Contents

10 Rotational Motion–I

We now turn to the motion of rotation, in which an object turns about an axis.

10.1 The Rotational Variables

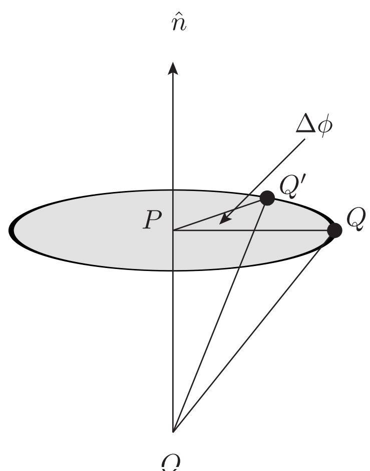
A rigid body is a body that can rotate with all its parts locked together and without any change of its shape. For a rigid body in pure rotation, every point of the body moves in a circle whose center lies on the **axis of rotation** (or **rotation axis**), and every point moves through the same angle during a particular time interval.

10.1.1 Angular Displacement Vector

We define the angular displacement vector $\Delta\vec{\phi}$ to be

$$\Delta\vec{\phi} = \Delta\phi\hat{n}$$

where $\hat{n}$ is unit vector along the the axis of rotation and $\Delta\phi$ is the angle that every point of the rigid body moves around $\hat{n}$ according to the **right-hand rule.** (Curl your right hand with the thumb pointing along the axis of rotation axis, your fingers then point to the positive direction of rotation angle.) In the following figure, the point Q rotates into $Q\prime$ through the angle $\Delta\phi$ and $\hat{n} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|}$ is the unit vector in the direction of the rotation axis.



For a small angle of rotation with $|\Delta\phi| \ll 1$, the displacement vector $\overrightarrow{QQ\prime}$ from Q to $Q\prime$ is almost perpendicular to $\overrightarrow{PQ}$ and the length of the displacement

$$\left|\overrightarrow{QQ\prime}\right| \simeq \left|\overrightarrow{PQ}\right| \Delta\phi = \left|\hat{n} \times \overrightarrow{OQ}\right| \Delta\phi = \left|\Delta\vec{\phi} \times \overrightarrow{OQ}\right|$$

It can be seen from the above figure that, when $|\Delta\phi| \ll 1$, $\overrightarrow{QQ\prime}$ is perpendicular to the plane containing $\hat{n}$ and $\overrightarrow{PQ}$ and points to the direction of $\Delta\vec{\phi} \times \overrightarrow{OQ}$. Therefore,

$$\overrightarrow{QQ\prime} \simeq \Delta\vec{\phi} \times \overrightarrow{OQ}$$

If we identify $\overrightarrow{OQ}$ as the position vector $\vec{x}$ (subject to the condition that the origin of the coordinate system O is on the rotation axis) and $\overrightarrow{OQ'}$ as $\vec{x}' = \vec{x} + \Delta\vec{x}$, then the linear displacement vector $\Delta\vec{x}$ is related to the angular displacement vector $\Delta\vec{\phi}$ by

$$\Delta\vec{x} \simeq \Delta\vec{\phi} \times \vec{x}$$

If Δt is the time interval that the rotation takes $\vec{x}$ to $\vec{x}'$, divide the above by Δt and we arrive at

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{x}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\phi}}{\Delta t} \times \vec{x} \tag{1}$$

10.1.2 Angular Velocity Vector

The angular velocity is defined as

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\phi}}{\Delta t} = \frac{d\vec{\phi}}{dt}$$

Then (1) gives us

$$\vec{v} = \vec{\omega} \times \vec{x} \tag{2}$$

which determines the velocity of a point $\vec{x}$ on the rigid boy that is rotating with angular velocity $\vec{\omega}$.

