and cos nx ($m \neq n$) for $a = \pi$ from the graph. For what m and n will you get orthogonality for $a = \pi/2, \pi/3, \pi/4$? Other a? Extend the experiment to cos mx sin nx and sin mx sin nx.

25. CAS EXPERIMENT. Order of Fourier Coefficients. The order seems to be $1/n$ if f is discontinous, and $1/n^2$ if f is continuous but $f' = df/dx$ is discontinuous, $1/n^3$ if f and f' are continuous but f'' is discontinuous, etc. Try to verify this for examples. Try to prove it by integrating the Euler formulas by parts. What is the practical significance of this?

11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

We now expand our initial basic discussion of Fourier series.

Orientation. This section concerns three topics:

1. Transition from period 2π to any period $2L$, for the function f, simply by a transformation of scale on the x-axis.
2. Simplifications. Only cosine terms if f is even ("Fourier cosine series"). Only sine terms if f is odd ("Fourier sine series").
3. Expansion of f given for $0 \leqq x \leqq L$ in two Fourier series, one having only cosine terms and the other only sine terms ("half-range expansions").

1. From Period 2π to Any Period $p = 2L$

Clearly, periodic functions in applications may have any period, not just 2π as in the last section (chosen to have simple formulas). The notation $p = 2L$ for the period is practical because L will be a length of a violin string in Sec. 12.2, of a rod in heat conduction in Sec. 12.5, and so on.

The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, as follows. Let $f(x)$ have period $p = 2L$. Then we can introduce a new variable v such that $f(x)$, as a function of v, has period 2π. If we set

$$\text{(1)} \qquad \text{(a)} \quad x = \frac{p}{2\pi} v, \qquad \text{so that} \qquad \text{(b)} \quad v = \frac{2\pi}{p} x = \frac{\pi}{L} x$$

then $v = \pm\pi$ corresponds to $x = \pm L$. This means that f, as a function of v, has period 2π and, therefore, a Fourier series of the form

$$\text{(2)} \qquad f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

with coefficients obtained from (6) in the last section

$$\text{(3)} \qquad \begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) dv, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \cos nv\, dv, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \sin nv\, dv. \end{aligned}$$

We could use these formulas directly, but the change to x simplifies calculations. Since

$$v = \frac{\pi}{L}x, \qquad \text{we have} \qquad dv = \frac{\pi}{L}\,dx \tag{4}$$

and we integrate over x from $-L$ to L. Consequently, we obtain for a function $f(x)$ of period $2L$ the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \tag{5}$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas** (π/L in dx cancels $1/\pi$ in (3))

$$\textbf{(6)} \qquad \begin{aligned} &\textbf{(0)} \quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ &\textbf{(a)} \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots \\ &\textbf{(b)} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots. \end{aligned}$$

Just as in Sec. 11.1, we continue to call (5) with any coefficients a **trigonometric series**. And we can integrate from 0 to $2L$ or over any other interval of length $p = 2L$.

EXAMPLE 1 Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 263)

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

Solution. From (6.0) we obtain $a_0 = k/2$ (verify!). From (6a) we obtain

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots, \qquad a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots.$$

From (6b) we find that $b_n = 0$ for $n = 1, 2, \cdots$. Hence the Fourier series is a **Fourier cosine series** (that is, it has no sine terms)

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$ ■

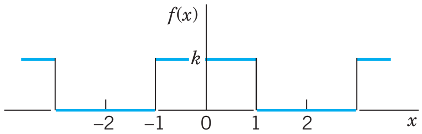


Fig. 263. Example 1

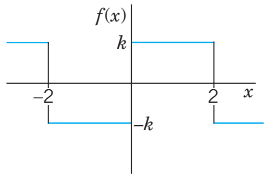


Fig. 264. Example 2

EXAMPLE 2 **Periodic Rectangular Wave. Change of Scale**

Find the Fourier series of the function (Fig. 264)

$$f(x) = \begin{cases} -k & \text{if} \quad -2 < x < 0 \\ k & \text{if} \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2.$$

Solution. Since $L = 2$, we have in (3) $v = \pi x/2$ and obtain from (8) in Sec. 11.1 with v instead of x, that is,

$$g(v) = \frac{4k}{\pi}\left(\sin v + \frac{1}{3}\sin 3v + \frac{1}{5}\sin 5v + \cdots\right)$$

the present Fourier series

$$f(x) = \frac{4k}{\pi}\left(\sin \frac{\pi}{2}x + \frac{1}{3}\sin \frac{3\pi}{2}x + \frac{1}{5}\sin \frac{5\pi}{2}x + \cdots\right).$$

