Contents

9 Center of Mass and Momentum

In this chapter, we discuss how complicated motion of a system of objects can be simplified if we determine a special point of the system—the *center of mass* of that system.

9.1 The Center of Mass

The center of mass of a system of particles in the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied here. To be more specific, for a system of N point particles of mass m_i at position $\vec{x}_i$, the total mass of the system is

$$M = \sum_{i=1}^{N} m_i$$

and the center of mass is defined as

$$\vec{x}_{cm} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{x}_i \tag{1}$$

In particular, the component equations for the above are

$$x_{cm} = \frac{\sum_{i=1}^{N} m_i x_i}{M}, y_{cm} = \frac{\sum_{i=1}^{N} m_i y_i}{M}, z_{cm} = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

An ordinary object contains so many particles that we can best treat it as a continuous distribution of matter. The "particles" then become differential mass elements dm, the sum of (1) becomes integrals

$$\vec{x}_{cm} = \frac{1}{M} \int \vec{x} dm \tag{2}$$

where

$$M = \int dm \tag{3}$$

The density ρ is the mass per unit volume

$$\rho = \frac{dm}{dV}$$

(2) and (2) can be rewritten as

$$M = \int dm = \int \frac{dm}{dV} dV = \int \rho dV$$

and

$$\vec{x}_{cm} = \frac{1}{M} \int \rho \vec{x} dV$$

If ρ is uniform, we have $M = \rho \int dV = \rho V$ and therefore

$$\vec{x}_{cm} = \frac{1}{V} \int \vec{x} dV \text{ if } \rho \text{ is uniform}$$

For a system of N particles, we may divide it into two groups–one with N_A particles, the other with the rest of $N_B = N - N_A$ particles, and

$$M = \sum_{i=1}^{N} m_i = \sum_{i=1}^{N_A} m_i + \sum_{i=N_A+1}^{N} m_i = M_A + M_B$$

where

$$\begin{aligned} M_A &= \sum_{i=1}^{N_A} m_i \\ M_B &= \sum_{i=N_A+1}^{N} m_i \end{aligned}$$

There will be two centers of mass $\vec{x}_{A,cm}$ and $\vec{x}_{B,cm}$, for these two systems of particles with masses M_A and M_B, and

$$\begin{aligned} \vec{x}_{A,cm} &= \frac{1}{M_A} \sum_{i=1}^{N_A} m_i \vec{x}_i \\ \vec{x}_{B,cm} &= \frac{1}{M_B} \sum_{i=N_A+1}^{N} m_i \vec{x}_i \end{aligned}$$

The center of mass of the original system is also equal to the center of mass for a system the two particles—one located at $\vec{x}_{A,cm}$ with mass mass M_A and the other at $\vec{x}_{B,cm}$ with mass mass M_B. This is because

$$\begin{aligned} & \frac{1}{M} \left(M_A \vec{x}_{A,cm} + M_B \vec{x}_{B,cm} \right) \qquad (4) \\ &= \frac{1}{M} \left(\sum_{i=1}^{N_A} m_i \vec{x}_i + \sum_{i=N_A+1}^{N} m_i \vec{x}_i \right) \\ &= \frac{1}{M} \sum_{i=1}^{N} m_i \vec{x}_i = \vec{x}_{cm} \end{aligned}$$

Note that in dividing the system into two groups, the above identity does not exclude the possibility that the position vector for a particle in group A may happen to be equal to the position vector of some particle in group

B. If there is an overlapping of position vectors, it is not even necessary that masses for the two particles with overlapping position are both positive. In fact, we may even allow a zero mass particle in the original system to be split into two particles at the same position with opposite masses and then assigned to group A and B, respectively. In the extreme case, (4) is still applicable with group B consists entirely of negative mass particles as long as the corresponding particles with positive masses are accounted for in group A.

