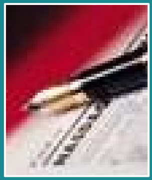


Chapter 7
Sampling

7.0 Introduction

Under certain conditions, a continuous-time signal can be completely represented by and recoverable form knowledge of its values, or samples, at points equally spaced in time. This somewhat surprising property follows from a basic result that is referred to as the sampling theorem.

Much of the importance of the sampling theorem also lies in its role as a bridge between continuous-time signals and discrete-time signals.

7.0 Introduction

本章將敘述一個重要的事實，在某些特定條件下，一個連續時間訊號可以完全由一組它的取樣序列還原。這個事實提供了以一個離散時間訊號，來表示一個連續時間訊號的架構。由於以離散時間訊號處理較具彈性，且離散時間系統價格不高、重量輕、可程式且易於重製，故常利用離散時間系統的實現來處理連續時間訊號。因此，取樣的觀念被廣泛運用。

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

In general, in the absence of any additional conditions or information, we would not expect that a signal could be uniquely specified by a sequence of equally spaced samples.

$$x_1(kT) = x_2(kT) = x_3(kT).$$

在沒有任何條件限制之下，一組等間距的取樣序列無法唯一描述一個訊號。

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

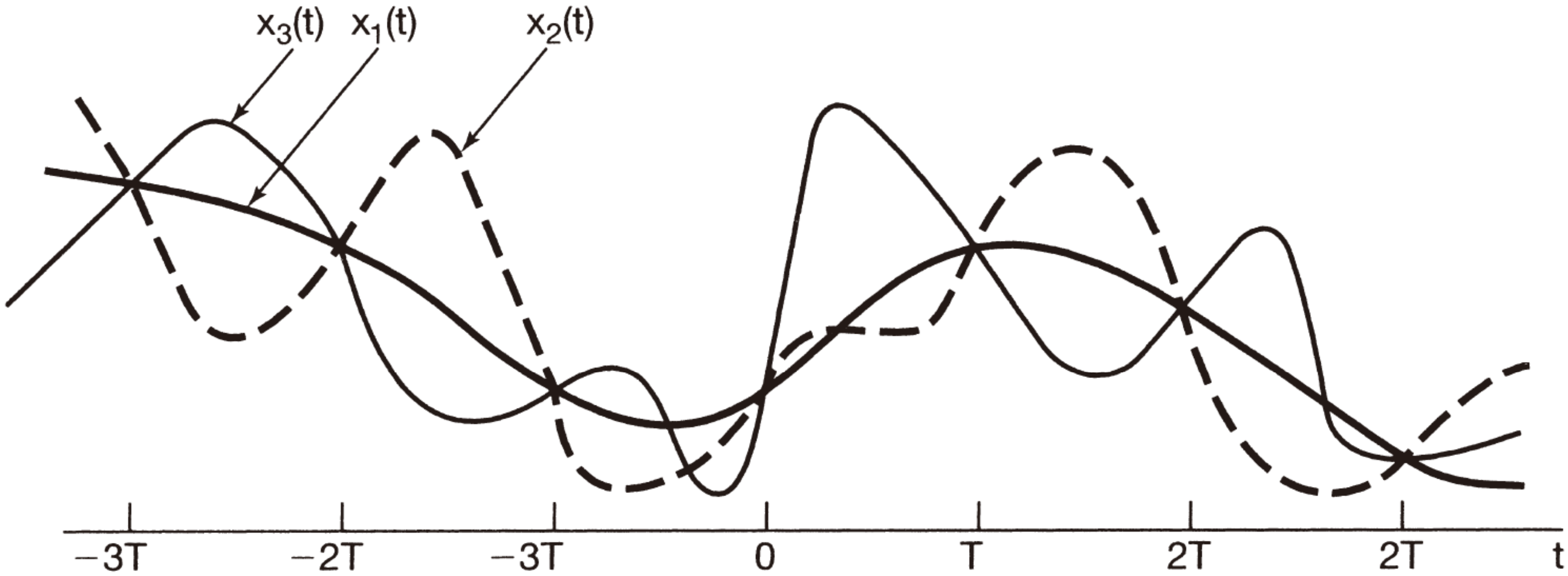


Figure 7.1 Three continuous-time signals with identical values at integer multiples of T.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

The periodic impulse train *p(t)* is referred to as the sampling function, the period *T* as the sampling period, and the fundamental frequency of *p(t)*, $\omega_s = 2\pi / T$, as the *sampling frequency*. In the time domain,

設*T*為取樣週期，取樣頻率 $\omega_s = 2\pi / T$ 。

$$x_p(t) = x(t)\,p(t), \tag{7.1}$$

*x(t)*經脈衝串取樣成$x_p(t)$。

where

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT). \tag{7.2}$$

取樣函數(單位脈衝串函數)

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

脈衝串取樣方塊圖

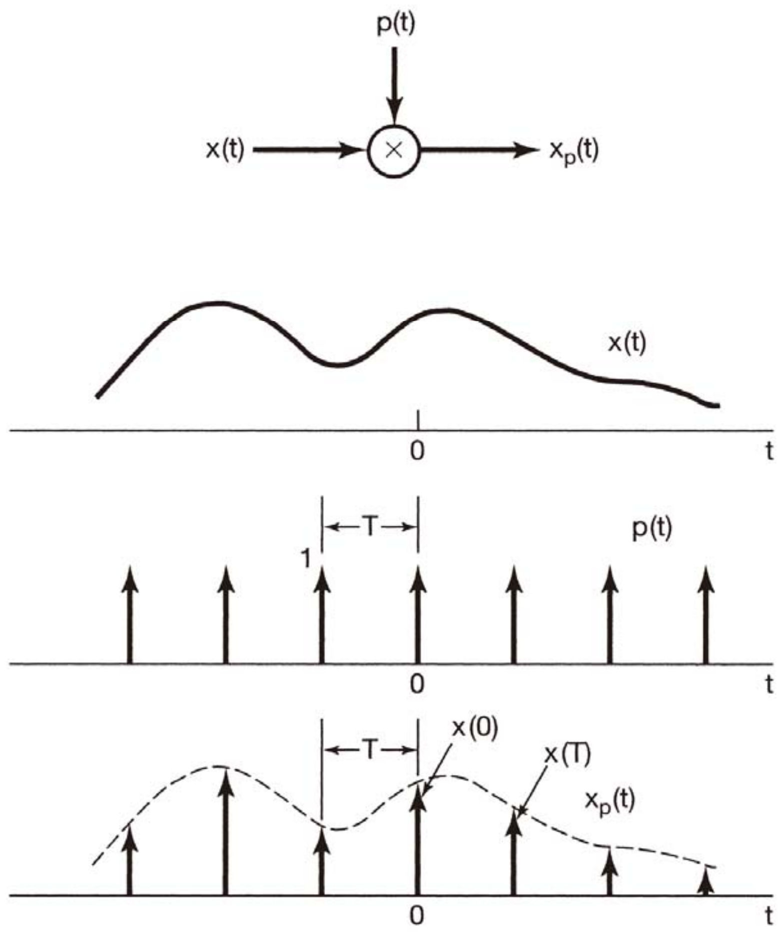


Figure 7.2 Impulse-train sampling.

取樣後的訊號$x_p(t)$，脈衝串的包絡線為原訊號*x(t)*。

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

the impulses equal to the samples of *x(t)* at intervals spaced by *T*; that is,

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT). \qquad (7.3)$$

From the multiplication property (Section 4.5), we know that

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)P(j(\omega-\theta))d\theta. \qquad (7.4)$$

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

and from Example 4.8,

$$P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s) \qquad (7.5)$$

單位脈衝串函數的傅立葉轉換

Since convolution with an impulse simply shifts a signal [i.e., $X(j\omega)*\delta(\omega-\omega_0)=X(j(\omega-\omega_0))$], it follows that

$$X_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X(j(\omega-k\omega_s)). \qquad (7.6)$$

取樣訊號與原訊號的傅立葉轉換關係

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

In Figure 7.3 (c), $\omega_M < (\omega_s - \omega_M)$ or equivalently, $\omega_s > 2\omega_M$, and thus there is no overlap between the shifted replicas of $X(j\omega)$, whereas in Figure 7.3(d), with $\omega_s < 2\omega_M$, there is overlap.

由圖7.3(c)可知；若$\omega_s > 2\omega_M$, 則$x_p(j\omega)$ 圖形沒有重疊。若$\omega_s < 2\omega_M$，則 $x_p(j\omega)$ 圖形有重疊。

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

原訊號頻譜

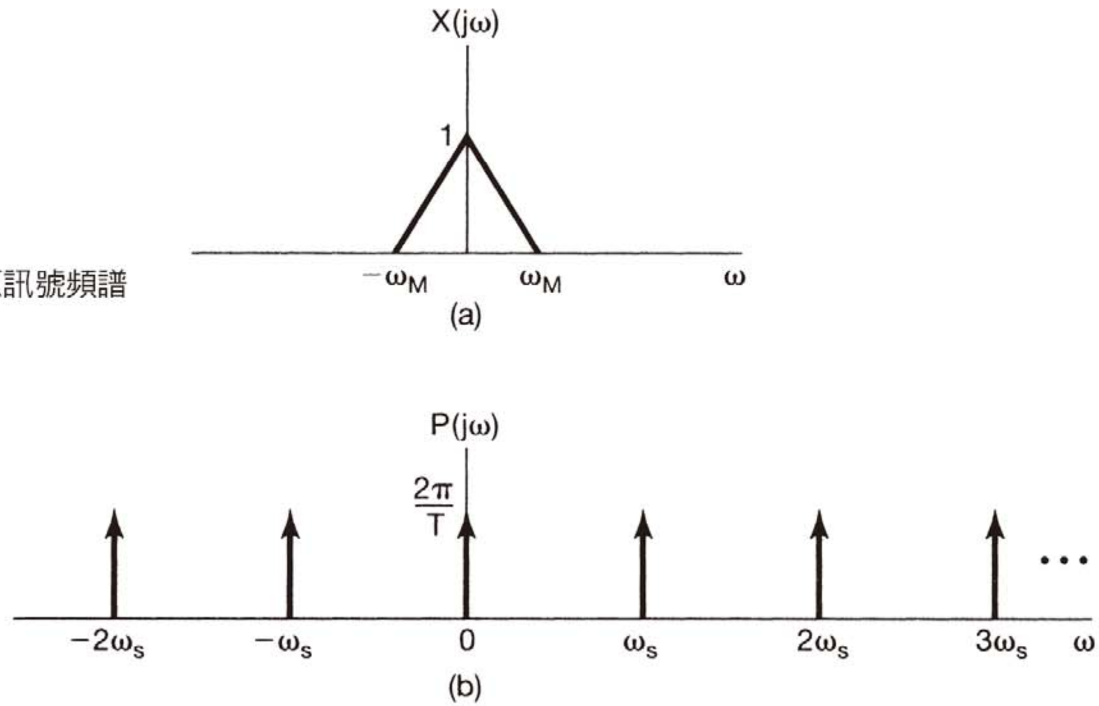
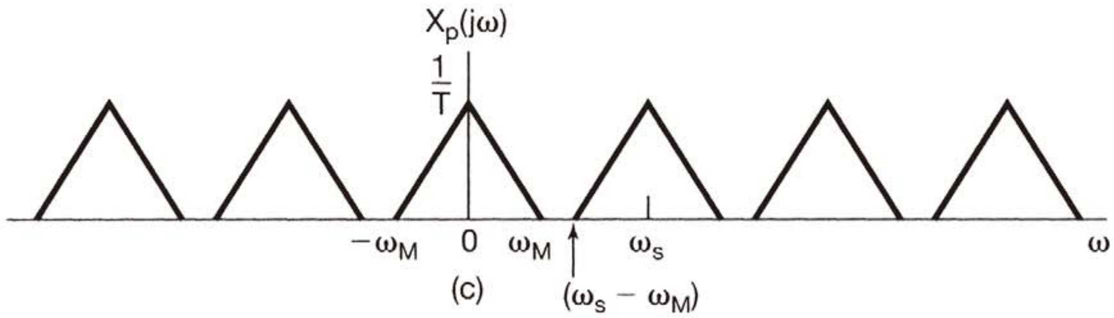


Figure 7.3 Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function;

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem



取樣訊號的頻譜。注意它具有與原訊號頻譜大小不同，但形狀相同的許多複製。

Figure 7.3 *Continued* (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

Sampling Theorem:

Let *x(t)* be a band-limited signal with $X(j\omega) = 0$ for $|\omega| \rangle \omega_M$. Then *x(t)* is uniquely determined by its samples *x(nT)*, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s \rangle 2\omega_M ,$$

where

$$\omega_s = \frac{2\pi}{T}.$$

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

Given these samples, we can reconstruct *x(t)* by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain *T* and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal *x(t)*.

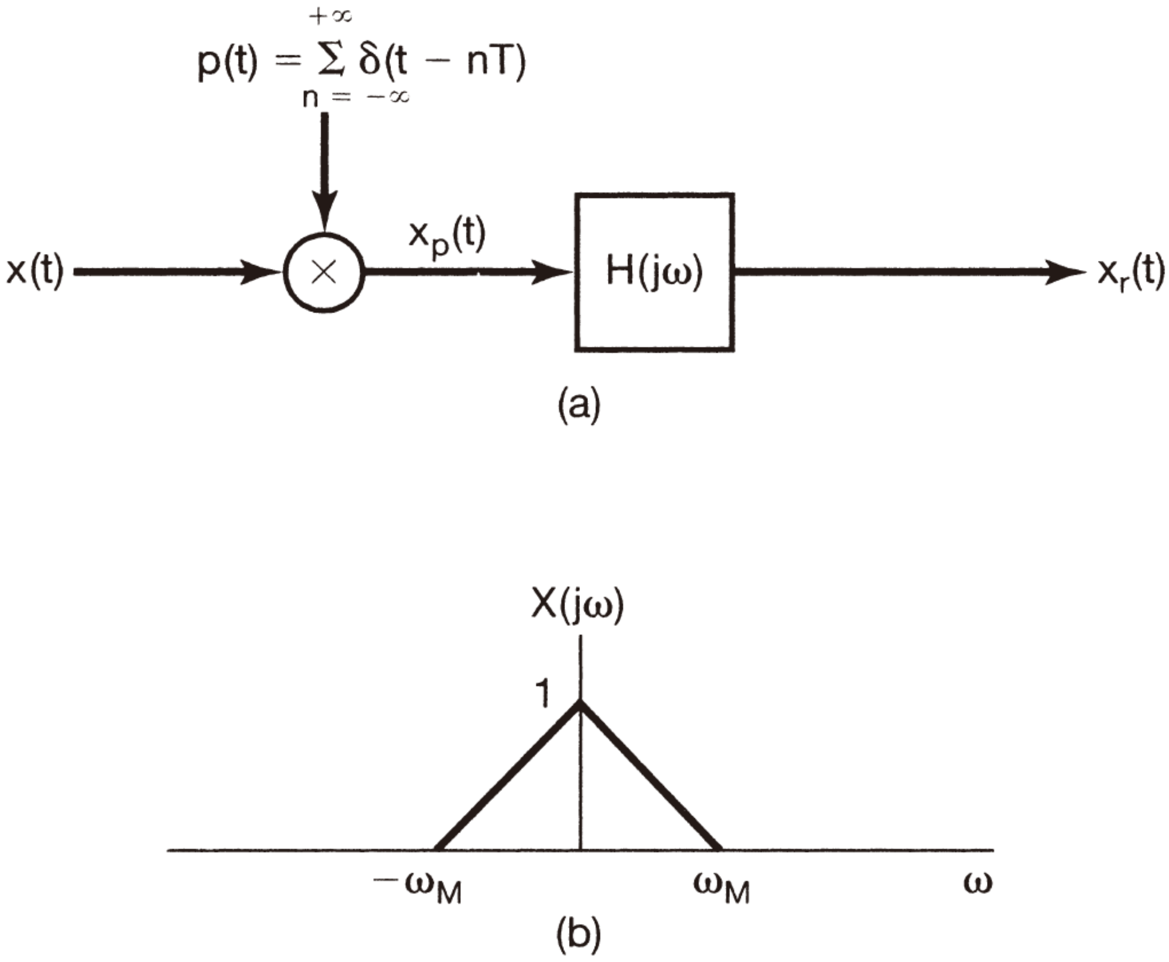
7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

取樣定理：

設*x(t)*為一有限頻帶訊號且在 $|\omega|\rangle\omega_M$ 時，*x(jω)* = 0。若 $\omega_s\rangle 2\omega_M$ ，其中 $\omega_s = 2\pi/T$ ，則*x(t)*可由其取樣*x(nT)*，*n* =0, ±1, ±2,…唯一決定。

將取樣後的脈衝串 $x_p(t)$ 經過一個截止頻率介於 ω_M 至 $\omega_s-\omega_M$ 之間的理想低通濾波器後，所得的輸出恰好等於*x(t)*。

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem



7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

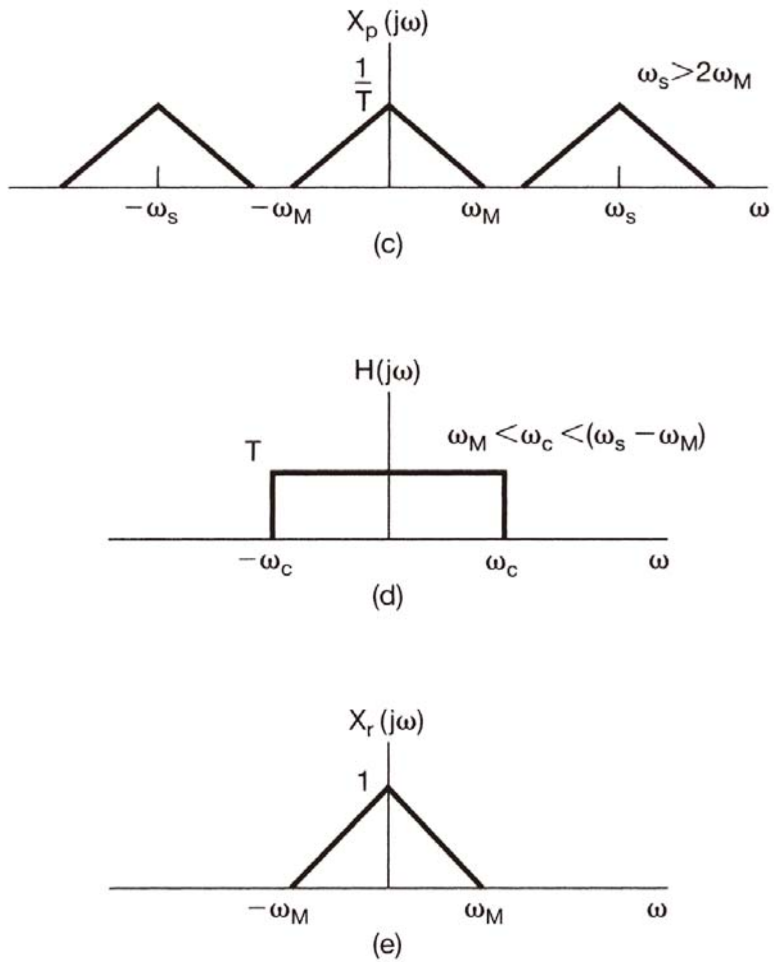


Figure 7.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter: (a) system for sampling and reconstruction; (b) representative spectrum for $x(t)$; (c) corresponding spectrum for $x_p(t)$; (d) ideal lowpass filter to recover $X(j\omega)$ from $X_p(j\omega)$; (e) spectrum of $x_r(t)$.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

The frequency $2\omega_M$, which, under the sampling theorem, must be exceeded by the sampling frequency, is commonly referred to as *Nyquist rate*.