10.1.3 Angular Acceleration

As in the case of linear motion that defines linear acceleration $a = \frac{d\vec{v}}{dt}$, the angular acceleration is defined similarly,

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\phi}}{dt^2}$$

From (2), the linear acceleration $\vec{a}$ for a point $\vec{x}$ on the rigid body is

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{x}) = \frac{d\vec{\omega}}{dt} \times \vec{x} + \vec{\omega} \times \frac{d\vec{x}}{dt} \\ &= \vec{\alpha} \times \vec{x} + \vec{\omega} \times \vec{v}\end{aligned}$$

Let $\vec{\omega} = \omega\hat{n}$ with $\hat{n}$ being a unit vector. Decompose

$$\vec{x} = \vec{x}_{\parallel} + \vec{x}_{\perp} \tag{3}$$

where

$$\vec{x}_{\parallel} = (\hat{n} \cdot \vec{x})\,\hat{n}$$

is the projection of $\vec{x}$ along $\hat{n}$ and thus

$$\begin{aligned} &\hat{n} \times \vec{x}_{\parallel} = 0, \hat{n} \times \vec{x} = \hat{n} \times \left(\vec{x} - \vec{x}_{\parallel}\right) = \hat{n} \times \vec{x}_{\perp}, \\ &\hat{n} \cdot \vec{x}_{\perp} = \hat{n} \cdot (\vec{x} - (\hat{n} \cdot \vec{x})\,\hat{n}) = 0 \end{aligned}$$

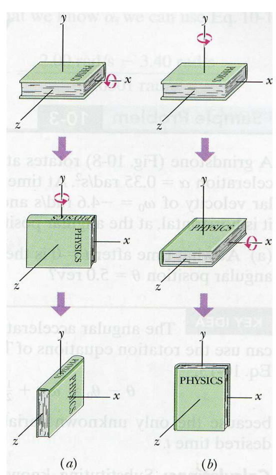
In particular, if the angular velocity $\vec{\omega}$ is constant with $\vec{\alpha} = 0$,

$$\begin{aligned} \vec{a} &= \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{x}) = \omega^2 \hat{n} \times (\hat{n} \times \vec{x}) \\ &= \omega^2 \hat{n} \times (\hat{n} \times \vec{x}_{\perp}) = -\omega^2 \vec{x}_{\perp} \end{aligned}$$

This is the well known centripetal acceleration for uniform circular motion.

10.1.4 Finite Rotations

The figure below gives an example that the sum of angular displacement vectors does not equal to the vector that represents the resulting angular displacement.



An initially horizontal book is given two 90° rotations, first in (a) in the order of rotation with the angular displacement vector $\frac{\pi}{2}\hat{\imath}$, then with the angular displacement $\frac{\pi}{2}\hat{\jmath}$; and then in (b) in the order of $\frac{\pi}{2}\hat{\jmath}$ and then $\frac{\pi}{2}\hat{\imath}$. The book end up with different orientations. Therefore unlike the linear displacement, the angular displacement vector for the resulting rotation can not be taken to be the sum of the two angular displacement vectors that describe the two consecutive rotations.

10.1.5 Infinitesimal Rotations

But for two infinitesimal rotations, it is possible to represent the resulting rotation, which is infinitesimal, by the sum of the two infinitesimal angular displacement vector. The is because under rotation 1 described by the angular displacement $\Delta\vec{\phi}_1$

$$\vec{x} \to \vec{x}' \simeq \vec{x} + \Delta\vec{\phi}_1 \times \vec{x}$$

and under rotation 2 with $\Delta\vec{\phi}_2$

$$\vec{x}' \to \vec{x}" \simeq \vec{x}' + \Delta\vec{\phi}_2 \times \vec{x}'$$

So the resulting rotation that rotates $\vec{x}$ to $\vec{x}"$ with

$$\begin{aligned} \vec{x}" &\simeq \vec{x}' + \Delta\vec{\phi}_2 \times \vec{x}' \simeq \vec{x} + \Delta\vec{\phi}_1 \times \vec{x} + \Delta\vec{\phi}_2 \times \left(\vec{x} + \Delta\vec{\phi}_1 \times \vec{x}\right) \\ &\simeq \vec{x} + \left(\Delta\vec{\phi}_1 + \Delta\vec{\phi}_2\right) \times \vec{x} \end{aligned}$$

can be described by the angular displacement $\Delta\vec{\phi}_1 + \Delta\vec{\phi}_2$.