We may divide the original system of particles into more than two groups and

(4) can be easily generalized. Specifically, suppose we divide the system of N particles into p groups.

$$M = \sum_{i=1}^{N} m_i = \sum_{k=1}^{p} M_k$$

The center of mass for kth group is $\vec{x}_{(k),cm}$. We have

$$\vec{x}_{cm} = \frac{1}{M} \sum_{k=1}^{p} M_k \vec{x}_{(k),cm}$$

9.1.1 Symmetry and the Center of Mass

We shall next show that if an object has a point, a line, or a plane of symmetry, the center of mass of such an object then lies at that point, on that line, or in that plane.

(i) If there is a symmetry point $\vec{x}_s$, a particle of mass m located at $\vec{x}$ must have a mirrored particle of equal mass and located at $\vec{x}'$ such that $\vec{x}_s = \frac{\vec{x}+\vec{x}'}{2}$. The center of mass for this paired particles symmetrical with respect to $\vec{x}_s$ is also at $\vec{x}_s$. The object can be considered to be composed of system of paired particles that are symmetric with respect to $\vec{x}_s$. Since the center of mass for every paired particles is $\vec{x}_s$, the entire object must also have its center of mass at $\vec{x}_s$.

(ii) If there is a line L of symmetry, the center of mass of the paired particles symmetric with respect to L must also reside on L. The center of mass of the entire system must be on L since it is the same as that for the system of paired particles, whose center of mass is on L. (Why is the center of mass of a system of particles on a line must also remain on the very line?.)

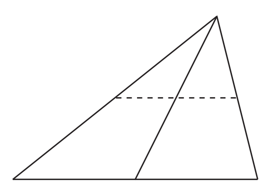
(iii) If there is a plane P of symmetry, the center of mass of the paired particles symmetric with respect to P must also reside in P. The center of mass of the entire system must be on P since it is the same as that for the system of paired particles, whose center of mass is on P. (Why is the center of mass of a system of particles on a plane must also remain on the very plane?)

Examples:

a. The center of mass of a uniform sphere is at the center of the sphere (which is the point of symmetry).

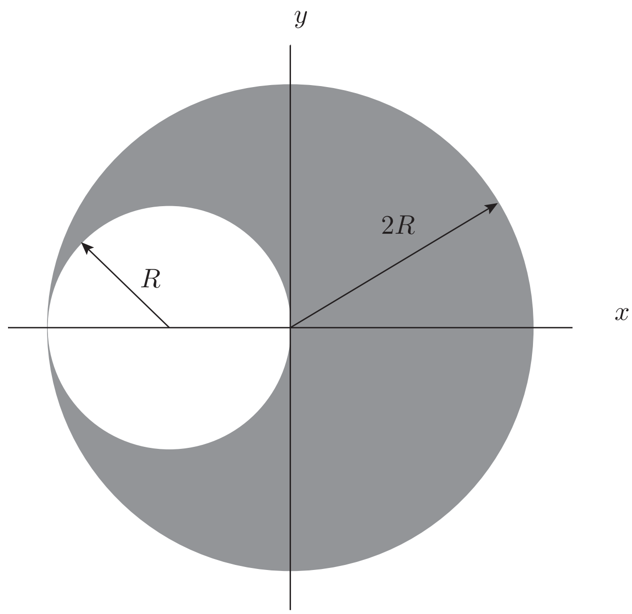
b. The center of mass of a uniform cone (whose axis is a line of symmetry) lies on the axis of the cone.

c. For a uniform triangular plate,



the center of mass must lie along the symmetrical dividing line. The only point that is common to all three lines emitting from the three different vertices of the triangle is the geometrical center, which must therefore also be the center of mass.

d. A uniform metal plate P of radius $2R$ from which a disk of radius R has been stamped out.



First, put the stamped-out disk (call it S) back into place to form the original plate (call it C). Because of its circular symmetry, the center of mass for C is at the $\vec{x}_{C,cm} = \vec{0}$ and for S is at $\vec{x}_{S,cm} = -R\hat{\imath}$. The mass of the plate $M_P = M_{-S} + M_C$ where M_{-S} is the negative of the mass of S, M_S. The center of mass for P is equal to

$$\begin{aligned}\vec{x}_{P,cm} &= \frac{1}{M_P}\left(M_{-S}\vec{x}_{S,cm} + M_C\vec{0}\right) = \frac{M_{-S}}{M_P}\left(-R\hat{\imath}\right) \\ &= \frac{M_S}{M_C - M_S}R\hat{\imath} = \frac{R^2}{\left(2R\right)^2 - R^2}R\hat{\imath} = \frac{R}{3}\hat{\imath}\end{aligned}$$

e. The center of mass of an object need not lie within the object. There is no dough at the center of mass of a doughnut.