$2\omega_M$ 的頻率常為「奈奎士速率」(另外， ω_M 稱為「奈奎士頻率」)。

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

Obviously, any such approximation in the lowpass filtering stage will lead to some discrepancy between $x(t)$ and $X_r(t)$ in Figure 7.4 or, equivalently, between $X(j\omega)$ and $X_r(j\omega)$.

若改用非理想低通濾波器，則圖7.4中的$x(t)$與$X_r(t)$將有些許差異。

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

The reconstruction of $x(t)$ from the output of a zero-order hold can again be carried out by lowpass filtering.

「零階保持」的動作為：在某瞬間取得$x(t)$的取樣值，並保持至下一個取樣瞬間為止。

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin\left(\omega T/2\right)}{\omega}\right], \tag{7.7}$$

this requires that

$$H_r(j\omega) = \frac{e^{j\omega T/2}H(j\omega)}{\dfrac{2\sin(\omega T/2)}{\omega}}. \tag{7.8}$$

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

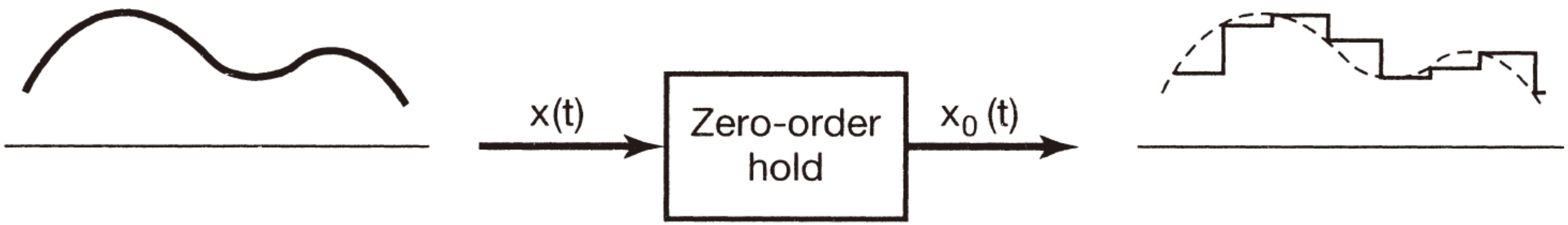


Figure 7.5 Sampling utilizing a zero-order hold.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

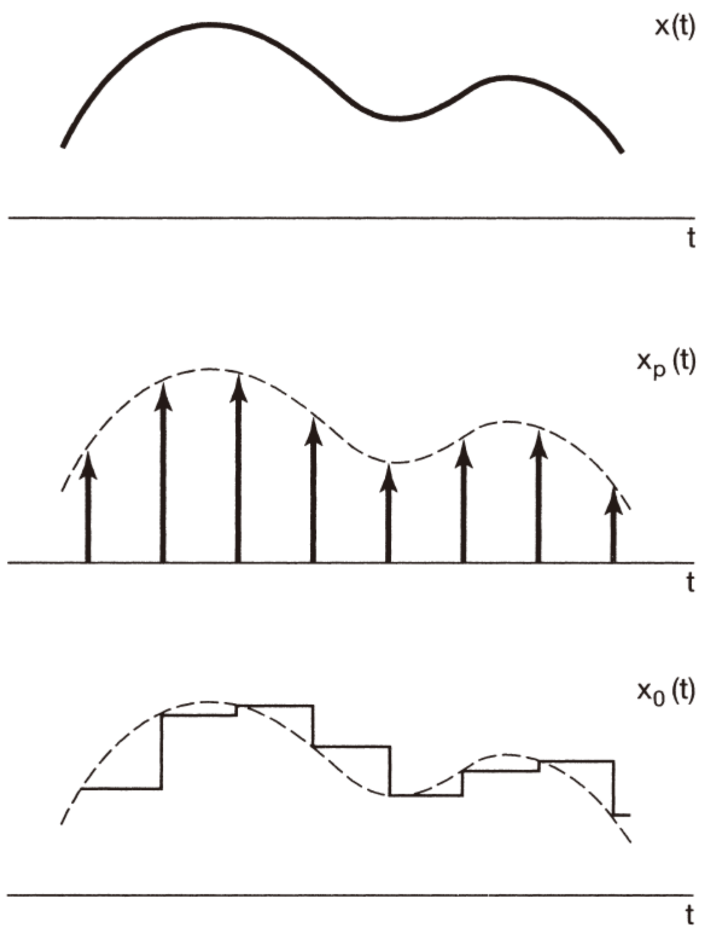


Figure 7.6 Zero-order hold as impulse-train sampling followed by an LTI system with a rectangular impulse response.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

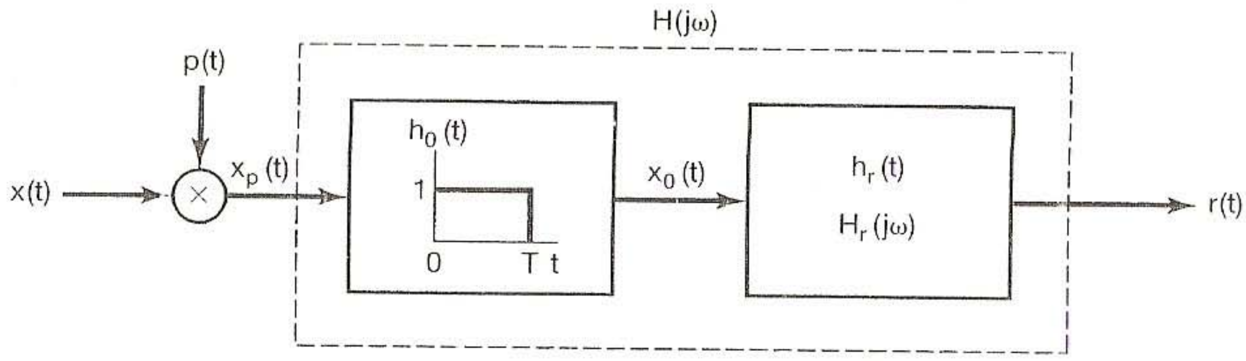


Figure 7.7 Cascade of the representation of a zero-order hold (Figure 7.6) with a reconstruction filter.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

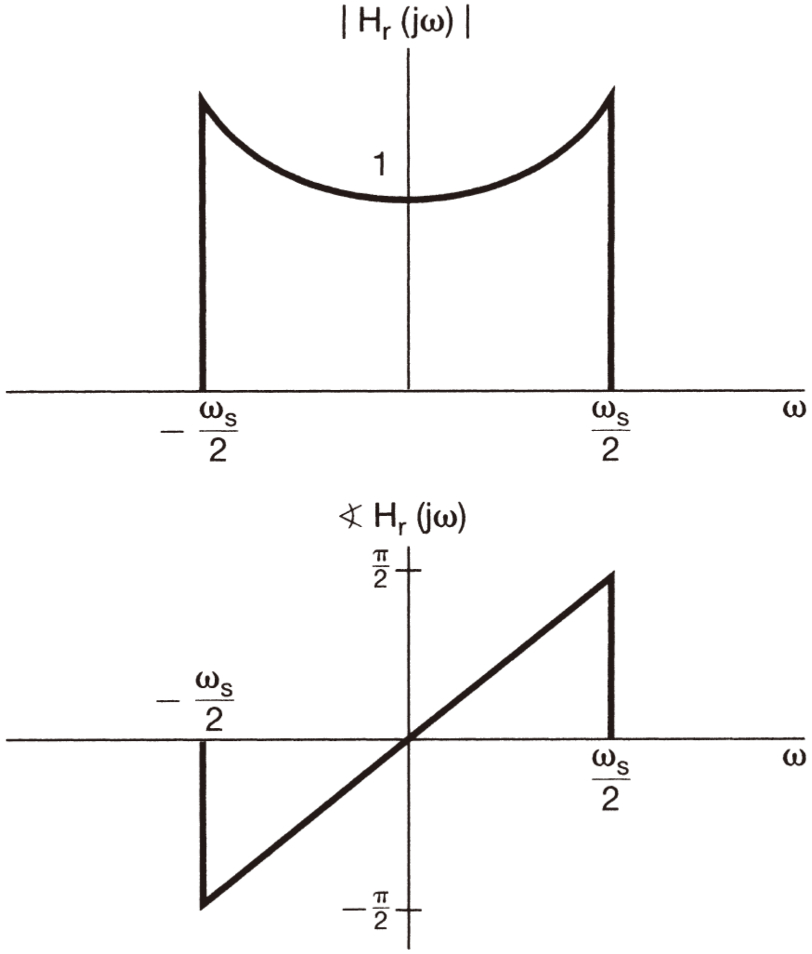


Figure 7.8 Magnitude and phase for the reconstruction filter for a zero-order hold.

7.1 Representation of A Continuous-Time Signal by ITS Samples: The Sampling Theorem

實用上，零階保持的輸出可視為原訊號的適當的近似。

某些應用上，可能在相鄰的取樣點之間，加上平滑的內插曲線。

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

「線性內插」為將相鄰的取樣點以直線連接作為內插公式。

Figure 7.9 Linear interpolation between sample points. The dashed curve represents the original signal and the solid curve the linear interpolation.

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

The interpretation of the reconstruction of *x(t)* as a process of interpolation becomes evident when we consider the effect in the time domain of the lowpass filter in Figure 7.4. In particular, the output is

$$x_r(t) = x_p(t) * h(t)$$

or, with $x_p(t)$ given by eq. (7.3),

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT) \qquad (7.9)$$

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

For the ideal lowpass filter *H(j ω)* in Figure 7.4,

$$h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t}, \qquad (7.10)$$

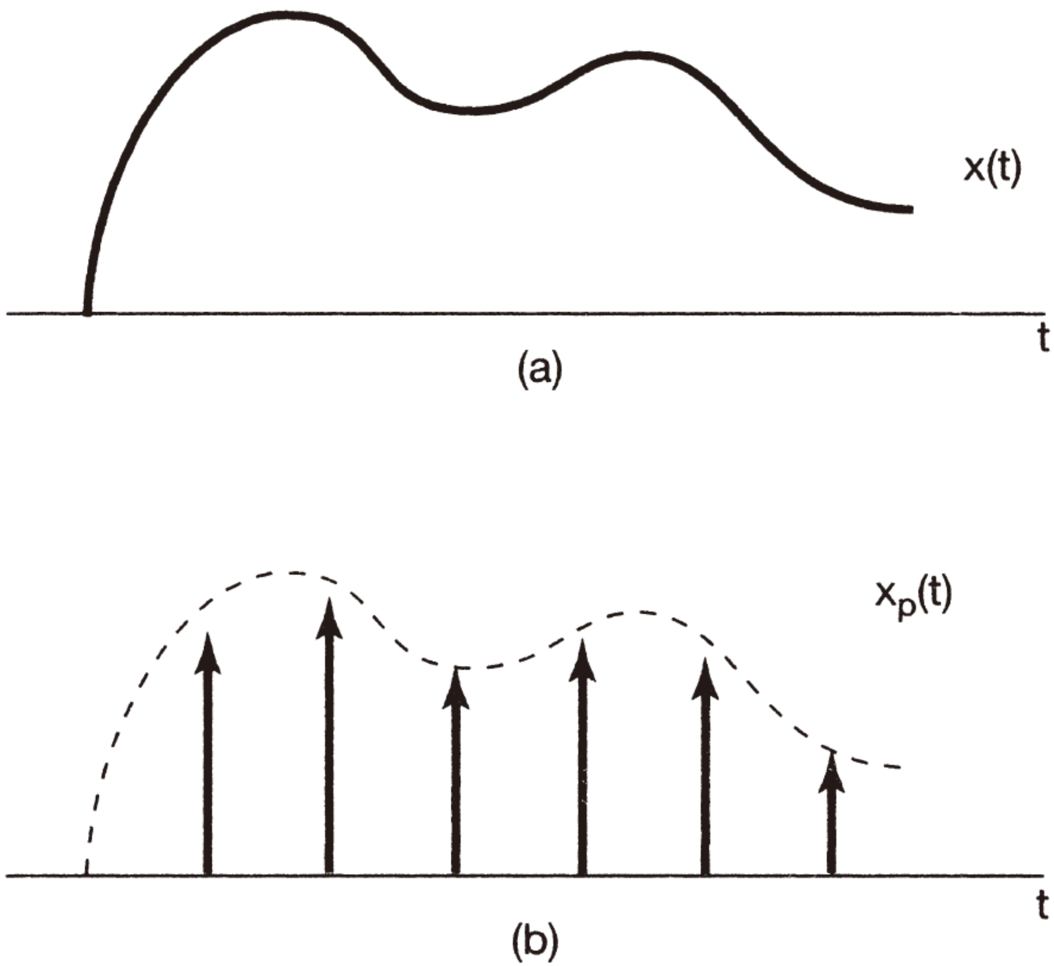
理想低通濾波器的脈衝響應

so that

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{\omega_c T}{\pi} \frac{\sin\left(\omega_c\left(t-nT\right)\right)}{\omega_c\left(t-nT\right)}. \qquad (7.11)$$

以理想低通濾波器的脈衝響應作為內插法則，稱為「有限頻帶內差法」。

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation



7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

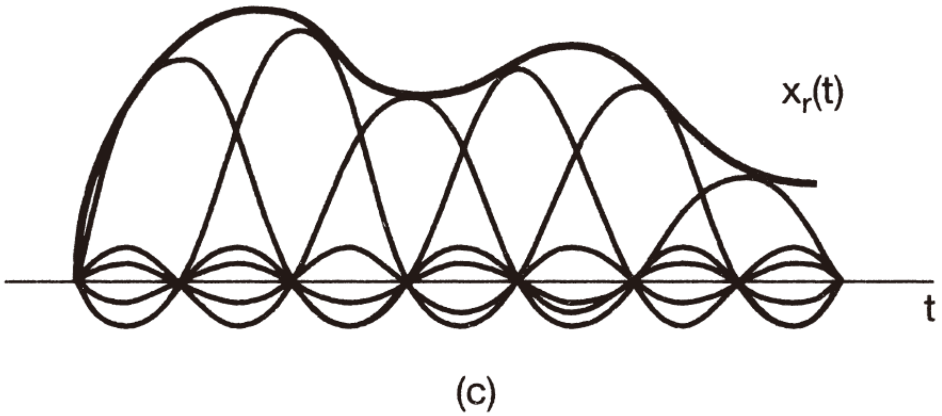


Figure 7.10 Ideal band-limited interpolation using the sinc function: (a) band-limited signal $x(t)$; (b) impulse train of samples of $x(t)$; (c) ideal band-limited interpolation in which the impulse train is replaced by a superposition of sinc functions [eq. (7.11)].

