Contents

16 Waves-I

Waves are one of the primary subjects in physics and play a crucial role in sound, light, and even quantum mechanics.

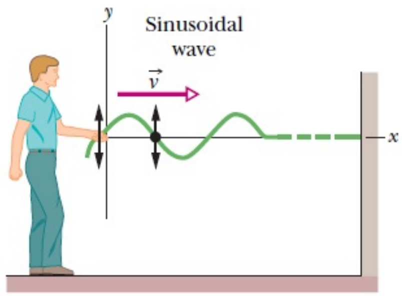
In this chapter we will focus on waves traveling along a stretched string.

16.1 Types of Waves

Mechanical waves: require a a material medium (water, air, rock, string)
Electromagnetic waves: visible and invisible light (x-rays, ultraviolet light, visible light, microwaves, radar waves, radio and TV waves,). Travels in vacuum at speed $c = 299,792,458\ m/s$.
Matter waves: small particles (electrons, protons, etc) can behave like waves $\rightarrow$ quantum mechanics

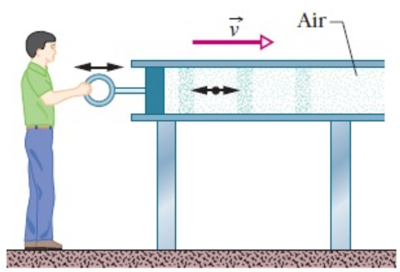
16.1.1 Transverse Waves

Displacement of every oscillating element, e.g., piece of string, is perpendicular to direction of wave travel.



16.1.2 Longitudinal Waves

Displacement of every oscillating element, e.g., air molecules, is parallel to direction of wave travel.



In general, a one dimensional wave is a displacement $y(x,t)$, which is function of the space x and time t, moves without changing its form. i.e., as $t \to t + \Delta t$, the amplitude at space point x is displaced to the point $x + \Delta x = x + v\Delta t$

$$y(x,t) = y(x + \Delta x, t + \Delta t) = y(x + v\Delta t, t + \Delta t)$$

where v is called the wave speed. Let $\Delta t = -t$, the above identity becomes

$$y(x,t) = y(x - vt, 0) = f(x - vt) \tag{1}$$

In other words, although the displacement $y(x,t)$ is a function of two variables x and t, it actually depends only the combination $x - vt$.

16.2 Wave Equation

By (1), the partial derivatives $\frac{\partial y}{\partial t}$ and $\frac{\partial y}{\partial x}$ can be related to the ordinary derivative of f. To see this, let $\tau = x - vt$. From the definition of partial derivative, we get

$$\begin{aligned}\frac{\partial y}{\partial x} &= \lim_{\Delta x \to 0} \frac{y(x+\Delta x, t) - y(x,t)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x - vt) - f(x - vt)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(\tau + \Delta x) - f(\tau)}{\Delta x} = \frac{df}{d\tau}|_{\tau = x - vt}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial y}{\partial t} &= \lim_{\Delta t \to 0} \frac{y(x, t+\Delta t) - y(x,t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x - vt - v\Delta t) - f(x - vt)}{\Delta t} \\ &= -v \lim_{\Delta t \to 0} \frac{f(\tau - v\Delta t) - f(\tau)}{-v\Delta t} = -v\frac{df}{d\tau}|_{\tau = x - vt}\end{aligned}$$

Since $\frac{df}{d\tau}$ is still a function of $\tau = x - vt$, we can similarly derive

$$\frac{\partial^2 y}{\partial^2 x} = \frac{d^2 f}{d\tau^2}|_{\tau = x - vt}, \frac{\partial^2 y}{\partial t^2} = v^2 \frac{d^2 f}{d\tau^2}|_{\tau = x - vt}$$

As a consequence, we have

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial^2 x} = 0 \tag{2}$$

which is identified as the wave equation.

