Chapter 5
Stresses in Beams

5.1 INTRODUCTION

(1)Deflection curve
(2)Coordinate axes
(3) Plane of bending

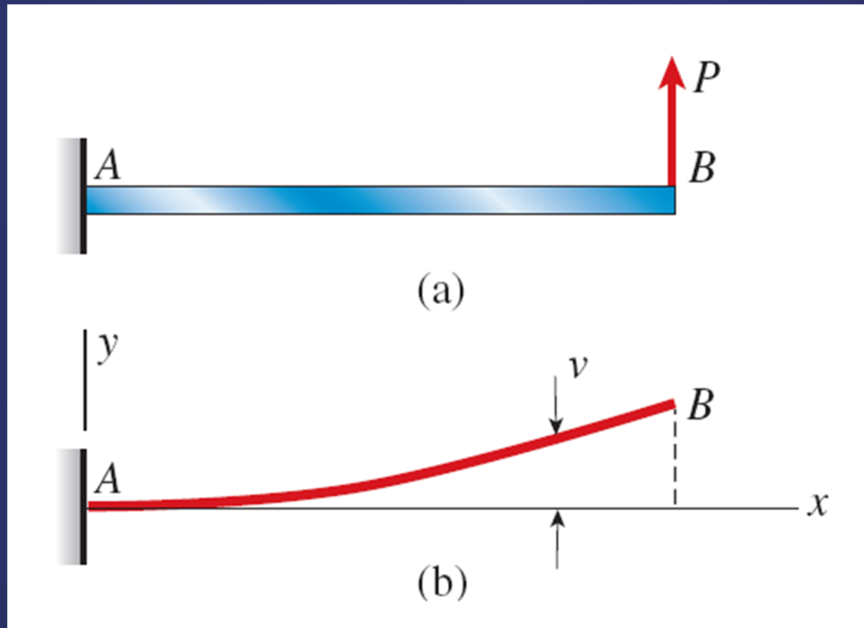


FIG. 5-1 Bending of a cantilever beam: (a) beam with load, and (b) deflection curve

5.2 PURE BENDING AND NONUNIFORM BENDING

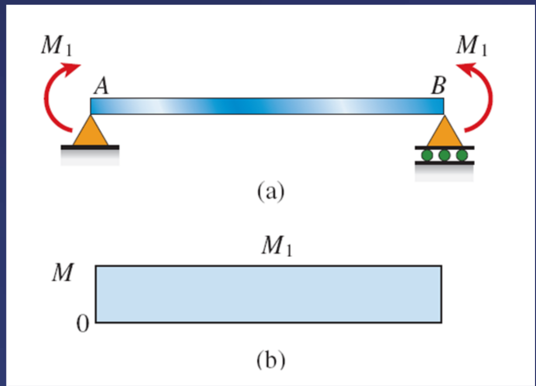


FIG. 5-2 Simple beam in pure bending (*M* *M*1)

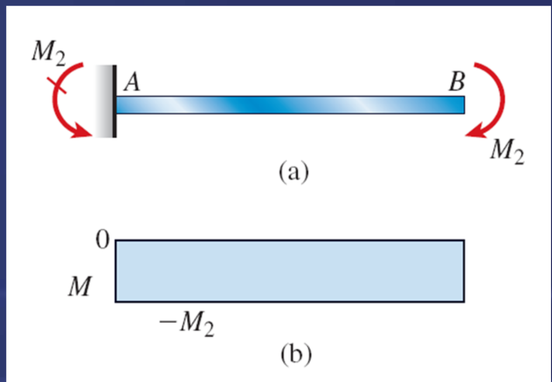


FIG. 5-3 Cantilever beam in pure bending (*M* *M*2)

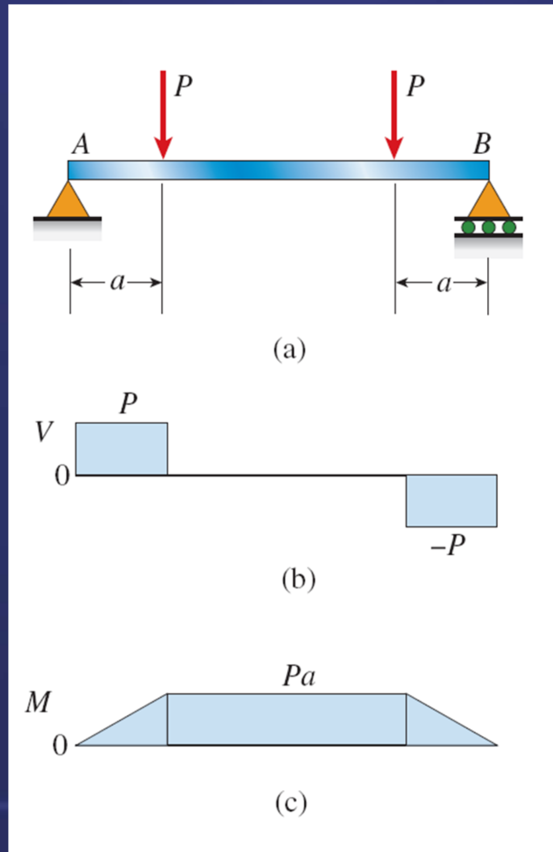


FIG. 5-4 Simple beam with central region in pure bending and end regions in nonuniform bending

(1) **Pure bending** refers to flexure of a beam under a constant bending moment.

(2) **Nonuniform bending** refers to flexure in the presence of shear forces

(1)Center of curvature
(2)Radius of curvature ρ
(3)Curvature *κ*

$$\kappa = \frac{1}{\rho} \qquad (5\text{-}1)$$

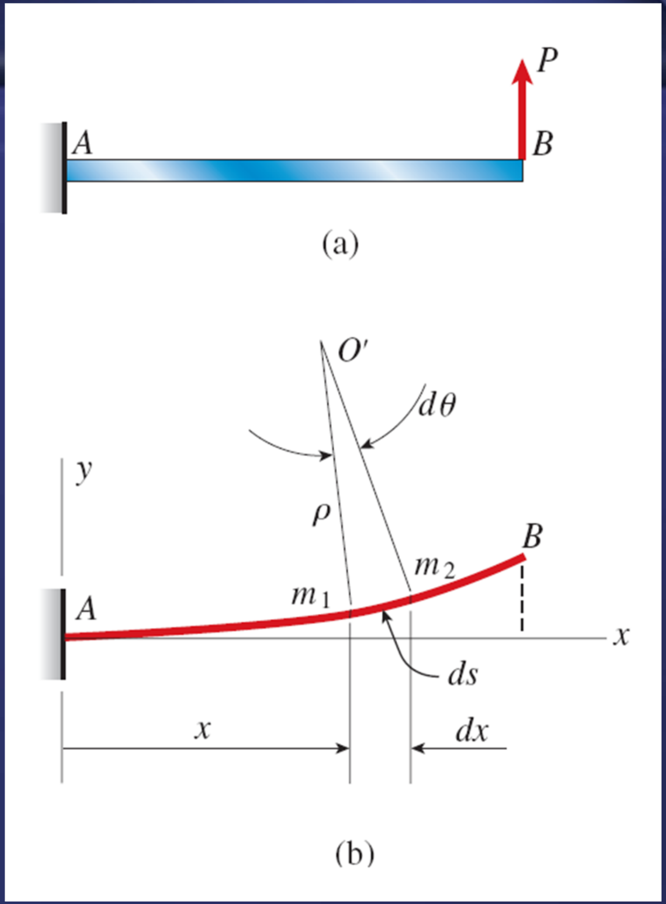


FIG. 5-5 Curvature of a bent beam:
(a) beam with load, and
(b) deflection curve

From the geometry of triangle $O'm_1m_2$ (Fig. 5-5b) we obtain

$$\rho d\theta = ds \qquad (a)$$

Combining Eq. (a) with Eq. (5-1), we get

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds} \qquad (5\text{-}2)$$

Sign convention for curvature

(1) When the beam is bent concave upward, the curvature is positive.
(2) When the beam is bent concave downward, the curvature is negative.