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

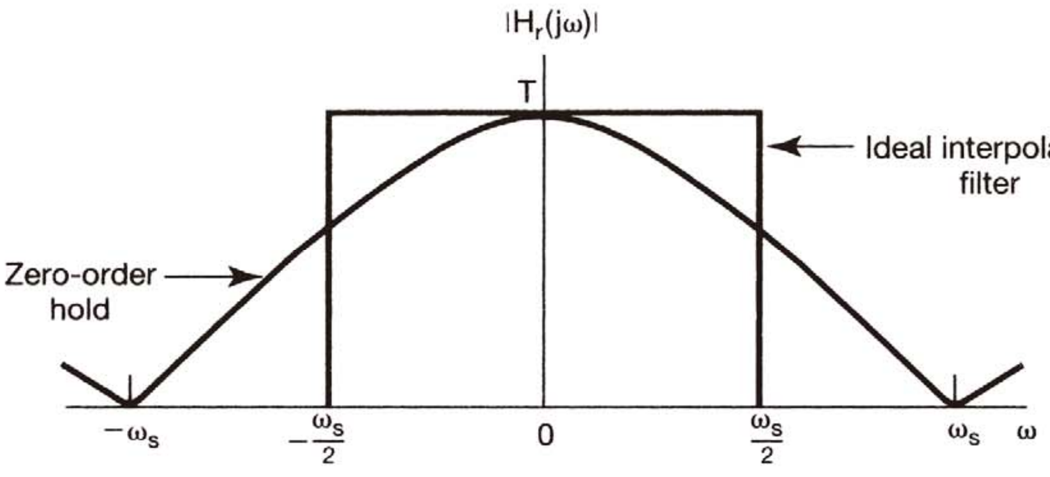
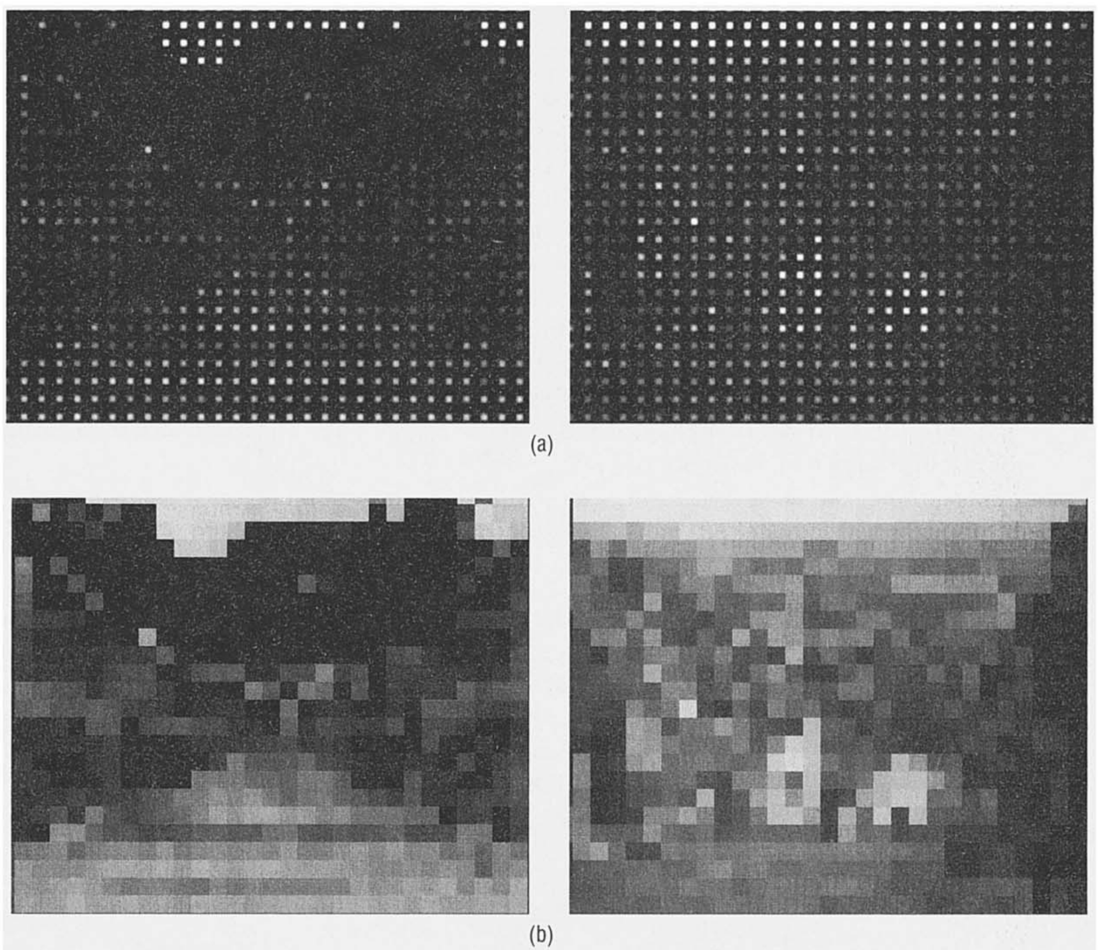


Figure 7.11 Transfer function for the zero-order hold and for the ideal interpolating filter.

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation



以脈衝取樣所得的影像。

加入零階保持處理的影像，呈現馬賽克效應（即產生如磁磚的拼貼效果）。因人類視覺系統有低通特性，馬賽克的不連續點會被平滑化。

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

(c)

將脈衝取樣的水平及垂直間距改為 (b) 圖的 1/4 倍的結果。

Figure 7.12 (a) The original pictures of Figures 6.2(a) and (g) with impulse sampling; (b) zero-order hold applied to the pictures in (a). The visual system naturally introduces lowpass filtering with a cutoff frequency that decreases with distance. Thus, when viewed at a distance, the discontinuities in the mosaic in Figure 7.12(b) are smoothed; (c) result of applying a zero-order hold after impulse sampling with one-fourth the horizontal and vertical spacing used in (a) and (b).

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

If the crude interpolation provided by the zero-order hold is insufficient, we can use a variety of smoother interpolation strategies, some of which are known collectively as higher order holds. In particular, the zero-order hold produces an output signal, as in Figure 7.5, that is discontinuous.

若零階保持的內插法無法滿足需求，則需更高階的保持。

零階保持的輸出為不連續函數。

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

$$H(j\omega)=\frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2. \qquad (7.12)$$

更高階的保持通常可重建出更高度的平滑程度。以二階為例，二階保持可提供一階微分的連續性及二階微分的不連續性的內插效果。

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

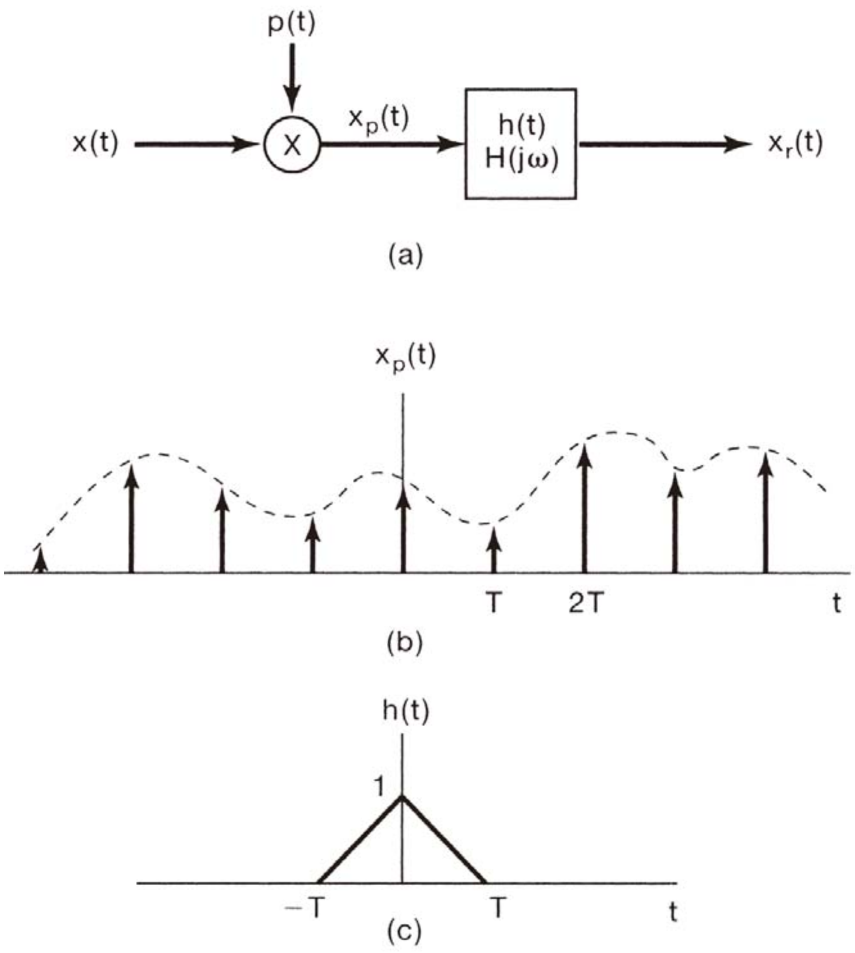


Figure 7.13 Linear interpolation (first-order hold) as impulse-train sampling followed by convolution with a triangular impulse response: (a) system for sampling and reconstruction; (b) impulse train of samples; (c) impulse response representing a first-order hold;

7.2 Reconstruction of A Signal From ITS Samples Using Interpolation

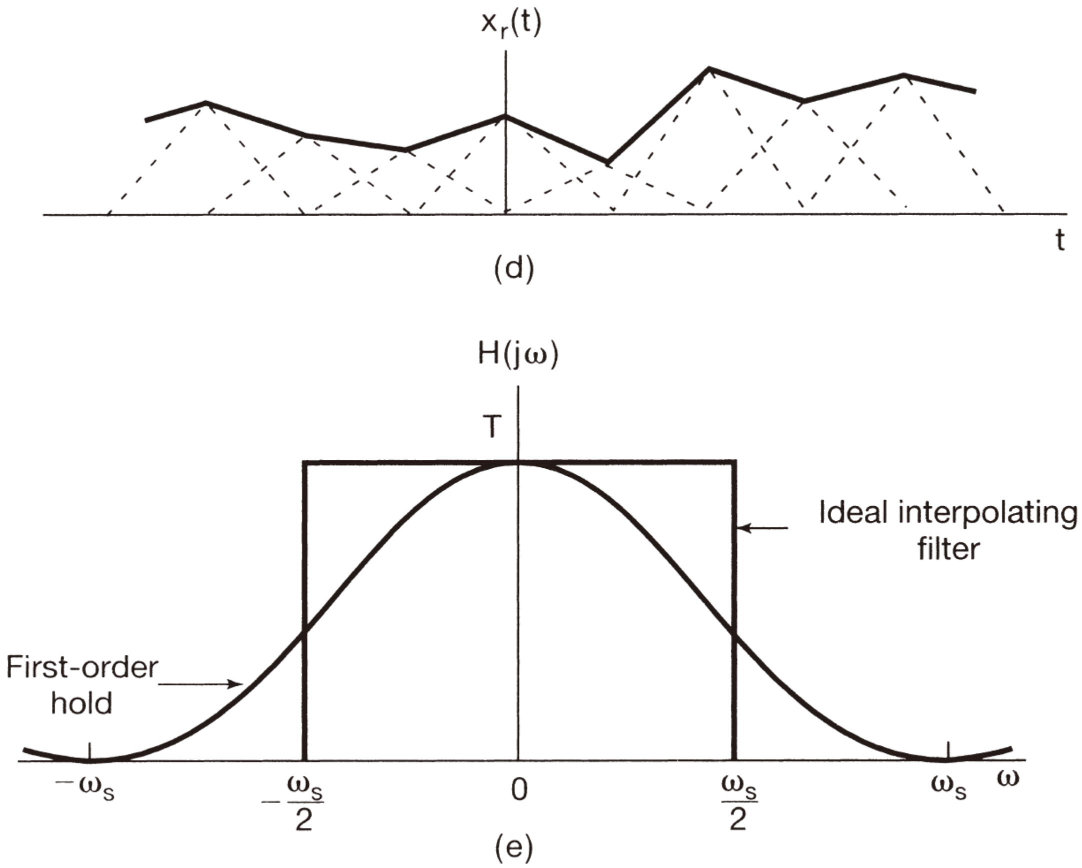
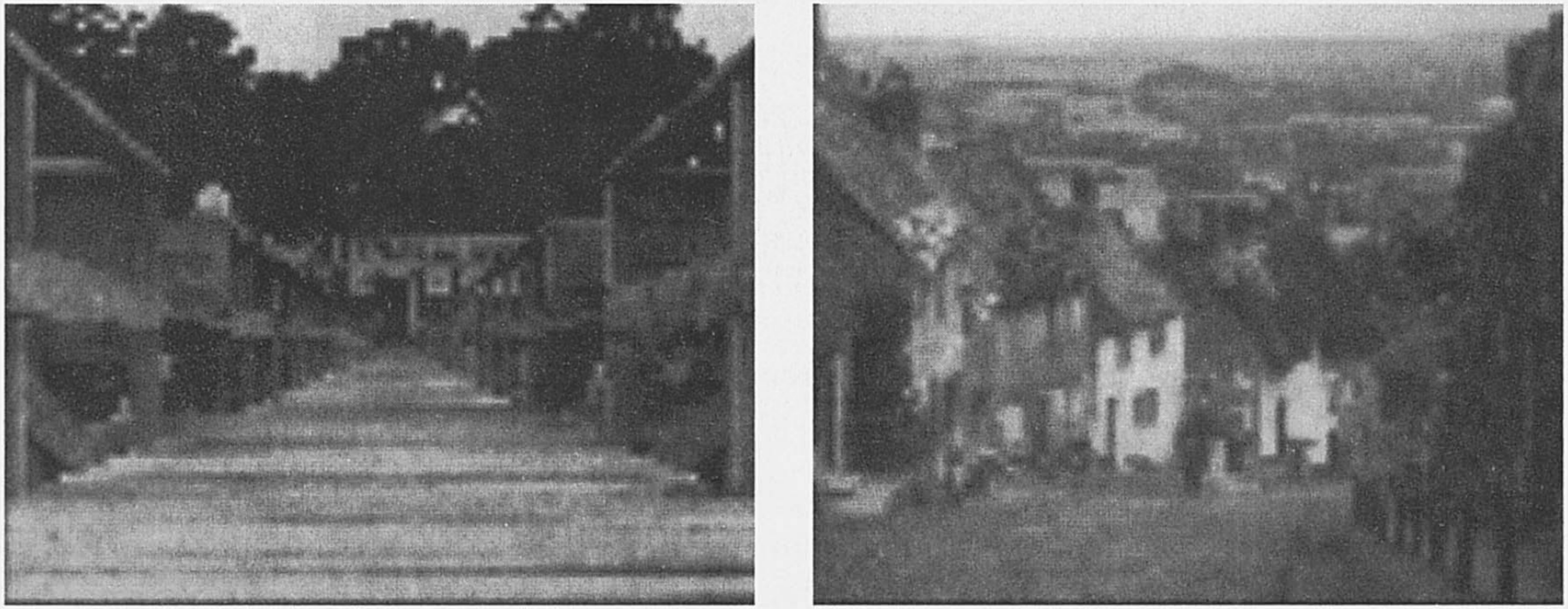


Figure 7.13 *Continued* (d) first-order hold applied to the sampled signal; (e) comparison of transfer function of ideal interpolating filter and first-order hold.

7.3 The Effect of Undersampling: Aliasing



(a) (b)

Figure 7.14 Result of applying a first-order hold rather than a zero-order hold after impulse sampling with one-third the horizontal and vertical spacing used in Figures 7.12(a) and (b).

7.3 The Effect of Undersampling: Aliasing

When $\omega_s < 2\omega_M$, *X(jω)*, the spectrum of *x(t)*, is no longer replicated in $X_p(j\omega)$ and thus is no longer recoverable by lowpass filtering.