16.3 Wavelength and Frequency

If the motion dictated by the displacement $y(x,t) = f(x - vt)$ at any fixed point x is SHM, the wave is called sinusoidal. In particluar, at $x = 0$, assume

$$y(0,t) = y_m \cos(\omega t + \phi).$$

Then

$$y(x,t) = f(x - vt) = y\left(0, t - \frac{x}{v}\right),$$

and we have

$$y(x,t) = y_m \cos\left(\omega\left(t - \frac{x}{v}\right) + \phi\right)$$

or

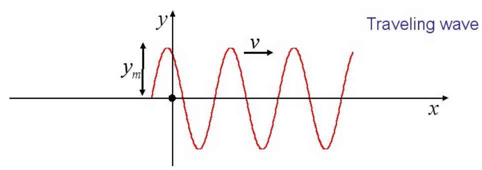
$$y(x,t) = y_m \cos(\omega t - kx + \phi) \tag{3}$$

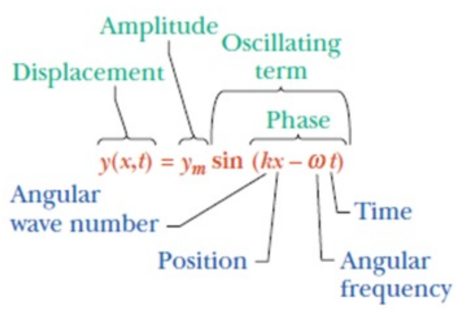
where

$$k = \frac{\omega}{v}$$

In particular, for $\phi = \frac{\pi}{2}$, we have

$$y(x,t) = y_m \sin(kx - \omega t)$$





The period T of SHM at any spatial is given by

$$\omega T = 2\pi, T = \frac{2\pi}{\omega}$$

The frequency

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

If we take a snap shot of the above $y(x,t)$ at any instant t, it is a function periodic in x with the wavelength (period in space) λ equal to

$$k\lambda = 2\pi, \lambda = \frac{2\pi}{k}$$

The wave velocity

$$v = \frac{\omega}{k} = \frac{\lambda}{T}$$

16.4 Speed for a Transverse Wave on a Stretched String

The tension force $\vec{T}$ points along the tangential direction on the wave displacement $y\left(x,t\right)$ treated as a function of x at any instant t.

$$\vec{T} = T\hat{n}$$

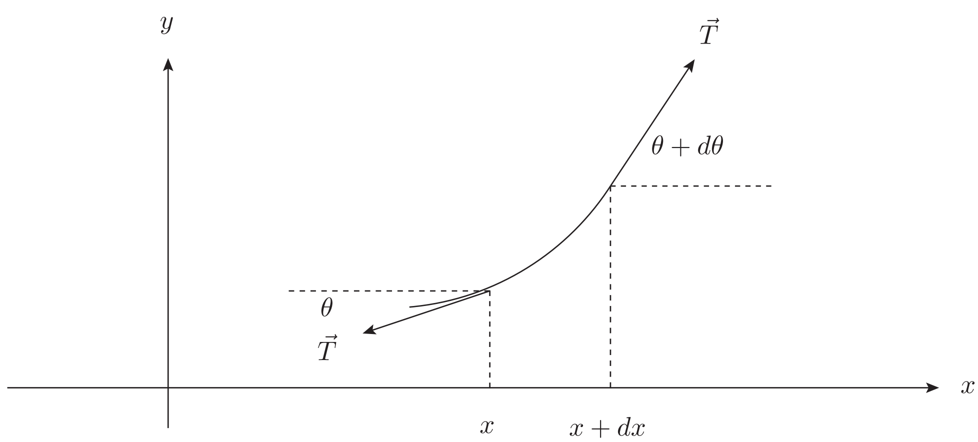
where

$$\hat{n} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

with $\tan\theta = \frac{\partial y}{\partial x}$. For the mass dm residing between x and $x + dx$, we have

$$dm = \mu dx$$

where μ is the linear mass density.



The motion in the transverse direction is determined by Newton's 2nd law of motion

$$dm\frac{\partial^2}{\partial t^2}y\left(x,t\right) = T\sin\theta\left(x + dx, t\right) - T\sin\theta\left(x,t\right)$$

If the amplitude of oscillation is small, $\theta \ll 1$ and

$$\begin{aligned} T\sin\theta\left(x + dx, t\right) - T\sin\theta\left(x,t\right) &\simeq T\left(\tan\theta\left(x + dx, t\right) - \tan\theta\left(x,t\right)\right) \\ &= T\left(\frac{\partial y}{\partial x}\left(x + dx, t\right) - \frac{\partial y}{\partial x}\left(x,t\right)\right) \\ &\simeq T dx \frac{\partial^2 y}{\partial x^2}\left(x,t\right) \end{aligned}$$

