2-6

… important in itself, this theory will also provide the foundation for our study of nonhomogeneous linear ODEs, whose theory and engineering applications form the content of the remaining four sections of this chapter.

PROBLEM SET 2.6

1. Derive (6*) from (6).

2–8 BASIS OF SOLUTIONS. WRONSKIAN

Find the Wronskian. Show linear independence by using quotients and confirm it by Theorem 2.

2. $e^{4.0x}, e^{-1.5x}$
3. $e^{-0.5x}, e^{-2.5x}$
4. $2x, 1/(4x)$
5. x^3, x^2
6. $e^{-x} \cos \omega x, e^{-x} \sin \omega x$
7. $\cosh \frac{a}{2}x, \sinh \frac{a}{2}x$
8. $x^k \cos (\ln x), x^k \sin (\ln x)$

9–15 ODE FOR GIVEN BASIS. WRONSKIAN. IVP

(a) Find a second-order homogeneous linear ODE for which the given functions are solutions. **(b)** Show linear independence by the Wronskian. **(c)** Solve the initial value problem.

9. $\cos 5x, \sin 5x, \quad y(0) = 3, \quad y'(0) = -5$
10. $x^{m_1}, x^{m_2}, \quad y(1) = -2, \quad y'(1) = 2m_1 - 4m_2$
11. $e^{-2.5x} \cos 0.5x, e^{-2.5x} \sin 0.5x, \quad y(0) = 1.5, \quad y'(0) = -2.0$
12. $x^2, x^2 \ln x, \quad y(1) = 4, \quad y'(1) = 6$
13. $1, e^{3x}, \quad y(0) = 2, \quad y'(0) = -1$
14. $e^{-kx} \cos \pi x, e^{-kx} \sin \pi x, \quad y(0) = 1, \quad y'(0) = -k - \pi$
15. $\cosh 1.8x, \sinh 1.8x, \quad y(0) = 14.20, \quad y'(0) = 16.38$

16. **TEAM PROJECT. Consequences of the Present Theory.** This concerns some noteworthy general properties of solutions. Assume that the coefficients p and q of the ODE (1) are continuous on some open interval I, to which the subsequent statements refer.

(a) Solve $y'' - y = 0$ (a) by exponential functions, (b) by hyperbolic functions. How are the constants in the corresponding general solutions related?

(b) Prove that the solutions of a basis cannot be 0 at the same point.

(c) Prove that the solutions of a basis cannot have a maximum or minimum at the same point.

(d) Why is it likely that formulas of the form (6*) should exist?

(e) Sketch $y_1(x) = x^3$ if $x \geqq 0$ and 0 if $x < 0$, $y_2(x) = 0$ if $x \geqq 0$ and x^3 if $x < 0$. Show linear independence on $-1 < x < 1$. What is their Wronskian? What Euler–Cauchy equation do y_1, y_2 satisfy? Is there a contradiction to Theorem 2?

(f) Prove **Abel's formula**[6]

$$W(y_1(x), y_2(x)) = c \exp \left[-\int_{x_0}^{x} p(t)\, dt \right]$$

where $c = W(y_1(x_0), y_2(x_0))$. Apply it to Prob. 6. *Hint:* Write (1) for y_1 and for y_2. Eliminate q algebraically from these two ODEs, obtaining a first-order linear ODE. Solve it.

2.7 Nonhomogeneous ODEs

2-7

7. $(D^2 - 4D + 3I)y = e^x - \frac{9}{2}x$
8. $(3D^2 + 27I)y = 3 \cos x + \cos 3x$
9. $(D^2 - 16I)y = 9.6e^{4x} + 30e^x$
10. $(D^2 + 2D + I)y = 2x \sin x$

11–18 NONHOMOGENEOUS LINEAR ODEs: IVPs

Solve the initial value problem. State which rule you are using. Show each step of your calculation in detail.

