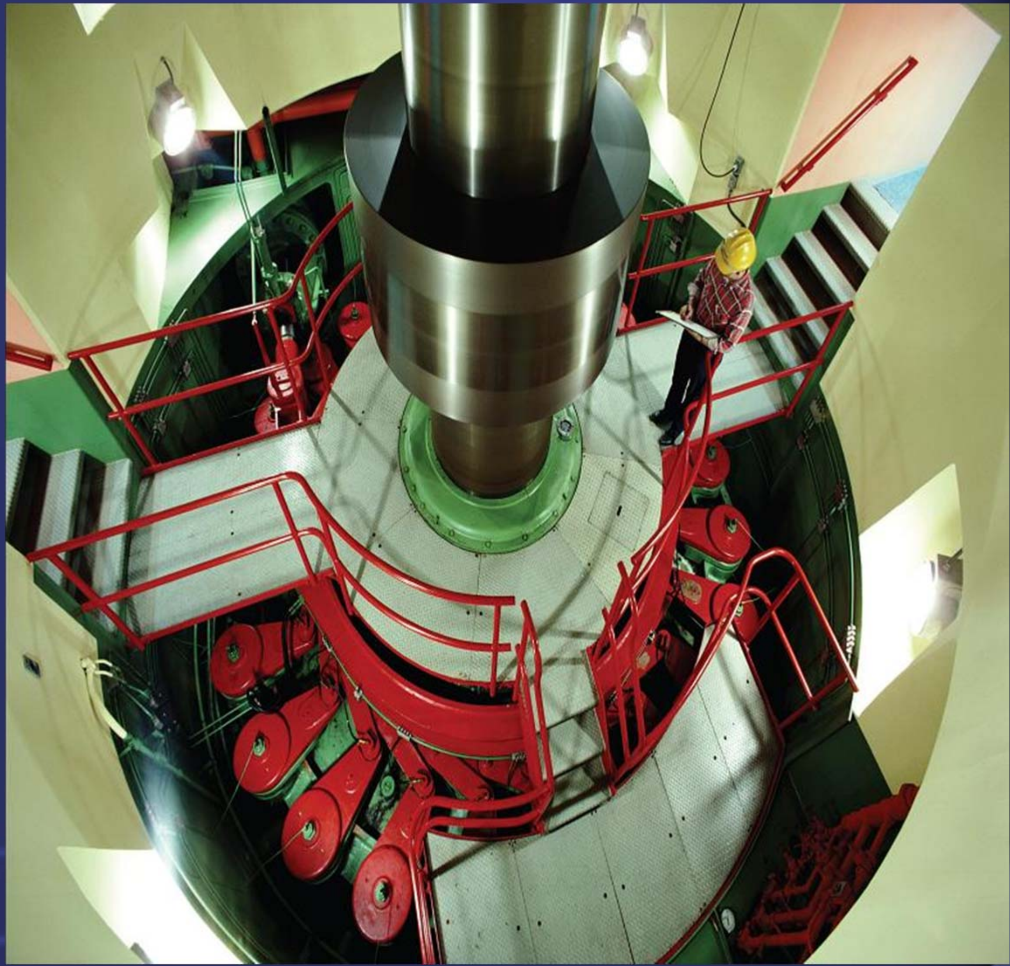
Chapter 3
Torsion



3.1 INTRODUCTION

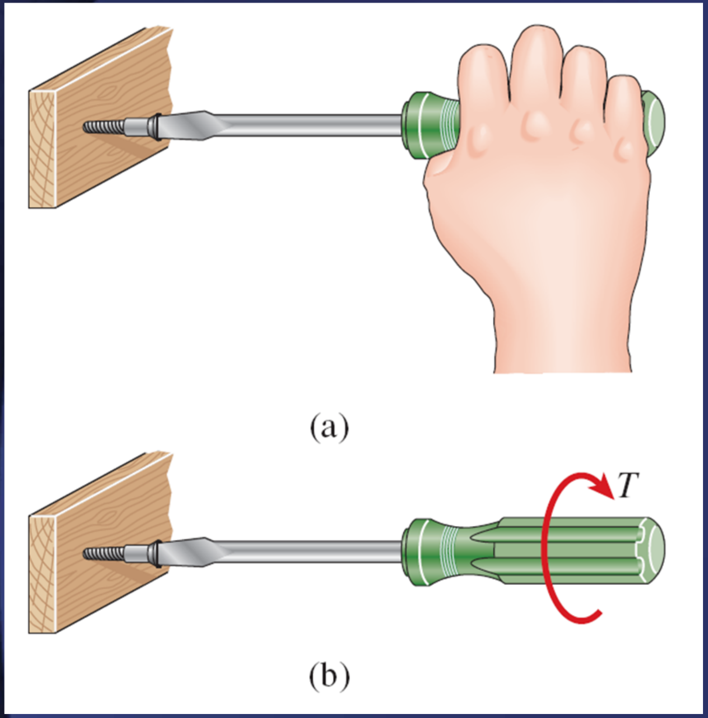


FIG. 3-1 Torsion of a screwdriver due to a torque T applied to the handle

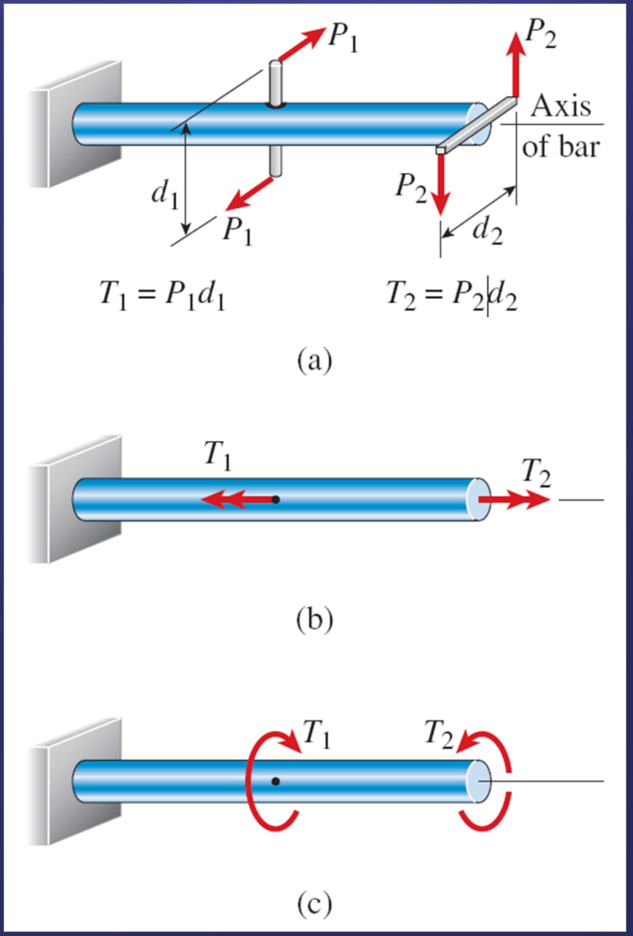


FIG. 3-2 Circular bar subjected to torsion by torques T_1 and T_2

3.2 TORSIONAL DEFORMATIONS OF A CIRCULAR BAR

If every cross section of the bar has the same radius and is subjected to the same torque (pure torsion), the angle $\Phi(x)$ will vary linearly between the ends.

