$F = S_1, S_2, S_3$ are shown in Fig. 269 in Sec. 11.2, and $F = S_{20}$ is shown in Fig. 279. Although $|f(x) - F(x)|$ is large at $\pm\pi$ (how large?), where f is discontinuous, F approximates f quite well on the whole interval, except near $\pm\pi$, where "waves" remain owing to the "Gibbs phenomenon," which we shall discuss in the next section.

Can you think of functions f for which E^* decreases more quickly with increasing N? ■

PROBLEM SET 11.4

1. **CAS Problem.** Do the numeric and graphic work in Example 1 in the text.

2–5 MINIMUM SQUARE ERROR

Find the trigonometric polynomial $F(x)$ of the form (2) for which the square error with respect to the given $f(x)$ on the interval $-\pi < x < \pi$ is minimum. Compute the minimum value for $N = 1, 2, \cdots, 5$ (or also for larger values if you have a CAS).

2. $f(x) = x \quad (-\pi < x < \pi)$
3. $f(x) = |x| \quad (-\pi < x < \pi)$
4. $f(x) = x^2 \quad (-\pi < x < \pi)$
5. $f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi \end{cases}$
6. Why are the square errors in Prob. 5 substantially larger than in Prob. 3?
7. $f(x) = x^3 \quad (-\pi < x < \pi)$
8. $f(x) = |\sin x| \quad (-\pi < x < \pi)$, full-wave rectifier
9. **Monotonicity.** Show that the minimum square error (6) is a monotone decreasing function of N. How can you use this in practice?
10. **CAS EXPERIMENT. Size and Decrease of E^*.** Compare the size of the minimum square error E^* for functions of your choice. Find experimentally the factors on which the decrease of E^* with N depends. For each function considered find the smallest N such that $E^* < 0.1$.

11–15 PARSEVALS'S IDENTITY

Using (8), prove that the series has the indicated sum. Compute the first few partial sums to see that the convergence is rapid.

11. $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots = \dfrac{\pi^2}{8} = 1.233700550$

    Use Example 1 in Sec. 11.1.

12. $1 + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots = \dfrac{\pi^4}{90} = 1.082323234$

    Use Prob. 14 in Sec. 11.1.

13. $1 + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \cdots = \dfrac{\pi^4}{96} = 1.014678032$

    Use Prob. 17 in Sec. 11.1.

14. $\displaystyle\int_{-\pi}^{\pi} \cos^4 x \, dx = \frac{3\pi}{4}$

15. $\displaystyle\int_{-\pi}^{\pi} \cos^6 x \, dx = \frac{5\pi}{8}$

11.5 Sturm–Liouville Problems. Orthogonal Functions

The idea of the Fourier series was to represent general periodic functions in terms of cosines and sines. The latter formed a *trigonometric system.* This trigonometric system has the desirable property of orthogonality which allows us to compute the coefficient of the Fourier series by the Euler formulas.

The question then arises, can this approach be generalized? That is, can we replace the trigonometric system of Sec. 11.1 by other *orthogonal systems* (*sets of other orthogonal functions*)? The answer is "yes" and will lead to generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series in Sec. 11.6.

To prepare for this generalization, we first have to introduce the concept of a Sturm–Liouville Problem. (The motivation for this approach will become clear as you read on.) Consider a second-order ODE of the form

$$\textbf{(1)} \qquad [p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

on some interval $a \leqq x \leqq b$, satisfying conditions of the form

$$(2) \qquad \begin{aligned} &\textbf{(a)} \quad k_1 y + k_2 y' = 0 \quad \text{at } x = a \\ &\textbf{(b)} \quad l_1 y + l_2 y' = 0 \quad \text{at } x = b. \end{aligned}$$

Here λ is a parameter, and k_1, k_2, l_1, l_2 are given real constants. Furthermore, at least one of each constant in each condition (2) must be different from zero. (We will see in Example 1 that, if $p(x) = r(x) = 1$ and $q(x) = 0$, then $\sin \sqrt{\lambda}x$ and $\cos \sqrt{\lambda}x$ satisfy (1) and constants can be found to satisfy (2).) Equation (1) is known as a **Sturm–Liouville equation**.[4] Together with conditions 2(a), 2(b) it is know as the **Sturm–Liouville problem**. It is an example of a boundary value problem.

A **boundary value problem** consists of an ODE and given boundary conditions referring to the two boundary points (endpoints) $x = a$ and $x = b$ of a given interval $a \leqq x \leqq b$.