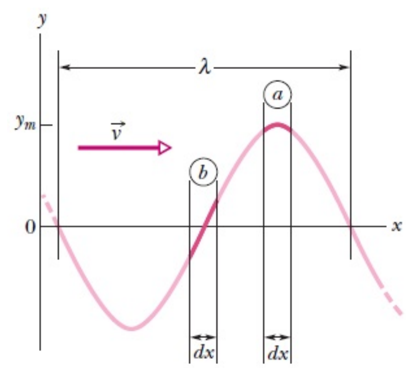
We thus have

$$dm\frac{\partial^2}{\partial t^2}y\left(x,t\right) \simeq Tdx\frac{\partial^2 y}{\partial x^2}\left(x,t\right)$$
$$\frac{1}{T}\frac{dm}{dx}\frac{\partial^2}{\partial t^2}y\left(x,t\right) - \frac{\partial^2 y}{\partial x^2}\left(x,t\right) = 0$$
$$\frac{\mu}{T}\frac{\partial^2}{\partial t^2}y\left(x,t\right) - \frac{\partial^2 y}{\partial x^2}\left(x,t\right) = 0$$

From the wave equation (2), the wave speed is found to be

$$v = \sqrt{\frac{T}{\mu}}$$

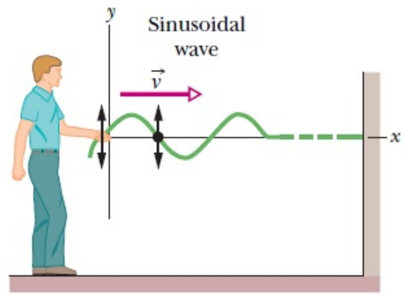
16.5 Energy and Power of a Traveling Wave Along a String



Kinetic Energy (K) Transverse motion of string: Maximum at (b) in the above figure. Minimum at (a)

Elastic Potential Energy (U) Energy in stretching string from equilibrium: Maximum at (a). Minimum at (b).

Energy Transport K and U are transported along the string. For example, the figure below shows the wave bringing energy to regions where $K = U = 0$ initially.



16.6 The Rate of Energy Transmission

The kinetic energy dK associated with string element of mass dm

$$dK = \frac{1}{2}dmv^2 = \frac{1}{2}dm\left(\frac{\partial y}{\partial t}\right)^2$$

For the sinusoidal wave (3),

$$\begin{aligned} dK &= \frac{1}{2}dm\left(y_m\frac{\partial\cos\left(\omega t - kx + \phi\right)}{\partial t}\right)^2 \\ &= \frac{1}{2}\mu y_m^2\omega^2\sin^2\left(\omega t - kx + \phi\right)dx \end{aligned}$$

where we have used $dm = \mu dx$. In time dt, the wave moves a distance $dx = vdt$ and the amount of dK energy carried by the wave passes through any particular spatial point is equal to

$$dK = \frac{1}{2}\mu v y_m^2\omega^2\sin^2\left(\omega t - kx + \phi\right)dt$$

Averaging dK with respect to time, we get

$$\left\langle\frac{dK}{dt}\right\rangle = \frac{1}{2}\mu v y_m^2\omega^2\left\langle\sin^2\left(\omega t - kx + \phi\right)\right\rangle = \frac{1}{4}\mu v y_m^2\omega^2$$

where we have utilized

$$\begin{aligned} &\left\langle\sin^2\left(\omega t - kx + \phi\right)\right\rangle \\ &= \left\langle\cos^2\left(\omega t - kx + \phi\right)\right\rangle \\ &= \frac{1}{2}\left\langle\sin^2\left(\omega t - kx + \phi\right) + \cos^2\left(\omega t - kx + \phi\right)\right\rangle = \frac{1}{2} \end{aligned}$$

For SHM, the average of $\langle K \rangle$ is equal to the average of $\langle U \rangle$. We thus get

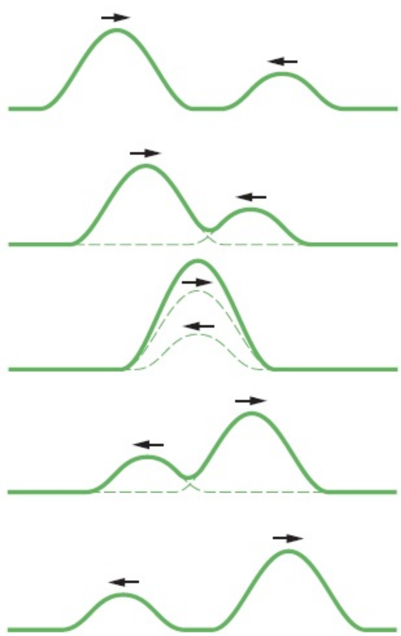
$$P = \left\langle \frac{dK}{dt} + \frac{dU}{dt} \right\rangle = 2 \left\langle \frac{dK}{dt} \right\rangle = \frac{1}{2} \mu v y_m^2 \omega^2$$