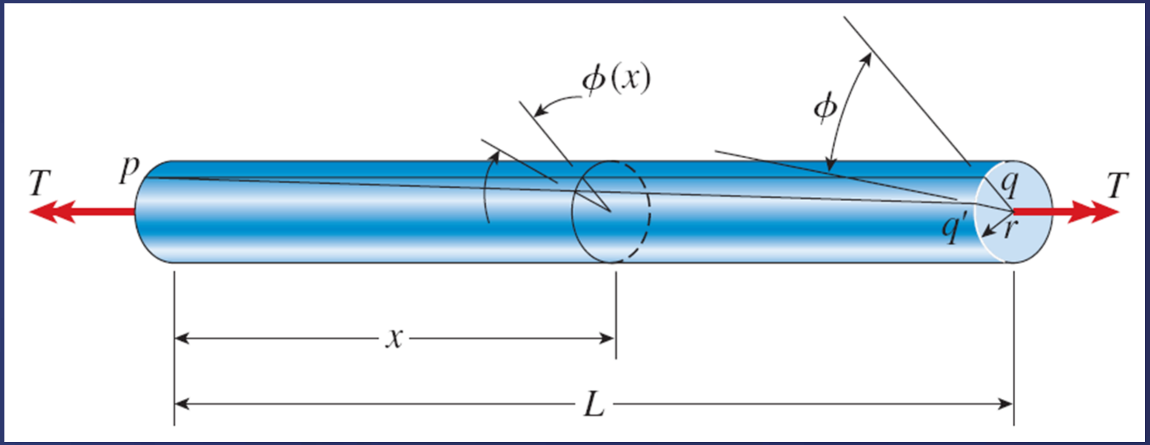


FIG. 3-3 Deformations of a circular bar in pure torsion

The shear strain γ_{max} (Fig. 3-4b) is

$$\gamma_{max} = \frac{bb'}{ab} \qquad \text{(a)}$$

$$\gamma_{max} = \frac{rd\phi}{dx} \qquad \text{(b)}$$

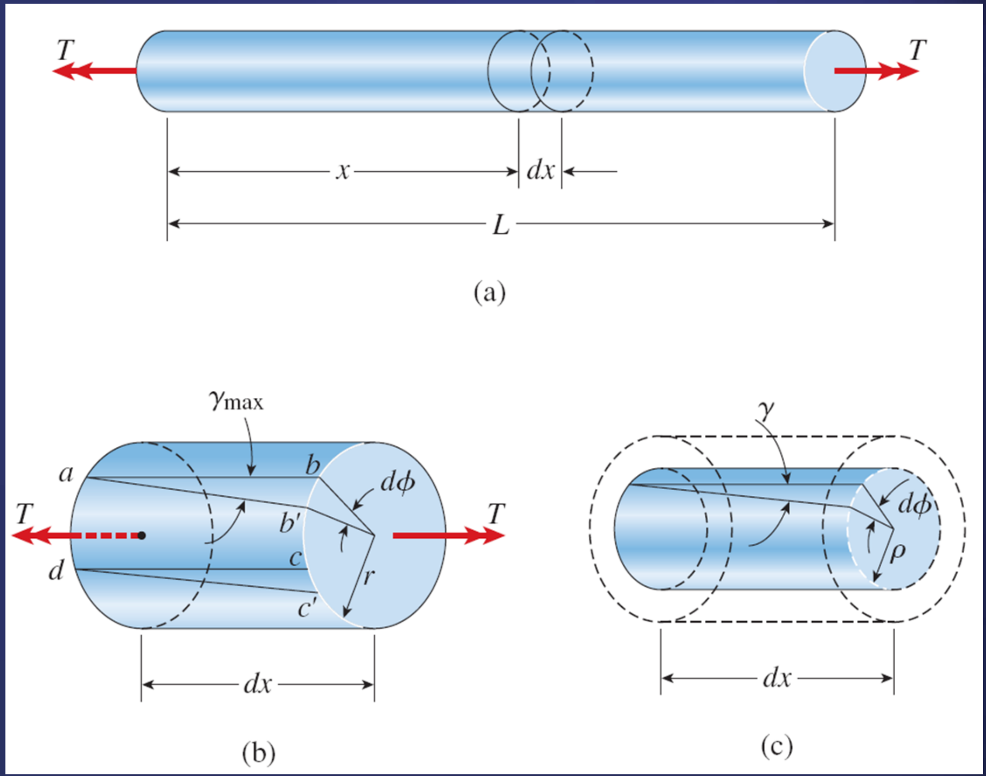


FIG. 3-4 Deformation of an element of length *dx* cut from a bar in torsion

The quantity *dΦ/dx* is the angle of twist per unit length, or the rate of twist :

$$\theta = \frac{d\phi}{dx} \qquad (3\text{-}1)$$

The shear strain at the outer surface (Eq. b) is

$$\gamma_{\max} = \frac{r d\phi}{dx} = r\theta \qquad (3\text{-}2)$$

For pure torsion only, we obtain

$$\gamma_{\max} = r\theta = \frac{r\phi}{L} \qquad (3\text{-}3)$$

Interior elements are also in pure shear with the corresponding shear strains given by the equation (compare with Eq. 3-2):

$$\gamma = \rho\theta = \frac{\rho}{r}\gamma_{\max} \qquad (3\text{-}4)$$

This equation shows that the shear strains in a circular bar vary linearly with the radial distance ρ from the center, with the strain being zero at the center and reaching a maximum value at the outer surface.

The minimum strain is related to the maximum strain by the equation

$$\gamma_{max} = \frac{r_2 \phi}{L} \qquad \gamma_{min} = \frac{r_1}{r_2}\gamma_{max} = \frac{r_1 \phi}{L} \qquad \text{(3-5a,b)}$$

in which r_1 and r_2 are the inner and outer radii, respectively, of the tube.

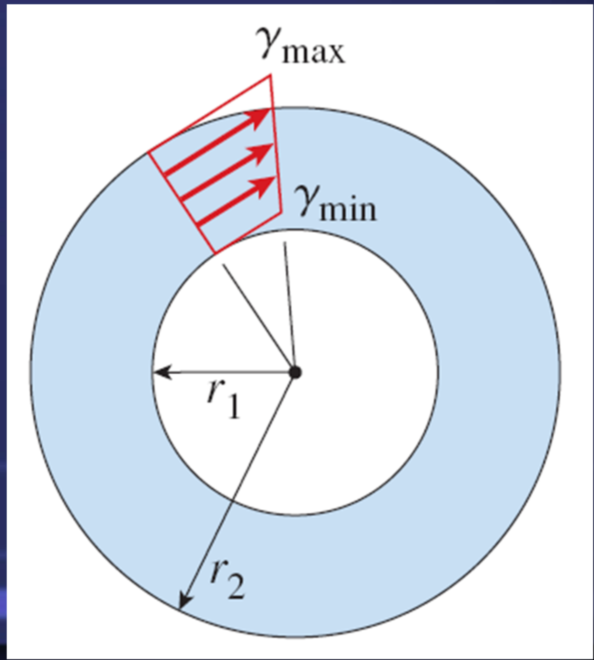


FIG. 3-5 Shear strains in a circular tube

3.3 CIRCULAR BARS OF LINEARLY ELASTIC MATERIALS

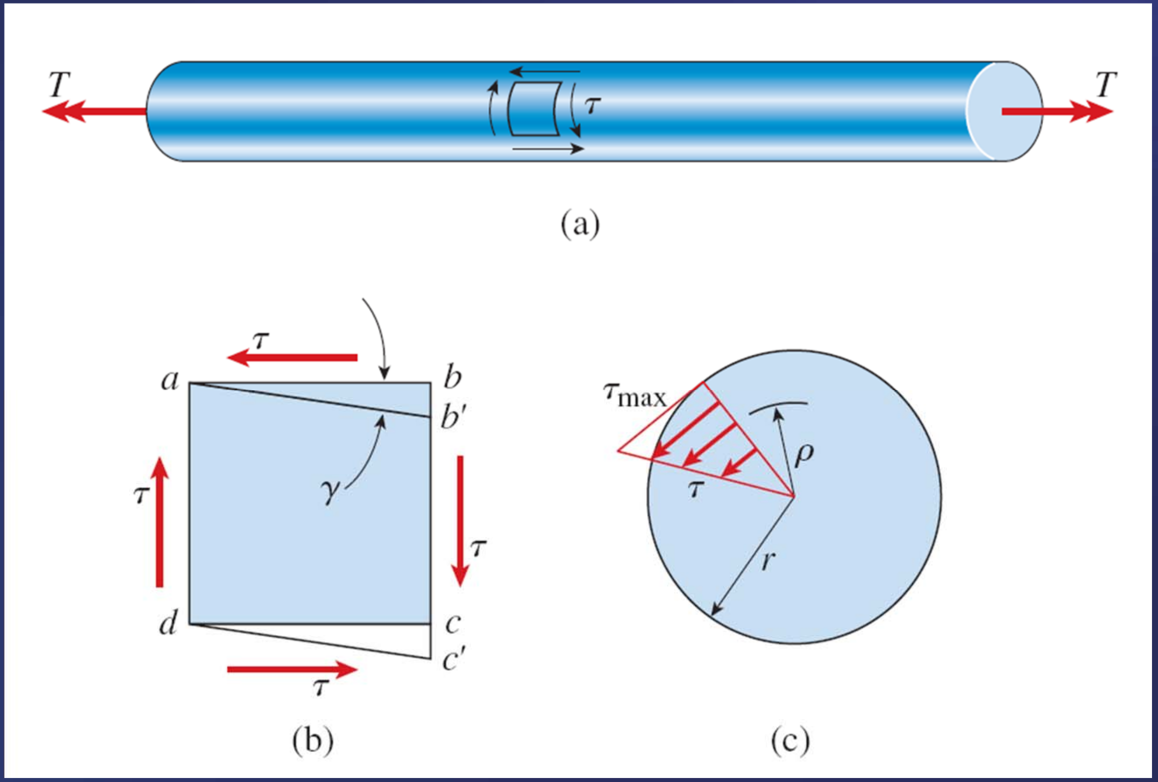


FIG. 3-6 Shear stresses in a circular bar in torsion

If the material is linearly elastic, we can use Hooke's law in shear (Eq. 1-14):

$$\tau = G\gamma \qquad (3\text{-}6)$$

in which G is the shear modulus of elasticity and γ is the shear strain in radians.

$$\tau_{\max} = Gr\theta \qquad \tau = G\rho\theta = \frac{\rho}{r}\tau_{\max} \qquad (3\text{-}7a,b)$$

in which τ_{max} is the shear stress at the outer surface of the bar (radius ρ), τ is the shear stress at an interior point (radius ρ), and θ is the rate of twist.

