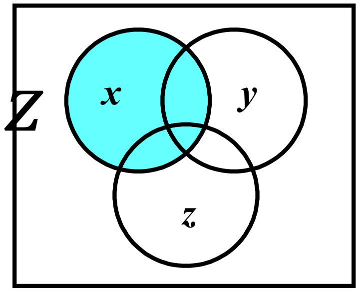
(g) $x \times \bar{y}$



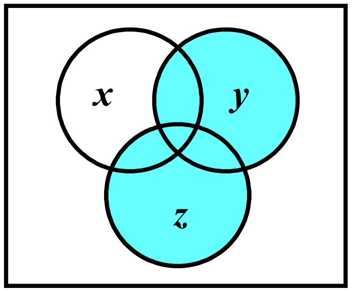
(h) $x \times y + z$

Verification of Distributive Property

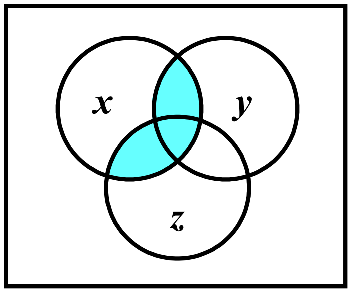
- $x \cdot (y + z) = x \cdot y + x \cdot z$



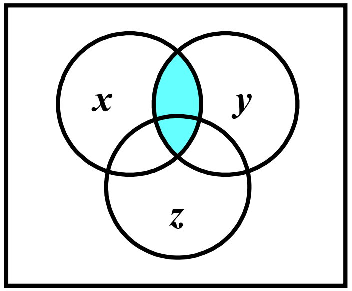
(a) x



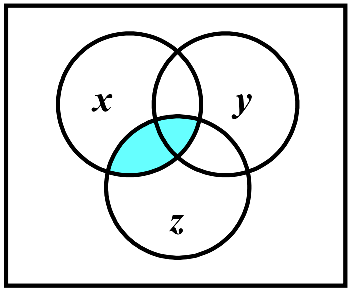
(b) $y + z$



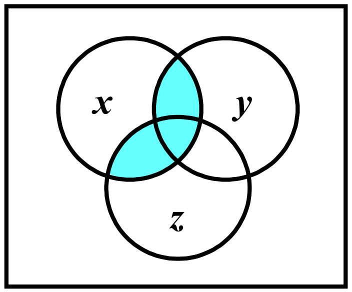
(c) $x \times (y + z)$



(d) $x \times y$



(e) $x \times z$



(f) $x \times y + x \times z$

Boolean v.s. Ordinary

Boolean Algebra

- Associate law not included (but still valid)
- Distributive law is valid
- No additive or multiplicative inverses
- Define complement in axiom 4
- No. of elements is not clearly defined
  - 2 for two-valued Boolean algebra

Ordinary Algebra

- Associate law included
- Distributive law may not valid
- Have additive and multiplicative inverses
- No complement operator
- Deal with real numbers
  - Infinite set of elements

Notation and Terminology

- Boolean algebra is based on the AND and OR operations. We have adopted the symbols · and + to denote these operations.
- Because of the similarity with the arithmetic addition and multiplication operation, the OR and AND operations are often called the *logical sum* and *logical product* operations, or to say simply *sum* and *product*.
- $x_1 \cdot \overline{x_2} \cdot x_3 + \overline{x_1} \cdot x_4 + x_2 \cdot x_3 \cdot \overline{x_4}$: *sum* of three product terms.
- $(\overline{x_1} + x_3) \cdot (x_1 + \overline{x_3}) \cdot (\overline{x_2} + x_3 + x_4)$: *product* of three sum terms.

Minterms and Maxterms

			Minterms		Maxterms	
x	y	z	Term	Name	Term	Name
0	0	0	x’y’z’	m_0	x+y+z	M_0
0	0	1	x’y’z	m_1	x+y+z’	M_1
0	1	0	x’yz’	m_2	x+y’+z	M_2
0	1	1	x’yz	m_3	x+y’+z’	M_3
1	0	0	xy’z’	m_4	x’+y+z	M_4
1	0	1	xy’z	m_5	x’+y+z’	M_5
1	1	0	xyz’	m_6	x’+y’+z	M_6
1	1	1	xyz	m_7	x’+y’+z’	M_7

