Contents

12 Equilibrium and Elasticity

In this chapter we will define equilibrium and find the conditions needed so that an object is at equilibrium.

We will then apply these conditions to a variety of practical engineering problems of static equilibrium.

We will also examine how a "rigid" body can be deformed by an external force. In this section we will introduce the following concepts:

Stress and strain Young's modulus (in connection with tension and compression) Shear modulus (in connection with shearing) Bulk modulus (in connection to hydraulic stress)

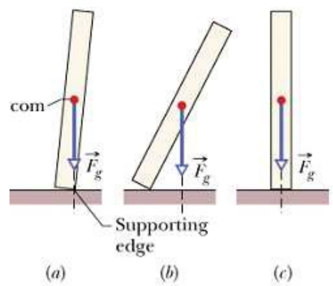
12.1 Equilibrium

For an object,
1. The linear momentum $\vec{P}$ of its center of mass is constant.
2. Its angular momentum $\vec{L}$ about its center of mass, or about any other point is also constant.

We say such an object is in equilibrium. The two requirements for equilibrium are then

$$\vec{P} = \text{ a constant and } \vec{L} = \text{ a constant.}$$

Our concern in this chapter is with situations in which $\vec{P} = 0$ and $\vec{L} = 0$. That is we are interested in objects that are not moving in any way (this includes translational as well as rotational motion) in the reference frame from which we observe them. Such objects are said to be in static equilibrium. In chapter 8 we differentiated between stable and unstable static equilibrium. If a body that is in static equilibrium is displaced slightly from this position, the forces on it may return it to its old position. In this case we say that the equilibrium is stable. If the body does not return to its old position, then the equilibrium is unstable.



An example of unstable equilibrium is shown in the figures above. In fig. a we balance a domino with the domino's center of mass vertically above the supporting edge. The torque of the gravitational force $\vec{F}_g$ about the supporting edge is zero because the line of action of $\vec{F}_g$ passes through the edge. Thus the domino is in equilibrium. Even a slight force on the domino ends the equilibrium. As the line of action of $\vec{F}_g$ moves to one side of the supporting edge (see fig. b) the torque due to $\vec{F}_g$ is non-zero and the domino rotates in the clockwise direction away from its equilibrium position of fig. a. The domino in fig. a is in a position of unstable equilibrium. The domino is fig. c is not quite as unstable. To topple the domino the applied force would have to rotate it through and beyond the position of fig. a. A flick of the finger against the domino can topple it.

12.2 The Conditions for Equilibrium

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} = 0 \text{ balance of forces}$$

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0 \text{ balance of torques}$$

The above two requirements is independent of the origin of the coordinate system. e.g. suppose

$$\vec{x} = \vec{x}' + \vec{a}$$

where $\vec{a}$ is a constant. Then

$$\vec{P} = \vec{P}'$$

$$\vec{F}'_{net} = \frac{d\vec{P}'}{dt} = \frac{d\vec{P}}{dt} = 0$$

$$\begin{aligned}\vec{L} &= \sum_i \vec{x}_i \times \vec{p}_i = \sum_i (\vec{x}'_i + \vec{a}) \times \vec{p}'_i \\ &= \sum_i \vec{x}'_i \times \vec{p}'_i + \vec{a} \times \sum_i \vec{p}'_i = \vec{L}' + \vec{a} \times \vec{P}'\end{aligned}$$

$$\vec{\tau}'_{net} = \frac{d\vec{L}'}{dt} = \frac{d\vec{L}}{dt} - \vec{a} \times \frac{d\vec{P}'}{dt} = 0$$

12.3 The Center of Gravity (cog)

The gravitational force acting on an extended body is the vector sum of the gravitational forces acting on the individual elements of the body.

$$\vec{F}_g = \sum_i \vec{F}_{gi} = \sum_i m_i \vec{g} = M\vec{g}$$

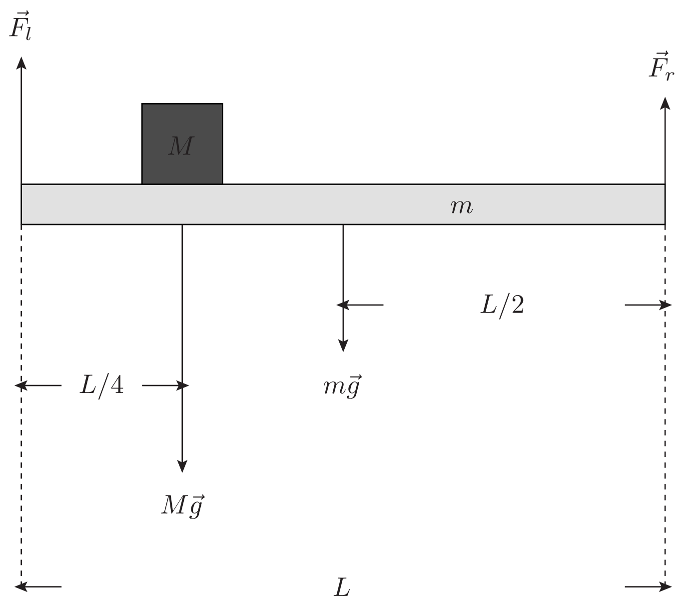
$$\begin{aligned}\vec{\tau}_{net} &= \sum_i \vec{x}_i \times \vec{F}_{gi} = \sum_i \vec{x}_i \times (m_i \vec{g}) \\ &= \sum_i \frac{m_i \vec{x}_i}{M} \times (M\vec{g}) = \vec{x}_{cm} \times (M\vec{g})\end{aligned}$$