The Torsion Formula

The moment of this force about the axis of the bar is

$$dM = \tau \rho dA = \frac{\tau_{\max}}{r} \rho^2 dA$$

The resultant moment (equal to the torque T) is

$$T = \int_A dM = \frac{\tau_{\max}}{r} \int_A \rho^2 dA = \frac{\tau_{\max}}{r} I_p \qquad (3\text{-}8)$$

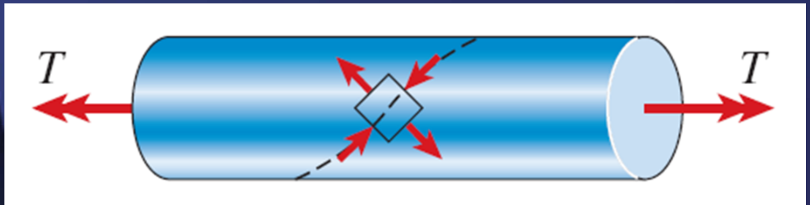


FIG. 3-8 Tensile and compressive stresses acting on a stress element oriented at 45° to the longitudinal axis

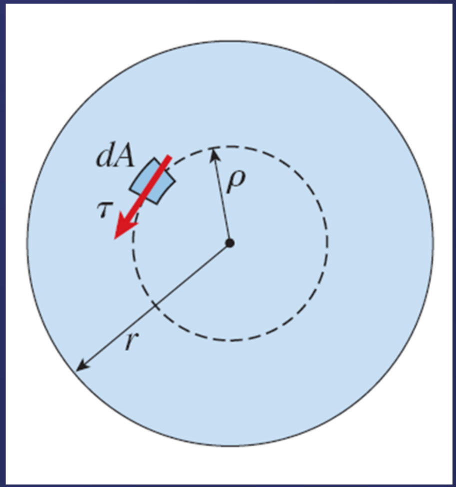


FIG. 3-9 Determination of the resultant of the shear stresses acting on a cross section

in which

$$I_p = \int_A \rho^2 dA \qquad (3\text{-}9)$$

For a circle of radius *r* and diameter *d*, the polar moment of inertia is

$$I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad (3\text{-}10)$$

The maximum shear stress can be obtained by

$$\tau_{\max} = \frac{Tr}{I_p} \qquad (3\text{-}11)$$

This equation, known as the torsion formula,

$$\tau_{\max} = \frac{16T}{\pi d^3} \qquad (3\text{-}12)$$

The shear stress at distance *r* from the center of the bar is

$$\tau = \frac{\rho}{r}\tau_{\max} = \frac{T\rho}{I_p} \qquad (3\text{-}13)$$

Angle of Twist

Combining Eq. (3-7a) with the torsion formula, we get

$$\theta = \frac{T}{GI_p} \qquad (3\text{-}14)$$

GI_P is known as the torsional rigidity of the bar.
For a bar in pure torsion, the total angle of twist Φ is

$$\phi = \frac{TL}{GI_p} \qquad (3\text{-}15)$$

in which Φ is measured in radians.

The quantity GI_P/L, called the torsional stiffness of the bar. The torsional flexibility is the reciprocal of the stiffness, or L/GI_P

$$k_T = \frac{GI_p}{L} \qquad f_T = \frac{L}{GI_p} \qquad (a,b)$$

Circular Tubes

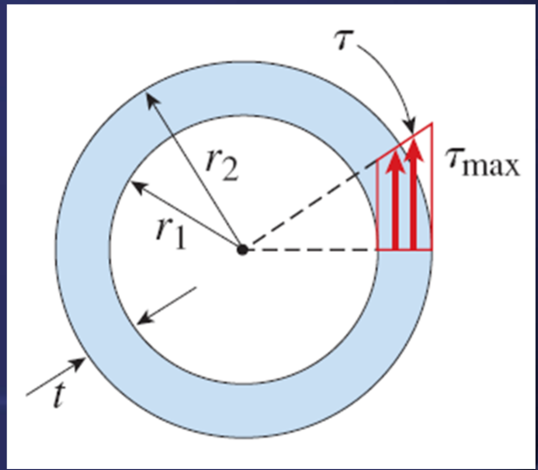
The polar moment of inertia of the cross-sectional area of a tube is

$$I_P = \frac{\pi}{2}(r_2^4 - r_1^4) = \frac{\pi}{32}(d_2^4 - d_1^4) \qquad (3\text{-}16)$$

or

$$I_P = \frac{\pi r t}{2}(4r^2 + t^2) = \frac{\pi d t}{4}(d^2 + t^2) \qquad (3\text{-}17)$$

If the wall thickness *t* (Fig. 3-10) is small compared to the radius, the following approximate formulas may be used for the polar moment of inertia:



$$I_P \approx 2\pi r^3 t = \frac{\pi d^3 t}{4} \qquad (3\text{-}18)$$

FIG. 3-10 Circular tube in torsion

Example 3-1

A solid steel bar of circular cross section (Fig. 3-11) has diameter d = 40 mm, length L = 1.3 m, and shear modulus of elasticity G = 80 GPa. The bar is subjected to torques T acting at the ends.

(a) If the torques have magnitude T = 340 Nm, what is the maximum shear stress in the bar? What is the angle of twist between the ends?

(b) If the allowable shear stress is 42 MPa and the allowable angle of twist is 2.5°, what is the maximum permissible torque?

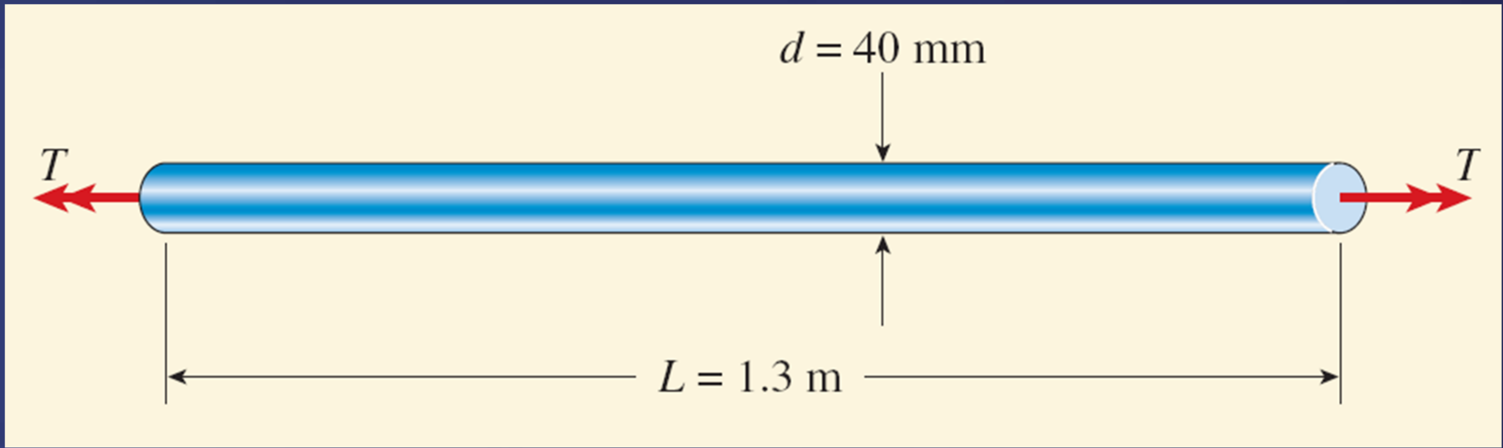


FIG. 3-11 Example 3-1. Bar in pure torsion

Solution

(a)The maximum shear stress from Eq. (3-12), as follows:

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16(340\ \mathrm{N\cdot m})}{\pi(0.04\ \mathrm{m})^3} = 27.1\ \mathrm{MPa}$$

The angle of twist is obtained

$$I_p = \frac{\pi d^4}{32} = \frac{\pi(0.04\ \mathrm{m})}{32} = 2.51\times10^{-7}\ \mathrm{m}^4$$

$$\phi = \frac{TL}{GI_p} = \frac{(340\ \mathrm{N\cdot m})(1.3\ \mathrm{m})}{(80\ \mathrm{GPa})(2.51\times10^{-9}\ \mathrm{m}^4)} = 0.02198\ \mathrm{rad} = 1.26^\circ$$

(b) By the allowable shear stress

$$T_1 = \frac{\pi d^3 \tau_{\text{allow}}}{16} = \frac{\pi}{16}(0.04\ \text{m})^3(42\ \text{MPa}) = 528\ \text{N}\cdot\text{m}$$

By the allowable angle of twist

$$T_2 = \frac{GI_P\phi_{\text{allow}}}{L} = \frac{(80\ \text{GPa})(2.51\times10^{-7}\ \text{m}^4)(2.5°)(\pi\ \text{rad}/180°)}{1.3\ \text{m}}$$

$$= 674\ \text{N}\cdot\text{m}$$

The maximum permissible torque is the smaller of T_1 and T_2:

$$T_{\max} = 528\ \text{N}\cdot\text{m}$$

Example 3-2

A steel shaft is to be manufactured either as a solid circular bar or as a circular tube (Fig. 3-12). The shaft is required to transmit a torque of 1200 Nm without exceeding an allowable shear stress of 40 MPa nor an allowable rate of twist of 0.75°/m. (The shear modulus of elasticity of the steel is 78 GPa.)

(a) Determine the required diameter d_0 of the solid shaft.

(b) Determine the required outer diameter d_2 of the hollow shaft if the thickness t of the shaft is specified as one-tenth of the outer diameter.

(c) Determine the ratio of diameters (that is, the ratio d_2/d_0) and the ratio of weights of the hollow and solid shafts.

