PROBLEM SET 4.1

1–6 MIXING PROBLEMS

1. Find out, without calculation, whether doubling the flow rate in Example 1 has the same effect as halfing the tank sizes. (Give a reason.)
2. What happens in Example 1 if we replace T_1 by a tank containing 200 gal of water and 150 lb of fertilizer dissolved in it?
3. Derive the eigenvectors in Example 1 without consulting this book.
4. In Example 1 find a "general solution" for any ratio a = (flow rate)/(tank size), tank sizes being equal. Comment on the result.
5. If you extend Example 1 by a tank T_3 of the same size as the others and connected to T_2 by two tubes with flow rates as between T_1 and T_2, what system of ODEs will you get?
6. Find a "general solution" of the system in Prob. 5.

7–9 ELECTRICAL NETWORK

In Example 2 find the currents:

7. If the initial currents are 0 A and −3 A (minus meaning that $I_2(0)$ flows against the direction of the arrow).
8. If the capacitance is changed to $C = 5/27$ F. (General solution only.)
9. If the initial currents in Example 2 are 28 A and 14 A.

10–13 CONVERSION TO SYSTEMS

Find a general solution of the given ODE **(a)** by first converting it to a system, **(b)**, as given. Show the details of your work.

10. $y'' + 4y' + 3y = 0$ **11.** $2y'' - 3y' - 2y = 0$

12. $y''' - 2y'' - y' + 2y = 0$

13. $y'' + y' - 12y = 0$

14. TEAM PROJECT. Two Masses on Springs. **(a)** Set up the model for the (undamped) system in Fig. 81. **(b)** Solve the system of ODEs obtained. *Hint.* Try $\mathbf{y} = \mathbf{x}e^{\omega t}$ and set $\omega^2 = \lambda$. Proceed as in Example 1 or 2. **(c)** Describe the influence of initial conditions on the possible kind of motions.

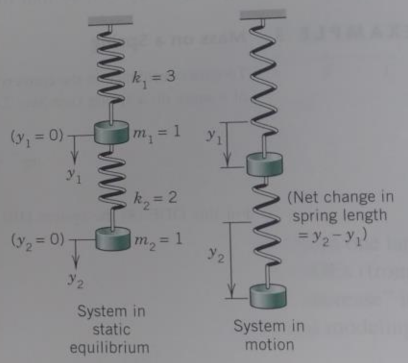

Fig. 81. Mechanical system in Team Project

15. CAS EXPERIMENT. Electrical Network. **(a)** In Example 2 choose a sequence of values of C that increases beyond bound, and compare the corresponding sequences of eigenvalues of $\mathbf{A}$. What limits of these sequences do your numeric values (approximately) suggest?

(b) Find these limits analytically.

(c) Explain your result physically.

(d) Below what value (approximately) must you decrease C to get vibrations?

PROBLEM SET 4.3

1–9 GENERAL SOLUTION

Find a real general solution of the following systems. Show the details.

1. $y_1' = 2y_1 - y_2$
   $y_2' = 3y_1 - 2y_2$

2. $y_1' = 6y_1 + 9y_2$
   $y_2' = y_1 + 6y_2$

3. $y_1' = -2y_1 + \frac{3}{2}y_2$
   $y_2' = -4y_1 + 3y_2$

4. $y_1' = -8y_1 - 2y_2$
   $y_2' = 2y_1 - 4y_2$

5. $y_1' = 2y_1 + 5y_2$
   $y_2' = 5y_1 + 12.5y_2$

6. $y_1' = 2y_1 - 2y_2$
   $y_2' = 2y_1 + 2y_2$

7. $y_1' = ay_2$
   $y_2' = -ay_1 + ay_3$
   $y_3' = -ay_2$
   $(a \neq 0)$

8. $y_1' = 8y_1 - y_2$
   $y_2' = y_1 + 10y_2$

9. $y_1' = 10y_1 - 10y_2 - 4y_3$
   $y_2' = -10y_1 + y_2 - 14y_3$
   $y_3' = -4y_1 - 14y_2 - 2y_3$

10–15 IVPs

Solve the following initial value problems.

