Contents

7 Work-Kinetic Energy Theorem

7.1 What is Energy?

Loosely speaking, energy is a number that we associate with a system of one or more objects. If a force changes one of the objects by, say, making it move, then the energy number changes. After countless experiments, scientists and engineers realized if the scheme by which we assign energy numbers is planned carefully, the number can be used to predict the outcomes of experiments and even more important, to build machines. This success is based on a wonderful property of our universe: Energy can be transformed from one type to another and transferred from one object to another, but the total amount is always the same (energy is *conserved*.) No exception to the *principle of energy conservation* has ever been found.

In this chapter, we focus on only one type of energy (*kinetic energy*) and on only one way in which energy can be transferred (*work*).

7.2 Kinetic Energy

Kinetic Energy K is energy associated with the state of motion of an object. For an object of mass m whose speed v is well below the speed of light c,

$$K = \frac{1}{2}mv^2$$

The SI unit of kinetic energy is the **joule** (J), named for James Prescott Joule, an English scientist of the 1800s. It is defined in terms of the units for mass and velocity

$$1\ joule = 1\ J = 1\ kg \cdot m^2/s^2$$

7.3 Work

If you accelerate an object to a greater speed by applying a force to the object, you increase the kinetic energy K ($\frac{1}{2}mv^2$) of the object. Similarly, if you decelerate the object to a lesser speed by applying a force, you decrease the kinetic energy of the object. We account for the change in kinetic energy by saying that your force has transferred energy to the object from yourself or from the object to yourself. In such a transfer of energy, work W is said to be *done on the object by the force.* If the force $\vec{F}$ is applied to an object which moves a distance specified by the displacement vector $\Delta\vec{x}$, the amount of work ΔW that is done on the object is defined to be

$$\Delta W = \vec{F} \cdot \Delta\vec{x} = \left|\vec{F}\right| |\Delta\vec{x}| \cos\phi$$

where ϕ is the angle between the direction of the displacement $\Delta\vec{x}$ and the force $\vec{F}$. Note that ΔW is positive if $0 \leq \phi < \frac{\pi}{2}$ or the force $\vec{F}$ has a vector component in the same direction as the displacement $\Delta\vec{x}$, is negative if $\frac{\pi}{2} < \phi \leq \pi$ or $\vec{F}$ has a vector component in the opposite direction of $\Delta\vec{x}$, and is zero if $\phi = \frac{\pi}{2}$ or $\vec{F} \perp \Delta\vec{x}$.

7.3.1 Net work done by several forces

When two or more forces act on a particle, the net work done on the particle is the sum of the works done by the individual forces. We can (1) find the work done by each force and then sum those works (2) find the net force $\vec{F}_{net}$ and then calculate the work done by $\vec{F}_{net}$. The work done by a force $\vec{F}_i$ is

$$\Delta W_i = \vec{F}_i \cdot \Delta\vec{x}$$

The net work is

$$\Delta W = \sum_i \Delta W_i = \sum_i \left(\vec{F}_i \cdot \Delta\vec{x} \right) = \left(\sum_i \vec{F}_i \right) \cdot \Delta\vec{x} = \vec{F}_{net} \cdot \Delta\vec{x}$$

Note for an object of extended size, the net work done on the object may not be equal to work done by the net force. This is because different parts of the object may experience different displacement $\Delta\vec{x}_i$ and no common displacement $\Delta\vec{x}$ may be defined.

$$\Delta W = \sum_i \Delta W_i = \sum_i \vec{F}_i \cdot \Delta\vec{x}_i \neq \left(\sum_i \vec{F}_i \right) \cdot \Delta\vec{x}$$

7.4 Work-Kinetic Energy Theorem

The work done on a particle which is subjected to a net force $\vec{F}$ is

$$\Delta W = \vec{F} \cdot \Delta\vec{x} = m\vec{a} \cdot \Delta\vec{x}$$

where we have use Newton's second law to identify $\vec{F} = m\vec{a}$. For an infinitesimal displacement,

