Contents

37 Relativity

Relativity is the field of study that measures events, where and when they happen, and by how much any two events are separated in space and in time. In addition, relativity has to do with transforming such measurements (and also measurements of energy and momentum) between reference frames that move relative to each other.

Einstein's special theory of relativity deals with inertial reference frames, which are frames in which Newton's laws are valid. His general theory of relativity treats the more challenging situation in which reference frames can undergo gravitational acceleration.

37.1 The postulates

1. The Relativity Postulate: The law of physics are the same in all inertial frames. No one frame is preferred over any other.
2. The Speed of Light Postulate: The speed of light in vacuum has the same value c in all directions and in all inertial frame.

37.1.1 The Ultimate Speed

This ultimate speed has been defined to be exactly

$$c = 299,792,458 m/s$$

37.2 Measuring an Event

An **event** is something that happens, and every event can be assigned three space coordinates and one time coordinate.
1. The Space Coordinates. We imagine the observer's coordinate system fitted with a close-packed, three-dimensional array of measuring rods, one set of rods parallel to each of the three coordinate axes. These rods provide a way to determine coordinates along the axes.
2. The Time Coordinate. For the time coordinate, we imagine that every point of intersection in the array of measuring rods includes a tiny clock, which the observer can read because the clock is illuminated by the light generated by the event.
3. The Space-time Coordinates. The observer can now assign space-time coordinates to an event by simply recording the time on the clock nearest the event and the position as measured on the nearest measuring rods. If there are two events, the observer computes their separation in time as the difference in the times on clocks near each and their separation in space from the differences in coordinates on rods near each.

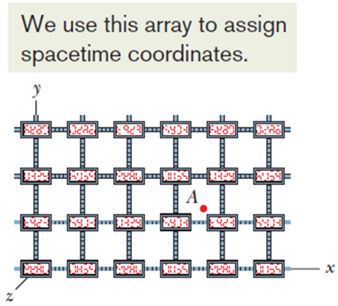
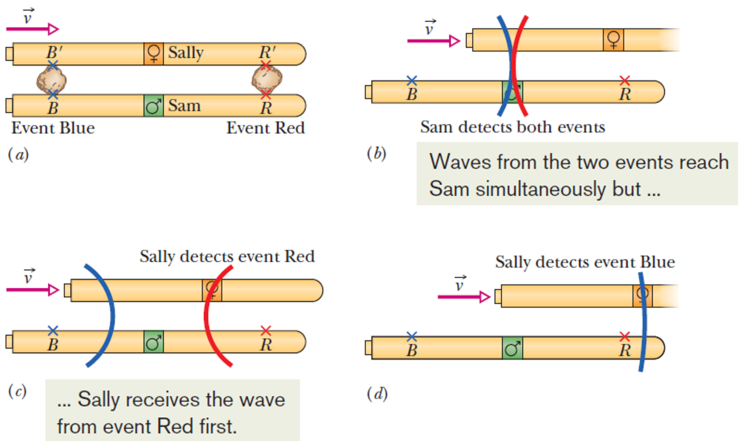


Fig. 37-3 One section of a three-dimensional array of clocks and measuring rods by which an observer can assign spacetime coordinates to an event, such as a flash of light at point A. The event's space coordinates are approximately $x = 3.6$ rod lengths, $y = 1.3$ rod lengths, and $z = 0$. The time coordinate is whatever time appears on the clock closest to A at the instant of the flash.

37.3 The Relativity of Simultaneity

If two observes are in relative motion, they will not, in general, agree to whether two events are simultaneous. If one observer finds them to be simultaneous, the other generally will not.

Simultaneity is not an absolute concept but rather a relative one, depending on the motion of the observer.



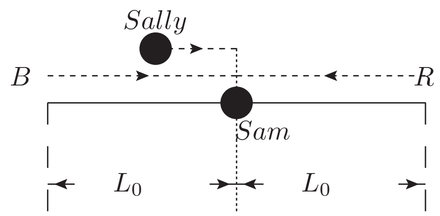
The spaceships of Sally and Sam and the occurrences of events from Sam's view. Sally's ship moves rightward with velocity $\vec{v}$. (a) Event Red occurs at positions RR' and the event Blue occurs at position BB'; each event sends out a wave of light. (b) Sam simultaneously detects the waves from event Red and event Blue. (c) Sally detects the wave from event Red. (d) Sally detects the wave from event Blue.
According to **Sam**: Light from event Red and light from Event Blue reached me at the same time. I find that I was standing halfway between the two sources. There event Red and event Blue are simultaneous events.

In Sam's reference frame, the distance between Sam and the R mark on the space ship is L_0 so is the distance between Sam and the B mark. Sally detected event Red after the lapse of time Δt_R (Sam's time) which must satisfy $c\Delta t_R + v\Delta t_R = L_0$. Sally detected event Blue after Δt_B with $c\Delta t_B - v\Delta t_B = L_0$. The distance travelled by Sally to meet event Red is $\Delta x_R = v\Delta t_R$ while the distance for event Blue is $\Delta x_B = v\Delta t_B$. Since

$$\Delta x_R = v\Delta t_R = \frac{vL_0}{c+v} < \frac{vL_0}{c-v} = \Delta x_B,$$

event Red was detected by Sally before event Blue.
According to **Sally**: Light from Event Red reached me before light from event Blue did. From the marks on my spaceship, I found that I too was standing halfway between the two sources. Therefore, the events were not simultaneous; event Red occurred first, followed by event Blue.