5.4 LONGITUDINAL STRAINS IN BEAMS

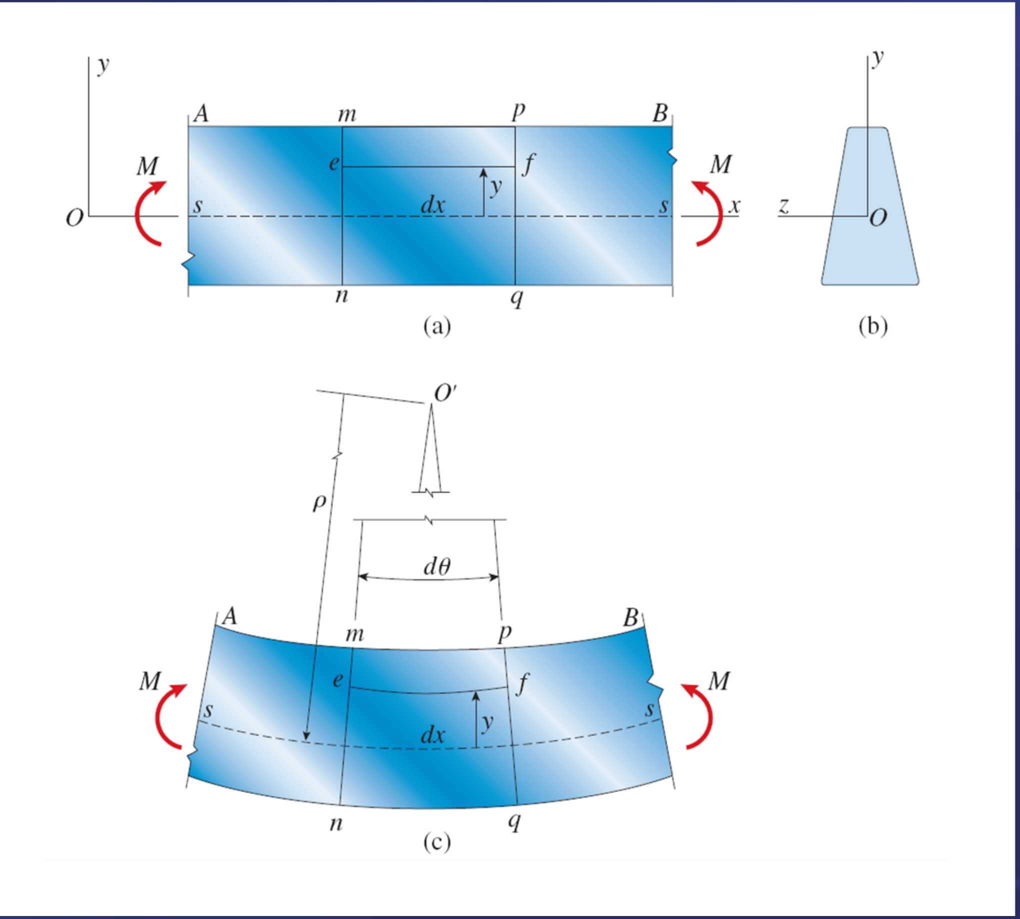


FIG. 5-7 Deformations of a beam in pure bending: (a) side view of beam, (b) cross section of beam, and (c) deformed beam

The length L_1 of line *ef* after bending takes place is

$$L_1 = (\rho - y)\,d\theta = dx - \frac{y}{\rho}\,dx$$

The corresponding longitudinal strain is

$$\epsilon_x = -\frac{y}{\rho} = -\kappa y \qquad (5\text{-}4)$$

where κ is the curvature

5.5 NORMAL STRESSES IN BEAMS (LINEARLY ELASTIC MATERIALS)

Hooke's law for uniaxial stress, we obtain

$$\sigma_x = E \in_x = -\frac{Ey}{\rho} = -E\kappa y \qquad (5\text{-}7)$$

Location of Neutral Axis

The force acting on the element is

$$\int_A \sigma_x dA = -\int_A E\kappa y dA = 0 \qquad \text{(a)}$$

We can drop them from the equation and obtain

$$\int_A y dA = 0 \qquad \text{(5-8)}$$

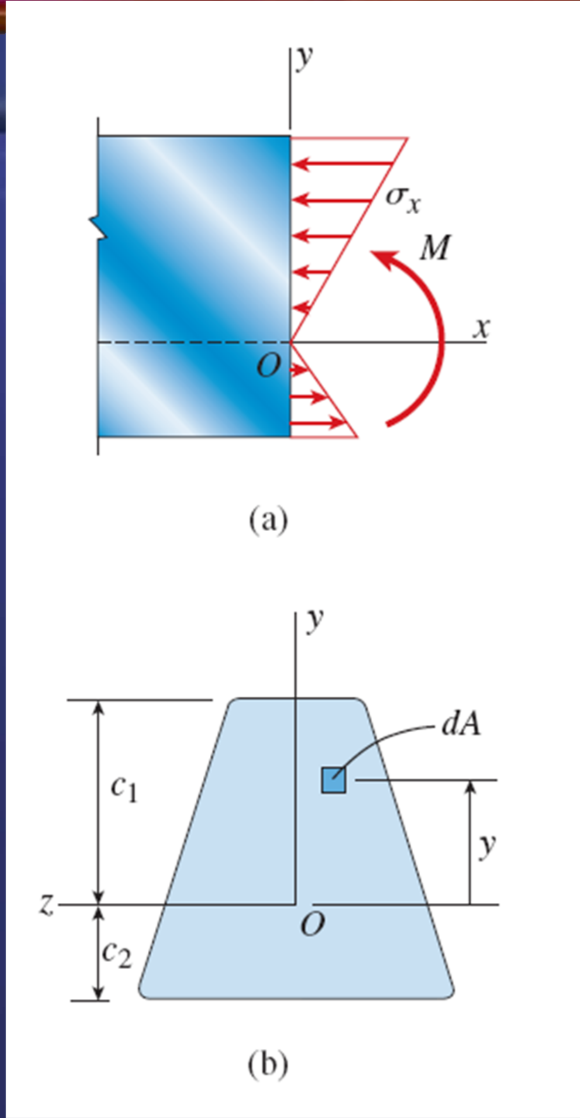


FIG. 5-9 Normal stresses in a beam of linearly elastic material: (a) side view of beam showing distribution of normal stresses, and (b) cross section of beam showing the z

Conclusion

The neutral axis passes through the centroid of the cross-sectional area when the material follows Hooke's law and there is no axial force acting on the cross section.

