Chapter 4 Systems of Differential Equations, Phase Plane, Qualitative Methods

4.0 Introduction: Vectors, Matrices, Eigenvalues

※ **matrix**

$$\mathbf{A} = \left[a_{jk}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \text{row}$$

column diagonal

$\mathbf{A}$: n x n matrix , a_{11}, a_{12} …. : entries

※ **column vector** with n components

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

※ **row vector** with n components

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

※ **equality**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$\mathbf{A} = \mathbf{B}$ if and only if $a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{21} = b_{21}$, $a_{22} = b_{22}$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{if and only if} \quad \begin{matrix} v_1 = x_1 \\ v_2 = x_2 \end{matrix}$$

※ **addition**

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

※ **scalar multiplication**

$$\mathbf{A}=\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix}, \qquad -7\mathbf{A}=\begin{bmatrix} -63 & -21 \\ 14 & 0 \end{bmatrix}$$

※ **matrix multiplication**

$\mathbf{A}=\lfloor a_{jk} \rfloor$ and $\mathbf{B}=\lfloor b_{jk} \rfloor$ are n x n matrix, if $\mathbf{C}=\mathbf{AB}$

then $\mathbf{C}=[c_{jk}]$, $\quad c_{jk}=\sum_{m=1}^{n} a_{jm}b_{mk} \qquad \begin{matrix} j=1,2,\cdots n \\ k=1,2,\cdots n \end{matrix}$

example:

$$\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}=\begin{bmatrix} 9\cdot 1+3\cdot 2 & 9\cdot(-4)+3\cdot 5 \\ -2\cdot 1+0\cdot 2 & (-2)\cdot(-4)+0\cdot 5 \end{bmatrix}=\begin{bmatrix} 15 & -21 \\ -2 & 8 \end{bmatrix}$$

Caution: Matrix multiplication is not commutative, $\mathbf{AB}\neq\mathbf{BA}$

※ **differentiation**

$$\mathbf{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\begin{bmatrix} e^{-2t} \\ \sin t \end{bmatrix} \quad \Rightarrow \quad \mathbf{y}'(t)=\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}=\begin{bmatrix} -2e^{-2t} \\ \cos t \end{bmatrix}$$

※ **transposition**

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^{\mathbf{T}}=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

※ Inverse of a mayrix

unit matrix of 3x3 : $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For a given $n \times n$ matrix **A** there is an $n \times n$ matrix **B** such that **A B** = **B A** = **I** , then **A** is called nonsingular and **B** is called inverse of **A**

And is denoted by $\mathbf{A}^{-1}$

$\Longrightarrow \quad \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

If **A** has no inverse, it is called singular.

Example: n = 2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

※ system of differential eqs. as vector equations

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2 \end{aligned} \Rightarrow \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Ay}$$

If $\mathbf{y} \equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$; $\mathbf{A} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

※ Eigenvalues, Eigenvectors

Let $\mathbf{A} = \lfloor a_{jk} \rfloor$ be an $n \times n$ matrix

Consider $\mathbf{Ax} = \lambda\, \mathbf{x}$, λ : a scalar (1)

If a scalar λ such that (1) holds for some $\mathbf{x} \neq \mathbf{0}$

λ **:** eigenvalue of **A** , **x** : eigenvector of **A**

(1) $\Rightarrow \mathbf{Ax} - \lambda\,\mathbf{x} = 0 \quad or \quad (\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{x} = 0$

for a non-zero solution $\mathbf{x} \neq \mathbf{0}$, the determinant of the coefficient matrix

$\mathbf{A} - \lambda\,\mathbf{I}$ must be zero. i.e.,

det $(\mathbf{A} - \lambda\,\mathbf{I}) = 0 \quad \rightarrow$ characteristic determinant of **A**

for n = 2 det $(\mathbf{A} - \lambda\,\mathbf{I}) =$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$\Longrightarrow$ characteristic equation of **A,** solutions are λ_1, λ_2

substitute into (1) can obtain the corresponding eigenvector $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

Example: $\mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$

Characteristic equation:

$$\det(\mathbf{A} - \lambda\,\mathbf{I}) = \begin{vmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{vmatrix} = \lambda^2 + 2.8\lambda + 1.6 = 0$$

$\Rightarrow \quad \lambda_1 = -2, \quad \lambda_2 = -0.8$

for $\lambda_1 = -2$, $\rightarrow \; (\mathbf{AX} - \lambda_1 \mathbf{X}) = \mathbf{0}$

