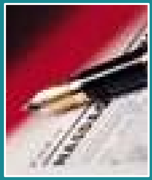


Chapter 5
The Discrete-Time Fourier Transform

5.0 Introduction

In the chapter, we take advantage of the similarities between continuous-time and discrete-time Fourier analysis by following a strategy essentially identical to that used in Chapter 4.

在本章後段將利用連續時間與離散時間傅立葉分析的相似特性，來進行與第4章相同的分析策略。

5.1.1 Development of the Discrete-Time Fourier Transform

Consider a general sequence *x[n]* that is of finite duration. That is, for some integers N_1 and N_2, *x[n]* = 0 outside the range $-N_1 \le n \le N_2$. A signal of this type is illustrated I Figure 5.1(a).

設*x[n]*為有限區間訊號。

令 $\tilde{x}[n]$ 為週期訊號，且一個週期恰與*x[n]*相同。

5.1.1 Development of the Discrete-Time Fourier Transform

$\tilde{x}[n]$的離散時間傅立葉級數

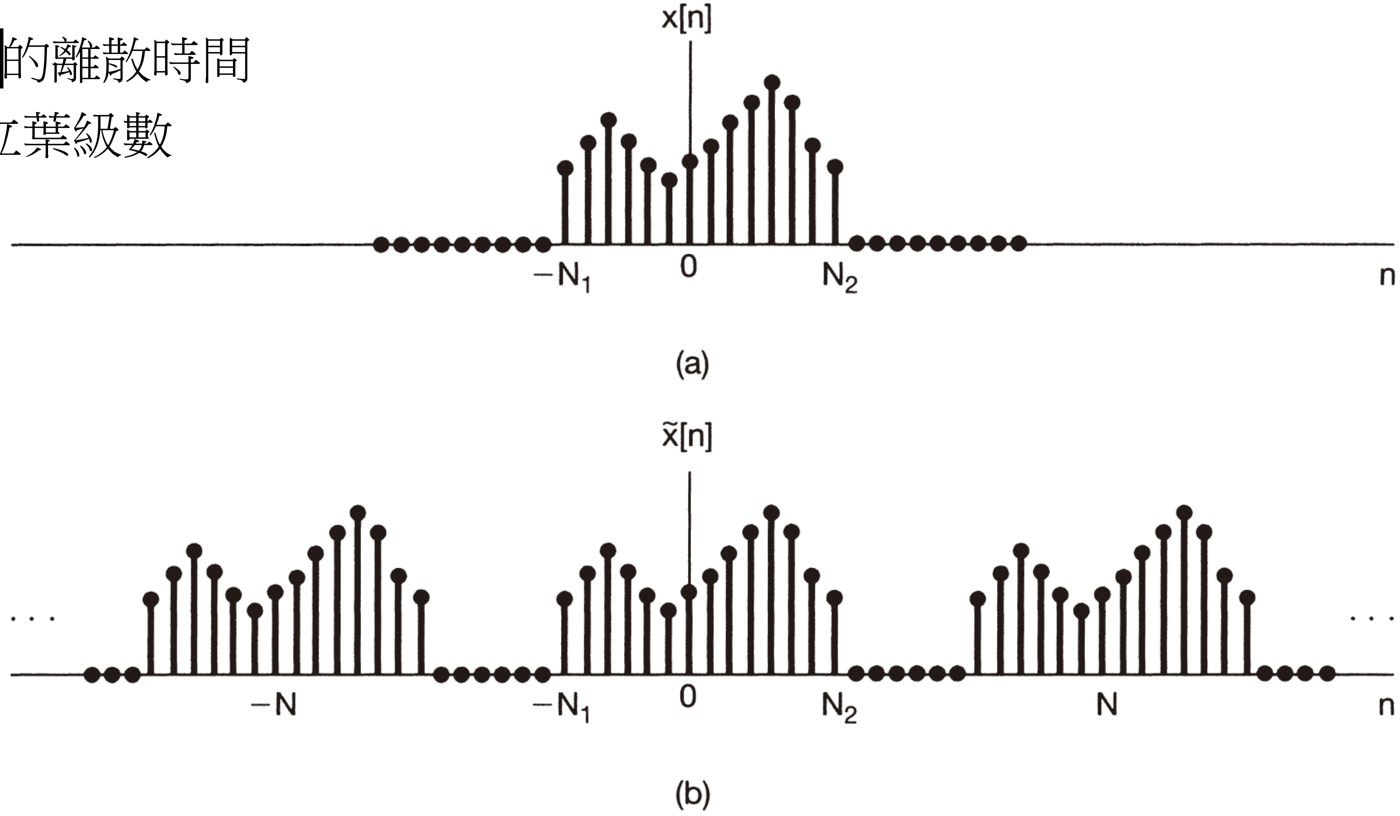


Figure 5.1 (a) Finite-duration signal $x[n]$; (b) periodic signal $\tilde{x}[n]$ constructed to be equal to $x[n]$ over one period.

