# Chaper 6 Laplace Transform

6.1 Laplace transform , inverse transform , linearity .

Def : if $f(t)$ is a function of $t \geq 0$, the Laplace transform of $f(t)$, denoted by $F(s)$ or $\mathcal{L}\{f\}$ is defined:

$$\mathcal{L}\{f(t)\} \equiv F(s) \equiv \int_0^\infty e^{-st} \cdot f(t)dt$$

also $f(t)$ is called the inverse Laplace transform $F(s)$ and denoted by $\mathcal{L}^{-1}\{F(s)\}$,i.e.

$$f(t) \equiv \mathcal{L}^{-1}\{F(s)\}$$

Ex : $f(t) = 1$ for $t \geq 0$

$$\mathcal{L}\{f(t)\} = \mathcal{L}(1) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \quad \text{for } s > 0$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s} \quad \text{for s} > 0}$$

Ex : $f(t) = e^{at}$ for $t \geq 0$. a = constant

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st}e^{at}dt = \frac{1}{a-s}e^{-(s-a)t}\Big|_0^\infty = \frac{1}{s-a} \quad , \quad s > a$$

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a} \quad \text{for s} > \text{a}}$$

Linearity

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ a、b constant.

Proof : $\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}[af(t) + bg(t)]dt$

$$= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt$$
$$= a\mathcal{L}(f) + b\mathcal{L}(g) = aF(s) + bG(s)$$

also $\mathcal{L}^{-1}\{aF(s) + bG(s)\} = af(t) + bg(t)$

Ex : $f(t) = \cosh at = \dfrac{(e^{at} + e^{-at})}{2}$

$$\mathcal{L}(\cosh at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}\quad(e^{-st}) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$= \dfrac{s}{s^2 - a^2}$ also $\mathcal{L}(\sinh at) = \dfrac{a}{s^2 - a^2}$

$$\boxed{\mathcal{L}\begin{Bmatrix}\cosh(at)\\ \sinh(at)\end{Bmatrix} = \begin{Bmatrix}\dfrac{s}{s^2-a^2}\\ \dfrac{a}{s^2-a^2}\end{Bmatrix} \quad \text{for s > a}}$$

Ex. Since $\mathcal{L}(e^{i\omega t}) = \dfrac{1}{s - i\omega} = \dfrac{s + i\omega}{(s - i\omega)(s + i\omega)} = \dfrac{s}{s^2 + \omega^2} + i\dfrac{\omega}{s^2 + \omega^2}$

Also $\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos\omega t + i\sin\omega t) = \mathcal{L}(\cos\omega t) + i\,\mathcal{L}(\sin\omega t)$

Hence we have:

$$\boxed{\mathcal{L}\begin{Bmatrix}\cos\omega t\\ \sin\omega t\end{Bmatrix} = \begin{Bmatrix}\dfrac{s}{s^2+\omega^2}\\ \dfrac{\omega}{s^2+\omega^2}\end{Bmatrix}}$$

Ex : Given $F(s) = \dfrac{1}{(s-a)(s-b)}$ $a \neq b$, Find $f(t)$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\cdot\frac{1}{s-b}\right\} = \mathcal{L}^{-1}\left[\frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)\right]$$

$$= \frac{1}{a-b}\left[\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-b}\right)\right] = \frac{1}{a-b}(e^{at} - e^{bt})$$

Ex. $\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a dt = \int_0^\infty e^{-\tau}\left(\dfrac{\tau}{s}\right)^a \dfrac{d\tau}{s} = \dfrac{1}{s^{a+1}}\int_0^\infty e^{-\tau}\tau^a d\tau$

$$\boxed{\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}}$$

if $a = \text{integer} = n \;\Rightarrow$

$$\boxed{\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \qquad \text{n=1.2.3...}}$$

Def : piecewise continuous :

a function $f(x)$ is said to be piecewise continuous ( P.C.) if it satisfies the following conditions:

1. *f(x)* has finite number of discontinuity in the interval
2. *f(x)* has finite number of maximum and minimum in the interval

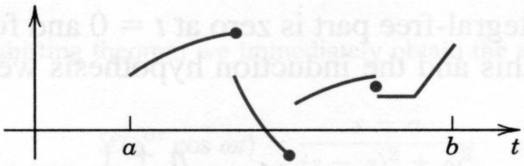



**Fig. 107.** Example of a piecewise continuous function $f(t)$
(The dots mark the function values at the jumps.)

Def : Exponential order

A function $f(t)$ is said to be of exponential order, if there are some finite number $\gamma, M, T$ such that

$|f(t)| \le Me^{\gamma t}$ for all $t > T$

Ex : $f(t) = e^{2t}$

$\because |e^{2t}| \le Me^{\gamma t}$ for $\gamma \ge 2$; M = 1; T = 0

→ $e^{2t}$ is a function of exponential order.

Ex : $f(t) = \sin(e^{t^2})$

since $\left|\sin(e^{t^2})\right| \le 1$ → exp.order.

however $f'(t) = 2t \cdot e^{t^2} \cos(e^{t^2}) \Rightarrow$ not exp.order.

note: $f(t)$ exp.order $\Rightarrow f'(t)$ : may no loger exp.order.

but $\int f(t)dt$ : still exp.order.

Theorem: (Sufficient condition)

If $f(t)$ is $\begin{cases} \text{1.piecewise continuous} \\ \text{2.of exponential order} \end{cases}$ ,then $\mathscr{L}\{f(t)\}$ exists.

Proof :

$(2) \Rightarrow |f(t)| < M_1 e^{\alpha t} \quad t > T$

$(1) \Rightarrow |f(t)| < M_2 \quad 0 \le t \le T$

choose $M = \max(M_1, M_2)$

then $|f(t)| < Me^{\alpha t} \quad 0 \le t \le \infty$

Hence $\mathscr{L}\ [f(t)] = \int_0^\infty f(t)e^{-st}dt \le \int_0^\infty |f(t)|e^{-st}dt \le M\int_0^\infty e^{\alpha t}e^{-st}dt$

$$= M\int_0^\infty e^{-(s-\alpha)t}dt = \frac{M}{s-\alpha} \qquad \text{when} \quad s > \alpha$$

i.e. $\mathscr{L}\{f(t)\}$ is finite when $s > \alpha$ → exists.