Canonical Form

- $F_1 = x'y'z+xy'z'+xyz = m_1+m_4+m_7$
- $F_1' = x'y'z'+x'yz'+x'yz+xy'z+xyz'$

  $\rightarrow F_1 = (x+y+z)(x+y'+z)(x+y'+z')$

  $(x'+y+z')(x'+y'+z)$

  $= M_0M_2M_3M_5M_6$

x	y	z	F_1	F_2
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

- Similarly:

  $F_2 = x'yz+xy'z+xyz'+xyz = m_3+m_5+m_6+m_7$

  $= (x+y+z)(x+y+z')(x+y'+z)(x'+y+z) = M_0M_1M_2M_4$
- Boolean functions expressed as a sum of minterms or product of maxterms are said to be in **canonical form**

Notation for Sum of Minterms

- F = A+B'C = ABC+ABC'+AB'C+AB'C'+A'B'C

  $= m_1+m_4+m_5+m_6+m_7$
- F(A, B, C) = Σ(1,4,5,6,7)
  - ∑: ORing of terms
  - Can be derived directly from the truth table

x	y	z	F_1		
0	0	0	0		
0	0	1	1	$\rightarrow$	m_1
0	1	0	0		
0	1	1	0		
1	0	0	1	$\rightarrow$	m_4
1	0	1	1	$\rightarrow$	m_5
1	1	0	1	$\rightarrow$	m_6
1	1	1	1	$\rightarrow$	m_7

Σ(1,4,5,6,7)

Conversion between Canonical Forms

- The complement of a function = the sum of minterms missing from the original function
- $F(A,B,C) = \Sigma(1,4,5,6,7)$
- $F'(A,B,C) = \Sigma(0,2,3) = m_0+m_2+m_3$
- From DeMorgan.s theorem:
  - $F = (m_0+m_2+m_3)' = m_0'm_2'm_3' = M_0M_2M_3 = \Pi(0,2,3)$
    - $m_j' = M_j$ are shown in Table 2-3
- To convert from one canonical from to another:
  - Interchange the symbol Σ and Π
  - List those numbers missing from the original form

Conversion of Canonical Form

- F = xy + x'z = x'y'z + x'yz + xyz' + xyz

x	y	z	F_1		
0	0	0	0		
0	0	1	1	$\rightarrow$	m_1
0	1	0	0		
0	1	1	1	$\rightarrow$	m_3
1	0	0	0		
1	0	1	0		
1	1	0	1	$\rightarrow$	m_6
1	1	1	1	$\rightarrow$	m_7

Σ(1,3,6,7)

- The missing numbers are 0, 2, 4, 5
  - F = Π(0,2,4,5)

Standard Forms

- The canonical forms are basic forms obtained from the truth table
  - Very seldom to have the least number of literals
- Standard forms : not required to have all variables in each term
  - **Sum of products** [ex: $F_1 = y' + xy + x'yz'$]
  - **Product of sums** [ex: $F_2 = x(y' + z)(x' + y + z')$]
- Results in a two-level gating structure

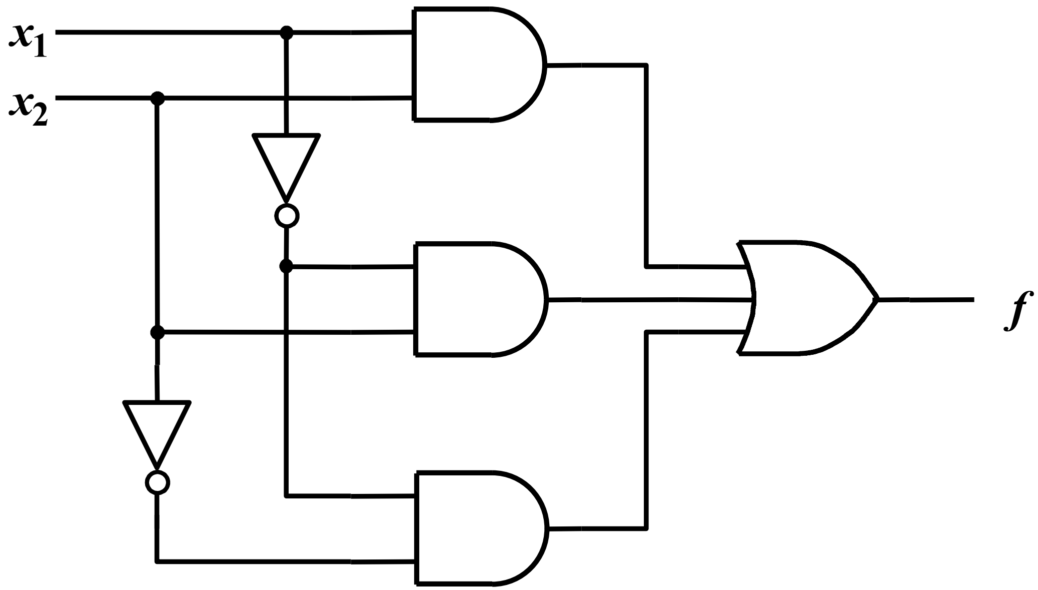
A Two-Variable Function to be Synthesized