5.1.1 Development of the Discrete-Time Fourier Transform

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n} \qquad (5.2)$$

Since $x[n] = \tilde{x}[n]$ over a period that includes the interval $-N_1 \le n \le N_2$, it is convenient to choose the interval of summation in eq. (5.2) to include this interval, to that $\tilde{x}[n]$ can be replaced by *x[n]* in the summation. Therefore,

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(2\pi/N)n}, \qquad (5.3)$$

5.1.1 Development of the Discrete-Time Fourier Transform

Where in the second equality in eq. (5.3) we have used the fact that *x[n]* is zero outside the interval $-N_1 \leq n \leq N_2$. Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \tag{5.4}$$

we see that the coefficients a_k are proportional to samples of $X(e^{j\omega})$, i.e.,

$$a_k = \frac{1}{N} X(e^{jk\omega_0}), \tag{5.5}$$

5.1.1 Development of the Discrete-Time Fourier Transform

Where $\omega_0 = 2\pi / N$ is the spacing of the samples in the frequency domain. Combining eqs. (5.1) and (5.5) yields

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}. \qquad (5.6)$$

Since $\omega_0 = 2\pi / N$, or equivalently, $1/N = \omega_0 / 2\pi$ eq. (5.6) can be rewritten as

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0. \qquad (5.7)$$

5.1.1 Development of the Discrete-Time Fourier Transform

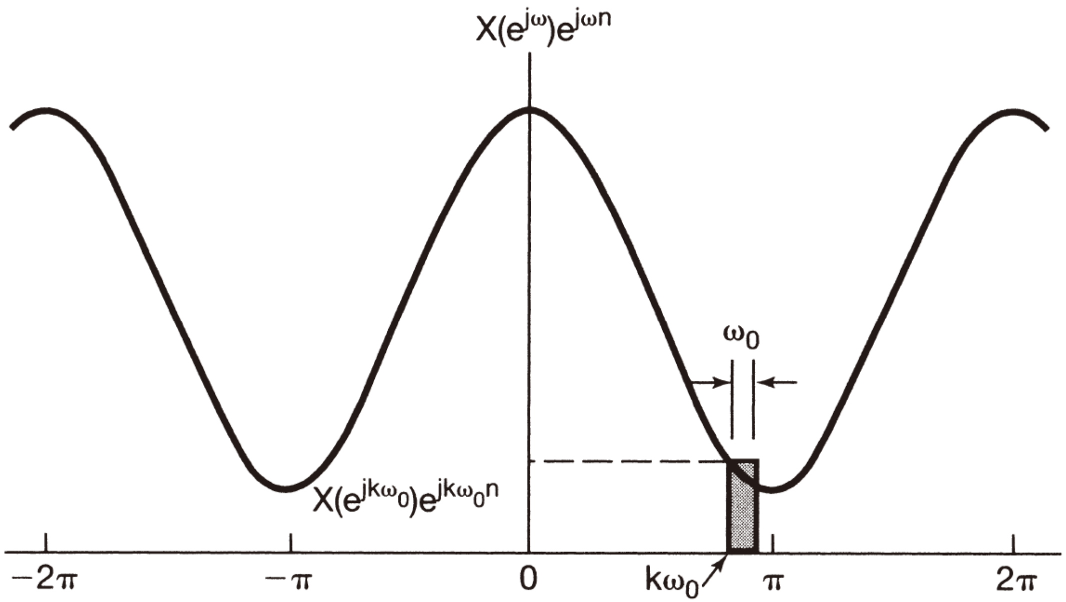


Figure 5.2 Graphical interpretation of eq. (5.7).

5.1.1 Development of the Discrete-Time Fourier Transform

As $N \to \infty$, $\tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega,$$

當 $N \to \infty, \tilde{x}[n] \to x[n]$，可得*x[n]*與$X(e^{j\omega})$關係式。

Since $X(j^{j\omega})e^{j\omega n}$ is periodic with period 2π, the interval of integration can be taken as any interval of length 2π.

離散時間傅立葉轉換對

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \tag{5.8}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}. \tag{5.9}$$

5.1.1 Development of the Discrete-Time Fourier Transform

The Fourier transform $X(e^{j\omega})$ will often be referred to as the spectrum of *x[n]*

$X(e^{j\omega})$ 常稱為*x[n]*的頻譜。

5.1.1 Development of the Discrete-Time Fourier Transform

As our derivation indicates, the discrete-time Fourier transform shares many similarities with the continuous-time case. The major differences between the two are the periodicity of the discrete-time transform $X(e^{j\omega})$ and the finite interval of integration in the synthesis equation.