10. $y_1' = -4y_1 + 5y_2$
    $y_2' = -y_1 + 2y_2$
    $y_1(0) = 0, \quad y_2(0) = 4$

11. $y_1' = -\frac{5}{4}y_1 + \frac{9}{4}y_2$
    $y_2' = -y_1 + 2y_2$
    $y_1(0) = -2, \quad y_2(0) = 0$

12. $y_1' = y_1 + 3y_2$
    $y_2' = \frac{1}{3}y_1 + y_2$
    $y_1(0) = 12, \quad y_2(0) = 2$

13. $y_1' = 2y_2$
    $y_2' = 2y_1$
    $y_1(0) = 0, \quad y_2(0) = 1$

14. $y_1' = -y_1 - y_2$
    $y_2' = y_1 - y_2$
    $y_1(0) = 1, \quad y_2(0) = 0$

15. $y_1' = y_1 + 2y_2$
    $y_2' = 2y_1 + y_2$
    $y_1(0) = 0.25, \quad y_2(0) = -0.25$

16–17 CONVERSION

Find a general solution by conversion to a single ODE.

16. The system in Prob. 8.

17. The system in Example 5 of the text.

18. **Mixing problem, Fig. 88.** Each of the two tanks contains 200 gal of water, in which initially 100 lb (Tank T_1) and 200 lb (Tank T_2) of fertilizer are dissolved. The inflow, circulation, and outflow are shown in Fig. 88. The mixture is kept uniform by stirring. Find the fertilizer contents $y_1(t)$ in T_1 and $y_2(t)$ in T_2.

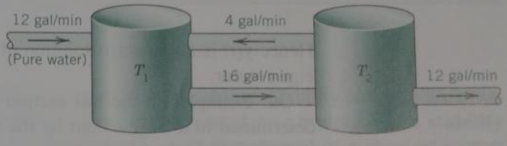

Fig. 88. Tanks in Problem 18

19. **Network.** Show that a model for the currents $I_1(t)$ and $I_2(t)$ in Fig. 89 is

$$\frac{1}{C}\int I_1\,dt + R(I_1 - I_2) = 0, \quad LI_2' + R(I_2 - I_1) = 0.$$

Find a general solution, assuming that $R = 3\ \Omega$, $L = 4$ H, $C = 1/12$ F.

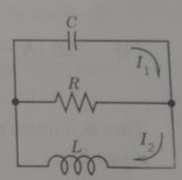

Fig. 89. Network in Problem 19

20. **CAS PROJECT. Phase Portraits.** Graph some of the figures in this section, in particular Fig. 87 on the degenerate node, in which the vector $\mathbf{y}^{(2)}$ depends on t. In each figure highlight a trajectory that satisfies an initial condition of your choice.

Then $y_2' = y'' = -(k/m)y_1 - (c/m)y_2$. Hence

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}\mathbf{y}, \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -k/m & -c/m - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

We see that $p = -c/m$, $q = k/m$, $\Delta = (c/m)^2 - 4k/m$. From this and Tables 4.1 and 4.2 we obtain the following results. Note that in the last three cases the discriminant Δ plays an essential role.

No damping. $c = 0$, $p = 0$, $q > 0$, a center.
Underdamping. $c^2 < 4mk$, $p < 0$, $q > 0$, $\Delta < 0$, a stable and attractive spiral point.
Critical damping. $c^2 = 4mk$, $p < 0$, $q > 0$, $\Delta = 0$, a stable and attractive node.
Overdamping. $c^2 > 4mk$, $p < 0$, $q > 0$, $\Delta > 0$, a stable and attractive node. ■

PROBLEM SET 4.4

1–10 TYPE AND STABILITY OF CRITICAL POINT

Determine the type and stability of the critical point. Then find a real general solution and sketch or graph some of the trajectories in the phase plane. Show the details of your work.