- $f(x_1, x_2) = m_0 + m_1 + m_3$
  $= x_1x_2 + \overline{x_1}\,\overline{x_2} + \overline{x_1}x_2$
  $= x_1x_2 + \overline{x_1}\,\overline{x_2} + \overline{x_1}x_2 + \overline{x_1}x_2$ (7a)
  $= (x_1 + \overline{x_1})\, x_2 + \overline{x_1}(\overline{x_2} + x_2)$ (12a)
  $= 1 \cdot x_2 + \overline{x_1} \cdot 1$ (8b)
  $= x_2 + \overline{x_1}$ (6a)

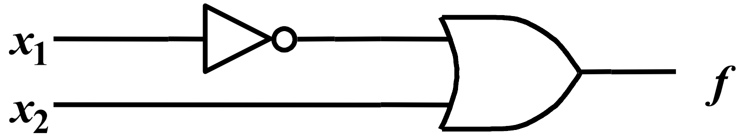
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

-

Synthesized Schematic



(a) Canonical sum-of-products



(b) Minimal-cost realization

Example 2.3

- $f(x_1, x_2, x_3) = \sum m(2,3,4,6,7)$
- $f = m_2 + m_3 + m_4 + m_6 + m_7$

$$= \overline{x_1}x_2\overline{x_3} + \overline{x_1}x_2x_3 + x_1\overline{x_2}\overline{x_3} + x_1x_2\overline{x_3} + x_1x_2x_3$$

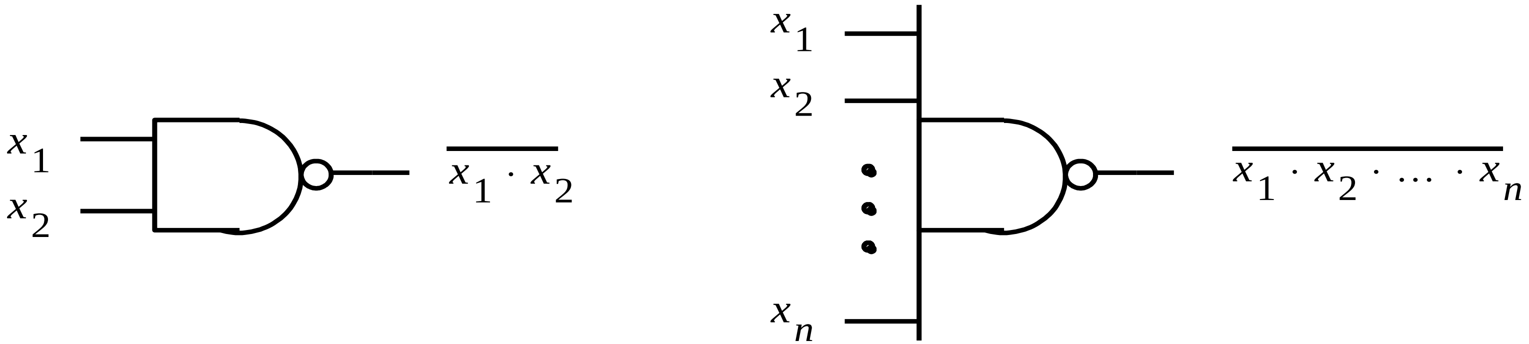
$$= \overline{x_1}x_2(\overline{x_3} + x_3) + x_1(\overline{x_2} + x_2)\overline{x_3} + x_1x_2(\overline{x_3} + x_3)$$