The gravitational force $\vec{F}_g$ on a body effectively acts at a single point known as the center of gravity of the body. Here effectively has the following meaning: If the individual gravitational forces on the elements of the body are turned off and replaced by $\vec{F}_g$ acting at the center of gravity, then the net force and the net torque about any point on the body does not change. We shall prove that if the acceleration of gravity $\vec{g}$ is the same for all the elements of the body then the center of gravity coincides with the center of mass. This is a reasonable approximation for objects near the surface of the earth because $\vec{g}$ changes very little.

12.4 Examples

12.4.1 Sample Problem 12 - 1

A uniform beam of length L and mass m is at rest on two scales. A uniform block of mass M is a rest on the beam at a distance $L/4$ from its left end. Calculate the scales readings. We choose to calculate the torque with respect to an axis through the left end of the beam (point O).



$$F_l\hat{\jmath} + F_r\hat{\jmath} - Mg\hat{\jmath} - mg\hat{\jmath} = 0$$

$$\vec{0} \times F_l\hat{\jmath} + L\hat{\imath} \times F_r\hat{\jmath} - \frac{L}{4}\hat{\imath} \times Mg\hat{\jmath} - \frac{L}{2}\hat{\imath} \times mg\hat{\jmath} = 0$$

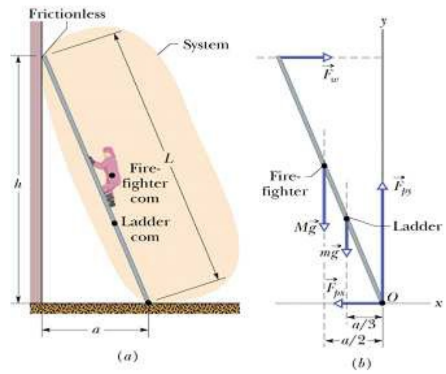
$$F_l + F_r - Mg - mg = 0$$

$$LF_r - \frac{L}{4}Mg - \frac{L}{2}mg = 0$$

$$Fr = \frac{1}{4}Mg + \frac{1}{2}mg$$

$$F_l = \frac{3}{4}Mg + \frac{1}{2}mg$$

12.4.2 Sample Problem 12 - 2



A ladder of length L and mass m leans against a frictionless wall. The ladder's upper end is at a height h above the pavement on which the lower end rests. The center of mass of the ladder is $L/3$ from the lower end. A firefighter of mass M climbs half way up the ladder. Find the forces exerted on the ladder by the wall and the pavement. Let us take the torques about an axis through O.

$$
\begin{aligned}
\vec{\tau}_{net} &= -\frac{a}{3}\hat{\imath} \times m\vec{g} - \frac{a}{2}\hat{\imath} \times M\vec{g} + h\hat{\jmath} \times \vec{F}_w \\
&= a\left(\frac{m}{3} + \frac{M}{2}\right)\hat{\imath} \times g\hat{\jmath} + h\hat{\jmath} \times F_w\hat{\imath} \\
&= \left(ag\left(\frac{m}{3} + \frac{M}{2}\right) - hF_w\right)\hat{k} = 0
\end{aligned}
$$

$$F_w = \frac{a}{h}\left(\frac{m}{3} + \frac{M}{2}\right)g$$

$$
\begin{aligned}
\vec{F}_{net} &= \vec{F}_w + \vec{F}_p + M\vec{g} + m\vec{g} \\
&= F_w\hat{\imath} - F_{px}\hat{\imath} + F_{py}\hat{\jmath} - Mg\hat{\jmath} - mg\hat{\jmath} = 0
\end{aligned}
$$