Note: this is a Sufficient condition but not necessary

Ex:

$\mathscr{L}(t^{-1/2}) = \frac{1}{\sqrt{s}}\Gamma(\frac{1}{2}) = \sqrt{\frac{\pi}{s}}$ exists for $s > 0$ but $(t^{-1/2})$ is infinite at $t = 0$

Uniqueness:

Given $f(t)$ → if $\mathscr{L}\{f(t)\}$ exists, it is unique.

Given $F(s)$→$\mathscr{L}^{-1}\{F(s)\}$ i.e. $f(t)$, is essentially unique for $t > 0$,

(i.e. they may differ at various isolated points.)

6.2 Transforms of derivatives and integrals

Theorem (Differential) :

If $f(t)$ is continuous and of exp. order.

and $\dfrac{df}{dt}$ is piecewise continuous .

Then $\mathcal{L}[f'(t)] = s\mathcal{L}[f] - f(0^+)$

Proof :

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st}\cdot f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$

$$= -f(0) + s\int_0^\infty f(t)e^{st}dt = s\mathcal{L}[f] - f(0)$$

also

$$\mathcal{L}[f''] = s\mathcal{L}(f') - f'(0) = s[s\mathcal{L}(f) - f(0)] - f'(0)$$
$$= s^2\mathcal{L}(f) - sf(0) - f'(0)$$
$$\mathcal{L}[f'''] = s^3\mathcal{L}(f) - s^2 f(0) - sf'(0) - f''(0)$$
$$\vdots$$

Thus $\Rightarrow$

$$\mathcal{L}[f^{(n)}] = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0)\cdots + sf^{(n-2)}(0) - f'(0)$$

provided that $f, f', f''\cdots f^{(n-1)}\ f^{(n)}$ are continuous and exp.order.

and $f^{(n)}$ is piecewise continuous .

Ex: $f(t) = \sin^2 t$ Find $\mathcal{L}(f)$

since $f(0) = 0$ ; $f'(t) = 2\sin t\cos t = \sin 2t$

→ $\mathcal{L}[f'(t)] = \mathcal{L}(\sin 2t) = \dfrac{2}{s^2 + 2^2} = s\mathcal{L}(f) - f(0)$

$\therefore\ \mathcal{L}(f) = \dfrac{2}{s(s^2 + 2^2)}$

- Application to Initial value problem

Consider $y'' + ay' + by = r(t)$ $y(0) = K_0$ $y'(0) = K_1$

$r(t)$ :input(applied fore) $y(t)$: output (respones)

Take the Laplace transform of the equation →:

$$\mathcal{L}\{y''+ay'+by\}=\mathcal{L}\{r(t)\}$$

by diffrential theorem : let $\mathcal{L}\{y(t)\}\equiv Y(s)$ $\quad \mathcal{L}\{r(t)\}\equiv R(s)$

$$\Rightarrow [s^2Y-sy(0)-y'(0)]+a[sY-y(0)]+bY=R$$

$$\Rightarrow (s^2+as+b)Y=(s+a)y(0)+y'(0)+R(s)$$

Hence

$$Y(s)=[(s+a)y(0)+y'(0)+R(s)]\cdot Q(s)$$

where $Q(s)=\dfrac{1}{s^2+as+b}$ $\leftarrow$ Transfer function

the solution of the equation can be obtained :

$$y(t)=\mathcal{L}^{-1}[Y(s)]$$

if the system is passive i.e. $y(0)=y'(0)=0$

$$\Rightarrow Y(s)=R(s)Q(s) \qquad \Rightarrow Q(s)=\frac{Y(s)}{R(s)}=\frac{\text{Laplace tranform of output}}{\text{Laplace tra}\cancel{\text{nform of in}}\text{tput}}$$

Note: since $Q(s)$ depends on $a,b$ only but not depends $r(t)$ $\Rightarrow$ it is a system characteristics.

$$\begin{tikzcd}
\boxed{\text{Given problem of of } y(t)} \arrow[r, "\mathcal{L}\{y(t)\}"'] & \boxed{\text{Subsidiary equation. } Y(s)} \arrow[d] \\
\boxed{\text{Solution of } y(t)} & \boxed{\text{Solution of } Y(s)} \arrow[l, "\mathcal{L}^{-1}[Y(s)]"]
\end{tikzcd}$$

t-space s-space

Ex : $y''+y=2t$

I.C. $\quad y(\frac{\pi}{4})=\frac{\pi}{2} \qquad y'(\frac{\pi}{4})=2-\sqrt{2}$

sol 1: $y = A\cos t + B\sin t + 2t$

$$y(\frac{\pi}{4}) = \frac{\pi}{2} \quad \Rightarrow \quad A\sqrt{2} + \sqrt{2}B = 0 \qquad \Rightarrow \quad A = -B$$

$$y'(\frac{\pi}{4}) = 2 - \sqrt{2} \quad \Rightarrow \quad -A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} + 2 = 2 - \sqrt{2}$$

$$\Rightarrow \quad B = -1 \quad ; \qquad A = 1$$

$$\therefore \quad y = \cos t - \sin t + 2t$$

sol 2: Take Laplace transform of the equation:

$$s^2 Y - sy(0) - y'(0) + Y = \frac{2}{s^2}$$

$$\Rightarrow \quad Y = \frac{2}{(s^2+1)s^2} + y(0)\frac{s}{s^2+1} + y'(0)\frac{1}{s^2+1}$$

$$= 2(\frac{1}{s^2} - \frac{1}{s^2+1}) + y(0)\frac{s}{s^2+1} + y'(0)\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}[Y(s)] \quad \Rightarrow$$

$$y(t) = 2t + y(0)\cos t + [y'(0) - 2]\sin t$$

$$= 2t + A\cos t + B\sin t \quad \rightarrow \text{sol. 1}$$

Theorem (integral)

If $f(t)$ is P.C and of exp.order.

then $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}\mathcal{L}\{f(t)\}$

Proof :

Since $\left|f(t)\right| \le Me^{\gamma t}$ Let $g(t) \equiv \int_0^t f(\tau)d\tau$

$$\therefore \quad |g(t)| \le \int_0^t |f(\tau)|d\tau \le M\int_0^t e^{\gamma\tau}d\tau = \frac{M}{\gamma}(e^{\gamma t} - 1) \le \frac{M}{\gamma}e^{\gamma t}$$

i.e. $g(t)$ is of exp.order and continuous.