$$= \overline{x_1}x_2 + x_1\overline{x_3} + x_1x_2$$

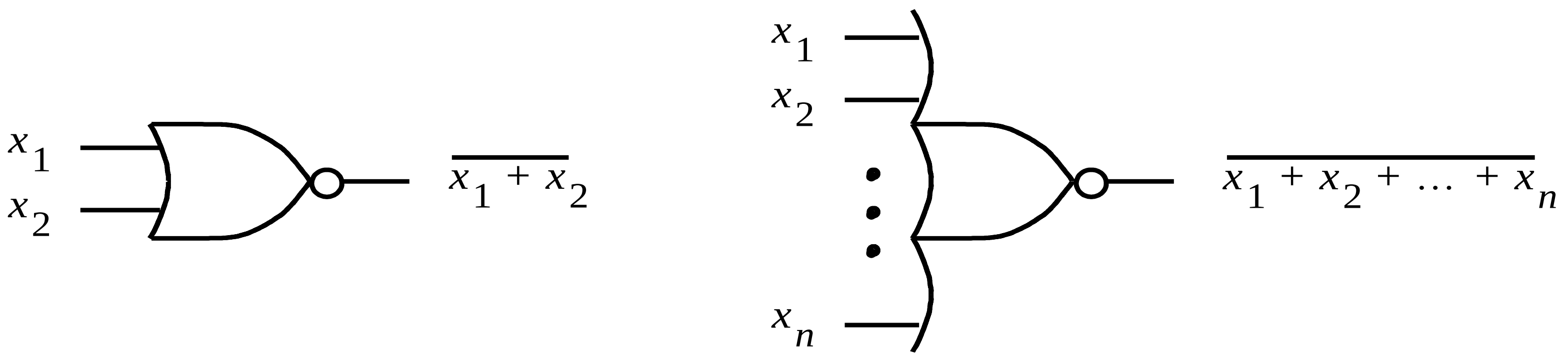
$$= (\overline{x_1} + x_1)\overline{x_2} + x_1\overline{x_3}$$