11. $y'' + 4y = 8x^2, \quad y(0) = -3, \quad y'(0) = 0$
12. $y'' + 4y = -12 \sin 2x, \quad y(0) = 1.8, \quad y'(0) = 5.0$
13. $8y'' - 6y' + y = 6 \cosh x, \quad y(0) = 0.2, \quad y'(0) = 0.05$
14. $y'' + 6y' + 9y = e^{-x} \cos 2x, \quad y(0) = 1, \quad y'(0) = -1$
15. $(x^2D^2 - 3xD + 3I)y = 3 \ln x - 4, \quad y(1) = 0, \quad y'(1) = 1; \quad y_p = \ln x$
16. $(D^2 - 2D)y = 6e^{2x} - 4e^{-2x}, \quad y(0) = -1, \quad y'(0) = 6$
17. $(D^2 + 0.4D + 0.4I)y = 2.25e^{0.25x}, \quad y(0) = 0.5, \quad y'(0) = -0.5$
18. $(D^2 + 2D + 10I)y = 17 \sin x - 37 \sin 3x, \quad y(0) = 6.6, \quad y'(0) = -2.2$

19. **CAS PROJECT. Structure of Solutions of Initial Value Problems.** Using the present method, find, graph, and discuss the solutions y of initial value problems of your own choice. Explore effects on solutions caused by changes of initial conditions. Graph $y_p, y, y - y_p$ separately, to see the separate effects. Find a problem in which **(a)** the part of y resulting from y_h decreases to zero, **(b)** increases, **(c)** is not present in the answer y. Study a problem with $y(0) = 0, y'(0) = 0$. Consider a problem in which you need the Modification Rule (a) for a simple root, (b) for a double root. Make sure that your problems cover all three Cases I, II, III (see Sec. 2.2).

20. **TEAM PROJECT. Extensions of the Method of Undetermined Coefficients.** **(a)** Extend the method to products of the function in Table 2.1, **(b)** Extend the method to Euler–Cauchy equations. Comment on the practical significance of such extensions.

2.8 Modeling: Forced Oscillations. Resonance

In Sec. 2.4 we considered vertical motions of a mass–spring system (vibration of a mass m on an elastic spring, as in Figs. 33 and 53) and modeled it by the *homogeneous* linear

2-8

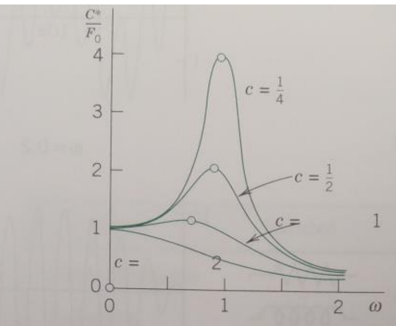


Fig. 57. Amplification C^*/F_0 as a function of ω for $m = 1, k = 1$, and various values of the damping constant c

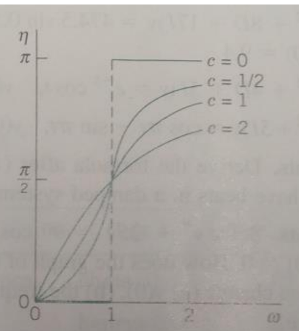


Fig. 58. Phase lag η as a function of ω for $m = 1, k = 1$, thus $\omega_0 = 1$, and various values of the damping constant c

PROBLEM SET 2.8

1. WRITING REPORT. Free and Forced Vibrations. Write a condensed report of 2–3 pages on the most important similarities and differences of free and forced vibrations, with examples of your own. No proofs.

2. Which of Probs. 1–18 in Sec. 2.7 (with x = time t) can be models of mass–spring systems with a harmonic oscillation as steady-state solution?

3–7 STEADY-STATE SOLUTIONS

Find the steady-state motion of the mass–spring system modeled by the ODE. Show the details of your work.

3. $y'' + 4y' + 3y = 4.5 \sin 2t$

4. $y'' + 2.5y' + 10y = -13.6 \sin 4t$

5. $(D^2 + D + 1.25I)y = 2.5 \cos 1.5t$

6. $(D^2 + 4D + 3I)y = \cos t + \frac{1}{3} \cos 3t$

7. $(4D^2 + 12D + 9I)y = 225 - 75 \sin 3t$

8–15 TRANSIENT SOLUTIONS

Find the transient motion of the mass–spring system modeled by the ODE. Show the details of your work.

8. $2y'' + 4y' + 6.5y = \cos 1.5t$

9. $y'' + 3y' + 3.25y = 3 \cos t - 1.5 \sin t$

10. $y'' + 16y = 56 \cos 4t$

11. $(D^2 + 9I)y = \cos 3t + \sin 3t$

12. $(D^2 + 2D + 5I)y = 4 \cos t + 8 \sin t$

13. $(D^2 + 4I)y = \sin \omega t, \ \omega^2 \neq 1$

14. $(D^2 + I)y = 5e^{-t} \cos t$

15. $(D^2 + 4D + 8I)y = 2 \cos 2t + \sin 2t$

16–20 INITIAL VALUE PROBLEMS

Find the motion of the mass–spring system modeled by the ODE and the initial conditions. Sketch or graph the solution curve. In addition, sketch or graph the curve of $y - y_p$ to see when the system practically reaches the steady state.