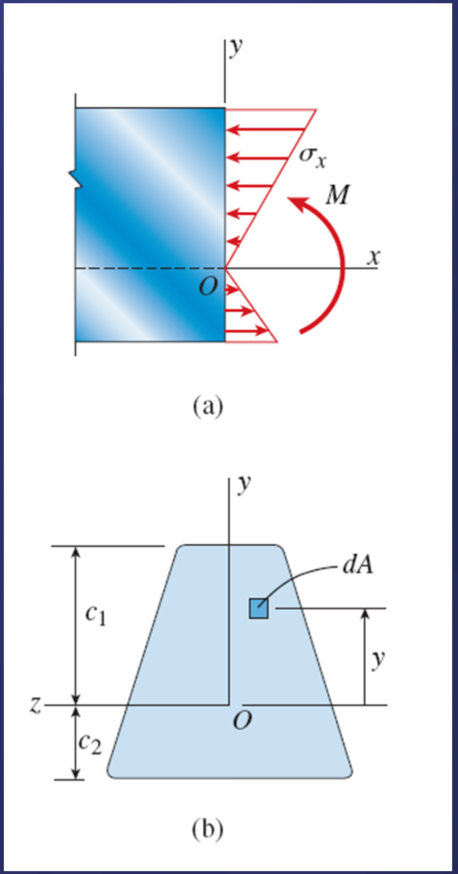


FIG. 5-9

Moment-Curvature Relationship

The increment *dM* in the bending moment is

$$dM = -\sigma_x y dA$$

The integral of all such elemental moments over the entire cross-sectional area *A must equal the bending moment:*

$$M = -\int_A \sigma_x y dA \qquad \text{(b)}$$

or, upon substituting for σ_x *from Eq. (5-7),*

$$M = \int_A \kappa E y^2 dA = \kappa E \int_A y^2 dA \qquad \text{(5-9)}$$

or

$$M = \kappa EI \qquad (5\text{-}10)$$

In which

$$I = \int_A y^2 dA \qquad (5\text{-}11)$$

The *curvature in* terms of the bending moment in the beam is

$$\kappa = \frac{1}{\rho} = \frac{M}{EI} \qquad (5\text{-}12)$$

Known as the **moment-curvature equation**, *EI* is called the **flexural rigidity** of the beam

Conclusion

Positive bending moment produces positive curvature and a negative bending moment produces negative curvature (see Fig. 5-10).

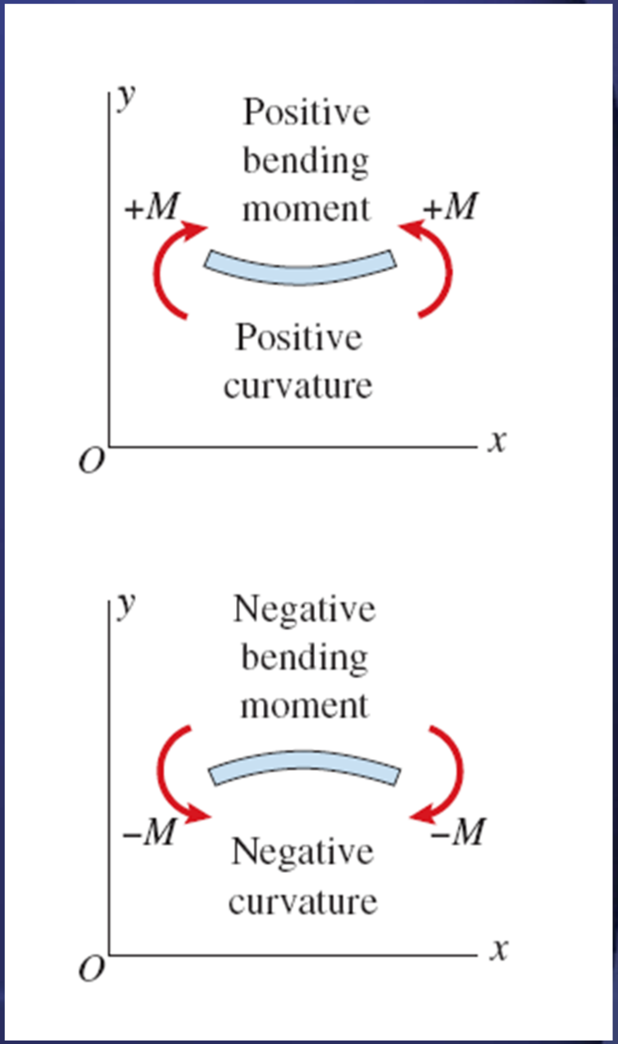


FIG. 5-10 Relationships between signs of bending moments and signs of curvatures

Flexure Formula

The expression for the stress σ_x is

$$\sigma_x = -\frac{My}{I} \qquad (5\text{-}13)$$

This equation called the **flexure formula**.

Stresses calculated from the flexure formula are called **bending stresses** or **flexural stresses**.

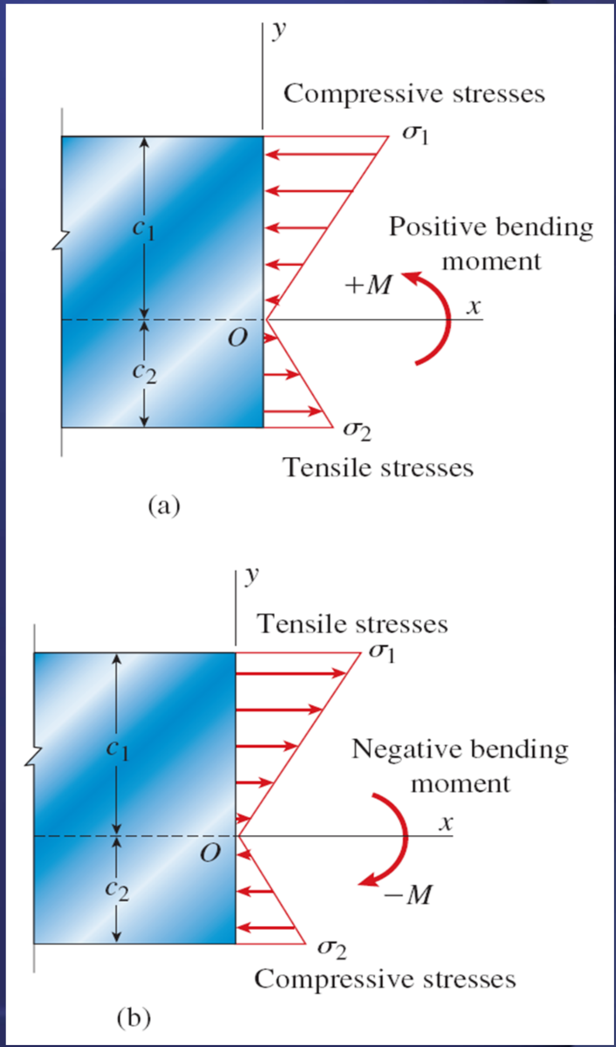


FIG. 5-11

Maximum Stresses at a Cross Section

Then the corresponding **maximum normal stresses** *σ_1 and σ_2 (from the flexure formula) are*

$$\sigma_1 = -\frac{Mc_1}{I} = -\frac{M}{S_1} \qquad \sigma_2 = -\frac{Mc_2}{I} = -\frac{M}{S_2} \qquad \text{(5-14a,b)}$$

in which

$$S_1 = \frac{I}{c_1} \qquad S_2 = \frac{I}{c_2} \qquad \text{(5-15a,b)}$$

The quantities S_1 and S_2 are known as the **section moduli** of the **crosssectional** area.