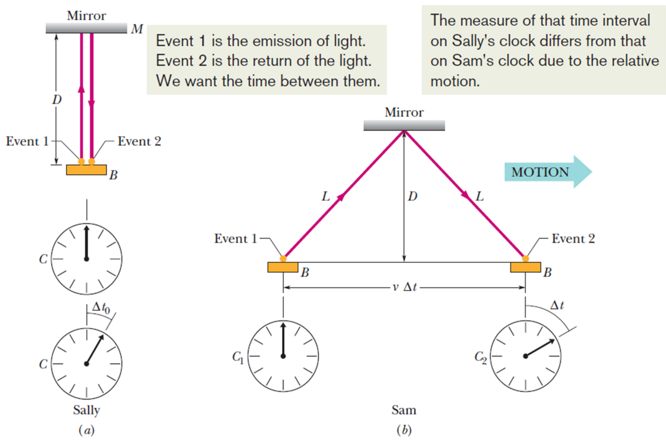
37.3.1 Another Method



Event R and event B occurred simultaneously and at equal distance L_0 from Sam in the rest frame of Sam. The two light signals met each other at the Sam's location. Suppose Sally who moved to the right with a velocity v relative to Sam also arrived at the location of Sam at the instant when the two light signals arrived at Sam's location. In Sally's frame, light from event R travelled a longer distance than the light from event B and both light signals also met each other at Sally's location. Since the velocities of the two light signals are the same, event R occurred earlier than event B did according to Sally.

37.4 The Relativity of Time

The time interval between two events depends on how far apart they occur in both space and time; that is, their spatial and temporal separations are entangled.



$$\Delta t_0 = \frac{2D}{c} \quad \text{(Sally)}$$

$$\Delta t = \frac{2L}{c} \quad \text{(Sam)}$$

$$\begin{aligned} L = \frac{1}{2}c\Delta t &= \sqrt{\left(\frac{1}{2}v\Delta t\right)^2 + D^2} \\ &= \sqrt{\left(\frac{1}{2}v\Delta t\right)^2 + \left(\frac{1}{2}c\Delta t_0\right)^2} \end{aligned}$$

$\Rightarrow$

$$\left(\frac{1}{2}c\Delta t\right)^2 = \left(\frac{1}{2}v\Delta t\right)^2 + \left(\frac{1}{2}c\Delta t_0\right)^2$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \Delta t_0 \quad \text{(time dilation)}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

and

$$\beta = \frac{v}{c}$$

γ is called the Lorentz factor.

When two events occur at the same location in an inertial frame, the time interval between them, measured in that frame, is called the proper time interval or the proper time. Measurements of the same time interval from any other inertial frame are always greater.

In the previous case, Sally measures a proper time interval, and Sam measures a greater time interval. The amount by which a measured time interval is greater than the corresponding proper time interval is called **time dilation**.

37.4.1 Example, Time dilation of spacecraft which returns to Earth

Your starship passes Earth with a relative speed of $0.9990c$. After traveling $10.0y$ (your time), you stop at lookout post LP13, turn, and then travel back to earth with the same relative speed. The trip back takes another $10.0y$ (your time). How long does the round trip take according to measurements made on Earth? (Neglect any effects due to the accelerations involved with sopping, turning, and getting back to speed.)

We begin by analyzing the outward trip:

1. This problem involves measurements made from two (inertial) reference frames, one attached to Earth and the other (your reference frame) attached to your ship.
2. The outward trip involves two events: the start of the trip at Earth and the end of trip at LP13.
3. Your measurement of $10.0y$ for the outward trip is the proper time Δt_0 between those two events, because the events occur at the same location in your reference frame — namely, on your ship.

4. The Earth-frame measurement of the time interval Δt for the outward trip must be greater than Δt_0 according to time dilation ($\Delta t = \gamma \Delta t_0$).

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{10.0y}{\sqrt{1 - (0.9990)^2}} = (22.37)\, 10.0y$$
$$= 224y$$

On the return trip, we have the same situation and the same data. Thus the round trip requires $20.0y$ of your time but

$$\Delta t_{total} = (2)\,(224y) = 448y$$

of Earth time. In other words,you have aged $20y$ while the Earth has aged $448y$, although you cannot travel into the past (as far as we know), you can travel into the future of, say, Earth, by using high-speed relative motion to adjust the rate at which time passes.

37.4.2 Example, Time dilation and travel distance for a relativistic particle

The elementary particle known as the positive kaon (K^+) is unstable in that it can decay (transform) into other particles. Although the decay occurs randomly, we find that, on average, a positive kaon has a lifetime of $0.1237\mu s$ when stationary–that is, when the lifetime is measured in the rest frame of the kaon. If a positive kaon has a speed of $0.990c$ relative to a laboratory reference frame when the kaon is produced, how far can it travel in that frame during its lifetime according to classical physics (which is a reasonable approximation for speeds much less than c) and according to special relativity (which is correct for all physically possible speed)?
Classical Physics: In classical physics we would find the same distance and time interval whether we measured them from the kaon frame or from the laboratory frame. Thus, we need not be careful about the frame in which the measurements are made. To find the kaon's travel distance d_{cp} according to classical physics, we first write

$$d_{cp} = v\Delta t$$

where Δt is the time interval between the two events in either frame. Then, substituting $0.990c$ for v and $0.1237\mu s$ for Δt, we find

$$\begin{aligned} d_{cp} &= (0.990c)\,(0.1237\mu s) \\ &= (0.990)\,(299792458m/s)\,(0.1237\mu s) \\ &36.7m \end{aligned}$$

Special Relativity: In Special Relativity we must be very careful that both the distance and the time interval are measured in the same reference frame—especially when the speed is close to c, as here. Thus, to find the actual travel distance d_{sr} of the kaon as measured from the laboratory frame and according to special relativity,we write

$$d_{sr} = v\Delta t$$

where Δt is the time interval between the two events as measured from the laboratory frame.

Before we can evaluate d_{sr}, we must find Δt. The $0.1237\mu s$ time interval is a proper time because the two events occur at the same location in the kaon frame— namely, at the kaon itself. Therefore, let Δt_0 represent this proper time interval. Then we can use ($\Delta t = \gamma \Delta t_0$) for time dilation to find the time interval Δt as measured from the laboratory frame.