Also $g'(t)$ is of exp.order and continuous.

Hence

$$\mathcal{L}[f(t)] = \mathcal{L}[g'(t)] = s\mathcal{L}(g(t)) - g(0)$$

$$\because \quad g(0) = 0$$

$$\therefore \quad \mathcal{L}(f(t)) = s\mathcal{L}(g(t)) = s\mathcal{L}(\int_0^t f(\tau)d\tau)$$

$$\rightarrow \mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

$$\therefore \mathcal{L}^{-1}\ \{\frac{1}{s}F(s)\} = \int_0^t f(\tau)d\tau$$

Ex : $\mathcal{L}(f) = \dfrac{1}{s(s^2+\omega^2)}$ Find $f(t)$

$$\because \quad \mathcal{L}^{-1}(\frac{1}{s^2+\omega^2}) = \mathcal{L}^{-1}(\frac{\omega}{s^2+\omega^2}\cdot\frac{1}{\omega}) = \frac{1}{\omega}\sin\omega t$$

$$\therefore \quad \mathcal{L}^{-1}\{\frac{1}{s(s^2+\omega^2)}\} = \frac{1}{\omega}\int_0^t \sin\omega\tau\cdot d\tau = \frac{1}{\omega^2}(1-\cos\omega t)$$

6.3 s-shifting t-shifting Unit step function

Theorem : ( s-shifting )

if $\mathcal{L}\{f(t)\} = F(s)$

then $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

Proof :

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^{\infty} \{e^{at} f(t)\} e^{-st} dt = \int_0^{\infty} f(t) e^{-(s-a)t} dt = F(s-a)$$

$$\therefore\ \mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)$$

Ex :

since $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ → $\mathcal{L}\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}}$

Since $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2 + \omega^2}$ → $\mathcal{L}\{e^{at} \cos \omega t\} = \dfrac{s-a}{(s-a)^2 + \omega^2}$

- Def : Unit step function : (Heaviside function)

$$H(t-a) \equiv U(t-a) \equiv \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases}$$

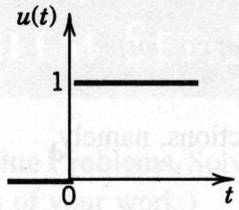



Fig. 110. Unit step function $u(t)$

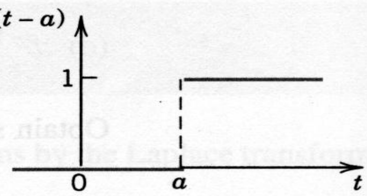
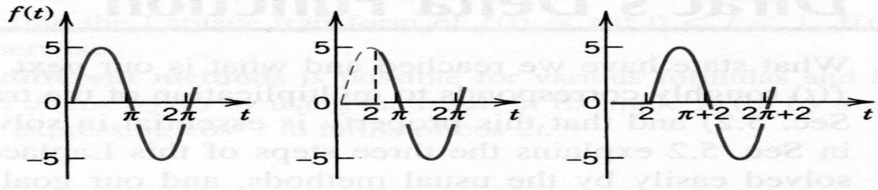



Fig. 111. Unit step function $u(t-a)$

$$\therefore \quad f(t-a)u(t-a) = \begin{cases} 0 & \text{when } t < a \\ f(t-a) & \text{when } t > a \end{cases}$$




(a) $f(t) = 5 \sin t$ (b) $f(t)u(t-2)$ (c) $f(t-2)u(t-2)$

Fig. 112. Effects of the unit step function: (a) Given function. (b) Switching off and on. (c) Shift.

Theorem :( t-shifting )

If $\mathcal{L}\{f(t)\} = F(s)$

then $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$

Proof:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty f(t-a)U(t-a)e^{-st}dt$$

$$= \int_a^\infty f(t-a)e^{-st}dt \qquad \text{let} \quad t-a=\tau$$

$$= \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau = e^{-as}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-as}F(s)$$

$$\therefore \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

Ex. $\mathcal{L}\{1\cdot\} = \frac{1}{s}$ $\rightarrow$ $\mathcal{L}\{u(t-a)\} = \frac{1}{s}e^{-as}$

Ex : Find $\mathcal{L}^{-1}\{e^{-3s}/s^3\}$

Since $\mathcal{L}^{-1}\{\frac{1}{s^3}\} = \frac{t^2}{2!} = \frac{t^2}{2}$

$$\therefore \quad \mathcal{L}^{-1}\{e^{-3s}/s^3\} = \frac{1}{2}(t-3)^2u(t-3) = \begin{cases} 0 & \text{when } t<3 \\ \frac{1}{2}(t-3)^2 & \text{when } t>3 \end{cases}$$

6.4 short impulses, Dirac's delta function, Partial Fractions

- Def. Dirac delta function:( Unit Impulse function)
  The Delta function $\delta(t-t_0)$ is defined (in generalized sense):

$$\boxed{\int_{-\infty}^{\infty}\delta(t-t_0)\cdot g(t)dt = g(t_0)}$$

For every ordinrary function $g(t)$

Ex. $g(t)\equiv t^2$

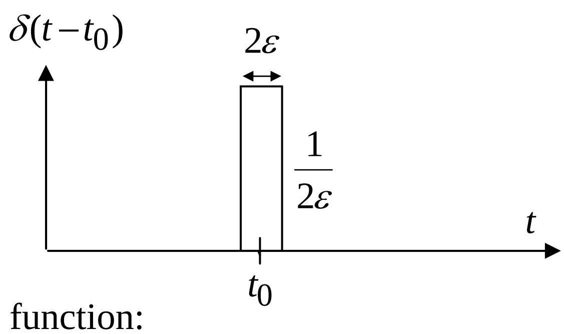
then $\int_{-\infty}^{\infty}\delta(t-3)\cdot g(t)dt = t^2\Big|_{t=3} = 9$

The Delta function can be visualized as:

$$\delta(t-t_0) = \lim_{\varepsilon\to 0}\begin{cases}\dfrac{1}{2\varepsilon} & \text{When } t_0-\varepsilon < t < t_0+\varepsilon \\ 0 & \text{otherwise}\end{cases}$$

Since $\int_{-\infty}^{\infty}\delta(t-t_0)dt = 1$

→ Unit impulse




The Laplace transform of Dirac delta function:

For $t_0 \ge 0$

$$\mathcal{L}\{\delta(t-t_o)\} = \int_{0}^{\infty}\delta(t-t_0)\cdot e^{-st}dt = \int_{-\infty}^{\infty}\delta(t-t_0)\cdot e^{-st}dt$$

$$= e^{-st}\Big|_{t=t_o} = e^{-st_o}$$

when $t_o = 0$ we have $\mathcal{L}\{\delta(t)\} = 1$

Ex: Consider the following two problems:

(i) $m\ddot{y}+\dot{c}y+ky=0$ I.C. $y(0)=0$, $\dot{y}(0)=V_0$

(ii) $m\ddot{y}+\dot{c}y+ky=mV_0\cdot\delta(t)$ I.C. $y(0)=0$, $\dot{y}(0)=0$

Sol:

(i) Take the Transform of the equation (i) , we have:

$$m\{s^2Y-sy(o)-\dot{y}(0)\}-c\{sY-y(0)\}+kY=0$$

$$Y=\frac{mV_0}{ms^2+cs+k} \rightarrow y(t)=mV_0\mathscr{L}^{-1}\{\frac{1}{ms^2+cs+k}\}$$

(ii) Take the Transform of the equation (ii) , we have:

$$m\{s^2Y-sy(o)-\dot{y}(0)\}-c\{sY-y(0)\}+kY=mV_0$$

$$Y=\frac{mV_0}{ms^2+cs+k}$$ → same as (i)

the physical sense of delta function $\delta(t)$ here, is a force such that it can produce a "unit" momentum change of the system at $t=0^+$.

Ex: $y''+y=\delta(t-2\pi)$, I.C. $y(0)=1$, $y'(0)=0$

Take the Laplace transform the equation, we have:

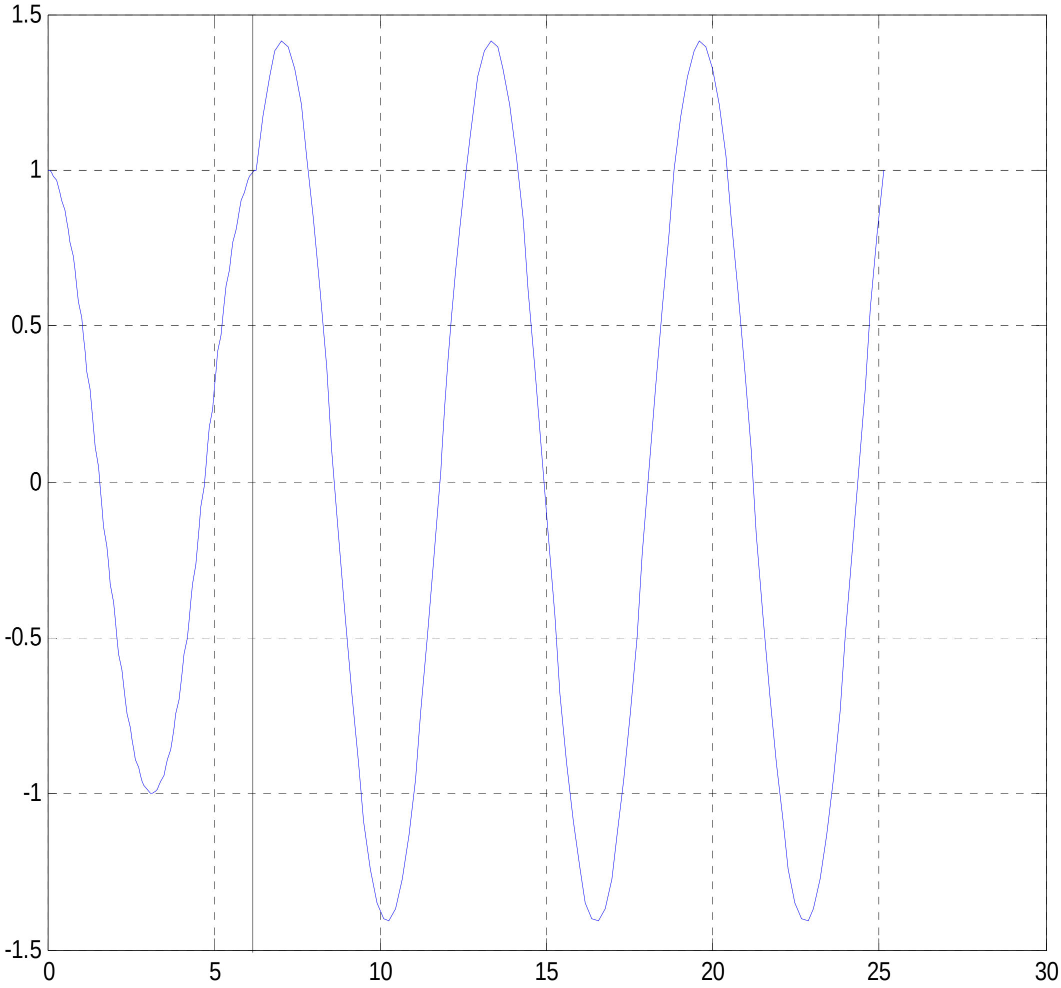
$$s^2Y-s+Y(s)=e^{-2\pi s}$$

$$\Rightarrow Y(s)=\frac{s}{s^2+1}+\frac{1}{s^2+1}e^{-2\pi s}$$

$$\Rightarrow y(t)=\cos t+\sin(t-2\pi)H(t-2\pi)$$

i.e.

$$y(t)=\begin{cases}\cos t & 0\le t\le 2\pi\\ \cos t+\sin(t-2\pi) & t>2\pi\end{cases}$$


1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30

※ Partial Fractions

Consider $Y(s) = \dfrac{F(s)}{G(s)}$ ------------------(1)

If $F(s)$ and $G(s)$ are polynomials of $s$ ,with real codfficients, and $G(s)$ is of higher degree than $F(s)$ .
Then $Y(s)$ can be expressed as a sum of Partial Fraction.

Case 1. When $(s-a)$ is a factor of $G(s)$. i.e. $G(s)$ has the form: $G(s) \equiv (s-a)\cdot r(s)$. Then (1) can be expressed as :

$$Y(s) = \frac{F(s)}{G(s)} = \frac{A}{s-a} + W(s) \text{ ----------------} (i)$$

where $W(s)$ denotes the sum of the partial factions corresponding to all the other factors of $G(s)$.