Confirm this by using (6) and integrating. ■

EXAMPLE 3 **Half-Wave Rectifier**

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave (Fig. 265). Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if} \quad -L < t < 0, \\ E \sin \omega t & \text{if} \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \qquad L = \frac{\pi}{\omega}.$$

Solution. Since $u = 0$ when $-L < t < 0$, we obtain from (6.0), with t instead of x,

$$a_0 = \frac{\omega}{2\pi}\int_0^{\pi/\omega} E \sin \omega t \, dt = \frac{E}{\pi}$$

and from (6a), by using formula (11) in App. A3.1 with $x = \omega t$ and $y = n\omega t$,

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} E \sin \omega t \cos n\omega t \, dt = \frac{\omega E}{2\pi}\int_0^{\pi/\omega} [\sin (1 + n)\, \omega t + \sin (1 - n)\, \omega t] \, dt.$$

If $n = 1$, the integral on the right is zero, and if $n = 2, 3, \cdots$, we readily obtain

$$a_n = \frac{\omega E}{2\pi}\left[-\frac{\cos (1 + n)\omega t}{(1 + n)\omega} - \frac{\cos (1 - n)\omega t}{(1 - n)\omega}\right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi}\left(\frac{-\cos (1 + n)\pi + 1}{1 + n} + \frac{-\cos (1 - n)\pi + 1}{1 - n}\right).$$

If n is odd, this is equal to zero, and for even n we have

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1 + n} + \frac{2}{1 - n}\right) = -\frac{2E}{(n - 1)(n + 1)\pi} \qquad (n = 2, 4, \cdots).$$

In a similar fashion we find from (6b) that $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$. Consequently,

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$

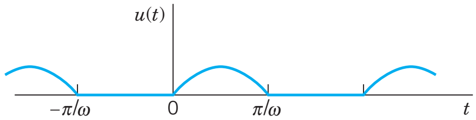


Fig. 265. Half-wave rectifier

2. Simplifications: Even and Odd Functions

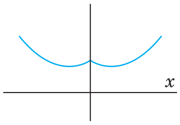


Fig. 266. Even function

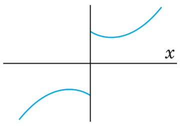


Fig. 267. Odd function

If $f(x)$ is an **even function**, that is, $f(-x) = f(x)$ (see Fig. 266), its Fourier series (5) reduces to a **Fourier cosine series**

$$\textbf{(5*)} \qquad \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x} \qquad (f \text{ even})$$

with coefficients (note: integration from 0 to L only!)

$$(6^*) \qquad a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots.$$

If $f(x)$ is an **odd function**, that is, $f(-x) = -f(x)$ (see Fig. 267), its Fourier series (5) reduces to a **Fourier sine series**

$$\textbf{(5**)} \qquad \boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x} \qquad (f \text{ odd})$$

with coefficients

$$(6^{**}) \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

These formulas follow from (5) and (6) by remembering from calculus that the definite integral gives the net area (= area above the axis minus area below the axis) under the curve of a function between the limits of integration. This implies

$$(7) \qquad \begin{aligned} &\text{(a)} \quad \int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx && \text{for even } g \\ &\text{(b)} \quad \int_{-L}^{L} h(x)\,dx = 0 && \text{for odd } h \end{aligned}$$

Formula (7b) implies the reduction to the cosine series (even f makes $f(x)\sin(n\pi x/L)$ odd since sin is odd) and to the sine series (odd f makes $f(x)\cos(n\pi x/L)$ odd since cos is even). Similarly, (7a) reduces the integrals in (6*) and (6**) to integrals from 0 to L. These reductions are obvious from the graphs of an even and an odd function. (Give a formal proof.)

Summary

Even Function of Period 2π. If f is even and $L = \pi$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx, \quad n = 1, 2, \cdots$$

Odd Function of Period 2π. If f is odd and $L = \pi$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx, \qquad n = 1, 2, \cdots .$$

EXAMPLE 4 Fourier Cosine and Sine Series

The rectangular wave in Example 1 is even. Hence it follows without calculation that its Fourier series is a Fourier cosine series, the b_n are all zero. Similarly, it follows that the Fourier series of the odd function in Example 2 is a Fourier sine series.

In Example 3 you can see that the Fourier cosine series represents $u(t) - E/\pi - \frac{1}{2}E \sin \omega t$. Can you prove that this is an even function? ■

Further simplifications result from the following property, whose very simple proof is left to the student.