9.2 Newton's Second Law for a System of Particles

$$\vec{v}_{cm} = \frac{d\vec{x}_{cm}}{dt}, \vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{d^2\vec{x}_{cm}}{dt^2}$$

$$
\begin{aligned}
M\vec{a}_{cm} = M\frac{d^2\vec{x}_{cm}}{dt^2} &= \frac{d^2}{dt^2}\left(\sum_i m_i\vec{x}_i\right) \\
&= \sum_i m_i\frac{d^2\vec{x}_i}{dt^2} = \sum_i \vec{F}_i = \vec{F}_{net}
\end{aligned}
$$

The vector equation that governs the motion of the center of mass of a system of particles is

$$\vec{F}_{net} = M\vec{a}_{cm} \tag{5}$$

Although the center of mass is just a point, it moves like a particle whose mass is equal to the total mass of the system; we can assign a position , a velocity, and an acceleration to it.

$\vec{F}_{net}$ is the net force of all the external forces that act on the system. By newton's third law, forces on one part of the system from another part of the system (internal forces) are not included in (5).

M is the total mass of the system. We assume that no mass enters or leaves the system as it moves, so M remains constant. The system is said to be closed.

$\vec{a}_{cm}$ is the acceleration of the center of mass of the system. (5) gives no information about the acceleration of any other point of the system.

9.3 Linear Momentum

The linear momentum of a particle is a vector $\vec{p}$ defined as

$$\vec{p} = m\vec{v}$$

The time rate change of the momentum of a particle is equal to the net force acting on the particle.

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

9.4 The Linear Momentum of a System of Particles

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

$$
\begin{aligned}
\vec{P}_{total} &= \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i = \sum_i m_i \frac{d\vec{x}_i}{dt} \\
&= M \frac{d}{dt} \sum_i \frac{m_i \vec{x}_i}{M} = M \frac{d\vec{x}_{cm}}{dt} = M\vec{v}_{cm}
\end{aligned}
$$

Now

$$
\frac{d\vec{P}_{total}}{dt} = M\frac{d\vec{v}_{cm}}{dt} = M\vec{a}_{cm}
$$

The Newton's second law for a system of particles can be written as

$$
\vec{F}_{net} = \frac{d\vec{P}_{total}}{dt} \text{ (system of particles)} \tag{6}
$$

9.5 Collision and Impulse

The momentum $\vec{p}$ of any particle-like body cannot change unless a net external force changes it. In a collision, the external force on the body is brief, has large magnitude, and suddenly changes the body's momentum. Consider a simple collision in which a moving particle-like body (a projectile) collides with a target.

9.5.1 Single Collision

Let the projectile be a ball and the target be a bat. The collision is brief, and ball experiences a force that is great enough to slow, stop, or even reverse its motion. The ball experiences a force $\vec{F}(t)$ that varies during the collision and changes the linear momentum $\vec{p}$ of the ball. That change is related to the force by Newton's second law written in the form $\vec{F} = \frac{d\vec{p}}{dt}$. This, in time interval dt, the change in the ball's momentum is

$$
d\vec{p} = \vec{F}(t)\, dt
$$

The net change in the ball's momentum due to the collision can be found by integrating both sides in the above from a time t_i just before the collision to a time t_f just after the collision:

$$
\int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)\, dt
$$

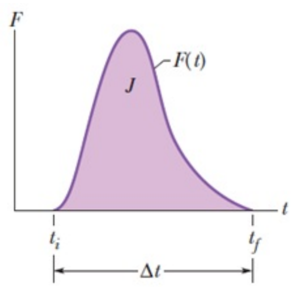
The left side of this equation gives us the change in momentum: $\vec{p}_f - \vec{p}_i = \Delta\vec{p}$. The right side, which is a measure of both the magnitudes and the duration of the collision force, is called the impulse $\vec{J}$ of the collision:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}\,(t)\,dt$$

Thus

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \vec{J}$$

If we have a function for $\vec{F}\,(t)$, we can evaluate $\vec{J}$ by integrating the function. If we have a plot of $\vec{F}$ versus t, we can evaluate $\vec{J}$ by finding the area between the curve and the t axis.