16.7 The Principle of Superposition of Waves

Overlapping waves algebraically add to produce a resultant wave (or net wave)

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

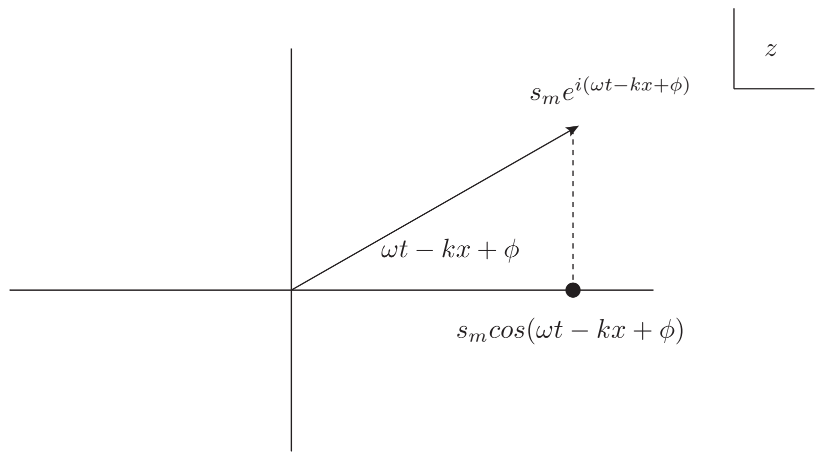
Principle of superposition: when several effects occur simultaneously, their net effect is the sum of the individual effects. Overlapping waves do not in any way alter the travel of each other



16.8 Interference of Waves

Observe that

$$s(x,t) = s_m \cos(\omega t - kx + \phi) = \mathrm{Re}\left(s_m e^{i(\omega t - kx + \phi)}\right)$$



The wave $s_m \cos(\omega t - kx + \phi)$ can be represented in the complex plane by the complex number $s_m e^{i(\omega t - kx + \phi)}$ whose real part is the physical wave displacement. Let

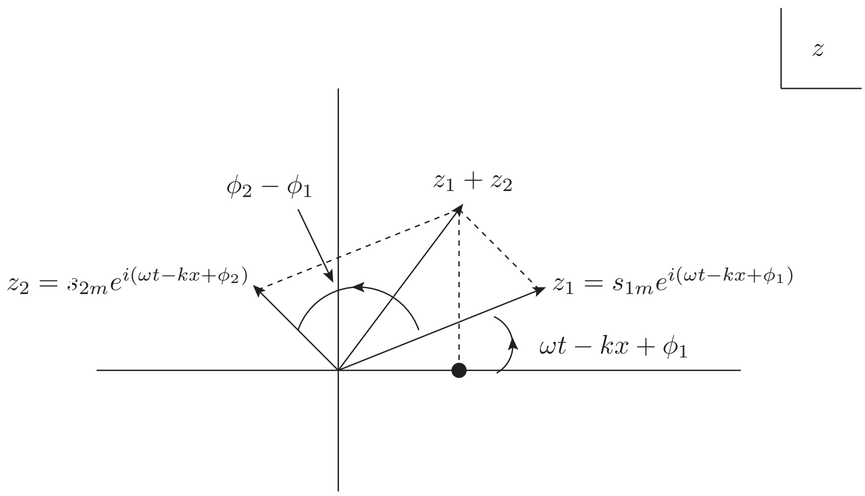
$$z_1 = s_{1m} e^{i(\omega t - kx + \phi_1)}$$

$$z_2 = s_{2m} e^{i(\omega t - kx + \phi_2)}$$

The sum of two waves:

$$\begin{aligned} & s_{1m} \cos(\omega t - kx + \phi_1) + s_{2m} \cos(\omega t - kx + \phi_2) \\ & = \mathrm{Re}\left(s_{1m} e^{i(\omega t - kx + \phi_1)} + s_{2m} e^{i(\omega t - kx + \phi_2)}\right) = \mathrm{Re}\,(z_1 + z_2) \end{aligned}$$