Doubly Symmetric Shapes

If the cross section of a beam is symmetric with respect to the *z axis as* well as the *y axis* (*doubly symmetric cross section*),*then*

$$\sigma_1 = -\sigma_2 = -\frac{Mc}{I} = -\frac{M}{S} \qquad \text{(5-16a,b)}$$

in which

$$S = \frac{I}{c} \qquad \text{(5-17)}$$

For a beam **of rectangular cross section** with width *b and height h* (Fig. 5-12a), the moment of inertia and section modulus are

$$I = \frac{bh^3}{12} \qquad S = \frac{bh^2}{6} \qquad \text{(5-18a,b)}$$

For a **circular cross section** of diameter *d (Fig. 5-12b), these properties are*

$$I = \frac{\pi d^4}{64} \qquad S = \frac{\pi d^3}{32} \qquad \text{(5-19a,b)}$$

5.6 DESIGN OF BEAMS FOR BENDING STRESSES

FIG. 5-17 Welder fabricating a large wide flange steel beam (Courtesy of AISC)

$$S = \frac{M_{\max}}{\sigma_{\text{allow}}} \qquad (5\text{-}24)$$

Example 5-6

A vertical post 2.5-meters high must support a lateral load *P =12 kN at its upper* end (Fig. 5-20). Two plans are proposed—a solid wood post and a hollow aluminum tube.

(a) What is the minimum required diameter d_1 *of the wood post if the allowable* bending stress in the wood is 15 MPa?

(b) What is the minimum required outer diameter d_2 *of the aluminum tube* if its wall thickness is to be one-eighth of the outer diameter and the allowable bending stress in the aluminum is 50 MPa?

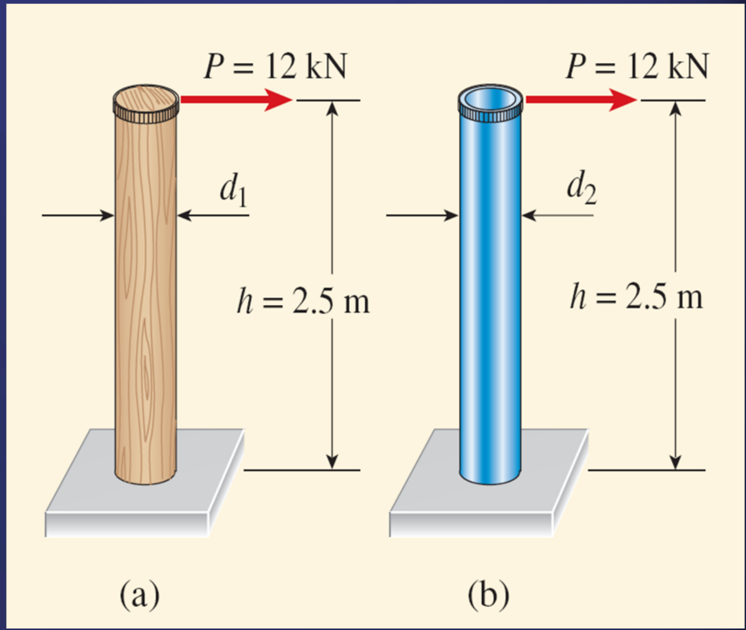


FIG. 5-20 Example 5-6. (a) Solid wood post, and (b) aluminum tube

Solution

Maximum bending moment. The maximum moment is

$$M_{\max} = Ph = (12\ \text{kN})(2.5\ \text{m}) = 30\ \text{kN}\cdot\text{m}$$

(a) Wood post.

$$S_1 = \frac{\pi d_1^3}{32} = \frac{M_{\max}}{\sigma_{\text{allow}}} = \frac{30\ \text{kN}\cdot\text{m}}{15\ \text{MPa}} = 0.0020\ \text{m}^3 = 2\times10^6\ \text{mm}^3$$

we get

$$d_1 = 273\ \text{mm}$$

(b) Aluminum tube.

$$I_2 = \frac{\pi}{64}\left[d_2{}^4 - \left(0.75d_2\right)^4\right] = 0.03356d_2{}^4$$

The section modulus of the tube is now obtained as follows

$$S_2 = \frac{I_2}{c} = \frac{0.03356d_2{}^4}{d_2/2} = 0.06712d_2{}^3$$

The required section modulus is obtained from Eq. (5-24)

$$S_2 = \frac{M_{\max}}{\sigma_{\text{allow}}} = \frac{30\ \text{kN}\cdot\text{m}}{50\ \text{MPa}} = 0.0006\ \text{m}^3 = 600\times10^3\ \text{mm}^3$$

The required outer diameter is

$$d_2 = \left(\frac{600\times10^3\ \text{mm}^3}{0.06712}\right)^{1/3} = 208\ \text{mm}$$

The corresponding inner diameter is 0.75(208 mm), or 156 mm.