In many situations, we do not know how the force varies with time but we do know the average force $\vec{F}_{avg}$ and the during $\Delta t = t_f - t_i$ of the collision. Then

$$\vec{J} = \vec{F}_{avg}\Delta t$$

Instead of the ball, we could have focused on the bat. At any instant, Newton's third law tells us that the force on the bat has the same magnitude but the opposite direction as the force on the ball. This means that the impulse on the bat has the same magnitude but the opposite direction as the impulse on the ball.

9.5.2 Series of Collision

Now let us consider the force on a body when it undergoes a series of identical, repeated collisions. Suppose there is a steady stream of identical projectile bodies, with identical mass m and linear momenta $m\vec{v}$, moving along an x axis and collides in a time interval Δt. Each projectile has initial momentum $m\vec{v}$ and undergoes a change $\Delta\vec{p}$ in linear momentum because of the collision.

The total change in linear momentum for n projectiles during interval is $n\Delta\vec{p}$. The resulting impulse $\vec{J}$ on the target during Δt is

$$\vec{J} = -n\Delta\vec{p}$$

The average force acting on the target during the collision is

$$\vec{F}_{avg} = \frac{\vec{J}}{\Delta t} = \frac{-n\Delta\vec{p}}{\Delta t} = -\frac{n}{\Delta t}m\Delta\vec{v}$$

This equation gives us $\vec{F}_{avg}$ in terms of $\frac{n}{\Delta t}$, the rate at which the projectiles collide with the target, and $\Delta\vec{v}$, the change in the velocity of those projectiles. In the time Δt, an amount of mass $\Delta m = nm$ collides with the target. With this result,

$$\vec{F}_{avg} = -\frac{\Delta m}{\Delta t}\Delta\vec{v}$$

which gives the average force in terms of $\frac{\Delta m}{\Delta t}$, the rate at which mass collides with the target.

9.6 Conservation of Linear Momentum

If $\vec{F}_{net} = 0$, by (6) we have $\frac{d\vec{P}_{total}}{dt} = 0$, or

$$\vec{P}_{total} = \text{ constant}$$

In other words, if no net external force acts on a system of particles, the total linear momentum $\vec{P}_{total}$ of the system cannot change. This result is called the **law of conservation of linear momentum**. The law says that, for a closed, isolated system.

$$\begin{pmatrix} \text{total linear momentum} \\ \text{at some initial time } t_i \end{pmatrix} = \begin{pmatrix} \text{total linear momentum} \\ \text{at some later time } t_f \end{pmatrix}$$

If the component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

9.7 Momentum and Kinetic Energy in Collisions

We now focus to a closed and isolated system. The total linear momentum $\vec{P}_{total}$ cannot change because there is no net force to change it. This is a very powerful rule because it can allows us to determine the results of a collision without knowing the details of the collision.

If the total kinetic energy of a system remains unchanged by the collision, then the kinetic energy of the system is conserved (it is the same before and after the collision). Such a collision is called an elastic collision. In everyday collisions of common bodies, such as two cars or a ball and a bat, some energy is always transferred from kinetic energy to other forms of energy, such as thermal energy or energy of sound. Then, the kinetic energy of the system is not conserved. Such a collision is called inelastic collision.

The inelastic collision of two bodies always involves a loss in the kinetic energy of the system. The greatest loss occurs if the bodies stick together, in which case the collision is called completely inelastic collision.