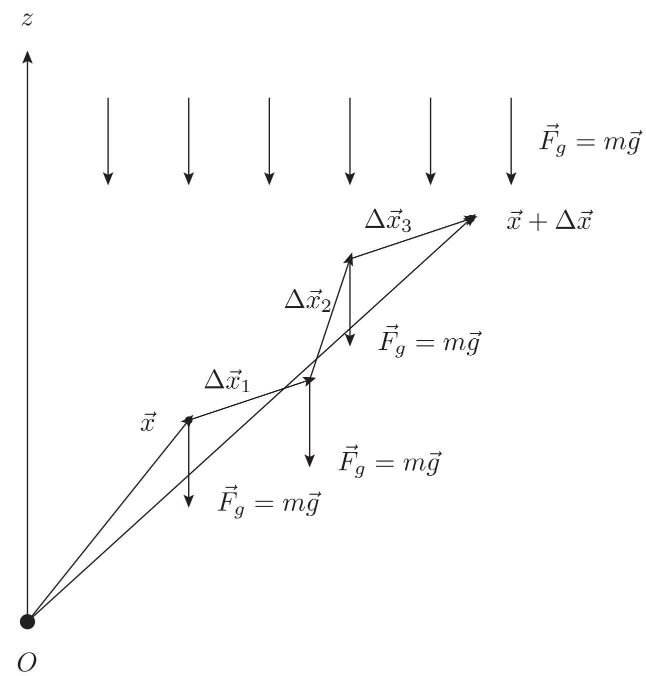
$$\Delta\vec{x} = \vec{v}\Delta t$$

Thus, we have

$$\begin{aligned}\Delta W &= m\vec{a} \cdot \vec{v}\Delta t = m\frac{d\vec{v}}{dt} \cdot \vec{v}\Delta t = \frac{1}{2}m\left(\frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt}\right)\Delta t \\ &= \frac{d}{dt}\left(\frac{1}{2}m\vec{v} \cdot \vec{v}\right)\Delta t = \Delta\left(\frac{1}{2}m\vec{v} \cdot \vec{v}\right) = \Delta K\end{aligned}$$

This is known traditionally as the work-kinetic energy theorem. It says that the amount of net work done on a particle is equal to the change of its kinetic energy. This statement holds for both positive and negative work. If the net work done on the particle is positive, then the particle's kinetic energy increases by that amount. If the net work done is negative, then the particle's kinetic energy decreases by the amount of the work.

7.5 Work Done by the Constant Gravitational Force



In the above figure, the work done by the constant gravitational force $\vec{F}_g = m\vec{g}$ to move the particle from $\vec{x}$ to $\vec{x}+\Delta\vec{x}$, where $\Delta\vec{x} = \Delta\vec{x}_1+\Delta\vec{x}_2+\Delta\vec{x}_3$, is equal to

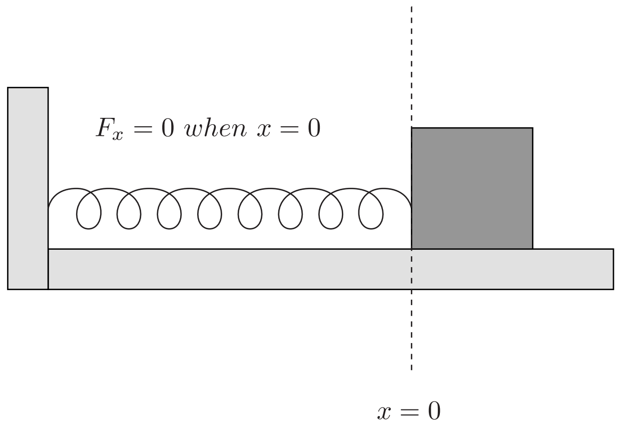
$$\begin{aligned}\Delta W &= \Delta W_1 + \Delta W_2 + \Delta W_3 \\ &= m\vec{g}\cdot\Delta\vec{x}_1 + m\vec{g}\cdot\Delta\vec{x}_2 + m\vec{g}\cdot\Delta\vec{x}_3 \\ &= m\vec{g}\cdot\Delta\vec{x}\end{aligned}$$

Now suppose the positive z-axis points in the vertical direction. $\vec{g} = -g\hat{k}$. Assume $\vec{x} = (x, y, z)$ and $\Delta\vec{x} = (\Delta x, \Delta y, \Delta z)$. Then

$$\Delta W = -mg\,(0, 0, 1)\cdot(\Delta x, \Delta y, \Delta z) = -mg\Delta z$$

The work done is equal to $-mg$ times the change of height of the particle. It is positive the particle is lowered and is negative if the particle is lifted. According the work-kinetic energy theorem, the kinetic energy (so is the speed) increases if the height of the particle is lowered otherwise the kinetic energy decreases if the particle's height is raised.