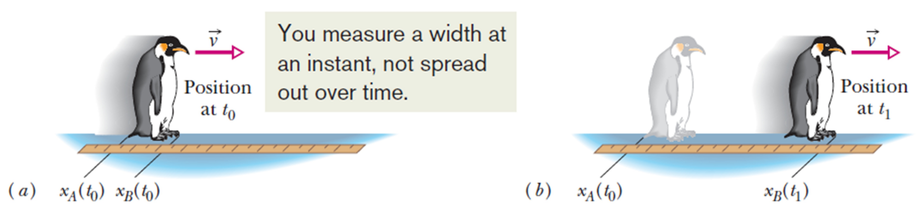
$$\Delta t = \frac{\Delta t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{0.1237 \times 10^{-6}s}{\sqrt{1-(0.990)^2}} = 8.769 \times 10^{-7}s$$

This is about seven times longer than the kaon's proper lifetime.

$$d_{sr} = v\Delta t = (0.990c)\left(8.769 \times 10^{-7}s\right) = 260m$$

This is about seven times d_{cp}. Experiments like the one outlined here, which verify special relativity, became routine in physics laboratories decades ago.

37.5 The relativity of Length



If you want to measure the front-to-back length of a penguin while it is moving, you must mark the positions of its front and back simultaneously (in your reference frame), as in (a), rather than at different times, as in (b).

The length L_0 of an object measured in the rest frame of the object is its proper length or rest length. Measurements of the length from any reference frame that is in relative motion parallel to the length are always less than the proper length.

Consider that both Sally, seated on a train moving through a station, and Sam, again on the station platform, want to measure the length of the platform.

Sam, using a tape measure, finds the length to be L_0, a proper length, because the platform is at rest with respect to him. Sam also notes that Sally, on the train, moves through this length in a time $\Delta t = L_0/v$, where v is the speed of the train.

Therefore,

$$L_0 = v\Delta t \quad \text{(Sam)}$$

For Sally, however, the platform is moving past her. She finds that the two events measured by Sam occur at the same place in her reference frame. She can time them with a single stationary clock, and so the interval t_0 that she measures is a proper time interval. To her, the length L of the platform is given by

$$L = v\Delta t_0 \quad \text{(Sally)}$$

Therefore,

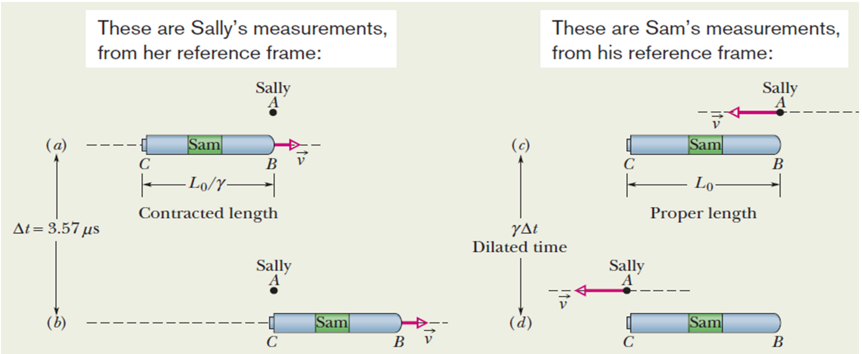
$$\frac{L}{L_0} = \frac{v\Delta t_0}{v\Delta t} = \frac{1}{\gamma}$$

Let L_0 be the length of a rod that you measure when the rod is stationary (meaning you and it are in the same reference frame, the rod's rest frame). If, instead, there is relative motion at speed v between you and the rod along the length of the rod, then with simultaneous measurements you obtain a length L given by

$$L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma} \quad \text{(length contraction)}$$

Since the Lorentz factor γ is always greater than unity if there is relative motion, L is less than L_0.

37.5.1 Example, Time dilation and length contraction



In the above figure, Sally (at point A) and Sam's spaceship (of proper length $L_0 = 230m$) pass each other with constant speed v. Sally measures a time interval of $3.57\mu s$ for the ship to pass her (from the passage of point B to the passage of point C). In terms of c, what is the relative speed v between Sally and the ship?

We are free to use either frame for the measurements. Because we know that the time interval Δt between the two events measured from Sally's frame is $3.57\mu s$, let us also use the distance L between the two events measured in her frame. Then

$$v = \frac{L}{\Delta t} = \frac{L_0/\gamma}{\Delta t} = \frac{L_0\sqrt{1-\frac{v^2}{c^2}}}{\Delta t}$$

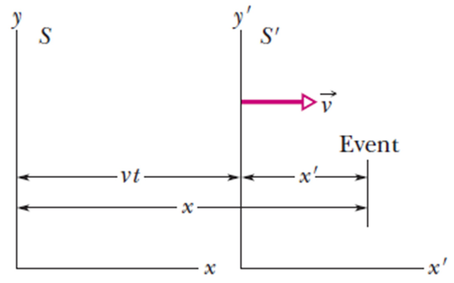
$$v^2c^2\Delta t^2 = L_0^2\left(c^2 - v^2\right)$$

$$v^2\left(c^2\Delta t^2 + L_0^2\right) = L_0^2c^2$$

$$v = \frac{L_0 c}{\sqrt{c^2\Delta t^2 + L_0^2}} = \frac{(230m)\,c}{\sqrt{((299792458m/s)\,3.57\times10^{-6}s) + (230m)^2}}$$

$$= 0.210c$$

37.6 The Galilean Transformation



Two inertial frames: frame S' has velocity $\vec{v}$ relative to frame S.