離散時間與連續時間有許多的相似性。但兩者之間的主要差異在於離散時間傅立葉轉換 $X(e^{j\omega})$ 為週期性的，且積分區間是有限範圍的。

$X(e^{j\omega})$ 的週期為2 π 。

5.1.1 Development of the Discrete-Time Fourier Transform

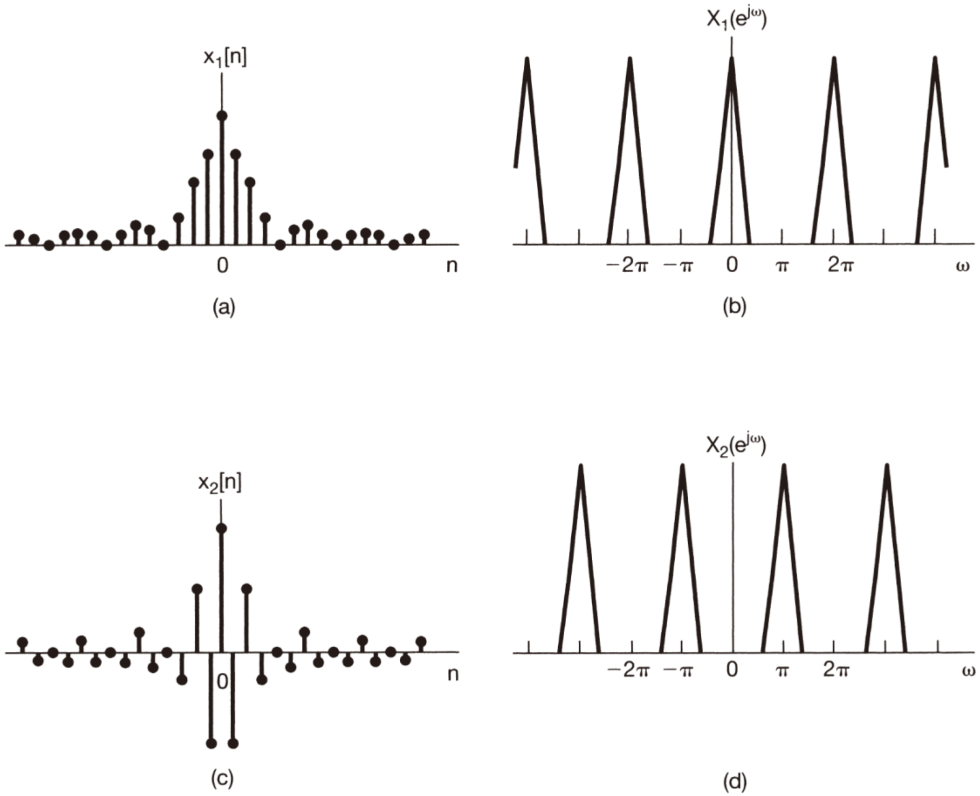


Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$.

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

Specifically, eq. (5.9) will converge either if *x[n]* is absolutely summable, that ,is

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty, \tag{5.13}$$

若*x[n]*為絕對可加，則其傅立葉轉換存在。

or if the sequence has finite energy, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty. \tag{5.14}$$

若*x[n]*具有有限能量，則其傅立葉轉換存在。

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

In particular, if we approximate an aperiodic signal *x[n]* by an integral of complex exponentials with frequencies taken over the interval $|\omega| \le W$, i.e.,

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})e^{j\omega n}d\omega. \qquad (5.15)$$

當*W*=π時，(5.15)式所得的$\hat{x}[n]$ 等於*x[n]*。

5.2 The Fourier Transform for Periodic Signals

As in the continuous-time case, discrete-time periodic signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an impulse train in the frequency domain.

由離散時間傅立葉轉換可知，一週期訊號在頻域上為一個脈衝串。

$$x[n] = e^{j\omega_0 n}. \tag{5.17}$$

5.2 The Fourier Transform for Periodic Signals

In tact, the Fourier transform of *x[n]* is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l), \qquad (5.18)$$

$e^{j\omega_0 n}$的傅立葉轉換

5.2 The Fourier Transform for Periodic Signals

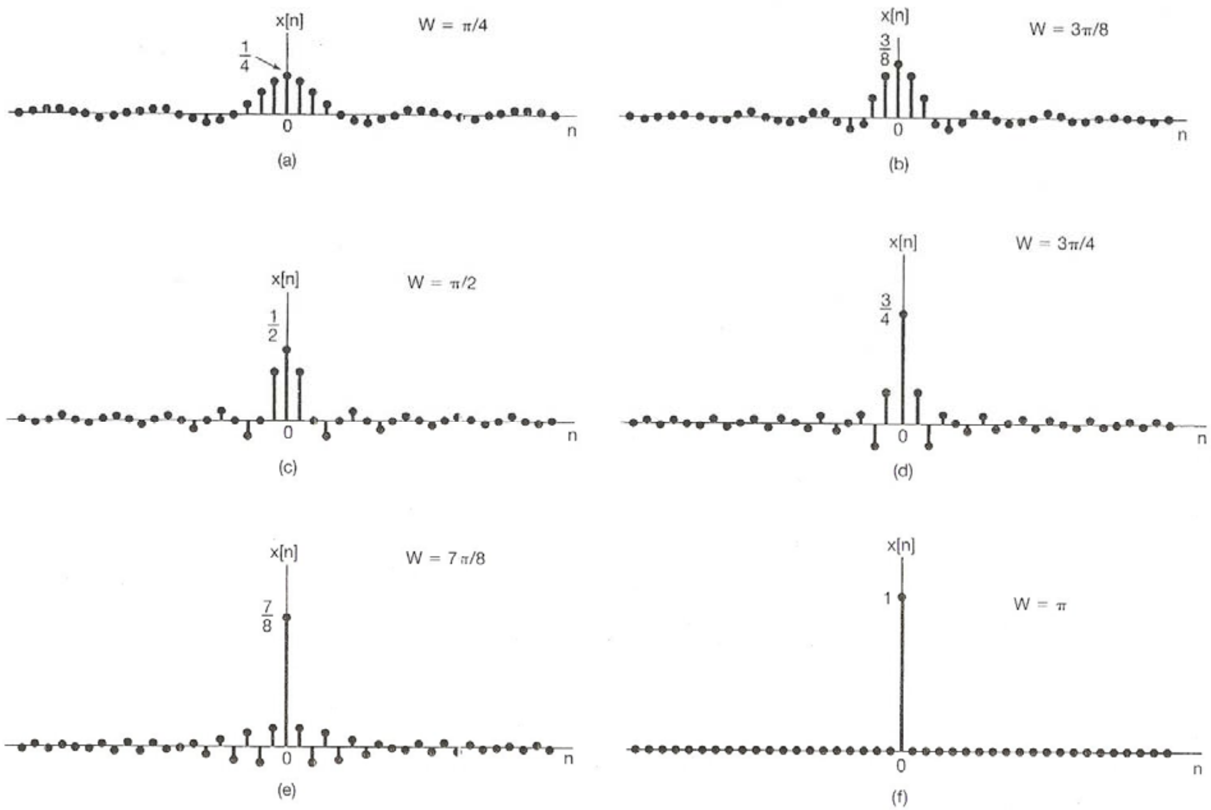


Figure 5.7 Approximation to the unit sample obtained as in eq. (5.16) using complex exponentials with frequencies $|\omega| \leq W$: (a) $W = \pi/4$; (b) $W = 3\pi/8$; (c) $W = \pi/2$; (d) $W = 3\pi/4$; (e) $W = 7\pi/8$; (f) $W = \pi$. Note that for $W = \pi$, $\hat{x}[n] = \delta[n]$.