1. $y_1' = y_1$
$y_2' = 0.5y_2$

2. $y_1' = -4y_1$
$y_2' = -3y_2$

3. $y_1' = y_2$
$y_2' = -4y_1$

4. $y_1' = 2y_1 + y_2$
$y_2' = 5y_1 - 2y_2$

5. $y_1' = -y_1 + y_2$
$y_2' = -y_1 - y_2$

6. $y_1' = -6y_1 - y_2$
$y_2' = -9y_1 - 6y_2$

7. $y_1' = -y_1 - y_2$
$y_2' = -4y_1 + 2y_2$

8. $y_1' = -y_1 + 4y_2$
$y_2' = 3y_1 - 2y_2$

9. $y_1' = 6y_1 + 3y_2$
$y_2' = -4y_1 - y_2$

10. $y_1' = y_2$
$y_2' = -5y_1 - 2y_2$

11–18 TRAJECTORIES OF SYSTEMS AND SECOND-ORDER ODEs. CRITICAL POINTS

11. Damped oscillations. Solve $y'' + 2y' + 5y = 0$. What kind of curves are the trajectories?

12. Harmonic oscillations. Solve $y'' + \frac{1}{9}y = 0$. Find the trajectories. Sketch or graph some of them.

13. Types of critical points. Discuss the critical points in (10)–(13) of Sec. 4.3 by using Tables 4.1 and 4.2.

14. Transformation of parameter. What happens to the critical point in Example 1 if you introduce $\tau = -t$ as a new independent variable?

15. Perturbation of center. What happens in Example 4 of Sec. 4.3 if you change $\mathbf{A}$ to $\mathbf{A} + 0.1\mathbf{I}$, where $\mathbf{I}$ is the unit matrix?

16. Perturbation of center. If a system has a center as its critical point, what happens if you replace the matrix $\mathbf{A}$ by $\widetilde{\mathbf{A}} = \mathbf{A} + k\mathbf{I}$ with any real number $k \neq 0$ (representing measurement errors in the diagonal entries)?

17. Perturbation. The system in Example 4 in Sec. 4.3 has a center as its critical point. Replace each a_{jk} in Example 4, Sec. 4.3, by $a_{jk} + b$. Find values of b such that you get **(a)** a saddle point, **(b)** a stable and attractive node, **(c)** a stable and attractive spiral, **(d)** an unstable spiral, **(e)** an unstable node.

18. CAS EXPERIMENT. Phase Portraits. Graph phase portraits for the systems in Prob. 17 with the values of b suggested in the answer. Try to illustrate how the phase portrait changes "continuously" under a continuous change of b.

19. WRITING PROBLEM. Stability. Stability concepts are basic in physics and engineering. Write a two-part report of 3 pages each (A) on general applications in which stability plays a role (be as precise as you can), and (B) on material related to stability in this section. Use your own formulations and examples; do not copy.

20. Stability chart. Locate the critical points of the systems (10)–(14) in Sec. 4.3 and of Probs. 1, 3, 5 in this problem set on the stability chart.

PROBLEM SET 4.5

1. **Pendulum.** To what state (position, speed, direction of motion) do the four points of intersection of a closed trajectory with the axes in Fig. 93b correspond? The point of intersection of a wavy curve with the y_2-axis?

2. **Limit cycle.** What is the essential difference between a limit cycle and a closed trajectory surrounding a center?

3. **CAS EXPERIMENT. Deformation of Limit Cycle.** Convert the van der Pol equation to a system. Graph the limit cycle and some approaching trajectories for $\mu = 0.2, 0.4, 0.6, 0.8, 1.0, 1.5, 2.0$. Try to observe how the limit cycle changes its form continuously if you vary μ continuously. Describe in words how the limit cycle is deformed with growing μ.

4–8 CRITICAL POINTS. LINEARIZATION

Find the location and type of all critical points by linearization. Show the details of your work.

4. $y_1' = 4y_1 - y_1^2$, $y_2' = y_2$

5. $y_1' = 2y_2$, $y_2' = -y_1 + \frac{1}{4}y_1^2$

6. $y_1' = y_2$, $y_2' = -y_1 - y_1^2$

7. $y_1' = -2y_1 + y_2 - y_2^2$, $y_2' = -y_1 - \frac{1}{2}y_2$

8. $y_1' = y_2 - y_2^2$, $y_2' = y_1 - y_1^2$

9–13 CRITICAL POINTS OF ODEs

Find the location and type of all critical points by first converting the ODE to a system and then linearizing it.