5.7 SHEAR STRESSES IN BEAMS OF RECTANGULAR CROSS SECTION

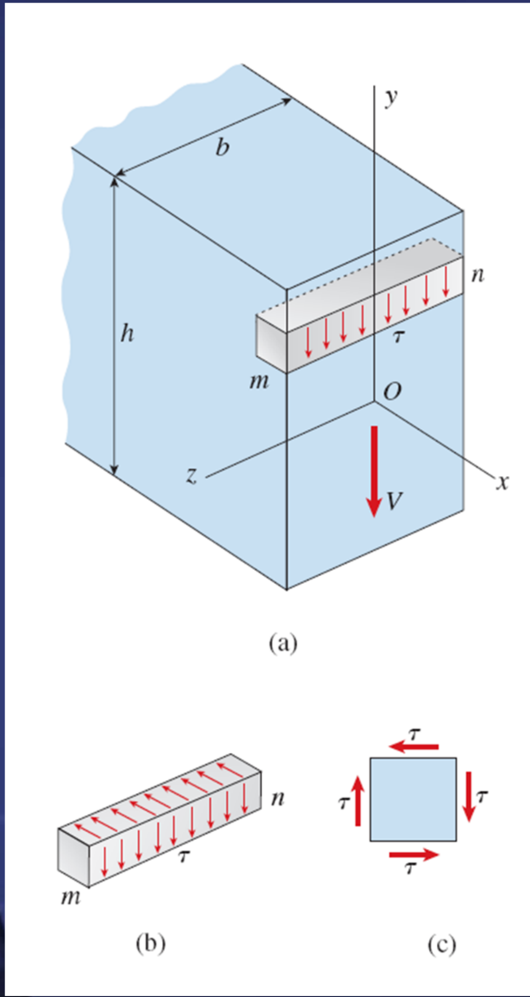


FIG. 5-23

Non-uniform bending

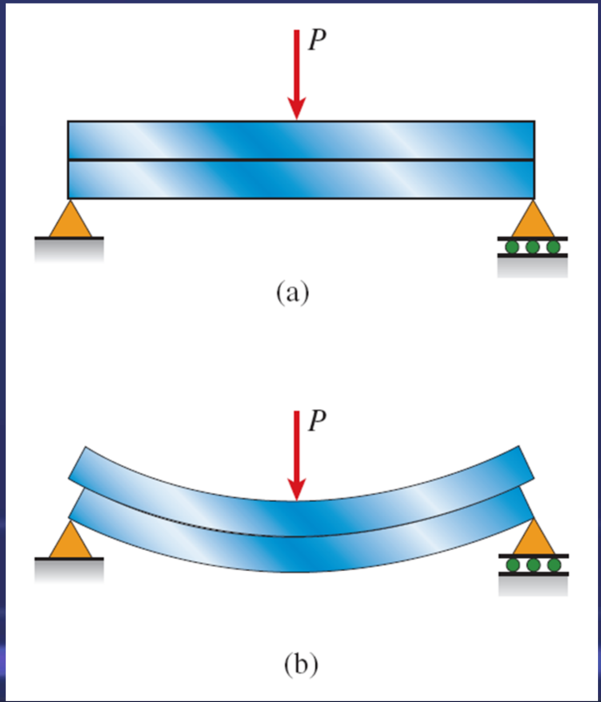


FIG. 5-24

Derivation of Shear Formula

On cross sections *mn and* m_1n_1 *the normal* stresses are, respectively,

$$\sigma_1 = -\frac{My}{I}$$

and

$$\sigma_2 = -\frac{(M + dM)\,y}{I}$$

(a,b)

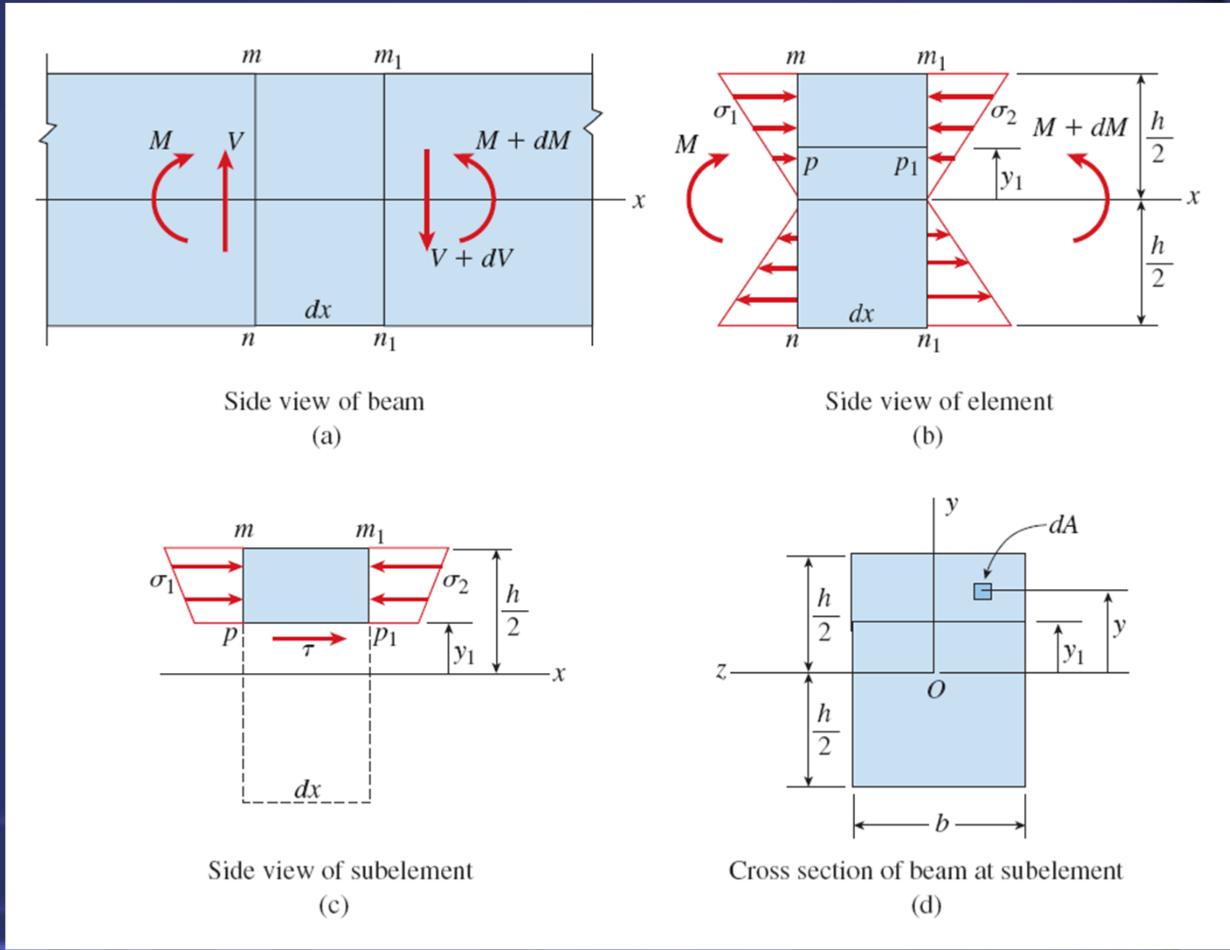


FIG. 5-25 Shear stresses in a beam of rectangular cross section

The element of force is

$$\sigma_1 dA = \frac{My}{I} dA$$

Summing these elements of force over the area of face *mp of the subelement (Fig. 5-28c)* gives the total horizontal force F_1 *acting on that face:*

$$F_1 = \int \sigma_1 dA = \int \frac{My}{I} dA \qquad \text{(c)}$$

In a similar manner, we find that the total force F_2 *acting on the* right-hand face m_1p_1 *of the subelement (Fig. 5-26 and Fig. 5-25c) is*

$$F_2 = \int \sigma_2 dA = \int \frac{(M + dM)y}{I} dA \qquad \text{(d)}$$

We sum forces in the *x* direction and obtain

$$F_3 = F_2 - F_1 \qquad \text{(e)}$$

The expression for the force F_3 becomes

$$F_3 = \frac{dM}{I} \int y dA \qquad \text{(5-30)}$$

If the shear stresses *τ are uniformly distributed across the width b of the* beam, the force *F_3 is also equal to the following:*

$$F_3 = \tau b dx \qquad (5\text{-}31)$$

The shear stress *τ, is*

$$\tau = \frac{dM}{dx}\left(\frac{1}{Ib}\right)\int y dA \qquad (5\text{-}32)$$

or

$$\tau = \frac{V}{Ib}\int y dA \qquad (5\text{-}33)$$

The integral is the first moment of the cross-sectional area above the level at which the shear stress τ is being evaluated. This first moment is usually denoted by the symbol *Q:*

$$Q = \int y\,dA \tag{5-34}$$

The shear stress becomes

$$\tau = \frac{VQ}{Ib} \tag{5-35}$$

Distribution of Shear Stresses in a Rectangular Beam

The first moment *Q of the* shaded part of the cross-sectional area is obtained by multiplying the area by the distance from its own centroid to the neutral axis:

$$Q = b\left(\frac{h}{2} - y_1\right)\left(y_1 + \frac{h/2 - y_1}{2}\right) = \frac{b}{2}\left(\frac{h^2}{4} - y_1{}^2\right)$$