5.2 The Fourier Transform for Periodic Signals

Which is illustrated in Figure 5.8. In order to check the validity of this expression, we must evaluate its inverse transform. Substituting eq. (5.18) into the synthesis equation (5.8), we find that

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)e^{j\omega n}d\omega.$$

5.2 The Fourier Transform for Periodic Signals

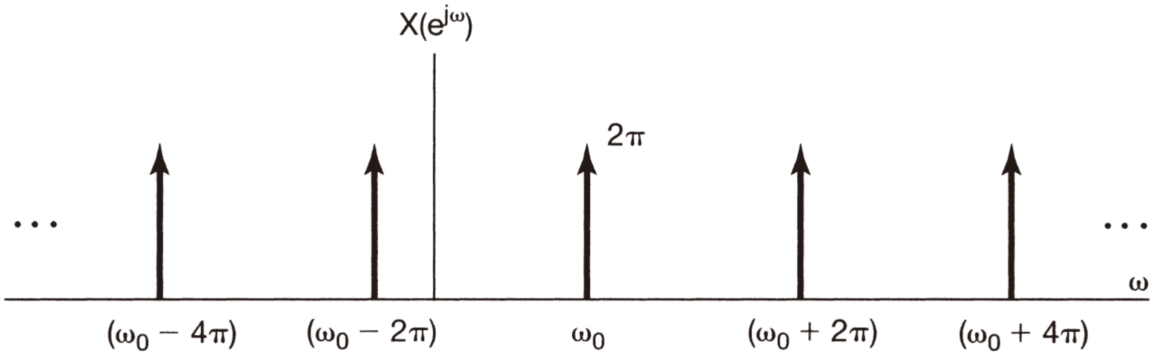


Figure 5.8 Fourier transform of $x[n] = e^{j\omega_0 n}$.

5.2 The Fourier Transform for Periodic Signals

If the interval of integration chosen includes the impulse located at $\omega_0 + 2\pi r$, then

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}.$$

5.2 The Fourier Transform for Periodic Signals

設*x[n]*為週期序列，週期為*N*。

Now consider a periodic sequence *x[n]* with period *N* and with the Fourier series representation

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}. \qquad (5.19)$$

*x[n]*的傅立葉級數表示法

In this case, the Fourier transform is

$$X(e^{j\omega})=\sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega-\frac{2\pi k}{N}\right), \qquad (5.20)$$

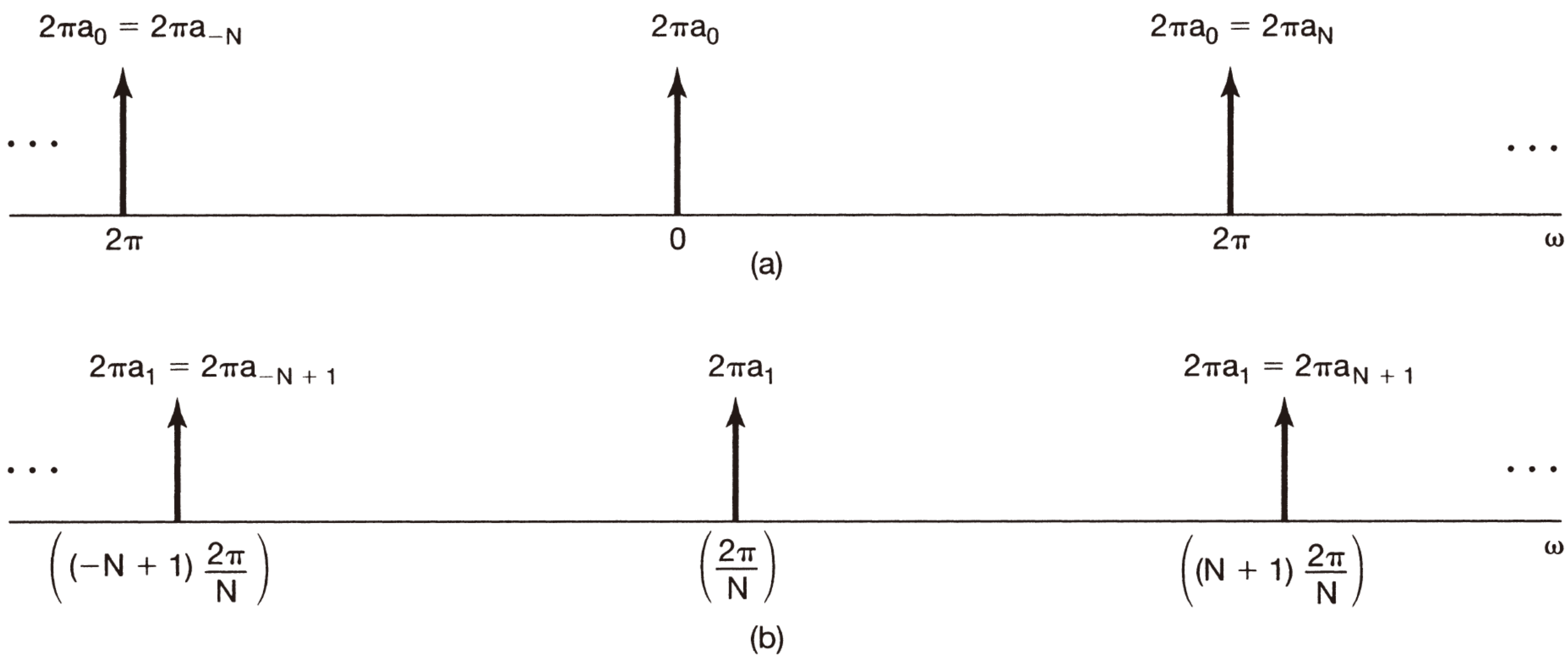
*x[n]*的傅立葉轉換(可由傅立葉係數直接建立)

