Instructor's Guide to Parallel Programming in C with MPI and OpenMP

Michael J. Quinn

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Contents

Preface

This booklet contains solutions to the "pencil and paper" exercises found in *Parallel Programming in C with MPI and OpenMP*. It does not contain solutions to the programming assignments, except for a few of the shortest ones. Likewise, it does not report results for the benchmarking exercises, since the times that students observe will vary greatly from system to system.

I apologize for any mistakes that have survived my proofreading. If you identify any errors in this solutions manual, or if you have any ideas for additional exercises, I would appreciate hearing from you.

Michael J. Quinn
Oregon State University
Department of Computer Science
Corvallis, OR 97331

`quinn@cs.orst.edu`

Chapter 1

Motivation and History

1.1 I like to do this exercise in class during the first lecture. It helps motivate many of the concepts discussed in the course, including speedup, contention, and scalability.

  (a) The best way to sort cards by hand can spark an entertaining debate. I believe it is faster to sort the cards initially by suit, then to use an insertion sort to order each suit in turn while holding it in your hands. (Knuth disagrees: see *The Art of Computer Programming, Volume 3, Sorting and Searching*, Addison-Wesley, 1973, pp. 170–171, for another opinion.)

  (b) Assuming no great variance in card-handling skills, the best way for p people to sort p decks of cards is to let each person sort one deck of cards. Assuming person i requires t_i time units to sort a deck of cards, the time needed for p persons to sort p decks of cards is $\max\{t_1, t_2, \ldots, t_p\}$.

  (c) The time needed for p people to sort 1 deck of cards depends heavily upon the parallel algorithm used. Students often use the following algorithm when $2 \leq p \leq 4$. Each person is assigned responsibility for one or two suits. The unsorted deck is divided among the people. Each person keeps the cards associated with his or her suit and tosses the other cards to the appropriate partner. After working through the unsorted deck, each person sorts the suits he or she has been assigned.

  One of my classes discovered an interesting parallel algorithm for six people that exhibits both pipelining and data parallelism. Persons 1 and 2 distribute the cards, while persons 3, 4, 5, and 6 are each responsible for sorting cards in a particular suit.

Person 1 starts with the deck of cards. He or she hands roughly one-half the cards to person 2. Persons 1 and 2 work through their cards, tossing each card to one of the other persons, depending upon the suit. Persons 3, 4, 5, and 6 take the cards tossed to them by persons 1 and 2 and use insertion sort to order the cards in their suit. When these people have finished sorting their cards, they hand them to person 1, who concatenates them, completing the sort.

1.2 It requires $\lceil \log_2 n \rceil$ folds to put n holes in a piece of paper with a single stroke of the hole punch.

Proof of correctness (by induction on number of holes). Let $f(i)$ represent the number of folds needed to put i holes in a piece of paper. Base case: With no folds of the paper, the punch can put one hole in it. Hence $f(1) = 0 = \lceil \log_2 1 \rceil$. Induction: Assume $f(k) = \lceil \log_2 k \rceil$ for $1 \leq n \leq k$. Prove true for $2k$. We want to make $2k$ holes. If we fold the paper in half $\lceil \log_2 k \rceil$ times, we can make k holes (induction hypothesis). Hence if we fold the paper in half once more we can make $2k$ holes. Suppose $k = 2^j + r$, where $r < 2^j$. Then $2k = 2^{j+1} + 2r$, where $2r < 2^{j+1}$. Hence

$$f(2k) = \lceil \log_2 k \rceil + 1 = (j + 1) + 1 = j + 2 = \lceil \log_2 2k \rceil$$

Optimality proof (by induction on number of folds): Let $h(i)$ denote the maximum number of holes that can be made with i folds of the paper. Base case: With no folds of the paper, the punch can put at most one hole in it. $h(0) = 2^0 = 1$. Induction: Assume $h(n) = 2^n$ for $0 \leq n \leq k$. Prove true for $k + 1$. Folding the paper once more allows us to double the number of holes. Hence $h(k+1) = 2h(k) = 2 \times 2^k = 2^{k+1}$.

Since at most 2^i holes can be made with i folds of the paper, an algorithm that makes j holes with $\lceil \log_2 j \rceil$ folds of the paper is optimal.

1.3 (a) Hand the cards to the accountant at the leftmost desk in the front row. The accountant keeps 25 cards and passes 975 cards to the accountant on her right. This accountant keeps 25 cards and passes 950 to the accountant on her right. This process continues until every accountant in the front row has 25 cards. Meanwhile, the front row accountants pass cards to the accountants behind them. The number of accountants receiving cards depends upon how many cards each accountant keeps. For example, if each accountant keeps 5 cards, then a total of 200 accountants (5 in each column) will receive cards.

(b) After each accountant has computed the total of the cards she controls, the subtotals are accumulated by reversing the flow used to distribute the cards. The last accountant in each column that received cards passes her subtotal to the accountant in front of her. This accountant adds the subtotal she received with her own subtotal and passes the sum to the accountant in front of her. Eventually the subtotals reach the front row. The accountants accumulate subtotals right to left until the leftmost accountant in the front row has the grand total.

(c) Increasing the number of accountants decreases the total computation time, but increases the total communication time. The overall time needed is the sum of these two times. For awhile, adding accountants reduces the overall time needed, but eventually the decrease in computation time is outweighed by the increase in communication time. At this point adding accountants increases, rather than decreases, overall execution time. The point at which this occurs depends upon your estimates of how long an addition takes and how long a communication takes. However, if 1,000 accountants are actually used, the computation must take longer than if 500 accountants are used, because there is nothing for the extra accountants to do other than waste time collecting single cards and then handing them back. (An accountant with only one card cannot do an addition.) For this reason the curve must slope up between 500 and 1,000 accountants.

(d) One accountant should require 10 times longer to add 10,000 numbers than 1,000 numbers. When adding 10,000 numbers, we should be able to accommodate many more accountants before overall execution time increases.

(e) One thousand accountants cannot perform the task one thousand time faster than one accountant because they spend time communicating with each other. This represents extra work that the single accountant does not have to do.

1.4 We will arrange the desks of the accountants into a binary tree. One accountant is at the root. The desks of two other accountants are near the desk of the "root" accountant. Each of these accountants have two accountants near them, and so on. The root accountant is given 1000 cards. She divides them in half, giving 500 to each of the two accountants near her. These accountants divide their portions in half, giving 250 cards to both of the accountants near them, and so on.

This communication pattern reduces the communication overhead, meaning that the overall execution time of the algorithm for a given number of accountants should be lower than the execution time of the algorithm described in the previous exercise.

(In Chapter 3 we'll see how a binomial tree is even better than a binary tree, because it avoids the problem of inactive interior nodes while computing gets done at the leaves.)

1.5 It takes m time units for the first task to be processed. After that, a new task is completed every time unit. Hence the time needed to complete n tasks is $m + n - 1$.

1.6 The time needed to copy k pages is $15 + 4 + (k - 1) = 18 + k$ seconds, or $(18 + k)/60$ minutes. We want to choose k such that

$$\frac{k}{(18+k)/60} = \frac{60k}{18+k} = 40$$

Solving for k, we get $k = 36$.

1.7 All of today's supercomputers are parallel computers. Today's supercomputers are capable of trillions of operations per second, and no single processor is that fast. Not every parallel computer is a supercomputer. A parallel computer with a few commodity CPUs is not capable of performing trillions of operations per second.

1.8 Performance increases exponentially. Since performance improves by a factor of 10 every 5 years, we know that $x^5 = 10$. Hence

$$5 \ln x = \ln 10 \Rightarrow \ln x = \ln 10/5 \Rightarrow \ln x = 0.46 \Rightarrow x = e^{0.46} \Rightarrow x = 1.58$$

We want to find y such that $x^y = 2$.

$$1.58^y = 2 \Rightarrow y \ln 1.58 = \ln 2 \Rightarrow y = 1.51$$

So the performance of CPUs doubles every 1.51 years (about every 18 months).

1.9 (a) The 64-CPU Cosmic Cube had a best case performance of 5–10 megaflops. This translates into a single-CPU speed of 0.078–0.156 megaflops. Caltech's Supercomputing '97 cluster had 140 CPUs executing at more than 10 gigaflops. This translates into a speed of at least 71.4 megaflops per CPU. The overall speed increase is in the range 458–915.

(b) We want to find x_1 such that ${x_1}^{15} = 458$.

$${x_1}^{15} \Rightarrow 15 \ln x_1 = \ln 458 \Rightarrow x_1 = 1.50$$

We want to find x_2 such that ${x_2}^{15} = 915$.

$${x_1}^{15} \Rightarrow 15 \ln x_1 = \ln 915 \Rightarrow x_1 = 1.57$$

The annual microprocessor performance improvement needed to account for the speed increase calculated in part (a) was between 50% and 57%.

1.10 The faster computer will be capable of performing 100 times as many operations in 15 hours as the existing computer. Let n denote the size of the problem that can be solved in 15 hours by the faster computer.

(a) $n/100,000 = 100 \Rightarrow n = 10,000,000$.

(b) $n \log_2 n/(100,000 \log_2 100,000) = 100 \Rightarrow n \approx 7,280,000$

(c) $n^2/(100,000)^2 = 100 \Rightarrow n^2 = 1 \times 10^{12} \Rightarrow n = 1,000,000$

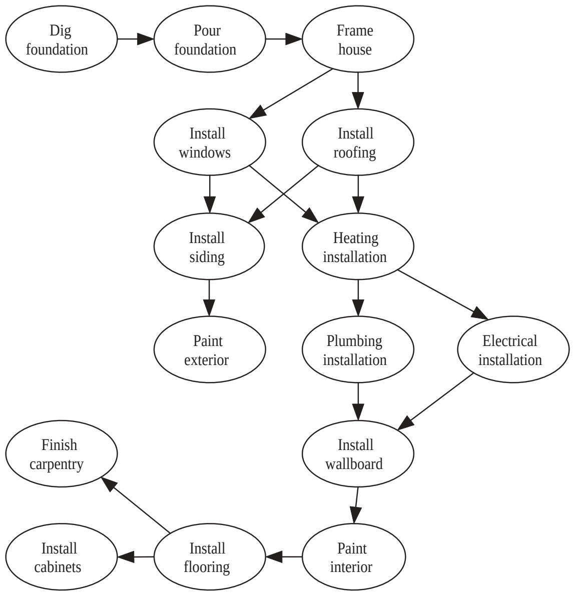
(d) $n^3/(100,000)^3 = 100 \Rightarrow n^3 = 1 \times 10^{17} \Rightarrow n = 464,159$

1.11 Advantages of commodity clusters: The latest CPUs appear in commodity clusters before commercial parallel computers. Commodity clusters take advantage of small profit margins on components, meaning the cost per megaflop is lower. Commodity clusters can be built entirely with freely available software. Commodity clusters have a low entry cost.

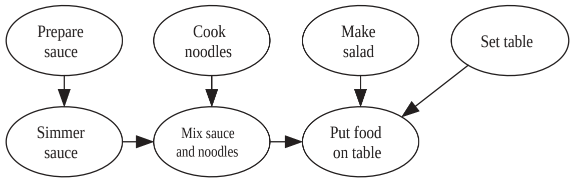
Advantages of commercial parallel computers: Commercial systems can have a higher performance interprocessor communication system that is better balanced with the speed of the processors.

1.12 Students will come up with many acceptable task graphs, of course. Here are examples.

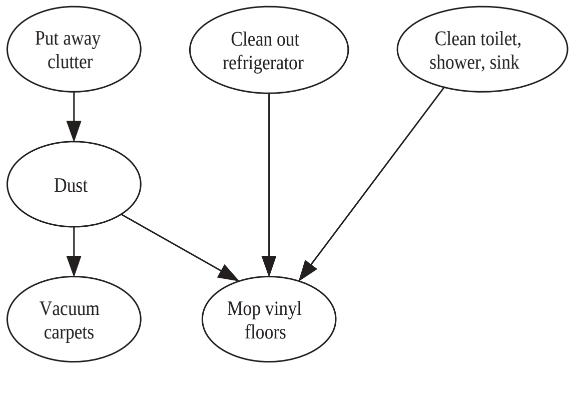
(a) Building a house



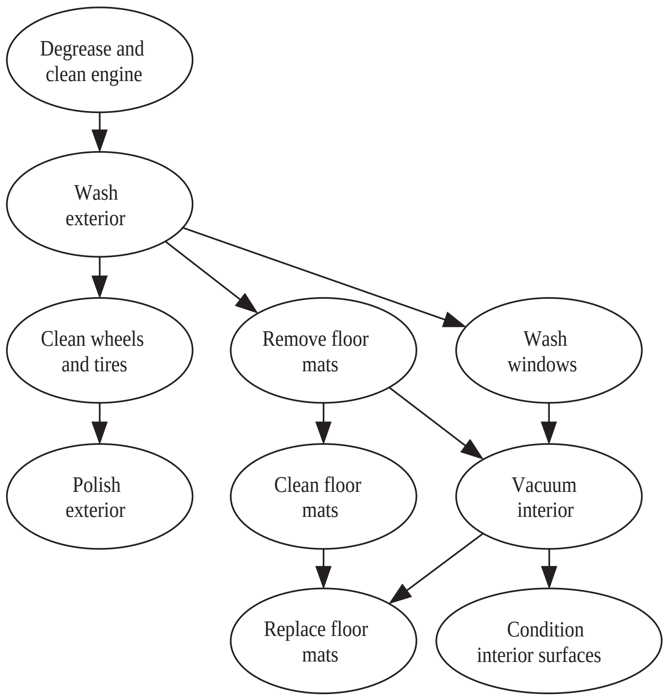
(b) Cooking and serving a spaghetti dinner



(c) Cleaning an apartment



(d) Detailing an automobile



1.13 The upper set of three nodes labeled A is a source of data parallelism. The lower let of three nodes labeled A is another source of data parallelism. B and C are functionally parallel. C and D are functionally parallel. D and the lower group of three nodes labeled A are functionally parallel.

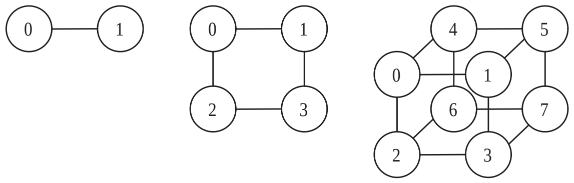
1.14 The advantage of preallocation is that it reduces the overhead associated with assigning documents to idle processors at run-time. The advantage of putting documents on a list and letting processors remove documents as fast as they could process them is that it balances the work among the processors. We do not have to worry about one processor being done with its share of documents while another processor (perhaps with longer documents) still has many to process.