若 $\omega_s < 2\omega_M$，則 $X_p(j\omega)$ 不再是由*X(jω)*複製而成，且無法由低通濾波還原。取樣訊號頻譜有重疊現象，稱為「頻率虛化」。

7.3 The Effect of Undersampling: Aliasing

Clearly, if the system of Figure 7.4 is applied to a signal with $\omega_s < 2\omega_M$, the reconstructed signal $x_r(t)$ will no longer be equal to *x(t)*.

$$x_r(nT) = x(nT), \qquad n = 0, \pm 1, \pm 2, \ldots \qquad (7.13)$$

若 $\omega_s < 2\omega_M$ ，雖還原後的 $x_r(t)$ 不等於原*x(t)*，但在取樣時間點上的值是相等的。

7.3 The Effect of Undersampling: Aliasing

Some insight into the relationship between x(t) and $x_r(t)$ when $\omega_s < 2\omega_M$ is provided by considering in more detail the comparatively simple case of a sinusoidal signal. Thus, let

$$x(t) = \cos \omega_0 t, \qquad (7.14)$$

7.3 The Effect of Undersampling: Aliasing

The lowpass filtered output $x_r(t)$ is given as follows:

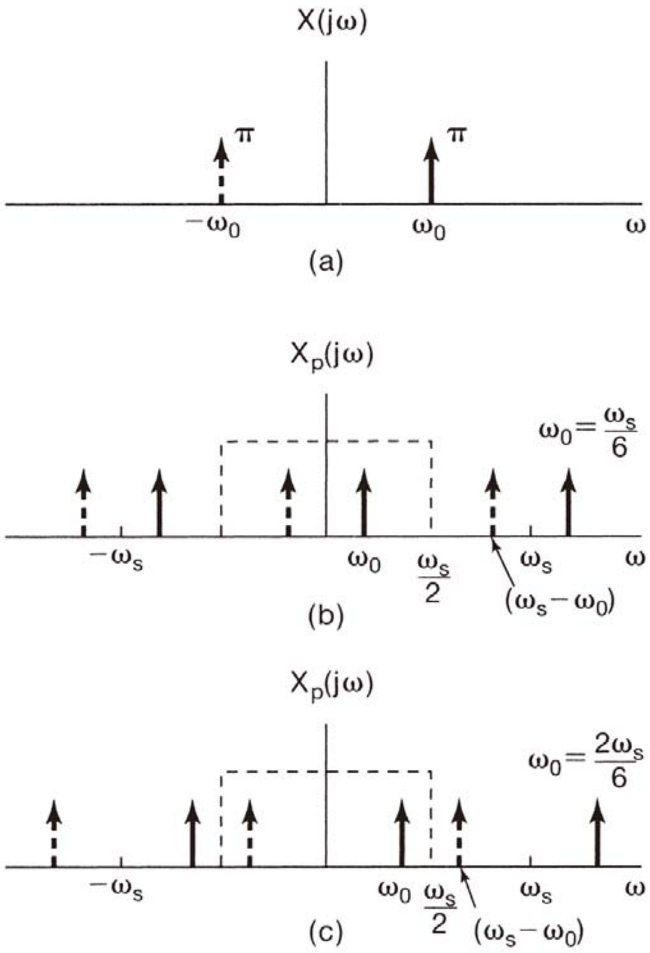
$$(a) \qquad \omega_0 = \frac{\omega_s}{6}; \qquad x_r(t) = \cos\omega_0 t = x(t)$$

$$(b) \qquad \omega_0 = \frac{2\omega_s}{6}; \qquad x_r(t) = \cos\omega_0 t = x(t)$$

$$(c) \qquad \omega_0 = \frac{4\omega_s}{6}; \qquad x_r(t) = \cos(\omega_s - \omega_0)t \neq x(t)$$

$$(d) \qquad \omega_0 = \frac{5\omega_s}{6}; \qquad x_r(t) = \cos(\omega_s - \omega_0)t \neq x(t).$$

7.3 The Effect of Undersampling: Aliasing



7.3 The Effect of Undersampling: Aliasing

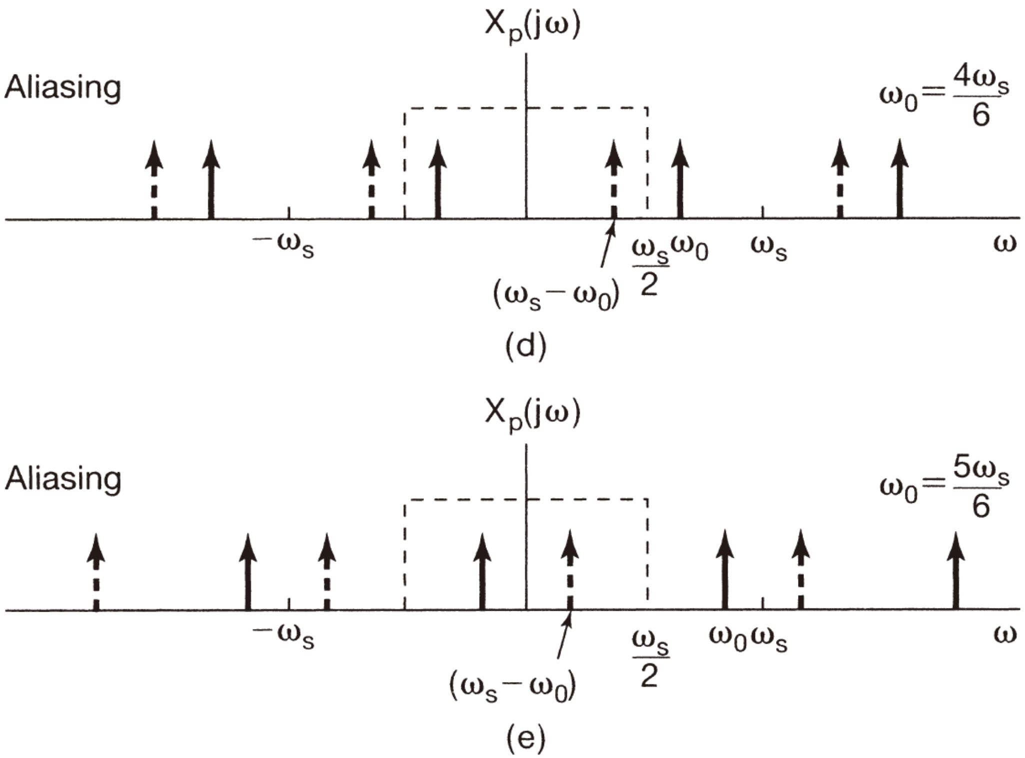


Figure 7.15 Effect in the frequency domain of oversampling and undersampling: (a) spectrum of original sinusoidal signal; (b), (c) spectrum of sampled signal with $\omega_s > 2\omega_0$; (d), (e) spectrum of sampled signal with $\omega_s < 2\omega_0$. As we increase ω_0 in moving from (b) through (d), the impulses drawn with solid lines move to the right, while the impulses drawn with dashed lines move to the left. In (d) and (e), these impulses have moved sufficiently that there is a change in the ones falling within the passband of the ideal lowpass filter.

7.3 The Effect of Undersampling: Aliasing

As a variation on the preceding examples, consider the signal

$$x(t) = \cos(\omega_0 t + \phi). \qquad (7.15)$$

7.3 The Effect of Undersampling: Aliasing

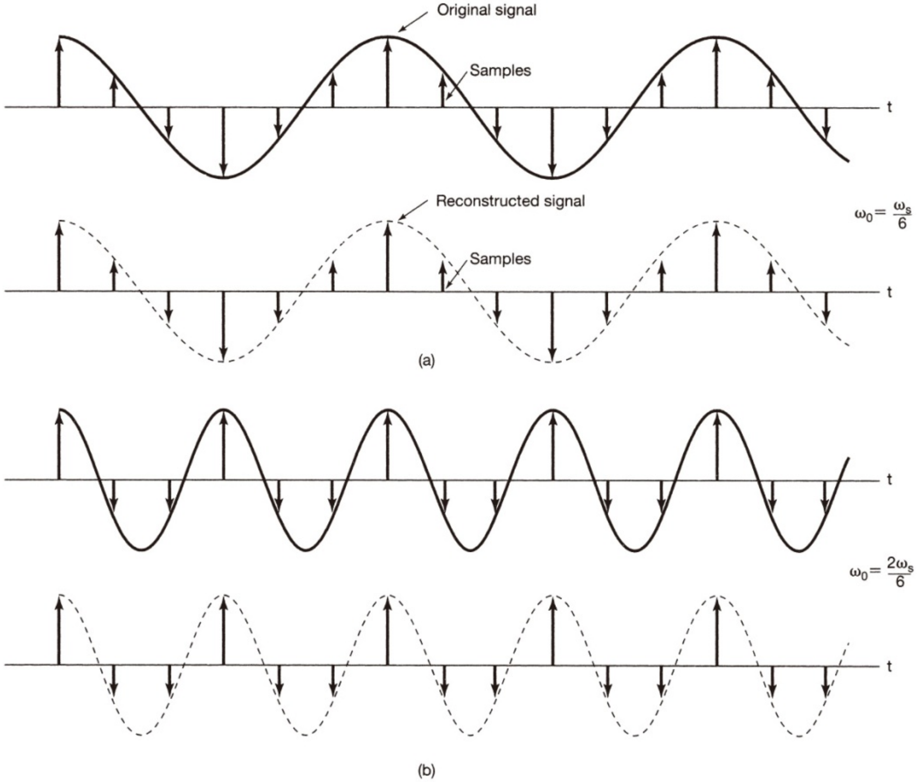


Figure 7.16 Effect of aliasing on a sinusoidal signal. For each of four values of ω_0, the original sinusoidal signal (solid curve), its samples, and the reconstructed signal (dashed curve) are illustrated: (a) $\omega_0 = \omega_s/6$; (b) $\omega_0 = 2\omega_s/6$; (c) $\omega_0 = 4\omega_s/6$; (d) $\omega_0 = 5\omega_s/6$. In (a) and (b) no aliasing occurs, whereas in (c) and (d) there is aliasing.

7.3 The Effect of Undersampling: Aliasing

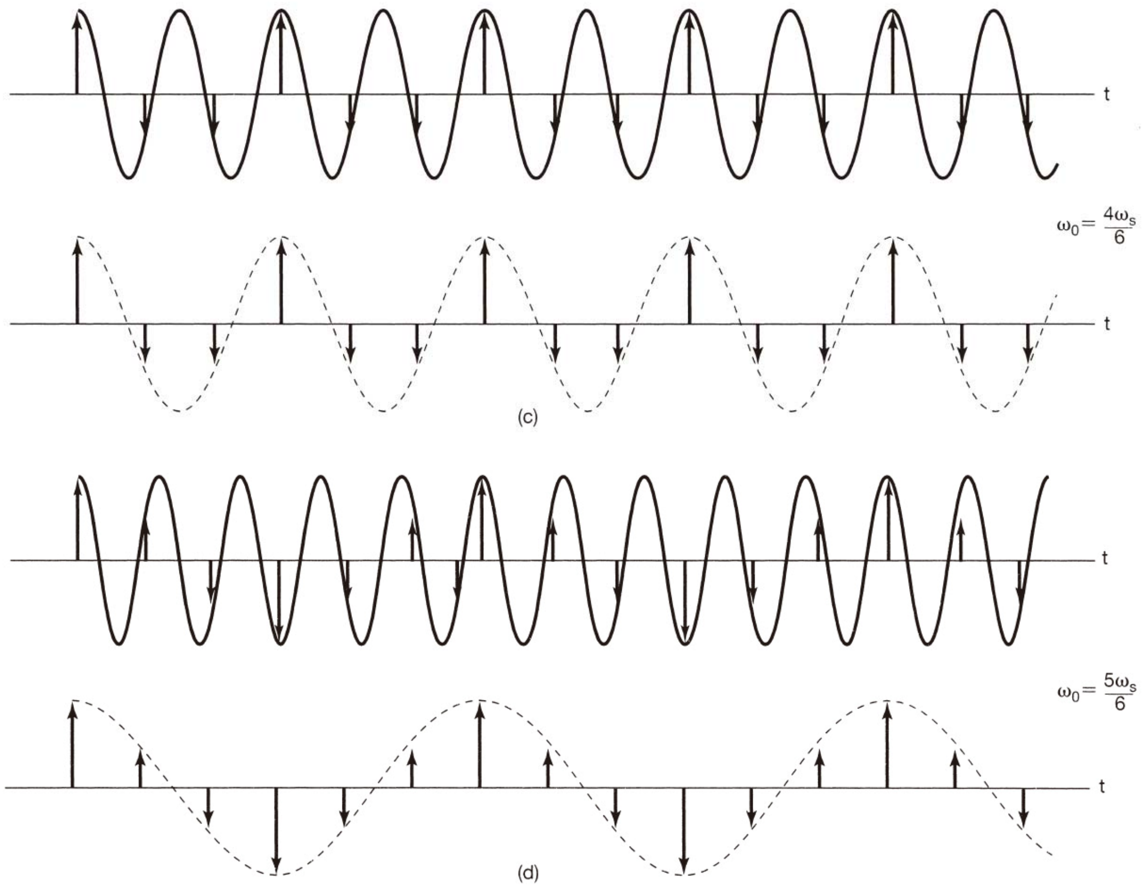


Figure 7.16 *Continued*

7.3 The Effect of Undersampling: Aliasing

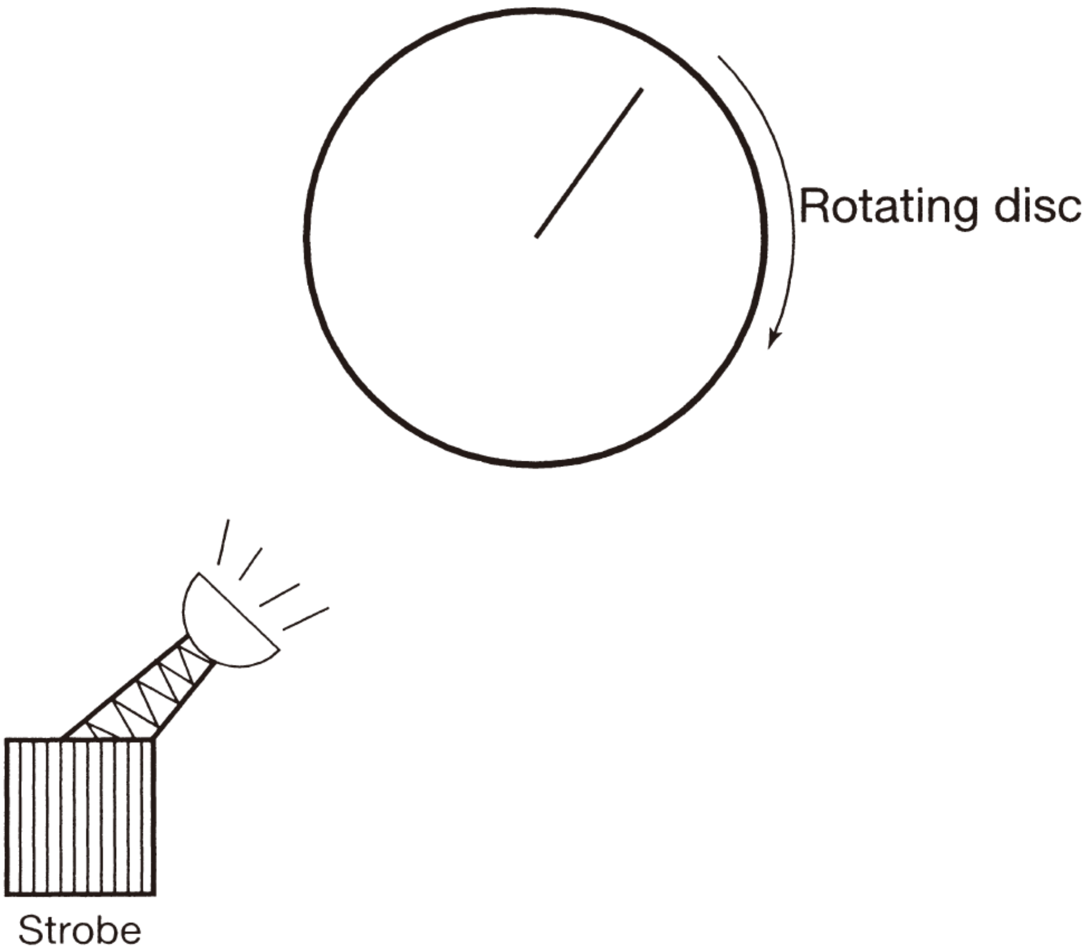


Figure 7.18 Strobe effect.

7.4 Discrete-Time Processing of Continuous-Time Signals

The continuous-time signal $x_c(t)$ is exactly represented by a sequence of instantaneous sample values $x_c(nT)$; that is, the discrete-time sequence is related to $x_d[n]$ by $x_c(t)$

$$x_d[n] = x_c(nT). \qquad (7.16)$$

7.4 Discrete-Time Processing of Continuous-Time Signals

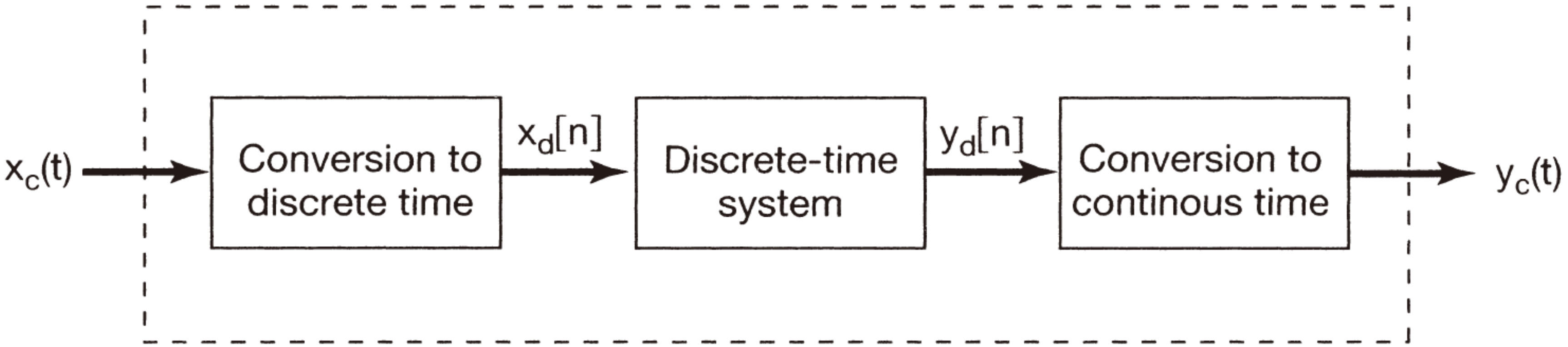


Figure 7.19 Discrete-time processing of continuous-time signals.

圖 7.19 顯示利用離散時間處理連續時間訊號的三個串接的動作。

7.4 Discrete-Time Processing of Continuous-Time Signals

The transformation of $x_c(t)$ to $x_d[n]$ corresponding to the first system in Figure 7.19 will be referred to as *continuous-to-discrete-time conversion* and will be abbreviated C/D. The reverse operation corresponding to the third system in Figure 7.19 will be abbreviated D/C, representing *discrete-time to continuous-time conversion*.

