機率與統計 Exercise4 Chapter4　B06505047 陳銘杰

Problem 4.3.4

$$F_X(x) = \int_{t=-\infty}^{x} f_X(t)dt = \begin{cases} \int_{t=0}^{x} a^2 t e^{\frac{-a^2t^2}{2}} dt, x > 0 \\ 0, otherwise \end{cases} = \begin{cases} -e^{\frac{-a^2t^2}{2}}, x > 0 \\ 0, otherwise \end{cases}$$

Problem 4.4.7

$$f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} \frac{1}{8}, -5 \le u < -3 \\ \frac{3}{8}, 3 \le u < 5 \\ 0, otherwise \end{cases}$$

(a) $\mathrm{E[U]} = \int_{u=-\infty}^{\infty} uf_U(u)du = \int_{u=-5}^{-3} \frac{u}{8} du + \int_{u=3}^{5} \frac{3u}{8} du$

$$= \frac{u^2}{16}\Big|_{u=-5}^{u=-3} + \frac{3u^2}{16}\Big|_{u=3}^{u=5}$$

$$= \frac{9-25}{16} + \frac{3(25-9)}{16}$$

$$= -1 + 3$$

$$= 2$$

$$\mathrm{Var[U]} = \int_{u=-\infty}^{\infty} (u-2)^2 f_U(u)du$$

$$= \int_{u=-5}^{-3} \frac{(u-2)^2}{8} du + \int_{u=3}^{5} \frac{3(u-2)^2}{8} du$$

$$= \int_{t=-7}^{-5} \frac{t^2}{8} dt + \int_{u=1}^{3} \frac{3t^2}{8} dt \ , (t = u-2)$$

$$= \frac{t^3}{24}\Big|_{u=-7}^{u=-5} + \frac{t^3}{8}\Big|_{u=1}^{u=3}$$

$$= \frac{-125+343}{24} + \frac{27-1}{8}$$

$$= \frac{218}{24} + \frac{26}{8} = \frac{296}{24} = \frac{37}{3}$$

(b) $E[2^U] = \int_{u=-\infty}^{\infty} 2^u f_U(u)du = \int_{u=-5}^{-3} \frac{2^u}{8} du + \int_{u=3}^{5} \frac{3\times 2^u}{8} du$

$$= \frac{2^u}{8 \times \ln(2)} \Big|_{u=-5}^{u=-3} + \frac{3 \times 2^u}{8 \times \ln(2)} \Big|_{u=3}^{u=5}$$

$$= \frac{\frac{1}{8} - \frac{1}{32} + 96 - 24}{8 \times \ln(2)}$$

$$= 13.00116$$

Problem 4.5.12

Uniform random variable:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, a \leq x < b \\ 0, otherwise \end{cases}$$

$$\mu_X = \frac{b+a}{2} = 7, Var[X] = \frac{(b-a)^2}{12} = 3$$

$$\mathrm{b} + \mathrm{a} = 14\,, \mathrm{b} - \mathrm{a} = 6 \;\; => \mathrm{a} = 4, \mathrm{b} = 10$$

$$f_X(x) = \begin{cases} \frac{1}{6}, 4 \leq x < 10 \\ 0, otherwise \end{cases}$$

Problem 4.6.14

(a) $f_X(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(x-k)^2}{2t}}$

$$E[V] = \int_{x=k}^{\infty} (x-k) \frac{1}{\sqrt{2\pi t}} e^{\frac{-(x-k)^2}{2t}} dx$$

$$= \int_{v=0}^{\infty} v \frac{1}{\sqrt{2\pi t}} e^{\frac{-v^2}{2t}} dv\,, v = x - k$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{v=0}^{\infty} v e^{\frac{-v^2}{2t}} dv$$

$$= \frac{t}{\sqrt{2\pi t}} \int_{0}^{\infty} e^{-w}\, dw = \sqrt{\frac{t}{2\pi}}\,, (w = \frac{v^2}{2t}\,, dw = \frac{v}{t} dv)$$

(b) $P[R>0]=P[V>d]$

$\mathrm{P}[R_0>0]=P[V>d_0], V=(X-k)^+, P[X>k]=\frac{1}{2}=P[X-k>0]$

$d_0=0$

(c) $\mathrm{E[R]=E[V]-d}, \sqrt{\frac{t}{2\pi}}-d_1=0.01d_1, d_1=\frac{100}{101}\sqrt{\frac{t}{2\pi}}$

(d)

Problem 4.7.6

$$f_X(x)=\begin{cases}\frac{1}{3}e^{-\frac{1}{3}x}, x\geq 0\\ 0, otherwise\end{cases}, F_X(x)=\begin{cases}1-e^{-\frac{1}{3}x}, x\geq 0\\ 0, otherwise\end{cases}$$

(a) $F_W(w)=\begin{cases}\frac{1}{2}+\frac{1}{2}F_X(w), w\geq 0\\ 0, otherwise\end{cases}$

$$=\begin{cases}1-\frac{1}{2}e^{-\frac{1}{180}w}, w\geq 0\\ 0, otherwise\end{cases}$$

(b) $f_W(w)=\begin{cases}\frac{1}{2}\delta(w)+\frac{1}{360}e^{-\frac{1}{180}w}, w\geq 0\\ 0, otherwise\end{cases}$

(c) $\mathrm{E[W]}=0+\frac{1}{2}\times 60\times 3=90$

$$\mathrm{Var[W]}=\mathrm{E}[W^2]-(E[W])^2=\frac{1}{2}\int_0^\infty v^2\frac{1}{180}e^{-\frac{1}{180}v}dv-90^2$$

$$=32400-8100=24300$$