Matrix Chain Product

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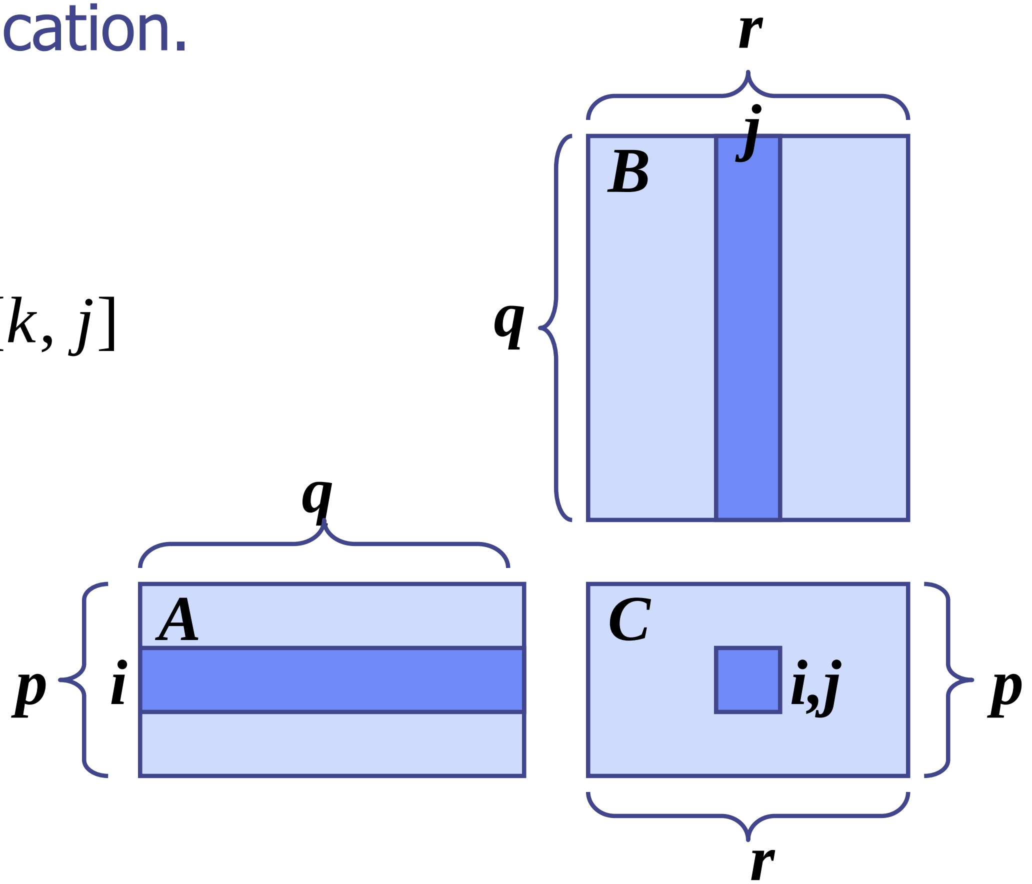
Matrix Chain Products (MCP)

- Review: Matrix Multiplication.
  - $C = A*B$
  - A is $p \times q$ and B is $q \times r$

$$C[i,j] = \sum_{k=0}^{q-1} A[i,k] * B[k,j]$$

  - $O(pqr)$ time

```
for (i=0; i<p; i++)
    for (j=0; j<r; j++){
        c[i,j]=0;
        for (k=0; k<q; k++)
            c[i,j]+=a[i,k]*b[k,j];
    }
```



Matrix Chain-Products

- Problem definition
  - Given *n* matrices $A_0, A_1, \ldots, A_{n-1}$, where A_i is of dimension $d_i \times d_{i+1}$
  - How to parenthesize $A_0 * A_1 * \ldots * A_{n-1}$ to minimize the overall cost?

Example of MCP

- The product A (2×3), B (3×5), C (5×2), D (2×4) can be fully parenthesized in 5 distinct ways:

  $(A\,(B\,(C\,D))) \rightarrow 5\times2\times4 + 3\times5\times4 + 2\times3\times4 = 124$

  $(A\,((B\,C)\,D)) \rightarrow 3\times5\times2 + 3\times2\times4 + 2\times3\times4 = 78$

  $((A\,B)\,(C\,D)) \rightarrow 2\times3\times5 + 5\times2\times4 + 2\times5\times4 = 110$

  $((A\,(B\,C))\,D) \rightarrow 3\times5\times2 + 2\times3\times2 + 2\times2\times4 = 58$

  $(((A\,B)\,C)\,D) \rightarrow 2\times3\times5 + 2\times5\times2 + 2\times2\times4 = 66$

- The way the chain is parenthesized can have a dramatic impact on the cost of evaluating the product.

An Enumeration Approach

- **Matrix Chain Product Alg.:**
  - Try all possible ways to parenthesize $A=A_0*A_1*\ldots*A_{n-1}$
  - Calculate total number of operations for each way
  - Pick the one that is best
- Running time:
  - The number of ways of parenthesizations is equal to the number of binary trees with n leave nodes
  - It is called the Catalan number, and it is almost 4^n ➔ **exponential**!