圖7.19中，由 $x_c(t)$ 至 $x_d[n]$ 為連續至離散時間轉換(C/D)；由 $y_d[n]$ 至 $y_c(t)$ 為離散至連續時間轉換(D/C)。

$$y_d[n] = y_c(nT).$$

7.4 Discrete-Time Processing of Continuous-Time Signals

In systems such as digital computers and digital systems for which the discrete-time signal is represented in digital form, the device commonly used to implement the C/D conversion is referred to as an *analog-to-digital* (A to-D) converter, and the device used to implement the D/C conversion is referred to as a *digital-to-analog* (D-to-A) converter.

C/D轉換的裝置為頻比至數位轉換器(ADC)：D/C轉換的裝置為數位至類比轉換器(DAC)。

7.4 Discrete-Time Processing of Continuous-Time Signals

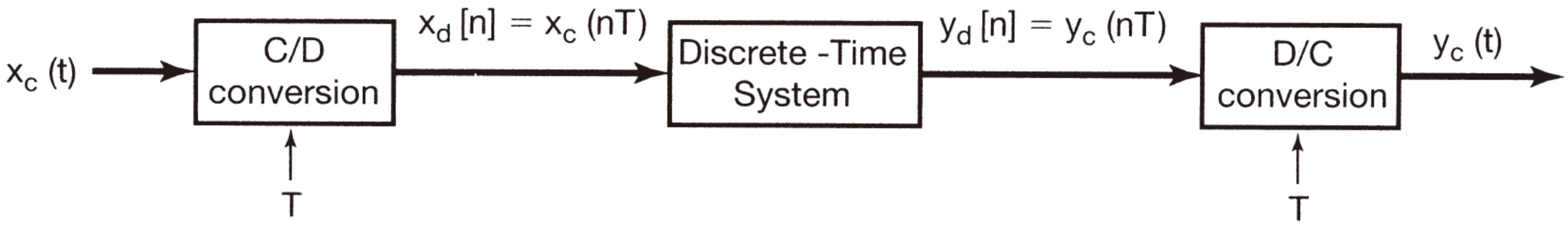
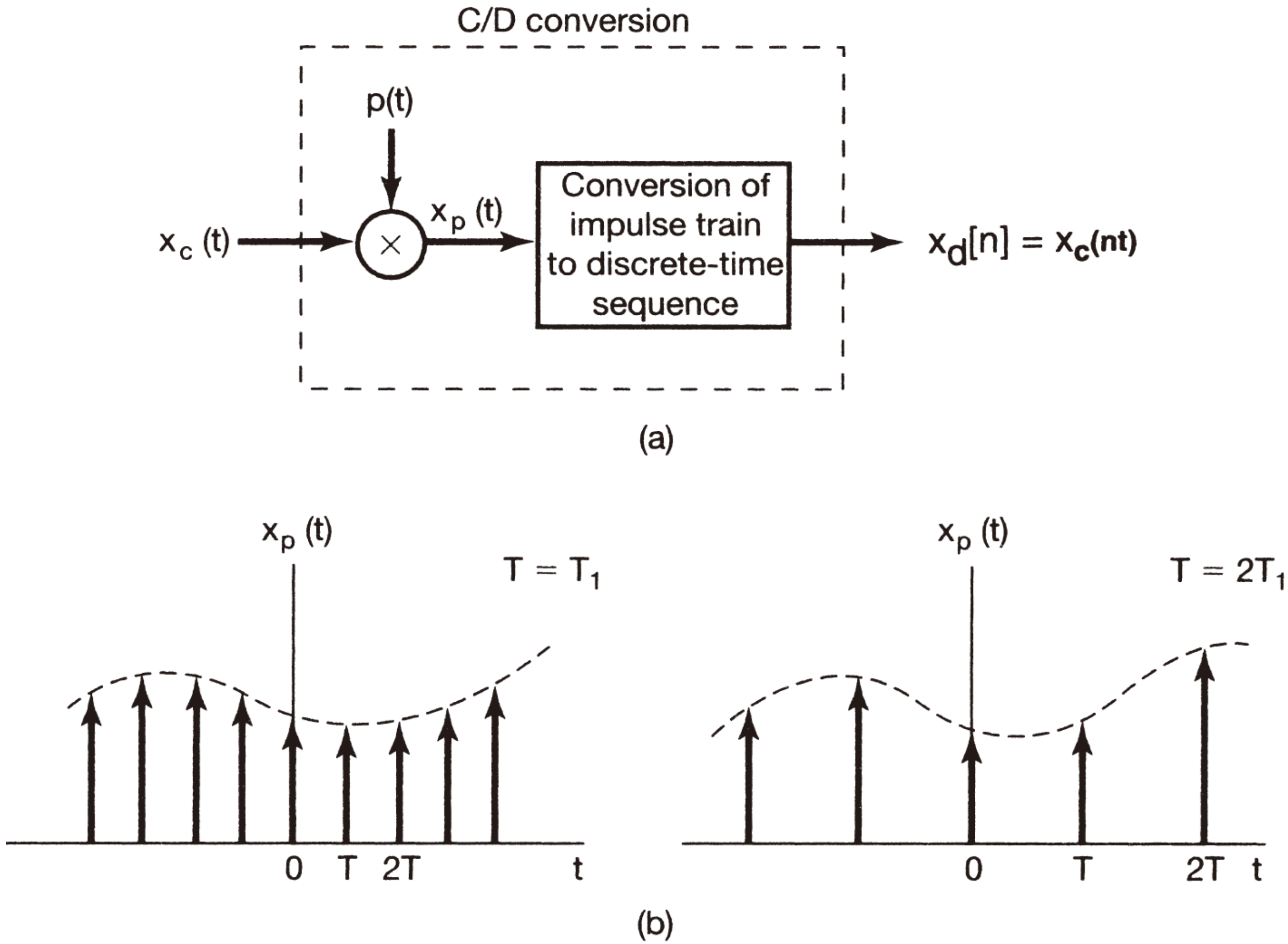


Figure 7.20 Notation for continuous-to-discrete-time conversion and discrete-to-continuous-time conversion. T represents the sampling period.

7.4 Discrete-Time Processing of Continuous-Time Signals



7.4 Discrete-Time Processing of Continuous-Time Signals

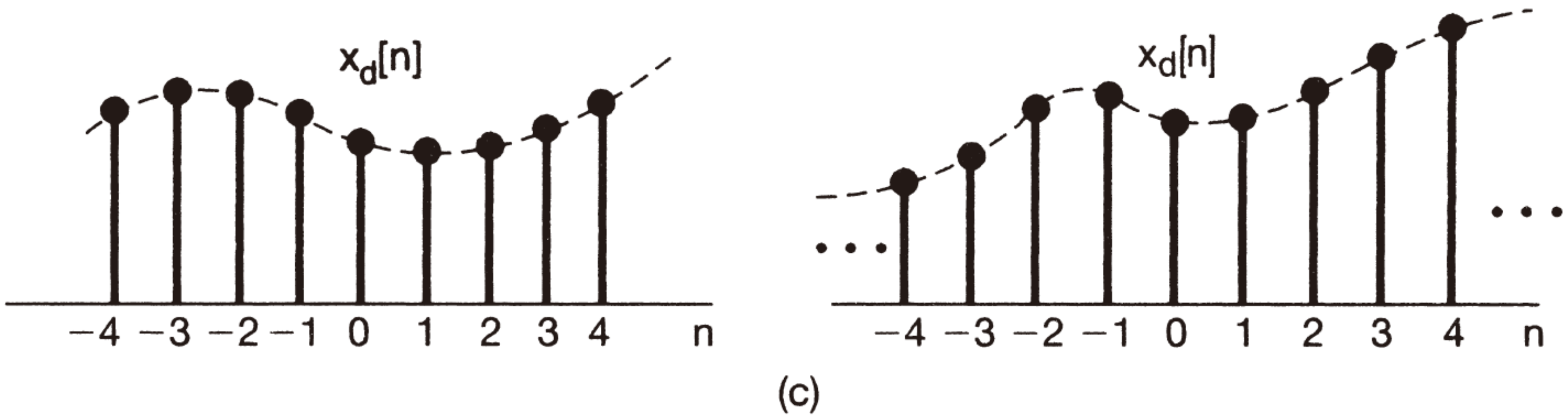


Figure 7.21 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence: (a) overall system; (b) $x_p(t)$ for two sampling rates. The dashed envelope represents $x_c(t)$; (c) the output sequence for the two different sampling rates.

7.4 Discrete-Time Processing of Continuous-Time Signals

To begin let us express $X_p(j\omega)$, the continuous-time Fourier transform of $x_p(t)$, in terms of the sample values of $x_c(t)$ by applying the Fourier transform to eq. (7.3). Since

$$x_p(t) = \sum x_c(nT)\delta(t - nT), \tag{7.17}$$

and since the transform of $\delta(t\text{-}nT)$ is $e^{-j\omega nT}$, it follows that

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-j\omega nT} \tag{7.18}$$

7.4 Discrete-Time Processing of Continuous-Time Signals

Now consider the discrete-time Fourier transform of $x_d[n]$, that is,

$$Xd\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{+\infty}x_d[n]e^{-j\Omega n}, \qquad (7.19)$$

or, using eq. (7.16),

$$X_d\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{+\infty}x_c(nT)e^{-j\Omega n}. \qquad (7.20)$$

7.4 Discrete-Time Processing of Continuous-Time Signals

Comparing eqs. (7.18) and (7.20), we see that $X_d\left(e^{j\Omega}\right)$ and $X_p[j\omega]$ are related through

$$X_d\left(e^{j\Omega}\right) = X_p\left(j\Omega / T\right). \qquad (7.21)$$

Also, recall that, as developed in eq. (7.6) and illustrated in Figure 7.3,

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(j(\omega - k\omega_s)\right). \qquad (7.22)$$

Consequently,

$$X_d\left(e^{j\Omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(j(\Omega - 2\pi k)/T\right). \qquad (7.23)$$

7.4 Discrete-Time Processing of Continuous-Time Signals

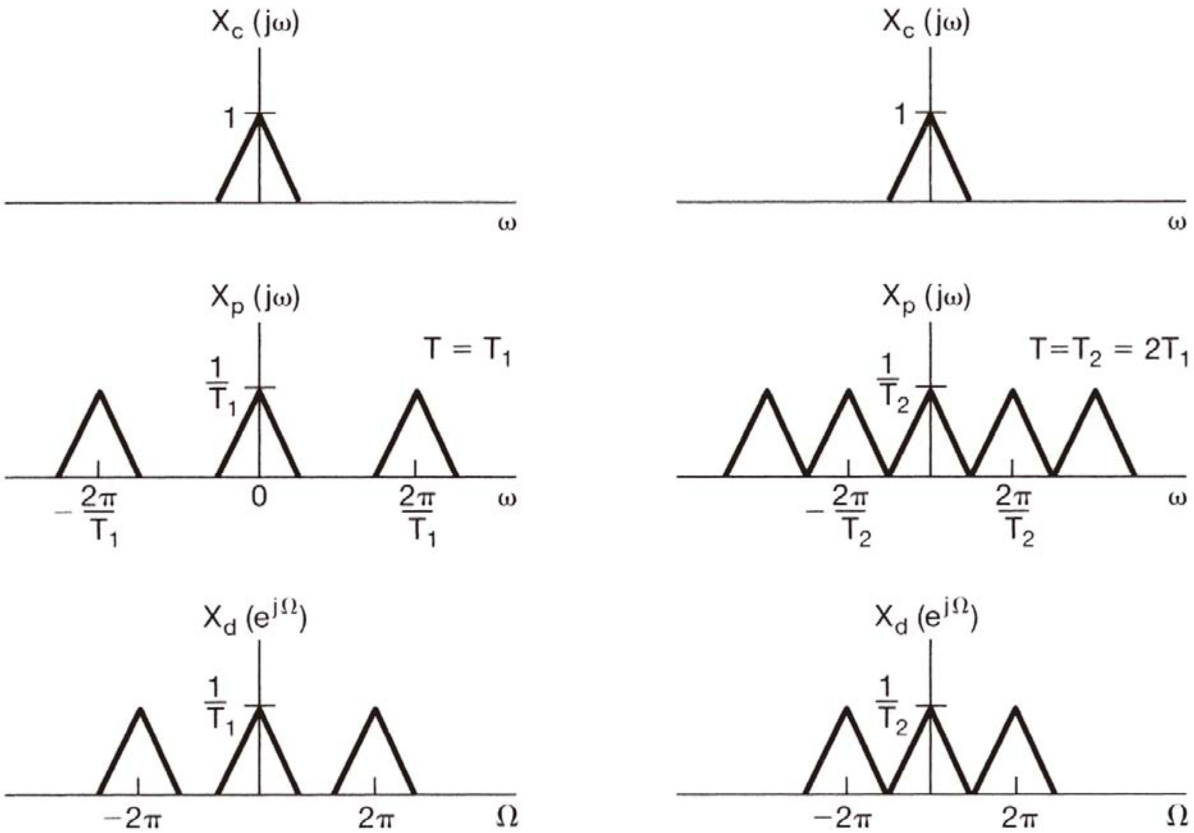


Figure 7.22 Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

7.4 Discrete-Time Processing of Continuous-Time Signals

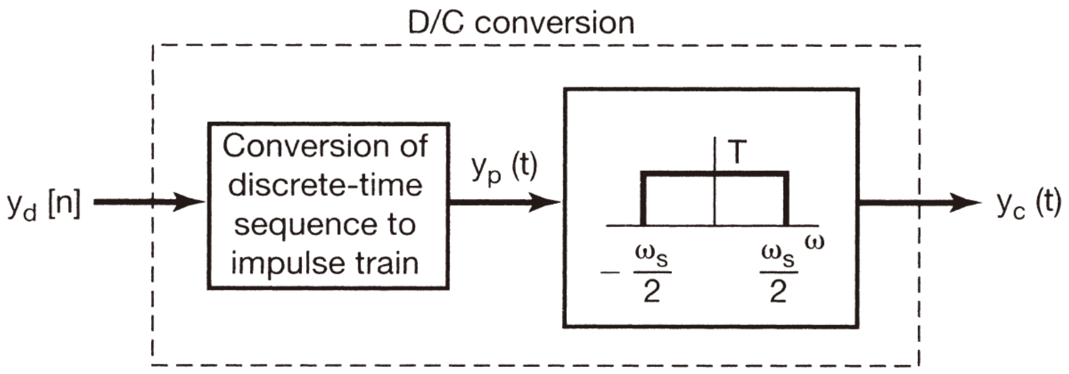


Figure 7.23 Conversion of a discrete-time sequence to a continuous-time signal.

7.4 Discrete-Time Processing of Continuous-Time Signals

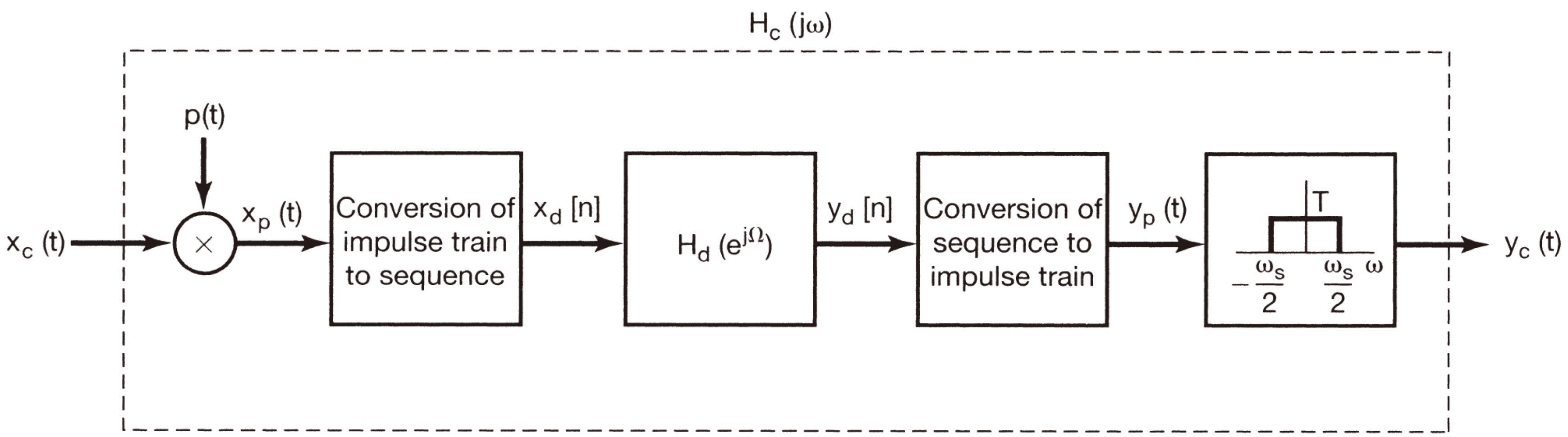


Figure 7.24 Overall system for filtering a continuous-time signal using a discrete-time filter.