The corresponding complex number representing the resulting wave is equal to the sum of the two complex numbers representing the two waves as shown below:



In fact,

$$\begin{aligned} z_1 + z_2 &= s_{1m}e^{i(\omega t - kx + \phi_1)} + s_{2m}e^{i(\omega t - kx + \phi_2)} \\ &= \left(s_{1m}e^{i\phi_1} + s_{2m}e^{i\phi_2}\right) e^{i(\omega t - kx)} \end{aligned}$$

Express $s_{1m}e^{i\phi_1} + s_{2m}e^{i\phi_2}$ in polar form

$$s_{1m}e^{i\phi_1} + s_{2m}e^{i\phi_2} = s_m e^{i\phi}$$

We then have

$$\mathrm{Re}\,(z_1 + z_2) = \mathrm{Re}\left(s_m e^{i(\omega t - kx + \phi)}\right) = s_m \cos(\omega t - kx + \phi)$$

The amplitude s_m of the resulting wave is equal to

$$s_m = \left|s_m e^{i\phi}\right| = \left|s_{1m}e^{i\phi_1} + s_{2m}e^{i\phi_2}\right| = |z_1 + z_2|$$

which is the distance to the origin for the two complex number $z_1 + z_2$. Note the resulting wave has the same wave frequency ω and the wave number k as those for the original component waves.

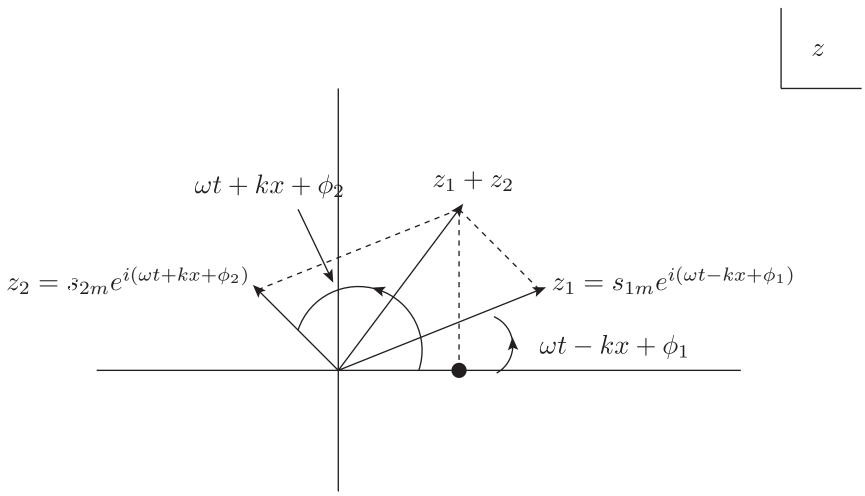
16.9 Standing Waves

Consider the linear superposition of two waves moving in opposite direction:

$$z_1 = s_{1m}e^{i(\omega t - kx + \phi_1)}$$

$$z_2 = s_{2m}e^{i(\omega t + kx + \phi_2)}$$

$$s_{1m}\cos(\omega t - kx + \phi_1) + s_{2m}\cos(\omega t + kx + \phi_2) = \mathrm{Re}\,(z_1 + z_2) \tag{4}$$



For a fixed position x, $z_1 + z_2$ still rotates with the angular frequency ω and therefore its projection $\mathrm{Re}\,(z_1 + z_2)$ on the real axis also oscillates with the very angular frequency ω and amplitude $|z_1 + z_2|$. But as we move to another spatial point, the angle between z_1 and z_2 varies as well and so does the amplitude

$$|z_1 + z_2| = \left| s_{1m} e^{i(\omega t - kx + \phi_1)} + s_{2m} e^{i(\omega t + kx + \phi_2)} \right| = \left| s_{1m} e^{i\phi_1} e^{-ikx} + s_{2m} e^{i\phi_2} e^{ikx} \right|$$