The coefficient $A$ can be determined as following:

$(s-a)\times(i) \Rightarrow$

$$\frac{F(s)}{G(s)}(s-a) = A + W(s-a) \qquad \text{taking limit } s \to a \text{ we have:}$$

$$A = \lim_{s \to a} \frac{(s-a)F(s)}{G(s)} \quad \text{or}$$

$$A = \lim_{s \to a} \frac{(s-a)F(s)}{G(s)} = F(a) \lim_{s \to a} \frac{(s-a)}{G(s)} = \frac{F(a)}{G'(a)}$$

Ex. $Y(s) = \dfrac{s+1}{s^3 + s^2 - 6s}$ Find $y(t)$

Since $\dfrac{s+1}{s^3 + s^2 - 6s} = \dfrac{s+1}{s(s+3)(s-2)}$

$\therefore Y(s)$ can be expressed as:

$$\frac{s+1}{s^3+s^2-6s}=\frac{A}{s}+\frac{B}{(s+3)}+\frac{C}{(s-2)}$$

since $F(s)\equiv s+1 \quad G(s)\equiv s^3+s^2-6s \quad G'(s)\equiv 3s^2+2s-6$

$$A=\frac{F(0)}{G'(0)}=-\frac{1}{6} \qquad B=\frac{F(-3)}{G'(-3)}=-\frac{2}{15} \qquad C=\frac{F(2)}{G'(2)}=\frac{3}{10}$$

$$\therefore Y(s)=-\frac{1}{6}\frac{1}{s}-\frac{2}{15}\frac{1}{(s+3)}+\frac{3}{10}\frac{1}{(s-2)}$$

$$\Rightarrow y(t)=-\frac{1}{6}-\frac{2}{15}e^{-3t}+\frac{3}{10}e^{2t}$$

Case 2. When $(s-a)^m$ is a factor of $G(s)$. i.e. $G(s)$ has the form: $G(s)\equiv(s-a)^m\cdot r(s)$. Then (1) can be expressed as :

$$Y(s)=\frac{F(s)}{G(s)}=\frac{A_1}{(s-a)}+\frac{A_2}{(s-a)^2}+....+\frac{A_m}{(s-a)^m}+W(s)-----(ii)$$

where $W(s)$ denotes the sum of the partial factions corresponding to all the other factors of $G(s)$.

The coefficient $A_1,A_2,...A_m$ can be determined as following:

$$(s-a)^m\times(ii)\Rightarrow$$

$$\frac{F(s)(s-a)^m}{G(s)}=A_1(s-a)^{m-1}+A_2(s-a)^{m-2}+......$$

$$+A_{m-1}(s-a)+A_m+W(s)(s-a)^m---(A)$$

let $Q_a(s)\equiv\dfrac{F(s)(s-a)^m}{G(s)}$ then (A) →

$$A_m=Q_a(a);\quad A_{m-1}=Q_a'(a);\quad A_{m-2}=\frac{1}{2!}Q_a''(a);\quad ...$$

$$A_1=\frac{1}{(m-1)!}\cdot\frac{d^{m-1}}{ds^{m-1}}Q_a(s)\bigg|_{s=a}$$

Ex. $Y(s)=\dfrac{s^2+2}{s(s+2)(s-4)^2}$ Find $y(t)$

$$\frac{s^2+2}{s(s+2)(s-4)^2}=\frac{A}{s}+\frac{B}{(s+2)}+\frac{C_1}{(s-4)}+\frac{C_2}{(s-4)^2}$$

we have

$$A=\lim_{s\to 0}\frac{(s^2+2)}{(s+2)(s-4)^2}=\frac{1}{16}$$

$$B=\lim_{s\to -2}\frac{(s^2+2)}{s(s-4)^2}=\frac{-1}{12}$$

since

$$Q_4(s)=\frac{s^2+2}{s(s+2)} \quad \Rightarrow$$

$$C_2=\lim_{s\to 4}Q_4(s)=\frac{3}{4}$$

$$C_1=\lim_{s\to 4}Q_4'(s)=\lim_{s\to 4}\frac{2s^2(s+2)-(s^2+2)(2s+2)}{s^2(s+2)^2}=\frac{1}{48}$$

$$\therefore Y(s)=\frac{1}{16}\frac{1}{s}-\frac{1}{12}\frac{1}{(s+2)}+\frac{1}{48}\frac{1}{(s-4)}+\frac{3}{4}\frac{1}{(s-4)^2}$$

$$\therefore y(t)=\frac{1}{16}-\frac{1}{12}e^{-2t}+\frac{1}{48}e^{4t}+\frac{3}{4}t\cdot e^{4t}$$

Case 3. When $(s^2+ps+q)^m$ is a factor of $G(s)$. i.e. $G(s)$ has the form: $G(s)\equiv(s^2+ps+q)^m\cdot r(s)$. Then (1) can be expressed as :

$$Y(s)=\frac{F(s)}{G(s)}=\frac{A_1s+B_1}{(s^2+ps+q)}+\frac{A_2s+B_2}{(s^2+ps+q)^2}+ \\ \dots+\frac{A_ms+B_m}{(s^2+ps+q)^m}+W(s)-----(iii)$$

where $W(s)$ denotes the sum of the partial factions corresponding to all the other factors of $G(s)$.

When $(iii)\times G(s)$ ; since $G(s)\equiv(s^2+ps+q)^m\cdot r(s)$ , we have:

$$F(s)=(A_1s+B_1)(s^2+ps+q)^{m-1}\cdot r(s)+(A_2s+B_2)(s^2+ps+q)^{m-2}\cdot r(s)$$
$$+\dots+(A_{m-1}s+B_{m-1})(s^2+ps+q)\cdot r(s)+(A_ms+B_m)\cdot r(s)$$
$$+(s^2+ps+q)^m\cdot r(s)W(s)------(A)$$

Comparing the coefficient of terms with same degree of $s$, we have $2m$ equations and can solve for $A_i, b_i$. ( $i=1,2\dots m$).