7.4 Discrete-Time Processing of Continuous-Time Signals

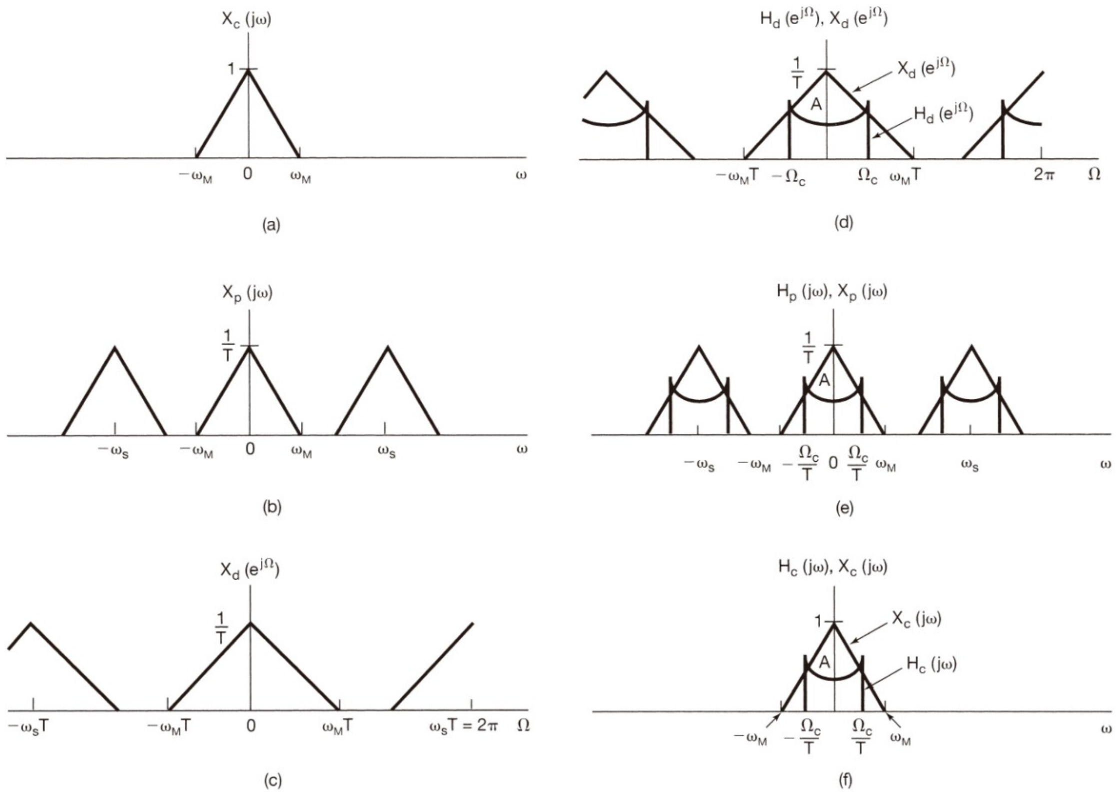


Figure 7.25 Frequency-domain illustration of the system of Figure 7.24: (a) continuous-time spectrum $X_c(j\omega)$; (b) spectrum after impulse-train sampling; (c) spectrum of discrete-time sequence $x_d[n]$; (d) $H_d(e^{j\Omega})$ and $X_d(e^{j\Omega})$ that are multiplied to form $Y_d(e^{j\Omega})$; (e) spectra that are multiplied to form $Y_p(j\omega)$; (f) spectra that are multiplied to form $Y_c(j\omega)$.

7.4 Discrete-Time Processing of Continuous-Time Signals

In comparing Figures 7.25(a) and (f), we see that

$$Y_c(j\omega) = X_c(j\omega)H_d(e^{j\omega T}). \qquad (7.24)$$

equivalent to a continuous-time LTI system with frequency response $H_c(j\omega)$ which is related to the discrete-time frequency response $H_d\left(e^{j\Omega}\right)$ through

$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}. \qquad (7.25)$$

7.4 Discrete-Time Processing of Continuous-Time Signals

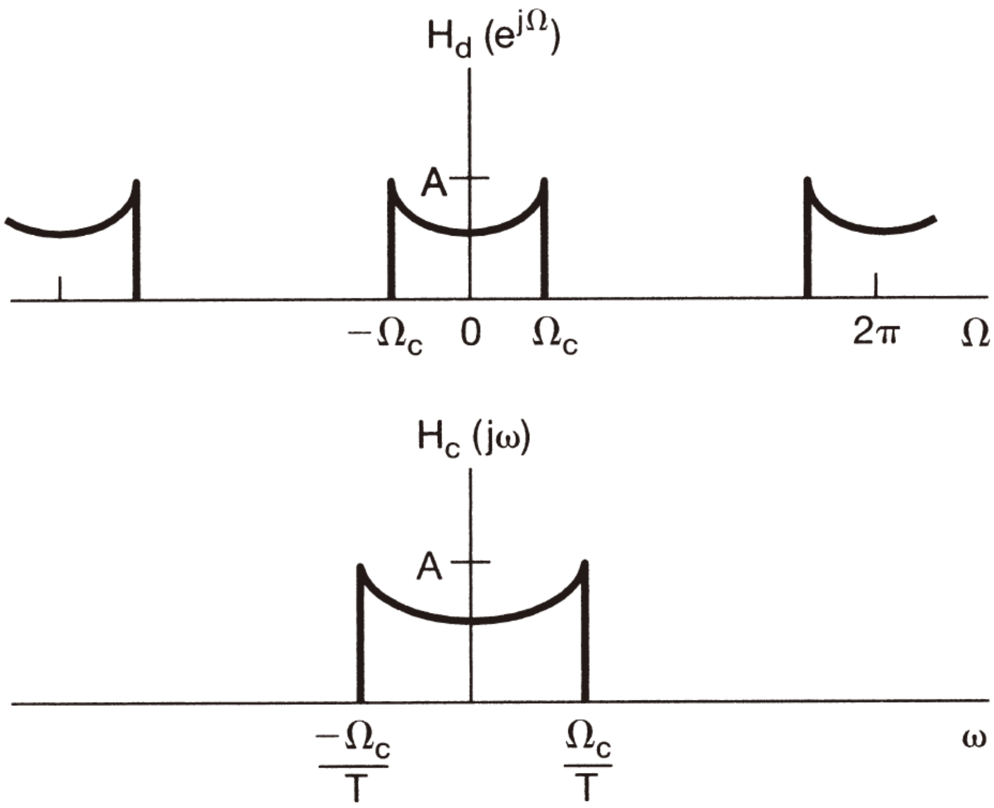


Figure 7.26 Discrete-time frequency response and the equivalent continuous-time frequency response for the system of Figure 7.24.

7.4.1 Digital Differentiator

Consider the discrete-time implementation of a continuous-time band-limited differentiating filter.

$$H_c(j\omega) = j\omega, \qquad (7.26)$$

連續時間微分濾波器的頻率響應

and that of a band-limited differentiator with cutoff frequency ω_c is

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \qquad (7.27)$$

截止頻率為 ω_c 的有限頻帶微分器

7.4.1 Digital Differentiator

As sketched in Figure 7.27. Using eq. (7.25) with a sampling frequency $\omega_s = 2\omega_c$, we see that the corresponding discrete-time transfer function is

$$H_d\left(e^{j\Omega}\right) = j\left(\frac{\Omega}{T}\right), \qquad |\Omega| < \pi, \tag{7.28}$$

在 $\omega_s = 2\omega_c$ 下所對應的離散時間轉移函數

7.4.1 Digital Differentiator

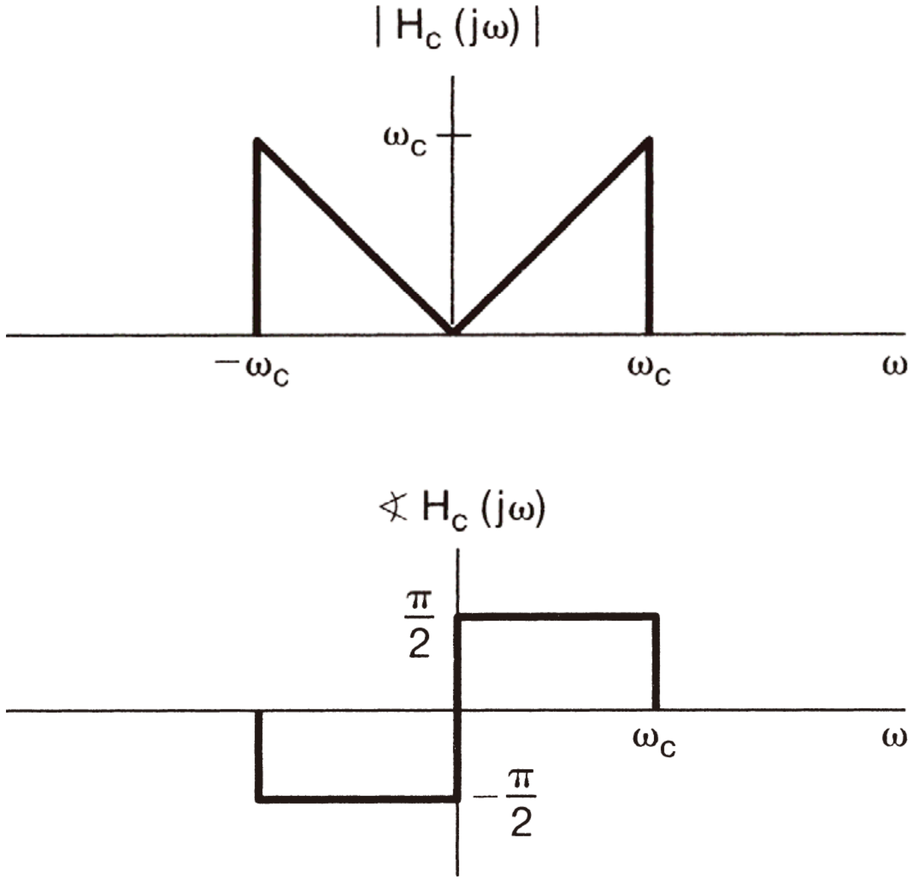


圖 7.27 爲 (7.27) 式的頻率響應圖。

Figure 7.27 Frequency response of a continuous-time ideal band-limited differentiator $H_c(j\omega) = j\omega,\ |\omega| < \omega_c$.

7.4.2 Half-Sample Delay

We require that the input and output of the overall system be related by

$$y_c(t) = x_c(t-\Delta) \tag{7.33}$$

From the time-shifting property derived in Section 4.3.2

$$Y_c(j\omega) = e^{-j\omega\Delta} X_c(j\omega).$$

7.4.2 Half-Sample Delay

From eq. (7.25), the equivalent continuous-time system to be implemented must be band limited. Therefore, we take

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & otherwise \end{cases}, \qquad (7.34)$$

有限頻寬連續時間延遲系統的頻率響應

7.4.2 Half-Sample Delay

With the sampling frequency ω_s taken as $\omega_s = 2\omega_c$, the corresponding discrete-time frequency response is

$$H_d\left(e^{j\Omega}\right) = e^{-j\Omega\Delta/T}, \qquad |\Omega| < \pi, \qquad (7.35)$$

半取樣延遲系統的頻率響應

7.4.2 Half-Sample Delay

圖7.29(b)為 $\omega_s = 2\omega_c$ 之下的離散時間半取樣延遲系統。

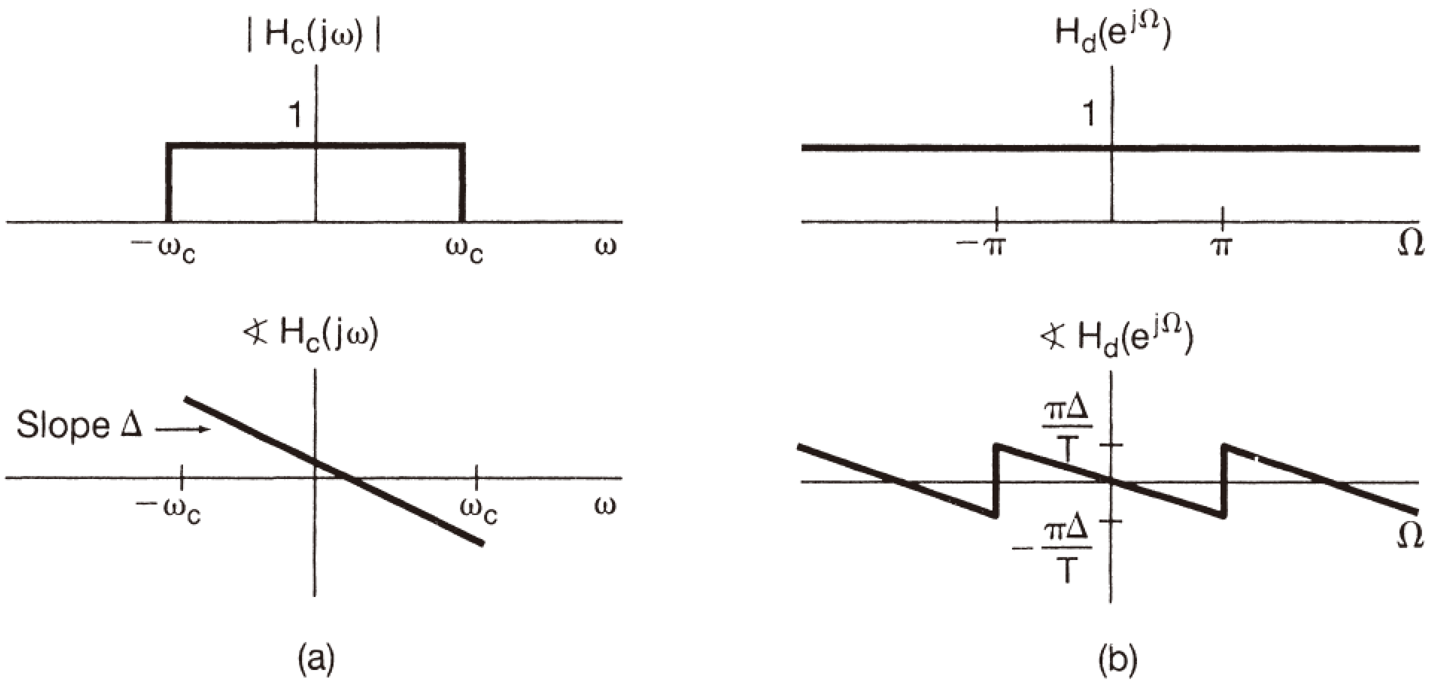


Figure 7.29 (a) Magnitude and phase of the frequency response for a continuous-time delay; (b) magnitude and phase of the frequency response for the corresponding discrete-time delay.

7.4.2 Half-Sample Delay

For Δ/T an integer, the sequence $y_d[n]$ is a delayed replica of $x_d[n]$; that is,

$$y_d[n] = x_d\left[n - \frac{\Delta}{T}\right]. \qquad (7.36)$$

若 Δ/T 為整數，則 $y_d[n]$ 為 $x_d[n]$ 的時間移位。

7.4.2 Half-Sample Delay

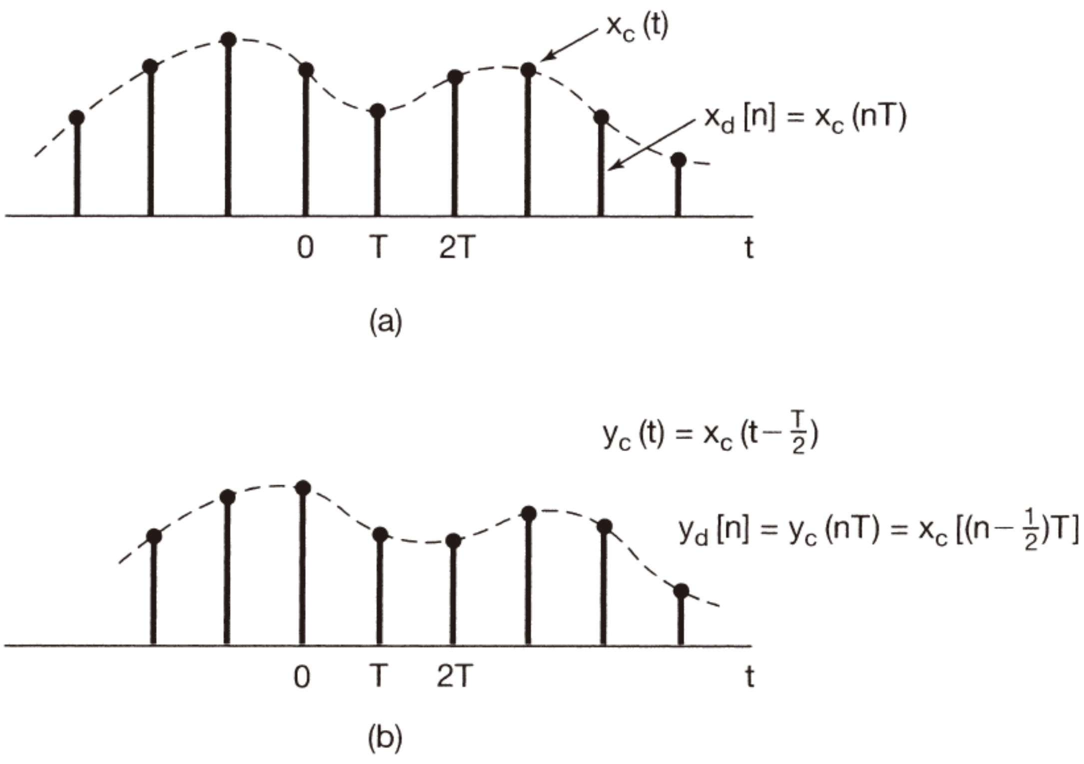


Figure 7.30 (a) Sequence of samples of a continuous-time signal $x_c(t)$; (b) sequence in (a) with a half-sample delay.