9. $y'' - 4y + y^3 = 0$

10. $y'' + y - y^3 = 0$

11. $y'' + \cos(2y) = 0$

12. $y'' + 9y + y^2 = 0$

The last term on the right is a solution of the homogeneous system. Hence we can absorb it into $\mathbf{y}^{(h)}$. We thus obtain as a general solution of the system (3), in agreement with (5*).

$$\mathbf{y} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-4t} - 2\begin{bmatrix}1\\1\end{bmatrix}te^{-2t} + \begin{bmatrix}-2\\2\end{bmatrix}e^{-2t}. \qquad (9)$$

PROBLEM SET 4.6

1. Prove that (2) includes every solution of (1).

2–7 GENERAL SOLUTION

Find a general solution. Show the details of your work.

2. $y_1' = y_1 + y_2 + 10\cos t$, $y_2' = 3y_1 - y_2 - 10\sin t$

3. $y_1' = y_2 + e^{2t}$, $y_2' = y_1 - 3e^{2t}$

4. $y_1' = 4y_1 - 8y_2 + 2\cosh t$, $y_2' = 2y_1 - 6y_2 + \cosh t + 2\sinh t$

5. $y_1' = 4y_1 + 3y_2 + t$, $y_2' = -2y_1 - y_2 - 2t$

6. $y_1' = 4y_2$, $y_2' = 4y_1 - 16t^2 + 2$

7. $y_1' = -y_1 - y_2 + 8t + 5$, $y_2' = -4y_1 + 2y_2 + 3e^{-t} - 15t - 2$

8. **CAS EXPERIMENT. Undetermined Coefficients.** Find out experimentally how general you must choose $\mathbf{y}^{(p)}$, in particular when the components of $\mathbf{g}$ have a different form (e.g., as in Prob. 7). Write a short report, covering also the situation in the case of the modification rule.

9. **Undetermined Coefficients.** Explain why, in Example 1 of the text, we have some freedom in choosing the vector $\mathbf{v}$.

10–15 INITIAL VALUE PROBLEM

Solve, showing details:

10. $y_1' = -3y_1 - 4y_2 + 5e^t$, $y_2' = 5y_1 + 6y_2 - 6e^t$, $y_1(0) = 19$, $y_2(0) = -23$

11. $y_1' = y_2 + 2e^t$, $y_2' = y_1 - 2e^t$, $y_1(0) = 0$, $y_2(0) = 1$

12. $y_1' = y_1 + 4y_2 - t^2 + 6t$, $y_2' = y_1 + y_2 - t^2 + t - 1$, $y_1(0) = 2$, $y_2(0) = -1$

13. $y_1' = -y_2 + 2\cos t$, $y_2' = 4y_1 - 8\sin t$, $y_1(0) = -1$, $y_2(0) = 2$

14. $y_1' = 4y_2 + 5e^t$, $y_2' = -y_1 - 20e^{-t}$, $y_1(0) = 1$, $y_2(0) = 0$

15. $y_1' = y_1 + e^{2t} - 4t$, $y_2' = 2y_1 - y_2 + 2 + t$, $y_1(0) = -1$, $y_2(0) = -2$

16. **WRITING PROJECT. Undetermined Coefficients.** Write a short report in which you compare the application of the method of undetermined coefficients to a single ODE and to a system of ODEs, using ODEs and systems of your choice.

17–20 NETWORK

Find the currents in Fig. 99 (Probs. 17–19) and Fig. 100 (Prob. 20) for the following data, showing the details of your work.

17. $R_1 = 2\ \Omega$, $R_2 = 8\ \Omega$, $L = 1$ H, $C = 0.5$ F, $E = 200$ V

18. Solve Prob. 17 with $E = 440 \sin t$ V and the other data as before.

19. In Prob. 17 find the particular solution when currents and charge at $t = 0$ are zero.

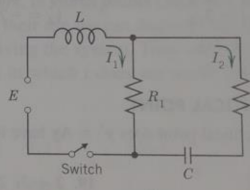

Fig. 99. Problems 17–19

20. $R_1 = 1\ \Omega$, $R_2 = 1.4\ \Omega$, $L_1 = 0.8$ H, $L_2 = 1$ H, $E = 100$ V, $I_1(0) = I_2(0) = 0$

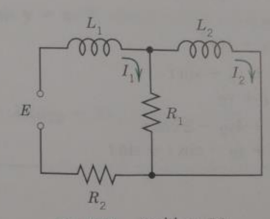

Fig. 100. Problem 20