$$= \overline{x_2} + x_1\overline{x_3}$$

NAND and NOR Gates

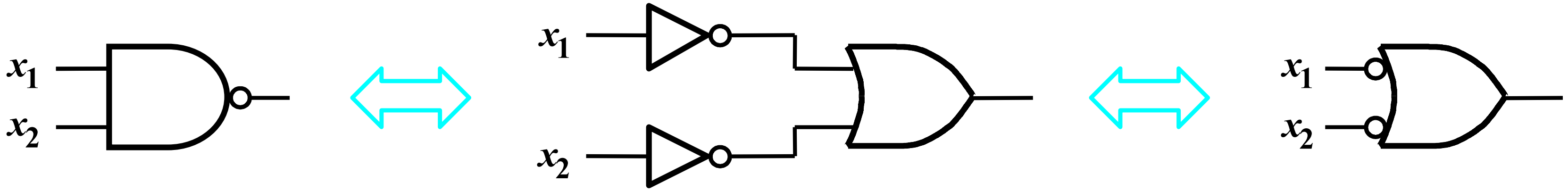


(a) NAND gates

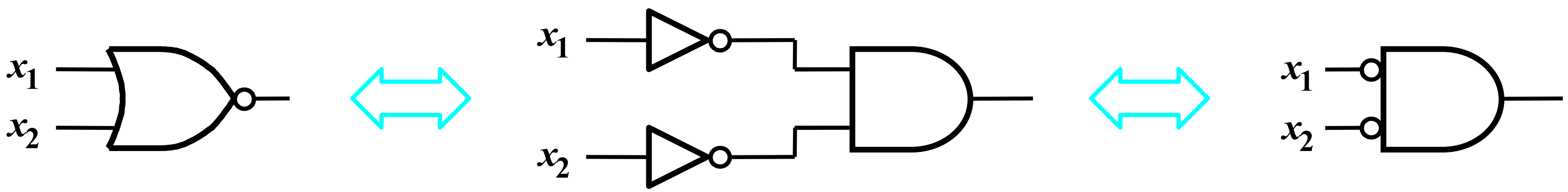


(b) NOR gates

Derivation with DeMorgan’s Theorem

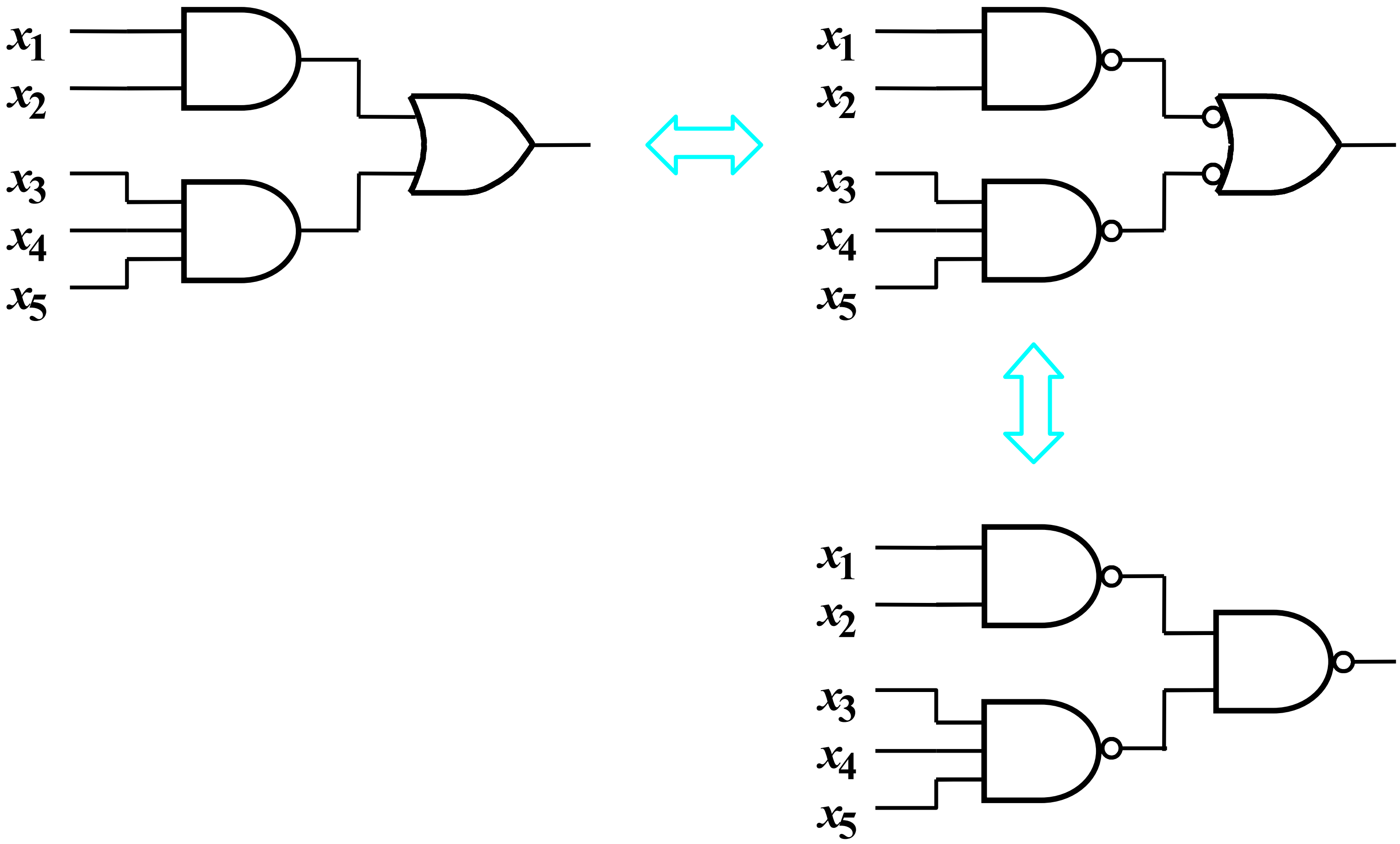


(a) $\overline{x_1 x_2} = \bar{x}_1 + \bar{x}_2$

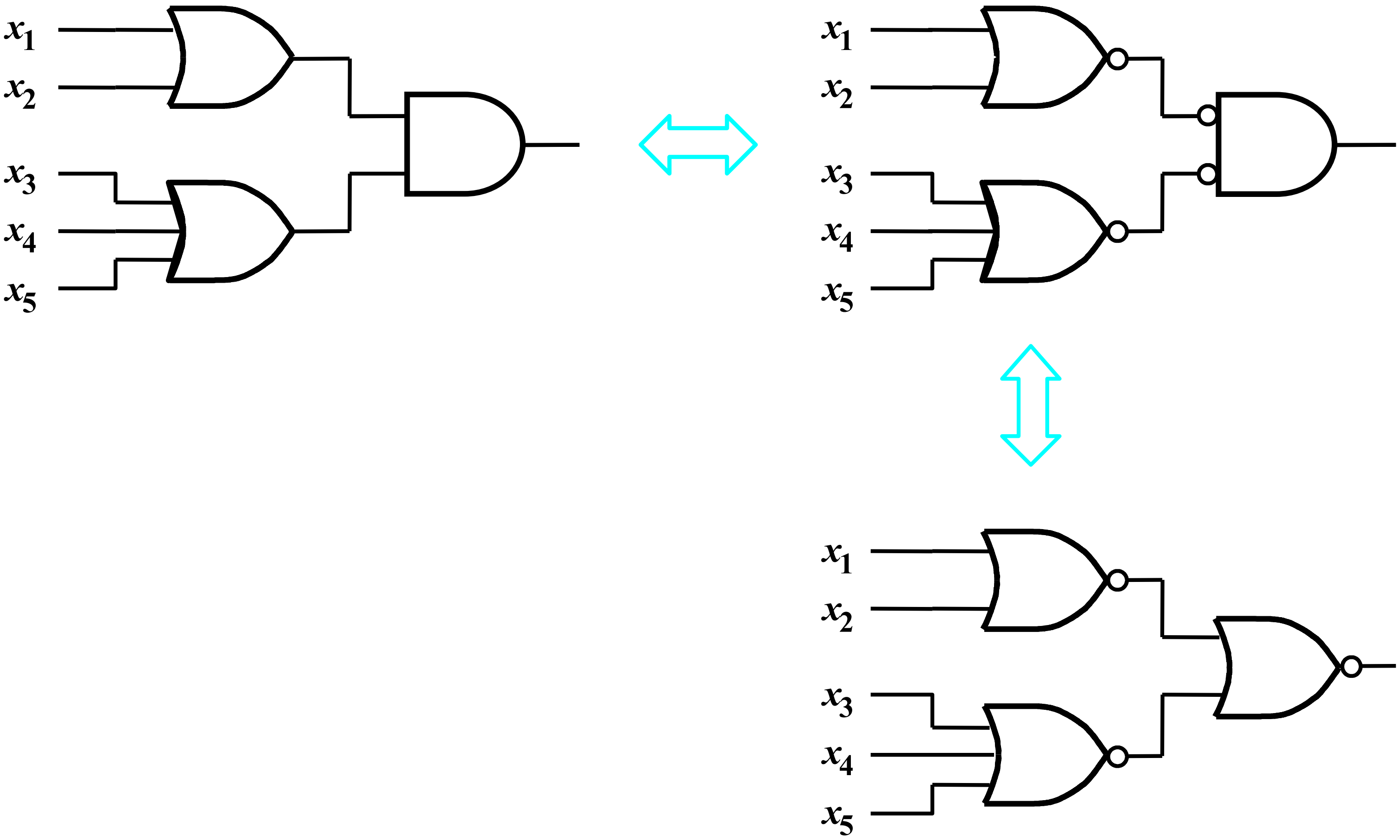