1.15 Before allocating any documents to processors we could add the lengths of all the documents and then use a heuristic to allocate documents to processors so that each processor is responsible for roughly the same total document length. (We need to use a heuristic, since optimally balancing the document lengths is an NP-hard problem.) This approach avoids massive imbalances in workloads. It also avoids run-time overhead associated with access to the shared list of unassigned documents.

Chapter 2

Parallel Architectures

2.1 Hypercube networks with 2, 4, and 8 nodes. A hypercube network with n nodes is a subgraph of a hypercube network with $2n$ nodes.



2.2 A 0-dimensional hypercube can be labeled 1 way. A d-dimensional hypercube can be labeled $2^d(\log_2 d)!$ ways, for $d \geq 1$. Explanation: We can give any of the 2^d nodes the label 0. Once this is done, we label the neighbors of node 0. We have $\log_2 d$ choices for the label 1, $\log_2 d - 1$ choices for the label 2, $\log_2 d - 2$ choices for the label 4, $\log_2 d - 3$ choices for the label 8, etc. In other words, there are $(\log_2 d)!$ ways to label the nodes adjacent to node 0. Once this is done, all the other labels are forced. Hence the number of different labelings is $2^d(\log_2 d)!$.

2.3 The number of nodes distance i from s in a d-dimensional hypercube is $\binom{d}{i}$ (d choose i).

2.4 If node u is distance i from node v in a hypercube, exactly i bits in their addresses (node numbers) are different. Starting at u and flipping one of these bits at a time yields the addresses of the nodes on a path from u to v. Since these bits can be flipped in any order, there are $i!$ different paths of length i from u to v.

2.5 If node u is distance i from node v in a hypercube, exactly i bits in their addresses (node numbers) are different. Starting at u and flipping one of these bits at a time yields the addresses of the nodes on a path from u to v. To generate paths that share no intermediate vertices, we order the bits that are different:

$$b_0, b_1, \ldots, b_{i-1}$$

We generate vertices on the first path by flipping the bits in this order:

$$b_0, b_1, \ldots, b_{i-1}$$

We generate vertices on the second path by flipping the bits in this order:

$$b_1, b_2, \ldots, b_{i-1}, b_0$$

We generate vertices on the third path by flipping the bits in this order:

$$b_2, b_3, \ldots, b_{i-1}, b_0, b_1$$

This generates i distinct paths from u to v.

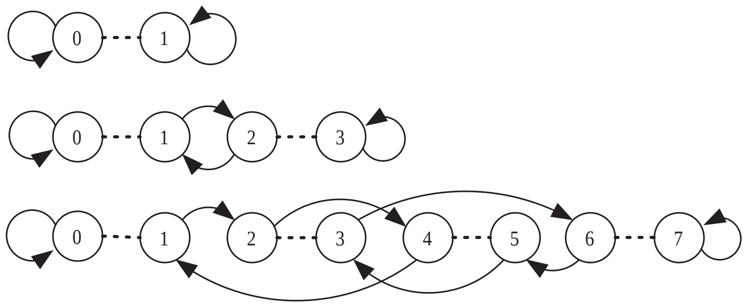
2.6 A hypercube is a bipartite graph. Nodes whose addresses contain an even number of 1s are connected to nodes whose addresses contain an odd number of 1s. A bipartite graph cannot have a cycle of odd length.

2.7 Compute the bitwise "exclusive or" operation on v and v. The number of 1 bits in the result represents the length of the shortest path from u to v. Since u and v have $\log n$ bits, there are no more than $\log n$ bits set, and the shortest path has length at most $\log n$. Number the indices from right to left, giving them values $0, 1, \ldots, \log n - 1$. Suppose the exclusive or results in k bits being set, and these bits are at indices $b_1, b_2, \ldots, b_k$. This algorithm prints the shortest path from u to v.

```
$w \leftarrow u$
print $w$
for $i \leftarrow 1$ to $k$
   $w \leftarrow w \otimes 2^{b_k}$          {exclusive or}
   print $w$
endfor
```

Example: Find a shortest path from node 2 to node 11 in a four-dimensional hypercube. $0010 \otimes 1011 = 1001$. Hence $i_1 = 0$ and $i_1 = 3$. First jump: $0010 \otimes 0001 = 0011$. Second jump: $0011 \otimes 1000 = 1011$. Path is $0010 - 0011 - 1011$.

2.8 Shuffle-exchange networks with 2, 4, and 8 nodes. Dashed lines denote exchange links; solid arrows denote shuffle links. A shuffle-exchange network with n nodes is not a subgraph of a shuffle-exchange network with $2n$ nodes.



2.9 Nodes u and v are exactly $2k-1$ link traversals apart if one is node 0 and the other is node $2^k - 1$. When this is the case, all k bits must be changed, which means there must be at least k traversals along exchange links. There must also be at least $k-1$ traversals along shuffle links to set up the bit reversals. The total number of link traversals is $2k-1$.

In all other cases, u and v have at least one bit in common, and following no more than $k-1$ shuffle links and $k-1$ exchange links is sufficient to reach v from u.

2.10 Algorithm to route message from u to v in an n-node shuffle-exchange network:

```
for $i \leftarrow 1$ to $\log_2 n$ do
    $b \leftarrow u \otimes v$        {exclusive or}
    if $b$ is odd then
        print “exchange”
    endif
    $u \leftarrow$ left_cyclic_rotate($u$)
    $v \leftarrow$ left_cyclic_rotate($v$)
    print “shuffle”
```

Example: Routing message from 6 to 5 in a 16-node shuffle-exchange network. $u = 0110$ and $v = 0101$. Algorithm produces the following route: exchange – shuffle – shuffle – shuffle – exchange – shuffle.

2.11 (a) $(n/2)\log_2 n$

(b) $\log_2 n$

(c) $(n/2)\log_2 n$

(d) 4

(e) No

2.12 Suppose the address of the destination is $b_{k-1}b_{k-2}\ldots b_1b_0$. Every switch has two edges leaving it, an "up" edge (toward node 0) and a "down" edge (away from node 0). The following algorithm indicates how to follow the edges to reach the destination node with the given address.

```
for $i \leftarrow k-1$ downto 0 do
  if $b_i = 0$ then
    print "up"
  else
    print "down"
  endif
endfor
```

2.13 Data parallelism means applying the same operation across a data set. A processor array is only capable of performing one operation at a time, but every processing element may perform that operation on its own local operands. Hence processor arrays are a good fit for data-parallel programs.

2.14 Let i be the vector length, where $1 \leq i \leq 50$. The total number of operations to be performed is i. The time required to perform these operations is $\lceil i/8 \rceil \times 0.1$ μsecond. Hence performance is

$$\frac{i}{\lceil i/8 \rceil} \times 10^7 \text{ operations/second}$$

For example, when vector length is 8, performance is 80 million operations per second. When vector length is 50 performance is $(50/7)\times 10^7$ operations per second, or about 71.4 million operations per second.

2.15 (a) The efficiency is $1/k$

(b) The efficiency is

$$\frac{\sum_{i=1}^{k} I_i P_i}{\sum_{i=1}^{k} I_i}$$

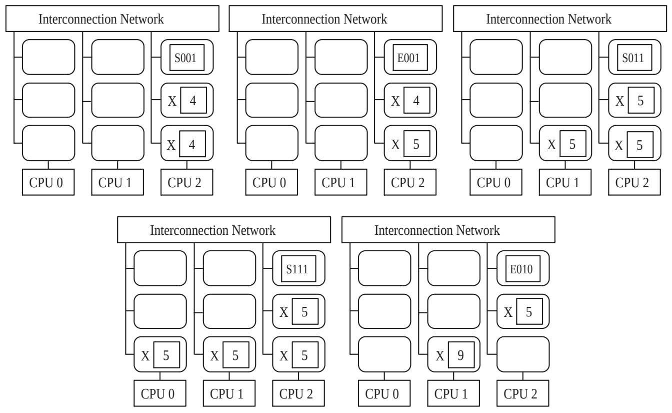
2.16 Large caches reduce the load on the memory bus, enabling the system to utilize more processors efficiently.

2.17 Even with large instruction and data caches, every processor still needs to access the memory bus occasionally. By the time the system contains a few dozen processors, the memory bus is typically saturated.

2.18 (a) The directory should be distributed among the multiprocessors' local memories so that accessing the directory does not become a performance bottleneck.

(b) If the contents of the directories were replicated, we would need to make sure the values were coherent, which is the problem we are trying to solve by implementing the directory.

2.19 Continuation of directory-based cache coherence protocol example.



2.20 Here is one set of examples.

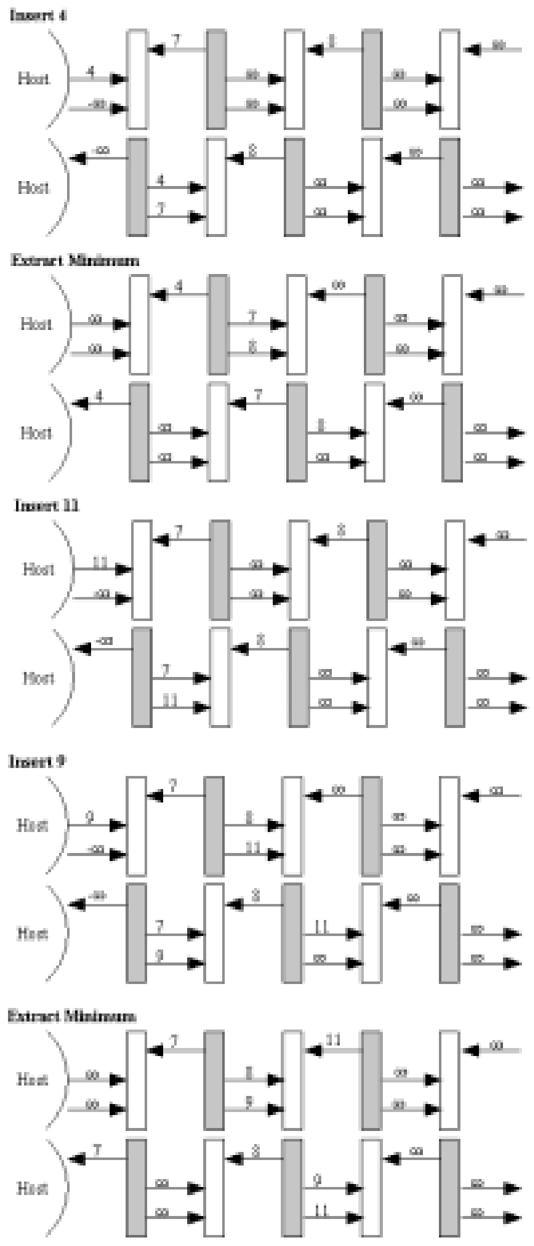
SISD: Digital Equipment Corporation PDP-11

SIMD: MasPar MP-1

MISD: Intel iWarp

MIMD: Sequent Symmetry

2.21 Continuation of systolic priority queue example.



2.22 Contemporary supercomputers must contain thousands of processors. Even distributed multiprocessors scale only to hundreds of processors. Hence supercomputers are invariably multicomputers. However, it is true that each node of a supercomputer is usually a multiprocessor. The typical contemporary supercomputer, then, is a collection of multiprocessors forming a multicomputer.

Chapter 3

Parallel Algorithm Design

3.1 Suppose two processors are available to find the sum of n values, which are initially stored on a single processor. If one processor does all the summing, the processor utilization is 50%, but there is no interprocessor communication. If two processors share the computation, the processor utilization is closer to 100%, but there is some interprocessor communication. One processor must send the other processor $n/2$ values and receive in return the sum of those values. The time spent in interprocessor communication prevents utilization from being 100%.

3.2 (a) $\log 3 \approx 1.58$, $\lfloor \log 3 \rfloor = 1$, $\lceil \log 3 \rceil = 2$

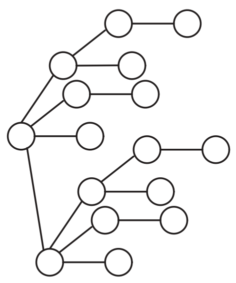
(b) $\log 13 \approx 3.70$, $\lfloor \log 13 \rfloor = 3$, $\lceil \log 13 \rceil = 4$

(c) $\log 32 = 5$, $\lfloor \log 32 \rfloor = 5$, $\lceil \log 32 \rceil = 5$

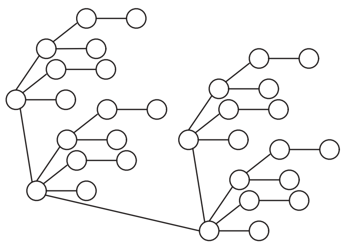
(d) $\log 123 \approx 6.94$, $\lfloor \log 123 \rfloor = 6$, $\lceil \log 123 \rceil = 7$

(e) $\log 321 \approx 8.33$, $\lfloor \log 321 \rfloor = 8$, $\lceil \log 321 \rceil = 9$

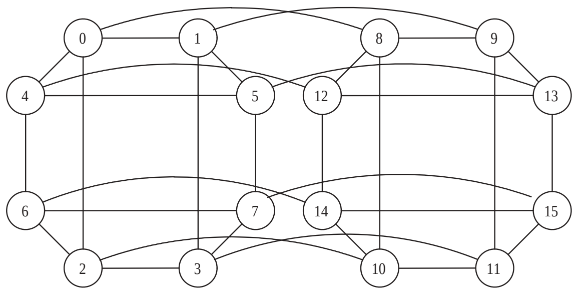
3.3 (a) Binomial tree with 16 nodes



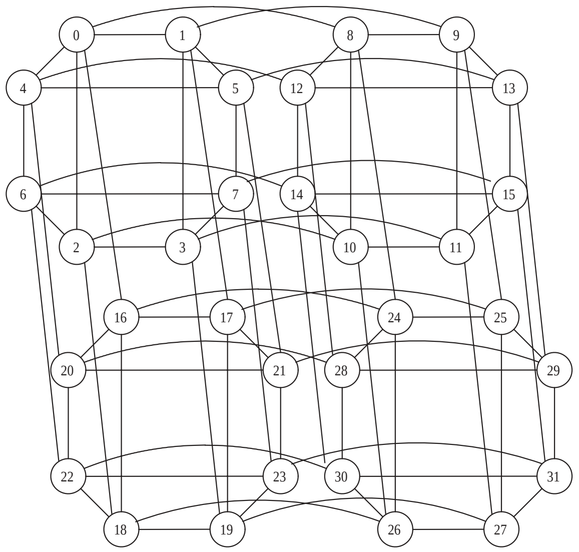
(b) Binomial tree with 32 nodes



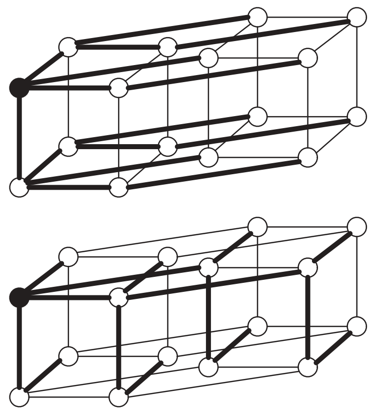
3.4 (a) Hypercube with 16 nodes



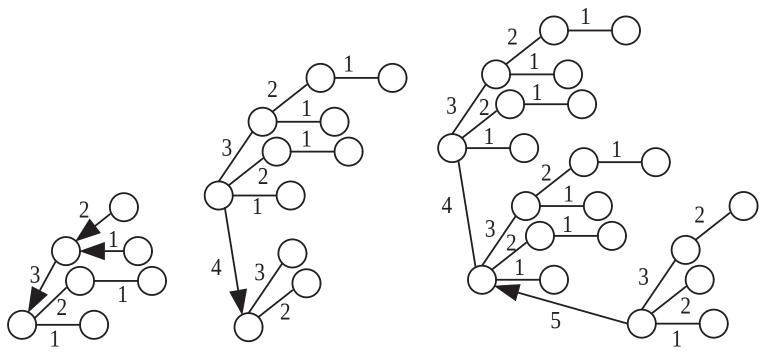
(b) Hypercube with 32 nodes



3.5 Here are four binomial trees embedded in a four-dimensional hypercube. All four binomial trees have the same hypercube node as their root.