16. $y'' + 16y = 4 \sin t, \quad y(0) = 1, \quad y'(0) = 1$

17. $(D^2 + 4I)y = \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t$, $y(0) = 0, \quad y'(0) = \frac{3}{35}$

18. $(D^2 + 8D + 17I)y = 474.5 \sin 0.5t, \quad y(0) = -5.4$, $y'(0) = 9.4$

19. $(D^2 + 4D + 5I)y = e^{-t} \cos t, \quad y(0) = 0, y'(0) = 1$

20. $(D^2 + 5I)y = \cos \pi t - \sin \pi t, \quad y(0) = 0, \quad y'(0) = 0$

21. Beats. Derive the formula after (12) from (12). Can we have beats in a damped system?

22. Beats. Solve $y'' + 25y = 99 \cos 4.9t, \quad y(0) = 1.5$, $y'(0) = 0$. How does the graph of the solution change if you change **(a)** $y(0)$, **(b)** the frequency of the driving force?

23. TEAM EXPERIMENT. Practical Resonance.
(a) Derive, in detail, the crucial formula (16).
(b) By considering dC^*/dc show that $C^*(\omega_{max})$ in-

24. Gun barrel. Solve $y'' + y = 1 - t^2/\pi^2$ if $t \leqq \pi$ and 0 if $t \to \infty$; here, $y(0) = 0, y'(0) = 0$. This models an undamped system on which a force F acts during some interval of time (see Fig. 59), for instance, the force on a gun barrel when a shell is fired, the barrel being braked by heavy springs (and then damped by a dashpot, which we disregard for simplicity). *Hint:* At π both y and y' must be continuous.

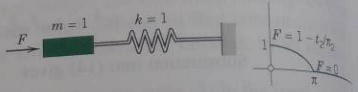
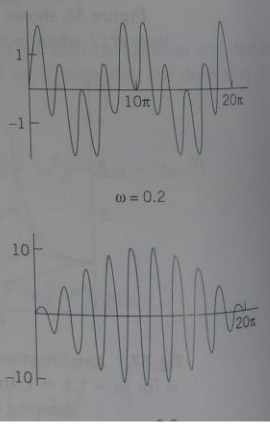


Fig. 59. Problem 24

25. CAS EXPERIMENT. Undamped Vibrations. **(a)** Solve the initial value problem $y'' + y = \cos \omega t$, $\omega^2 \neq 1, y(0) = 0, y'(0) = 0$. Show that the solution can be written

$$y(t) = \frac{2}{1 - \omega^2} \sin\left[\tfrac{1}{2}(1 + \omega)t\right] \sin\left[\tfrac{1}{2}(1 - \omega)t\right]$$

(b) Experiment with the solution by changing ω to see the change of the curves from those for small ω (>0) to beats, to resonance, and to large values of ω (see Fig. 60).



2.9 Modeling:

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11. $R = 24\ \Omega, L = 1.2\text{ H}, C = \frac{1}{90}\text{ F}$,
$E = 220 \sin 5t$ V

12. $R = 0.2\ \Omega, L = 0.1\text{ H}, C = 2\text{ F}, E = 220 \sin 314t$ V

13. $R = 16\ \Omega, L = 2.0\text{ H}, C = \frac{1}{200}\text{ F}, E = 120 \sin 50t$ V

14. Prove the claim in the text that if $R \neq 0$ (hence $R > 0$), then the transient current approaches I_p as $t \to \infty$.

15. Cases of damping. What are the conditions for an *RLC*-circuit to be (I) overdamped, (II) critically damped, (III) underdamped? What is the critical resistance R_{crit} (the analog of the critical damping constant $2\sqrt{mk}$)?

16–18 Solve the **initial value problem** for the *RLC*-circuit in Fig. 61 with the given data, assuming zero initial current and charge. Graph or sketch the solution. Show the details of your work.