7.5.1 Impulse-Train Sampling

The new sequence $x_p[n]$ resulting from the sampling process is equal to the original sequence *x[n]* at integer multiples of the sampling period *N* and is zero at the intermediate samples;

$$x_p[n] = \begin{cases} x[n], & \text{if n = an integer multiple of N} \\ 0, & \text{otherwise} \end{cases} \qquad (7.38)$$

$x_p[n]$ 為*x[n]*以取樣週期*N*取值，而取樣點之間的訊號值均為0所得的序列。

7.5.1 Impulse-Train Sampling

As with continuous-time sampling in Section 7.1, the effect in the frequency domain of discrete-time sampling is seen by using the multiplication property developed in Section 5.5. Thus, with

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n-kN], \qquad (7.39)$$

離散時間取樣訊號與原訊號的時域關係

7.5.1 Impulse-Train Sampling

We have, in the frequency domain,

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right) X\left(e^{j(\omega-\theta)}\right) d\theta. \quad (7.40)$$

離散時間取樣訊號與原訊號的頻域關係

7.5.1 Impulse-Train Sampling

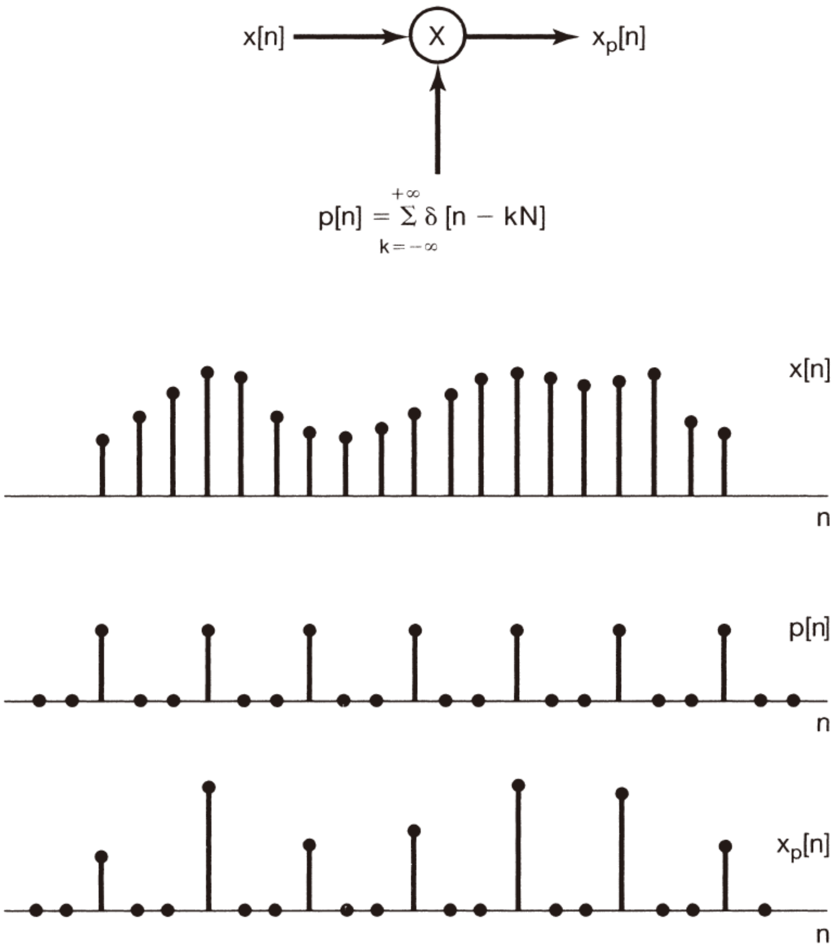


Figure 7.31 Discrete-time sampling.

7.5.1 Impulse-Train Sampling

As in Example 5.6, the Fourier transform of the sampling sequence *p[n]* is

$$P\left(e^{j\omega}\right)=\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s), \qquad (7.41)$$

where ω_s, the sampling frequency, equals $2\pi/N$. Combining eqs. (7.40) and (7.41), we have

$$X_p\left(e^{j\omega}\right)=\frac{1}{N}\sum_{k=0}^{N-1}X\left(e^{j(\omega-k\omega_s)}\right). \qquad (7.42)$$

7.5.1 Impulse-Train Sampling

離散時間取樣訊號的傅立葉轉換
類似(7.6)式的連續時間情形。

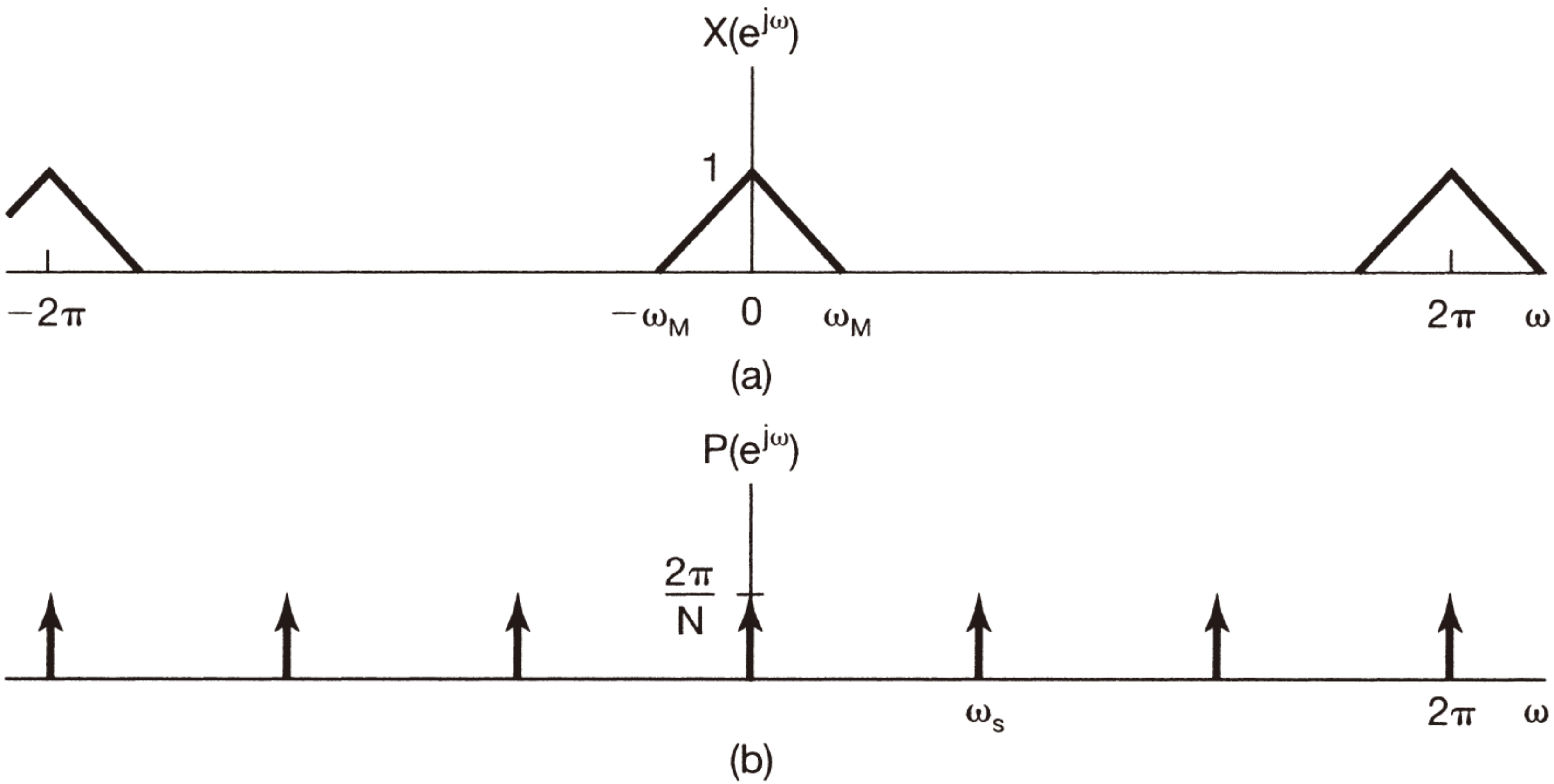
配合圖7.32：

若 $\omega_s > 2\omega_M$ ，則沒有頻率虛化；

若 $\omega_s < 2\omega_M$ ，則造成頻率虛化。

在無頻率虛化之下，$x_p[n]$ 可經由增益為*N*且截止頻率介於 ω_M 及 $\omega_s - \omega_M$ 的低通濾波器還原。

7.5.1 Impulse-Train Sampling



7.5.1 Impulse-Train Sampling

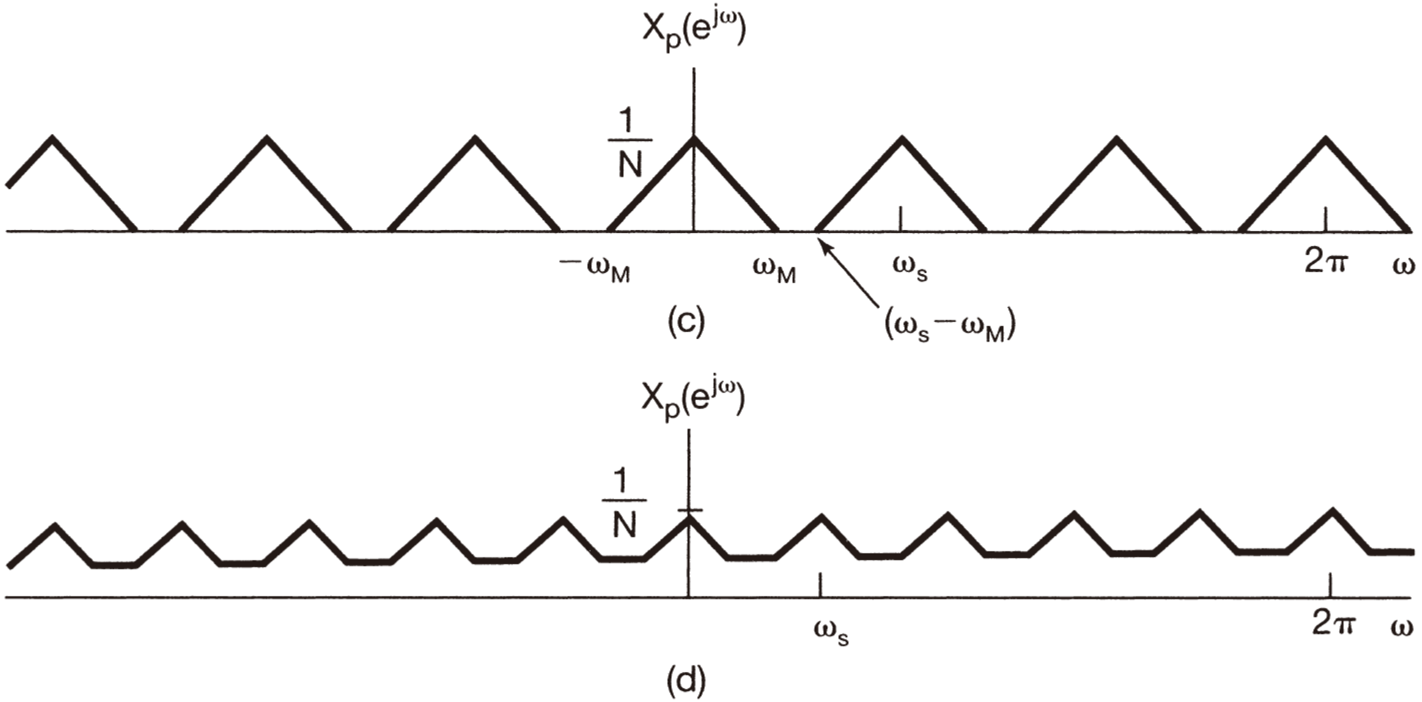


Figure 7.32 Effect in the frequency domain of impulse-train sampling of a discrete-time signal: (a) spectrum of original signal; (b) spectrum of sampling sequence; (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$. Note that aliasing occurs.

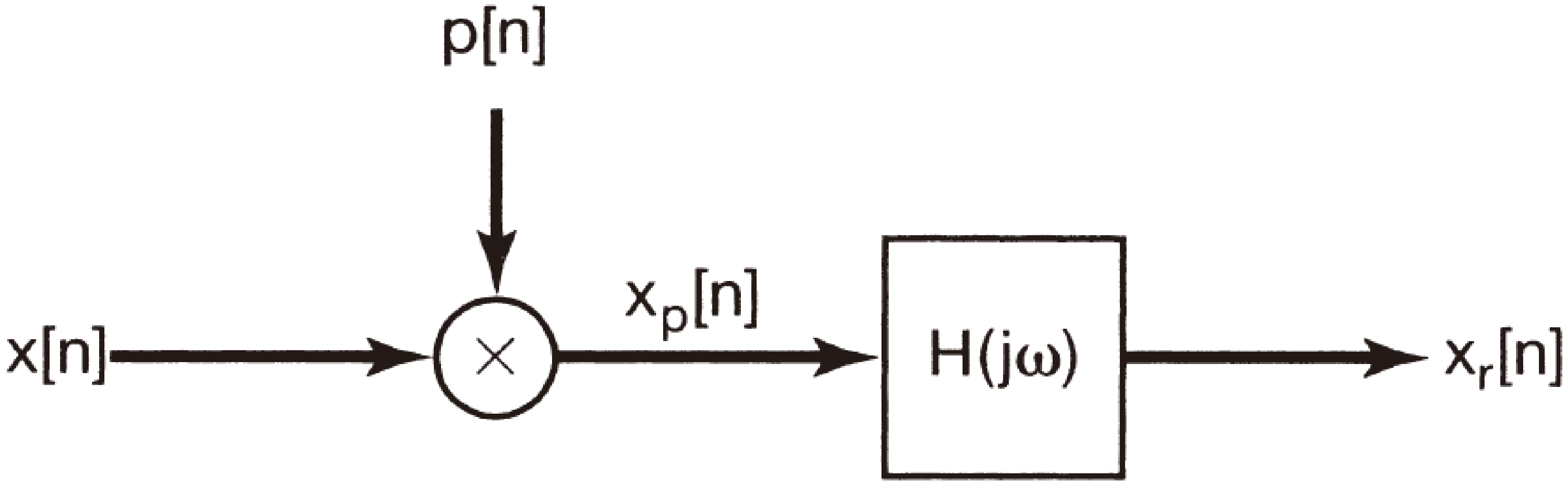
7.5.1 Impulse-Train Sampling

If the overall system of Figure 7.33(a) is aplied to a sequence for which $\omega_s < 2\omega_M$, to that aliasing results, $x_r[n]$ will no longer be equal to *x[n]*. However, as wit continuous-time sampling, the two sequences will be equal at multiples of the sampling period; that is, corresponding to eq. (7.13), we have

$$x_r[kN] = x[kN], \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{7.43}$$

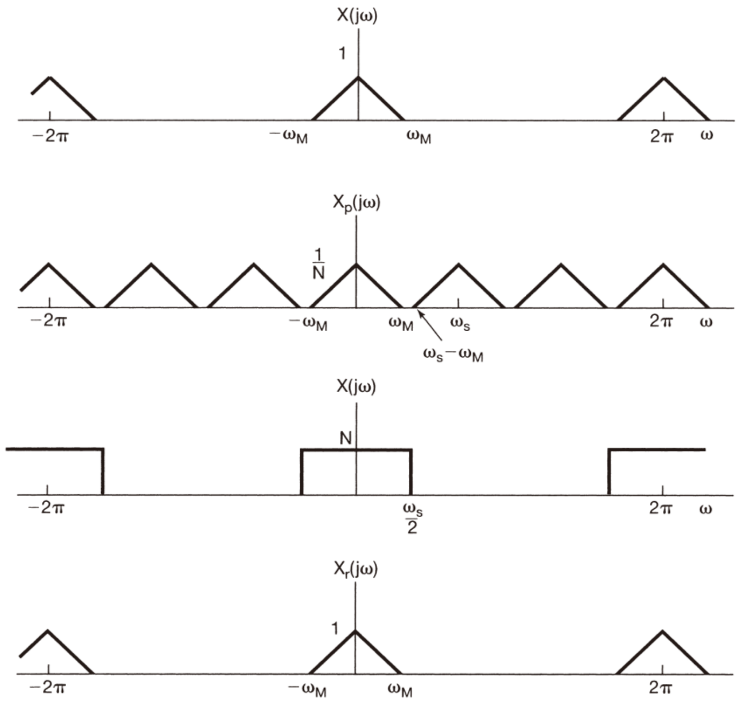
若 $\omega_s < 2\omega_M$ ，則 $x_r[n]$ 無法還原成*x[n]*，但在取樣點處是相等的。

7.5.1 Impulse-Train Sampling



(a)

7.5.1 Impulse-Train Sampling



(b)

Figure 7.33 Exact recovery of a discrete-time signal from its samples using an ideal lowpass filter: (a) block diagram for sampling and reconstruction of a band-limited signal from its samples; (b) spectrum of the signal $x[n]$; (c) spectrum of $x_p[n]$; (d) frequency response of an ideal lowpass filter with cutoff frequency $\omega_s/2$; (e) spectrum of the reconstructed signal $x_r[n]$. For the example depicted here $\omega_s > 2\omega_M$ so that no aliasing occurs and consequently $x_r[n] = x[n]$.