(f)

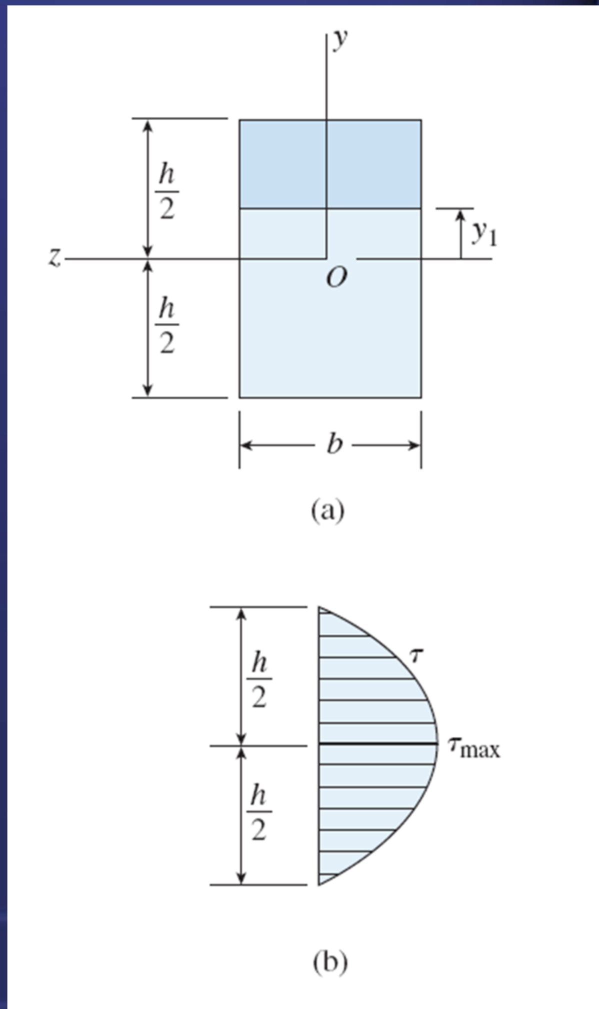


FIG. 5-27

The result is

$$Q = \int y \, dA = \int_{y_1}^{h/2} yb \, dy = \frac{b}{2}\left(\frac{h^2}{4} - y_1^{\ 2}\right) \quad \text{(g)}$$

The shear stress is

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^{\ 2}\right) \quad \text{(5-36)}$$

The maximum value of the shear stress occurs at the neutral axis is

$$\tau_{\max} = \frac{Vh^2}{8I} = \frac{3V}{2A} \quad \text{(5-37)}$$

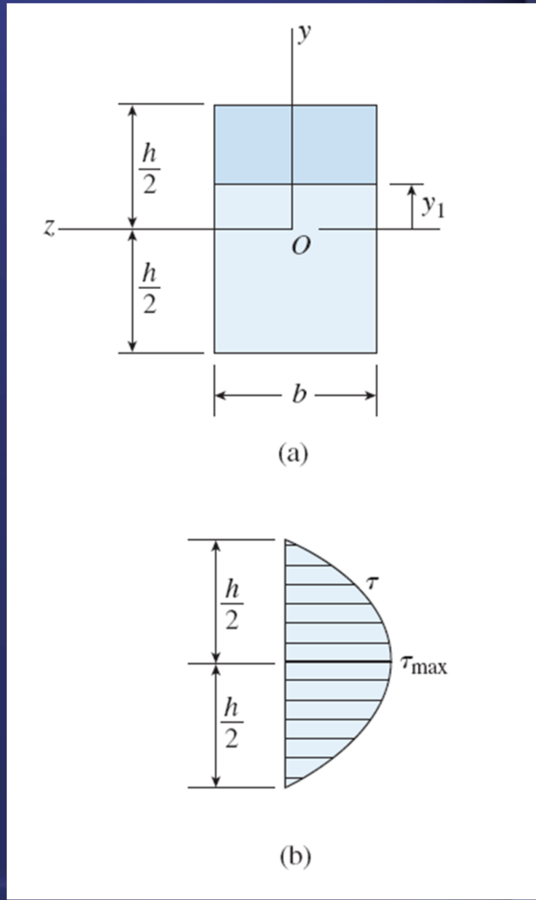


FIG. 5-27

Example 5-10

A wood beam *AB supporting two concentrated loads P (Fig. 5-30a) has a* rectangular cross section of width b = *100 mm and height* h = *150 mm* (Fig. 5-30b). The distance from each end of the beam to the nearest load is $a = 0.5$ *m.*

Determine the maximum permissible value P_{max} *of the loads if the allowable* stress in bending is $\sigma_{allow} = 11$ MPa *(for both tension and compression) and* the allowable stress in horizontal shear is $\tau_{allow} = 1.2$ MPa. *(Disregard the* weight of the beam itself.)

Note: Wood beams are much weaker in horizontal shear (shear parallel to the longitudinal fibers in the wood) than in *cross-grain shear (shear on the cross* sections). Consequently, the allowable stress in horizontal shear is usually considered in design.