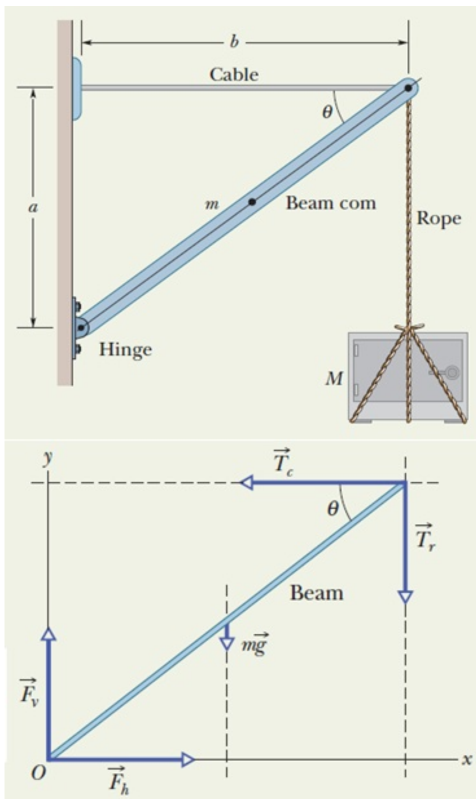
$$F_{px} = F_w$$

$$F_{py} = (M + m)\,g$$

12.4.3 Sample Problem 12 - 3

A safe of mass M hangs by a rope from a boom with dimensions a and b. The beam of the boom has mass m

Find the tension T_c in the cable and the force exerted on the beam by the hinge.



We calculate the net torque about an axis normal to the page that passes through point O.

$$
\begin{aligned}
\vec{\tau}_{net} &= \left(b\hat{\imath} + a\hat{\jmath}\right) \times \left(\vec{T}_c + \vec{T}_r\right) + \frac{(b\hat{\imath} + a\hat{\jmath})}{2} \times m\vec{g} \\
&= (b\hat{\imath} + a\hat{\jmath}) \times \left(-T_c\hat{\imath} - T_r\hat{\jmath} - \frac{m}{2}g\hat{\jmath}\right) \\
&= \left(-b\left(T_r + \frac{m}{2}g\right) + aT_c\right)\hat{k} = 0
\end{aligned}
$$

$$b\left(T_r + \frac{m}{2}g\right) = aT_c$$

$$T_r = Mg$$

$$T_c = \frac{b}{a}\left(M + \frac{m}{2}\right)g$$

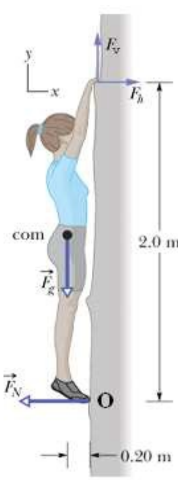
$$\vec{F}_{net} = \vec{F}_h + \vec{F}_v + \vec{T}_c + \vec{T}_r + m\vec{g} = 0$$

$$F_h = T_c$$

$$F_v = (M + m)\,g$$

12.4.4 Sample Problem 12 - 4

A rock climber hangs by the crimp hold of one hand. Her feet touch the rock directly below her fingers. Assume that the force from the horizontal ledge supporting her fingers is equally shared by the four fingers. Calculate the horizontal and vertical components of the force on each fingertip.



We calculate the net torque about an axis that is perpendicular to the page and passes through point O. Let $a = 0.20m$, $b = 2.0m$.

$$\begin{aligned} 0 = \vec{\tau}_{net} &= -a\hat{\imath} \times m\vec{g} + b\hat{\jmath} \times 4\vec{F}_h \\ &= (amg - 4bF_h)\,\hat{k} = 0 \end{aligned}$$

$$F_h = \frac{a}{4b}mg$$

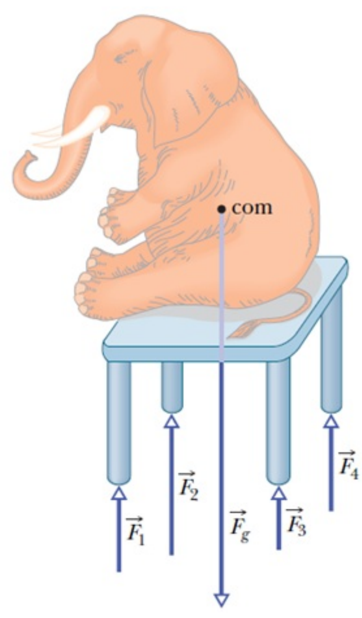
$$0 = \vec{F}_{net} = 4\vec{F}_h + 4\vec{F}_v + \vec{F}_N + m\vec{g} = 0$$
$$F_N = 4F_h$$
$$F_v = \frac{1}{4}mg$$

12.5 Indeterminate Structures

For the problems in this chapter we have the following equations at our disposal:

$$\vec{F}_{net} = 0$$
$$\vec{\tau}_{net} = 0$$

If the problem has too many unknowns, we cannot solve it. We can solve a statics problem for a table with three legs but not for one with four legs. Problems like these are called indeterminate. An example is given in the following figure.