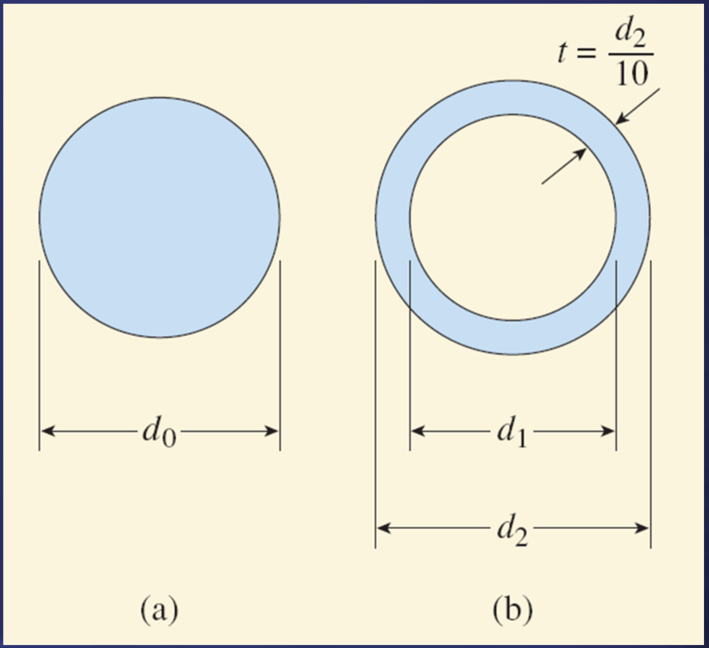


FIG. 3-12 Example 3-2. Torsion of a steel shaft

Solution

(a) *Solid shaft.*

In the case of the allowable shear stress we rearrange Eq. (3-12) and obtain

$$d_0^3 = \frac{16T}{\pi\tau_{\text{allow}}} = \frac{16(1200\ \text{N}\cdot\text{m})}{\pi(40\ \text{MPa})} = 152.8\times10^{-6}\ \text{m}^3$$

we get

$$d_0 = 0.0535\ \text{m} = 53.5\ \text{mm}$$

In the case of the allowable rate of twist, we start by finding the required polar moment of inertia (see Eq. 3-14):

$$I_P = \frac{T}{G\theta_{\text{allow}}} = \frac{1200\ \text{N}\cdot\text{m}}{(78\ \text{GPa})(0.75°/\text{m})(\pi\ \text{rad}/180°)} = 1175\times10^{-9}\ \text{m}^4$$

The required diameter is

$$d_0^4 = \frac{32 I_P}{\pi} = \frac{32(1175 \times 10^{-9} \text{ m}^4)}{\pi} = 11.97 \times 10^{-6} \text{ m}^4$$

The required diameter of the solid shaft is

$$d_0 = 58.8 \text{ mm}$$

(b) *Hollow shaft*.

The inner diameter is

$$d_1 = d_2 - 2t = d_2 - 2(0.1d_2) = 0.8d_2$$

The polar moment of inertia (Eq. 3-16) is

$$I_p = \frac{\pi}{32}(d_2^4 - d_1^4) = \frac{\pi}{32}\left[d_2^4 - (0.8d_2)^4\right] = \frac{\pi}{32}(0.5904d_2^4) = 0.05796d_2^4$$

In the case of the allowable shear stress

$$\tau_{\text{allow}} = \frac{Tr}{I_P} = \frac{T(d_2/2)}{0.05796d_2^4} = \frac{T}{0.1159d_2^3}$$

Rearranging, we get

$$d_2^3 = \frac{T}{0.1159\tau_{\text{allow}}} = \frac{1200\ \text{N}\cdot\text{m}}{0.1159(40\ \text{MPa})} = 258.8\times10^{-6}\ \text{m}^3$$

$$d_2 = 0.0637\ \text{m} = 63.7\ \text{mm}$$

In the case of the allowable rate of twist

$$\theta_{\text{allow}} = \frac{T}{G(0.05796 d_2^4)}$$

$$\tau_{\text{allow}} = \frac{Tr}{I_P} = \frac{T(d_2/2)}{0.05796 d_2^4} = \frac{T}{0.1159 d_2^3}$$

Solving for d_2 gives

$$d_2 = 0.0671 \text{ m} = 67.1 \text{ mm}$$

Comparing the two values of d_2

$$d_2 = 67.1 \text{ mm}$$

(c) *Ratios of diameters and weights*.

$$\frac{d_2}{d_0} = \frac{67.1 \text{ mm}}{58.8 \text{ mm}} = 1.14$$

Since the weights of the shafts are proportional to their cross-sectional areas, we can express the ratio of the weight of the hollow shaft to the weight of the solid shaft as follows:

$$\frac{W_{\text{hollow}}}{W_{\text{solid}}} = \frac{A_{\text{hollow}}}{A_{\text{solid}}} = \frac{\pi(d_2^2 - d_1^2)/4}{\pi d_0^2/4} = \frac{d_2^2 - d_1^2}{d_0^2}$$

$$= \frac{(67.1 \text{ mm})^2 - (53.7 \text{ mm})^2}{(58.8 \text{ mm})^2} = 0.47$$

3.4 NONUNIFORM TORSION

Case 1. *Bar consisting of prismatic segments with constant torque throughout each segment* (Fig. 3-14).

From Fig. 3-14

$$T_{CD} = -T_1 - T_2 + T_3 \quad T_{BC} = -T_1 - T_2 \quad T_{AB} = -T_1 \qquad \text{(a,b,c)}$$

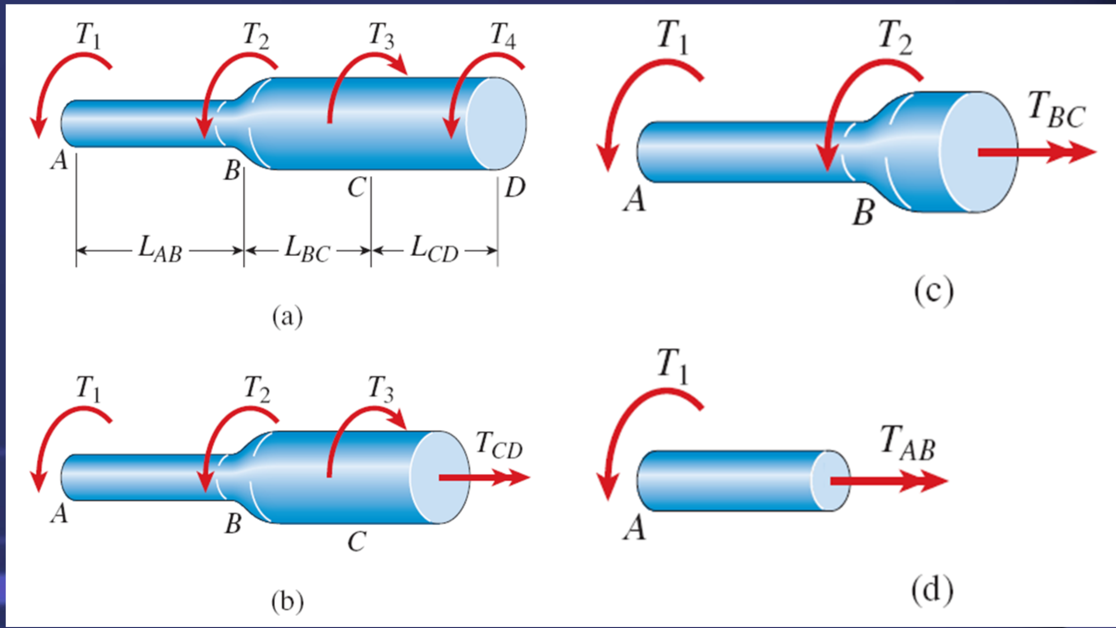


FIG. 3-14 Bar in nonuniform torsion (Case 1)

The total angle of twist of one end of the bar with respect to the other is then obtained by algebraic summation, as follows:

$$\phi = \phi_1 + \phi_2 + ... + \phi_n \qquad (3\text{-}19)$$

or we can write the general formula

$$\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i (I_p)_i} \qquad (3\text{-}20)$$

Case 2. *Bar with continuously varying cross sections and constant torque* (Fig. 3-15).

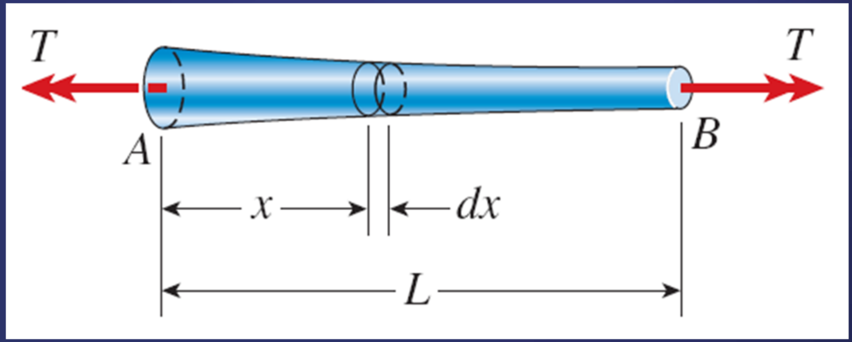


FIG. 3-15 Bar in nonuniform torsion (Case 2)

The differential angle of rotation $d\Phi$ for this element is

$$d\phi = \frac{Tdx}{GI_P(x)} \qquad \text{(d)}$$

The angle of twist for the entire bar is the summation of the differential angles of rotation:

$$\phi = \int_0^L d\phi = \int_0^L \frac{Tdx}{GI_P(x)} \qquad \text{(3-21)}$$

Case 3. *Bar with continuously varying cross sections and continuously varying torque* (Fig. 3-16).

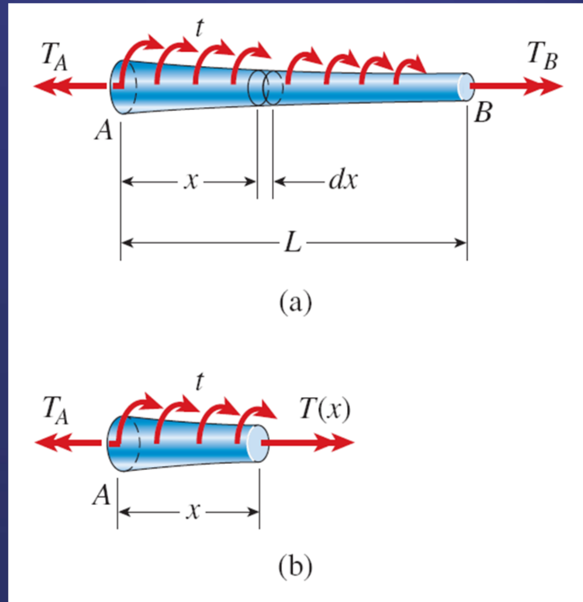


FIG. 3-16 Bar in nonuniform torsion (Case 3)

The equation for the angle of twist becomes

$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x)dx}{GI_P(x)} \qquad \text{(3-22)}$$

Example 3-4

A solid steel shaft *ABCDE* (Fig. 3-17) having diameter d = 30 mm turns freely in bearings at points *A* and *E*. The shaft is driven by a gear at *C*, which applies a torque T_2 = 450 Nm in the direction shown in the figure. Gears at *B* and *D* are driven by the shaft and have resisting torques T_1 = 275 Nm and T_3 = 175 Nm, respectively, acting in the opposite direction to the torque T_2. Segments *BC* and *CD* have lengths L_{BC} = 500 mm and L_{CD} = 400 mm, respectively, and the shear modulus G = 80 GPa.