3.6 Three reductions, each performed with $\lceil \log n \rceil$ communication steps.



3.7 A C program that describes the communications performed by a task participating in a reduction.

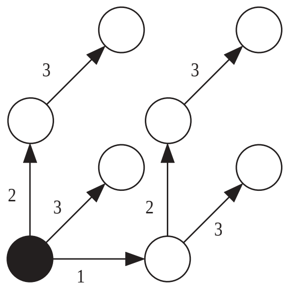
```
int main (int argc, char *argv[])
{
   int i;
   int id;  /* Task ID */
   int p;   /* Number of tasks */
   int pow; /* ID diff between sending/receiving procs */

   if (argc != 3) {
      printf ("Command line: %s <p> <id>\n", argv[0]);
      exit (-1);
   }
   p = atoi(argv[1]);
   id = atoi(argv[2]);
   if ((id < 0) || (id >= p)) {
      printf ("Task id must be in range 0..p-1\n");
      exit (-1);
   }
   if (p == 1) {
      printf ("No messages sent or received\n");
      exit (-1);
   }
   pow = 1;
   while(2*pow < p) pow *= 2;
   while (pow > 0) {
      if ((id < pow) && (id + pow <= p))
         printf ("Message received from task %d\n",
            id + pow);
      else if (id >= pow) {
         printf ("Message sent to task %d\n", id - pow);
         break;
      }
      pow /= 2;
   }
}
```

3.8 From the discussion in the chapter we know that we can reduce n values in time $\Theta(\log n)$ using n tasks. The only question, then, is whether we can improve the execution time by using fewer tasks.

Suppose we use only k tasks. Each task reduces about n/k values, and then the tasks communicate to finish the reduction. The time complexity of this parallel algorithm would be $\Theta(n/k + \log k)$. Let's find the value of k that minimizes this function. If $f(k) = n/k + \log k$, then $f'(k) = -n/k^2 + 1/k \ln 2$. $f'(k) = 0$ when $k = n \ln 2$; i.e., when $k = \Theta(n)$. Hence the complexity of the parallel reduction is $\Omega(n/n + \log n) = \Omega(\log n)$.

3.9 Our broadcast algorithm will be based on a binomial tree communication pattern. Data values flow in the opposite direction they do for a reduction.

(a) Here is a task/channel graph showing how the root task (shown in black) broadcasts to 7 other tasks in three steps. In general, use of a binomial tree allows a root task to broadcast to $p-1$ other tasks in $\lceil \log p \rceil$ communication steps.

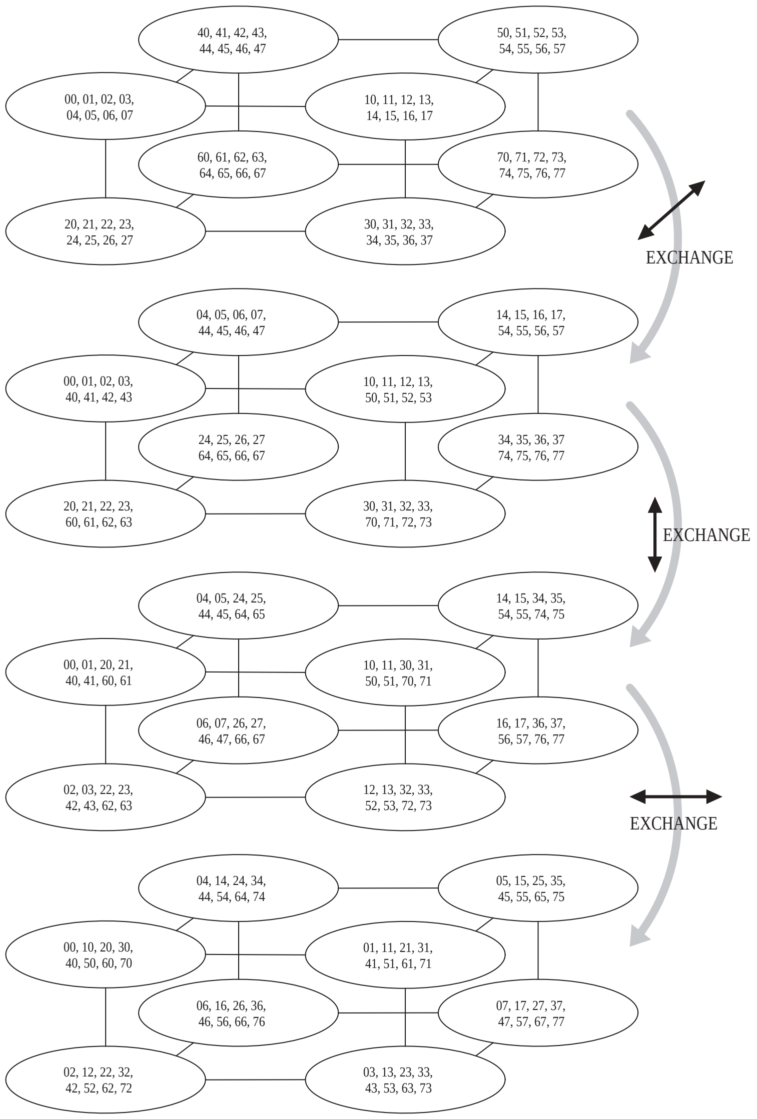


(b) We prove by induction that broadcast among p tasks (1 root task broadcasting to $p-1$ other tasks) requires at least $\lceil \log p \rceil$ communication steps. Basis. Suppose $p = 1$. No communications are needed. $\lceil \log 1 \rceil = 0$. Induction. Suppose we can broadcast among p tasks in $\lceil \log p \rceil$ communication steps, for $1 \leq p \leq k$. Then $\lceil \log k \rceil + 1$ communication steps are needed to broadcast among $2k$ tasks, because $\lceil \log k \rceil$ communication steps allow k tasks to get the value, and 1 additional step allows each of the k tasks to communicate the value to one of the k tasks that needs it. $\lceil \log k \rceil + 1 = \lceil \log 2k \rceil$. Hence the algorithm devised in part (a) is optimal.

3.10 The all-gather algorithm is able to route more values in the same because its utilization of the available communication channels is higher. During each step of the all-gather algorithm every task sends and receives a message. In the scatter algorithm the tasks are engaged gradually. About half of the tasks never send a message, and these tasks only receive a message in the last step of the scatter.

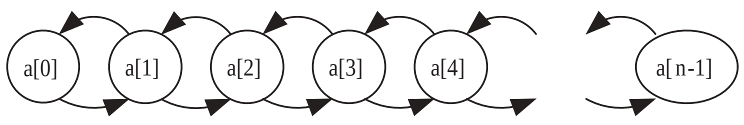
3.11 We organize the tasks into a logical hypercube. The algorithm has $\log p$ iterations. In the first iteration each task in the lower half of the hypercube swaps $p/2$ elements with a partner task in the upper half of the hypercube. After this step all elements destined for tasks in the upper half of the hypercube are held by tasks in the upper half of the hypercube, and all elements destined for tasks in the lower half of the hypercube are held by tasks in the lower half of the hypercube. At this point the algorithm is applied recursively to the lower and upper halves of the hypercube (dividing the hypercube into quarters). After $\log p$ swap steps, each process controls the correct elements.

The following picture illustrates the algorithm for $p = 8$:

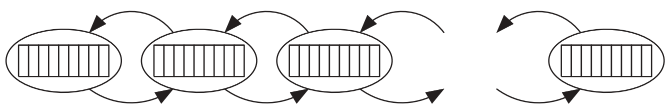


During each step each process sends and receives $p/2$ elements. There are $\log p$ steps. Hence the complexity of the hypercube-based algorithm is $\Theta(p \log p)$.

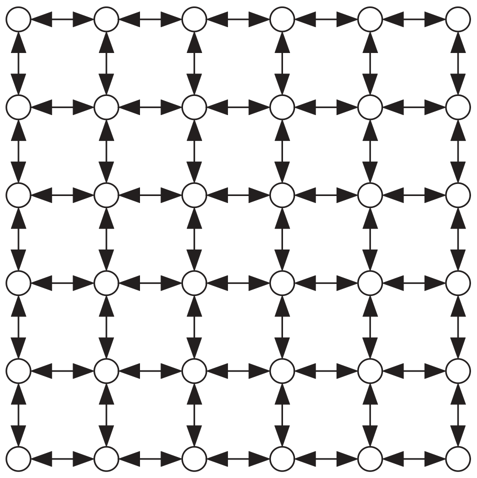
3.12 Initially we create a primitive task for each key. The algorithm iterates through two steps. In the first step the tasks associated with even-numbered keys send their keys to the tasks associated with the successor keys. The receiving tasks keep the larger value and return the smaller value. In the second step the tasks associated with the odd-numbered keys send their keys to the tasks associated with the successor keys. The receiving tasks keep the larger value and return the smaller value. After $n/2$ iterations the list must be sorted. The time complexity of this algorithm is $\Theta(n)$ with n concurrent tasks.



After agglomeration we have p tasks. The algorithm iterates through two steps. In the first step each task makes a pass through its subarray, from low index to high index, comparing adjacent keys. It exchanges keys it finds to be out of order. In the second step each task (except the last) passes its largest key to its successor task. The successor task compares the key it receives with it smallest key, returning the smaller value to the sending task. The computational complexity of each step is $\Theta(n/p)$. The communication complexity of each step is $\Theta(1)$. After at most n iterations the list is sorted. The overall complexity of the parallel algorithm is $\Theta(n^2/p + n)$ with p processors.

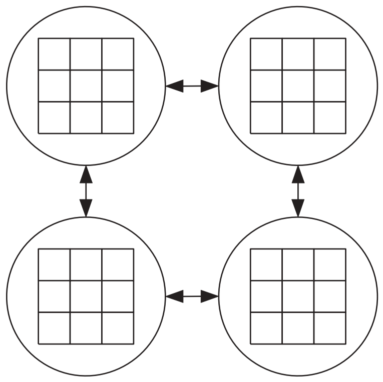


3.13 Initially we associate a primitive task with each pixel. Every primitive task communicates with tasks above it, below it, to its left, and to its right, if these tasks exist. Interior tasks have four neighbors, corner tasks has two neighbors, and edge tasks have three neighbors.

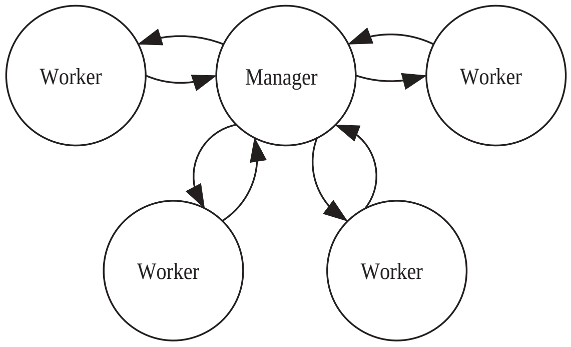


Each task associated with a 1 pixel at position (i, j) sets its component number to the unique value $n \times i + j$. At each step of the algorithm each task sends its current component number to its neighbors and receives from its neighbors their component numbers. The minimum of the component numbers received from a neighboring task is less than the task's current component number, then the task replaces its component number with this minimum value. In fewer than n^2 iterations the component numbers have stabilized.

A good agglomeration maximizes the computation/communication ratio by grouping primitive tasks into rectangles that are as square as possible.



3.14 We set up one "manager" task and a bunch of "worker" tasks. A pair of channels connects the manager with each worker. All tasks have a copy of the dictionary.



The manager partially fills out a crossword puzzle and then assigns it to a worker to complete. By assigning partially-completed puzzles to the workers, the manager ensures no two workers are doing the same thing. A worker sends a message back to the manager either when it has successfully filled in a puzzle or when it has determined there is no way to solve the partially filled-in puzzle it received. When the manager receives a solved puzzle from a worker it tells all the workers to stop.