5.2 The Fourier Transform for Periodic Signals

In particular, suppose that we choose the interval of summation in eq. (5.19) as k =0,1,…, N-1, so that

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \ldots + a_{N-1} e^{j(N-1)(2\pi/N)n}. \quad (5.21)$$

5.2 The Fourier Transform for Periodic Signals



5.2 The Fourier Transform for Periodic Signals

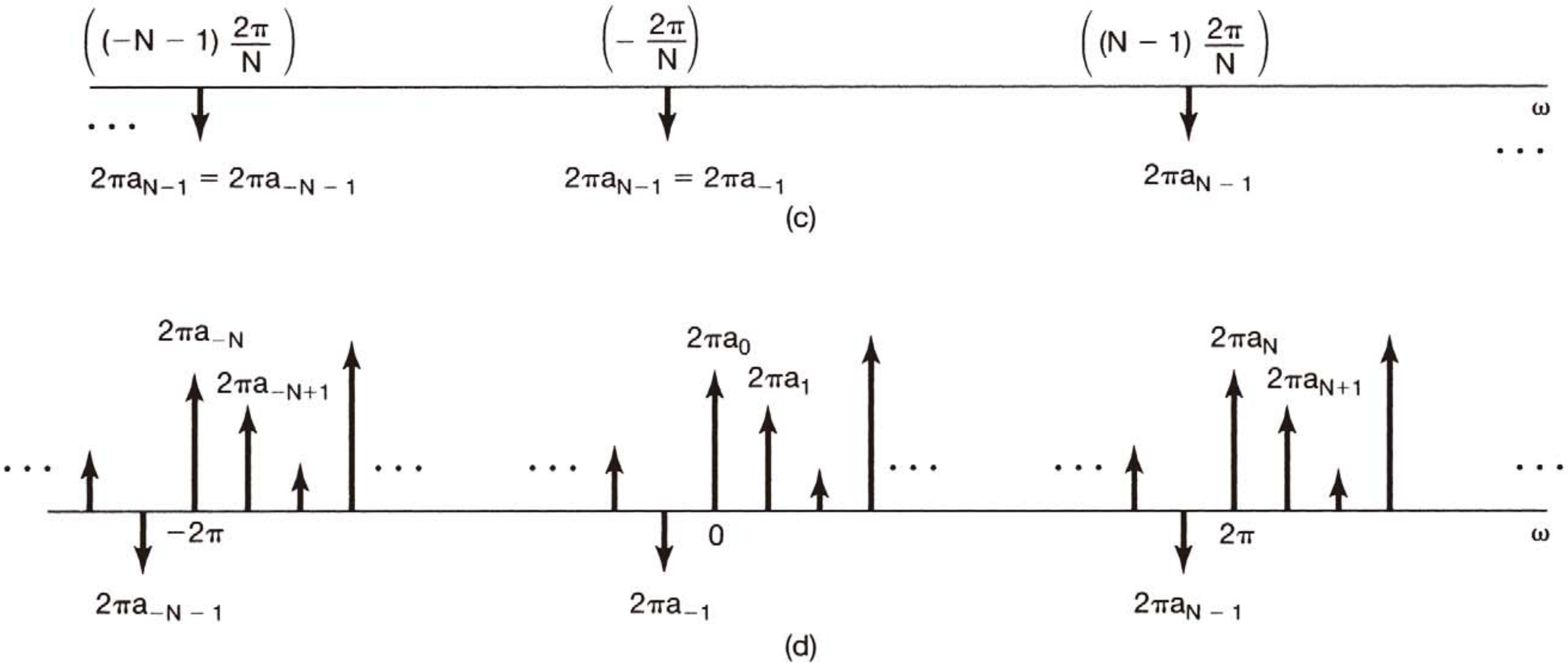


Figure 5.9 Fourier transform of a discrete-time periodic signal: (a) Fourier transform of the first term on the right-hand side of eq. (5.21); (b) Fourier transform of the second term in eq. (5.21); (c) Fourier transform of the last term in eq. (5.21); (d) Fourier transform of $x[n]$ in eq (5.21).

5.3 Properties of the Discrete-Time Fourier Transform

By comparing this table with Table 4.1, we can get a clear picture of some of the similarities and differences between continuous-time and discrete-time Fourier transform properties.

比較表4.1及5.1可瞭解連續時間與離散時間傅立葉轉換的一些異同點。

5.3 Properties of the Discrete-Time Fourier Transform

In the following discussions, it will be convenient to adopt notation similar to that used in Section 4.3 to indicate the pairing of a signal and its transform. That is,

傅立葉轉換

$$X(e^{j\omega}) = F\{x[n]\},$$

反傅立葉轉換

$$x[n] = F^{-1}\{X(e^{j\omega})\},$$

訊號與其轉換關係符號

$$x[n] \xleftrightarrow{F} X(e^{j\omega}).$$

5.3.1 Periodicity of the Discrete-Time Fourier Transform

As we discussed in Section 5.1, the discrete-time Fourier transform is always periodic in ω with period 2π; i.e.,

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega}). \qquad (5.28)$$

$X(e^{j\omega})$必為週期函數，周期為2π。

5.3.2 Linearity of the Fourier Transform

If

$$x_1[n] \xleftrightarrow{F} X_1(e^{j\omega})$$

And

$$x_2[n] \xleftrightarrow{F} X_2(e^{j\omega}),$$

Then

$$ax_1[n] + bx_2[n] \xleftrightarrow{F} aX_1(e^{j\omega}) + bX_2(e^{j\omega}). \tag{5.29}$$

