5.7 Sturm-Liouville Problem (Orthogonal function)

Consider the equation of the form:

$[r(x)y']'+[q(x)+\lambda\ p(x)]y=0, \quad a\le x\le b ------(1)$

where $r(x), p(x), q(x)$ are known continuous function and $p(x)>0$

λ is a parameter .

and the boundary conditions:

$$\begin{cases} k_1 y(a)+k_2 y'(a)=0 \\ \ell_1 y(b)+\ell_2 y'(b)=0 \end{cases} ---------(2)$$

where k_1, k_2 (also l_1, l_2) are given constants which are not both zero.

Equation (1) is called a Sturm-Liouville equation.

Equation (1) with the boundary conditions (2) is called a Sturm-Liouville Problem.

For a specified parameter λ, if one can find the a non-trivial solution of (1) satisfying (2), this nontrivial solution is called an eigenfunction of the problem, and this λ is called a corresponding eigenvalue.

e.g.

Legendre's equation:

$(1-x^2)y''-2xy'+n(n+1)y=0 \quad -1<x<1$

→ $[(1-x^2)y']'+n(n+1)y=0$

i.e. $r(x)\equiv(1-x^2) \quad q(x)\equiv 0 \quad p(x)\equiv 1 \quad \lambda=n(n+1)$

Bessel equation(of order v with parameter λ)

$x^2y''+xy'+(\lambda^2x^2-v^2)y=0,$

→ $xy''+y'+(-\frac{v^2}{x}+\lambda^2x)y=0,$

→ $[xy']'+(-\frac{v^2}{x}+\lambda x)y=0$

i.e. $r(x)\equiv x \quad q(x)\equiv -\frac{v^2}{x} \quad p(x)\equiv x \quad \lambda=\lambda$

Example.

Equation: $y''+\lambda y=0 \qquad 0<x<2\pi$

Boundary Condition : $\begin{cases} y(0)=0 \\ y(2\pi)=0 \end{cases}$

※ Orthogonality:

A set of function $\{y_1, y_2, \ldots\ldots, y_m, \ldots\}$ defined on $a \le x \le b$ are said to be orthogonal on $a \le x \le b$ if

$$\int_a^b y_m(x) \cdot y_n(x) dx = 0 \qquad \text{for all } m \ne n$$

They are said to to be orthogonal on $a \le x \le b$ with respect to the weighted function $p(x)$ if $p(x) \ge 0$ for all x in (a,b) and

$$\int_a^b p(x) \cdot y_m(x) \cdot y_n(x) dx = 0 \qquad \text{for all } m \ne n$$

the norm of $y_m(x)$ (denoted by $\|y_m\|$) is defined :

$$\|y_m\| = \sqrt{\int_a^b y_m^2(x)\,dx} \quad \text{or} \quad \|y_m\| = \sqrt{\int_a^b p(x) y_m^2(x)\,dx}$$

If $\|y_m\| = 1$ for all m ,this function set is said to be orthonormal

※ Orthogonality of Eigenfunctions:

Theorem:

Suppose that the function $r(x), p(x), q(x)$ in the Sturm-Liouville equation (1) are continuous real valued function and and $p(x) \ge 0$ on the interval $a \le x \le b$.

Let y_m and y_n be eigenfunctions of the Sturm-Liouville problem (1) and (2) that correspond to distinct eigenvalues λ_m and λ_n, respectively. Then y_m and y_n are orthogonal on that interval with respect to the weight function $p(x)$.

Proof :

since y_m and y_n are the solutions associated with two distinct values of λ_m and λ_n. →

$$\frac{d}{dx}[r(x)y_m'] + [q(x) + \lambda_m\ p(x)]y_m = 0 ------(3)$$

$$\frac{d}{dx}[r(x)y_n'] + [q(x) + \lambda_n\ p(x)]y_n = 0 ------(4)$$

$$(3) \times y_n - (4) \times y_m$$

$$\Rightarrow (\lambda_m - \lambda_n) p y_m y_n = y_m \frac{d}{dx}[r(x)y_n'] - y_n \frac{d}{dx}[r(x)y_m'] ----(5)$$