Ex. $Y(s)=\dfrac{2s^2-s}{(s^2+4)^2}$ Find $y(t)$

$$\frac{2s^2-s}{(s^2+4)^2}=\frac{A_1s+B_1}{(s^2+4)}+\frac{A_2s+B_2}{(s^2+4)^2}$$

$$2s^2-s=(A_1s+B_1)(s^2+4)+(A_2s+B_2)$$
$$=A_1s^3+B_1s^2+(4A_1+A_2)s+(4B_1+B_2)$$

$$\left.\begin{aligned}A_1&=0\\ B_1&=0\\ 4A_1+A_22&=-1\\ 4B_1+B_2&=0\end{aligned}\right\}\Rightarrow A_1=0;\quad A_2=-1;\ B_1=2;\ \mathrm{B}_2=-8;$$

$$Y(s)=\frac{2}{(s^2+4)}+\frac{-s-8}{(s^2+4)^2}$$

$$y(t)=\mathcal{L}^{-1}\{\frac{2}{(s^2+4)}\}+\mathcal{L}^{-1}\{\frac{-s}{(s^2+4)^2}\}+\mathcal{L}^{-1}\frac{-8}{(s^2+4)^2}$$

since

$$\mathcal{L}\{\sin 2t\}=\frac{2}{(s^2+4)}\qquad;\mathcal{L}\{t\cdot\sin 2t\}=-\frac{d}{ds}[\frac{2}{(s^2+4)}]=\frac{4s}{(s^2+4)^2}$$

$$\mathcal{L}\{\sin 2t*\sin 2t\}=\frac{4}{(s^2+4)^2}$$

$$\therefore\ y(t)=\sin 2t-\frac{1}{4}t\cdot\sin 2t-2\int_0^t \sin 2\tau\cdot\sin 2(t-\tau)d\tau$$

$$=\sin 2t-\frac{1}{4}t\cdot\sin 2t-\frac{1}{2}\cdot[\sin 2t-2t\cos 2t]$$

$$=\frac{1}{2}\sin 2t-\frac{1}{4}t\cdot\sin 2t+t\cos 2t$$

6.5 convolution

The convolution integral of $f(t)$ and $g(t)$ is defined:

$$(f*g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \equiv \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau \equiv (g*f)(t)$$

if $f(t)$ and $g(t)$ is a function such that

$f(t) = g(t) \equiv 0$ when $t<0$, the convolution integral can be written as:

$$f*g \equiv \int_0^t f(\tau)g(t-\tau)d\tau \equiv \int_0^t f(t-\tau)g(\tau)d\tau = g*f$$

Theorem : if $f(t)$ and $g(t)$ are P.C. and of Exp. Order ,

also $f(t) = g(t) \equiv 0$ when $t<0$

then

$\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$

Proof:

$$\mathcal{L}\{f*g\} = \mathcal{L}\int_0^t f(t-\lambda)g(\lambda)d\lambda = \int_0^\infty [\int_0^t f(t-\lambda)g(\lambda)d\lambda]e^{-st}dt ---(A)$$

$$\because u(t-\lambda) = \begin{cases} 1 & \text{when} \quad \lambda < t \\ 0 & \text{when} \quad \lambda > t \end{cases}$$

$\therefore$ (A) can be rewritten as :

$$\mathcal{L}\{f*g\} = \int_0^\infty [\int_0^\infty f(t-\lambda)g(\lambda)u(t-\lambda)d\lambda]e^{-st}dt$$

$$= \int_0^\infty g(\lambda)\int_0^\infty [f(t-\lambda)u(t-\lambda)e^{-st}dt]d\lambda$$

(let $t-\lambda = \tau$ in the inner integral)

$$= \int_0^\infty g(\lambda)[\int_{-\lambda}^\infty f(\tau)u(\tau)e^{-s(\tau+\lambda)}d\tau]d\lambda$$

$$= \int_0^\infty g(\lambda)e^{-s\lambda}\int_{-\lambda}^\infty [f(\tau)u(\tau)e^{-s\tau}d\tau]d\lambda = \int_0^\infty g(\lambda)e^{-s\lambda}[\int_0^\infty f(\tau)e^{-s\tau}d\tau]d\lambda$$

$$= \int_0^\infty g(\lambda)e^{-s\lambda}d\lambda \cdot \int_0^\infty f(\tau)e^{-s\tau}d\tau = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$$

Ex: Find $\mathcal{L}^{-1}\{\frac{1}{(s^2+k^2)^2}\}$

Let $F(s) \equiv \frac{1}{s^2+k^2}$ $G(s) \equiv \frac{1}{s^2+k^2}$

$$\because \mathcal{L}\{\frac{1}{k}\sin kt\} = \frac{1}{s^2+k^2} = F(s) = G(s)$$

$$\mathcal{L}^{-1}\frac{1}{\left(s^2+k^2\right)^2}$$

$$=\mathcal{L}^{-1}\{F(s)G(s)\} = f * g = \frac{1}{k^2}\int_0^t \sin k\tau \cdot \sin k(t-\tau)d\tau$$

$$= \frac{1}{2k^2}\int_0^t [\cos k(2\tau - t) - \cos kt]d\tau = \frac{1}{2k^2}[\frac{1}{2k}\sin k(2\tau - t) - \tau\cos kt]_0^t$$

$$= \frac{\sin kt - kt\cos kt}{2k^3}$$

Ex: (Differential Equations)

Consider the following two problem:

(i) $m\ddot{y} + c\dot{y} + ky = r(t)$ I.C. $y(0) = \dot{y}(0) = 0$

(ii) $m\ddot{y} + c\dot{y} + ky = \delta(t)$ I.C. $y(0) = \dot{y}(0) = 0$

(i) take the Laplace transform and by the Homogeneous I.C.of (i) ,we have:

$$Y(s) = \frac{R(s)}{ms^2+cs+k} = R(s)\cdot H(s)$$

where $H(s) = \frac{1}{ms^2+cs+k}$ $R(s) \equiv \mathcal{L}\{r(t)\}$

let $h(t) \equiv \mathcal{L}^{-1}\{H(s)\}$

then

$$y(t) = \mathcal{L}^{-1}(Y) = \int_0^t h(t-\tau)\cdot r(\tau)d\tau ----------(A)$$

(ii) take the Laplace transform and by the Homogeneous I.C.of (ii) ,we have

$$Y(s) = \frac{1}{ms^2 + cs + k} = H(s)$$

i.e. the solution of (ii) is $y(t) = h(t) - - - - - - - - - - (B)$

Compare (A) and (B), we conclude that : the response $y(t)$ of (i) due to input $r(t)$, can be computed by the convolution integral of $r(t)$, and $h(t)$, the response of the system due to unit impulse.