THEOREM 1 Sum and Scalar Multiple

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

EXAMPLE 5 Sawtooth Wave

Find the Fourier series of the function (Fig. 268)

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

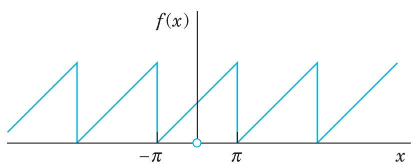


Fig. 268. The function $f(x)$. Sawtooth wave

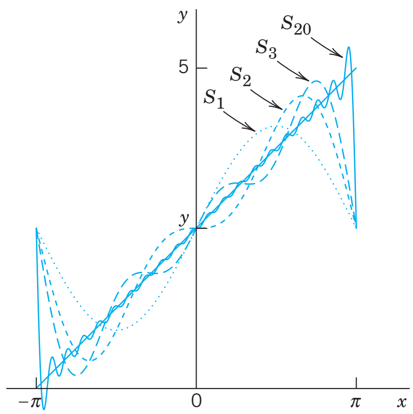


Fig. 269. Partial sums S_1, S_2, S_3, S_{20} in Example 5

Solution. We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$. The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π. Hence, by Theorem 1, the Fourier coefficients a_n, b_n are those of f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx.$$

Integrating by parts, we obtain

$$b_n = \frac{2}{\pi}\left[\frac{-x\cos nx}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right] = -\frac{2}{n}\cos n\pi.$$

Hence $b_1 = 2$, $b_2 = -\frac{2}{2}$, $b_3 = \frac{2}{3}$, $b_4 = -\frac{2}{4}, \cdots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right). \qquad \text{(Fig. 269)}$$

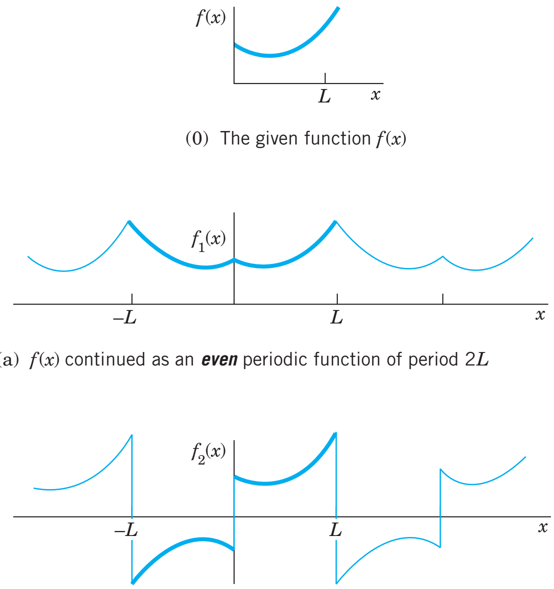
3. Half-Range Expansions

Half-range expansions are Fourier series. The idea is simple and useful. Figure 270 explains it. We want to represent $f(x)$ in Fig. 270.0 by a Fourier series, where $f(x)$ may be the shape of a distorted violin string or the temperature in a metal bar of length L, for example. (Corresponding problems will be discussed in Chap. 12.) Now comes the idea.

We could extend $f(x)$ as a function of period L and develop the extended function into a Fourier series. But this series would, in general, contain *both* cosine *and* sine terms. We can do better and get simpler series. Indeed, for our given f we can calculate Fourier coefficients from (6*) or from (6**). And we have a choice and can take what seems more practical. If we use (6*), we get (5*). This is the **even periodic extension** f_1 of f in Fig. 270a. If we choose (6**) instead, we get (5**), the **odd periodic extension** f_2 of f in Fig. 270b.

Both extensions have period $2L$. This motivates the name **half-range expansions**: f is given (and of physical interest) only on half the range, that is, on half the interval of periodicity of length $2L$.

Let us illustrate these ideas with an example that we shall also need in Chap. 12.



(0) The given function $f(x)$

(a) $f(x)$ continued as an ***even*** periodic function of period $2L$

(b) $f(x)$ continued as an ***odd*** periodic function of period $2L$

Fig. 270. Even and odd extensions of period $2L$

EXAMPLE 6 "Triangle" and Its Half-Range Expansions

Find the two half-range expansions of the function (Fig. 271)

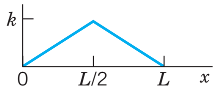


Fig. 271. The given function in Example 6

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\[2ex] \dfrac{2k}{L}(L - x) & \text{if} \quad \dfrac{L}{2} < x < L. \end{cases}$$

Solution. **(a)** ***Even periodic extension.*** From (6*) we obtain

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L - x)\,dx\right] = \frac{k}{2},$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_L^{L/2} x \cos\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L - x)\cos\frac{n\pi}{L}x\,dx\right].$$