線性性質(重疊性質)

5.3.3 Time Shifting and Frequency Shifting

If

$$x[n] \xleftrightarrow{F} X(e^{j\omega}),$$

Then

$$x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega}) \qquad (5.30)$$

時間移位性質

And

$$e^{j\omega_0 n} x[n] \xleftrightarrow{F} X(e^{j(\omega-\omega_0)}). \qquad (5.31)$$

頻率移位性質

5.3.4 Conjugating and Conjugate Symmetry

If

$$x[n] \xleftrightarrow{F} X(e^{j\omega}),$$

Then

$$x^*[n] \xleftrightarrow{F} X^*(e^{-j\omega}). \tag{5.35}$$

Also, if *x[n]* is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \qquad [x[n] real]. \tag{5.36}$$

若*x[n]*為實值訊號，則$X(e^{j\omega})$具有共軛對稱性質。

5.3.4 Conjugating and Conjugate Symmetry

From this, it follows that $\Re e\{x(e^{j\omega})\}$ is an even function of ω and $\mathcal{G}m\{X(e^{j\omega})\}$ is an odd function of ω. Similarly, the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function. Furthermore,

$$\mathcal{E}v\{x[n]\} \stackrel{F}{\longleftrightarrow} \Re e\{X(e^{j\omega})\}$$

and

$$\mathcal{O}d\{x[n]\} \stackrel{F}{\longleftrightarrow} j\,\mathcal{G}m\{X(e^{j\omega})\},$$

5.3.5 Differencing and Accumulation

From the linearity and time-shifting properties, the Fourier transform pair for the first-difference signal $x[n]-x[n-1]$ is given by

$$x[n]-x[n-1] \overset{F}{\longleftrightarrow} (1-e^{-j\omega})X(e^{j\omega}). \tag{5.37}$$

差分性質

Next, consider the signal

$$y[n]=\sum_{m=-\infty}^{n} x[m]. \tag{5.38}$$

5.3.5 Differencing and Accumulation

Since $y[n]-y[n-1]=x[n]$, we might conclude that the transform of *y[n]* should be related to the transform of *x[n]* by division by $(1-e^{-j\omega})$. This is partly correct, but as with the continuous-tie integration property given by eq. (4.32), there is more involved. The precise relationship is

$$\sum_{m=-\infty}^{n} x[m] \overset{F}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega-2\pi k).$$

(5.39)

5.3.6 Time Reversal

Let *x[n]* be a signal with spectrum $X(e^{j\omega})$, and consider the transform $Y(e^{j\omega})$ of $y[n] = x[-n]$. From eq. (5.9),

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n]e^{-j\omega n}. \qquad (5.40)$$

Substituting *m* = - *n* into eq.(5.40), we obtain

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega}). \qquad (5.41)$$

That is,

$$x[-n] \stackrel{F}{\longleftrightarrow} X(e^{-j\omega}). \qquad (5.42)$$

時間倒轉性質

5.3.7 Time Expansion

In Section 4.3.5 we derived the continuous=time property

$$x(at) \stackrel{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right). \tag{5.43}$$

There is a result that does closely parallel eq. (5.43), however. Let *k* be a positive integer, and define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases} \tag{5.44}$$

see that the Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}.$$

5.3.7 Time Expansion

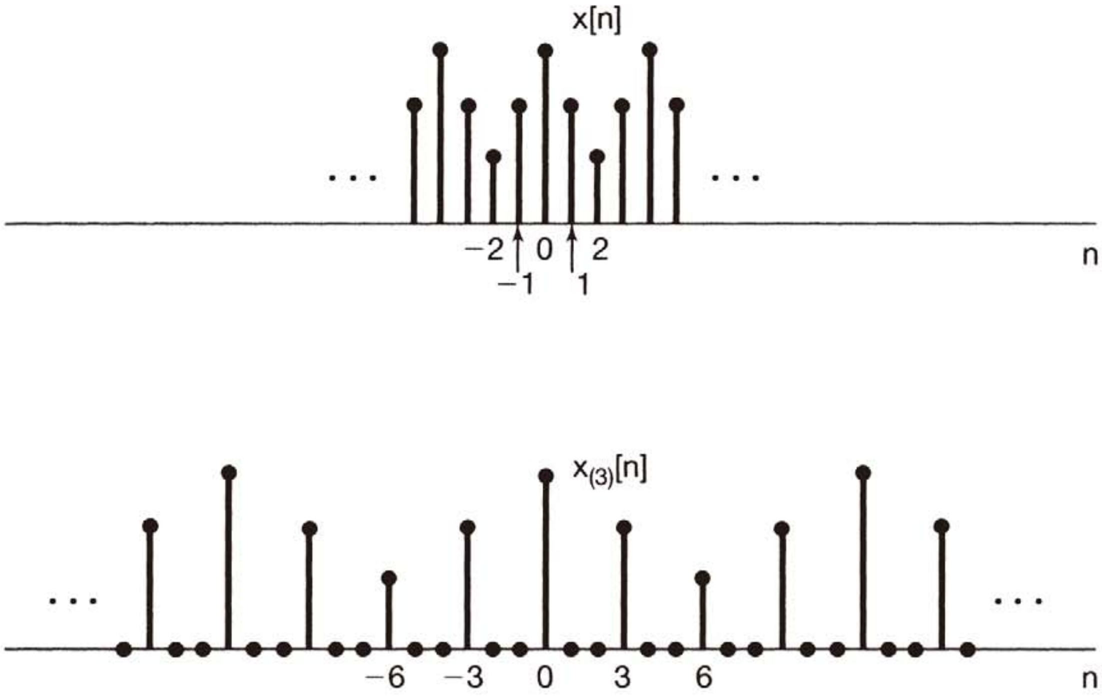


Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

5.3.7 Time Expansion

Furthermore, since $x_{(k)}[rk] = x[r]$, we find that

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r] e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is,