The Galilean Transformation Equations:

$$\begin{aligned} x' &= x - vt \\ t' &= t \end{aligned}$$

37.7 The Lorentz Transformation Equations

Consider two inertial reference frames K and K' with a relative velocity $\vec{v}$ between them. If

$$c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0,$$

we must also have

$$c^2 (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = 0,$$

and vice versa where Δt is the time difference and $\Delta \vec{x}$ is the spatial difference between the coordinates of two events. Suppose $[c\Delta t', \Delta x', \Delta y', \Delta z']$ is linearly related to $[c\Delta t, \Delta x, \Delta y, \Delta z]$. *i.e.*,

$$\begin{bmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

where the dimensionless elements of the matrix

$$\begin{bmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{bmatrix}$$

are space-time independent and depend only on the dimensionless parameter $\frac{\vec{v}}{c}$. Then

$$\begin{aligned} & c^2 \left(\Delta t'\right)^2 - \left(\Delta x'\right)^2 - \left(\Delta y'\right)^2 - \left(\Delta z'\right)^2 \\ & = \left(L_{00} c\Delta t + L_{01}\Delta x + ...\right)^2 - \left(L_{10} c\Delta t + L_{11}\Delta x + ...\right)^2 - ... \\ & = Ac^2 \left(\Delta t\right)^2 + B\left(\Delta x\right)^2 + C\left(\Delta y\right)^2 + D\left(\Delta z\right)^2 \\ & + c\Delta t\left(E_1 \Delta x + E_2 \Delta y + E_3 \Delta z\right) \\ & + F\Delta x\Delta y + G\Delta x\Delta z + H\Delta y\Delta z \end{aligned} \tag{1}$$

where the coefficients $(A, B, C, D, E_1, E_2, E_3, F, G, H)$ are dimensionless and depend only on $\frac{\vec{v}}{c}$

1. Assume $[c\Delta t, \Delta x, \Delta y, \Delta z] = s\,[1, 1, 0, 0]$. Then from(1) we have

$$0 = A + B + E_1 \tag{2}$$

2. Set $[c\Delta t, \Delta x, \Delta y, \Delta z] = s\,[1, -1, 0, 0]$. (1) yields

$$0 = A + B - E_1 \tag{3}$$

Combining (2) and (3), we get

$$E_1 = 0$$

$$B = -A$$

Similarly, we can show that

$$E_2 = E_3 = 0$$

$$C = D = -A$$

3. Let $[c\Delta t, \Delta x, \Delta y, \Delta z] = s\left[\sqrt{2}, 1, 1, 0\right]$. yields the condition

$$F = 0$$

In a similar manner, we may derive that

$$G = H = 0$$

To summarize, the universality of the speed of light c gives us

$$\begin{aligned} & c^2 (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \\ &= \lambda\left(\frac{\vec{v}}{c}\right)\left(c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\right) \end{aligned} \tag{4}$$

where $\lambda = A$ is dimensionless and depends only on $\frac{\vec{v}}{c}$. From the symmetry of relativity, we also have

$$\begin{aligned} & \left(c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\right) \\ &= \lambda\left(-\frac{\vec{v}}{c}\right)\left(c^2 (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2\right). \end{aligned}$$

Therefore

$$\lambda\left(\frac{\vec{v}}{c}\right)\lambda\left(-\frac{\vec{v}}{c}\right) = 1$$

If the space is isotropic, there is no preferred direction, we may assume $\lambda\left(\frac{\vec{v}}{c}\right) = \lambda\left(\frac{|\vec{v}|}{c}\right)$. Consequently

$$\lambda^2 = 1, \text{ and } \lambda = \pm 1$$

In the limit $v \to 0$, K becomes essentially identical to K'. So $\lambda(0) = 1$. If $\lambda\left(\frac{\vec{v}}{c}\right)$ is a continuous function of $\vec{v}$, $\lambda = 1$ is the only choice.

$$\lambda = 1 \quad \text{(to satisfy continuity condition)}$$

Thus we arrive at

$$\begin{aligned} & c^2 (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \\ &= c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \end{aligned} \tag{5}$$

37.7.1 Lorentz Transformation

The length Δs along a direction perpendicular to the boost velocity $\vec{v}$ between K and K' ($\vec{v}$ is the velocity of the origin of K' observed in K) are invariant

$$\Delta s' = \Delta s$$

This is because of the symmetry of relativity. A direction perpendicular to $\vec{v}$ is indistinguishable from a direction perpendicular to $-\vec{v}$. $\Delta s'$ is the length in K' moving with velocity $\vec{v}$ relative to K and Δs may be treated as the length in the frame K which moves with velocity $-\vec{v}$ relative to K'. In particular, if the boost velocity is in the x direction. we have $\Delta y' = \Delta y, \Delta z' = \Delta z$ and

$$(c\Delta t')^2 - (\Delta x')^2 = (c\Delta t)^2 - (\Delta x)^2 \tag{6}$$

Assume the linear transformation for the coordinates are

$$\begin{bmatrix} c\Delta t' \\ \Delta x' \end{bmatrix} = \begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix}$$

or

$$\begin{aligned} c\Delta t' &= L_{00} c\Delta t + L_{01}\Delta x \\ \Delta x' &= L_{10} c\Delta t + L_{11}\Delta x \end{aligned}$$

Thus

$$\begin{aligned} &(c\Delta t')^2 - (\Delta x')^2 \\ &= (L_{00} c\Delta t + L_{01}\Delta x)^2 - (L_{10} c\Delta t + L_{11}\Delta x)^2 \\ &= \left((L_{00})^2 - (L_{10})^2\right)(c\Delta t)^2 - \left((L_{11})^2 - (L_{01})^2\right)\Delta x^2 \\ &+ 2\left(L_{00}L_{01} - L_{10}L_{11}\right) c\Delta t \Delta x \end{aligned}$$

Substituting the above into (6), we get

$$\begin{aligned} (L_{00})^2 - (L_{10})^2 &= 1 \\ (L_{11})^2 - (L_{01})^2 &= 1 \\ L_{00}L_{01} - L_{10}L_{11} &= 0 \end{aligned} \tag{7}$$