Determine the maximum shear stress in each part of the shaft and the angle of twist between gears *B* and *D*.

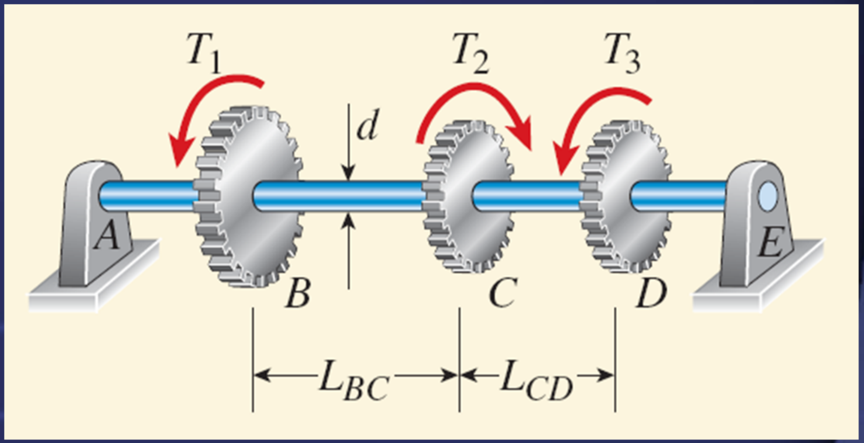


FIG. 3-17 Example 3-4. Steel shaft in torsion

Solution

Torques acting in the segments.
From equilibrium of the free body, we obtain

$$T_{CD} = T_2 - T_1 = 450\ \mathrm{N \cdot m} - 275\ \mathrm{N \cdot m} = 175\ \mathrm{N \cdot m}$$

$$T_{BC} = -T_1 = -275\ \mathrm{N \cdot m}$$

Shear stresses. The maximum shear stresses in segments *BC* and *CD* are found from the modified form of the torsion formula (Eq. 3-12); thus,

$$\tau_{BC} = \frac{16T_{BC}}{\pi d^3} = \frac{16(275\ \mathrm{N \cdot m})}{\pi (30\ \mathrm{mm})^3} = 51.9\ \mathrm{MPa}$$

$$\tau_{CD} = \frac{16T_{CD}}{\pi d^3} = \frac{16(175\ \mathrm{N \cdot m})}{\pi (30\ \mathrm{mm})^3} = 33.0\ \mathrm{MPa}$$

Angles of twist.

$$\phi_{BD} = \phi_{BC} + \phi_{CD}$$

The moment of inertia of the cross section:

$$I_P = \frac{\pi d^4}{32} = \frac{\pi(30\ \text{mm})^4}{32} = 79{,}520\ \text{mm}^4$$

The angles of twist are

$$\phi_{BC} = \frac{T_{BC}L_{BC}}{GI_P} = \frac{(-275\ \text{N}\cdot\text{m})(500\ \text{mm})}{(80\ \text{GPa})(79{,}520\ \text{mm}^4)} = -0.0216\ \text{rad}$$

$$\phi_{CD} = \frac{T_{CD}L_{CD}}{GI_P} = \frac{(175\ \text{N}\cdot\text{m})(400\ \text{mm})}{(80\ \text{GPa})(79{,}520\ \text{mm}^4)} = 0.0110\ \text{rad}$$

The total angle of twist is

$$\phi_{BD} = \phi_{BC} + \phi_{CD} = -0.0216 + 0.0110 = -0.0106\ \text{rad} = -0.61^\circ$$

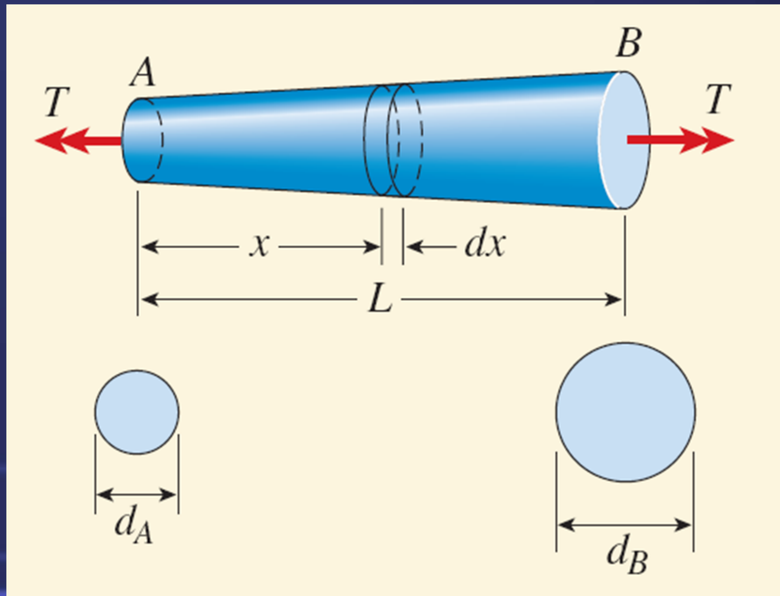
Example 3-5

A tapered bar *AB* of solid circular cross section is twisted by torques *T* applied at the ends (Fig. 3-19). The diameter of the bar varies linearly from d_A at the lefthand end to d_B at the right-hand end, with d_B assumed to be greater than d_A.

(a) Determine the maximum shear stress in the bar.

(b) Derive a formula for the angle of twist of the bar.

FIG. 3-19 Example 3-5. Tapered bar in torsion



Solution

(a) *Shear stresses.*

The maximum shear stress at end *A is*

$$\tau_{\max} = \frac{16T}{\pi d_A^3}$$

(b) *Angle of twist.*

the polar moment of inertia is :

$$I_P(x) = \frac{\pi d^4}{32} = \frac{\pi}{32}\left(d_A + \frac{d_B - d_A}{L}x\right)^4 \qquad (3\text{-}24)$$

The angle of twist is

$$\phi = \int_0^L \frac{T dx}{G I_P(x)} = \frac{32T}{\pi G}\int_0^L \frac{dx}{\left(d_A + \frac{d_B - d_A}{L}x\right)^4} \qquad (3\text{-}25)$$

We note that

$$\int \frac{dx}{(a+bx)^4} = -\frac{1}{3b(a+bx)^3}$$

Thus, the integral in Eq. (3-25) equals

$$\frac{L}{3(d_B - d_A)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right) \quad \text{(g)}$$

Replacing the integral in Eq. (3-25) with this expression, we obtain

$$\phi = \frac{32TL}{3\pi G(d_B - d_A)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right) \quad \text{(3-26)}$$

A convenient form in which to write the preceding equation is

$$\phi = \frac{TL}{G(I_P)_A}\left(\frac{\beta^2 + \beta + 1}{3\beta^3}\right) \quad \text{(3-27)}$$

in which

$$\beta = \frac{d_B}{d_A} \qquad (I_P)_A = \frac{\pi d_A^4}{32} \quad \text{(3-28)}$$

3.5 STRESSES AND STRAINS IN PURE SHEAR

This element is in a state of pure shear, because the only stresses acting on it are the shear stresses *t* on the four side faces (see the discussion of shear stresses in Section 1.6.)

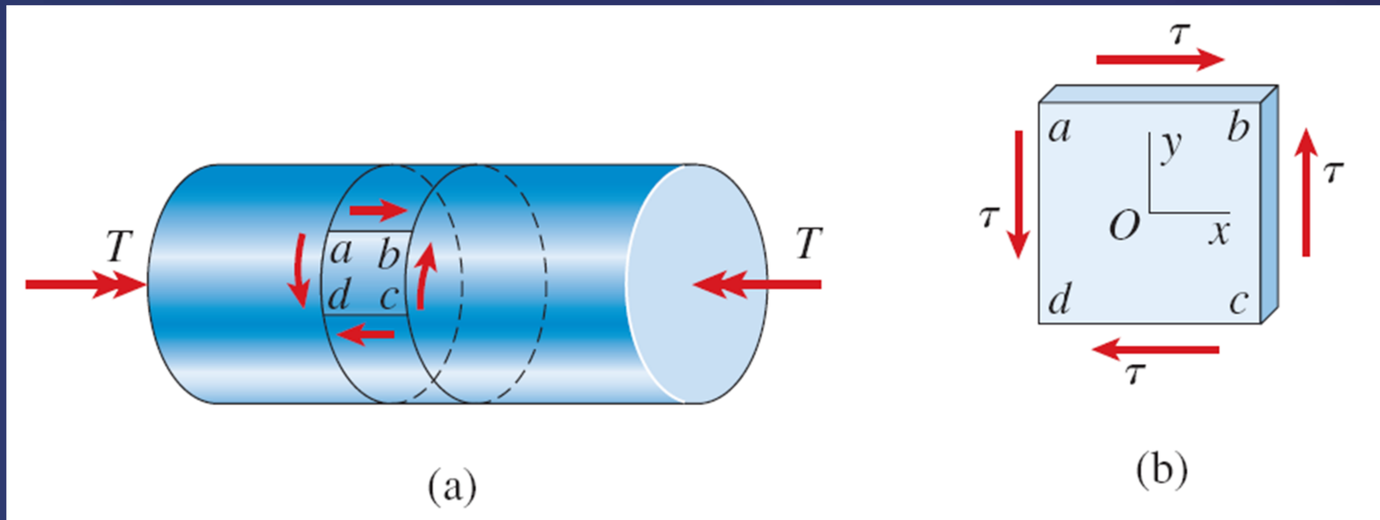


FIG. 3-20 Stresses acting on a stress *t* element cut from a bar in torsion (pure shear)

Stresses on Inclined Planes

A shear stress acting on a positive face of an element is positive, if it acts in the positive direction of one of the coordinate axes and negative if it acts in the negative direction of an axis.