$$\int_a^b (5)\, dx \quad \Rightarrow (\lambda_m - \lambda_n)\int_a^b p y_m y_n dx$$

$$= \int_a^b y_m \frac{d}{dx}[r(x) y_n']\, dx - \int_a^b y_n \frac{d}{dx}[r(x) y_m']\, dx ----(6)$$

$$\because \quad \int_a^b y_m \frac{d}{dx}[r(x) y_n']\, dx = r\, y_m y_n' \Big|_a^b - \int_a^b r(x) y_m' y_n'\, dx$$

$$\int_a^b y_n \frac{d}{dx}[r(x) y_m']\, dx = r\, y_n y_m' \Big|_a^b - \int_a^b r(x) y_m' y_n'\, dx$$

$$(6) \Rightarrow (\lambda_m - \lambda_n)\int_a^b p y_m y_n dx = r(x)(y_m y_n' - y_n y_m')\Big|_a^b$$

i.e.

$$(\lambda_m - \lambda_n)\int_a^b p y_m y_n dx = r(b)\begin{vmatrix} y_m(b) & y_n(b) \\ y_m'(b) & y_n'(b) \end{vmatrix} - r(a)\begin{vmatrix} y_m(a) & y_n(a) \\ y_m'(a) & y_n'(a) \end{vmatrix} ---(7)$$

since y_m and y_n must both satisfy the boundary conditions

$$\begin{cases} k_1 y(a) + k_2 y'(a) = 0 --- (A) \\ \ell_1 y(b) + \ell_2 y'(b) = 0 --- (B) \end{cases}$$

(A)→

$$\left.\begin{aligned} k_1 y_m(a) + k_2 y'_m(a) = 0 \\ k_1 y_n(a) + k_2 y'_n(a) = 0 \end{aligned}\right\} ----\text{(C)}$$

because k_1, k_2 are not both zero →

$$\begin{vmatrix} y_m(a) & y_n(a) \\ y_m'(a) & y_n'(a) \end{vmatrix} = 0; \quad \text{similarly, we have} \quad \begin{vmatrix} y_m(b) & y_n(b) \\ y_m'(b) & y_n'(b) \end{vmatrix} = 0$$

also since $\lambda_m \neq \lambda_n$, therefore (7) →

$$\int_a^b p(x) \cdot y_m \cdot y_n \cdot dx = 0$$

Note: the theorem can be modified as following:

(i) If $r(a)=0$, then the boundary condition at $x=a$ is not necessarily homogeneous.
(ii) if $r(b)=0$, then the boundary condition at $x=b$ is not necessarily homogeneous.
(iii) if $r(a)=r(b)=0$, then the boundary condition both at $x=a$ and $x=b$ are not necessarily homogeneous.
(iv) if $r(a)=r(b)\neq 0$, then the boundary condition (2) can be replaced by the periodic boundary conditions:

$$\begin{cases} y(a)=y(b) \\ y'(a)=y'(b) \end{cases}$$

Example.

$$y''+\lambda y=0 \qquad 0<x<2\pi$$

B.C. : $\begin{cases} y(0)=0 \\ y(2\pi)=0 \end{cases}$

The eigenvalues are $\lambda=m^2 \qquad m=1,2,3...$

The corresponding eigenfunctions are: $y_m=\sin mx$

Also $\int_0^{2\pi}\sin mx\cdot\sin nx\cdot dx=0$ when $m\neq n$

Example: Legendre’s equation

$$[(1-x^2)y']'+n(n+1)y=0 \qquad -1<x<1$$

since $r(x)=1-x^2 \rightarrow r(-1)=r(1)=0$

if n is an integer, since the Legendre’s polynomial $P_n(x)$ is a solution,

we have

$$\int_{-1}^{1} p_m(x)p_n(x)dx=0 \qquad \text{when } m\neq n$$

Example: Bessel equation(of order n with parameter λ)

$$x^2y''+xy'+(k^2x^2-n^2)y=0, \qquad x<0<R$$

B.C. $y(R)=0$

Since the equation can be rewritten as :

$$[xy']' + (-\frac{n^2}{x} + k^2 x)y = 0$$

i.e. $r(x) \equiv x \quad q(x) \equiv -\frac{n^2}{x} \quad p(x) \equiv x \quad \lambda = k^2$