→ Duhamel's Formula

Ex: (Integral Equation)

$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)d\tau$$

$$\Rightarrow y(t) = t + y * \sin(t)$$

take the Laplace transform of the equation

$$\Rightarrow Y = \frac{1}{s^2} + Y \cdot \frac{1}{s^2+1}$$

$$\Rightarrow Y(1 - \frac{1}{s^2+1}) = \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$\therefore y(t) = \mathcal{L}^{-1}(Y) = t + \frac{1}{6}t^3$$

6.6 Differentiation and Integration of transforms

Theorem (differentiation of $F(s)$)

If $\mathcal{L}\{f(t)\} = F(s)$

then $\mathcal{L}\{\mathrm{t} \cdot \mathrm{f}\} = -F'(s)$

Proof:

$$\because F(s) = \int_0^\infty f(t)e^{-st}dt \quad \Rightarrow \quad F'(s) = \int_0^\infty f(t)(-t)e^{-st}dt = -\ \mathcal{L}\{\mathrm{t} \cdot \mathrm{f}\}$$

General $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$

Ex:

$$\mathcal{L}\{y\} = \ln\frac{s+1}{s-1} \quad \Rightarrow \quad y(t) = ?$$

$$\because \ \mathcal{L}\{-t \cdot y\} = \frac{d}{ds}(\ln\frac{s+1}{s-1}) = \frac{d}{ds}\{\ln(s+1) - \ln(s-1)\} = \{\frac{1}{s+1} - \frac{1}{s-1}\}$$

$$\therefore \{-t \cdot y\} = \{e^{-t} - e^{t}\} \quad \rightarrow y(t) = \frac{e^{t} - e^{-t}}{t} = \frac{2\sinh t}{t}$$

Theorem ( Integration of $F(s)$)

If $\mathcal{L}\{f(t)\} = F(s)$, and $\dfrac{f(t)}{t}$ exists as $t \to 0$

then $\mathcal{L}\{\dfrac{f(t)}{t}\} = \displaystyle\int_s^\infty F(\bar{s})\, d\bar{s}$

Proof:

$$\int_s^\infty F(\bar{s})d\bar{s} = \int_s^\infty [\int_0^\infty f(t) \cdot e^{-\bar{s}t}dt]d\bar{s} = \int_0^\infty [\int_s^\infty e^{-\bar{s}t} f(t)d\bar{s}]dt = \int_0^\infty f(t)[\int_s^\infty e^{-\bar{s}t}d\bar{s}]dt$$

$$= \int_0^\infty f(t)\frac{e^{-st}}{t}dt = \mathcal{L}\{\frac{f(t)}{t}\}$$

Ex. Find $\mathcal{L}\{\dfrac{\sin \omega t}{t}\}$

(i)By integral theorem: $\mathcal{L}\{\dfrac{f(t)}{t}\} = \displaystyle\int_s^\infty F(\bar{s})d\bar{s}$

let $f(t) \equiv \sin\omega t \quad \because \mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2} = F(s)$

$\therefore \quad \mathcal{L}\{\dfrac{\sin\omega t}{t}\} = \int_s^\infty \dfrac{\omega}{\bar{s}^2+\omega^2} d\bar{s} = \tan^{-1}\dfrac{s}{\omega}\Big|_s^\infty = \dfrac{\pi}{2} - \tan^{-1}\dfrac{s}{\omega}$

since $\dfrac{\pi}{2} - \tan^{-1}\dfrac{s}{\omega} = \tan^{-1}\dfrac{\omega}{s}$

we have $\quad \mathcal{L}\{\dfrac{\sin\omega t}{t}\} = \tan^{-1}\dfrac{\omega}{s}$

(ii) By differential theorem: $\mathcal{L}\{\mathrm{t}\cdot\mathrm{f}\} = -F'(s)$

let $f(t) \equiv \dfrac{\sin wt}{t}$

$\rightarrow \mathcal{L}\{\mathrm{t}\cdot\dfrac{\sin\omega t}{t}\} = -F'(s) \quad \rightarrow -F'(s) = \dfrac{\omega}{s^2+\omega^2}$

$\rightarrow -\int_0^s F'(\bar{s})d\bar{s} = \int_0^s \dfrac{\omega}{\bar{s}^2+\omega^2} d\bar{s}$

$\rightarrow -F(s) + F(0) = \tan^{-1}\dfrac{s}{\omega}$

since $F(0) = \int_0^\infty \dfrac{\sin\omega t}{t} e^{-st} dt \Big|_{s=0} = \int_0^\infty \dfrac{\sin\omega t}{t} dt = \int_0^\infty \dfrac{\sin\tau}{\tau} d\tau = \dfrac{\pi}{2}$

$\therefore F(s) = \dfrac{\pi}{2} - \tan^{-1}\dfrac{s}{\omega} = \tan^{-1}\dfrac{\omega}{s}$

Ex. Differential equation with variable Coefficients

$$xy'' + y' + 4xy = 0 \qquad \text{I.C.} \quad y(0) = 3 \ , \quad y'(0) = 0$$

since

$\mathcal{L}\{xy\} = -\dfrac{d}{ds}\mathcal{L}\{y\} = -Y'$

$\mathcal{L}\{y'\} = sY - y(0)$

$\mathcal{L}\{xy''\} = -\dfrac{d}{ds}\mathcal{L}\{y''\} = -\dfrac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2Y' + y(0)$

Take the Laplace transform of the equation, we have:

$[-2sY - s^2Y' + y(0)] + [sY - y(0)] - 4Y' = 0$

$\Rightarrow (s^2+4)Y'+sY=0 \quad \Rightarrow \quad Y(s)=\dfrac{c}{\sqrt{s^2+4}}$

since $\mathcal{L}\{J_0(ax)\}=\dfrac{1}{\sqrt{a^2+s^2}}$

$\therefore y(x)=\mathcal{L}^{-1}\{\dfrac{c}{\sqrt{s^2+4}}\}=cJ_0(2x)$

$\because y(0)=3 \ \rightarrow\ cJ_0(0)=3 \quad \Rightarrow \quad c=3$ since $J_0(0)=1$