Recall that

$$\begin{aligned} &\cosh\xi = \frac{e^{\xi} + e^{-\xi}}{2}, \\ &\sinh\xi = \frac{e^{\xi} - e^{-\xi}}{2}, \\ &\cosh^2\xi - \sinh^2\xi = 1, \end{aligned}$$

and that

$$
\begin{aligned}
&\cosh\xi_1 \sinh\xi_2 + \cosh\xi_2 \sinh\xi_1 \\
&= \frac{e^{\xi_1} + e^{-\xi_1}}{2} \frac{e^{\xi_2} - e^{-\xi_2}}{2} + \frac{e^{\xi_2} + e^{-\xi_2}}{2} \frac{e^{\xi_1} - e^{-\xi_1}}{2} \\
&= \frac{e^{\xi_1+\xi_2} - e^{-(\xi_1+\xi_2)}}{2} = \sinh(\xi_1 + \xi_2)
\end{aligned}
$$

The first two equations in (7) allow the identification

$$L_{00} = \cosh\xi, L_{10} = -\sinh\xi,$$

$$L_{11} = \cosh\xi', L_{01} = -\sinh\xi',$$

The third equation of (7) gives

$$\cosh\xi \sinh\xi' - \cosh\xi' \sinh\xi = 0$$

which leads to

$$\sinh(\xi - \xi') = 0$$

or $\xi = \xi'$. ξ will be called rapidity, and the transformation may be written as

$$\begin{bmatrix} c\Delta t' \\ \Delta x' \end{bmatrix} = \begin{bmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix}$$

The 2nd row of the above matrix equation is

$$\Delta x' = -\sinh\xi\,(c\Delta t) + \cosh\xi\,(\Delta x)$$

By definition, the origin of the moving frame K' has the trajectory $\Delta x' = 0 = -\sinh\xi\,(c\Delta t) + \cosh\xi\,(\Delta x)$ and $\Delta x = v\Delta t = \beta\,(c\Delta t)$ in frame K where $\beta = \frac{v}{c}$. Thus

$$\beta = \tanh\xi$$

So

$$\cosh\xi = \sqrt{\frac{\cosh^2\xi}{\cosh^2\xi - \sinh^2\xi}} = \sqrt{\frac{1}{1 - \tanh^2\xi}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

$$\sinh\xi = \cosh\xi \tanh\xi = \gamma\beta$$

and we arrive at the following Lorentz transformation for boost along x-axis:

$$\begin{bmatrix} c\Delta t' \\ \Delta x' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix}$$

which is equivalent to

$$c\Delta t' = \gamma\left(c\Delta t - \beta\Delta x\right)$$
$$\Delta x' = \gamma\left(\Delta x - \beta c\Delta t\right)$$

or

$$\Delta t' = \gamma\left(\Delta t - \frac{\beta}{c}\Delta x\right) = \frac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$\Delta x' = \gamma\left(\Delta x - \beta c\Delta t\right) = \frac{\Delta x - v\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}$$

Another Derivation of Lorentz Transformation Assume

$$\begin{bmatrix} c\Delta t' \\ \Delta x' \end{bmatrix} = \begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix}\begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix}.$$

According to frame K, the origin of K' moves in the x-direction with velocity $v = c\beta$. Therefore, we must have

$$\begin{aligned} \begin{bmatrix} c\Delta t' \\ 0 \end{bmatrix} &= \begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix}\begin{bmatrix} c\Delta t \\ \beta c\Delta t \end{bmatrix} \\ &= \begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix}\begin{bmatrix} 1 \\ \beta \end{bmatrix}(c\Delta t) \\ &= \begin{bmatrix} L_{00}+\beta L_{01} \\ L_{10}+\beta L_{11} \end{bmatrix}(c\Delta t) \end{aligned}$$

Thus

$$L_{10} + \beta L_{11} = 0 \rightarrow L_{10} = -\beta L_{11} \tag{8}$$
$$\Delta t' = \left(L_{00} + \beta L_{01}\right)\Delta t$$

According to time dilation, $\Delta t' = \frac{\Delta t}{\gamma}$ when $\Delta x' = 0$. We are led to

$$\frac{1}{\gamma} = L_{00} + \beta L_{01} \rightarrow L_{01} = \frac{1}{\beta}\left(\frac{1}{\gamma} - L_{00}\right) \tag{9}$$

On the other hand, the origin of K moves with velocity $-v$ relative to K', $\Delta x = 0 \Leftrightarrow \Delta x' = -v\Delta t' = -\beta\left(c\Delta t'\right)$. Thus

$$\begin{bmatrix} c\Delta t' \\ -\beta\left(c\Delta t'\right) \end{bmatrix} = \begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix}\begin{bmatrix} c\Delta t \\ 0 \end{bmatrix}.$$

$$c\Delta t' = L_{00}\left(c\Delta t\right),$$
$$-\beta\left(c\Delta t'\right) = L_{10}\left(c\Delta t\right),$$

$$L_{10} = -\beta L_{00} \to L_{11} = L_{00} \tag{10}$$

The time dilation for two events at the same spatial location at K frame with $\Delta x = 0$ is $\Delta t = \frac{\Delta t'}{\gamma}$. We have accordingly

$$\left[\begin{array}{c} c\Delta t' \\ \Delta x' \end{array}\right] = \left[\begin{array}{cc} L_{00} & \frac{1}{\beta}\left(\frac{1}{\gamma} - L_{00}\right) \\ -\beta L_{00} & L_{00} \end{array}\right]\left[\begin{array}{c} c\frac{\Delta t'}{\gamma} \\ 0 \end{array}\right],$$

of which the equation for the 1st row gives us

$$c\Delta t' = L_{00}c\frac{\Delta t'}{\gamma} \to L_{00} = \gamma. \tag{11}$$

(9) becomes

$$L_{01} = \frac{1}{\beta}\left(\frac{1}{\gamma} - L_{00}\right) = \frac{1}{\beta}\left(\frac{1}{\gamma} - \gamma\right) = -\frac{\gamma}{\beta}\left(1 - \frac{1}{\gamma^2}\right) = -\gamma\beta \tag{12}$$