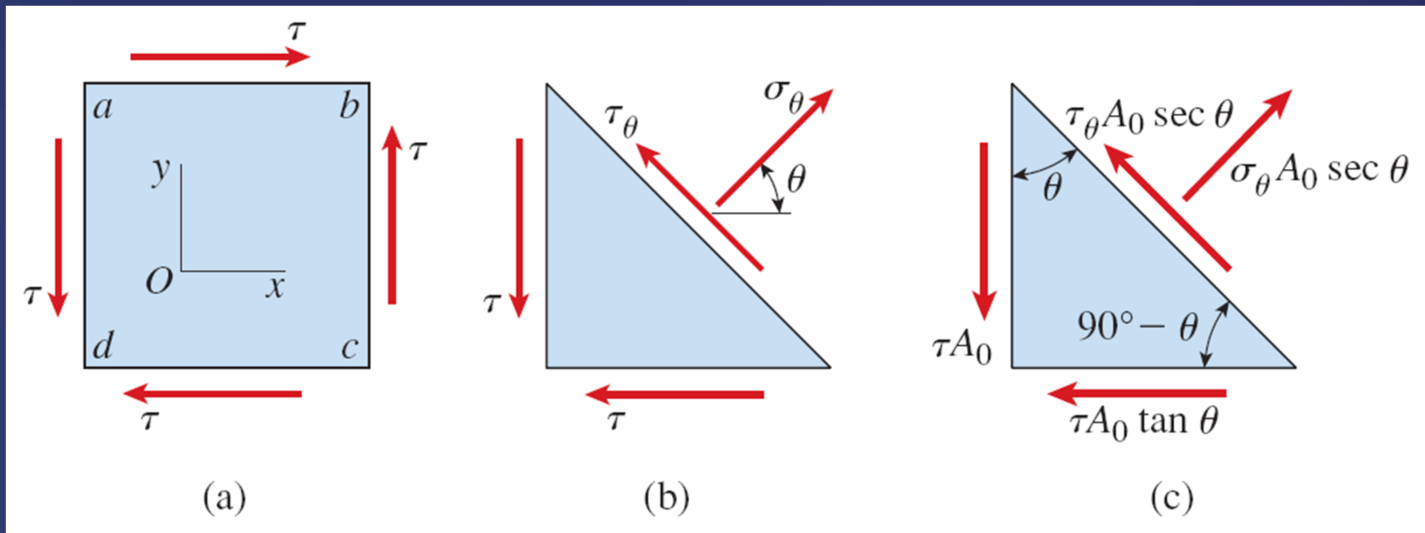


FIG. 3-21 Analysis of stresses on inclined planes:
(a) element in pure shear,
(b) stresses acting on a triangular stress element, and
(c) forces acting on the triangular stress element (free-body diagram)

The stresses acting on the inclined face may be determined from the equilibrium of the triangular element.

The first equation, obtained by summing forces in the direction of σ_θ, is

$$\sigma_\theta A_0 \sec\theta = \tau A_0 \sin\theta + \tau A_0 \tan\theta \cos\theta$$

or

$$\sigma_\theta = 2\tau \sin\theta \cos\theta \qquad \text{(3-29a)}$$

The second equation is obtained by summing forces in the direction of τ_θ:

$$\tau_\theta A_0 \sec\theta = \tau A_0 \cos\theta - \tau A_0 \tan\theta \sin\theta$$

or

$$\tau_\theta = \tau(\cos^2\theta - \sin^2\theta) \qquad \text{(3-29b)}$$

or

$$\sigma_\theta = \tau \sin 2\theta \qquad \tau_\theta = \tau \cos 2\theta \qquad \text{(3-30a,b)}$$

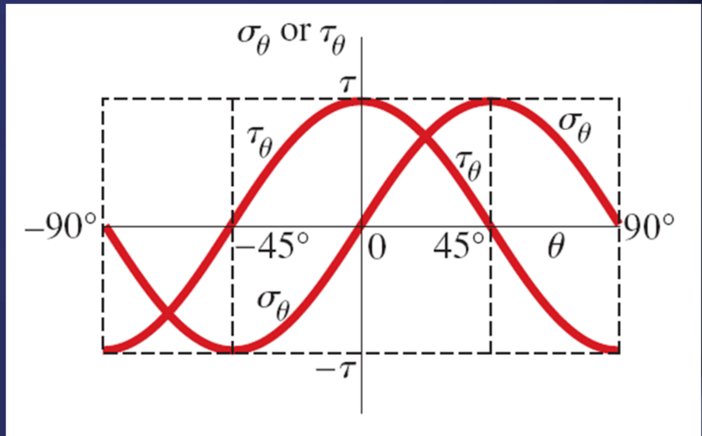


FIG. 3-22 Graph of normal stresses σ_θ and shear stresses τ_θ versus angle θ of the inclined plane

FIG. 3-23 Stress elements oriented at θ = 0 and θ = 45° for pure shear

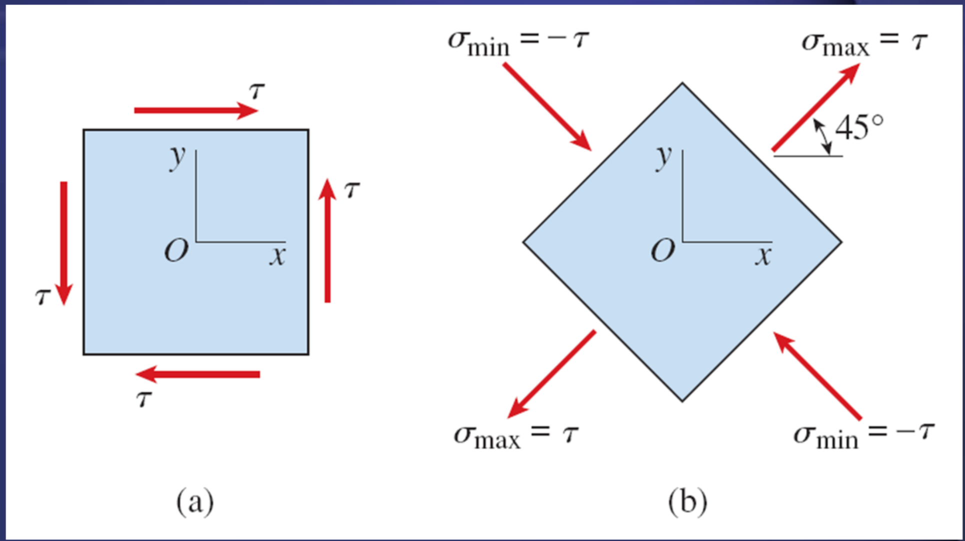
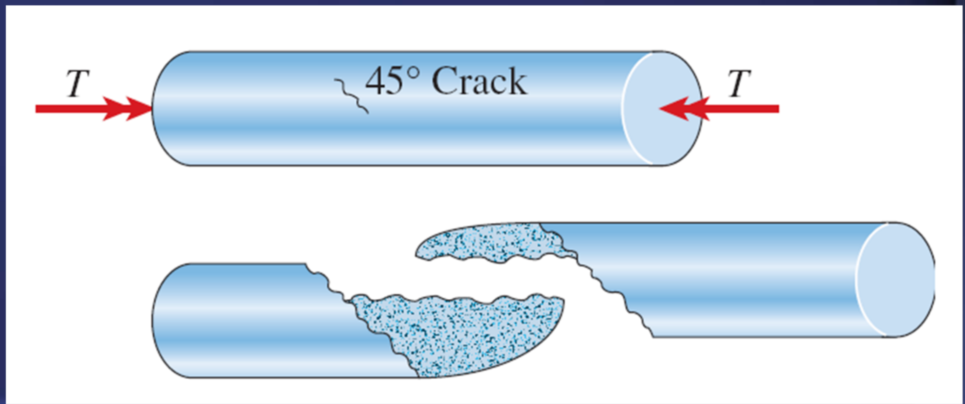


FIG. 3-24 Torsion failure of a brittle material by tension cracking along a 45° helical surface



Strains in Pure Shear

Therefore, the element changes its shape from a rectangular parallelepiped (Fig. 3-23a) to an oblique parallelepiped (Fig. 3-25a). This change in shape is called a shear distortion.

