14. TEAM PROJECT. Orthogonality on the Entire Real Axis. Hermite Polynomials.[8] These orthogonal polynomials are defined by $He_0(1) = 1$ and

$$(19) \quad He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}), \qquad n = 1, 2, \cdots.$$

REMARK. As is true for many special functions, the literature contains more than one notation, and one sometimes defines as Hermite polynomials the functions

$$H_0^* = 1, \qquad H_n^*(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

This differs from our definition, which is preferred in applications.

(a) Small Values of *n*. Show that

$$He_1(x) = x, \qquad He_2(x) = x^2 - 1,$$
$$He_3(x) = x^3 - 3x, \qquad He_4(x) = x^4 - 6x^2 + 3.$$

(b) Generating Function. A generating function of the Hermite polynomials is

$$(20) \qquad e^{tx - t^2/2} = \sum_{n=0}^{\infty} a_n(x) t^n$$

because $He_n(x) = n!\, a_n(x)$. Prove this. *Hint*: Use the formula for the coefficients of a Maclaurin series and note that $tx - \frac{1}{2}t^2 = \frac{1}{2}x^2 - \frac{1}{2}(x - t)^2$.

(c) Derivative. Differentiating the generating function with respect to x, show that

$$(21) \qquad He_n'(x) = nHe_{n-1}(x).$$

(d) Orthogonality on the *x*-Axis needs a weight function that goes to zero sufficiently fast as $x \to \pm\infty$, (Why?) Show that the Hermite polynomials are orthogonal on $-\infty < x < \infty$ with respect to the weight function $r(x) = e^{-x^2/2}$. *Hint*. Use integration by parts and (21).

(e) ODEs. Show that

$$(22) \qquad He_n'(x) = xHe_n(x) - He_{n+1}(x).$$

Using this with $n - 1$ instead of n and (21), show that $y = He_n(x)$ satisfies the ODE

$$(23) \qquad y'' = xy' + ny = 0.$$

Show that $w = e^{-x^2/4}y$ is a solution of **Weber's equation**

$$(24) \quad w'' + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)w = 0 \qquad (n = 0, 1, \cdots).$$

15. CAS EXPERIMENT. Fourier–Bessel Series. Use Example 2 and $R = 1$, so that you get the series

$$(25) \qquad f(x) = a_1 J_0(\alpha_{0,1}x) + a_2 J_0(\alpha_{0,2}x) + a_3 J_0(\alpha_{0,3}x) + \cdots$$

With the zeros $\alpha_{0,1}\alpha_{0,2}, \cdots$ from your CAS (see also Table A1 in App. 5).

(a) Graph the terms $J_0(\alpha_{0,1}x), \cdots, J_0(\alpha_{0,10}x)$ for $0 \leqq x \leqq 1$ on common axes.

(b) Write a program for calculating partial sums of (25). Find out for what $f(x)$ your CAS can evaluate the integrals. Take two such $f(x)$ and comment empirically on the speed of convergence by observing the decrease of the coefficients.

(c) Take $f(x) = 1$ in (25) and evaluate the integrals for the coefficients analytically by (21a), Sec. 5.4, with $\nu = 1$. Graph the first few partial sums on common axes.

11.7 Fourier Integral

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Sections 11.2 and 11.3 first illustrated this, and various further applications follow in Chap. 12. Since, of course, many problems involve functions that are ***nonperiodic*** *and are of interest on the whole x-axis,* we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

In Example 1 we start from a special function f_L of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$. Then we do the same for an *arbitrary* function f_L of period $2L$. This will motivate and suggest the main result of this section, which is an integral representation given in Theorem 1 below.

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[8]CHARLES HERMITE (1822–1901), French mathematician, is known for his work in algebra and number theory. The great HENRI POINCARÉ (1854–1912) was one of his students.

EXAMPLE 1 Rectangular Wave

Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L. \end{cases}$$

The left part of Fig. 280 shows this function for $2L = 4, 8, 16$ as well as the nonperiodic function $f(x)$, which we obtain from f_L if we let $L \to \infty$,

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now explore what happens to the Fourier coefficients of f_L as L increases. Since f_L is even, $b_n = 0$ for all n. For a_n the Euler formulas (6), Sec. 11.2, give

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \qquad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\,\frac{\sin(n\pi/L)}{n\pi/L}.$$

This sequence of Fourier coefficients is called the **amplitude spectrum** of f_L because $|a_n|$ is the maximum amplitude of the wave $a_n \cos(n\pi x/L)$. Figure 280 shows this spectrum for the periods $2L = 4, 8, 16$. We see that for increasing L these amplitudes become more and more dense on the positive w_n-axis, where $w_n = n\pi/L$. Indeed, for $2L = 4, 8, 16$ we have 1, 3, 7 amplitudes per "half-wave" of the function $(2 \sin w_n)/(Lw_n)$ (dashed in the figure). Hence for $2L = 2^k$ we have $2^{k-1} - 1$ amplitudes per half-wave, so that these amplitudes will eventually be everywhere dense on the positive w_n-axis (and will decrease to zero).