The goal is to solve these type of problems. To do so, we have to consider

Eigenvalues, Eigenfunctions

Clearly, $y \equiv 0$ is a solution—the "**trivial solution**"—of the problem (1), (2) for any λ because (1) is homogeneous and (2) has zeros on the right. This is of no interest. We want to find **eigenfunctions** $y(x)$, that is, solutions of (1) satisfying (2) without being identically zero. We call a number λ for which an eigenfunction exists an **eigenvalue** of the Sturm–Liouville problem (1), (2).

Many important ODEs in engineering can be written as Sturm–Liouville equations. The following example serves as a case in point.

EXAMPLE 1 Trigonometric Functions as Eigenfunctions. Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$(3) \qquad y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0.$$

This problem arises, for instance, if an elastic string (a violin string, for example) is stretched a little and fixed at its ends $x = 0$ and $x = \pi$ and then allowed to vibrate. Then $y(x)$ is the "space function" of the deflection $u(x, t)$ of the string, assumed in the form $u(x, t) = y(x)w(t)$, where t is time. (This model will be discussed in great detail in Secs, 12.2–12.4.)

Solution. From (1) nad (2) we see that $p = 1, q = 0, r = 1$ in (1), and $a = 0, b = \pi, k_1 = l_1 = 1$, $k_2 = l_2 = 0$ in (2). For negative $\lambda = -\nu^2$ a general solution of the ODE in (3) is $y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x}$. From the boundary conditions we obtain $c_1 = c_2 = 0$, so that $y \equiv 0$, which is not an eigenfunction. For $\lambda = 0$ the situation is similar. For positive $\lambda = \nu^2$ a general solution is

$$y(x) = A \cos \nu x + B \sin \nu x.$$

[4] JACQUES CHARLES FRANÇOIS STURM (1803–1855) was born and studied in Switzerland and then moved to Paris, where he later became the successor of Poisson in the chair of mechanics at the Sorbonne (the University of Paris).

JOSEPH LIOUVILLE (1809–1882), French mathematician and professor in Paris, contributed to various fields in mathematics and is particularly known by his important work in complex analysis (Liouville's theorem; Sec. 14.4), special functions, differential geometry, and number theory.

From the first boundary condition we obtain $y(0) = A = 0$. The second boundary condition then yields

$$y(\pi) = B \sin \nu\pi = 0, \qquad \text{thus} \qquad \nu = 0, \pm 1, \pm 2, \cdots.$$

For $\nu = 0$ we have $y \equiv 0$. For $\lambda = \nu^2 = 1, 4, 9, 16, \cdots$, taking $B = 1$, we obtain

$$y(x) = \sin \nu x \qquad (\nu = \sqrt{\lambda} = 1, 2, \cdots).$$

Hence the eigenvalues of the problem are $\lambda = \nu^2$, where $\nu = 1, 2, \cdots$, and corresponding eigenfunctions are $y(x) = \sin \nu x$, where$\nu = 1, 2 \cdots$.

Note that the solution to this problem is precisely the trigonometric system of the Fourier series considered earlier. It can be shown that, under rather general conditions on the functions p, q, r in (1), the Sturm–Liouville problem (1), (2) has infinitely many eigenvalues. The corresponding rather complicated theory can be found in Ref. [All] listed in App. 1.

Furthermore, if p, q, r, and p' in (1) are real-valued and continuous on the interval $a \leqq x \leqq b$ and r is positive throughout that interval (or negative throughout that interval), then all the eigenvalues of the Sturm–Liouville problem (1), (2) are real. (Proof in App. 4.) This is what the engineer would expect since eigenvalues are often related to frequencies, energies, or other physical quantities that must be real.

The most remarkable and important property of eigenfunctions of Sturm–Liouville problems is their ***orthogonality***, which will be crucial in series developments in terms of eigenfunctions, as we shall see in the next section. This suggests that we should next consider orthogonal functions.

Orthogonal Functions

Functions $y_1(x), y_2(x), \cdots$ defined on some interval $a \leqq x \leqq b$ are called **orthogonal** on this interval with respect to the **weight function** $r(x) > 0$ if for all m and all n different from m,

$$(4) \qquad (y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x)\, dx = 0 \qquad (m \neq n).$$

(y_m, y_n) is a ***standard notation*** for this integral. **The norm** $\|y_m\|$ of y_m is defined by

$$(5) \qquad \|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x) y_m^2(x)\, dx}.$$

Note that this is the square root of the integral in (4) with $n = m$.