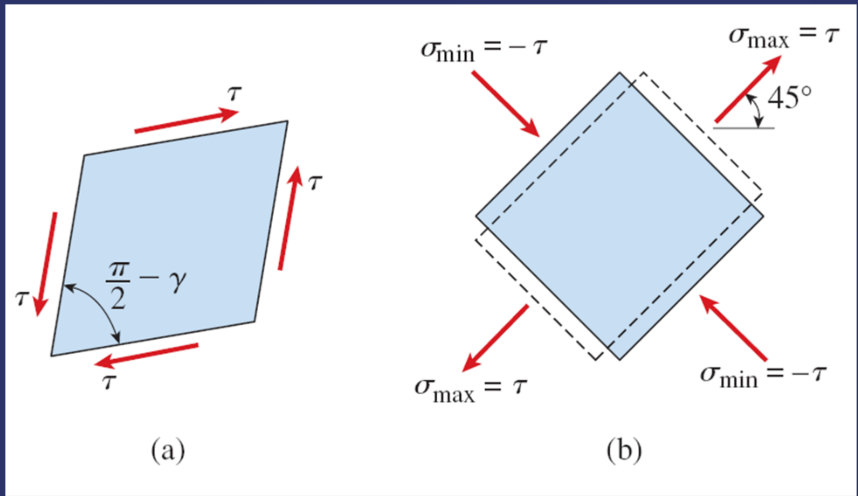


FIG. 3-25 Strains in pure shear:
(a) shear distortion of an element oriented at $\theta = 0$, and
(b) distortion of an element oriented at $\theta = 45^{\circ}$

The shear strain for the element oriented at $\theta = 0$ by Hooke's law is :

$$\gamma = \frac{\tau}{G} \quad (3\text{-}31)$$

The normal strain in the 45° direction is

$$\epsilon_{\max} = \frac{\tau}{E} + \frac{\nu\tau}{E} = \frac{\tau}{E}(1+\nu) \quad (3\text{-}32)$$

The normal strain ε_{max} in the 45° direction is:

$$\epsilon_{\max} = \frac{\gamma}{2} \quad (3\text{-}33)$$

3.6 RELATIONSHIP BETWEEN MODULI OF ELASTICITY *E* AND *G*

In the 45° direction the length of diagonal *bd* is

$$L_{bd} = \sqrt{2}h(1 + \in_{max}) \qquad \text{(a)}$$

This length can be related to the shear strain *γ* by considering the geometry of the deformed element.

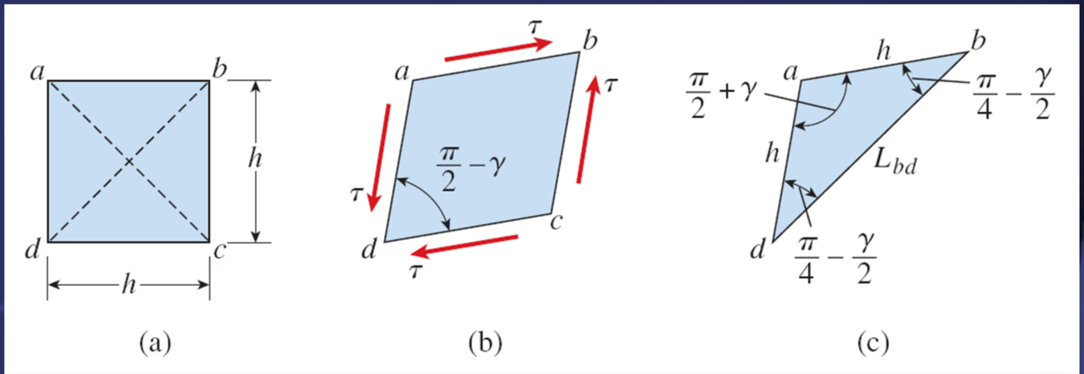


FIG. 3-28 Geometry of deformed element in pure shear

Using the law of cosines (see Appendix C) for triangle *abd*, we get

$$L_{bd}^2 = h^2 + h^2 - 2h^2 \cos(\frac{\pi}{2} + \gamma)$$

Substituting for L_{bd} from Eq. (a) and simplifying, we get

we obtain $$(1+ \in_{max})^2 = 1 - \cos(\frac{\pi}{2} + \gamma)$$

$$1 + 2 \in_{max} + \in_{max}^2 = 1 + \sin \gamma$$

we can disregard ε^2_{max} in comparison with $2\varepsilon_{max}$ and we can replace sin γ by γ. The resulting expression is

$$\in_{max} = \frac{\gamma}{2} \qquad (3\text{-}34)$$

in Eq. (3-34) yields

$$G = \frac{E}{2(1+\nu)} \qquad (3\text{-}35)$$

3.7 TRANSMISSION OF POWER BY CIRCULAR SHAFTS

In general, the work W done by a torque of constant magnitude is equal to the product of the torque and the angle through which it rotates; that is,

$$W = T\psi \tag{3-36}$$

Power is the *rate* at which work is done, or

$$P = \frac{dW}{dt} = T\frac{d\psi}{dt} \tag{3-37}$$

in which P is the symbol for power and t represents time.

The rate of change $d\Psi/dt$ of the angular displacement Ψ is the angular speed ω.

Therefore,

$$P = T\omega \qquad (\omega = \text{rad/s}) \qquad \text{(3-38)}$$

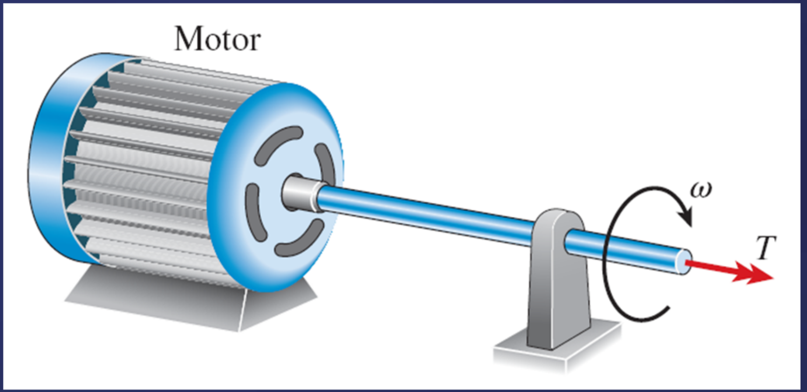


FIG. 3-29 Shaft transmitting a constant torque *T* at an angular speed *ω*

If the torque *T* is expressed in newton meters, then the power is expressed in watts (W). One watt is equal to one newton meter per second (or one joule per second).

Angular speed is often expressed as the frequency *f* of rotation, which is

$$\omega = 2\pi f \qquad (\omega = \text{rad/s},\ f = \text{Hz} = s^{-1}) \tag{3-39}$$

The expression for power (Eq. 3-38) then becomes

$$P = 2\pi f T \qquad (f = \text{Hz} = s^{-1}) \tag{3-40}$$

Or

$$n = 60 f \tag{3-41}$$

$$P = \frac{2\pi n T}{60} \qquad (n = \text{rpm}) \tag{3-42}$$

In U.S. engineering practice, power is sometimes expressed in horsepower (hp), a unit equal to 550 ft-lb/s. Therefore,

$$H = \frac{2\pi n T}{60(550)} = \frac{2\pi n T}{33{,}000} \qquad (n = \text{rpm},\ T = \text{lb-ft},\ H = \text{hp}) \tag{3-43}$$

$$1\ \text{hp} = 746\ \text{watts}$$

Example 3-7

A motor driving a solid circular steel shaft transmits 30 kW to a gear at *B* (Fig. 3-30).

The allowable shear stress in the steel is 42 MPa.

(a) What is the required diameter *d* of the shaft if it is operated at 500 rpm?

(b) What is the required diameter *d* if it is operated at 4000 rpm?

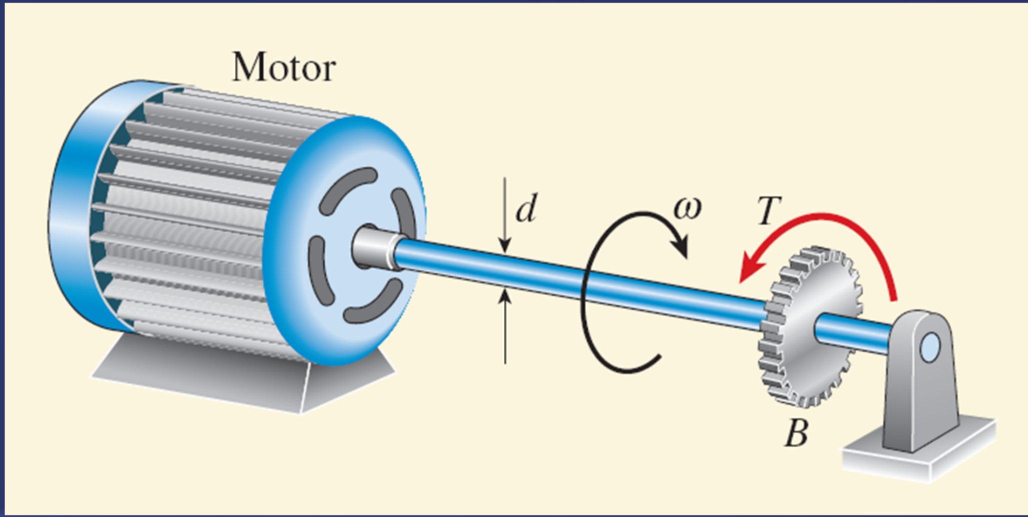


FIG. 3-30 Example 3-7. Steel shaft in torsion

Solution

(a) *Motor operating at 500 rpm*.