9.8 Inelastic Collisions in One Dimension

For a collision of two bodies, velocities and momenta before the collision are indicated with the subscript i and those after the collision are indicated with the subscript f. The law of conservation of linear momentum gives us

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

Because the motion is one dimensional, we can drop the overhead arrow for vectors and use only components along the axis, indicating direction with a sign. Thus,

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

9.8.1 Completely inelastic Collision

After a completely inelastic collision of two bodies, the two bodies stick together with velocity V.

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{P}_{total}$$

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)\, V$$

$$V = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$

9.8.2 Velocity of the Center of Mass

In a closed, isolated system, the velocity $\vec{v}_{cm}$of the center of mass of the system cannot be changed by a collision because, with the system isolated, there is no net force yo change it. Since $\vec{P}_{total} = M\vec{v}_{cm}$, we have

$$M\vec{v}_{cm} = \vec{P}_{total} = \vec{p}_{1i} + \vec{p}_{2i}$$

$$\vec{v}_{cm} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}$$

9.9 Elastic Collisions in One Dimension

In an elastic collision, the kinetic energy of each colliding body change, but the total kinetic energy of the system does not change.

9.9.1 Stationary Target:

For stationary target, $v_{2i} = 0$,

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Rewrite the above two equations as

$$m_1 \left(v_{1i} - v_{1f}\right) = m_2 v_{2f} \tag{7}$$

$$m_1 \left(v_{1i} - v_{1f}\right)\left(v_{1i} + v_{1f}\right) = m_2 v_{2f}^2 \tag{8}$$

The ration $\frac{(8)}{(7)}$ gives us

$$v_{1i} + v_{1f} = v_{2f}$$

which can be substituted into (7) to yield

$$m_1 \left(v_{1i} - v_{1f}\right) = m_2 \left(v_{1i} + v_{1f}\right)$$

or

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

Then

$$v_{2f} = v_{1i} + v_{1f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

9.9.2 Moving Target:

$v_{2i} \neq 0$, and we have

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Rewrite the above two equations as

$$m_1 \left(v_{1i} - v_{1f}\right) = -m_2 \left(v_{2i} - v_{2f}\right) \tag{9}$$

$$m_1 \left(v_{1i} - v_{1f}\right)\left(v_{1i} + v_{1f}\right) = -m_2 \left(v_{2i} - v_{2f}\right)\left(v_{2i} + v_{2f}\right) \tag{10}$$

Divide (10) by (9), we get

$$v_{1i} + v_{1f} = v_{2i} + v_{2f} \tag{11}$$

(9) and (11) can be combined into a 2x2 matrix equation

$$\begin{bmatrix} 1 & -1 \\ m_1 & m_2 \end{bmatrix} \begin{bmatrix} v_{1f} \\ v_{2f} \end{bmatrix} = \begin{bmatrix} v_{2i} - v_{1i} \\ m_1 v_{1i} + m_2 v_{2i} \end{bmatrix}$$

By Cramer's rule, we have

$$v_{1f} = \frac{\det \begin{bmatrix} v_{2i} - v_{1i} & -1 \\ m_1 v_{1i} + m_2 v_{2i} & m_2 \end{bmatrix}}{\det \begin{bmatrix} 1 & -1 \\ m_1 & m_2 \end{bmatrix}} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \tag{12}$$

$$v_{2f} = \frac{\det \begin{bmatrix} 1 & v_{2i} - v_{1i} \\ m_1 & m_1 v_{1i} + m_2 v_{2i} \end{bmatrix}}{\det \begin{bmatrix} 1 & -1 \\ m_1 & m_2 \end{bmatrix}} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