The functions $y_1, y_2, \cdots$ are called **orthonormal** on $a \leqq x \leqq b$ if they are orthogonal on this interval and all have norm 1. Then we can write (4), (5) jointly by using the **Kronecker symbol**[5] δ_{mn}, namely,

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x)\, dx = \delta_{mn} = \begin{cases} 0 & \text{if} \quad m \neq \boldsymbol{n} \\ 1 & \text{if} \quad m = \boldsymbol{n}. \end{cases}$$

[5]LEOPOLD KRONECKER (1823–1891). German mathematician at Berlin University, who made important contributions to algebra, group theory, and number theory.

If $r(x) = 1$, we more briefly call the functions *orthogonal* instead of orthogonal with respect to $r(x) = 1$; similarly for orthognormality. Then

$$(y_m, y_n) = \int_a^b y_m(x) y_n(x)\, dx = 0 \quad (m \neq n), \qquad \|y_m\| = \sqrt{(y_m, y_n)} = \sqrt{\int_a^b y_m^2(x)\, dx}.$$

The next example serves as an illustration of the material on orthogonal functions just discussed.

EXAMPLE 2 **Orthogonal Functions. Orthonormal Functions. Notation**

The functions $y_m(x) = \sin mx$, $m = 1, 2, \cdots$ form an orthogonal set on the interval $-\pi \leqq x \leqq \pi$, because for $m \neq n$ we obtain by integration [see (11) in App. A3.1]

$$(y_m, y_n) = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (m - n)x\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos (m + n)x\, dx = 0, \quad (m \neq n).$$

The norm $\|y_m\| = \sqrt{(y_m, y_m)}$ equals $\sqrt{\pi}$ because

$$\|y_m\|^2 = (y_m, y_m) = \int_{-\pi}^{\pi} \sin^2 mx\, dx = \pi \qquad (m = 1, 2, \cdots)$$

Hence the corresponding orthonormal set, obtained by division by the norm, is

$$\frac{\sin x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \quad \frac{\sin 3x}{\sqrt{\pi}}, \quad \cdots.$$

Theorem 1 shows that for any Sturm–Liouville problem, the eigenfunctions associated with these problems are orthogonal. This means, in practice, if we can formulate a problem as a Sturm–Liouville problem, then by this theorem we are guaranteed orthogonality.

THEOREM 1 **Orthogonality of Eigenfunctions of Sturm–Liouville Problems**

Suppose that the functions p, q, r, and p' *in the Sturm–Liouville equation* (1) *are real-valued and continuous and* $r(x) > 0$ *on the interval* $a \leqq x \leqq b$. *Let* $y_m(x)$ *and* $y_n(x)$ *be eigenfunctions of the Sturm–Liouville problem* (1), (2) *that correspond to different eigenvalues* λ_m *and* λ_n, *respectively. Then* y_m, y_n *are orthogonal on that interval with respect to the weight function* r, *that is,*

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x)\, dx = 0 \qquad (m \neq n). \tag{6}$$

If $p(a) = 0$, *then* (2a) *can be dropped from the problem. If* $p(b) = 0$, *then* (2b) can be dropped. [It is then required that y and y' remain bounded at such a point, and the problem is called **singular**, as opposed to a **regular problem** in which (2) is used.]

If $p(a) = p(b)$, *then* (2) *can be replaced by the* "**periodic boundary conditions**"

$$y(a) = y(b), \qquad y'(a) = y'(b). \tag{7}$$

The boundary value problem consisting of the Sturm–Liouville equation (1) and the periodic boundary conditions (7) is called a **periodic Sturm–Liouville problem**.

PROOF By assumption, y_m and y_n satisfy the Sturm–Liouville equations

$$(py_m')' + (q + \lambda_m r)y_m = 0$$
$$(py_n')' + (q + \lambda_n r)y_n = 0$$

respectively. We multiply the first equation by y_n, the second by $-y_m$, and add,

$$(\lambda_m - \lambda_n)ry_m y_n = y_m(py_n')' - y_n(py_m')' = [(py_n')y_m - [(py_m')y_n]'$$

where the last equality can be readily verified by performing the indicated differentiation of the last expression in brackets. This expression is continuous on $a \leqq x \leqq b$ since p and p' are continuous by assumption and y_m, y_n are solutions of (1). Integrating over x from a to b, we thus obtain

$$(\lambda_m - \lambda_n)\int_a^b ry_m y_n\,dx = [p(y_n' y_m - y_m' y_n)]_a^b \qquad (a < b). \tag{8}$$

The expression on the right equals the sum of the subsequent Lines 1 and 2,

$$\begin{aligned} &p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] &&\text{(Line 1)} \\ &-p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)] &&\text{(Line 2).} \end{aligned} \tag{9}$$

Hence if (9) is zero, (8) with $\lambda_m - \lambda_n \neq 0$ implies the orthogonality (6). Accordingly, we have to show that (9) is zero, using the boundary conditions (2) as needed.