$$x_{(k)}[n] \xleftrightarrow{F} X(e^{jk\omega}).$$

時間延展性質

當$k > 1$時，$x_{(k)}[n]$ 將延展開來而使變化較緩，故傅立葉轉換受到壓縮。

5.3.7 Time Expansion

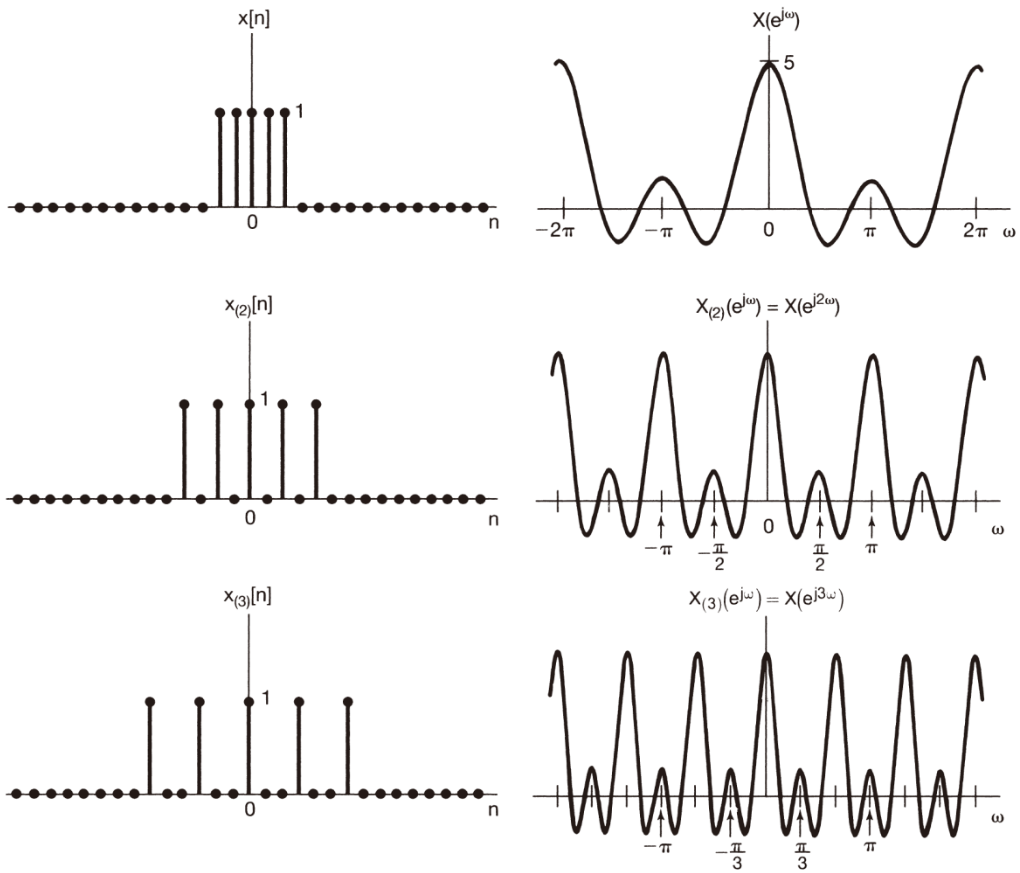


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

5.3.8 Differentiation in Frequency

Again, let

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}).$$

If we use the definition of $X(e^{j\omega})$ in the analysis equation (5.9) and differentiate both sides, we obtain

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

The right-hand side of this equation is the Fourier transform of $-jnx[n]$. Therefore, multiplying both sides by j, we see that

$$nx[n] \xleftrightarrow{\mathcal{F}} j\frac{dX(e^{j\omega})}{d\omega}. \qquad (5.46)$$

頻域微分性質

5.3.8 Differentiation in Frequency

If *x[n]* and $X(e^{j\omega})$ are a Fourier transform pair, then

$$\sum_{n=-\infty}^{+\infty} \left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega. \qquad (5.47)$$

Parseval's relation states that this energy can also be determined by integrating the energy per unit frequency, $\left|X(e^{j\omega})\right|^2 / 2\pi$, over a full 2 π interval of distinct discrete-time frequencies.

巴斯瓦關係式說明了訊號的能量亦可在頻域中積分求得。

5.4 The convolution Property

Specifically, if *x[n]*,*h[n]*, and *y[n]* are the input, impulse response, and output, respectively, of an LTI system, so that

對於一LTI系統：

$$y[n] = x[n] * h[n],$$

輸出、輸入及脈衝響應的迴旋積分關係(時域)

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}), \tag{5.48}$$

輸出、輸入的傅立葉轉換與頻率響應關係(頻域)

5.5 The Multiplication Property

Consider *y[n]* equal to the product of $x_1[n]$ and $x_2[n]$, with $Y(e^{j\omega})$, $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$ denoting the corresponding Fourier transforms. Then

若 $y[n] = x_1[n]x_2[n]$ (在時域中為相乘)

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n},$$

or since

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X1(e^{j\theta})e^{j\theta n}d\theta, \tag{5.60}$$

it follows that

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-j\omega n}. \tag{5.61}$$

5.5 The Multiplication Property

Interchanging the order of summation and integration, we obtain

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})\left[\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n}\right]d\theta. \qquad (5.62)$$

The bracketed summation is $X_2(e^{j(\omega-\theta)})$, and consequently, eq. (5.62) becomes

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2(e^{j(\omega-\theta)})d\theta. \qquad (5.63)$$

則 $Y(e^{j\omega})$ 如 (5.63)式(在頻域中為迴旋運算)。

5.5 The Multiplication Property

Equation (5.63) corresponds to a periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, and the integral in this equation can be evaluated over any interval of length 2π.