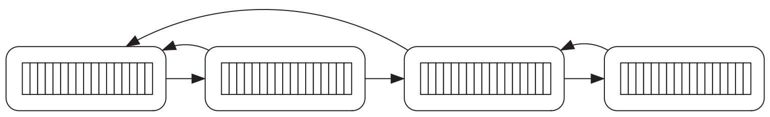
3.15 Each primitive task is responsible for the coordinates of one of the houses. All primitive tasks have the coordinates of the train station. Each task computes the euclidean distance from its house to the train station. Each task i contributes a pair (d_i, i) to a reduction in which the operation is minimum. The result of the reduction is the pair (d_k, k), where d_k is the minimum of all the submitted values of d_i, and k is the index of the task submitting d_k. The communication pattern is a binomial tree, allowing the reduction to take place in logarithmic time. A sensible agglomeration assigns about n/p coordinate pairs to each of p tasks. Each task find the closest house among the n/p houses it has been assigned, then participates in the reduction step with the other tasks.

The "before" and "after" task/channel graphs should resemble Figure 3.16.

3.16 The text is divided into p segments, one per task. Each task is responsible for searching its portion of the text for the pattern. We need to be able to handle the case where the pattern is split between two tasks' sections. Suppose the pattern has k letters. Each of the first $p-1$ tasks must pass the last $k-1$ letters of its portion of the text to the subsequent task. Now all occurrences of the pattern can be detected.

After the tasks have completed their portions of the search, a gather communication brings together information about all occurrences of the pattern.

In the following task/channel graph, the straight arrows represent channels used to copy the last final $k-1$ letters of each task's section to its successor task. The curved arrows represent channels used for the gather step.



3.17 If the pattern is likely to appear multiple times in the text, and if we are only interested in finding the first occurrence of the pattern in the text, then the allocation of contiguous blocks of text to the tasks is probably unacceptable. It makes it too likely that the first task will find the pattern, making the work of the other tasks superfluous. A better strategy is to divide the text into jp portions and allocate these portions in an interleaved fashion to the tasks. Each task ends up with j segments from different parts of the text.

0	1	2	3	0	1	2	3	0	1	2	3

As before, the final $k-1$ letters in each segment must be copied into the successor segment, in case the pattern is split across two segments.

Communication tasks enable one task to inform the other tasks as soon as it has found a match.

3.18 Associate a primitive task with each key. In the first step the tasks perform a max-reduction on the keys. In the second step the maximum key is broadcast to the tasks. In the third step each task compares its key with the maximum. In the fourth step the tasks perform another max-reduction. Every task submits a candidate key to the second max-reduction except the task whose key is equal to the maximum key.

We can reduce the communication complexity of the parallel algorithm without increasing the computation time on p processors by agglomerating groups of n/p primitive tasks into larger tasks.

3.19 This solution is similar to the previous solution, except that in the max-reduction step each task submits the pair (k_i, i), where k_i is the value of the key and i is its index in the list. The result of the max-reduction is the pair (m_i, i), where m_i is the smallest key submitted and i is its list position. This pair is broadcast to all the primitive tasks. In the fourth step the tasks perform another max-reduction. Every task submits a candidate key except the task whose index is equal to the index received in the broadcast step.

We can reduce the communication complexity of the parallel algorithm without increasing the computation time on p processors by agglomerating groups of n/p primitive tasks into larger tasks.

Chapter 4

Message-passing Programming

4.1 (a) Give the pieces of work the numbers $0, 1, 2, \ldots, n-1$. Process k gets the pieces of work $k, k+p, k+2p$, etc.,

(b) Piece of work j is handled by process $j \bmod p$.

(c) $\lceil n/p \rceil$

(d) If n is an integer multiple of p, then all processes have the same amount of work. Otherwise, $n \bmod p$ processes have more pieces of work—$\lceil n/p \rceil$. These are processes $0, 1, \ldots, (n \bmod p) - 1$.

(e) $\lfloor n/p \rfloor$

(f) If n is an integer multiple of p, then all processes have the same amount of work. Otherwise, $p - (n \bmod p)$ processes have less pieces of work—$\lfloor n/p \rfloor$. These are processes $(n \bmod p), (n \bmod p) + 1, \ldots, p-1$.

4.2 (a) 241

(b) Overflow. In C, get 128 as result.

(c) 99

(d) 13

(e) 127

(f) 0

(g) 1

(h) 1

4.3 Modified version of function `check_circuit`.

```
int check_circuit (int id, int z) {
   int v[16];        /* Each element is a bit of z */
   int i;

   for (i = 0; i < 16; i++) v[i] = EXTRACT_BIT(z,i);
   if ((v[0]||v[1]) && (!v[1]||!v[3]) && (v[2]||v[3])
      && (!v[3] || !v[4]) && (v[4] || !v[5])
      && (v[5] || !v[6]) && (v[5] || v[6])
      && (v[6] || !v[15]) && (v[7] || !v[8])
      && (!v[7] || !v[13]) && (v[8] || v[9])
      && (v[8] || !v[9]) && (!v[9] || !v[10])
      && (v[9] || v[11]) && (v[10] || v[11])
      && (v[12] || v[13]) && (v[13] || !v[14])
      && (v[14] || v[15])) {
      printf ("%d) %d%d%d%d%d%d%d%d%d%d%d%d%d%d%d%d\n",
         id,v[0],v[1],v[2],v[3],v[4],v[5],v[6],v[7],
         v[8],v[9],v[10],v[11],v[12],v[13],v[14],v[15]);
      fflush (stdout);
      return 1;
   } else return 0;
}
```

4.4 See Figure 4.6.

4.5 (a) The circuit can be represented by a logical expression using the same syntax as the C programming language. Parentheses are used as necessary to indicate the order in which operators should be applied.

(b) Creating a syntax-free grammar for logical expressions is easy. We could write a top-down or a bottom-up parser to construct a syntax tree for the circuit.

(c) The circuit would be represented by an expression tree. The leaves of the tree would be the inputs to the circuit. The interior nodes of the tree would be the logical gates. A *not* gate would have one child; *and* and *or* gates would have two children.

4.6 Parallel variant of Kernighan and Ritchie's “hello, world” program.

```
#include "mpi.h"
#include <stdio.h>

int main (int argc, char *argv[]) {
   int id;               /* Process rank */

   MPI_Init (&argc, &argv);
   MPI_Comm_rank (MPI_COMM_WORLD, &id);

   printf ("hello, world, from process %d\n", id);
   fflush (stdout);
   MPI_Finalize();
   return 0;
}
```

Chapter 5

The Sieve of Eratosthenes

5.1 (a) The first $p-1$ processes get $\lceil n/p \rceil$ elements and the last process does not get any elements when $n = k(p-1)$, where k is an integer greater than 1 and less than p. Here are the cases where this happens for $n \leq 10$: $n = 4$ and $p = 3$; $n = 6$ and $p = 4$; $n = 8$ and $p = 5$; $n = 9$ and $p = 4$; and $n = 10$ and $p = 6$.

(b) In order for $\lfloor p/2 \rfloor$ processes to get no values, p must be odd and n must be equal to $p+1$. Here are the cases where this happens for $n \leq 10$: $n = 4$ and $p = 3$; $n = 6$ and $p = 5$; $n = 8$ and $p = 7$; and $n = 10$ and $p = 9$.

5.2 We list the number of elements held by each process for each scheme. For example, (4, 3, 3) means that the first process has four elements and the second and third processes have three elements each.

(a) Scheme 1: (4, 4, 4, 3); Scheme 2: (3, 4, 4, 4)

(b) Scheme 1: (3, 3, 3, 2, 2, 2); Scheme 2: (2, 3, 2, 3, 2, 3)

(c) Scheme 1: (4, 3, 3, 3, 3); Scheme 2: (3, 3, 3, 3, 4)

(d) Scheme 1: (5, 5, 4, 4); Scheme 2: (4, 5, 4, 5)

(e) Scheme 1: (4, 4, 3, 3, 3, 3); Scheme 2: (3, 3, 4, 3, 3, 4)

(f) Scheme 1: (4, 4, 3, 3, 3, 3, 3); Scheme 2: (3, 3, 3, 4, 3, 3, 4)

5.3 This table shows the predicted execution time of the first parallel implementation of the Sieve of Eratosthenes on 1–16 processors.

Processors	*Time*
1	24.910
2	12.727
3	8.846
4	6.770
5	5.796
6	4.966
7	4.373
8	3.928
9	3.854
10	3.577
11	3.350
12	3.162
13	3.002
14	2.865
15	2.746
16	2.643

5.4 This table compares the predicted versus actual execution time of the second parallel implementation of the Sieve of Eratosthenes on a commodity cluster.

Processors	*Predicted Time*	*Actual Time*	*Error (%)*
1	12.455	12.237	1.78
2	6.499	6.609	1.66
3	4.695	5.019	6.46
4	3.657	4.072	10.19
5	3.305	3.652	9.50
6	2.890	3.270	11.62
7	2.594	3.059	15.20
8	2.371	2.856	16.98

The average error in the predictions for 2–7 processors is 10.23%.

5.5 This table compares the predicted versus actual execution time of the third parallel implementation of the Sieve of Eratosthenes on a commodity cluster.

Processors	*Predicted Time*	*Actual Time*	*Error (%)*
1	12.457	12.466	0.07
2	6.230	6.378	2.32
3	4.154	4.272	2.76
4	3.116	3.201	2.66
5	2.494	2.559	2.54
6	2.078	2.127	2.30
7	1.782	1.820	2.09
8	1.560	1.585	1.58

The average error in the predictions for 2–7 processors is 2.32%.

5.10 The first disadvantage is a lack of scalability. Since every process must set aside storage representing the integers from 2 through n, the maximum value of n cannot increase when p increases. The second disadvantage is that the OR-reduction step reduces extremely large arrays. Communicating these arrays among the processors will require a significant amount of time. The third disadvantage is that the workloads among the processors are unbalanced. For example, in the case of three processors, the processor sieiving with 2, 7, 17, etc. will take significantly to mark multiples of its primes than the processor sieving with 5, 13, 23, etc.

Chapter 6

Floyd's Algorithm

6.1 Suppose $n = kp + r$, where k and r are nonnegative integers and $0 \leq r < p$. Note that this implies $k = \lceil n/p \rceil$.

If r (the remainder) is 0, then all processes are assigned $n/p = \lceil n/p \rceil$ elements. Hence the last process is responsible for $\lceil n/p \rceil$ elements.

If $r > 0$, then the last process is responsible for elements $\lfloor (p-1)n/p \rfloor$ through $n - 1$.

$$\begin{aligned}
\lfloor (p-1)n/p \rfloor &= \lfloor (p-1)(kp+r)/p \rfloor \\
&= \lfloor p(kp+r)/p - kp/p - r/p \rfloor \\
&= (kp+r) - k - \lfloor r/p \rfloor \\
&= kp + r - k - 1 \\
&= n - (k+1)
\end{aligned}$$

The total number of elements the last process is responsible for is

$$(n-1) - (n - (k+1)) + 1 = k + 1 = \lceil n/p \rceil$$

6.2 Process 0 only needs to allocate $\lfloor n/p \rfloor$ elements to store its portion of the file. Since it may need to input and pass along $\lceil n/p \rceil$ elements to some of the processes, process 0 cannot use its file storage buffer as a temporary buffer holding the elements being passed along to the other processes. (Room for one extra element must be allocated.) In contrast, process 3 will eventually be responsible for $\lceil n/p \rceil$ elements of the file. Since no process stores more elements, the space allocated for process 3's portion of the file is sufficient to be used as a temporary message buffer.

We see this played out in Figure 6.6, in which process 0 is responsible for $\lfloor 14/4 \rfloor = 3$ elements, while process 3 is responsible for $\lceil 14/4 \rceil = 4$ elements.

6.3 We need to consider both I/O and communications within Floyd's algorithm itself. File input will be more complicated, assuming the distance matrix is stored in the file in row-major order. One process is responsible for accessing the file. It reads in one row of the matrix at a time and then scatters the row among all the processes. After n read-and-scatter steps, each process has its portion of the distance matrix.

During iteration k of Floyd's algorithm, the process controlling column k must broadcast this column to the other processes. No other communications are needed to perform the computation.

In order to print the matrix, one process should be responsible for all of the printing. Each row is gathered onto the one process that prints the row before moving on to print the next row.

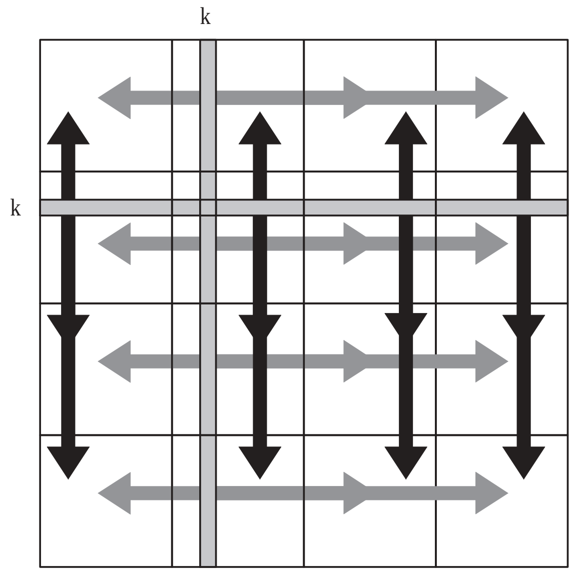
6.4 Several changes are needed when switching from a rowwise block striped decomposition to a rowwise interleaved striped decomposition.

First, the process that inputs the matrix from the file can only read one row at a time. In each of n iterations, the process reads row i and sends it to process $i \bmod p$ (except when it keeps rows it is responsible for). Hence this decomposition results in more messages being sent by the process responsible for reading the matrix from the file.

Second, in the performance of Floyd's algorithm, the computation of the process broadcasting row k is different. In the block decomposition the process controlling row k is $\lfloor p(k+1)-1)/n \rfloor$. In the interleaved decomposition the process controlling row k is $k \bmod p$.

Third, when the matrix is printed, processes send their rows to the printing process one at a time, instead of all at once. The printing process must print the first row held by process 0, the first row held by process 1, ..., the first row held by process $p-1$, the second row held by process 0, etc. Hence this decomposition results in more messages being received by the process responsible for printing the matrix.

6.5 (a) The processes are logically organized into a square mesh. During each iteration of the outer loop of the algorithm, a process needs access to `a[i][k]` and `a[k][j]` in order to update `a[i][j]`. Hence every process controlling a portion of row `k` must broadcast it to other processes in the same column of the process mesh. Similarly, every process controlling a portion of column `k` must broadcast it to other processes in the same row of the process mesh.



Note that the $\sqrt{p}$ broadcasts across rows of the process mesh may take place simultaneously. The $\sqrt{p}$ broadcasts across columns of the process mesh may also take place simultaneously.