7.6 Spring Force



The above figure shows a spring in its relaxed mode—that is, neither compressed nor extended. One end is fixed, and an object is attached to the other, free end. If we stretch the spring by pulling the object to the right, the spring pulls on the object toward the left. If we compress the spring by pushing the object to the left, the spring now pushes on the object to the right.

To a good approximation for many springs, the restoring force $\vec{f_s}$ from a spring is proportional to the displacement $\vec{d}$ of the free end from its position when the spring is in the relaxed state and is given by

$$\vec{f_s} = -k\vec{d} \quad \text{(Hooke's law)}$$

The minus sign in the above indicates that the direction of the spring force is always opposite the direction of the displacement of the spring's free end. The constant k is called the **spring constant** (or **force constant**) and is a measure of the spring's stiffness. The larger k is, the stiffer the spring; that is the larger k is, the stronger the spring's pull or push for a given displacement.

Assume the x-axis has been placed parallel to the length of the spring, with its origin ($x = 0$) at the position of the free end when the spring is in its relaxed state. For this arrangement, $\vec{f_s} = F_x\hat{\imath}$ and we have

$$F_x = -kx \ \text{(Hooke's law)}$$

Note that this spring force is a variable force because it is a function of x, the position of the free end and F_x can be symbolized as $F(x)$.

$$\vec{f_s} = F(x)\,\hat{\imath} = -kx\hat{\imath}$$

7.6.1 The Work Done by a Spring Force

To find the work done by the spring, we use calculus. Let the object's initial position be x_i and final position be x_f. Then divide the distance between these two positions into many segments, each of tiny length Δx. Label these segments starting from x_i as segments 1,2, and so on. As the object moves through a segment, the spring force hardly varies because the segment is so short that x hardly varies. Thus we may approximate the force as being constant within the segment. Label these forces as F_{x1} in segment 1, F_{x2} in segment 2 and so on. Then the work done is $F_{x1}\Delta x$ in segment 1, $F_{x2}\Delta x$ in segment 2 and so on. The net work W_s done by the spring from x_i to x_f, is the sum of all these works.

$$W_s = \sum_i F_{xi} \Delta x$$

In the limit as Δx goes to zero, the above becomes

$$\begin{aligned} W_s &= \int_{x_i}^{x_f} F(x)\, dx = -k \int_{x_i}^{x_f} x dx \\ &= -\frac{1}{2} k x^2 |_{x_i}^{x_f} = \frac{1}{2} k x_i^2 - \frac{1}{2} k x_f^2 \end{aligned}$$

W_s is positive if the object ends up closer to the relaxed position ($x = 0$) than it was initially. It is negative if the object end up farther away from $x = 0$.

7.6.2 The Work Done by an Applied Force

Suppose we displace the object attached to the spring along the x-axis by applying a force $\vec{F}_a$ continuously. By work-kinetic energy theorem, the change ΔK in the kinetic energy is

$$\Delta K = K_f - K_i = W_s + W_a$$

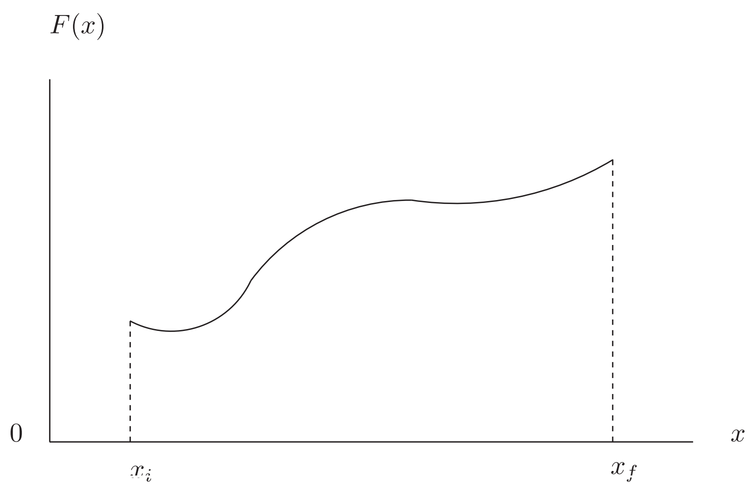
If the object that is attached to the spring is stationary before and after the displacement, $K_i = K_f = 0$, then