We consider a_n. For the first integral we obtain by integration by parts

$$\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx = \frac{Lx}{n\pi}\sin\frac{n\pi}{L}x\,\Bigg|_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

Similarly, for the second integral we obtain

$$\int_{L/2}^{L}(L - x)\cos\frac{n\pi}{L}x\,dx = \frac{L}{n\pi}(L - x)\sin\frac{n\pi}{L}x\,\Bigg|_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi}{L}x\,dx$$

$$= \left(0 - \frac{L}{n\pi}\left(L - \frac{L}{2}\right)\sin\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right).$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 272a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) ***Odd periodic extension.*** Similarly, from (6**) we obtain

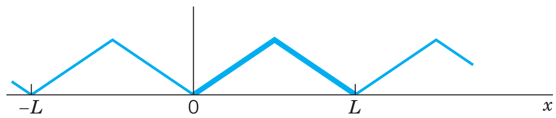
$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 272b)

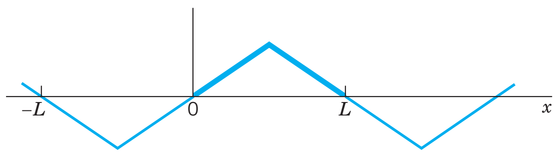
$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

The series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5.



(a) Even extension



(b) Odd extension

Fig. 272. Periodic extensions of $f(x)$ in Example 6

PROBLEM SET 11.2

1–7 EVEN AND ODD FUNCTIONS

Are the following functions even or odd or neither even nor odd?

1. e^x, $\quad e^{-|x|}$, $\quad x^3 \cos nx$, $\quad x^2 \tan \pi x$, $\quad \sinh x - \cosh x$

2. $\sin^2 x$, $\quad \sin(x^2)$, $\quad \ln x$, $\quad x/(x^2 + 1)$, $\quad x \cot x$

3. Sums and products of even functions

4. Sums and products of odd functions

5. Absolute values of odd functions

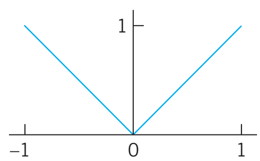
6. Product of an odd times an even function

7. Find all functions that are both even and odd.

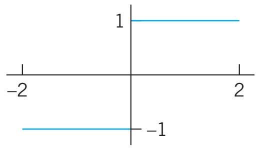
8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

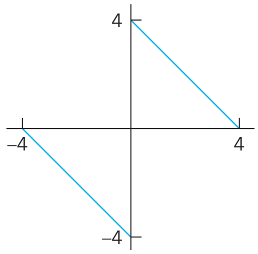
8.



9.



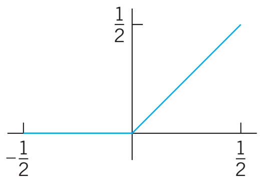
10.



11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

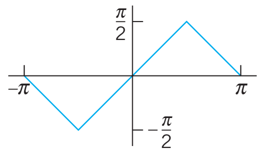
12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$

13.



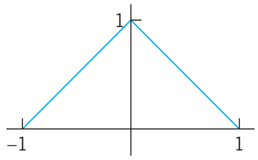
14. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$

15.



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$

17.



18. Rectifier. Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.

19. Trigonometric Identities. Show that the familiar identities $\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$ and $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.

20. Numeric Values. Using Prob. 11, show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2$.

21. CAS PROJECT. Fourier Series of $2L$-Periodic Functions. (a) Write a program for obtaining partial sums of a Fourier series (5).

(b) Apply the program to Probs. 8–11, graphing the first few partial sums of each of the four series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well. Compare and comment.

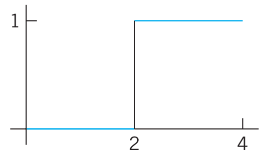
22. Obtain the Fourier series in Prob. 8 from that in Prob. 17.

23–29 HALF-RANGE EXPANSIONS

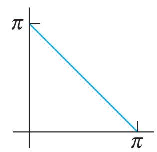
Find **(a)** the Fourier cosine series, **(b)** the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

23.

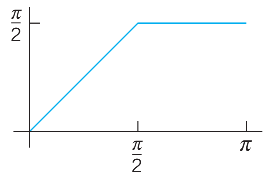
24.



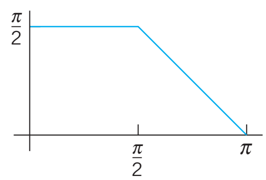
25.



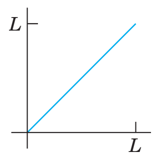
26.



27.



28.



29. $f(x) = \sin x \; (0 < x < \pi)$

30. Obtain the solution to Prob. 26 from that of Prob. 27.