(b) $\overline{x_1 + x_2} = \bar{x}_1 \bar{x}_2$

Sum of Product Realization with NAND Gates



Product of Sum Realization with NOR Gates

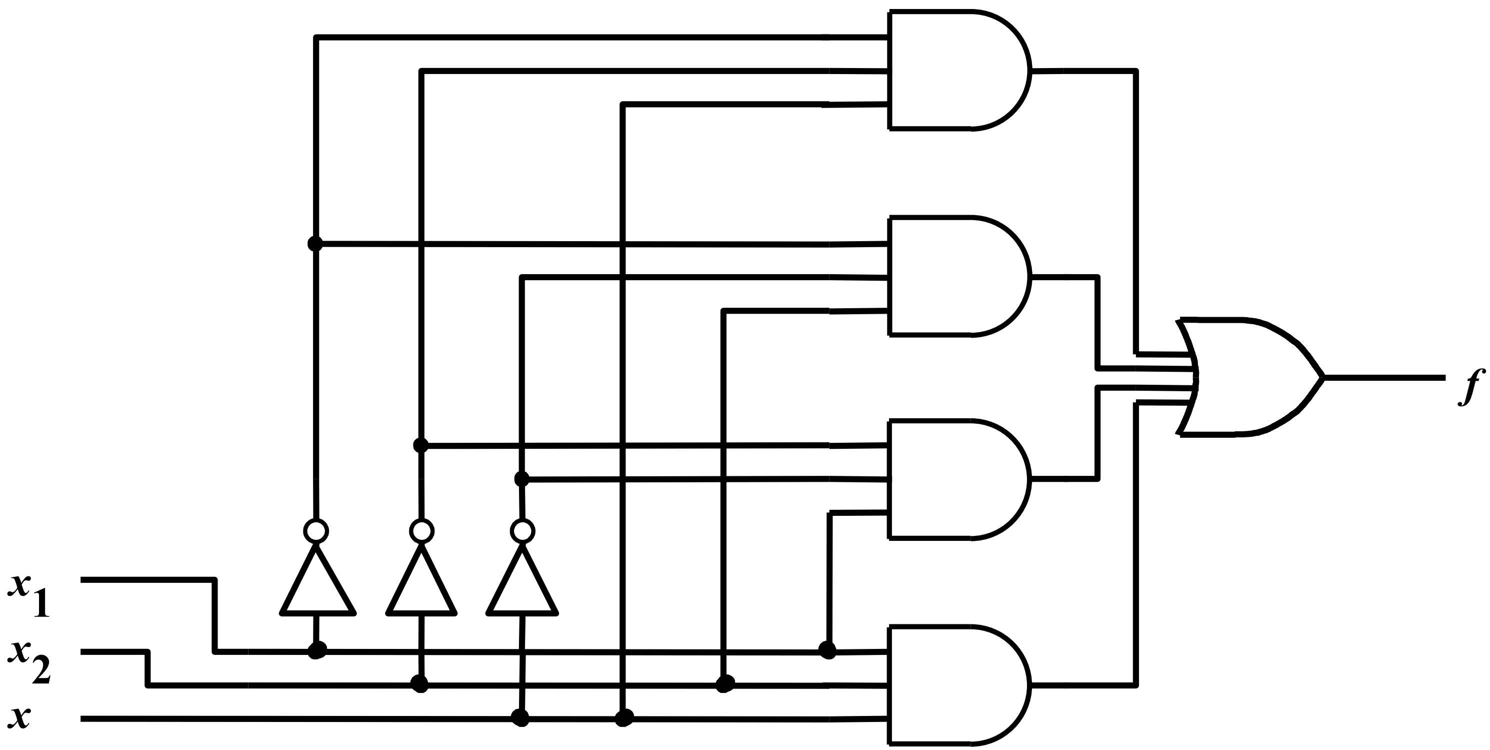


Three Way Light Switch Control

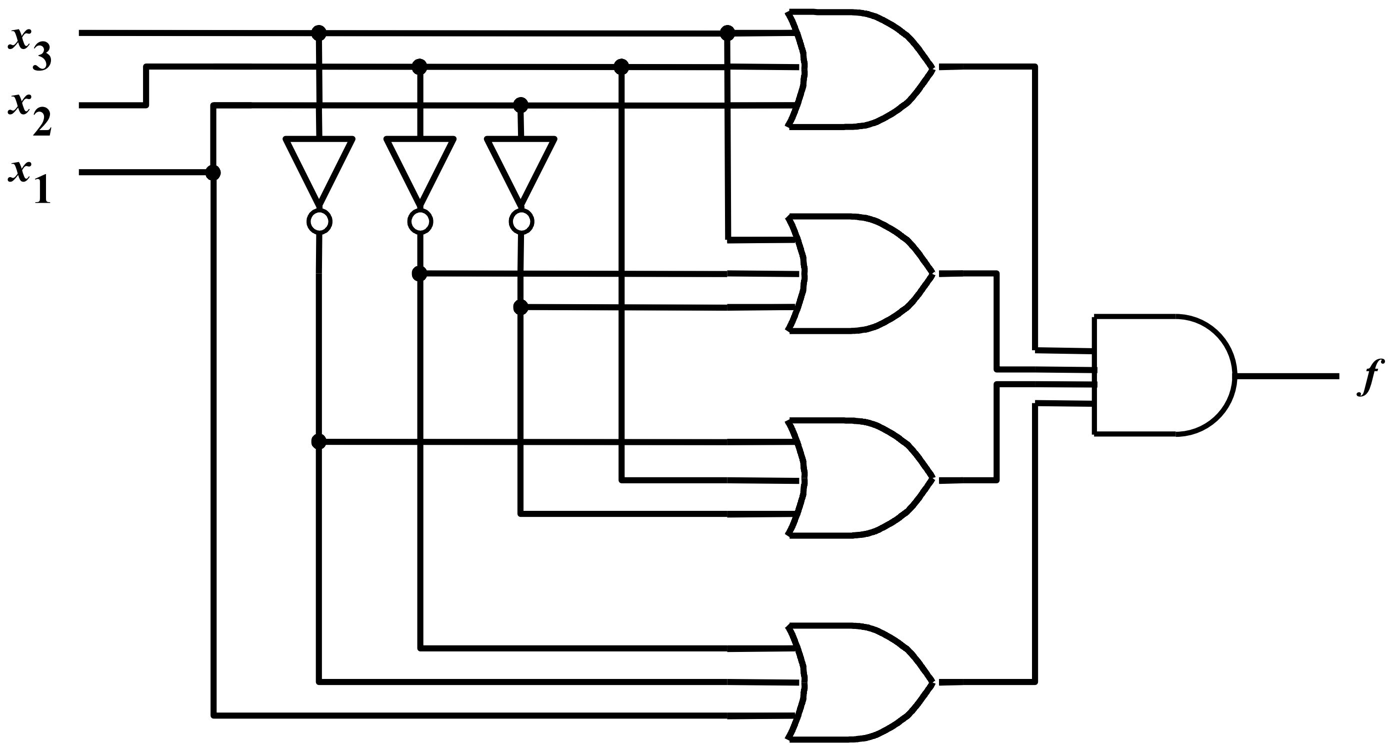
- $f = m_1 + m_2 + m_4 + m_7$
  $= \overline{x_1 x_2} x_3 + \overline{x_1} x_2 \overline{x_3} + x_1 \overline{x_2 x_3} + x_1 x_2 x_3$