The outcome of this example gives an intuitive impression of what about to expect if we turn from our special function to an arbitrary one, as we shall do next. ■

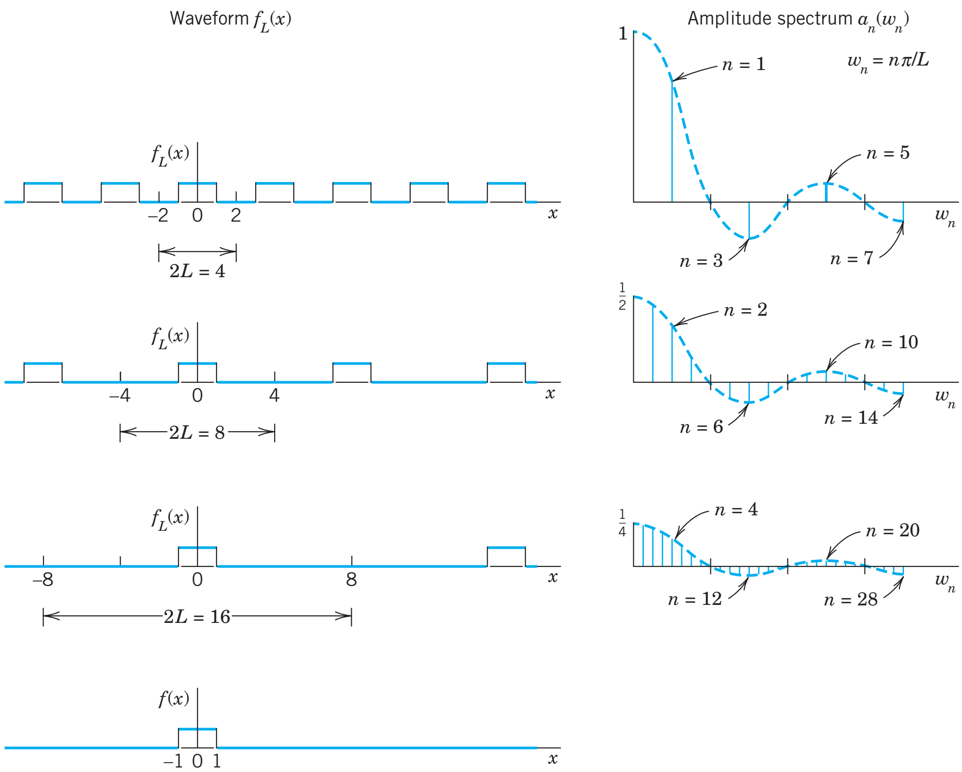

Fig. 280. Waveforms and amplitude spectra in Example 1

From Fourier Series to Fourier Integral

We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \qquad w_n = \frac{n\pi}{L}$$

and find out what happens if we let $L \to \infty$. Together with Example 1 the present calculation will suggest that we should expect an integral (instead of a series) involving $\cos wx$ and $\sin wx$ with w no longer restricted to integer multiples $w = w_n = n\pi/L$ of π/L but taking *all* values. We shall also see what form such an integral might have.

If we insert a_n and b_n from the Euler formulas (6), Sec. 11.2, and denote the variable of integration by v, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v\,dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big].$$

We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

Then $1/L = \Delta w/\pi$, and we may write the Fourier series in the form

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos w_n x)\,\Delta w \int_{-L}^{L} f_L(v)\cos w_n v\,dv + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big]. \tag{1}$$

This representation is valid for any fixed L, arbitrarily large, but finite.

We now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is **absolutely integrable** on the x-axis; that is, the following (finite!) limits exist:

$$\lim_{a\to-\infty}\int_{a}^{0} |f(x)|\,dx + \lim_{b\to\infty}\int_{0}^{b} |f(x)|\,dx \quad \left(\text{written } \int_{-\infty}^{\infty} |f(x)|\,dx\right). \tag{2}$$

Then $1/L \to 0$, and the value of the first term on the right side of (1) approaches zero. Also $\Delta w = \pi/L \to 0$ and it seems ***plausible*** that the infinite series in (1) becomes an integral from 0 to ∞, which represents $f(x)$, namely,

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv\, dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv\, dv\right] dw. \tag{3}$$

If we introduce the notations

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv \tag{4}$$

we can write this in the form

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\, dw. \tag{5}$$

This is called a representation of $f(x)$ by a **Fourier integral**.