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

$\boxed{((A_0(A_1A_2))A_3) \rightarrow \text{binary tree}}$

Observations Leading to DP

- Define **subproblems**:
  - Find the best parenthesization of $A_i*A_{i+1}*\dots*A_j$.
  - Let $N_{i,j}$ denote the minimum number of operations required by this subproblem.
  - The optimal solution for the whole problem is $N_{0,n-1}$.
- **Subproblem optimality**: The optimal solution can be defined in terms of optimal subproblems
  - There has to be a final multiplication (root of the expression tree) for the optimal solution.
  - Say, the final multiply is at index i: $(A_0*\dots*A_i)*(A_{i+1}*\dots*A_{n-1})$.
  - Then the optimal solution $N_{0,n-1}$ is the sum of two optimal subproblems, $N_{0,i}$ and $N_{i+1,n-1}$ plus the time for the last multiply.

Three-Step DP Formula

- To solve matrix chain-product with DP
  - Optimum-value function
    - $N_{i,j}$: the minimum number of operations required by parenthesizing $A_i*A_{i+1}*\ldots*A_j$.
  - Recurrent equation

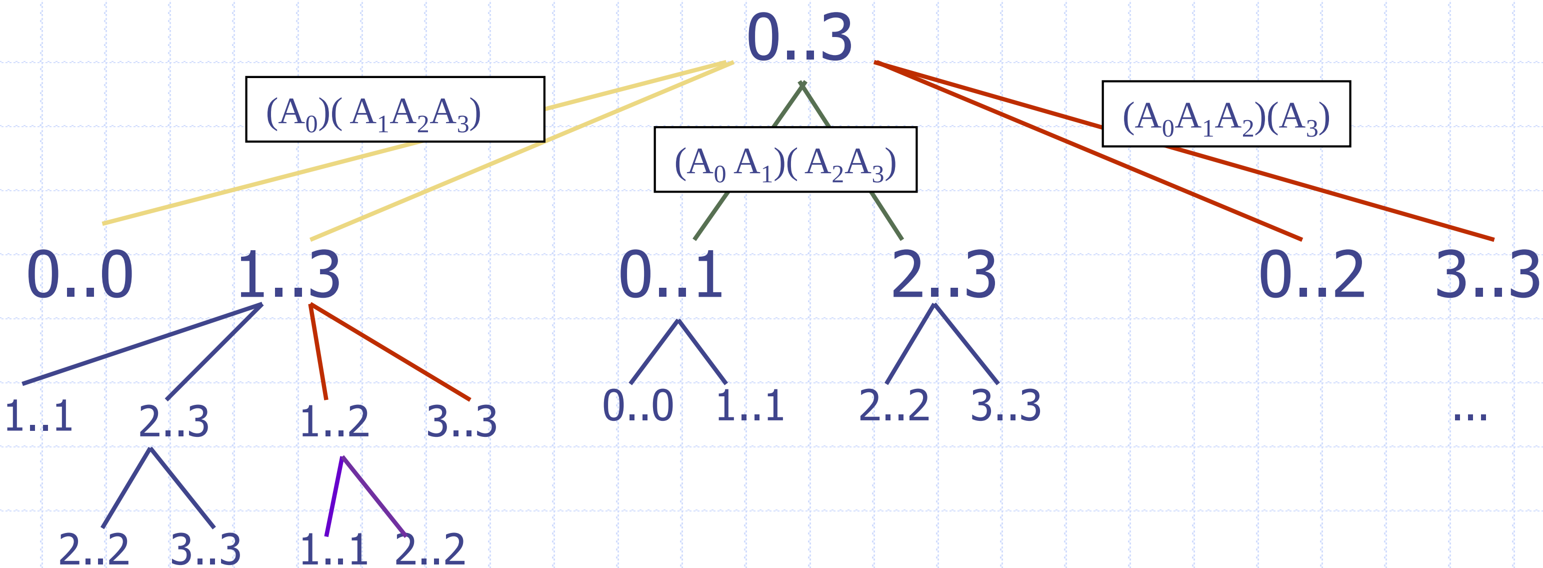
$$N_{i,j} = \min_{i \le k < j}\{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$
$$\text{with } N_{i,i} = 0, \forall i$$