$\Rightarrow \quad (-4.0 + 2.0)\, x_1 + 4.0 x_2 = 0 \quad \Rightarrow x_1 = 2,\; x_2 = 1$

also the same for $-1.6\, x_1 + (1.2 + 2.0) x_2 = 0$

The eigenvector of **A** corresponding to $\lambda_1 = -2$ is

$$\mathbf{x}^{(1)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} .$$

Similarly for $\lambda_2 = -0.8 \quad \Rightarrow \quad (\mathbf{AX} - \lambda_2 \mathbf{X}) = \mathbf{0}$

$$\Rightarrow \left.\begin{array}{l} (-4.0 + 0.8)\, x_1 + 4.0 x_2 = 0 \\ -1.6\, x_1 + (1.2 + 0.8) x_2 = 0 \end{array}\right\}$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

4.1 Introductory Examples

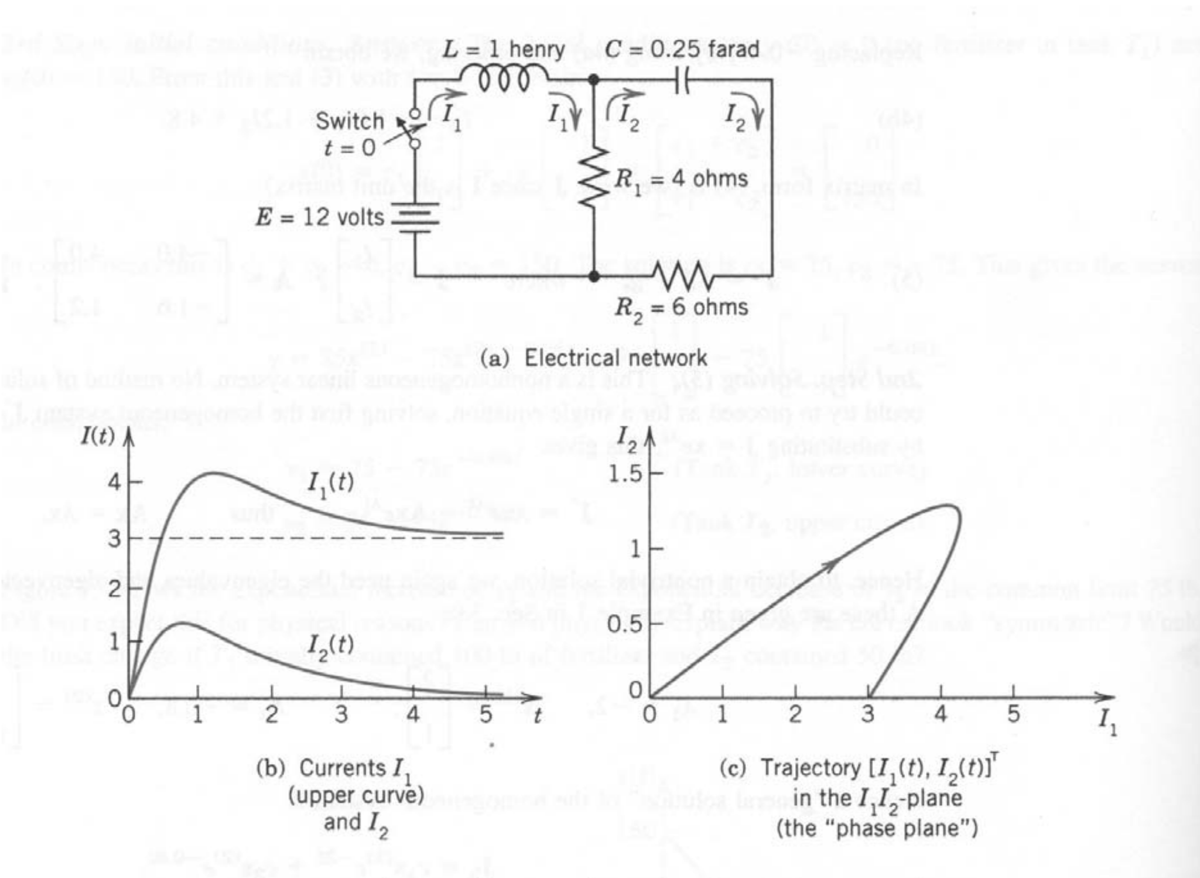


Fig. 76. Electrical network and currents in Example 2

Find the currents $I_1(t)$ and $I_2(t)$ in the network. Assuming that all charges and currents are zero at t =0, the instant when the switch is closed.