It is clear that our naive approach merely ***suggests*** the representation (5), but by no means establishes it; in fact, the limit of the series in (1) as Δw approaches zero is not the definition of the integral (3). Sufficient conditions for the validity of (5) are as follows.

THEOREM 1

Fourier Integral

If $f(x)$ is piecewise continuous (see Sec. 6.1) *in every finite interval and has a right-hand derivative and a left-hand derivative at every point* (see Sec 11.1) *and if the integral* (2) *exists, then $f(x)$ can be represented by a Fourier integral* (5) *with A and B given by* (4). *At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point* (see Sec. 11.1). (Proof in Ref. [C12]; see App. 1.)

Applications of Fourier Integrals

The main application of Fourier integrals is in solving ODEs and PDEs, as we shall see for PDEs in Sec. 12.6. However, we can also use Fourier integrals in integration and in discussing functions defined by integrals, as the next example.

EXAMPLE 2 **Single Pulse, Sine Integral. Dirichlet's Discontinuous Factor. Gibbs Phenomenon**

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases} \qquad \text{(Fig. 281)}$$

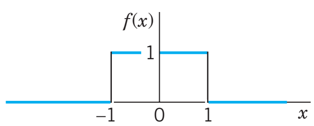

Fig. 281. Example 2

Solution. From (4) we obtain

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv = \frac{1}{\pi}\int_{-1}^{1}\cos wv\,dv = \frac{\sin wv}{\pi w}\bigg|_{-1}^{1} = \frac{2\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1}\sin wv\,dv = 0$$

and (5) gives the *answer*

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\cos wx\sin w}{w}\,dw. \tag{6}$$

The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $(1 + 0)/2$, that is, $\frac{1}{2}$.

Furthermore, from (6) and Theorem 1 we obtain (multiply by $\pi/2$)

$$\int_0^{\infty}\frac{\cos wx\sin w}{w}\,dw = \begin{cases}\pi/2 & \text{if} \quad 0 \leqq x < 1,\\ \pi/4 & \text{if} \quad x = 1,\\ 0 & \text{if} \quad x > 1.\end{cases} \tag{7}$$

We mention that this integral is called **Dirichlet's discontinous factor**. (For P. L. Dirichlet see Sec. 10.8.)

The case $x = 0$ is of particular interest. If $x = 0$, then (7) gives

$$\int_0^{\infty}\frac{\sin w}{w}\,dw = \frac{\pi}{2}. \tag{8*}$$

We see that this integral is the limit of the so-called **sine integral**

$$\mathrm{Si}(u) = \int_0^{u}\frac{\sin w}{w}\,dw \tag{8}$$

as $u \to \infty$. The graphs of $\mathrm{Si}(u)$ and of the integrand are shown in Fig. 282.

In the case of a Fourier series the graphs of the partial sums are approximation curves of the curve of the periodic function represented by the series. Similarly, in the case of the Fourier integral (5), approximations are obtained by replacing ∞ by numbers a. Hence the integral

$$\frac{2}{\pi}\int_0^{a}\frac{\cos wx\sin w}{w}\,dw \tag{9}$$

approximates the right side in (6) and therefore $f(x)$.

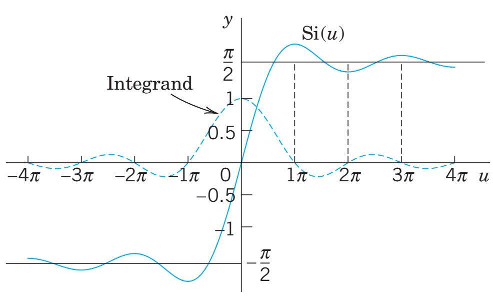

Fig. 282. Sine integral Si(*u*) and integrand

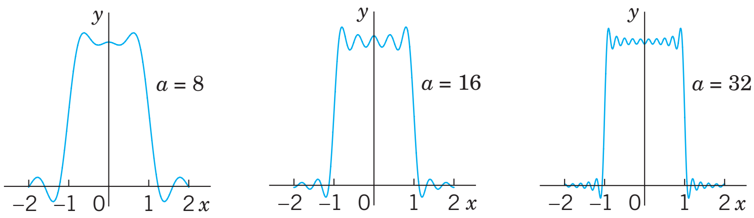

Fig. 283. The integral (9) for $a = 8$, 16, and 32, illustrating the development of the Gibbs phenomenon