10.2 Kinetic Energy of Rotation

Let $\vec{\omega} = \omega\hat{n}$. For a point $\vec{x}_i$, its linear velocity is

$$\vec{v}_i = \vec{\omega} \times \vec{x}_i = \omega\hat{n} \times \vec{x}_{\perp i}$$

where $\vec{x}_{\perp i} = \vec{x}_i - (\vec{x}_i \cdot \hat{n})\,\hat{n}$. Thus

$$|\vec{v}_i|^2 = \omega^2 \left|\hat{n} \times \vec{x}_{\perp i}\right|^2 = \omega^2 \left|\vec{x}_{\perp i}\right|^2 = \omega^2 r_{\perp i}^2$$

where $r_{\perp i} = |\vec{x}_{\perp i}|$ is the length of $\vec{x}_{\perp i}$, the distance from the point $\vec{x}_{\perp i}$ to the rotation axis. The total kinetic energy of rotation is

$$K = \sum_i \frac{1}{2} m_i \left|\vec{v}_i\right|^2 = \sum_i \frac{1}{2} m_i r_{\perp i}^2 \omega^2 = \frac{1}{2} I_{\hat{n}} \omega^2$$

where

$$I_{\hat{n}} = \sum_i m_i r_{\perp i}^2$$

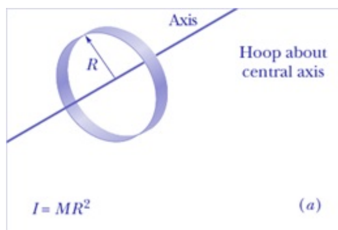
is the **rotational inertia** (or **moment of inertia**) of the body with respect to the axis of rotation $\hat{n}$. We note that $I_{\hat{n}}$ of a rotation body involves not only its mass but also how that mass is distributed.

10.3 Calculating the Rotational Inertia

In the continuum limit

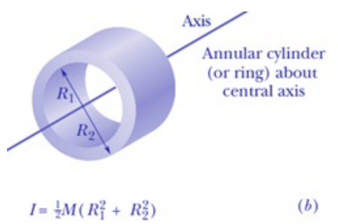
$$I_{\hat{n}} = \int r_{\perp}^2 dm$$

10.3.1 (a) Hoop about central axis



Every point on the body has the same distance R to the rotation axis, $I = \sum m_i R^2 = MR^2$

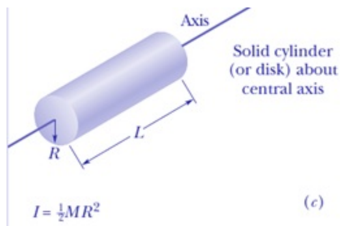
10.3.2 (b) Annular cylinder about central axis



$$dm = M \frac{2\pi r dr}{\pi \left(R_2^2 - R_1^2\right)}$$

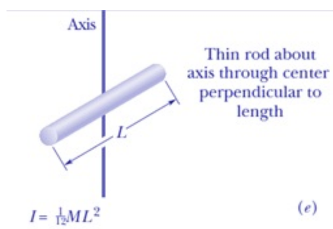
$$\begin{aligned} I &= \int dI = \int r^2 dm = M \int_{R_1}^{R_2} r^2 \frac{2\pi r dr}{\pi \left(R_2^2 - R_1^2\right)} \\ &= \frac{1}{2} \frac{M r^4 |_{R_1}^{R_2}}{\left(R_2^2 - R_1^2\right)} = \frac{1}{2} \frac{M \left(R_2^4 - R_1^4\right)}{\left(R_2^2 - R_1^2\right)} = \frac{1}{2} M \left(R_2^2 + R_1^2\right) \end{aligned}$$

10.3.3 (c) Solid cylinder about central axis



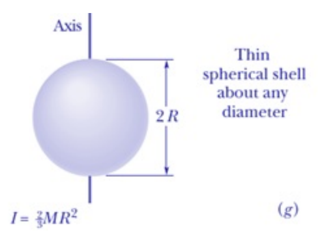
This corresponds to $R_2 = R$ and $R_1 = 0$ in (b). Thus $I = \frac{1}{2}MR^2$

10.3.4 (e) Thin rod about axis through center perpendicular to length



$$I = \int dI = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M}{L} x^2 dx = \frac{2}{3}\frac{M}{L}\left(\frac{L}{2}\right)^3 = \frac{1}{12}ML^2$$

10.3.5 (g) The spherical shell about any diameter



Without losing generality, assume the rotation axis is the z axis, then

$$I = \int \left(x^2 + y^2\right) dm$$

Due to complete spherical symmetry, the rotational inertia is the same the one respect the the x axis or the y axis and we have

$$I = \int \left(x^2 + y^2\right) dm = \int \left(y^2 + z^2\right) dm = \int \left(x^2 + z^2\right) dm$$