Depending on the position x, $|z_1 + z_2|$ will assume the maximum $s_{1m} + s_{2m}$ when z_1 and z_2 are parallel and the minimum $|s_{1m} - s_{2m}|$. Furthermore, if we move $x \to x + \frac{\lambda}{2}$ $\left(\lambda = \frac{2\pi}{k}\right)$ by half a wave length,

$$s_{1m} e^{i\phi_1} e^{-ikx} \to s_{1m} e^{i\phi_1} e^{-ik\left(x + \frac{\pi}{k}\right)} = -s_{1m} e^{i\phi_1} e^{-ikx}$$
$$s_{2m} e^{i\phi_2} e^{-ikx} \to s_{2m} e^{i\phi_2} e^{ik\left(x + \frac{\pi}{k}\right)} = -s_{2m} e^{i\phi_2} e^{ikx}$$

$$|z_1 + z_2| \to |-z_1 - z_2| = |z_1 + z_2|$$

The oscillation amplitudes at $x + \frac{n}{2}\lambda$, $n = 0, 1, 2, ...$ are all equal. In other words, the locations for maximum oscillation are fixed in space and spaced evenly by the distance $\frac{\lambda}{2}$. So does the locations for minimum oscillation. Such wave motion (4) is called a standing wave. In the particular case of $s_{1m} = s_{2m}$, the two waves constituting the standing wave are of equal amplitude,

$$|z_1 + z_2| = s_{1m} \left| e^{i\phi_1} e^{-ikx} + e^{i\phi_2} e^{ikx} \right| = s_{1m} \left| 1 + e^{i(\phi_2 - \phi_1 + 2kx)} \right|$$

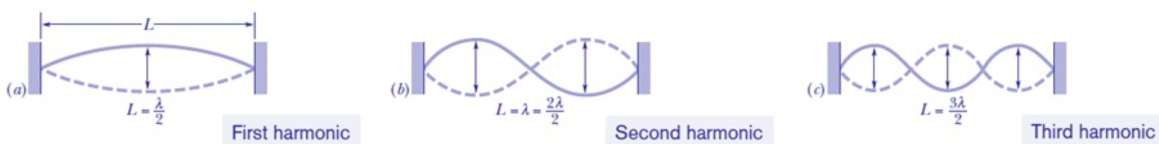
A minimum of $|z_1 + z_2|$ occurs at point x_n with $\phi_2 - \phi_1 + 2kx_n = (2n+1)\,\pi$ where n is an integer. At

$$\begin{aligned} x_n &= \frac{(2n+1)\,\pi - \phi_2 + \phi_1}{2k} = \frac{(2n+1)\,\pi - \phi_2 + \phi_1}{4\pi}\lambda \\ &= \frac{n}{2}\lambda + \frac{\pi - \phi_2 + \phi_1}{4\pi}\lambda, \end{aligned}$$

$|z_1 + z_2| = 0$. The wave at x_n always remains stationary and is called a node. As before, the distance between two nearby nodes is $\frac{\lambda}{2}$.

16.10 Standing Waves and Resonances

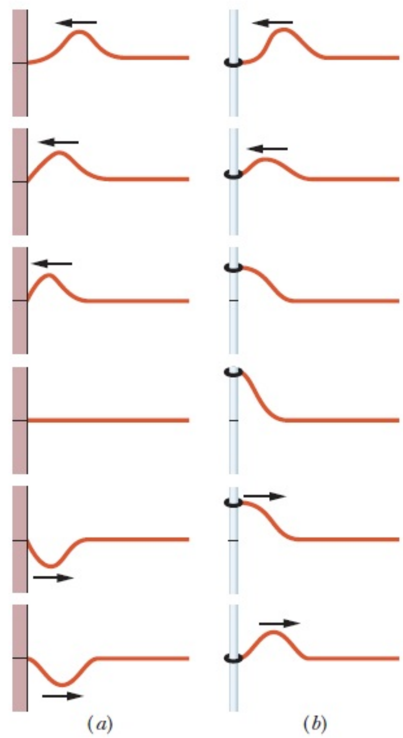
For a string clamped at both ends, standing wave is at resonance when nodes occur at both ends. Other oscillation modes that are not at the resonant frequency do not produce strong standing waves.



Standing waves are set up if

$$L = n\frac{\lambda}{2}, n = 1, 2, 3...$$

16.11 Reflections at a Boundary



(a) String tied at end: “hard” reflection, node at end.
Incident and reflected pulses must have opposite signs to cancel at node.
(b) String free at end: “soft” reflection, antinode at end.
Incident and reflected pulses must have same sign to add at antinode.