(b) During each iteration there is are two broadcast phases. The communication time is the same for both phases. A total of $n/\sqrt{p}$ data items, each of length 4 bytes, are broadcast among a group of $\sqrt{p}$ processes. The time needed for the broadcast is

$$\lceil \log \sqrt{p} \rceil (\lambda + 4n/(\sqrt{p}\beta)$$

Since there are two broadcast phases per iteration and n iterations, the total communication time is

$$2n\lceil \log \sqrt{p} \rceil (\lambda + 4n/(\sqrt{p}\beta))$$

(c) Since $2n\lceil \log \sqrt{p} \rceil = n\lceil \log p \rceil$, the number of messages sent by both algorithms is the same. However, the parallel version of

Floyd's algorithm that assigns each process a square block of the elements of **A** has its message transmission time reduced by a factor of $\sqrt{p}$. Hence it is superior.

6.6 We need to compute

$$n^2 \lceil n/p \rceil \chi + n \lceil \log p \rceil (lambda + 4n/\beta)$$

where $n = 1000, p = 16$, $\chi = 25.5$ nanosec, $\lambda = 250\ \mu$sec, and $\beta = 10^7$ bytes/sec. The result is 4.19 seconds.

6.7 This table predicts the execution times (in seconds) when executing Floyd's algorithm on problems of size 500 and 2,000 with $1, 2, \ldots, 8$ processors. The computer has the same values of χ, λ, and β as the system used for benchmarking in this chapter.

Processors	$n = 500$	$n = 2000$
1	3.19	204.0
2	1.72	102.5
3	1.32	69.0
4	1.05	52.0
5	1.01	42.3
6	0.91	35.6
7	0.83	30.7
8	0.78	27.0

Chapter 7

Performance Analysis

7.1 The speedup formula is

$$\psi(n,p) \leq \frac{\sigma(n)+\varphi(n)}{\sigma(n)+\varphi(n)/p+\kappa(n,p)}$$

We want to find p_0 such that $p > p_0 \Rightarrow \psi(n,p) < \psi(n,p_0)$. From the formula we see this is the same as saying we want to find p_0 such that $p > p_0 \Rightarrow \varphi(n)/p + \kappa(n,p) > \varphi(n)/p_0 + \kappa(n,p_0)$. We are told that $\kappa(n,p) = c \log p$. Let

$$f(p) = \varphi(n)/p + c \log p$$

Then

$$f'(p) = -\varphi(n)/p^2 + c/(p \ln 2)$$

Finding the point where the slope is 0:

$$f'(p) = 0 \Rightarrow p = \varphi(n) \ln 2/c$$

Let

$$p_0 = \lceil \varphi(n) \ln 2/c \rceil$$

Since $f'(p) > 0$ when $p > p_0$, $f(p)$ is increasing for $p > p_0$. Therefore, there exists a p_0 such that $p > p_0 \Rightarrow \psi(n,p) < \psi(n,p_0)$.

7.2 The definition of efficiency is

$$\varepsilon \leq \frac{\sigma(n)+\varphi(n)}{p\sigma(n)+\varphi(n)+p\kappa(n,p)}$$

Since $p\sigma(n)$ and $\kappa(n,p)$ are both nondecreasing functions of p, the denominator is a nondecreasing function of p, meaning that $\varepsilon(n,p)$ is a nonincreasing function of p. Hence $p' > p \Rightarrow \varepsilon(n,p') \leq \varepsilon(n,p)$.

7.3 Sequential execution time is 0.01 seconds. Speedup is parallel execution time divided by sequential execution time.

Processors	*Speedup*
1	1.00
2	1.96
3	2.83
4	3.70
5	4.35
6	5.08
7	5.79
8	6.45
9	6.62
10	7.14
11	7.64
12	8.11
13	8.55
14	8.97
15	9.37
16	9.76

7.4 By Amdahl's Law,

$$\psi \le \frac{1}{f + (1-f)/p} = \frac{1}{0.05 + 0.95/10} = 6.90$$

Maximum achievable speedup is 6.90.

7.5 Using Amdahl's Law,

$$10 \le \frac{1}{0.06 + .94/p} \Rightarrow p \ge 24$$

Hence at least 24 processors are needed to achieve a speedup of 10.

7.6 Starting with Amdahl's Law:

$$\psi \le \frac{1}{f + (1-f)/p}$$

we take the limit as $p \to \infty$ and find that

$$\psi \le \frac{1}{f} \Rightarrow f \le \frac{1}{\psi}$$

In order to achieve a speedup of 50, no more than 1/50th (2%) of a program's execution time can be spent in inherently sequential operations.

7.7 We begin with Amdahl's Law and solve for f:

$$\begin{aligned}\psi &\leq \frac{1}{f+(1-f)/p}\\ \Rightarrow 9 &\leq \frac{1}{f+(1-f)/10}\\ \Rightarrow 9 &\leq \frac{1}{f+0.1-0.1f}\\ \Rightarrow 9f &+ 0.9 - 0.9f \leq 1\\ \Rightarrow 8.1f &\leq 0.1\\ \Rightarrow f &\leq 0.0123\end{aligned}$$

At most 1.23% of the computations performed by the program are inherently sequential.

7.8 Gustafson-Barsis's Law states that

$$\psi \leq p + (1-p)s$$

In this case $s = 9/242 = 0.037$. Hence the scaled speedup on 16 processors is $16 + (1-16)0.0372 = 15.4$.

7.9 By Gustafson-Barsis's Law, scaled speedup is $40+(1-40)0.01 = 39.6$.

7.10 This problem asks us to apply the Karp-Flatt metric.

(I) The answer is (B). The experimentally determined serial fraction is 0.2 for $2 \leq p \leq 8$. If it remains at 0.2, speedup on 16 processors will 4, only 20% higher than on 8 processors.

(II) The answer is (C). The experimentally determined serial fraction is growing linearly with p. If it continues to increase at this rate, speedup on 16 processors will be 4, which is 8% lower than on 8 processors.

(III) The answer is (A). The experimentally determined serial fraction is holding steady at 0.06. If it remains at 0.06, speedup on 16 processors will be 8.42, or 49% higher than on 8 processors.

(IV) The answer is (A). The experimentally determined serial fraction is growing logarithmically with p. If it has the value 0.05 when $p = 16$, speedup will be 9.14, or 46% higher than on 8 processors.

(V) The answer is (B). The experimentally determined serial fraction is growing very slowly, but it is quite large.

(VI) The answer is (A). The experimentally determined serial fraction is small and growing slowly.

7.11 In Amdahl's Law, when you fix f and increase p, the problem size does not change. This is because the focus of Amdahl's Law is on the effects of applying more processors to a particular problem. Hence as $p \to \infty$, the maximum speedup converges on $1/f$.

In contrast, with Gustafson-Barsis' Law, when you fix s and increase p, the proportion of inherently sequential operations actually drops. For example, suppose $s = 0.1$ and $p = 2$. Then two processors are busy 90% of the time and one processor is busy 10% of the time. The number of inherently sequential operations represents just over 5% of the total computation. In contrast, suppose $s = 0.1$ and $p = 20$. Now twenty processors are busy 90% of the time and one processor is busy 10% of the time. The number of inherently sequential operations represents just over 0.5% of the total computation.

Because the proportion of inherently sequential operations drops when fixing s and increasing p, the scaled speedup predicted by Gustafson-Barsis' Law increases without bound.

7.12 Here are three reasons why it may be impossible to solve the problem within a specified time limit, even if you have access to al the processors you care to use.

First, the problem may not be amenable to a parallel solution.

Second, even if the bulk of the problem can be solved by a parallel algorithm, there may be a sequential component (such as file I/O) that limits the speedup achievable through parallelism.

Third, even if there is no significant sequential component, interprocessor communication overhead may limit the number of processors that can be applied before execution time increases, rather than decreases, when more processors are added.

7.13 Good news! If students can solve this problem, they can perform the isoefficiency analysis of *every* parallel algorithm in the book!

System a: $n \geq Cp$ and $M(n) = n^2$

$$M(Cp)/p = C^2p^2/p = C^2p$$

System b: $n \geq C\sqrt{p}\log p$ and $M(n) = n^2$

$$M(C\sqrt{p}\log p)/p = C^2p\log^2 p/p = C^2\log^2 p$$

System c: $n \geq C\sqrt{p}$ and $M(n) = n^2$

$$M(C\sqrt{p})/p = C^2 p/p = C^2$$

System d: $n \geq Cp\log p$ and $M(n) = n^2$

$$M(Cp\log p)/p = C^2 p^2 \log^2 p/p = C^2 p \log^2 p$$

System e: $n \geq Cp$ and $M(n) = n$

$$M(Cp)/p = Cp/p = C$$

System f: $n \geq p^C$, $1 < C < 2$, and $M(n) = n$

$$M(p^C)/p = p^C/p = p^{C-1} = O(p)$$

System g: $n \geq p^C$, $2 < C < 3$, and $M(n) = n$

$$M(p^C)/p = p^C/p = p^{C-1} = O(p^2)$$

Most scalable	c, e
	b
	f
	a
	d
Least scalable	g

Chapter 8

Matrix-vector Multiplication

8.2 The expected execution time of the first matrix-vector multiplication program on $9, 10, \ldots, 16$ processors, given the model and machine parameters described in Section 8.4. Predicted times for $1, 2, \ldots, 8$ and 16 processors are in Table 8.1.

Processors	*Time (sec)*
9	.0089
10	.0081
11	.0075
12	.0071
13	.0066
14	.0063
15	.0060

8.7 A C code segment that partitions the process grid into column communicators.

```
MPI_Comm grid_comm;
MPI_Comm grid_coords[2];
MPI_Comm col_com;

MPI_Comm_split (grid_comm, grid_coords[1],
                grid_coords[0], &col_comm);
```

8.8 The answer shown below gives more detail than is necessary. Not only does it show how to use `read_block_vector` to open a file and distribute it among a column of processes in a a grid communicator, it also shows how the variables used in this function call are declared and initialized.

```
int main (int argc, char *argv[])
{

double   *b;             /* Where to store vector */
MPI_Comm  col_comm;      /* Communicator shared by
                            processes in same column */
MPI_Comm  grid_comm;     /* Logical grid of procs */
int       grid_coords[2]; /* Coords of process */
int       grid_id;       /* Rank in grid comm */
int       grid_size[2];  /* Number of procs per dim */
int       p;             /* Number of processes */
int       periodic[2];   /* Wraparound flag */
int       nprime;        /* Size of vector */

MPI_Init (&argc, &argv);
MPI_Comm_size (MPI_COMM_WORLD, &p);
grid_size[0] = grid_size[1] = 0;
MPI_Dims_create (p, 2, grid_size);
periodic[0] = periodic[1] = 0;
MPI_Cart_create (MPI_COMM_WORLD, 2, grid_size, periodic,
                 1, &grid_comm);
MPI_Comm_rank (grid_comm, &grid_id);
MPI_Cart_coords (grid_comm, grid_id, 2, grid_coords);
MPI_Comm_split (grid_comm, grid_coords[1],
                grid_coords[0], &col_comm);
if (grid_coords[1] == 0)
   nprime = read_block_vector ("Vector", (void *) &b,
                               MPI_DOUBLE, col_comm);
...
}
```

8.9 (c) C code that redistributes vector **b**. It works for all values of p.

```
/* Variable 'is_square' true iff 'p' is square */

if (is_square) {
      /* Proc at (i,0) sends subvector to proc at
         (0,i). Proc at (0,0) just does a copy. */

      if ((grid_coords[0] == 0) &&
          (grid_coords[1] == 0)) {
         for (i = 0; i < recv_els; i++)
            btrans[i] = b[i];
      } else if ((grid_coords[0] > 0) &&
                 (grid_coords[1] == 0)) {
         send_els =
           BLOCK_SIZE(grid_coords[0],grid_size[0],n);
         coords[0] = 0;
         coords[1] = grid_coords[0];
         MPI_Cart_rank(grid_comm,coords, &dest);
         MPI_Send (b, send_els, mpitype, dest, 0,
            grid_comm);
      } else if ((grid_coords[1] > 0) &&
                 (grid_coords[0] == 0)) {
         coords[0] = grid_coords[1];
         coords[1] = 0;
         MPI_Cart_rank(grid_comm,coords, &src);
         MPI_Recv (btrans, recv_els, mpitype, src, 0,
                   grid_comm, &status);
      }
   } else {

      /* Process at (0,0) gathers vector elements
         from procs in column 0 */

      if (grid_coords[1] == 0) {
         recv_cnt = (int *)
            malloc (grid_size[0] * sizeof(int));
         recv_disp = (int *)
            malloc (grid_size[0] * sizeof(int));
         for (i = 0; i < grid_size[0]; i++)
            recv_cnt[i] =
               BLOCK_SIZE(i, grid_size[0], n);
         recv_disp[0] = 0;
         for (i = 1; i < grid_size[0]; i++)
            recv_disp[i] =
               recv_disp[i-1] + recv_cnt[i-1];
      MPI_Gatherv (b, BLOCK_SIZE(grid_coords[0],
         grid_size[0], n), mpitype, tmp, recv_cnt,
         recv_disp, mpitype, 0, col_comm);
   }

   /* Process at (0,0) scatters vector elements to
      other processes in row 0 of process grid */

   if (grid_coords[0] == 0) {
      if (grid_size[1] > 1) {
         send_cnt = (int *)
            malloc (grid_size[1] * sizeof(int));
         send_disp = (int *)
            malloc (grid_size[1] * sizeof(int));
         for (i = 0; i < grid_size[1]; i++) {
            send_cnt[i] =
               BLOCK_SIZE(i,grid_size[1],n);
         }
         send_disp[0] = 0;
         for (i = 1; i < grid_size[1]; i++) {
            send_disp[i] =
               send_disp[i-1] + send_cnt[i-1];
         }
         MPI_Scatterv (tmp, send_cnt, send_disp,
            mpitype, btrans, recv_els, mpitype,
            0, row_comm);
      } else {
         for (i = 0; i < n; i++)
            btrans[i] = tmp[i];
      }
   }
}

/* Row 0 procs broadcast their subvectors to
   procs in same column */

MPI_Bcast (btrans,recv_els, mpitype, 0, col_comm);
```

Chapter 9

Document Classification

9.1 (a) Efficiency is maximized when all workers are able to begin and end execution at the same time. This is the case whenever

$$\sum_{i=0}^{k-1} t_i = (p-1)r$$

and there exists a partitioning of the k tasks into $p-1$ subsets of tasks such that the sum of the execution times of the tasks within each subset is r. The simplest example is the case when there are $p-1$ tasks with identical execution times. Since the manager process is not performing any of the tasks, the maximum efficiency possible is $(p-1)/p$.