16. $R = 8\ \Omega, L = 0.2\text{ H}, C = 12.5 \cdot 10^{-3}\text{ F}$,
$E = 100 \sin 10t$ V

17. $R = 18\ \Omega, L = 1\text{ H}, C = \frac{1}{250}\text{ F}$,
$E = 250 (\cos t + \sin t)$

18. $R = 14\ \Omega, L = 1\text{ H}, C = \frac{1}{40}\text{ F}, E = 220 \cos 4t$ V

19. WRITING REPORT. Mechanic-Electric Analogy. Explain Table 2.2 in a 1–2 page report with examples, e.g., the analog (with $L = 1$ H) of a mass–spring system of mass 5 kg, damping constant 10 kg/sec, spring constant 60 kg/sec^2, and driving force $220 \cos 10t$ kg/sec.

20. Complex Solution Method. Solve $L\tilde{I}'' + R\tilde{I}' + \tilde{I}/C = E_0 e^{i\omega t}$, $i = \sqrt{-1}$, by substituting $I_p = Ke^{i\omega t}$ (K unknown) and its derivatives and taking the real part I_p of the solution $\tilde{I}_p$. Show agreement with (2), (4). *Hint:* Use (11) $e^{i\omega t} = \cos \omega t + i \sin \omega t$; cf. Sec. 2.2, and $i^2 = -1$.

2.10 Solution by Variation of Parameters

We continue our discussion of nonhomogeneous linear ODEs, that is

$$y'' + p(x)y' + q(x)y = r(x). \tag{1}$$

In Sec. 2.6 we have seen that a general solution of (1) is the sum of a general solution y_h of the corresponding homogeneous ODE and any particular solution y_p of (1). To obtain y_p when $r(x)$ is not too complicated, we can often use the *method of undetermined coefficients*, as we have shown in Sec. 2.7 and applied to basic engineering models in Secs. 2.8 and 2.9.

However, since this method is restricted to functions $r(x)$ whose derivatives are of a form similar to $r(x)$ itself (powers, exponential functions, etc.), it is desirable to have a method valid for more general ODEs (1), which we shall now develop. It is called the **method of variation**

2-10

Since y_1, y_2 form a basis, we have $W \neq 0$ (by Theorem 2 in Sec. 2.6) and can

$$u' = -\frac{y_2 r}{W}, \qquad v' = \frac{y_1 r}{W}. \tag{10}$$

By integration,

$$u = -\int \frac{y_2 r}{W}\, dx, \qquad v = \int \frac{y_1 r}{W}\, dx.$$

These integrals exist because $r(x)$ is continuous. Inserting them into (5) gives (2) and completes the derivation. ■

PROBLEM SET 2.10

1–13 **GENERAL SOLUTION**

Solve the given nonhomogeneous linear ODE by variation of parameters or undetermined coefficients. Show the details of your work.

1. $y'' + 4y = \cos 2x$
2. $y'' + 9y = \csc 3x$
3. $x^2y'' - xy' - 3y = x^2$
4. $y'' - 4y' + 5y = e^{2x} \csc x$
5. $y'' + y = \cos x - \sin x$
6. $(D^2 + 6D + 9I)y = 16e^{-3x}/(x^2 + 1)$
7. $(D^2 - 2D + I)y = 6x^2 e^{-x}$
8. $(D^2 + 4I)y = \cosh 2x$
9. $(D^2 - 2D + I)y = 35x^{3/2}e^x$
10. $(D^2 + 2D + 2I)y = 4e^{-x} \sec^3 x$
11. $(x^2D^2 - 4xD + 6I)y = 21x^{-4}$
12. $(D^2 - I)y = 1/\sinh x$
13. $(x^2D^2 + xD - 9I)y = 48x^5$
14. **TEAM PROJECT. Comparison of Methods. Invention.** The undetermined-coefficient method should be used whenever possible because it is simpler. Compare it with the present method as follows.

    (a) Solve $y'' + 4y' + 3y = 65 \cos 2x$ by both methods, showing all details, and compare.

    (b) Solve $y'' - 2y' + y = r_1 + r_2$, $r_1 = 35x^{3/2}e^x$ $r_2 = x^2$ by applying each method to a suitable function on the right.

    (c) Experiment to invent an undetermined-coefficient method for nonhomogeneous Euler–Cauchy equations.

CHAPTER 2 REVIEW QUESTIONS AND PROBLEMS

7–18 GENERAL SOLUTION