7.5.1 Impulse-Train Sampling

With *h[n]* denoting the impulse response of the lowpass filter, we have

$$h[n]=\frac{N\omega_c}{\pi}\frac{\sin\omega_c n}{\omega_c n}. \tag{7.44}$$

The reconstructed sequence is then

$$x_r[n]=x_p[n]*h[n], \tag{7.45}$$

7.5.1 Impulse-Train Sampling

or equivalently,

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN] \frac{N\omega_c}{\pi} \frac{\sin \omega_c (n-kN)}{\omega_c (n-kN)}. \qquad (7.46)$$

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN] h_r[n-kN], \qquad (7.47)$$

7.5.2 Discrete-Time Decimation and Interpolation

The sampled sequence is typically replaced by a new sequence $x_b[n]$, which is simply every Nth value of $x_p[n]$;that is,

在許多應用中，因$x_p[n]$在取樣瞬間之間為0，故直接以$x_p[n]$處理是沒有效率的。

令 $$x_b[n] = x_p[nN]. \tag{7.48}$$

Also, equivalently,

即 $$x_b[n] = x[nN], \text{ (每}N\text{個值取一值)。} \tag{7.49}$$

則可簡化離散時間取樣的處理，此一動作稱為「抽離」。

7.5.2 Discrete-Time Decimation and Interpolation

To determine the effect in the frequency domain of decimation, we wish to determine the relationship between $X_b(e^{j\omega})$ —the Fourier transform of $x_b[n]$ — and $X(e^{j\omega})$. To this end, we note that

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b[k] e^{-j\omega k}, \tag{7.50}$$

or, using eq. (7.48),

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_p[kN] e^{-j\omega k}. \tag{7.51}$$

7.5.2 Discrete-Time Decimation and Interpolation

If we let $n = kN$, or equivalently $k = n/N$, we can write

$$X_b(e^{j\omega}) = \sum_{\substack{n = \text{integer} \\ \text{Multiple of N}}} x_p[n] e^{-j\omega n/N}$$

and since $x_p[n] = 0$ when n is not an integer multiple of N, we can also write

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_p[n] e^{-j\omega n/N}. \qquad (7.52)$$

7.5.2 Discrete-Time Decimation and Interpolation

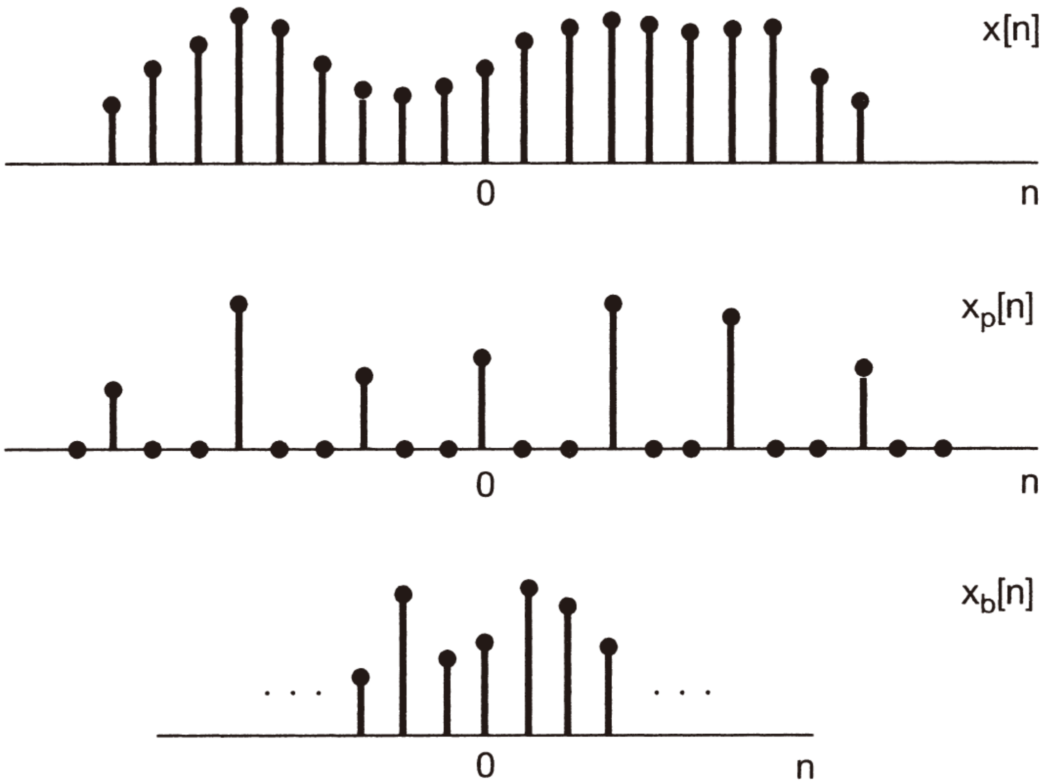


Figure 7.34 Relationship between $x_p[n]$ corresponding to sampling and $x_b[n]$ corresponding to decimation.

7.5.2 Discrete-Time Decimation and Interpolation

Furthermore, we recognize the right-hand side of eq.(7.52) as the Fourier transform of $x_p[n]$; that is,

$$\sum_{n=-\infty}^{+\infty} x_p[n] e^{-j\omega n/N} = X_p(e^{j\omega/N}). \qquad (7.53)$$

Thus, from eqs. (7.52) and (7.53), we conclude that

$$X_b(e^{j\omega}) = X_p(e^{j\omega/N}). \qquad (7.54)$$

抽離訊號與離散時間取樣訊號的傅立葉轉換關係式

7.5.2 Discrete-Time Decimation and Interpolation

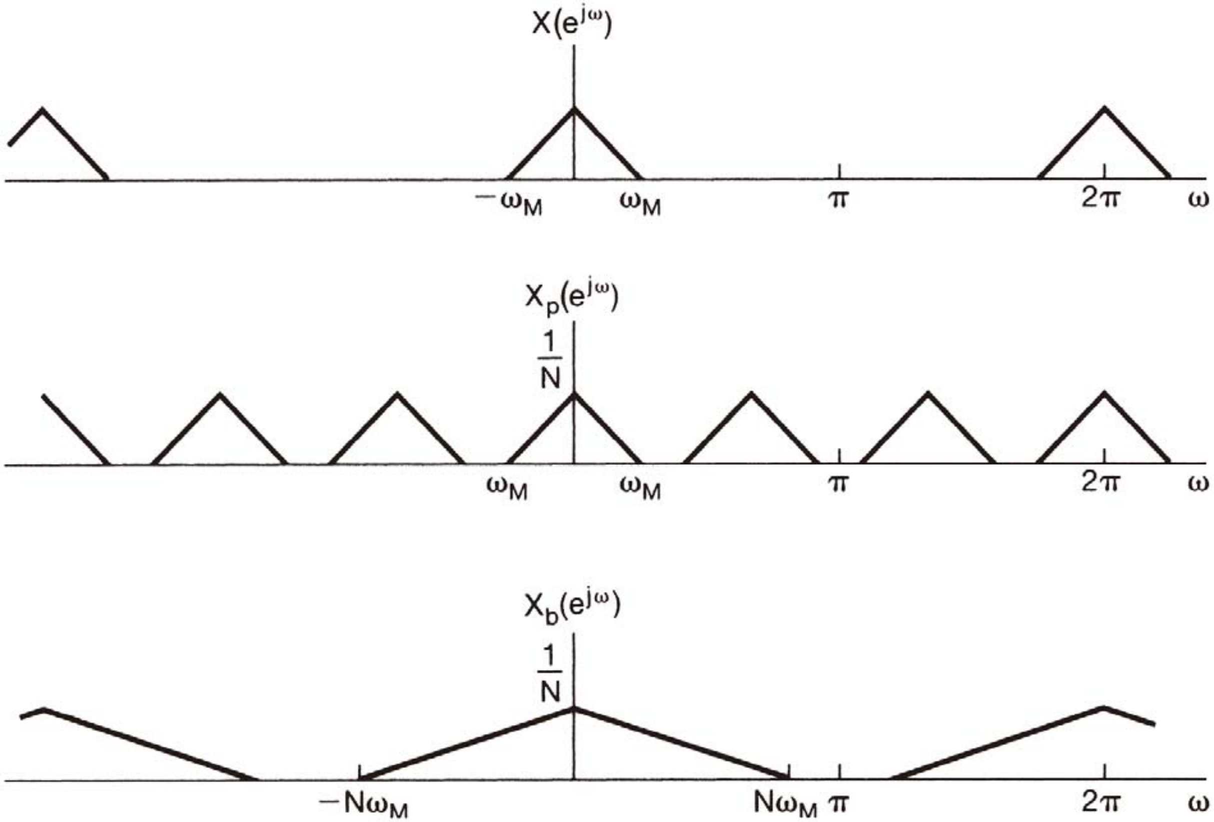


Figure 7.35 Frequency-domain illustration of the relationship between sampling and decimation.

7.5.2 Discrete-Time Decimation and Interpolation

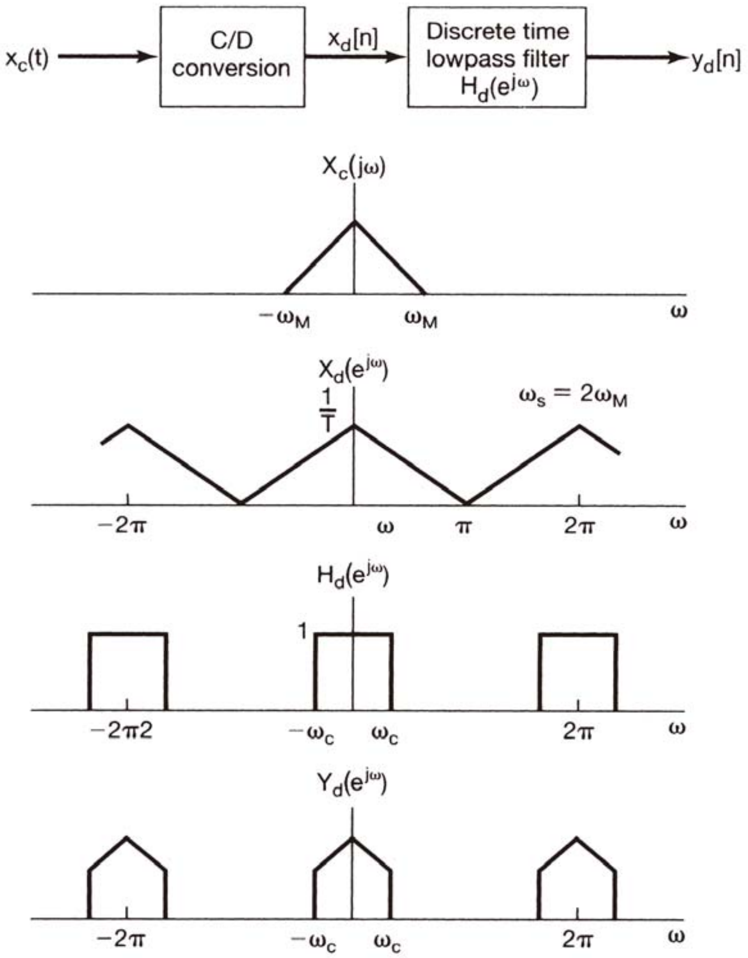


Figure 7.36 Continuous-time signal that was originally sampled at the Nyquist rate. After discrete-time filtering, the resulting sequence can be further downsampled. Here $X_c(j\omega)$ is the continuous-time Fourier transform of $x_c(t)$, $X_d(e^{j\omega})$ and $Y_d(e^{j\omega})$ are the discrete-time Fourier transforms of $x_d[n]$ and $y_d[n]$ respectively, and $H_d(e^{j\omega})$ is the frequency response of the discrete-time lowpass filter depicted in the block diagram.

7.5.2 Discrete-Time Decimation and Interpolation

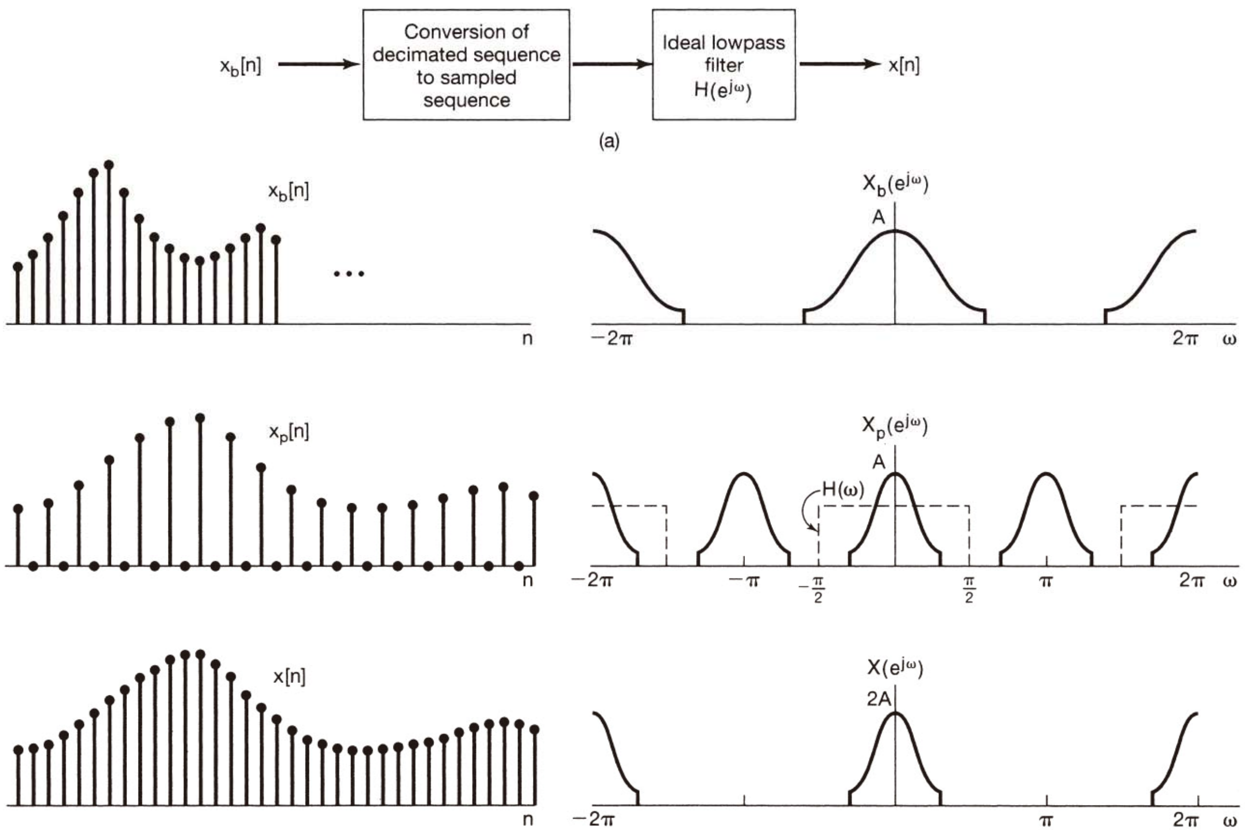


Figure 7.37 Upsampling: (a) overall system; (b) associated sequences and spectra for upsampling by a factor of 2.

7.6 Summary

在本章中，我們發展了取樣的概念。由此概念，一個連續時間或離散時間訊號，可用一個時間間隔相等的取樣值的序列來表示。此一訊號是否可以由取樣值完全還原的條件，則隱藏在取樣定理之中。對於完全的重建，這個定理要求此一被取樣的訊號必須是有限頻寬，且取樣頻率必須大於被取樣的訊號最大頻率的兩倍。

7.6 Summary

在這些條件下，原訊號的完全重建可以利用理想低通濾波器來達成。這種理想重建的過程在時域上的意義，常常可解釋為理想的有限頻寬內插法。在實際應用上，低通濾波器只是近似的實現，而時域中的內插法不再是完全（重建）。在某些實例中，簡單的內插法過程，諸如零階保持或線性內插法（一階保持）就已足夠了。

7.6 Summary

如果一個訊號為不完全取樣（即如果取樣頻率小於取樣定理的要求值），則由理想的有限頻寬內插法所重建的訊號與原訊號的關係將是一種失真的型式，稱為「頻率假化」。在許多的實例中，選擇取樣頻率以避免頻率假化是很重要的。然而，有許多不同種類的重要案例，例如閃光測頻器，頻率假化卻是可以善加利用的。

7.6 Summary

取樣有很多重要的應用。其中一種特別顯著的例子，是有關於在離散時間系統中，利用取樣來處理連續時間訊號，這些可以透過小型電腦、微處理器，或是任何針對離散時間訊號處理的裝置來達成。

取樣的基本理論對於連續時間與離散時間訊號兩者而言，是很相似的。在離散時間的情形中，有一個極為相關的概念，稱為「抽離」，由此，抽離後的序列可由原序列的等間隔抽取值而得。

7.6 Summary

取樣的基本理論對於連續時間與離散時間訊號兩者而言，是很相似的。在離散時間的情形中，有一個極為相關的概念，稱為「抽離」，由此，抽離後的序列可由原序列的等間隔抽取值而得。取樣與抽離的不同之處在於：對於取樣序列而言，取樣值之間的訊號值為零；而在抽離序列中，這些零值是棄而不用的，因而序列在時間上是壓縮的。抽離的相反處理就是內插。抽離和內插的概念起因於訊號與系統種種重要的實際應用，包括通訊系統、數位音訊、高畫質電視以及其他許多的應用。