Solving that equation for *T*, we get

$$T = \frac{60P}{2\pi n} = \frac{60(30\ \text{kW})}{2\pi(500\ \text{rpm})} = 573\ \text{N}\cdot\text{m}$$

The maximum shear stress in the shaft is;

$$\tau_{\text{max}} = \frac{16T}{\pi d^3}$$

we get

$$d^3 = \frac{16T}{\pi\tau_{\text{allow}}} = \frac{16(573\ \text{N}\cdot\text{m})}{\pi(42\ \text{MPa})} = 69.5\times10^{-6}\ \text{m}^3$$

from which

$$d = 41.1\ \text{mm}$$

(b) *Motor operating at 4000 rpm*.
we obtain

$$T = \frac{60P}{2\pi n} = \frac{60(30\ \text{kW})}{2\pi(4000\ \text{rpm})} = 71.6\ \text{N}\cdot\text{m}$$

$$d^3 = \frac{16T}{\pi\tau_{\text{allow}}} = \frac{16(71.6\ \text{N}\cdot\text{m})}{\pi(42\ \text{MPa})} = 8.68\times10^{-6}\ \text{m}^3$$

$$d = 20.55\ \text{mm}$$

3.8 STATICALLY INDETERMINATE TORSIONAL MEMBERS

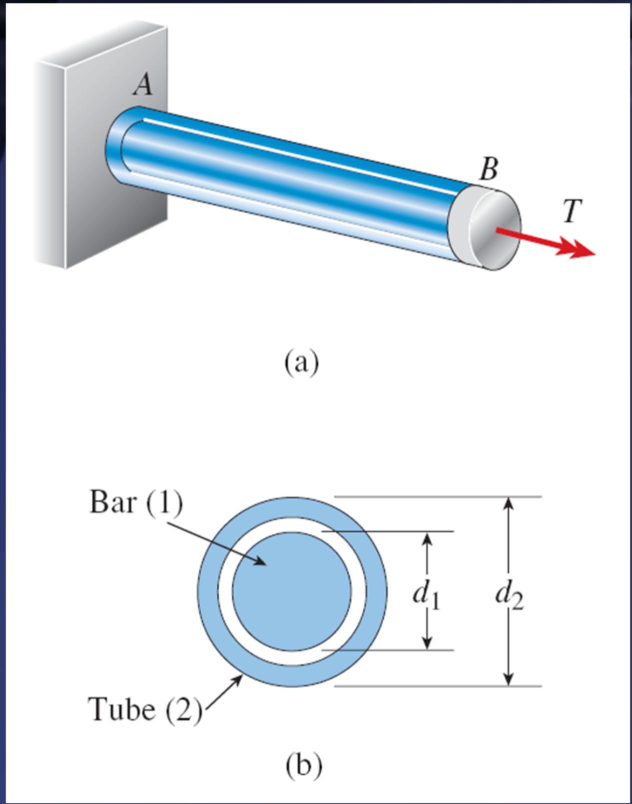
(1) Statically determinate

(2) Statically indeterminate

(3) Equations of equilibrium

(4) Equations of compatibility

(5) Torque-displacement relations



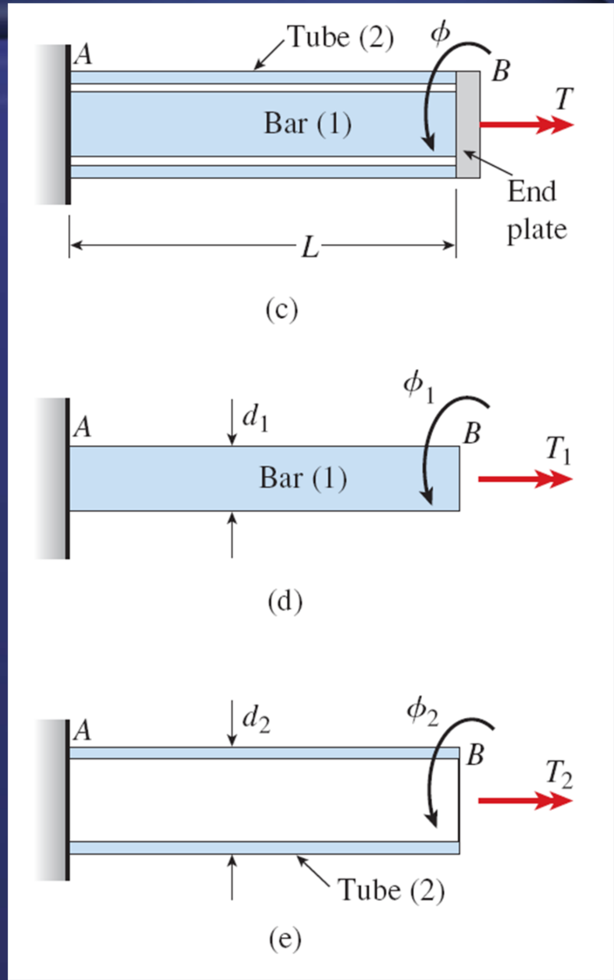


FIG. 3-32 Statically indeterminate bar in torsion

The *equation of equilibrium* is

$$T_1 + T_2 = T \qquad \text{(a)}$$

The *equation of compatibility* is

$$\phi_1 = \phi_2 \qquad \text{(b)}$$

By the *torque-displacement relations*

$$\phi_1 = \frac{T_1 L}{G_1 I_{P1}} \qquad \phi_2 = \frac{T_2 L}{G_2 I_{P2}} \qquad \text{(c,d)}$$

The equation of compatibility becomes

$$\frac{T_1 L}{G_1 I_{P1}} = \frac{T_2 L}{G_2 I_{P2}} \qquad \text{(e)}$$

The results are

$$T_1 = T\left(\frac{G_1 I_{P1}}{G_1 I_{P1} + G_2 I_{P2}}\right) \qquad T_2 = T\left(\frac{G_2 I_{P2}}{G_1 I_{P1} + G_2 I_{P2}}\right) \qquad \text{(3-44a,b)}$$

Example 3-9

The bar *ACB* shown in Figs. 3-33a and b is fixed at both ends and loaded by a torque T_0 at point *C*. Segments *AC* and *CB* of the bar have diameters d_A and d_B, lengths L_A and L_B, and polar moments of inertia I_{PA} and I_{PB}, respectively.

The material of the bar is the same throughout both segments.

Obtain formulas for

(a) the reactive torques T_A and T_B at the ends,

(b) the maximum shear stresses τ_{AC} and τ_{CB} in each segment of the bar, and

(c) the angle of rotation Φ_C at the cross section where the load T_0 is applied.

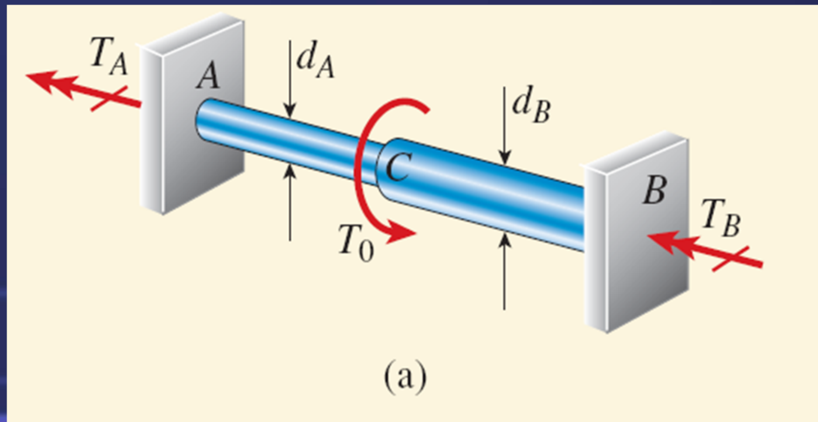


FIG. 3-33 Example 3-9. Statically indeterminate bar in torsion

Solution

Equation of equilibrium. From the equilibrium of the bar we obtain

$$T_A + T_B = T_0 \qquad \text{(f)}$$

The equation of compatibility is

$$\phi_1 + \phi_2 = 0 \qquad \text{(g)}$$

The equations are as follows:

$$\phi_1 = \frac{T_0 L_A}{G I_{PA}} \qquad \phi_2 = -\frac{T_B L_A}{G I_{PA}} - \frac{T_B L_B}{G I_{PB}} \qquad \text{(h,i)}$$

We now substitute the angles of twist (Eqs. h and i) into the compatibility equation (Eq. g) and obtain

$$\frac{T_0 L_A}{G I_{PA}} - \frac{T_B L_A}{G I_{PA}} - \frac{T_B L_B}{G I_{PB}} = 0$$

or

$$\frac{T_B L_A}{I_{PA}} + \frac{T_B L_B}{I_{PB}} = \frac{T_0 L_A}{I_{PA}} \qquad \text{(j)}$$

Solution of equations.
The results are

$$T_A = T_0\left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}}\right) \qquad T_B = T_0\left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}}\right) \qquad \text{(3-45a,b)}$$

Maximum shear stresses. The maximum shear stresses in each part of the bar are

$$\tau_{AC} = \frac{T_A d_A}{2 I_{PA}} \qquad \tau_{CB} = \frac{T_B d_B}{2 I_{PB}}$$

Substituting from Eqs. (3-45a) and (3-45b) gives

$$\tau_{AC} = \frac{T_0 L_B d_A}{2\left(L_B I_{PA} + L_A I_{PB}\right)} \qquad \tau_{CB} = \frac{T_0 L_A d_B}{2\left(L_B I_{PA} + L_A I_{PB}\right)} \qquad \text{(3-47a,b)}$$

Angle of rotation.
We obtain

$$\phi_C = \frac{T_A L_A}{G I_{PA}} = \frac{T_B L_B}{G I_{PB}} = \frac{T_0 L_A L_B}{G\left(L_B I_{PA} + L_A I_{PB}\right)} \qquad (3\text{-}48)$$

In the special case of a prismatic bar ($I_{PA} = I_{PB} = I_P$), the angle of rotation at the section where the load is applied is

$$\phi_C = \frac{T_0 L_A L_B}{G L I_P} \qquad (3\text{-}49)$$

The End of Chap. 3