Figure 283 shows oscillations near the points of discontinuity of $f(x)$. We might expect that these oscillations disappear as a approaches infinity. But this is not true; with increasing a, they are shifted closer to the points $x = \pm 1$. This unexpected behavior, which also occurs in connection with Fourier series (see Sec. 11.2), is known as the **Gibbs phenomenon**. We can explain it by representing (9) in terms of sine integrals as follows. Using (11) in App. A3.1, we have

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^a \frac{\sin(w + wx)}{w}\,dw + \frac{1}{\pi}\int_0^a \frac{\sin(w - wx)}{w}\,dw.$$

In the first integral on the right we set $w + wx = t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x+1)a$. In the last integral we set $w - wx = -t$. Then $dw/w = dt/t$, and $0 \leqq w \leqq a$ corresponds to $0 \leqq t \leqq (x-1)a$. Since $\sin(-t) = -\sin t$, we thus obtain

$$\frac{2}{\pi}\int_0^a \frac{\cos wx \sin w}{w}\,dw = \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t}\,dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t}\,dt.$$

From this and (8) we see that our integral (9) equals

$$\frac{1}{\pi}\operatorname{Si}(a[x+1]) - \frac{1}{\pi}\operatorname{Si}(a[x-1])$$

and the oscillations in Fig. 283 result from those in Fig. 282. The increase of a amounts to a transformation of the scale on the axis and causes the shift of the oscillations (the waves) toward the points of discontinuity -1 and 1. ■

Fourier Cosine Integral and Fourier Sine Integral

Just as Fourier *series* simplify if a function is even or odd (see Sec. 11.2), so do Fourier *integrals*, and you can save work. Indeed, if f has a Fourier integral representation and is *even*, then $B(w) = 0$ in (4). This holds because the integrand of $B(w)$ is odd. Then (5) reduces to a **Fourier cosine integral**

(10) $$f(x) = \int_0^\infty A(w)\cos wx\,dw \qquad \text{where} \qquad A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv.$$

Note the change in $A(w)$: for even f the integrand is even, hence the integral from $-\infty$ to ∞ equals twice the integral from 0 to ∞, just as in (7a) of Sec. 11.2.

Similarly, if f has a Fourier integral representation and is *odd*, then $A(w) = 0$ in (4). This is true because the integrand of $A(w)$ is odd. Then (5) becomes a **Fourier sine integral**

(11) $$f(x) = \int_0^\infty B(w)\sin wx\,dw \qquad \text{where} \qquad B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\,dv.$$

Note the change of $B(w)$ to an integral from 0 to ∞ because $B(w)$ is even (odd times odd is even).

Earlier in this section we pointed out that the main application of the Fourier integral representation is in differential equations. However, these representations also help in evaluating integrals, as the following example shows for integrals from 0 to ∞.

EXAMPLE 3 Laplace Integrals

We shall derive the Fourier cosine and Fourier sine integrals of $f(x) = e^{-kx}$, where $x > 0$ and $k > 0$ (Fig. 284). The result will be used to evaluate the so-called Laplace integrals.

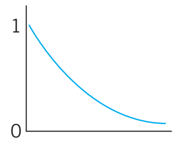

Fig. 284. $f(x)$ in Example 3

Solution. **(a)** From (10) we have $A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv} \cos wv \, dv$. Now, by integration by parts,

$$\int e^{-kv} \cos wv \, dv = -\frac{k}{k^2 + w^2} e^{-kv} \left(-\frac{w}{k} \sin wv + \cos wv \right).$$

If $v = 0$, the expression on the right equals $-k/(k^2 + w^2)$. If v approaches infinity, that expression approaches zero because of the exponential factor. Thus $2/\pi$ times the integral from 0 to ∞ gives

$$A(w) = \frac{2k/\pi}{k^2 + w^2}. \tag{12}$$

By substituting this into the first integral in (10) we thus obtain the Fourier cosine integral representation

$$f(x) = e^{-kx} = \frac{2k}{\pi} \int_0^\infty \frac{\cos wx}{k^2 + w^2} \, dw \qquad (x > 0, \quad k > 0).$$

From this representation we see that

$$\boxed{\int_0^\infty \frac{\cos wx}{k^2 + w^2} \, dw = \frac{\pi}{2k} e^{-kx}} \qquad (x > 0, \quad k > 0). \tag{13}$$