$(A_i*A_{i+1}*\ldots*A_k)(A_{k+1}*A_{k+2}*\ldots*A_j)$

$\boxed{d_i \times d_{k+1}}$ $\boxed{d_{k+1} \times d_{j+1}}$

  - Answer
    - $N_{0,\ n-1}$

Subproblem Overlap



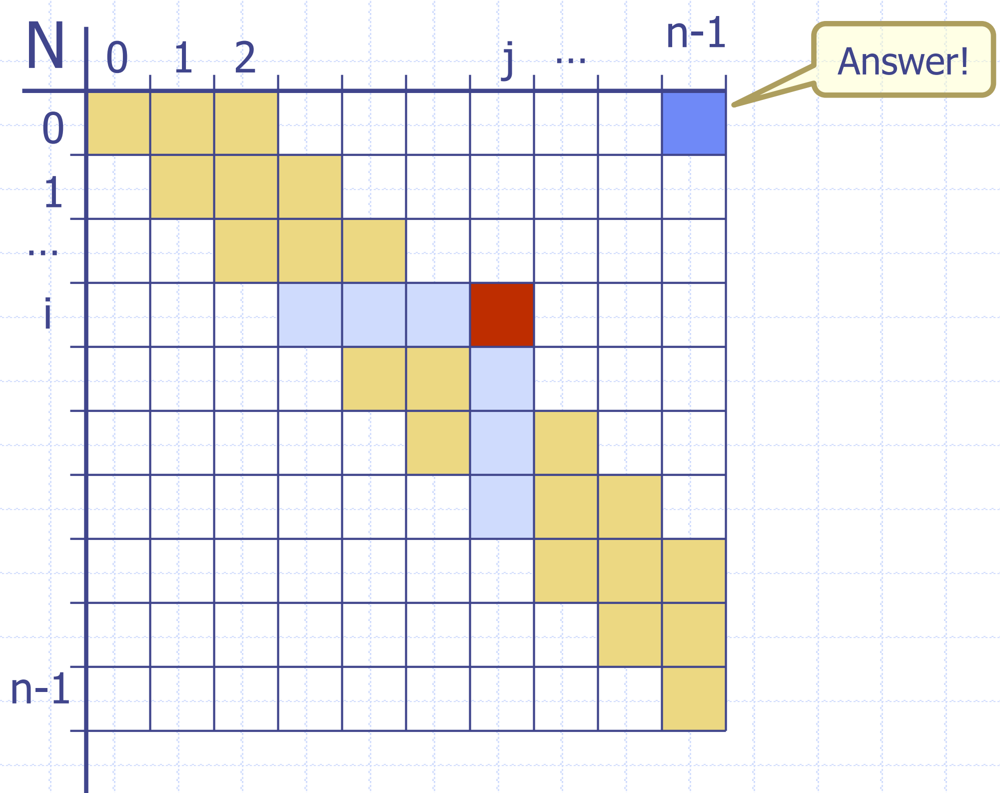
Due to the overlap, we need to keep track of previous results

Table Filling for DP

- The bottom-up approach fills in the upper-triangle of the n×n array by diagonals, starting from $N_{i,i}$'s.
- $N_{i,j}$ gets values from pervious entries in row i and column j.
- Filling in each entry in the N table takes O(n) time → Total time $O(n^3)$
- Actual parenthesization can be found by storing the best "k" for each entry

Easy for back tracking

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$



Hint: IJ-mode vs. XY-mode

Walkthrough of an MCP Example

Quiz!

- Product of A_0 (2×3), A_1 (3×5), A_2 (5×2), A_3 (2×4)

	A_0 2×3	A_1 3×5	A_2 5×2	A_3 2×4
A_0 2×3	0 2×3	30 2×5 k=0	42 2×2 k=0	58 2×4 k=2
A_1 3×5		0 3×5	30 3×2 k=1	54 3×4 k=2
A_2 5×2			0 5×2	40 5×4 k=2
A_3 2×4				0 2×4

Optimum value of k
(for back tracking)

$$N_{i,j} = \min_{i \le k < j}\{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

$$N_{0,2} = \min\begin{Bmatrix} N_{0,0} + N_{1,2} + 2\times 3\times 2 \\ N_{0,1} + N_{2,2} + 2\times 5\times 2 \end{Bmatrix} = \min\begin{Bmatrix} 0+30+12 \\ 30+0+20 \end{Bmatrix} = 42$$

$$N_{1,3} = \min\begin{Bmatrix} N_{1,1} + N_{2,3} + 3\times 5\times 4 \\ N_{1,2} + N_{3,3} + 3\times 2\times 4 \end{Bmatrix} = \min\begin{Bmatrix} 0+40+60 \\ 30+0+24 \end{Bmatrix} = 54$$

$$N_{0,3} = \min\begin{Bmatrix} N_{0,0} + N_{1,3} + 2\times 3\times 4 \\ N_{0,1} + N_{2,3} + 2\times 5\times 4 \\ N_{0,2} + N_{3,3} + 2\times 2\times 4 \end{Bmatrix} = \min\begin{Bmatrix} 0+54+24 \\ 30+40+40 \\ 42+0+16 \end{Bmatrix} = 58$$

Solution (after back tracking) →
$(A_0A_1A_2)(A_3)=(A_0(A_1A_2))(A_3)$

Exercise

Quiz!

- Product of A_0 (2×3), A_1 (3×5), A_2 (5×2), A_3 (2×4), A_4 (4×1)

	A_0 2×3	A_1 3×5	A_2 5×2	A_3 2×4	A_4 4×1
A_0 2×3	0 2×3	30 2×5 k=0	42 2×2 k=0	58 2×4 k=2	2×1 k=
A_1 3×5		0 3×5	30 3×2 k=1	54 3×4 k=2	3×1 k=
A_2 5×2			0 5×2	40 5×4 k=2	5×1 k=
A_3 2×4				0 2×4	2×1 k=3
A_4 4×1					0 4×1

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

Solution →