



CHAPTER 12

Partial Differential Equations (PDEs)

A PDE is an equation that contains one or more partial derivatives of an unknown function that depends on at least two variables. Usually one of these deals with time t and the remaining with space (spatial variable(s)). The most important PDEs are the wave equations that can model the vibrating string (Secs. 12.2, 12.3, 12.4, 12.12) and the vibrating membrane (Secs. 12.8, 12.9, 12.10), the heat equation for temperature in a bar or wire (Secs. 12.5, 12.6), and the Laplace equation for electrostatic potentials (Secs. 12.6, 12.10, 12.11). PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics. They have a much wider range of applications than ODEs, which can model only the simplest physical systems. Thus PDEs are subjects of many ongoing research and development projects.

Realizing that modeling with PDEs is more involved than modeling with ODEs, we take a gradual, well-planned approach to modeling with PDEs. To do this we carefully derive the PDE that models the phenomena, such as the one-dimensional wave equation for a vibrating elastic string (say a violin string) in Sec. 12.2, and then solve the PDE in a separate section, that is, Sec. 12.3. In a similar vein, we derive the heat equation in Sec. 12.5 and then solve and generalize it in Sec. 12.6.

We derive these PDEs from physics and consider methods for solving initial and boundary value problems, that is, methods of obtaining solutions which satisfy the conditions required by the physical situations. In Secs. 12.7 and 12.12 we show how PDEs can also be solved by Fourier and Laplace transform methods.

COMMENT. *Numerics for PDEs* is explained in Secs. 21.4–21.7, which, for greater teaching flexibility, is designed to be independent of the other sections on numerics in Part E.

Prerequisites: Linear ODEs (Chap. 2), Fourier series (Chap. 11).

Sections that may be omitted in a shorter course: 12.7, 12.10–12.12.

References and Answers to Problems: App. 1 Part C, App. 2.

12.1 Basic Concepts of PDEs

A **partial differential equation (PDE)** is an equation involving one or more partial derivatives of an (unknown) function, call it u , that depends on two or more variables, often time t and one or several variables in space. The order of the highest derivative is called the **order** of the PDE. Just as was the case for ODEs, second-order PDEs will be the most important ones in applications.

Just as for ordinary differential equations (ODEs) we say that a PDE is **linear** if it is of the first degree in the unknown function u and its partial derivatives. Otherwise we call it **nonlinear**. Thus, all the equations in Example 1 are linear. We call a *linear* PDE **homogeneous** if each of its terms contains either u or one of its partial derivatives. Otherwise we call the equation **nonhomogeneous**. Thus, (4) in Example 1 (with f not identically zero) is nonhomogeneous, whereas the other equations are homogeneous.

EXAMPLE 1 Important Second-Order PDEs

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|-----|--|---|
| (1) | $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ | <i>One-dimensional wave equation</i> |
| (2) | $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ | <i>One-dimensional heat equation</i> |
| (3) | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ | <i>Two-dimensional Laplace equation</i> |
| (4) | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ | <i>Two-dimensional Poisson equation</i> |
| (5) | $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$ | <i>Two-dimensional wave equation</i> |
| (6) | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ | <i>Three-dimensional Laplace equation</i> |

Here c is a positive constant, t is time, x, y, z are Cartesian coordinates, and *dimension* is the number of these coordinates in the equation. ■

A **solution** of a PDE in some region R of the space of the independent variables is a function that has all the partial derivatives appearing in the PDE in some domain D (definition in Sec. 9.6) containing R , and satisfies the PDE everywhere in R .

Often one merely requires that the function is continuous on the boundary of R , has those derivatives in the interior of R , and satisfies the PDE in the interior of R . Letting R lie in D simplifies the situation regarding derivatives on the boundary of R , which is then the same on the boundary as it is in the interior of R .

In general, the totality of solutions of a PDE is very large. For example, the functions

$$(7) \quad u = x^2 - y^2, \quad u = e^x \cos y, \quad u = \sin x \cosh y, \quad u = \ln(x^2 + y^2)$$

which are entirely different from each other, are solutions of (3), as you may verify. We shall see later that the unique solution of a PDE corresponding to a given physical problem will be obtained by the use of **additional conditions** arising from the problem. For instance, this may be the condition that the solution u assume given values on the boundary of the region R (“**boundary conditions**”). Or, when time t is one of the variables, u (or $u_t = \partial u / \partial t$ or both) may be prescribed at $t = 0$ (“**initial conditions**”).

We know that if an ODE is linear and homogeneous, then from known solutions we can obtain further solutions by superposition. For PDEs the situation is quite similar:

THEOREM 1

Fundamental Theorem on Superposition

If u_1 and u_2 are solutions of a **homogeneous linear** PDE in some region R , then

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 and c_2 is also a solution of that PDE in the region R .

The simple proof of this important theorem is quite similar to that of Theorem 1 in Sec. 2.1 and is left to the student.

Verification of solutions in Probs. 2–13 proceeds as for ODEs. Problems 16–23 concern PDEs solvable like ODEs. To help the student with them, we consider two typical examples.

EXAMPLE 2 Solving $u_{xx} - u = 0$ Like an ODE

Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y .

Solution. Since no y -derivatives occur, we can solve this PDE like $u'' - u = 0$. In Sec. 2.2 we would have obtained $u = Ae^x + Be^{-x}$ with constant A and B . Here A and B may be functions of y , so that the answer is

$$u(x, y) = A(y)e^x + B(y)e^{-x}$$

with arbitrary functions A and B . We thus have a great variety of solutions. Check the result by differentiation. ■

EXAMPLE 3 Solving $u_{xy} = -u_x$ Like an ODE

Find solutions $u = u(x, y)$ of this PDE.