$J_n(kx)$ is a solution of the equation.

the boundary condition → $J_n(kR) = 0$

If $\alpha_{1n}, \alpha_{2n}, \alpha_{3n}, \ldots$ are zeros of $J_n(x)$,

i.e. $J_n(\alpha_{mn}) = 0 \quad m = 1,2,3...$

→ the parameter k of the equation should be

$$kR = \alpha_{mn} \quad \Rightarrow \quad k = k_{mn} = \alpha_{mn}/R \qquad m = 1, 2, 3, \cdots ---- (A)$$

Hence, the eigenvalues and the corresponding eigenfunctions of the problem are :

$$\left.\begin{aligned} k &= \frac{\alpha_{mn}}{R} \\ y_m(x) &= J_n(\frac{\alpha_{mn}}{R}x) \end{aligned}\right\} \quad m = 1, 2, 3, \cdots ------ (B)$$

where α_{mn} are the mth zero of $J_n(x)$

since $r(0) = 0$ →

these eigenfunctions are orthogonal (with respected to the weight function $p(x) \equiv x$ in $0 < x < R$, i.e.

$$\int_0^R x \cdot J_n(\frac{\alpha_{in}}{R}x) \cdot J_n(\frac{\alpha_{jn}}{R}x)dx = 0 \quad \text{when i} \neq \text{j}$$

5.8 Orthogonal eigenfunction expansions

Consider a function set $\{y_m(x)\}$, i .e. $\{y_1(x), y_2(x) .... y_n(x) \cdots\}$

If there is a non-negative function $p(x)$ in (a,b) interval, such that

$$(y_m, y_n) \equiv \int_a^b p(x) y_m(x) y_n(x) dx = 0 \qquad \text{when } m \neq n$$

this function set is called an orthogonal set.(with respected to $p(x)$)

the norm of $y_m(x)$ is defined as:

$$\|y_m\| \equiv \sqrt{\int_a^b p(x) {y_m}^2(x) dx}$$

if each function in $\{y_m(x)\}$ is normalized by its norm, i.e.

$$\left\{ \frac{y_m(x)}{\|y_m\|} \right\}$$

since

$$\left(\frac{y_m(x)}{\|y_m\|} \cdot \frac{y_n(x)}{\|y_n\|} \right) \equiv \int_a^b p(x) \frac{y_m(x)}{\|y_m\|} \cdot \frac{y_n(x)}{\|y_n\|} dx$$

$$= \frac{1}{\|y_m\|} \cdot \frac{1}{\|y_n\|} \int_a^b p(x) y_m(x) y_n(x) dx = \delta_{mn} \equiv \begin{cases} 0 & \text{when m} \neq \text{n} \\ 1 & \text{when m} = \text{n} \end{cases}$$

where δ_{mn} is called Kronecker delta

in this case the function set $\left\{ \frac{y_m(x)}{\|y_m\|} \right\}$ is said to be Orthonormal.

- Orthogonal expansion (or Generalized Fourier series)

If $\{y_m(x)\}$ is an orthogonal set with respect to $p(x)$ on $a \leq x \leq b$.

If $f(x)$ is a given function in (a,b) and expanded in terms of $\{y_m(x)\}$, i.e.

$$\boxed{f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + a_2 y_2(x) + \cdots -- (1)}$$

(1) is called an orthogonal expansion (or generalized Fourier series) of $f(x)$.

If $\{y_m(x)\}$ is a set formed by the eigenfunctions of a

Sturm-Liouville problem, (1) is an eigenfunction expansion of $f(x)$.