Summarizing (8)-(12), we get

$$\left[\begin{array}{c} c\Delta t' \\ \Delta x' \end{array}\right] = \gamma\left[\begin{array}{cc} 1 & -\beta \\ -\beta & 1 \end{array}\right]\left[\begin{array}{c} c\Delta t \\ \Delta x \end{array}\right].$$

37.7.2 Light Cone

A fruitful concept in special relativity is the idea of the light cone and "space-like" and "timelike" separations between two events. Consider the invariant separation or interval s_{12} between two events $P_1\left(t_1, \vec{x}_1\right)$ and $P_2\left(t_2, \vec{x}_2\right)$ in space-time. The square of the Lorentz invariant interval is

$$s_{12}^2 = c^2\left(t_2 - t_1\right)^2 - \left|\vec{x}_2 - \vec{x}_1\right|^2$$

There three possibilities:(1) $s_{12}^2 > 0$ (2) $s_{12}^2 = 0$ (3) $s_{12}^2 < 0$. Let us rotate the coordinate axes such that $\vec{x}_2 - \vec{x}_1$ is along the x direction. Then

$$s_{12}^2 = c^2\left(t_2 - t_1\right)^2 - \left|x_2 - x_1\right|^2$$

If $s_{12}^2 > 0$, the events are said to have a timelike separation. Let

$$\beta = \frac{x_2 - x_1}{c\,(t_2 - t_1)} = \frac{\Delta x}{c\Delta t}$$

be the relative velocity between K and K'. Note that $\beta = \left|\frac{\Delta x}{c\Delta t}\right| < 1$. Then

$$\Delta x' = \gamma\,(\Delta x - \beta\,(c\Delta t)) = 0$$

$$\begin{aligned} c\Delta t' &= \gamma\,(c\Delta t - \beta\Delta x) = \gamma\left(1 - \beta^2\right) c\Delta t \\ &= \sqrt{1 - \left|\frac{\Delta x}{c\Delta t}\right|^2}\, c\Delta t = \frac{c\Delta t}{\gamma} \end{aligned}$$

and thus it is always possible to find a Lorentz transformation to a new coordinate frame K', such that $x_1' = x_2'$ $(\Delta x' = 0)$, and

$$s_{12}^2 = c^2\,(t_2' - t_1')^2 = (c\Delta t')^2$$

In the frame K', the two events occur at the same spatial point, but are separated in time.

If $s_{12}^2 < 0$, the events are said to have a spacelike separation. Under this circumstance, set

$$\beta = \frac{c\,(t_2 - t_1)}{x_2 - x_1} = \frac{c\Delta t}{\Delta x}$$

and $\beta = \left|\frac{c\Delta t}{\Delta \vec{x}}\right| < 1$. For the frame K' moving with this $\vec{\beta}$ relative to K, we have

$$c\Delta t' = \gamma\,(c\Delta t - \beta\Delta x) = 0$$

$$\begin{aligned} \Delta x' &= \gamma\,(\Delta x - \beta\,(c\Delta t)) = \gamma\left(\Delta x - \beta^2\Delta x\right) \\ &= \gamma\left(1 - \beta^2\right)\Delta x = \frac{\Delta x}{\gamma} \end{aligned}$$

The final possibility $s_{12}^2 = 0$, implies a lightlike separation. The events lie on the light cone with respect to each other and can be connected only by light signal.

37.7.3 Proper Time

Consider a particle moving with an instantaneous velocity $\vec{u}\,(t)$ relative to some inertial frame K. In a time interval dt its position changes by $d\vec{x} = \vec{u}dt$. The square of the corresponding infinitesimal ds is

$$ds^2 = c^2dt^2 - |d\vec{x}|^2 = c^2dt^2\left(1-\beta^2\right)$$

where $\beta = \left|\vec{\beta}\right| = \left|\frac{\vec{u}}{c}\right|$. In the frame K' where the particle is instantaneously at rest the space-time increments are $dt' = d\tau, d\vec{x}' = 0$. Thus the invariant interval is

$$ds = cd\tau = \sqrt{1-\beta^2\,(t)}cdt$$

The increment $d\tau$ in the instantaneous rest frame is thus a Lorentz invariant quantity that takes the form

$$d\tau = \sqrt{1-\beta^2\,(t)}dt = \frac{dt}{\gamma\,(t)}$$

The time τ is called the proper time of the particle. It is the time as seen in the rest frame of the system. From the above, it follows

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2\,(\tau)}} = \int_{\tau_1}^{\tau_2} \gamma\,(\tau)\,d\tau$$

which expresses the phenomenon known as time dilation.

37.7.4 Four Vector

Identify $\langle \Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3 \rangle = \langle c\Delta t, \Delta x, \Delta y, \Delta z \rangle = \langle \Delta x^\mu \rangle$. Note that we have defined that $x_0 = ct$.The Lorentz transformation mentioned in the above describes the transformation of the coordinates from one inertial frame to another. By analogy we speak of a 4-vector $\langle A^0, A^1, A^2, A^3 \rangle$ which transforms like $\langle \Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3 \rangle$. The invariance of (4) (note $\lambda = 1$) for the coordinates has its counter part for the 4-vector

$$\left(A'^0\right)^2 - \left|\vec{A}'\right|^2 = \left(A^0\right)^2 - \left|\vec{A}\right|^2$$

Let $\langle B^0, B^1, B^2, B^3 \rangle$ be another 4-vector.