9.10 Center of Mass Frame

For the reference frame, in which the origin is the center of mass, $\vec{v}_{cm} = 0$ and the total linear momentum is zero, $\vec{P}_{total} = 0$. Before and after the collision of two bodies, $\vec{P}_{total}$ remains zero and we have

$$0 = m_1 \vec{v}'_{1i} + m_2 \vec{v}'_{2i}, \vec{v}'_{2i} = -\frac{m_1}{m_2} \vec{v}'_{1i}$$

$$0 = m_1 \vec{v}'_{1f} + m_2 \vec{v}'_{2f}, \vec{v}'_{2f} = -\frac{m_1}{m_2} \vec{v}'_{1f}$$

where $\vec{v}'$ is the velocity seen from the center of mass frame. Thus,

$$\frac{1}{2}m_2 v'^2_{2i} = \frac{1}{2}\frac{m_1^2}{m_2}v'^2_{1i}, \frac{1}{2}m_2 v'^2_{2f} = \frac{1}{2}\frac{m_1^2}{m_2}v'^2_{1f} \tag{13}$$

The total kinetic energy is not changed for elastic collision, so

$$\frac{1}{2}m_1 v'^2_{1i} + \frac{1}{2}m_2 v'^2_{2i} = \frac{1}{2}m_1 v'^2_{1f} + \frac{1}{2}m_2 v'^2_{2f}$$

Combine the above identity with (13), we get

$$\frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right)v'^2_{1i} = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right)v'^2_{1f}$$

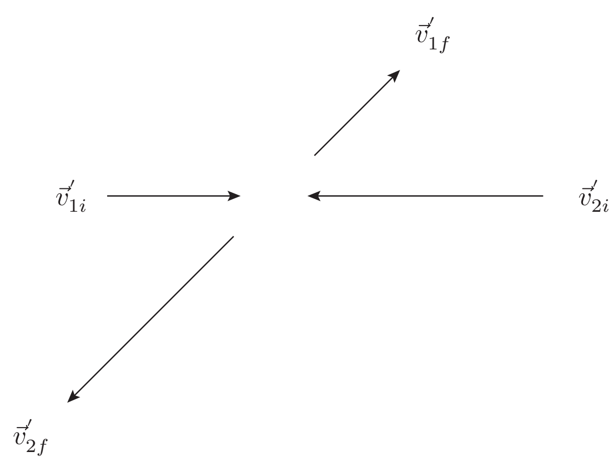
or

$$|\vec{v}'_{1i}| = \left|\vec{v}'_{1f}\right|$$

which also leads to

$$|\vec{v}'_{2i}| = \left|\vec{v}'_{2f}\right|,$$

This means that if we view the collision in the center of mass frame, the velocities of two colliding bodies may change their directions but not their magnitudes after the collision.



In one-dimensional case, we are left with the only option that

$$v'_{1f} = -v'_{1i}, v'_{2f} = -v'_{2i}$$

In the laboratory frame,
$$v_{1i} = v_{cm} + v'_{1i}$$
and
$$\begin{aligned}
v_{1f} &= v_{cm} + v'_{1f} = v_{cm} - v'_{1i} = v_{cm} - (v_{1i} - v_{cm}) \\
&= 2v_{cm} - v_{1i} = 2\frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} - v_{1i} \\
&= \frac{2m_2}{m_1 + m_2} v_{2i} + \frac{m_1 - m_2}{m_1 + m_2} v_{1i}
\end{aligned}$$
We arrive at the same result as (12).

9.11 Collisions in Two or Three Dimensions

For an elastic collision between two bodies, the magnitude of final velocity is equal to that of the initial velocity for either of the two bodies in the center of mass frame. To completely determine a final velocity, we also need the information on the direction of the final velocity. If the collision is two-dimensional, the direction can be specified by the angle (one parameter) between the initial velocity and the final velocity for either of the two colliding bodies. The angle, in general, depends on the interaction details between the two colliding bodies. For three dimensional collision, we need two parameters to specify the final direction relative to the initial one.