The coefficients a_m in (1) can be determined as following:

$$\int_a^b (1) p(x) y_n(x) dx \Rightarrow$$

$$\int_a^b p(x) f(x) y_n(x) dx = \int_a^b p(x) \left[\sum_{m=0}^{\infty} a_m y_m \right] y_n dx$$

$\Rightarrow$

$$(f, y_n) = \sum_{m=0}^{\infty} a_m (y_m, y_n) = a_n \|y_n\|^2 \quad \text{since } (y_m, y_n) = 0, \text{when m} \neq \text{n}$$

→ the coefficients can be obtained:

$$\boxed{a_n = \frac{1}{\|y_n\|^2} \int_a^b p(x) f(x) y_n(x) dx}$$

※ **Fourier series**

We firstly consider the Sturm-Liouville equation with periodic boundary :

$$y'' + \lambda^2 y = 0 \qquad -p < x < p \text{-------------} (A.1)$$

$$\text{B.C.} \quad \left. \begin{aligned} y(-p) &= y(p) \\ y'(-p) &= y'(p) \end{aligned} \right\} \text{--------} (A.2)$$

Compare with the standard form:

$[r(x)y']' + [q(x) + \lambda\ p(x)]y = 0, \quad a \le x \le b$

i.e. (A.1) is a special form of the Sturm-Liouville equation with

$r(x) \equiv 1; \quad q(x) \equiv 0; \quad p(x) \equiv 1; \quad (a,b) \equiv (-p, p); \quad \lambda \to \lambda^2$

since $r(-p) = r(p) \equiv 1$,

→ the eigenfunctions of (A.1),(A.2) form an orthogonal set.

In the following, we find the non-trivial solutions of (A.1),(A.2):

Sine the general solution of (A.1) is:

$$y = A\cos\lambda x + B\sin\lambda x \text{-------------} (A.3)$$

substitute (A.3) into(A.2)→

$$\left.\begin{array}{l}(A\cos\lambda p - B\sin\lambda p) - (A\cos\lambda p + B\sin\lambda p) = 0\\ (-\lambda A\sin\lambda p + \lambda B\cos\lambda p) - (\lambda A\sin\lambda p + \lambda B\cos\lambda p) = 0\end{array}\right\}$$

$$\rightarrow \left.\begin{array}{l}B\sin\lambda p = 0\\ A\sin\lambda p = 0\end{array}\right\}$$

if A, B not both zero → $\sin\lambda p = 0$ → $\lambda p = m\pi$ m = 0,1,2,3...

Hence the problem (A.1)(A.2) will have non-trivial solution when

$\lambda = \dfrac{m\pi}{p}$ m = 0,1,2,.... → the eigenvalues

and the corresponding eigenfunctions are:

$$y_m = A\cos\frac{m\pi x}{p} + \mathrm{B}\sin\frac{m\pi x}{p} \quad \mathrm{m} = 0,1,2,3...$$

where A, B are the arbitrary constants

if we choose $A = 1, B = 0$ we have $y_m = \cos\dfrac{m\pi x}{p}$ m = 0,1,2,3...

if we choose $A = 0, B = 1$ we have $y_m = \sin\dfrac{m\pi x}{p}$ m = 0,1,2,3...

according to the theorem, this set of eigenfunctions, i.e.

$\left\{1, \cos\dfrac{m\pi x}{p}, \sin\dfrac{m\pi x}{p} \quad \mathrm{m} = 1,2,3...\right\}$ form an orthogonal set in the interval $-p < x < p$.

If $f(x)$ is a function in $-p < x < p$, and is expanded by this orthogonal set, i.e.

$$f(x) = a_0 + \sum_{m=1}^{\infty}\left\{a_m \cos\frac{m\pi}{p}x + b_m \sin\frac{m\pi}{p}x\right\} - - - - - (2)$$

(2) is called the Fourier series of $f(x)$.

a_m, b_m are called the Fourier coefficients of $f(x)$

sinec the norm of each function in the set are:

$$\|1\|^2 = \int_{-p}^{p} 1^2\, dx = 2p$$

$$\left\|\cos\frac{m\pi x}{p}\right\|^2 = \int_{-p}^{p}\left[\cos\frac{m\pi x}{p}\right]^2 dx = p$$

$$\left\| \sin \frac{m\pi x}{p} \right\|^2 = \int_{-p}^{p} \left[\sin \frac{m\pi x}{p} \right]^2 dx = p$$

using the formula:

$$\boxed{a_n = \frac{1}{\|y_n\|^2} \int_a^b p(x) f(x) y_n(x) dx}$$

Hence the Fourier coefficients of (2) is given by:

$$\left. \begin{aligned} a_0 &= \frac{1}{2p} \int_{-p}^{p} f(x) dx \\ \begin{Bmatrix} a_m \\ b_m \end{Bmatrix} &= \frac{1}{p} \int_{-p}^{p} f(x) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \frac{m\pi x}{p} dx \end{aligned} \right\} ------(3)$$