$$W_a = -W_s$$

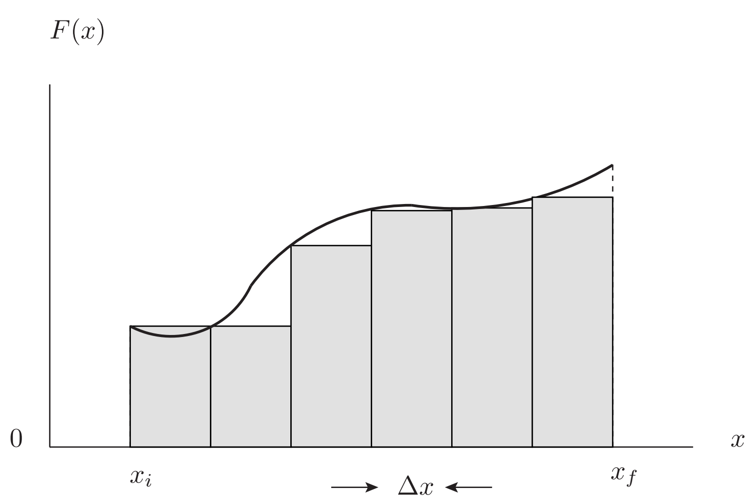
the work done by the applied force is the negative of the work done by the spring force.

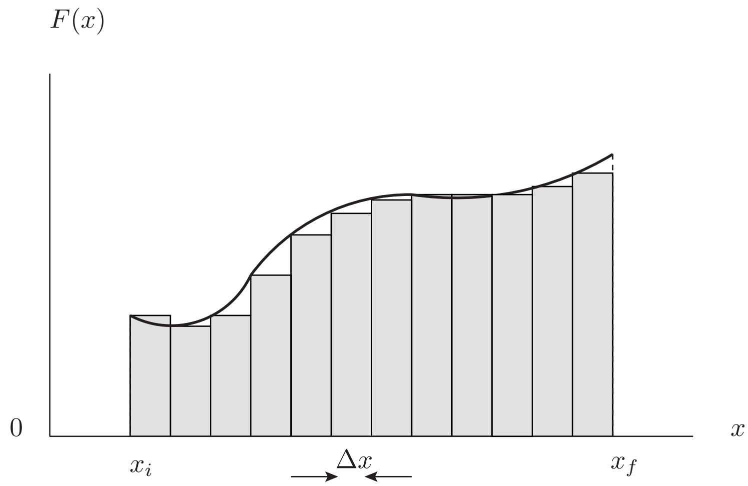
7.7 Work Done by a General Variable Force

7.7.1 One-Dimensional Analysis



The above figure shows a one-dimensional variable force. We want an expression for the work done on the particle by this particle as the particle moves from an initial point x_i to a final point x_f. We divide the area the area under the curve into a small number of narrow strips of width Δx. We choose Δx small enough to permit us to take the force $F(x)$ as becoming reasonably constant over that interval.





Let $F_{j,avg}$ be the average value of $F(x)$ within the jth interval. With $F_{j,avg}$ considered constant, the increment of work ΔW_j done by the force in the jth interval is now approximately given by

$$\Delta W_j = F_{j,avg} \Delta x$$

The total work W done by the force as the particle moves from x_i to x_f is calculated by adding the ares of all the strips between x_i and x_f.

$$W = \sum_j \Delta W_j = \sum_j F_{j,avg} \Delta x$$

We can make the approximation better by reducing the strip width Δx and using more strips. In the limit, we let the strip width approach zero; the number of strips then becomes infinitely large and we have, as an exact result,

$$W = \lim_{\Delta x \to 0} \sum_j F_{j,avg} \Delta x$$

This limit is exactly what we mean by the integral of the function $F(x)$ between x_i and x_f.

$$W = \int_{x_i}^{x_f} F(x)\, dx$$

7.7.2 Three-Dimensional Analysis

Consider now a particle that is acted on by a three-dimensional force

$$\vec{F} = F_x\,(x, y, z)\,\hat{\imath} + F_y\,(x, y, z)\,\hat{\jmath} + F_z\,(x, y, z)\,\hat{k}$$