(b) Efficiency is minimized when there is only a single task. The minimum possible efficiency is $1/p$.

9.2 Nonblocking sends and receives can reduce execution time by allowing a computer to overlap communications and computations. This is particularly helpful when a system has a dedicated communications coprocessor. Posting a receive before a message arrives allows the run-time system to save a copy operation by transferring the contents of the incoming message directly into the destination buffer rather than a temporary system buffer.

9.3 The first way: A worker's initial request for work is a message of length 0. The manager could distinguish this request from a message containing a document profile vector by looking at the length of the message.

The second way: The first message a manager receives from any worker is its request for work. The manager could keep track of how many messages it has received from each worker. When a manager received a message from a worker it could check this count. If the count of previously received messages is 0, the manager would know it was an initial request for work.

Both of these solutions are more cumbersome and obscure than using a special message tag. Hence tagging the message is preferable.

9.4 Typically a call to `MPI_Wait` is placed immediately before the buffer transmitted by `MPI_Isend` is reused. However, in this case there is no such buffer, since a message of length zero was sent.

9.5 If the manager set aside a profile vector buffer for each worker, it could initiate a nonblocking receive every time it sent a task to a worker. It could then replace the call to `MPI_Recv` with a call to `MPI_Testsome` to see which document vectors had been received. This change would eliminate the copying of profile vectors from system buffers to user buffers.

The call to `MPI_Send` in which the manager sends a file name to a worker could also be changed to the nonblocking `MPI_Isend`. Since the file name will never be overwritten, there is no need for a corresponding call to `MPI_Wait`.

9.6 In a hash table that uses chaining to resolve collisions, the table is an array of pointers to singly linked lists. It is easy enough for one worker to broadcast this array to the other workers, because these pointers are stored in a contiguous group of memory locations. However, the linked lists must also be broadcast. The structures on the linked list are allocated from the heap, and they are not guaranteed to occupy a contiguous block of memory. Even if they did, we cannot guarantee that the same memory locations are available on the other processors? In short, we cannot broadcast element allocated from the heap. A way to solve this problem is to allocate a large, static array and do all dynamic memory allocations from this array. In other words, we create a user-managed heap. We can broadcast the hash table by broadcasting this array along with the array of pointers.

Chapter 10

Monte Carlo Methods

10.1 If you use a linear congruential random number generator, the ordered tuples $(x_i, x_{i+1}, \ldots, x_{i+19})$ in the 19-dimensional space form a lattice structure, limiting the precision of the answer that can be computed by this method.

10.2 The principal risk associated with assigning each process the same linear congruential generator with different seed values is that the sequences generated by two or more processes may be correlated. In the worst case, the sequences of two or more processes will actually overlap.

10.3 Here is a C function (with accompanying static variables) that uses the Box-Muller transformation to return a double-precision floating point value from the normal distribution.

```
int      active = 0;  /* Set to 1 when g2 has a sample */
double   g2;          /* Normal variable */
unsigned short xi[3]; /* Random number seed */

double box_muller (void)
{
   double f, g1, r, v1, v2;   /* See book's pseudocode */

   if (active) {
      active = 0;
      return g2;
   } else {
      do {
            v1 = 2.0 * erand48(xi) - 1.0;
            v2 = 2.0 * erand48(xi) - 1.0;
            r = v1 * v1 + v2 * v2;
         } while ((r <= 0.0) || (r >= 1.0));
         f = sqrt(-2.0 * log(r)/r);
         g1 = f * v1;
         g2 = f * v2;
         active = 1;
         return g1;
      }
   }
```

10.4 The volume is 7.7718.

10.5 The volume is 2040.0.

10.6 The volume is 5000.1.

10.8 This table is based on 10 million neutrons per width H.

H	*Reflected (%)*	*Absorbed (%)*	*Transmitted (%)*
1	25.31	46.13	28.56
2	27.19	64.89	7.92
3	27.31	70.55	2.14
4	27.33	72.10	0.58
5	27.32	72.53	0.15
6	27.34	72.62	0.04
7	27.32	72.66	0.01
8	27.31	72.69	0.00
9	27.32	72.68	0.00
10	27.30	72.70	0.00

10.9 The temperature at the middle of the plate is 25.0°.

10.12 Average number of stalls occupied by cars: 73. Probability of a car being turned away: 8%.

10.13 Probability of waiting at each entrance:

- North: 39.2%
- West: 40.9%
- South: 21.7%
- East: 45.7%

Average queue length at each entrance:

- North: 0.66
- West: 0.73
- South: 0.28
- East: 0.89

Chapter 11

Matrix Multiplication

11.1 (a) $\mathbf{C} = \begin{pmatrix} -10 & 12 \\ -6 & -21 \\ 0 & 10 \end{pmatrix}$

(b) $A_{00}B_{00} + A_{01}B_{10} = (4) + (-14) = (-10)$

$A_{00}B_{01} + A_{01}B_{11} = (1) + (11) = (12)$

$$A_{10}B_{00} + A_{11}B_{10} = \begin{pmatrix} -5 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$$

$$A_{10}B_{01} + A_{11}B_{11} = \begin{pmatrix} -10 \\ -5 \end{pmatrix} + \begin{pmatrix} -11 \\ 15 \end{pmatrix} = \begin{pmatrix} -21 \\ 10 \end{pmatrix}$$

11.2 For reasonable performance we want to ensure that each process's portions of **A** and **B** fit in primary memory. If both **A** and **B** are divided into pieces and distributed among the processors, then larger problems can be solved when more processors are available. If we replicate **B** across all processes, then the maximum problem size is limited by the amount of memory available on each processor. Larger problems cannot be solved when more processors are available. Put another way, a parallel program based on this decomposition is not scalable.

11.3 (a) Cannon's algorithm is a better match for the recursive sequential matrix multiplication algorithm because it divides the factor matrices into square (or nearly square) pieces.

(b) When the block of **B** is too large to fit in cache, the recursive sequential matrix multiplication algorithm described in the book breaks the block into four pieces by dividing it horizontally and vertically. We modify the algorithm so that it only breaks the block into two pieces. It considers the dimensions of the block and divides it in half, making either a vertical cut or a horizontal cut. It breaks the block along the dimension which is largest. For example, if the block has more columns than rows, it makes a vertical cut.

11.4 (a) Computation time per iteration is $\chi n^3/p^2$. Communication time per iteration is $\lambda + n^2/(p\beta)$. Since $p = 16$, $\chi = 10$ nanoseconds, $\lambda = 250$ microseconds, and $\beta = 1.5$ million elements per second, communication time is less than computation time for all $n \geq 1073$.

(b) Computation time per iteration is $\chi n^3/p^{3/2}$. Communication time per iteration is $2(\lambda + n^2/(p\beta))$. Since $p = 16$, $\chi = 10$ nanoseconds, $\lambda = 250$ microseconds, and $\beta = 1.5$ million elements per second, communication time is less than computation time for all $n \geq 545$.

Chapter 12

Solving Linear Systems

12.1 Using back substitution to solve the diagonal system equations at the end of Section 12.4.1:

$$\begin{array}{rrrrcr} 4x_0 & +6x_1 & +2x_2 & -2x_3 & = & 8 \\ & -3x_1 & +4x_2 & -1x_3 & = & 0 \\ & & 1x_2 & +1x_3 & = & 9 \\ & & & 3x_3 & = & 6 \end{array}$$

We can solve the last equation directly, since it has only a single unknown. After we have determined that $x_3 = 2$, we can simplify the other equations by removing their x_3 terms and adjusting their values of $\mathbf{b}$:

$$\begin{array}{rrrrcr} 4x_0 & +6x_1 & +2x_2 & & = & 12 \\ & -3x_1 & +4x_2 & & = & 2 \\ & & 1x_2 & & = & 7 \\ & & & 3x_3 & = & 6 \end{array}$$

Now the third equation has only a single unknown, and can see that $x_2 = 7$. Again, we use this information to simplify the two equations above it:

$$\begin{array}{rrrrcr} 4x_0 & +6x_1 & & & = & -2 \\ & -3x_1 & & & = & -26 \\ & & 1x_2 & & = & 7 \\ & & & 3x_3 & = & 6 \end{array}$$

We have simplified the second equation to contain only a single unknown, and dividing b_1 by $a_{1,1}$ yields $x_1 = 26/3$. After subtracting $x_1 \times a_{0,1}$ from b_0 we have

$$\begin{array}{cccccc} 4x_0 & & & & = & -54 \\ & -3x_1 & & & = & -26 \\ & & 1x_2 & & = & 7 \\ & & & 3x_3 & = & 6 \end{array}$$

and it is easy to see that $x_0 = -54/4 = -27/2$.

Hence $\mathbf{x} = \{-27/2, 26/3, 7, 2\}$.

12.2 Forward substitution algorithm in pseudocode.

Forward Substitution:

```
  $a[0..n-1][0..n-1]$ — coefficient matrix
  $b[0..n-1]$ — constant vector
  $x[0..n-1]$ — solution vector

for $i \leftarrow 0$ to $n-2$ do
  $x[i] \leftarrow b[i]/a[i,i]$
  for $j \leftarrow i+1$ to $n-1$ do
    $b[j] \leftarrow b[j] - x[i] \times a[j,i]$
    $a[j,i] \leftarrow 0$ {This line is optional}
  endfor
endfor
```

12.6 (a) Assume the execution time of the sequential algorithm is χn^3. With interleaved rows, the work is divided more or less evenly among the processors, so we can assume the computation time of the parallel program is $\chi n^3/p$.

During each iteration of the parallel algorithm a tournament communication determines the pivot row. If this "all reduce" is implemented as a reduction followed by a broadcast, the total communication time is about $2\lceil \log p \rceil \lambda$. (The messages are so short that the communication time is dominated by the message latency.)

After the tournament one process broadcasts the pivot row to the other processes. As the algorithm progresses, the number of elements that must be broadcast continues to shrink. Over the course of the algorithm the average number of elements broadcast is about $n/2$. If β measures bandwidth in terms of elements per second, then the total communication time for broadcast steps is $n\lceil\log p\rceil(\lambda + n/(2\beta))$.

The overall estimated execution time of the parallel program is

$$\chi n^3/p + \lceil\log p\rceil(3\lambda + n/(2\beta))$$

(b) Assume the execution time of the sequential algorithm is χn^3. With interleaved columns, the work is divided more or less evenly among the processors, so we can assume the parallel program requires time $\chi n^3/p$ to perform these computations.

Also, a single process is responsible for determining the pivot element each iteration. Assuming it takes χ time units to check each pivot row, and given an average of $n/2$ rows to check per iteration, the time needed to determine the pivot row is $\chi n^2/2$.

After the tournament one process broadcasts column i to the other processes. This column does not shrink as the algorithm progresses, because the unmarked rows are scattered throughout the matrix. If β measures bandwidth in terms of elements per second, then the total communication time for the column broadcasts is $n\lceil\log p\rceil(\lambda + n/beta)$.

The overall estimated execution time of the parallel program is

$$\chi(n^3/p + n^2/2) + \lceil\log p\rceil(\lambda + n/\beta)$$

12.9 During each iteration $\sqrt{p}$ processes are involved determining the pivot row. Each process finds its candidate row. This has time complexity $\Theta(n/\sqrt{p})$. Then these processes participate in a tournament. The tournament has time complexity $\Theta(\log(\sqrt{p})) = \Theta(\log p)$. Up to $\sqrt{p}$ processes control portions of the pivot row that must be broadcast. (The number of processes decreases as the algorithm progresses.) Each process controlling a portion of the pivot row broadcasts it to the other processes in the same column of the virtual process grid. This broadcast has message latency $\Theta(\log\sqrt{p}) = \Theta(\log p)$ and message transmission time $\Theta(\log p(n/\sqrt{p}))$. Combining both communication steps, we see that the checkerboard-oriented parallel gaussian elimination algorithm has overall message latency $\Theta(n\log p)$ and overall message transmission time $\Theta(n^2\log p/\sqrt{p})$.

Next we consider the computation time. As the algorithm progresses, we manipulate elements in fewer and fewer columns. Hence processes gradually drop out of the computation. The processes associated with the rightmost columns stay active until the end. They determine the computation time of the program. Each process is responsible for n^2/p elements. If the process manipulates all of these elements for all n iterations, the time complexity of these computations is $\Theta(n^3/p)$.

Now we can determine the isoefficiency of the algorithm. The total communication overhead across p processes is $\Theta(n^2 p \log p/\sqrt{p}) = \Theta(n^2 \log p \sqrt{p})$. Hence

$$n^3 = Cn^2\sqrt{p}\log p) \Rightarrow n = C\sqrt{p}\log p$$

For this algorithm $M(n) = n^2$. Hence the scalability function is

$$M(C\sqrt{p}\log p)/p = C^2 p \log^2 p/p = C^2 \log^2 p$$

This algorithm is much more scalable than the row-oriented and column-oriented algorithms developed in the chapter. It is not as scalable as the pipelined, row-oriented algorithm.

12.12 (a) The first element of the file is a integer n indicating the number of rows in the matrix. The second element of the file is an integer w indicating the semibandwidth of the matrix. The file contains $n(2w+1)$ additional elements, representing the $2w+1$ elements in each of the n rows of the matrix. Note that first w rows and the last w rows of the banded matrix do not have $2w+1$ elements (see Figure 12.14). The first w and the last w rows in the file are "padded" with 0s for these nonexistent elements. This ensures that the $w+1$st element of each row in the file is the element on the main diagonal of the matrix.

Chapter 13

Finite Difference Methods

13.1 Function u has only three unique partial derivatives with order 2, because $u_{xy} = u_{yx}$. In other words, taking the partial derivative with respect to x and then with respect to y yields the same result as taking the partial derivative with respect to y and then taking the partial derivative with respect to x.

13.6 (a) When $k = 1$, each process sends four messages and receives four messages. All messages have length $n/\sqrt{p}$. We assume sends and receives proceed simultaneously. Communication time per iteration is $4(\lambda + (n/\sqrt{p})/\beta)$. There are no redundant computations. Hence communication cost per iteration is $4(\lambda + (n/\sqrt{p})/\beta)$.