***Case 1.* $p(a) = p(b) = 0$.** Clearly, (9) is zero, and (2) is not needed.
***Case 2.* $p(a) \neq 0, p(b) = 0$.** Line 1 of (9) is zero. Consider Line 2. From (2a) we have

$$k_1 y_n(a) + k_2 y_n'(a) = 0,$$
$$k_1 y_m(a) + k_2 y_m'(a) = 0.$$

Let $k_2 \neq 0$. We multiply the first equation by $y_m(a)$, the last by $-y_n(a)$ and add,

$$k_2[y_n'(a)y_m(a) - y_m'(a)y_n(a)] = 0.$$

This is k_2 times Line 2 of (9), which thus is zero since $k_2 \neq 0$. If $k_2 = 0$, then $k_1 \neq 0$ by assumption, and the argument of proof is similar.

***Case 3.* $p(a) = 0, p(b) \neq 0$.** Line 2 of (9) is zero. From (2b) it follows that Line 1 of (9) is zero; this is similar to Case 2.
***Case 4.* $p(a) \neq 0, p(b) \neq 0$.** We use both (2a) and (2b) and proceed as in Cases 2 and 3.
***Case 5.* $p(a) = p(b)$.** Then (9) becomes

$$p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b) - y_n'(a)y_m(a) + y_m'(a)y_n(a)].$$

The expression in brackets $[\cdots]$ is zero, either by (2) used as before, or more directly by (7). Hence in this case, (7) can be used instead of (2), as claimed. This completes the proof of Theorem 1. ■

EXAMPLE 3 **Application of Theorem 1. Vibrating String**

The ODE in Example 1 is a Sturm–Liouville equation with $p = 1$, $q = 0$, and $r = 1$. From Theorem 1 it follows that the eigenfunctions $y_m = \sin mx$ $(m = 1, 2, \cdots)$ are orthogonal on the interval $0 \leqq x \leqq \pi$. ■

Example 3 confirms, from this new perspective, that the trigonometric system underlying the Fourier series is orthogonal, as we knew from Sec. 11.1.

EXAMPLE 4 Application of Theorem 1. Orthogonlity of the Legendre Polynomials

Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ may be written

$$[(1 - x^2)y']' + \lambda y = 0 \qquad \lambda = n(n + 1).$$

Hence, this is a Sturm–Liouville equation (1) with $p = 1 - x^2$, $q = 0$, and $r = 1$. Since $p(-1) = p(1) = 0$, we need no boundary conditions, but have a "***singular*** " ***Sturm–Liouville problem*** on the interval $-1 \leqq x \leqq 1$. We know that for $n = 0, 1, \cdots$, hence $\lambda = 0, 1 \cdot 2, 2 \cdot 3, \cdots$, the Legendre polynomials $P_n(x)$ are solutions of the problem. Hence these are the eigenfunctions. From Theorem 1 it follows that they are orthogonal on that interval, that is,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad (m \neq n). \tag{10}$$

What we have seen is that the trigonometric system, underlying the Fourier series, is a solution to a Sturm–Liouville problem, as shown in Example 1, and that this trigonometric system is orthogonal, which we knew from Sec. 11.1 and confirmed in Example 3.

PROBLEM SET 11.5

1. **Proof of Theorem 1.** Carry out the details in Cases 3 and 4.

2–6 ORTHOGONALITY

2. **Normalization of eigenfunctions** y_m of (1), (2) means that we multiply y_m by a nonzero constant c_m such that $c_m y_m$ has norm 1. Show that $z_m = cy_m$ with *any* $c \neq 0$ is an eigenfunction for the eigenvalue corresponding to y_m.
3. **Change of x.** Show that if the functions $y_0(x), y_1(x), \cdots$ form an orthogonal set on an interval $a \leqq x \leqq b$ (with $r(x) = 1$), then the functions $y_0(ct + k), y_1(ct + k), \cdots, c > 0$, form an orthogonal set on the interval $(a - k)/c \leqq t \leqq (b - k)/c$.
4. **Change of x.** Using Prob. 3, derive the orthogonality of $1, \cos \pi x, \sin \pi x, \cos 2\pi x, \sin 2\pi x, \cdots$ on $-1 \leqq x \leqq 1$ $(r(x) = 1)$ from that of $1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots$ on $-\pi \leqq x \leqq \pi$.
5. **Legendre polynomials.** Show that the functions $P_n(\cos\theta)$, $n = 0, 1, \cdots$, from an orthogonal set on the interval $0 \leqq \theta \leqq \pi$ with respect to the weight function $\sin\theta$.
6. **Tranformation to Sturm–Liouville form.** Show that $y'' + fy' + (g + \lambda h)\,y = 0$ takes the form (1) if you set $p = \exp(\int f\,dx)$, $q = pg$, $r = hp$. Why would you do such a transformation?