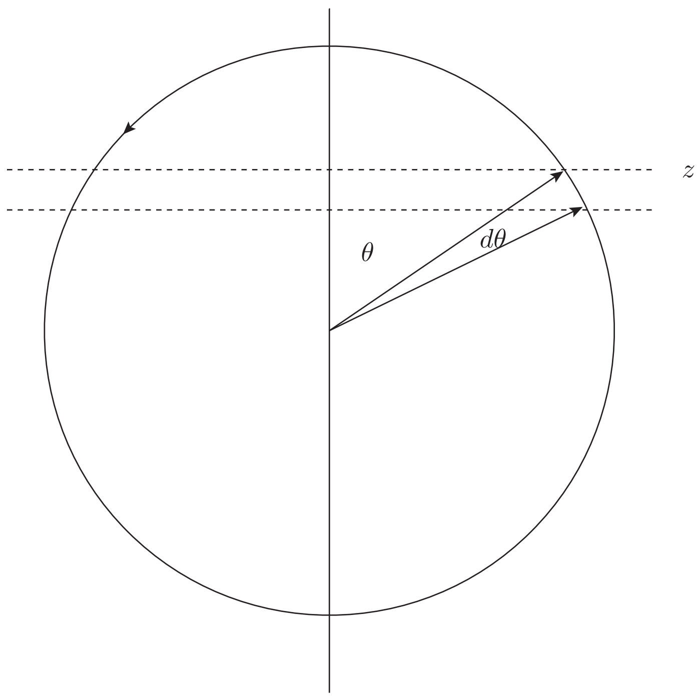
The average of the above yields

$$
\begin{aligned}
I &= \frac{1}{3}\int \left[\left(x^2+y^2\right)+\left(y^2+z^2\right)+\left(x^2+z^2\right)\right] dm \\
&= \frac{2}{3}\int \left(x^2+y^2+z^2\right) dm = \frac{2}{3}R^2\int dm = \frac{2}{3}MR^2
\end{aligned}
$$

Note, for this spherical shell, any point (x, y, z) is at a distance R from the center of the sphere and

$$x^2+y^2+z^2 = R^2$$

Alternatively, with z-axis as the rotation axis, in spherical coordinates,



θ is the angle from the z-axis and we have $z = R\cos\theta$. the portion of the spherical surface with height between z and $z + dz$ with area

$$dA = 2\pi R \sin\theta \, (R d\theta)$$

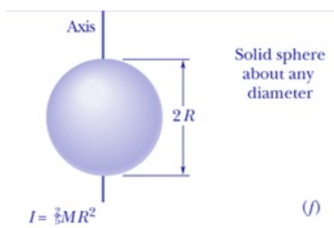
gives us

$$dI = \left(R^2 \sin^2\theta\right) dm = \left(R^2 \sin^2\theta\right) \frac{M}{4\pi R^2} dA$$
$$= \left(R^2 \sin^2\theta\right) \frac{M}{4\pi R^2} 2\pi R \sin\theta \left(R d\theta\right) = \frac{MR^2}{2} \sin^3\theta d\theta$$

Thus

$$I = \int dI = \frac{MR^2}{2} \int_{\theta=0}^{\theta=\pi} \sin^3\theta d\theta = -\frac{MR^2}{2} \int_{\cos\theta=1}^{\cos\theta=-1} \sin^2\theta d\cos\theta$$
$$= \frac{MR^2}{2} \int_{\cos\theta=-1}^{\cos\theta=1} \left(1-\cos^2\theta\right) d\cos\theta = \frac{MR^2}{2} \int_{-1}^{1} \left(1-t^2\right) dt$$
$$= \frac{MR^2}{2} \left(t - \frac{t^3}{3}\right) |_{-1}^{1} = \frac{2}{3} MR^2$$

10.3.6 (f) Solid sphere about any diameter



Treat the solid sphere as the union of thin spherical shells of radius r with thickness dr with mass

$$dm = \frac{M}{\frac{4}{3}\pi R^3} dV = \frac{M}{\frac{4}{3}\pi R^3} 4\pi r^2 dr = \frac{3M}{R^3} r^2 dr$$