(b) Similarly, from (11) we have $B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv} \sin wv \, dv$. By integration by parts,

$$\int e^{-kv} \sin wv \, dv = -\frac{w}{k^2 + w^2} e^{-kv} \left(\frac{k}{w} \sin wv + \cos wv \right).$$

This equals $-w/(k^2 + w^2)$ if $v = 0$, and approaches 0 as $v \to \infty$. Thus

$$B(w) = \frac{2w/\pi}{k^2 + w^2}. \tag{14}$$

From (14) we thus obtain the Fourier sine integral representation

$$f(x) = e^{-kx} = \frac{2}{\pi} \int_0^\infty \frac{w \sin wx}{k^2 + w^2} \, dw.$$

From this we see that

$$\boxed{\int_0^\infty \frac{w \sin wx}{k^2 + w^2} \, dw = \frac{\pi}{2} e^{-kx}} \qquad (x > 0, \quad k > 0). \tag{15}$$

The integrals (13) and (15) are called the **Laplace integrals**. ■

PROBLEM SET 11.7

1–6 EVALUATION OF INTEGRALS

Show that the integral represents the indicated function. *Hint*. Use (5), (10), or (11); the integral tells you which one, and its value tells you what function to consider. Show your work in detail.

1. $\displaystyle\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\,dx = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$

2. $\displaystyle\int_0^\infty \frac{\sin \pi w \sin xw}{1-w^2}\,dw = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \leqq x \leqq \pi \\ 0 & \text{if } x>\pi \end{cases}$

3. $\displaystyle\int_0^\infty \frac{1-\cos \pi w}{w}\sin xw\,dw = \begin{cases} \frac{1}{2}\pi & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

4. $\displaystyle\int_0^\infty \frac{\cos \frac{1}{2}\pi w}{1-w^2}\cos xw\,dw = \begin{cases} \frac{1}{2}\pi\cos x & \text{if } 0<|x|<\frac{1}{2}\pi \\ 0 & \text{if } |x| \geqq \frac{1}{2}\pi \end{cases}$

5. $\displaystyle\int_0^\infty \frac{\sin w - w\cos w}{w^2}\sin xw\,dw = \begin{cases} \frac{1}{2}\pi x & \text{if } 0<x<1 \\ \frac{1}{4}\pi & \text{if } x=1 \\ 0 & \text{if } x>1 \end{cases}$

6. $\displaystyle\int_0^\infty \frac{w^3\sin xw}{w^4+4}\,dw = \frac{1}{2}\pi e^{-x}\cos x \quad \text{if } x>0$

7–12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (10).

7. $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

9. $f(x) = 1/(1+x^2)$ [$x>0$. *Hint*. See (13).]

10. $f(x) = \begin{cases} a^2-x^2 & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

11. $f(x) = \begin{cases} \sin x & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

12. $f(x) = \begin{cases} e^{-x} & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

13. CAS EXPERIMENT. Approximate Fourier Cosine Integrals. Graph the integrals in Prob. 7, 9, and 11 as functions of x. Graph approximations obtained by replacing ∞ with finite upper limits of your choice. Compare the quality of the approximations. Write a short report on your empirical results and observations.

14. PROJECT. Properties of Fourier Integrals

(a) Fourier cosine integral. Show that (10) implies

(a1) $\displaystyle f(ax) = \frac{1}{a}\int_0^\infty A\left(\frac{w}{a}\right)\cos xw\,dw$

$(a>0)$ (*Scale change*)

(a2) $\displaystyle xf(x) = \int_0^\infty B^*(w)\sin xw\,dw,$

$\displaystyle B^* = -\frac{dA}{dw}, \qquad A \text{ as in (10)}$

(a3) $\displaystyle x^2f(x) = \int_0^\infty A^*(w)\cos xw\,dw,$

$\displaystyle A^* = -\frac{d^2A}{dw^2}.$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x)=1$ if $0<x<a$ and $f(x)=0$ if $x>a$.

(d) Fourier sine integral. Find formulas for the Fourier sine integral similar to those in (a).

15. CAS EXPERIMENT. Sine Integral. Plot $\mathrm{Si}(u)$ for positive u. Does the sequence of the maximum and minimum values give the impression that it converges and has the limit $\pi/2$? Investigate the Gibbs phenomenon graphically.

16–20 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

16. $f(x) = \begin{cases} x & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

17. $f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

18. $f(x) = \begin{cases} \cos x & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$

19. $f(x) = \begin{cases} e^x & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$

20. $f(x) = \begin{cases} e^{-x} & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$