9.11.1 Elastic Collision in Two Dimensions

Let us consider a two-dimensional collision between a projectile body and a target body initially at rest.
$$\vec{p}_{1i} = \vec{p}_{1f} + \vec{p}_{2f} \tag{14}$$
If the collision is elastic,
$$K_{1i} = K_{1f} + K_{2f}$$
Using $K = \frac{|\vec{p}|^2}{2m}$, the conservation of kinetic energy becomes
$$\frac{|\vec{p}_{1i}|^2}{2m_1} = \frac{|\vec{p}_{1f}|^2}{2m_1} + \frac{|\vec{p}_{2f}|^2}{2m_2} \tag{15}$$

If the $m_1 = m_2$, (14) and (15) become

$$\vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f} \tag{16}$$

$$|\vec{v}_{1i}|^2 = |\vec{v}_{1f}|^2 + |\vec{v}_{2f}|^2 \tag{17}$$

(16) shows that the three vectors $\vec{v}_{1f}, \vec{v}_{2f}$ and $\vec{v}_{1i}$ form the three sides of a triangle. Furthermore, (17) shows that it is a right triangle according to Pythagorean theorem with $\vec{v}_{1f} \perp \vec{v}_{2f}$.

9.12 System with Varying Mass: A Rocket

Assume that we are at rest relative to an inertial frame, watching a rocket accelerates through deep space with no gravitational or atmospheric drag forces acting on it.

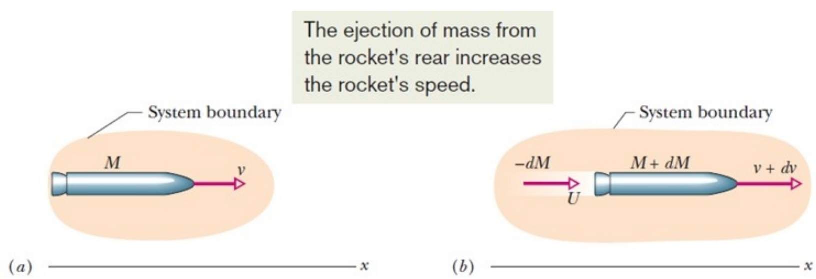


Fig. 9-22 (*a*) An accelerating rocket of mass M at time t, as seen from an inertial reference frame. (*b*) The same but at time $t + dt$. The exhaust products released during interval dt are shown.

Let M be the mass of the rocket and $\vec{v}\,(t)$ its velocity at an arbitrary time t. At a time interval dt later, the rocket now has velocity $\vec{v} + d\vec{v}$ and mass $M + dM$, where the change in mass dM is a negative quantity. The exhaust products released by the rocket during time interval dt have mass $-dM$ and velocity $\vec{U}$ relative to the inertial reference frame.

Our system consists of the rocket and the exhaust products released during interval dt. The system is closed and isolated, so the linear momentum of the system must be conserved during dt; that is

$$\vec{P}_i = \vec{P}_f$$

where the subscript i and f indicate the values at the beginning and end of time interval dt. The above equation can be rewritten as

$$M\vec{v} = -dM\vec{U} + (M + dM)(\vec{v} + d\vec{v}) \tag{18}$$

where the first term on the right is the linear momentum of the exhaust products released during interval dt and the second term is the linear momentum of the rocket at the end of interval dt. Let $\vec{v}_{rel}$ be the velocity of the exhaust products relative to the rocket. Then

$$\vec{U} = \vec{v} + \vec{v}_{rel}$$

and (18) becomes

$$0 = -\vec{v}_{rel}dM + Md\vec{v} \tag{19}$$

Let us also assume the rocket is moving forward in the x direction, $\vec{v} = v\hat{\imath}, d\vec{v} = dv\hat{\imath}$ and the exhaust products move backwards relative to the rocket, $\vec{v}_{rel} = -v_{rel}\hat{\imath}$. (18) can be rewritten as

$$-v_{rel}dM = Mdv \tag{20}$$

Dividing each side by dt, we get the first rocket equation:

$$-\frac{dM}{dt}v_{rel} = Rv_{rel} = M\frac{dv}{dt}$$

where $R = -\frac{dM}{dt}$ is the mass rate of the fuel consumption. (20) can also be written as

$$dv = -v_{rel}\frac{dM}{M}$$

Integrating the above equation

$$\int_{v_i}^{v_f} dv = -v_{rel}\int_{M_i}^{M_f} \frac{dM}{M}$$

leads to the second rocket equation:

$$v_f - v_i = v_{rel}\ln\frac{M_i}{M_f}$$