$$\left(B'^0\right)^2 - \left|\vec{B}'\right|^2 = \left(B^0\right)^2 - \left|\vec{B}\right|^2$$

$\left\langle A^0 + B^0, \vec{A} + \vec{B} \right\rangle$. is also a 4-vector. Thus

$$\left(A'^0 + B'^0\right)^2 - \left|\vec{A}' + \vec{B}'\right|^2 = \left(A^0 + B^0\right)^2 - \left|\vec{A} + \vec{B}\right|^2$$

The above three equations yield

$$A'^0 B'^0 - \vec{A}' \cdot \vec{B}' = A^0 B^0 - \vec{A} \cdot \vec{B}$$

The inner product for two 4-vectors $\langle A^\mu \rangle$ and $\langle B^\mu \rangle$ in this 4-dimensional space is defined as

$$A \cdot B \equiv A^0 B^0 - \vec{A} \cdot \vec{B}$$

The 4 dimensional space equipped with this inner product is called a Minkowski space in which the inner product is invariant under Lorentz transformations:

$$A' \cdot B' = A \cdot B$$

37.7.5 Relativistic Doppler Effect

Consider a plane wave of frequency ω and a wave vector $\vec{k}$ in the inertial frame K with a complex amplitude that is proportional to

$$e^{-i\left(\omega t - \vec{k} \cdot \vec{x}\right)}$$

In the moving frame K' this wave will have, in general, a different frequency ω' and wave vector $\vec{k}'$, but the phase of the wave is an invariant:

$$\phi = \omega t - \vec{k} \cdot \vec{x} = \omega' t' - \vec{k}' \cdot \vec{x}'$$

Since the scalar product of the 4-vectors $(ct, \vec{x})$ with another 4-vector is Lorentz invariant, $\left(\frac{\omega}{c}, \vec{k}\right)$ transforms like a 4-vector under Lorentz transformation. *.i.e.*, for a boost in the x direction with $\vec{\beta} = \beta \hat{\imath} = \frac{v}{c} \hat{\imath}$

$$\begin{aligned}
\frac{\omega'}{c} &= \gamma \left(\frac{\omega}{c} - \beta k_x\right) \\
k'_x &= \gamma \left(k_x - \beta \frac{\omega}{c}\right) \\
k'_y &= k_y, k'_z = k_z
\end{aligned}$$

For light waves, $\left(\frac{\omega}{c},\vec{k}\right)\cdot\left(\frac{\omega}{c},\vec{k}\right)=\left(\frac{\omega}{c}\right)^2-\left|\vec{k}\right|^2=0$, $\omega=c\left|\vec{k}\right|=ck$. The above transformation may be expressed in the more familiar form of the Doppler shift formulas

$$\begin{aligned}\omega' &= \gamma\omega\left(1-\beta\cos\theta\right) \\ \omega'\cos\theta' &= \gamma\omega\left(\cos\theta-\beta\right)\end{aligned} \tag{13}$$

where θ is the angle between $\vec{k}$ and $\vec{\beta}$ and θ' is the angle between $\vec{k}'$ and $\vec{\beta}$. ($\vec{k}\cdot\hat{\beta}=k_x=\frac{\omega}{c}\cos\theta, \vec{k}'\cdot\hat{\beta}=k'_x=\frac{\omega'}{c}\cos\theta'$) The ratio of the last two equations gives

$$\cos\theta'=\frac{\cos\theta-\beta}{1-\beta\cos\theta}$$

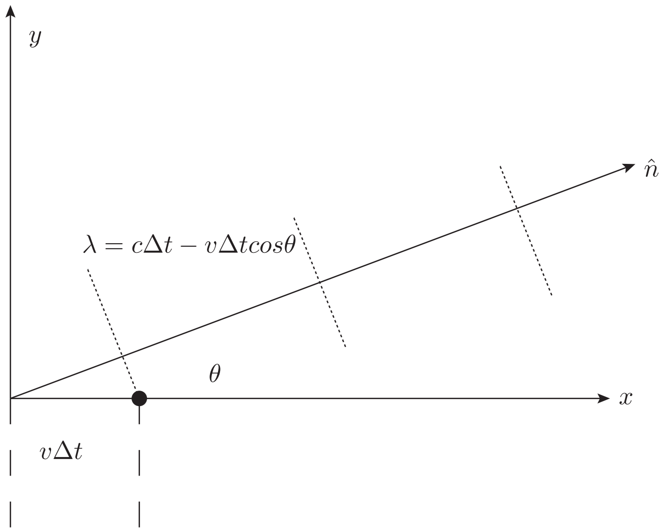
So

$$\begin{aligned}\sin\theta' &= \sqrt{1-\left(\frac{\cos\theta-\beta}{1-\beta\cos\theta}\right)^2} \\ &= \frac{\sqrt{\left(1-\beta\cos\theta\right)^2-\left(\cos\theta-\beta\right)^2}}{1-\beta\cos\theta}=\frac{\sin\theta}{\gamma\left(1-\beta\cos\theta\right)}\end{aligned}$$

and

$$\tan\theta'=\frac{\sin\theta}{\gamma\left(\cos\theta-\beta\right)}$$

Intuitive Derivation A light source moving in the x direction emits wave with period $\Delta\tau$ in the moving frame. The period becomes $\Delta t=\gamma\Delta\tau$ in the space frame but ticks at different location separated by $\Delta x=v\Delta t$. During the consecutive period Δt the light have travelled a distance $c\Delta t\hat{n}$ in the $\hat{n}$ direction and the source have also squeezed a distance $v\Delta t\hat{n}\cdot\hat{\imath}$ along the direction $\hat{n}$. (Imagine that the light source is a plane with normal $\hat{n}$)