7–15 STURM–LIOUVILLE PROBLEMS

Find the eigenvalues and eigenfunctions. Verify orthogonality. Start by writing the ODE in the form (1), using Prob. 6. Show details of your work.

7. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(10) = 0$
8. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$
9. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$
10. $y'' + \lambda y = 0, \quad y(0) = y(1), \quad y'(0) = y'(1)$
11. $(y'/x)' + (\lambda + 1)y/x^3 = 0, \quad y(1) = 0, \quad y(e^{\pi}) = 0.$ (Set $x = e^t$.)
12. $y'' - 2y' + (\lambda + 1)y = 0, \quad y(0) = 0, \quad y(1) = 0$
13. $y'' + 8y' + (\lambda + 16)y = 0, \quad y(0) = 0, \quad y(\pi) = 0$
14. **TEAM PROJECT. Special Functions. Orthogonal polynomials** play a great role in applications. For this reason, Legendre polynomials and various other orthogonal polynomials have been studied extensively; see Refs. [GenRef1], [GenRef10] in App. 1. Consider some of the most important ones as follows.

(a) Chebyshev polynomials[6] of the first and second kind are defined by

$$T_n(x) = \cos(n \arccos x)$$
$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}$$

respectively, where $n = 0, 1, \cdots$. Show that

$$\begin{aligned} &T_0 = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \\ &T_3(x) = 4x^3 - 3x, \\ &U_0 = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \\ &U_3(x) = 8x^3 - 4x. \end{aligned}$$

Show that the Chebyshev polynomials $T_n(x)$ are orthogonal on the interval $-1 \leqq x \leqq 1$ with respect to the weight function $r(x) = 1/\sqrt{1-x^2}$. (*Hint.* To evaluate the integral, set $\arccos x = \theta$.) Verify that $T_n(x)$, $n = 0, 1, 2, 3$, satisfy the **Chebyshev equation**

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

(b) Orthogonality on an infinite interval: Laguerre polynomials[7] are defined by $L_0 = 1$, and

$$L_n(x) = \frac{e^x}{n!}\frac{d^n(x^n e^{-x})}{dx^n}, \qquad n = 1, 2, \cdots.$$

Show that

$$L_n(x) = 1 - x, \qquad L_2(x) = 1 - 2x + x^2/2,$$
$$L_3(x) = 1 - 3x + 3x^2/2 - x^3/6.$$

Prove that the Laguerre polynomials are orthogonal on the positive axis $0 \leqq x < \infty$ with respect to the weight function $r(x) = e^{-x}$. *Hint.* Since the highest power in L_m is x^m, it suffices to show that $\int e^{-x}x^k L_n\,dx = 0$ for $k < n$. Do this by k integrations by parts.

11.6 Orthogonal Series. Generalized Fourier Series

Fourier series are made up of the trigonometric system (Sec. 11.1), which is orthogonal, and orthogonality was essential in obtaining the Euler formulas for the Fourier coefficients. Orthogonality will also give us coefficient formulas for the desired generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series. This generalization is as follows.

Let $y_0, y_1, y_2, \cdots$ be orthogonal with respect to a weight function $r(x)$ on an interval $a \leqq x \leqq b$, and let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots. \tag{1}$$

This is called an **orthogonal series**, **orthogonal expansion**, or **generalized Fourier series**. If the y_m are the eigenfunctions of a Sturm–Liouville problem, we call (1) an **eigenfunction expansion**. In (1) we use again m for summation since n will be used as a fixed order of Bessel functions.

Given $f(x)$, we have to determine the coefficients in (1), called the **Fourier constants** *of* $f(x)$ *with respect to* $y_0, y_1, \cdots$. Because of the orthogonality, this is simple. Similarly to Sec. 11.1, we multiply both sides of (1) by $r(x)y_n(x)$ (n ***fixed***) and then integrate on

[6]PAFNUTI CHEBYSHEV (1821–1894), Russian mathematician, is known for his work in approximation theory and the theory of numbers. Another transliteration of the name is TCHEBICHEF.

[7]EDMOND LAGUERRE (1834–1886), French mathematician, who did research work in geometry and in the theory of infinite series.