(5.63)式為一週期性迴旋積分，積分區間寬度為 2π。

5.7 Duality

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM 表 5.1 離散時間傅立葉轉換的性質

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n=\text{multiple of } k\\0, & \text{if } n\neq\text{multiple of } k\end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases}X(e^{j\omega})=X^*(e^{-j\omega})\\\mathcal{Re}\{X(e^{j\omega})\}=\mathcal{Re}\{X(e^{-j\omega})\}\\\mathcal{Im}\{X(e^{j\omega})\}=-\mathcal{Im}\{X(e^{-j\omega})\}\\\lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert\\\sphericalangle X(e^{j\omega})=-\sphericalangle X(e^{-j\omega})\end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n]=\mathcal{Ev}\{x[n]\}$ [$x[n]$ real] $x_o[n]=\mathcal{Od}\{x[n]\}$ [$x[n]$ real]	$\mathcal{Re}\{X(e^{j\omega})\}$ $j\mathcal{Im}\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty}|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2d\omega$$

5.7.1 Duality in the Discrete-Time Fourier Series

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS 表 5.2 基本的離散時間傅立葉轉換對

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \|a\| < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1) a^n u[n], \quad \|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r-1)!} a^n u[n], \quad \|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

5.7.1 Duality in the Discrete-Time Fourier Series

To see this in more detail, consider two periodic sequences with period N, related through the summation

$$f[m] = \frac{1}{N} \sum_{r=\langle N \rangle} g[r] e^{-jr(2\pi/N)m}. \qquad (5.65)$$

If we let $m = k$ and $r = n$, eq. (5.65) becomes

$$f[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk(2\pi/N)n}.$$

5.7.1 Duality in the Discrete-Time Fourier Series

See that the sequence *f[k]* corresponds to the Fourier series coefficients of the signal *g[n]*.

$$x[n] \xleftrightarrow{FS} a_k$$

its set of Fourier coefficients, the two periodic sequences related through eq. (5.65) satisfy

$$g[n] \xleftrightarrow{FS} f[k]. \qquad (5.66)$$

若*g[n]*的傅立葉係數為*f[k]*

5.7.1 Duality in the Discrete-Time Fourier Series

Alternatively, if we let $m = n$ and $r = -k$, eq. (5.65) becomes

$$f[n] = \sum_{k=\langle N \rangle} \frac{1}{N} g[-k] e^{jk(2\pi/N)n}.$$

Comparing this with eq. (3.94), we find that *(1/N)g[-k]* corresponds to the sequence of Fourier series coefficients of *f[n]*. That is,

$$f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k]. \qquad (5.67)$$

則*f[n]*的傅立葉係數為 $\frac{1}{N} g[-k]$ 。

5.7.1 Duality in the Discrete-Time Fourier Series

See that the pair of properties

$$x[n-n_0] \xleftrightarrow{FS} a_k e^{-jk(2\pi/N)n_0} \qquad (5.68)$$

and

$$e^{jm(2\pi/N)n} x[n] \xleftrightarrow{FS} a_{k-m} \qquad (5.69)$$

are dual. Similarly, from the same table, we can extract another pair of dual properties;

$$\sum_{r=\langle N \rangle} x[r]y[n-r] \xleftrightarrow{FS} N a_k b_k \qquad (5.70)$$

and

$$x[n]y[n] \xleftrightarrow{FS} \sum_{l=\langle N \rangle} a_l b_{k-l}. \qquad (5.71)$$

5.7.2 Duality between the discrete-Time Fourier Transform and the Continuous-Time Fourier Series

We repeat these equations here for convenience:

[eq. (5.8)]
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad (5.73)$$

[eq. (5.9)]
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \qquad (5.74)$$

[eq. (3.38)]
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \qquad (5.75)$$

[eq. (3.39)]
$$a_k = \frac{1}{T}\int_{T} x(t)e^{-jk\omega_0 t}dt. \qquad (5.76)$$

5.7.2 Duality between the discrete-Time Fourier Transform and the Continuous-Time Fourier Series

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

duality

5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

A general linear constant-coefficient difference equation for an LTI system with input *x[n]* and output *y[n]* is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \qquad (5.78)$$

*N*階線性常係數差分方程一般式

5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

求取上述LTI系統的頻率響應方法有二：

其一為利用特徵函數輸入$x[n]=e^{j\omega n}$，可得 $Y[n]=H(e^{j\omega})e^{j\omega n}$，比較可得 $H(e^{j\omega})$。

其二為利用迴旋運算、線性和時間移位性質求得 $H(e^{j\omega})$。

The convolution property, eq. (5.48), of the discrete-time Fourier transform then implies that

$$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}. \qquad (5.79)$$

5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

Applying the Fourier transform to both sides of eq. (5.78) and using the linearity and time-shifting properties, we obtain the expression

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}),$$

or equivalently,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}. \qquad (5.80)$$

頻率響應函數與差分方程係數的關係式

5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

Specifically, after expanding $H(e^{j\omega})$ by the method of partial fractions, we can find the inverse transform of each term by inspection. The same approach can be applied to the frequency response of any LTI system described by a linear constant-coefficient difference equation in order to determine the system impulse response.

若將 $H(e^{j\omega})$部份分式展開後，可得其反傅立葉轉換，亦即此LTI系統的脈衝響應。