$\therefore y(x)=3J_0(2x)$

also $x\cdot\{xy''+y'+4xy\}=0$ → $x^2y''+xy'+4x^2y=0$

→ $x^2y''+xy'+[(2x)^2-n^2]=0$ with $n=0$

→ the general solution is
$y=AJ_0(2x)+BY_0(2x)$
I.C. $y(0)=3$ , $y'(0)=0$

$$\begin{cases} AJ_0(0)+BY_0(0)=3 \\ 2AJ_0'(0)+BY_0'(0)=0 \end{cases}$$

since $J_0(0)=1$; $Y_0(0)=-\infty$; $J_0'(0)=0$; $Y_0'(0)=\infty$
we have $A=3 \qquad B=0$

→ the solution of the problem is $y=3J_0(2x)$

6.7 System of Differential Equations

Ex. $\begin{cases} \dot{y}_1 = 2y_1 + 3y_2 \\ \dot{y}_2 = 2y_1 + y_2 \end{cases}$ I.C. $\begin{cases} y_1(0) = 4 \\ y_2(0) = 1 \end{cases}$

$$\begin{matrix} sY_1 - 4 = 2Y_1 + 3Y_2 \\ sY_2 - 1 = 2Y_1 + Y_2 \end{matrix} \quad \Rightarrow \quad \begin{matrix} (s-2)Y_1 - 3Y_2 = 4 \\ -2Y_1 + (s-1)Y_2 = 1 \end{matrix}$$

$$\Rightarrow Y_1 = \frac{\begin{vmatrix} 4 & -3 \\ 1 & (s-1) \end{vmatrix}}{\begin{vmatrix} (s-2) & -3 \\ -2 & (s-1) \end{vmatrix}} = \frac{4s-1}{s^2 - 3s - 4} = \frac{4s-1}{(s-4)(s+1)} = \frac{3}{(s-4)} + \frac{1}{(s+1)}$$

$$\Rightarrow y_1 = 3e^{4t} + e^{-t} \quad \text{similarly} \quad \Rightarrow y_2 = 2e^{4t} - e^{-t}$$

Ex.

$$\begin{cases} m_1 y_1'' = -ky_1 + k(y_2 - y_1) \\ m_2 y_2'' = -k(y_2 - y_1) - ky_2 \end{cases}$$

I.C. :

$$\begin{cases} y_1(0) = 1 & y_1'(0) = \sqrt{3k} \\ y_2(0) = 1 & y_2'(0) = -\sqrt{3k} \end{cases},$$

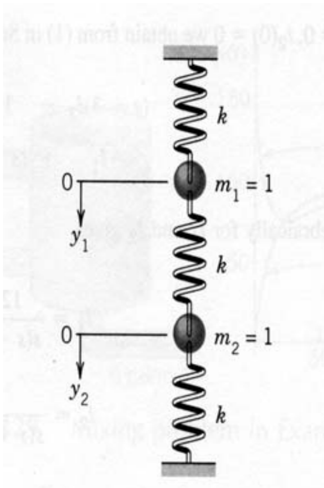


Fig. 131. Example 3

Take the Laplace transform of the equations and by using the I.C., we have:

$$\begin{cases} s^2 Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1) \\ s^2 Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2 \end{cases}$$

$$\Rightarrow \begin{cases} Y_1 = \dfrac{(s + \sqrt{3k})(s^2 + 2k) + k(s - 3k)}{(s^2 + 2k)^2 - k^2} = \dfrac{s}{s^2 + k} + \dfrac{\sqrt{3k}}{s^2 + 3k} \\ Y_2 = \dfrac{(s + 2k)(s^2 - \sqrt{3k}) + k(s + \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \dfrac{s}{s^2 + k} - \dfrac{\sqrt{3k}}{s^2 + 3k} \end{cases}$$

$$\therefore y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos\sqrt{k}t + \sin\sqrt{3k}t$$

$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos\sqrt{k}t - \sin\sqrt{3k}t$$

6.8 Periodic function

Theorem. The Laplace transform of a piecewise continuous periodic function $f(t)$ with period $p$ is ;

$$\mathcal{L}\{f\} = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)dt$$

Proof:

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)dt = \int_0^p e^{-st} f(t)dt + \int_p^{2p} e^{-st} f(t)dt + \cdots\cdots$$

$\because f(t)$ is periodic with period $p$ .i.e.

$$f(t) = f(t+p) = f(t+2p) = ... = f(t+np)$$

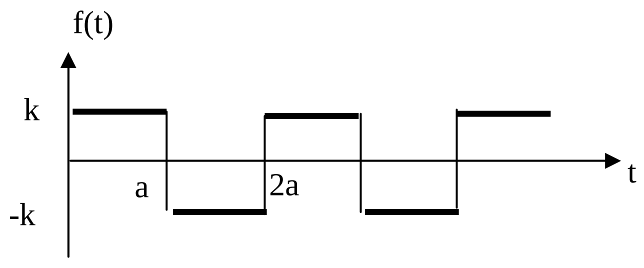
set $t = \tau + p,\ t = \tau + 2p \cdots\cdots$ in the integral, we have

$$\mathcal{L}(f) = \int_0^p e^{-s\tau} f(\tau)d\tau + \int_0^p e^{-s(\tau+p)} f(\tau)d\tau + \int_0^p e^{-s(\tau+2p)} f(\tau)d\tau + \cdots$$

$$= [1 + e^{-sp} + e^{-2sp} + \cdots]\int_0^p e^{-s\tau} f(\tau)d\tau$$

$$= \frac{1}{1-e^{-sp}} \int_0^p e^{-s\tau} f(\tau)d\tau$$

Ex. $f(t) = \begin{cases} k & \text{when } 0 < t < a \\ -k & \text{when } a < t < 2a \end{cases}$ and $f(t) = f(t+2a)$




$$\mathcal{L}\{f\} = \frac{1}{1-e^{-2as}} \int_0^{2a} f(t)e^{-st}dt = \frac{1}{1-e^{-2as}}[\int_0^a k \cdot e^{-st}dt + \int_a^{2a} (-k)e^{-st}dt]$$

$$= \frac{k}{s}\frac{1-e^{-as}}{1+e^{-as}} = \frac{k}{s}\tanh(\frac{as}{2})$$