When $k \geq 2$, each process must send eight messages and receive eight messages, because elements are needed from diagonally adjacent blocks. Choose some $k \geq 2$. The total number of elements each process sends (and the total number of elements each process receives) is

$$4\Big(k(n/\sqrt{p}) + \sum_{i=1}^{k-1} i\Big) = 4\Big(kn/\sqrt{p} + k(k-1)/2\Big)$$

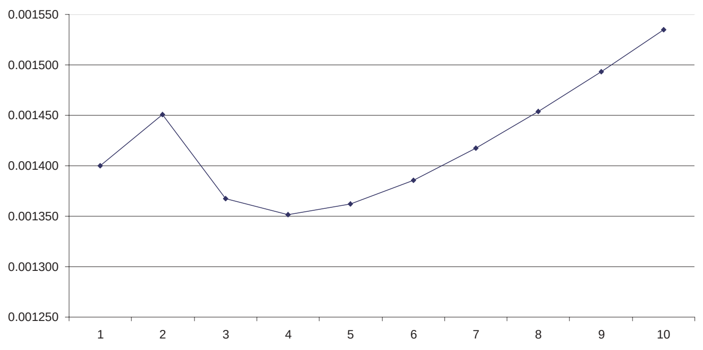
The number of redundant computations is the same as the number of elements communicated for $k - 1$. Hence the number of redundant computations is

$$4\Big((k-1)n/\sqrt{p} + (k-1)(k-2)/2\Big)$$

The average communication cost per iteration is the total communication time plus the redundant computation time, divided by k, or

$$\frac{8\lambda + 4k\Big(n/\sqrt{p} + (k-1)/2\Big)/\beta + 4(k-1)\chi\Big(n/\sqrt{p} + (k-2)/2\Big)}{k}$$

(b) This chart plots communication cost per iteration as a function of k, where $n = 5000$, $p = 16$, $\chi = 100$ nanosec, $\lambda = 100$ μsec, and $\beta = 5 \times 10^6$ elements/sec.



Chapter 14

Sorting

14.1 Quicksort is not a stable sorting algorithm.

14.2 **Note: In the first printing of the text there are a couple of typos in this exercise. First, n should be 100,000, not 100 million. Second, answers should be computed only for p equal to powers of 2: 1, 2, 4, 8, and 16.**

This is a difficult problem. I came up with these numbers by actually writing an MPI program implementing the first parallel quicksort algorithm described in the book, then performing a max-reduction to find the maximum partition size. The values in the table represent the average over 10 executions of the program.

(a)

p	*Maximum List Size*	Max list size / $\lceil n/p \rceil$
1	100,000	1.00
2	71,019	1.42
4	56,428	2.26
8	39,474	3.158
16	36,241	5.799

(b) Parallel execution time is the sum of the computation time and the communication time.

The computation time has two components: partition time and quicksort time. At each iteration of the algorithm each processor must partition its list of values into two pieces: the piece it is keeping and the piece it is passing. This time is about

$$m(\log p)(0.000000070 \text{ sec})$$

where m is the maximum list size. The quicksort time is

$$(0.000000070 \text{ sec})(m \log m - 2.846 \log m)$$

(see Baase and Van Gelder [5] for the expected number of comparisons in sequential quicksort).

The communication time is the time spent exchanging elements. We assume sending and receiving happen concurrently. We estimate the total communication time by the processor with the most elements concurrently sends half of these elements and receives half of these elements each iteration. The total communication time is

$$\log p\big(0.0002 \text{ sec} + (m/2)(4 \text{ byte})/(10^7 \text{ byte/sec})\big)$$

p	Time (sec)	Speedup
1	0.141	1.00
2	0.116	1.21
4	0.106	1.33
8	0.083	1.70
16	0.086	1.64

14.3 **Note: In the first printing of the text there are a couple of typos in this exercise. First, n should be 100,000, not 100 million. Second, answers should be computed only for p equal to powers of 2: 1, 2, 4, 8, and 16.**

This is a difficult problem. I came up with these numbers by actually writing an MPI program implementing the hyperquicksort algorithm described in the book, then performing a max-reduction to find the maximum partition size. The values in the table represent the average over 10 executions of the program.

(a)

p	*Maximum List Size*	Max list size / $\lceil n/p \rceil$
1	100,000	1.000
2	50,089	1.002
4	25,151	1.006
8	12,688	1.015
16	6,723	1.076

(b) Parallel execution time is the sum of the computation time and the communication time. The execution time formulas are the same as in the answer to exercise 14.2.

p	Time (sec)	Speedup
1	0.1412	1.00
2	0.0658	1.77
4	0.0307	3.16
8	0.0143	5.62
16	0.0070	9.41

14.4 We have $n = 10^8$, $\chi = 70$ nanosec, $\lambda = 200$ μsec, and $\beta = 2.5 \times 10^6$ elements/sec. In the first step each processor sorts n/p elements. The time required for this is

$$\chi(1.386(n/p)\log(n/p) - 2.846(n/p))$$

The time to gather the pivots is dominated by the message latency:

$$\lambda(\lceil \log p \rceil)$$

Next one process sorts the p^2 random samples. Using quicksort, the estimated time is

$$\chi(1.386(p^2)\log(p^2) - 2.846p^2$$

Broadcasting $p-1$ pivots to the other processes is dominated by the message latency:

$$\lambda(\lceil \log p \rceil)$$

Next comes the all-to-all communication step. We assume each process sends $p-1$ messages and receives $p-1$ messages. These happen concurrently. The total communication time for this step is

$$(p-1)\lambda + n(p-1)/(p\beta)$$

Finally, each process merges p sorted sublists. If each step requires $2\log p$ comparisons, the time needed for this step is

$$2\chi \log p(1.03n/p)$$

The total estimated execution time is

$$\begin{aligned}&\chi(1.386((n/p)\log(n/p) + 2p^2\log p) - 2.846((n/p) + p^2)\\&\quad + 2\log p(1.03n/p)) +\\&\lambda(2\lceil \log p \rceil + (p-1)) + n(p-1)/(p\beta)\end{aligned}$$

Processors	*Time (sec)*	*Speedup*
1	169.9	1.00
2	88.7	1.92
3	62.5	2.72
4	47.2	3.60
5	40.2	4.23
6	33.8	5.02
7	29.3	5.79
8	26.0	6.53
9	24.6	6.91
10	22.4	7.58
11	20.7	8.22
12	19.2	8.85
13	18.0	9.46
14	16.9	10.0
15	16.0	10.6
16	15.2	11.2

14.5 The PSRS algorithm is less disrupted when given a sorted list.

In the first iteration of the hyperquicksort algorithm, the root process sorts its sublist, then broadcasts the median of its sublist to the other processes. All processes use this value to divide their lists into the lower "half" and the upper "half." Since the root process has the smallest n/p keys, its estimate of the median will be very poor. The root process will send half of its keys to its partner process in the upper half of the hypercube. The other processes in the lower half of the hypercube will send all of their keys to their partner processes in the upper half of the hypercube. This severe workload imbalance will significantly increase the execution time of the algorithm.

In contrast, in the PSRS algorithm each process contributes p samples from its sorted sublist. After sorting the contributed samples, the root process selects regularly spaced pivot values. After receiving the pivot values, each process will partition its sorted sublist into p disjoint pieces. The largest partition held by process i will be partition i. When each process i keeps its ith partition and sends the jth partition to process j, relatively few values will be communicated. Each process should have nearly the same number of elements for the final mergesort step.

14.6 We are going to perform a p-way merge of p sorted sublists. We build a heap from the first element of each sorted sublist. This takes time $\Theta(p)$. The root of the heap is the element with the smallest value. It is the first element of the merged lists. We remove the heap element and replace it with the next member of that element's originating sublist. This can be done in constant time. We then perform a "heapify" operation to restore the heap property. This step takes $\Theta(\log p)$ time.

Now the root of the heap is the second element of the merged lists. We repeat the previous delete-replace-heapify procedure.

When an originating sublist is exhausted, we insert a key with a very large value into the heap, ensuring it will never be chosen before an element from one of the other sublists.

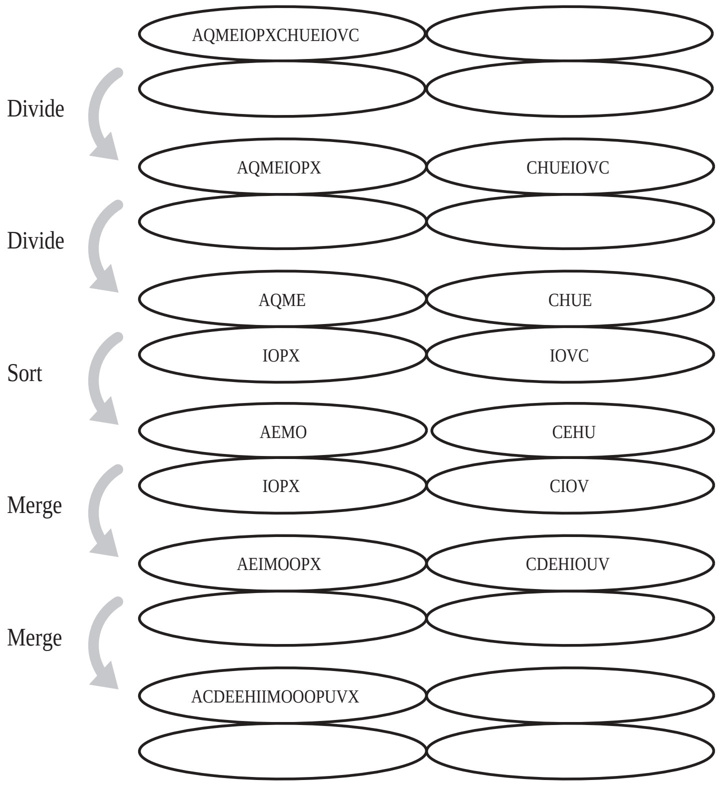
We continue until we have deleted from the heap a number of elements equal to the original total length of all the sublists. If there were $\Theta(n/p)$ elements total in the p sublists, the total time required to perform the merge is $\Theta((n/p)\log p)$.

14.9 (a) We organize the processes into a binomial tree. In the initial $\log p$ steps of the algorithm, each process with a list of elements divides its list in half and sends half of its list to a partner process. After $\log p$ steps, every process has n/p keys.

Next each process uses sequential mergesort to sorts its sublist.

Reversing the communication pattern of the first step, during each of $\log p$ steps half the processes with a sorted sublist send their sublist to another process with a sorted sublist, and the receiving processes merges the lists. After $\log p$ of these steps, a single process has the entire sorted list.

This figure illustrates parallel mergesort.



(b) Distributing the unsorted subarrays takes $\log p$ steps. In the first step 1 process sends $n/2$ values, in the second step 2 processes concurrently send $n/4$ values, and so on. Hence the communication complexity of this step is $\Theta(\log p + n)$. The time needed for each process to sort its section of n/p values is $\Theta((n/p)\log(n/p))$. The communication time needed for the merge step is $\Theta(\log p + n)$. The computation time for the merge step is $\Theta(n)$. Hence the overall time complexity of parallel mergesort is $\Theta(\log p + n + (n/p)\log(n/p))$.

(c) The communication overhead of the algorithm is p times the communication complexity, or $\Theta(p(\log p + n))$. When n is large the message transmission time dominates the message latency, so we'll simplify the communication overhead to $\Theta(p \log p)$. Hence the isoefficiency function is

$$n \log n \geq Cp \log p$$

When we double p we must slightly more than double n in order to maintain the same level of efficiency. Since p is much smaller than n, the algorithm is not perfectly scalable.

14.10 (a) We divide the interval $[0, k)$ into p buckets (subintervals). Each process is responsible for one of these buckets, and every process knows the interval handled by each process. In step one every process divides its n/p keys into p groups, one per bucket. In step two an all-to-all communication routes groups of keys to the correct processes. In step three each process sorts the keys in its bucket using quicksort.

(b) Step one has time complexity $\Theta(n/p)$. Since n is much larger than p, communication time is dominated by the time needed to transmit and receive the elements. We expect each processor to send about $n(p-1)/p$ elements to other processors and receive about the same number. Assuming the network has sufficient bandwidth to allow each processor to be sending and receiving a message concurrently, the time needed for step two is $\Theta(n/p)$. Step three has time complexity $\Theta((n/p)\log(n/p))$. The overall computational complexity of the parallel bucket sort is $\Theta((n/p)\log(n/p))$, and the communication complexity is $\Theta(n/p)$.

(c) The communication overhead is p times the communication complexity, or $\Theta(n)$. If there are p buckets, the sequential algorithm requires time $\Theta(n)$ to put the values in the buckets and time $\Theta((n/p)\log(n/p))$ to sort the elements in each bucket. With p buckets, the complexity of the sequential algorithm is $\Theta(n\log(n/p))$.

Hence the scalability function is

$$n\log(n/p) \geq Cn \Rightarrow \log(n/p) \geq C \Rightarrow n/p \geq 2^C \Rightarrow n \geq 2^C p$$

The parallel bucket sort algorithm is not highly scalable.

Chapter 15

The Fast Fourier Transform

15.1 (a) $1 = 1e^{0i}$

(b) $i = 1i^{(\pi/2)i}$

(c) $4 + 3i = 5e^{0.93i}$

(d) $-2 - 5i = 5.39e^{4.33i}$

(e) $2 - i = 2.24e^{5.82i}$

15.2 Principal 2d root of unity is $-1 = 1e^{\pi i}$.

Principal 3d root of unity is $-0.5 + 0.87i = 1e^{(2\pi/3)i}$.

Principal 4th root of unity is $i = 1e^{(\pi/2)i}$.

Principal 5th root of unity is $0.31 + 0.95i = 1e^{(2\pi/5)i}$.

15.3 (a) $(18, -4)$

(b) $(72, -6 + 6i, -8, -6 - 6i)$

(c) $(28, -5.83 - 1.59i, 4 + 8i, -0.17 + 4.41i, -8, -0.17 - 4.41i, 4 - 8i, -5.83 + 1.59i)$

15.4 (a) $(0.5, 2.5)$

(b) $(1, 2, 3, 4)$

(c) $(1, 1.82, 3.25, 2.03, 3.00, 2.18, -0.25, 0.97)$

15.8 Our first parallel quicksort algorithm and hyperquicksort exhibit a butterfly communication pattern. A butterfly communication pattern is also an appropriate way to perform an all-to-all exchange when the communication time is dominated by message latency. An all-reduce may be performed using a butterfly communication pattern.