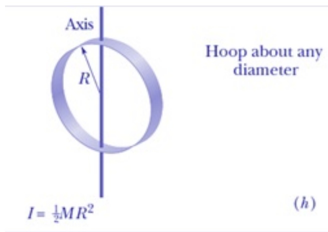
According to (g)

$$dI = \frac{2}{3} r^2 dm = \frac{2}{3} r^2 \frac{3M}{R^3} r^2 dr = \frac{2M}{R^3} r^4 dr$$

Thus

$$I = \int_0^R \frac{2M}{R^3} r^4 dr = \frac{2}{5} MR^2$$

10.3.7 (h) Hoop about any diameter



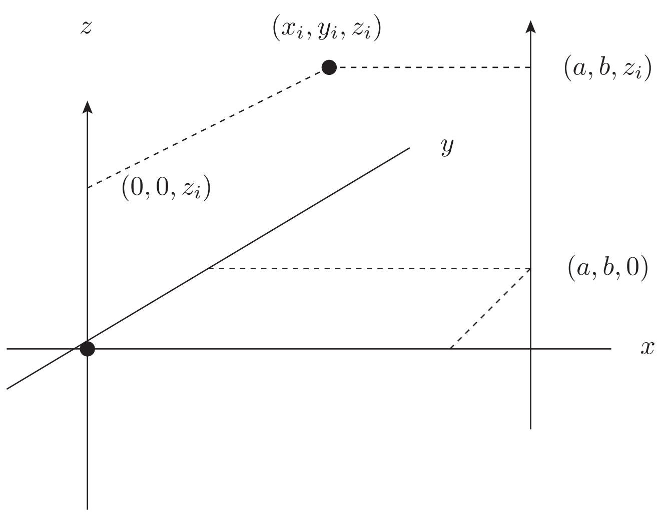
Parametrize the hoop by angular variable θ in the range of $(0, 2\pi)$. Then

$$I = \int (R\cos\theta)^2 \, dm = \frac{M}{2\pi}\int_0^{2\pi} R^2\cos^2\theta d\theta = \frac{M}{2}R^2$$

Note that we have utilized that

$$\int_0^{2\pi} \cos^2\theta d\theta = \int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \frac{\cos^2\theta + \sin^2\theta}{2} d\theta = \pi$$

10.3.8 Parallel-Axis Theorem



Without the loss of generality, we may assume the first rotation axis is the z axis passing through the center of mass $(x_{cm}, y_{cm}, z_{cm}) = (0, 0, 0)$ which is chosen to be origin and the second rotation axis that is parallel to the first one is the axis passing through the point $(a, b, 0)$ on the x-y plane. The distance between these two parallel axes is $h = \sqrt{a^2+b^2}$. The rotational inertia about the first axis passing through the center of mass is

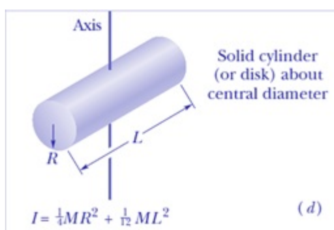
$$I_{cm} = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

while that about the second axis is

$$\begin{aligned} I &= \sum_i m_i \left((x_i - a)^2 + (y_i - b)^2\right) \\ &= \sum_i m_i \left(x_i^2 + y_i^2\right) - 2\sum_i \left(am_i x_i + bm_i y_i\right) + \sum_i m_i \left(a^2 + b^2\right) \\ &= I_{cm} - 2M\left(ax_{cm} + by_{cm}\right) + M\left(a^2+b^2\right) \\ &= I_{cm} + Mh^2 \end{aligned}$$

This is called the parallel-axis theorem for rotational inertia.