The wavelength observed in the direction $\hat{n}$ is thus

$$\lambda = c\Delta t - v\Delta t\hat{n}\cdot\hat{\imath} = c\left(1-\beta\cos\theta\right)\Delta t = \gamma\left(1-\beta\cos\theta\right)c\Delta\tau = \gamma\left(1-\beta\cos\theta\right)\lambda'$$

where λ' is the wavelength in the frame where light source is stationary. Thus

$$\omega = \frac{1}{\gamma\left(1-\beta\cos\theta\right)}\omega'$$

(I) Forward direction, $\theta' = 0 \Leftrightarrow \theta = 0, \omega = \frac{1}{\gamma(1-\beta)}\omega' = \sqrt{\frac{1+\beta}{1-\beta}}\omega'$
(II) Backward $\theta' = \pi \Leftrightarrow \theta = \pi, \omega = \frac{1}{\gamma(1+\beta)}\omega' = \sqrt{\frac{1-\beta}{1+\beta}}\omega'$
(III) Vertical in moving frame $\theta' = -\frac{\pi}{2}, \cos\theta = \beta, \sin\theta = -\frac{1}{\gamma}, \omega = \gamma\omega'$
(IV) Vertical in the space frame $\theta = -\frac{\pi}{2}, \omega = \frac{1}{\gamma}\omega'$

37.8 Addition of Velocities, 4-Velocity

Suppose a particle has the velocity $u'\hat{\imath}$ in the inertial frame K' and the frame K' is moving with velocity $\vec{v} = c\beta\hat{\imath}$ in the positive x direction with respect to the inertial frame K. We wish to know the velocity $\vec{u} = u\hat{\imath}$ for the particle as seen from K, From the Lorentz transformation, we have

$$\begin{aligned} dx_0 &= \gamma_v \left(dx_0' + \beta dx'\right) \\ dx &= \gamma_v \left(dx' + \beta dx_0'\right) \\ dy &= dy' \\ dz &= dz' \end{aligned}$$

where we have put a subscript on γ to distinguish it below from $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$. Since $u' = c\frac{dx'}{dx_0'}$ and $u = c\frac{dx}{dx_0}$, Thus

$$u = c\frac{dx}{dx_0} = c\frac{dx' + \beta dx_0'}{dx_0' + \beta dx'} = \frac{c\frac{dx'}{dx_0'} + \beta c}{1 + \beta\frac{dx'}{dx_0'}} = \frac{u' + \beta c}{1 + \beta\frac{u'}{c}}$$

or

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}.$$

There is a 4-vector closely related to the ordinary velocity. The proper time τ is a scalar so is $d\tau$. Thus $\left(U_0, \vec{U}\right) = \left(\frac{dx_0}{d\tau}, \frac{d\vec{x}}{d\tau}\right)$ is a 4-vector which is called the 4-velocity:

$$U_0 = \frac{dx_0}{d\tau} = \frac{cdt}{\sqrt{dt^2 - \frac{|d\vec{x}|^2}{c^2}}} = \frac{c}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma_u c$$

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} = \gamma_u \vec{u}$$

Note

$$\left(U_0, \vec{U}\right) \cdot \left(U_0, \vec{U}\right) = \left(\frac{dx_0}{d\tau}\right)^2 - \left(\frac{d\vec{x}}{d\tau}\right)^2 = \frac{\left(dx_0\right)^2 - |d\vec{x}|^2}{\left(d\tau\right)^2} = c^2$$

is a Lorentz scalar. Note also in the limit of small speed $u \ll c$, $d\tau \simeq dt$ and $\vec{U} \simeq \frac{d\vec{x}}{dt} = \vec{u}$.

37.9 Relativistic momentum and Energy

The 4-vector defined by

$$\left(P_0, \vec{P}\right) = m\left(U_0, \vec{U}\right) = \left(\gamma_u mc, \gamma_u m\vec{u}\right)$$

is called the 4-momentum. For a particle with speed small compared to light, $u \ll c$,

$$\vec{P} = m\vec{U} \simeq m\vec{u}$$

is the non-relativistic spatial momentum of the particle and

$$P_0 = mU_0 = mc\frac{dt}{d\tau} = \frac{mc}{\sqrt{1 - \frac{u^2}{c^2}}} \simeq \frac{1}{c}\left(mc^2 + \frac{1}{2}mu^2\right)$$

The non-relativistic limit of cP_0 is identified as the energy E which is the kinetic energy $\frac{1}{2}mu^2$ plus the constant mc^2.

Knowing that

$$\begin{aligned} P_0^2 - \left|\vec{P}\right|^2 &= \left(P_0, \vec{P}\right) \cdot \left(P_0, \vec{P}\right) \\ &= m^2 \left(U_0, \vec{U}\right) \cdot \left(U_0, \vec{U}\right) = m^2c^2, \end{aligned}$$

we have

$$P_0^2 = \left|\vec{P}\right|^2 + m^2c^2$$

$\Rightarrow$

$$E^2 = c^2P_0^2 = \left|\vec{P}\right|^2 c^2 + m^2c^4$$

37.9.1 Example, Energy and momentum of a relativistic electron

(a) What is the total energy of an electron with $2.53Mev$ kinetic energy?

The total energy E is the sum of electron's mass energy mc^2 and its kinetic energy:

$$E = mc^2 + K.$$

The electron's mass energy mc^2 is equal to

$$\begin{aligned} mc^2 &= \left(9.109 \times 10^{-31}kg\right)(299792458m/s)^2 \\ &= 8.187 \times 10^{-14}J \\ &= 0.511Mev \end{aligned}$$

$$E = 0.511Mev + 2.53Mev = 3.04Mev$$

What is the magnitude p of the electron's momentum?

$$E^2 = c^2P_0^2 = p^2c^2 + m^2c^4$$

$$
\begin{aligned}
p &= \frac{\sqrt{E^2 - (mc^2)^2}}{c} = \frac{\sqrt{(3.04Mev)^2 - (0.511Mev)^2}}{c} \\
&= \frac{3.00Mev}{c}
\end{aligned}
$$