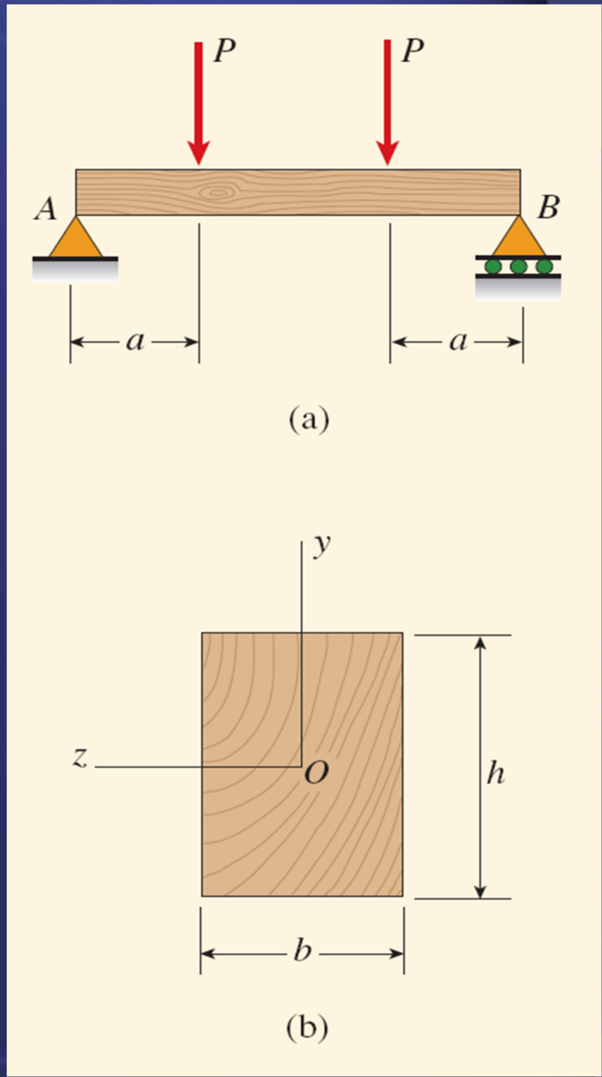


FIG. 5-30 Example 5-10. Wood beam with concentrated loads

Solution

The maximum shear force and the maximum bending moment are

$$V_{\max} = P \qquad M_{\max} = Pa$$

The section modulus *S and cross-sectional area A are*

$$S = \frac{bh^2}{6} \qquad A = bh$$

The maximum normal and shear stresses in the beam are

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{6Pa}{bh^2} \qquad \tau_{\max} = \frac{3V_{\max}}{2A} = \frac{3P}{2bh}$$

The maximum permissible values of the load *P in bending and shear,* respectively, are

$$P_{\text{bending}} = \frac{\sigma_{\text{allow}} bh^2}{6a} \qquad P_{\text{shear}} = \frac{2\tau_{\text{allow}} bh}{3}$$

Substituting numerical values into these formulas, we get

$$P_{\text{bending}} = \frac{(11\ \text{MPa})(100\ \text{mm})(150\ \text{mm})^2}{6(0.5\ \text{m})} = 8.25\ \text{kN}$$

$$P_{\text{shear}} = \frac{2(1.2\ \text{MPa})(100\ \text{mm})(150\ \text{mm})}{3} = 12.0\ \text{kN}$$

The bending stress governs the design, and the maximum permissible load is

$$P_{\max} = 8.25\ \text{kN}$$

5.8 SHEAR STRESSES IN BEAMS OF CIRCULAR CROSS SECTION

For use in the shear formula, we need the following properties pertaining to a circular cross section having radius *r:*

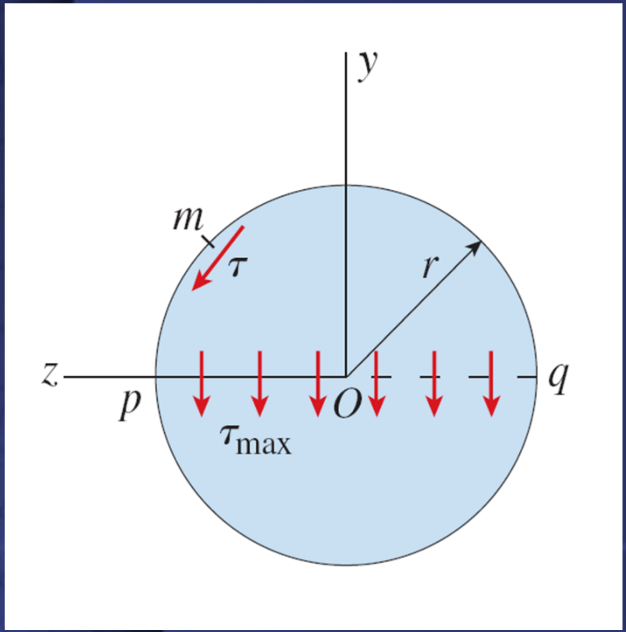


FIG. 5-31 Shear stresses acting on the cross section of a circular beam

$$I = \frac{\pi r^4}{4}$$

$$Q = A\bar{y} = \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right) = \frac{2r^3}{3}$$

$$b = 2r$$

(5-38a,b)

Substituting these expressions into the shear formula, we obtain

$$\tau_{\max} = \frac{VQ}{Ib} = \frac{V\left(2r^3/3\right)}{\left(\pi r^4/4\right)\left(2r\right)} = \frac{4V}{3\pi r^2} = \frac{4V}{3A} \qquad \text{(5-39)}$$

If a beam has a **hollow circular cross section.**
The required properties for a hollow circular section are

$$I = \frac{\pi}{4}\left(r_2^4 - r_1^4\right) \qquad Q = \frac{2}{3}\left(r_2^3 - r_1^3\right) \qquad b = 2\left(r_2 - r_1\right)$$

(5-40a,b,c)

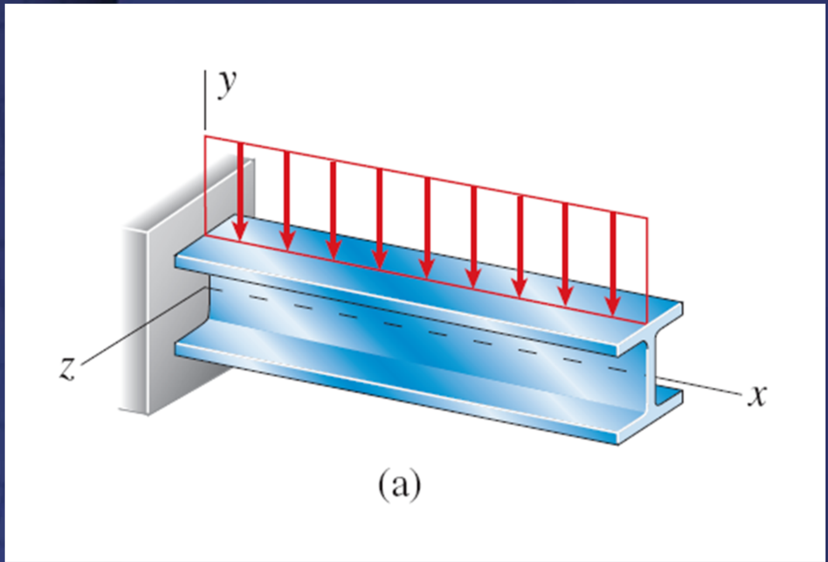
The maximum stress is

$$\tau_{\max} = \frac{VQ}{Ib} = \frac{4V}{3A}\left(\frac{r_2^{\,2} + r_2 r_1 + r_1^{\,2}}{r_2^{\,2} + r_1^{\,2}}\right) \qquad (5\text{-}41)$$