A big elephant sits on a wobbly table. If the table does not collapse it will deform so that all four legs touch the floor. The upward forces exerted on the legs by the floor assume definite and different values. How can we calculate the values of these forces? The net torque with respect to the origin at the center of the table is

$$\begin{aligned}\vec{\tau}_{net} &= a\left(-\hat{\imath}-\hat{\jmath}\right)\times F_1\hat{k}+a\left(-\hat{\imath}+\hat{\jmath}\right)\times F_2\hat{k}+a\left(\hat{\imath}-\hat{\jmath}\right)\times F_3\hat{k}+a\left(\hat{\imath}+\hat{\jmath}\right)\times F_4\hat{k}\\ &= a\left(-F_1+F_2-F_3+F_4\right)\hat{\imath}-a\left(-F_1-F_2+F_3+F_4\right)\hat{\jmath}=0\end{aligned}$$

$$F_1+F_3=F_2+F_4$$

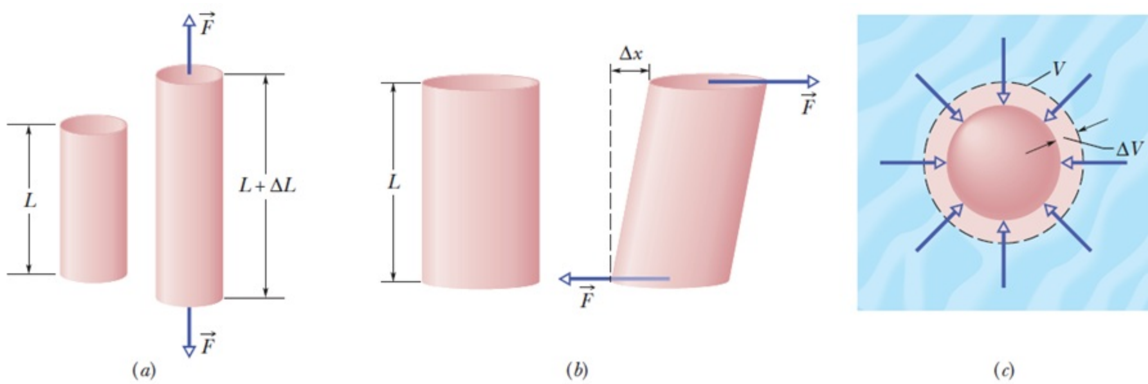
$$F_1+F_2=F_3+F_4$$

The vanishing total net force gives

$$F_g=F_1+F_2+F_3+F_4$$

To solve such an indeterminate equilibrium problem we must supplement the three equilibrium equations with some knowledge of elasticity, the branch of physics and engineering that describes how real bodies deform when forces are applied to them.

12.6 Elasticity



In the three figures above we show the three ways in which a solid might change its dimensions under the action of external deforming forces. In fig. a, the cylinder is stretched by forces acting along the cylinder axis. In fig. b, the cylinder is deformed by forces perpendicular to its axis. In fig. c a solid placed in a fluid under high pressure is compressed uniformly on all sides. All three deformation types have stress in common (defined as deforming force per unit area). These stresses are known as tensile/compressive for fig. a, shearing for fig. b, and hydraulic for fig. c. The application of stress on a solid results in strain which take different form for the three types of strain. Strain is related to stress via the equation:

$$stress = modulus \ \times \ strain.$$

(a) Tensile stress is defined as the ratio $\frac{F}{A}$ where A is the solid area. Strain (symbol S) is defined as the ratio $\frac{\Delta L}{L}$ where ΔL is the change in the length L of the cylindrical solid. For a wide range of applied stresses the stress - strain relation is linear and the solid returns to its original length when the stress is removed. This is known as the elastic range. If the stress is increased beyond a maximum value known as the yield strength S_y the cylinder becomes permanently deformed. If the stress continues to increase the cylinder breaks at a stress value known as ultimate strength S_u. For stresses below S_y (elastic range) stress and strain are connected via the equation

$$\frac{F}{A} = E\frac{\Delta L}{L}$$

where the constant E (modulus) is known as the*Young's modulus*. Note: Young's modulus is almost the same for tension and compression. The ultimate strength S maybe different.
(b) Shearing: In the case of shearing deformation strain is defined as the dimensionless ratio $\frac{\Delta x}{L}$. The stress/strain equation has the form:

$$\frac{F}{A} = G\frac{\Delta x}{L}$$

The constant G is known as the *shear modulus*.
(c) Hydraulic Stress: The stress is this case is the pressure $p = \frac{F}{A}$ the surrounding fluid exerts on the immersed object. Here A is the area of the object. In this case strain is defined as the dimensionless ratio$\frac{\Delta V}{V}$ where V is the volume of the object and ΔV the change in the volume due to the change of fluid pressure. The stress/strain equation has the form:

$$\Delta p = -B\frac{\Delta V}{V}$$

The constant B is known as the bulk modulus of the material.