Solution. Setting $u_x = p$, we have $p_y = -p$, $p_y/p = -1$, $\ln |p| = -y + \tilde{c}(x)$, $p = c(x)e^{-y}$ and by integration with respect to x ,

$$u(x, y) = f(x)e^{-y} + g(y) \quad \text{where} \quad f(x) = \int c(x) dx,$$

here, $f(x)$ and $g(y)$ are arbitrary. ■

PROBLEM SET 12.1

- Fundamental theorem.** Prove it for second-order PDEs in two and three independent variables. *Hint.* Prove it by substitution.

2–13 VERIFICATION OF SOLUTIONS

Verify (by substitution) that the given function is a solution of the PDE. Sketch or graph the solution as a surface in space.

2–5 Wave Equation (1) with suitable c

- $u = x^2 + t^2$
- $u = \cos 4t \sin 2x$
- $u = \sin kct \cos kx$
- $u = \sin at \sin bx$

6–9 Heat Equation (2) with suitable c

- $u = e^{-t} \sin x$
- $u = e^{-\omega^2 c^2 t} \cos \omega x$
- $u = e^{-9t} \sin \omega x$
- $u = e^{-\pi^2 t} \cos 25x$

10–13 Laplace Equation (3)

- $u = e^x \cos y, e^x \sin y$
- $u = \arctan(y/x)$
- $u = \cos y \sinh x, \sin y \cosh x$

- $u = x/(x^2 + y^2), y/(x^2 + y^2)$

14. TEAM PROJECT. Verification of Solutions

(a) **Wave equation.** Verify that $u(x, t) = v(x + ct) + w(x - ct)$ with any twice differentiable functions v and w satisfies (1).

(b) **Poisson equation.** Verify that each u satisfies (4) with $f(x, y)$ as indicated.

$$\begin{array}{ll} u = y/x & f = 2y/x^3 \\ u = \sin xy & f = (x^2 + y^2) \sin xy \\ u = e^{x^2 - y^2} & f = 4(x^2 + y^2)e^{x^2 - y^2} \\ u = 1/\sqrt{x^2 + y^2} & f = (x^2 + y^2)^{-3/2} \end{array}$$

(c) **Laplace equation.** Verify that

$u = 1/\sqrt{x^2 + y^2 + z^2}$ satisfies (6) and $u = \ln(x^2 + y^2)$ satisfies (3). Is $u = 1/\sqrt{x^2 + y^2}$ a solution of (3)? Of what Poisson equation?

(d) Verify that u with any (sufficiently often differentiable) v and w satisfies the given PDE.

$$\begin{array}{ll} u = v(x) + w(y) & u_{xy} = 0 \\ u = v(x)w(y) & uu_{xy} = u_x u_y \\ u = v(x + 2t) + w(x - 2t) & u_{tt} = 4u_{xx} \end{array}$$

- Boundary value problem.** Verify that the function $u(x, y) = a \ln(x^2 + y^2) + b$ satisfies Laplace's equation

(3) and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

16–23 PDEs SOLVABLE AS ODEs

This happens if a PDE involves derivatives with respect to one variable only (or can be transformed to such a form), so that the other variable(s) can be treated as parameter(s). Solve for $u = u(x, y)$:

16. $u_{yy} = 0$

17. $u_{xx} + 16\pi^2 u = 0$

18. $25u_{yy} - 4u = 0$ 19. $u_y + y^2 u = 0$

20. $2u_{xx} + 9u_x + 4u = -3 \cos x - 29 \sin x$

21. $u_{yy} + 6u_y + 13u = 4e^{3y}$

22. $u_{xy} = u_x$ 23. $x^2 u_{xx} + 2xu_x - 2u = 0$

24. **Surface of revolution.** Show that the solutions $z = z(x, y)$ of $yz_x = xz_y$ represent surfaces of revolution. Give examples. *Hint.* Use polar coordinates r, θ and show that the equation becomes $z_\theta = 0$.

25. **System of PDEs.** Solve $u_{xx} = 0, u_{yy} = 0$

12.2 Modeling: Vibrating String, Wave Equation

In this section we model a vibrating string, which will lead to our first important PDE, that is, equation (3) which will then be solved in Sec. 12.3. *The student should pay very close attention to this delicate modeling process and detailed derivation starting from scratch*, as the skills learned can be applied to modeling other phenomena in general and in particular to modeling a vibrating membrane (Sec. 12.7).

We want to derive the PDE modeling small transverse vibrations of an elastic string, such as a violin string. We place the string along the x -axis, stretch it to length L , and fasten it at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, call it $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$; see Fig. 286.

$u(x, t)$ will be the solution of a PDE that is the model of our physical system to be derived. This PDE should not be too complicated, so that we can solve it. Reasonable simplifying assumptions (just as for ODEs modeling vibrations in Chap. 2) are as follows.

Physical Assumptions

1. The mass of the string per unit length is constant (“homogeneous string”). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Under these assumptions we may expect solutions $u(x, t)$ that describe the physical reality sufficiently well.

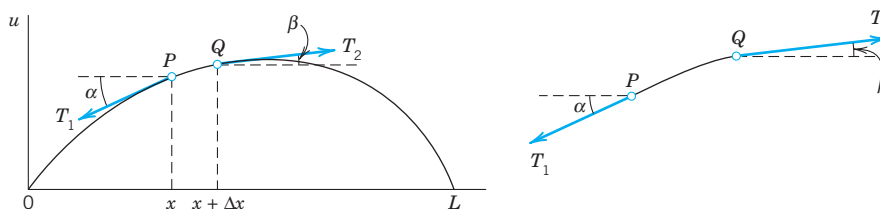


Fig. 286. Deflected string at fixed time t . Explanation on p. 544