- $f = M_0 \cdot M_3 \cdot M_5 \cdot M_6$
  $= (x_1 + x_2 + x_3)(x_1 + \overline{x_2} + \overline{x_3})$
  $(\overline{x_1} + x_2 + \overline{x_3})(\overline{x_1} + \overline{x_2} + x_3)$

x_1	x_2	x_3	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

SOP Realization

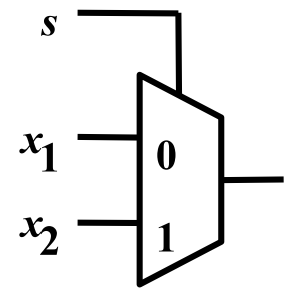


POS Realization

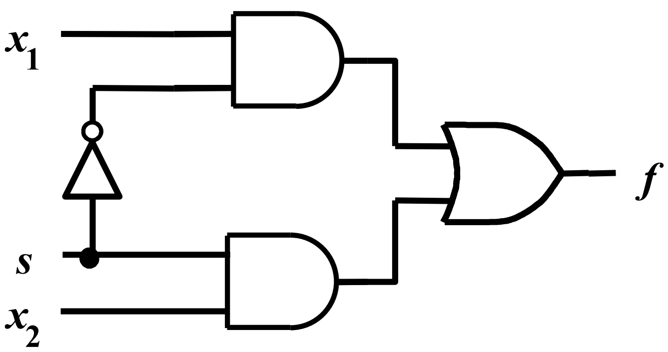


Multiplexer Circuit

- $f(s, x_1, x_2) = \bar{s}x_1\overline{x_2} + \bar{s}x_1x_2$
  $+s\overline{x_1}x_2 + sx_1x_2$
  $= \bar{s}x_1(\overline{x_2} + x_2) + s(\overline{x_1} + x_1)x_2$
  $= \bar{s}x_1 \cdot 1 + s \cdot 1 \cdot x_2 \ (8b)$
  $= \bar{s}x_1 + sx_2 \ (6a)$



$s\,x_1x_2$	$f\,(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1



s	$f\,(s, x_1, x_2)$
0	x_1
1	x_2