Example: Given a periodic $f(x) = \begin{cases} -1 & if \quad -\pi < x < 0 \\ 1 & if \quad 0 < x < \pi \end{cases}$ and $f(x) = f(x + 2\pi)$, Expand $f(x)$ by the orthogonal set $\left\{ 1, \cos \frac{m\pi x}{p}, \sin \frac{m\pi x}{p} \quad \text{m} = 1,2,3... \right\}$ with $p = \pi$

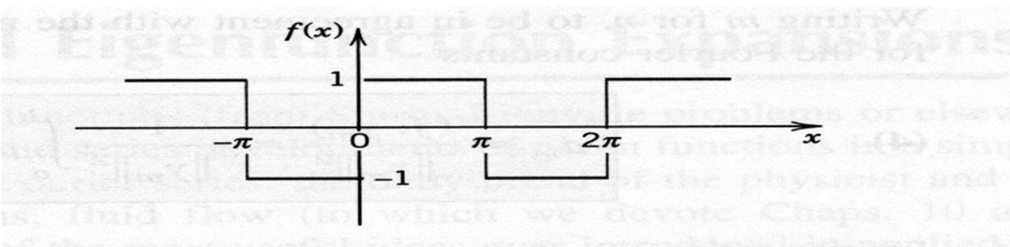

Fig. 106. Periodic square wave in Example 1

By using (2) the Fourier series of $f(x)$ is :

$$f(x) = a_0 + \sum_{m=1}^{\infty} \{ a_m \cos mx + b_m \sin mx \} ------(A)$$

then

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(x) \cdot 1 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot 1 dx = 0$$

$$a_m = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{m\pi}{p}x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cdot\cos mx\,dx + \int_{0}^{\pi}(1)\cdot\cos mx\,dx\right] = 0$$

$$b_m = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{m\pi}{p}x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cdot\sin mx\,dx + \int_{0}^{\pi}(1)\cdot\sin mx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left.\frac{\cos mx}{m}\right|_{-\pi}^{0} - \left.\frac{\cos mx}{m}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi m}[1 - 2\cos m\pi + 1] = \frac{2}{m\pi}[1-(-1)^m] = \begin{cases}\dfrac{4}{m\pi} & m = 1, 3, 5, \cdots \\ 0 & m = 2, 4, 6, \cdots\end{cases}$$

$$\therefore \quad f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\cdots\right)$$

Example: Fourier-Legendre series

When $f(x)$ is a function in $-1 < x < 1$, and expanded in Legendre Polynomial:

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots$$

$$= a_0 + a_1 x + a_2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \cdots$$

since Legendre polynomial $P_m(x)$ are the eigenfunctions of a Sturm-Liouville problem, and orthogonal on $-1 \le x \le 1$, and

$$\left\|P_m\right\|^2 \equiv \int_{-1}^{1} P_m^2(x)dx = \frac{2}{2m+1}$$

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)p_m(x)dx$$

Example: Fourier-Bessel series

$\left\{J_n(\frac{\alpha_{mn}}{R}x) \quad m = 1,2,3...\right\}$ are orthogonal on an interval $0 \le x \le R$ with respect to weight x. If a function $f(x)$ in $(0, R)$ is expanded by this orthogonal set. The corresponding Fourier-Bessel series is

$$f(x) = \sum_{m=1}^{\infty} a_m J_n\left(\frac{\alpha_{mn} x}{R}\right)$$

since

$$\left\| J_n\left(\frac{\alpha_{mn}}{R} x\right) \right\|^2 \equiv \int_0^R x \cdot \left[J_n\left(\frac{\alpha_{mn}}{R} x\right)\right]^2 dx = \frac{R^2}{2}\left[J_{n+1}(\alpha_{mn})\right]^2$$

Hence the coefficients are given by:

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{mn})} \int_0^R x f(x) J_n\left(\frac{\alpha_{mn}}{R} x\right) dx, \quad m = 1, 2, \cdots$$