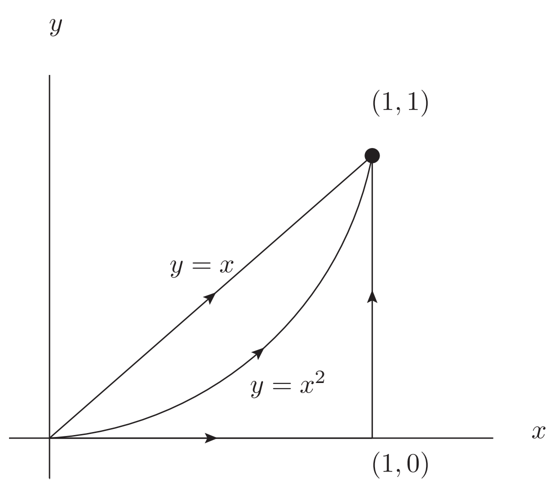
Let the particle move through an incremental displacement

$$d\vec{x} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$$

The work done by $\vec{F}$ while the particle moves from an initial position $\vec{x}_i = (x_i, y_i, z_i)$ to a final position $\vec{x}_f = (x_f, y_f, z_f)$ is then

$$\begin{aligned} W &= \int_{\vec{x}_i}^{\vec{x}_f} dW = \int_{\vec{x}_i}^{\vec{x}_f} \vec{F} \cdot d\vec{x} \\ &= \int_{\vec{x}_i}^{\vec{x}_f} (F_x dx + F_y dy + F_z dz) \end{aligned}$$

As an example, consider the work done by the force field $\vec{F} = y^2\hat{\imath} + x\hat{\jmath}$ for three different paths as show below:



Path 1: The particle moves from $(0,0,0)$ to $(1,1,0)$ along the straight line $x = y$, $z = 0$. We may parametrize the path by setting

$$x = t, y = t, z = 0 \text{ with } 0 \le t \le 1$$

Then $dW = \vec{F} \cdot d\vec{x} = y^2 dx + x dy = t^2 dt + t dt = (t^2 + t)\, dt$ and

$$W = \int_0^1 \left(t^2 + t\right) dt = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

Path 2: The particle moves from $(0,0,0)$ to $(1,1,0)$ along the curve $y = x^2$, $z = 0$. We may parametrize the path by setting

$$x = t, y = t^2, z = 0 \text{ with } 0 \leq t \leq 1$$

Then $dW = \vec{F} \cdot d\vec{x} = y^2 dx + x dy = t^4 dt + t dt^2 = (t^4 + 2t^2)\, dt$ and

$$W = \int_0^1 \left(t^4 + 2t^2\right) dt = \frac{1}{5} + \frac{2}{3} = \frac{13}{15}$$

Path 3: The particle moves from $(0,0,0)$ to $(1,0,0)$ along the straight line $y = 0$, $z = 0$ and then from $(1,0,0)$ to $(1,1,0)$ along the line $x = 1$, $z = 0$. We may parametrize the path by setting

$$x = t, y = 0, z = 0 \text{ with } 0 \leq t \leq 1$$

and

$$x = 1, y = t - 1, z = 0 \text{ with } 1 \leq t \leq 2$$

Then $dW = \vec{F} \cdot d\vec{x} = y^2 dx = 0$ along the x-axis and $dW = x dy = dt$ along the vertical line $x = 1$. Thus

$$W = \int_1^2 dt = 1$$

7.8 Power

The time rate at which work is done by a force is said to be the power due to the force. The average power due to the force during that time interval is

$$P_{avg} = \frac{\Delta W}{\Delta t}$$

The **instantaneous power** P is the instantaneous time rate of doing work, which we can write as

$$P = \frac{dW}{dt}$$

The SI unit of power is joule per second or **watt** (W), after James Watt, who greatly improved the rate at which steam engines could do work. In British system, the unit of power is the foot-pound per second. Often the horsepower is used. They are related by

$$1\ watt = 1\ W = 1\ J/s = 0.738\ ft \cdot lb/s$$

and

$$1\ horsepower = 1\ hp = 550\ ft \cdot lb/s = 746\ W$$

Work can sometimes expressed as power multiplied by time, as in the common unit kilowatt-hour

$$1\ kilowatt - hour = 1\ kW \cdot h = 3.60 \times 10^6\ J$$

We can also express the rate at which a force does work on a particle in terms of that force and particle's velocity.

$$P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{x}}{dt} = \vec{F} \cdot \vec{v}$$