10.3.9 (d) Solid cylinder about central diameter



For a uniform solid disk with mass M and radius R with rotation axis in the plane of disk and through the center, using (h), we have

$$I_{cm} = \int dI = \int \frac{1}{2} r^2 dm = \int_0^R \frac{1}{2} r^2 M \frac{2\pi r dr}{\pi R^2} = \frac{1}{4} MR^2$$

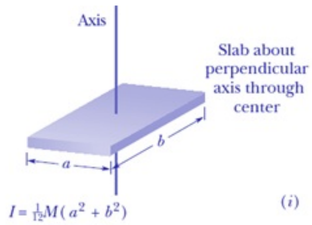
Let the central diameter point in the x direction. For the slice of disk between x and $x + dx$, we have, from parallel-axis theorem and (c)

$$dI = dI_{cm} + x^2 dm = \left(\frac{1}{4}R^2 + x^2\right) dm = \left(\frac{1}{4}R^2 + x^2\right) \frac{M}{L} dx$$

Thus

$$\begin{aligned} I &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{1}{4}R^2 + x^2\right) \frac{M}{L} dx = 2\int_0^{\frac{L}{2}} \left(\frac{1}{4}R^2 + x^2\right)\frac{M}{L} dx = 2\left(R^2 \frac{L}{8} + \frac{1}{3}\frac{L^3}{8}\right)\frac{M}{L} \\ &= \frac{1}{4}MR^2 + \frac{1}{12}ML^2 \end{aligned}$$

10.3.10 (i) Slab about perpendicular axis through center



For the thin rod parallel to y axis and lying between $(x, x + dx)$, we have

$$dI = dI_{cm} + x^2 dm = \left(\frac{1}{12}b^2 + x^2\right) dm = \left(\frac{1}{12}b^2 + x^2\right) \frac{M}{a} dx$$

$$I = \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\frac{1}{12}b^2 + x^2\right) \frac{M}{a} dx = \frac{1}{12} M \left(a^2 + b^2\right)$$

10.4 Torque

We define the torque

$$\vec{\tau}_i = \vec{x}_i \times \vec{F}_i$$

for a force $\vec{F}_i$ at the position $\vec{x}_i$. The total torque is the sum

$$\vec{\tau}_{total} = \sum_i \vec{\tau}_i$$

The internal forces do not contribute to the total torque if the strong version of the Newton's 3rd law is satisfied. Specifically, let $\vec{F}_{ij}$ be the internal force acting on particle i at $\vec{x}_i$ from particle j and

$$\vec{F}_{ij} + \vec{F}_{ji} = 0$$

The sum of the torques due to $\vec{F}_{ij}$ and $\vec{F}_{ji}$ is

$$\vec{x}_i \times \vec{F}_{ij} + \vec{x}_j \times \vec{F}_{ji} = (\vec{x}_i - \vec{x}_j) \times \vec{F}_{ij} = 0$$

which vanishes because $(\vec{x}_i - \vec{x}_j)$ is parallel or anti-parallel to $\vec{F}_{ji}$. Thus only external forces may contribute to the total torque for system.

$$\vec{\tau}_{total} = \sum_i \vec{x}_i \times \vec{F}_i^{(external)}$$

10.5 Newton's Second Law for Rotation

$$\begin{aligned} dW &= \sum_i \vec{F}_i \cdot d\vec{x}_i = \sum_i \vec{F}_i \cdot \left(d\vec{\phi} \times \vec{x}_i\right) \\ &= \sum_i \left(\vec{x}_i \times \vec{F}_i\right) \cdot d\vec{\phi} = \vec{\tau}_{total} \cdot d\vec{\phi} \end{aligned} \tag{4}$$

By work-kinetic energy theorem,

$$d\left(\frac{1}{2}I\omega^2\right) = dK = dW = \vec{\tau}_{total} \cdot d\vec{\phi}$$

or

$$\frac{d\left(\frac{1}{2}I\omega^2\right)}{dt} = \vec{\tau}_{total} \cdot \frac{d\vec{\phi}}{dt} = \vec{\tau}_{total} \cdot \vec{\omega} = \omega\vec{\tau}_{total} \cdot \hat{n} \tag{5}$$

where

$$\vec{\omega} = \omega\hat{n}$$

Let us assume the rotating system is a rigid body and the rotation axis $\hat{n}$ is fixed. Then

$$\frac{dI}{dt} = 0, \frac{d\hat{n}}{dt} = 0$$

and (5) becomes

$$I\omega\frac{d\omega}{dt} = I\omega\alpha = \omega\vec{\tau}_{total} \cdot \hat{n}$$

which yields

$$I\alpha = \vec{\tau}_{total} \cdot \hat{n}$$

10.6 Work and Rotational Kinetic Energy

By (4), the power exerting on a rotating body is

$$P = \frac{dW}{dt} = \vec{\tau}_{total} \cdot \vec{\omega}$$