15.9 The sequential versions of both hyperquicksort and the fast Fourier transform have time complexity $\Theta(n \log n)$. Both parallel algorithms have $\Theta(\log p)$ communication steps, each step transmitting $\Theta(n/p)$ values. Both parallel algorithms use a butterfly communication pattern that enables data starting on any processor to reach any other processor in no more than $\log p$ message-passing steps.

Chapter 16

Combinatorial Search

16.1 The decomposition step cannot be performed in parallel. The time needed for decompositions alone is

$$n + \frac{n}{k} + \frac{n}{k^2} + \ldots + \frac{n}{k^{\log_k n}}$$

Since this sum is $> n$ and $< nk/(k-1)$, a good lower bound on the complexity of the parallel algorithm is $\Omega(n)$.

16.2 There are two fundamentally different strategies for limiting the number of nodes in the state space tree. I do not know which one is more effective for this problem.

The first approach is to always fill in the word that would cause the most unassigned letters in the puzzle to be given a value. By assigning as many letters as possible at each node in the state space tree, this strategy aims to minimize the depth of the tree.

The second approach is to always fill in the word with the fewest unassigned letters. In case of a tie, we would choose the longest word with the fewest unassigned letters. This strategy aims to minimize the number of branches at each node in the state space tree.

The algorithm described in the book represents a middle point between the two strategies I have just described.

16.3 It is unlikely that a simple backtrack search of the state space tree for the 8-puzzle will yield a solution, because there is a good probability that before the search finds a solution node, it will encounter a node whose board position is identical to that of a previously-examined node. If this happens, it means that the search has run into a cycle.

Assuming that the legal moves from a given position are always examined in the same order, encountering a cycle guarantees that the search will never leave the cycle, and hence a solution will never be found.

Here are two ways to modify the backtrack search to guarantee it will eventually find a solution.

One way to avoid falling into an infinite cycle is to check for cycles every time a move is considered. If a move results in a board position already represented by one of the ancestor nodes in the state space tree, the move is not made.

Another way to avoid falling into an infinite cycle is to use iterative deepening; i.e., limiting the search to a certain depth. First we perform backtrack search to a fixed level d. If no solution is found, we perform a backtrack search to level $d+1$. We continue in this fashion until a solution is found.

16.4 The lower bound associated with the right node of the root is 7, because the eight tiles are out of place by a total of 6 positions (using the Manhattan distance calculation), and 1 move has already been taken. $6+1=7$. (Tile 5 is 1 position out of place, tile 3 is 2 positions out of place, tile 2 is 2 positions out of place, and tile 6 is 1 position out of place.)

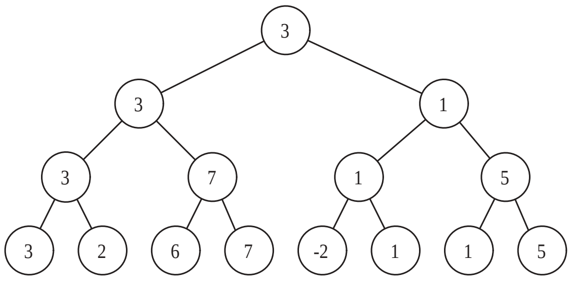
16.5 The number of nodes at each level of the state space tree is $1, 3, 8, 24, 64, \ldots$. (Multiply by 3, then multiply by $8/3$, multiply by 3, multiply by $8/3$, etc.) A breadth-first search examines all nodes in the first five levels of the state space tree, or $1+3+8+24+64=100$ nodes. A breadth-first search also examines all nodes up to and including the solution node at the sixth level of the tree. There are 96 nodes to the left of the solution node. The total number of nodes examined by a breadth-first search of the state space tree is 197.

16.6 In case of a tie, we want to pick the node deeper in the state space tree, since its lower bound is based more on definite information (the number of constraints added so far), and less on an estimate (the lower bound function).

16.7 The advantage of the proposed best-first branch-and-bound algorithm is that it has very little communication overhead. The disadvantage is that the workloads among the processes are likely to be highly imbalanced. The proposed algorithm is unlikely to achieve higher performance than the algorithm presented in the text.

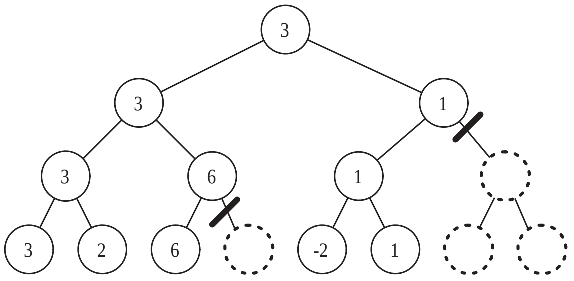
16.8 If a perfect evaluation function existed, the best move could be found simply by evaluating the position resulting from each legal move and choosing the position with the highest value.

16.9 Here is the result of a minimax evaluation of the game tree:

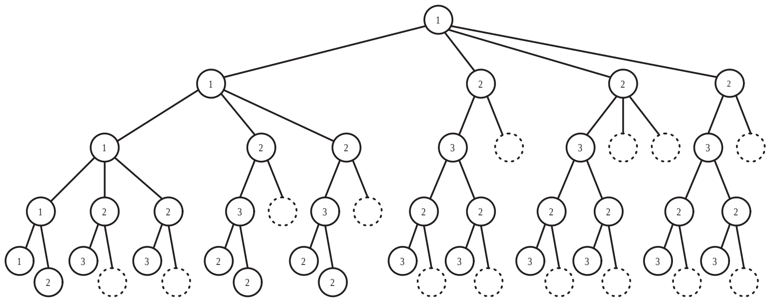


16.10 The alpha-beta algorithm prunes the game tree of Figure 16.19 at the node labeled B because this is a min node, and the first value returned is -9. The opponent is able to choose the move. Since the opponent is trying to minimize the value, we know that the ultimate value of the node is less than or equal to -9. However, the parent of this node is a max node. Since the player already has a move with leading to a position with value -7, there is no way the player will make a move leading to a position with value less than or equal to -9, so there is no point in continuing the search. The search tree is pruned at this point.

16.11 Here is the result of an alpha-beta search of the game tree:



16.12 Extending the perfectly ordered game tree.



16.13 When DTS search reaches a point where there is only a single processor allocated to a node, it executes the standard sequential alpha-beta search algorithm. Hence alpha-beta search is a special case of DTS—the case where $p = 1$.

16.14 Improving the pruning effectiveness of the underlying alpha-beta algorithm reduces the speedup achieved by the DTS algorithm on a particular application. Given a uniform game tree with branching factor b, if alpha-beta pruning reduces the effective branching factor to b^x (where $0.5 \leq x \leq 1$), then speedup achieved by DTS with p processors and breadth-first allocation will be reduced to $O(p^x)$.

Chapter 17

Shared-memory Programming

17.1 The two OpenMP function that have closest analogs in MPI functions are `omp_get_num_threads` and `omp_get_thread_num`. Function `omp_get_num_threads` is similar to `MPI_Comm_size`. Function `omp_get_thread_num` is similar to `MPI_Comm_rank`.

17.2 (a) `#pragma omp parallel for`

(b) This code segment is not suitable for execution because the control expression of the `for` loop is not in canonical form. The number of iterations cannot be determined at loop entry time.

(c) This loop can be executed in parallel assuming there are no side effects inside function `foo`. The correct pragma is

```
#pragma omp parallel for
```

(d) This loop can be executed in parallel assuming there are no side effects inside function `foo`. The correct pragma is

```
#pragma omp parallel for
```

(e) This loop is not suitable for parallel execution, because it contains a `break` statement.

(f) `#pragma omp parallel for reduction(+:dotp)`

(g) `#pragma omp parallel for`

(h) This loop is not suitable for parallel execution if `n` is greater than or equal to `2*k`, because in that case there is a dependence between the first iteration of the loop that assigns a value to `a[k]` and the (`k`+ 1)st iteration, that references the value.

17.3 Here is a C program segment that computes π using a private variable and the `critical` pragma.

```
double area, pi, subarea, x;
int i, n;

area = 0.0;
#pragma omp parallel private(x, subarea)
{
   subarea = 0.0;
#pragma omp for nowait
   for (i = 0; i < n; i++) {
      x = (i+0.5)/n;
      subarea += 4.0/(1.0 + x*x);
   }
#pragma omp critical
   area += subarea;
}
pi = area / n;
```

17.4 Matrices in C are stored in row-major order. There is a good chance that the end of one matrix row and the beginning of the next matrix row are stored in the same cache block. Increasing the chunk size reduces the number of times one thread is responsible for one matrix row and another thread is responsible for the next matrix row. Hence increasing the chunk size reduces the number of times two different threads share the same cache block and increases the cache hit rate.

17.5 Here is an example of a simple parallel `for` loop that would probably execute faster with an interleaved static scheduling than by giving each task a single contiguous chunk of iterations.

```
for (i = 0; i < 100000; i++) {
   if (i < 10000) {
      a[i] = (sin(i) + cos(i))*(sin(i) + cos(i));
   } else {
      a[i] = 0.0;
   }
}
```

17.6 A good example would be a loop with a small ratio of iterations to threads in which the variance in the execution times of each iteration is large and cannot be predicted at compile time.

17.7 Two threads can end up processing the same task if both threads are executing function `get_next_task` simultaneously and both threads execute statement

```
answer = (*job_ptr)->task;
```

before either thread executes the following statement:

```
*job_ptr = (*job_ptr)->next;
```

That is why we add the `critical` pragma to ensure the function executes atomically.

17.8 Another OpenMP implementation of the Monte Carlo algorithm to compute π.

```
#include <stdlib.h>
#include <stdio.h>

int main (int argc, char *argv[])
{
   int count;            /* Points inside unit circle */
   int i;
   int local_count;      /* This thread's subtotal */
   int samples;          /* Points to generate */
   unsigned short xi[3]; /* Random number seed */
   double x, y;          /* Coordinates of point */

   samples = atoi(argv[1]);
   omp_set_num_threads (atoi(argv[2]));
   count = 0;
#pragma omp parallel private(xi,x,y,local_count)
   {
   local_count = 0;
   xi[0] = 1;
   xi[1] = 1;
   xi[2] = omp_get_thread_num();
#pragma omp for nowait
```

```
   for (i = 0; i < samples; i++) {
      x = erand48(xi);
      y = erand48(xi);
      if (x*x + y*y <= 1.0) local_count++;
   }
#pragma omp critical
   count += local_count;
   }
   printf ("Estimate of pi: %7.5f\n", 4.0*count/samples);
}
```

17.9 Code segment from Winograd's matrix multiplication algorithm, augmented with OpenMP directives.

```
#pragma omp parallel
{
    #pragma omp for private(j) nowait
    for (i = 0; i < m; i++) {
       rowterm[i] = 0.0;
       for (j = 0; j < p; j++)
       rowterm[i] += a[i][2*j] * a[i][2*j+1];
    }
    #pragma omp for private(j) nowait
    for (i = 0; i < q; i++) {
       colterm[i] = 0.0;
       for (j = 0; j < p; j++)
          colterm[i] += b[2*j][i] * b[2*j+1][i];
    }
}
}
```

Alternatively, we could have used the `sections` pragma to enable both `for` loops indexed by `i` to execute concurrently.

```
#pragma omp parallel
{
    #pragma omp sections
    {
    #pragma omp section
    for (i = 0; i < m; i++) {
       rowterm[i] = 0.0;
```

```
      for (j = 0; j < p; j++)
      rowterm[i] += a[i][2*j] * a[i][2*j+1];
   }
   #pragma omp section
   for (i = 0; i < q; i++) {
      colterm[i] = 0.0;
      for (j = 0; j < p; j++)
         colterm[i] += b[2*j][i] * b[2*j+1][i];
   }
   }
}
```

Some compilers may allow both parallelizations. However, the Sun compiler does not allow a `for` pragma to be nested inside a `sections` pragma.

Chapter 18

Combining MPI and OpenMP

18.1 We consider the suitability of each of the program's functions for parallelization using threads.

`main`—The function contains no `for` loops. It has no large blocks of code that two or more threads could execute concurrently. It is not suitable for parallelization using threads.

`manager`—The only `for` loop in the function is the one that copies the document profile vector from `buffer` to `vector[assigned[src]]`. If the vector is long enough, this step could benefit from execution by concurrent threads. Before doing this, however, the `for` loop should be changed to a `memcpy` operation. There are no large blocks of code that two or more threads could execute concurrently.

`worker`—The function contains no `for` loops in canonical form. It has no large blocks of code that two or more threads could execute concurrently. It is not suitable for parallelization using threads.

`build_2d_array`—This function takes little time to execute.

`get_names`—This function searches a directory structure for names of documents to be profiled. The speed of this search could be improved by dividing subdirectories among threads.

`write_profiles`—This function writes profile vectors to a single file. It is not suitable for concurrent execution.

`build_hash_table`—This function initializes an array of pointers and takes little time to execute.

`make_profile`—This function builds a profile vector from the words in a document. We can imagine most of the program's execution time is spent inside this function. Assuming the profile vector for an entire document can be created from profile vectors of the document's pieces, this function is a good candidate for parallelization via threads.

`read_dictionary`—This function reads words from a single file. It is not a good candidate for parallelization.

From our analysis it appears the functions that are the best candidates for parallelization with OpenMP pragmas are `make_profile` and `get_names`.

18.2 The book describes three situations in which mixed MPI/OpenMP programs can outperform MPI-only programs.

The first situation is when the mixed MPI/OpenMP program has lower communication overhead than the MPI-only program. Monte Carlo methods typically have very little communication among the processes. Hence this situation probably does not hold.

The second situation is when it is practical to parallelize some portions of the computation via lighter weight threads, but not via heavier weight processes. In the case of a Monte Carlo program, the entire computation is completely parallelizable—every process is executing the simulation with a different stream of random numbers. Hence this situation probably does not hold.

The third situation is when some MPI processes are idle when others are busy, due to message-passing or other I/O operations. This is most likely not the case for Monte Carlo simulations.

The conclusion is that there are poor prospects for significantly improving the performance of the Monte Carlo methods by converting them into MPI/OpenMP programs.