in which

$$A = \pi\left(r_2^{\,2} - r_1^{\,2}\right)$$

5.9 SHEAR STRESSES IN THE WEBS OF BEAMS WITH FLANGES



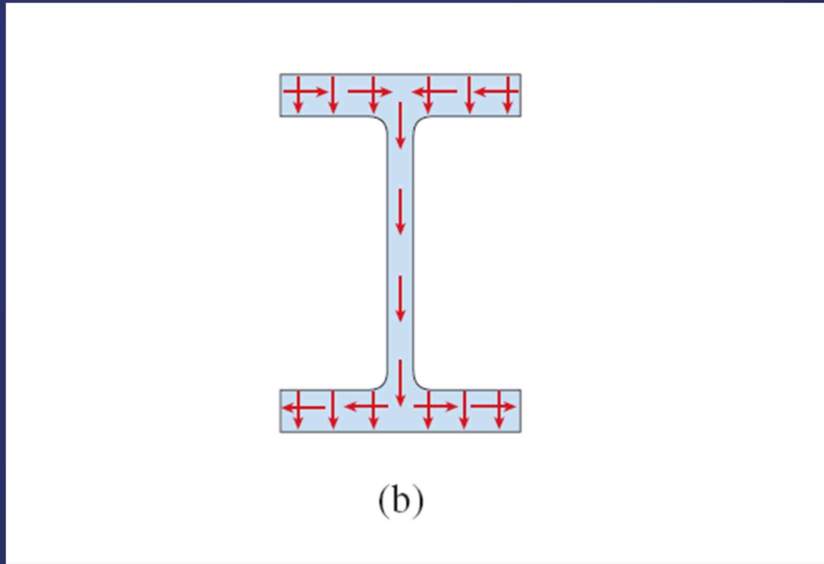


FIG. 5-34 (a) Beam of wide-flange shape, and (b) directions of the shear stresses acting on a cross section

Shear Stresses in the Web

The first rectangle is the upper flange itself, which has area

$$A_1 = b\left(\frac{h}{2} - \frac{h_1}{2}\right) \qquad \text{(a)}$$

The second rectangle is the part of the web between *ef and the flange, that is, rectangle efcb, which has area*

$$A_2 = t\left(\frac{h_1}{2} - y_1\right) \qquad \text{(b)}$$

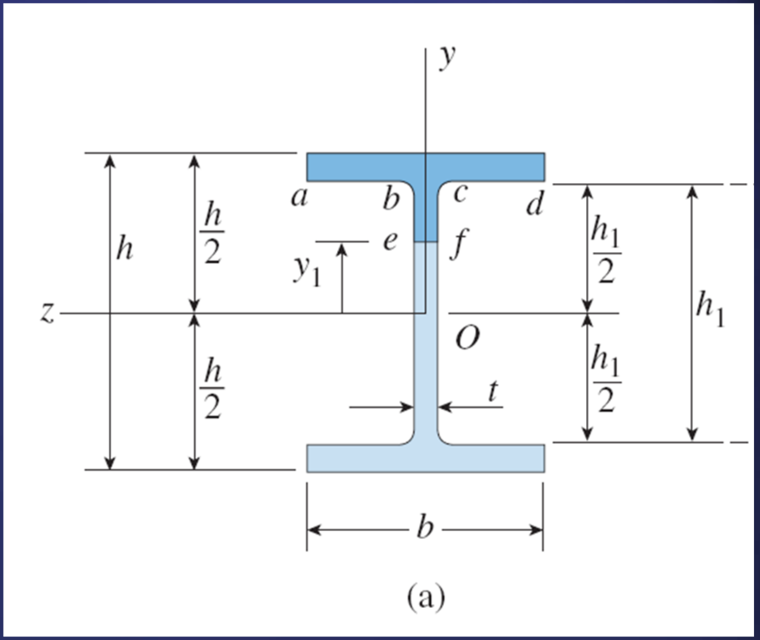


FIG. 5-35

The first moments gives the first moment *Q of the combined area:*

$$Q = A_1\left(\frac{h_1}{2} + \frac{h/2 - h_1/2}{2}\right) + A_2\left(y_1 + \frac{h_1/2 - y_1}{2}\right)$$

Upon substituting for A_1 *and* A_2 *from Eqs. (a) and (b) and then simplifying,*we get

$$Q = \frac{b}{8}\left(h^2 - h_1^2\right) + \frac{t}{8}\left(h_1^2 - 4y_1^2\right)$$

(5-42)

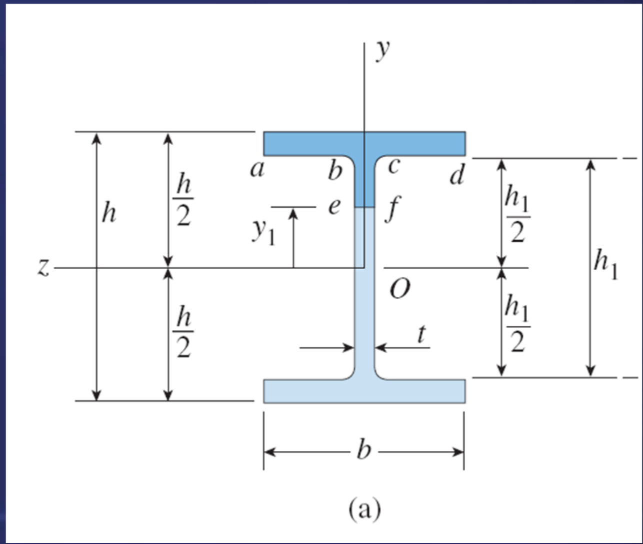


FIG. 5-35

Therefore, the shear stress *τ in the web of the beam at distance y_1 from* the neutral axis is

$$\tau = \frac{VQ}{It} = \frac{V}{8It}\left[b\left(h^2 - h_1^2\right) + t\left(h_1^2 - 4y_1^2\right)\right] \tag{5-43}$$

In which the moment of inertia of the cross section is

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}\left(bh^3 - bh_1^3 + th_1^3\right) \tag{5-44}$$

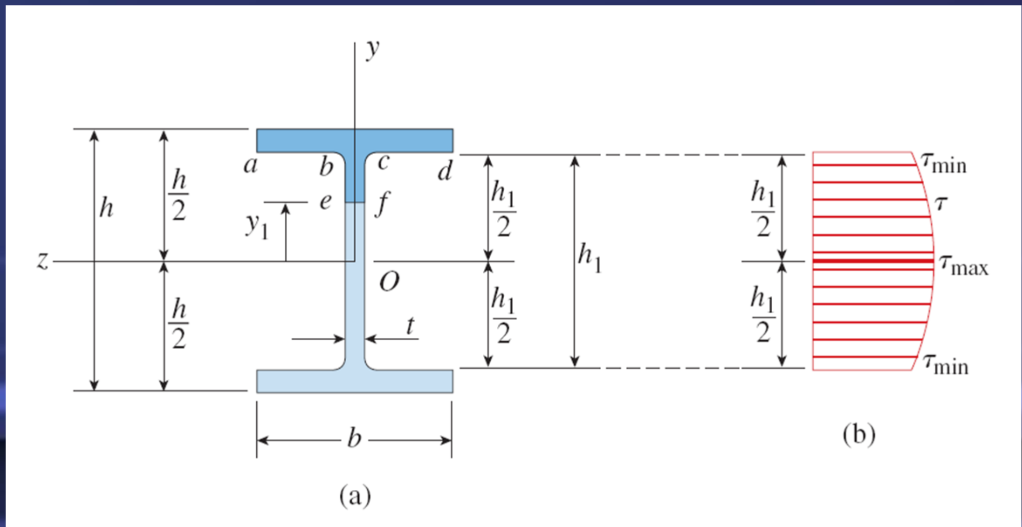


FIG. 5-35

The End of Chap. 5