Solve: by Kirchhoff's voltage law

Left loop: $I_1' + 4(I_1 - I_2) = 12$ (1)

Right loop: $6I_2 + 4(I_2 - I_1) + 4\int I_2\, dt = 0$ (2)

(1) $\Rightarrow I_1' = -4I_1 + 4I_2 + 12$ (3)

$\dfrac{d(2)}{dt}\Big/10 \quad \Rightarrow I_2' - 0.4I_1' + 0.4I_2 = 0$ (4)

(3) in (4) $\Rightarrow I_2' = -1.6I_1 + 1.2I_2 + 4.8$ (5)

$$(3)\&(5) \Rightarrow \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}' = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 12.0 \\ 4.8 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{J}' = \mathbf{A}\mathbf{J} + \mathbf{g} \qquad (6)$$

First consider the **homogeneous** equation (**homogeneous solution**)

by substituting $\mathbf{J} = \mathbf{x}\, e^{\lambda t}, \quad \mathbf{J}' = \lambda\, \mathbf{x}\, e^{\lambda t} = \mathbf{A}\mathbf{x}\, e^{\lambda t} \quad \Rightarrow \quad \mathbf{A}\mathbf{x} = \lambda\, \mathbf{x}$

the eigenvalues and eigenvectors are

$$\lambda_1 = -2, \quad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad ; \qquad \lambda_2 = -0.8, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

General solution for the **homogeneous** equation is

$$\mathbf{J}_{\mathrm{h}} = c_1 \mathbf{x}^{(1)}\, e^{-2t} + c_2 \mathbf{x}^{(2)}\, e^{-0.8t}$$

particular solution:

since **g** is constant $\rightarrow$ try particular solution $\mathbf{J}_p = \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

(6) $\Rightarrow \mathbf{A}\mathbf{a} + \mathbf{g} = \mathbf{0}$, in components,

$$\begin{aligned} -4.0\, a_1 + 4.0\, a_2 + 12.0 &= 0 \\ -1.6\, a_1 + 1.2\, a_2 + 4.8 &= 0 \end{aligned} \quad \Rightarrow \quad a_1 = 3, \; a_2 = 0$$

the general solution is

$$\mathbf{J} = \mathbf{J}_{\mathrm{h}} + \mathbf{J}_p = c_1 \mathbf{x}^{(1)}\, e^{-2t} + c_2 \mathbf{x}^{(2)}\, e^{-0.8t} + \mathbf{a}$$

in components:
$$\begin{aligned} I_1 &= 2c_1 e^{-2t} + c_2 e^{-0.8t} + 3 \\ I_2 &= c_1 e^{-2t} + 0.8 c_2 e^{-0.8t} \end{aligned}$$

I.C.
$$\begin{aligned} I_1(0) &= 2c_1 + c_2 + 3 = 0 \\ I_2(0) &= c_1 + 0.8 c_2 = 0 \end{aligned} \qquad c_1 = -4, \quad c_2 = 5$$

The solution is
$$\begin{aligned} I_1 &= -8e^{-2t} + 5e^{-0.8t} + 3 \\ I_2 &= -4e^{-2t} + 4e^{-0.8t} \end{aligned}$$

※ Conversion of an nth order differential equation to a system of 1st order differential equation

An nth order differential equation

$$y^{(n)} = F(t, y', y'', \cdots, y^{(n-1)})$$

can always be reduced to a system of n first order differential equations by setting

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \quad \cdots, \quad y_n = y^{(n-1)}$$

then obtains

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ &\vdots \\ y'_{n-1} &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n) \end{aligned}$$

Example: mass on a spring

$$y'' + \frac{c}{m} y' + \frac{k}{m} y = 0$$

$$\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = -\dfrac{k}{m} y_1 - \dfrac{c}{m} y_2 \end{cases}, \quad \text{set} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\text{then} \quad \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \mathbf{y}$$

the characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

assume that m = 1, c = 2, and k = 0.75. then $\lambda^2 + 2\lambda + 0.75 = 0$

this give the eigenvalues $\lambda_1 = -0.5, \quad \lambda_2 = -1.5$

Eigenvectors are $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$

The solution is $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-0.5t} + c_2 \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} e^{-1.5t}$

First component is the expect solution

$$y = y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$$

4.2 Basic concepts and Theory

consider first order systems(more general system)

$$
\begin{aligned}
y_1' &= f_1(t, y_1, y_2, \cdots, y_n) \\
y_2' &= f_2(t, y_1, y_2, \cdots, y_n) \\
y_3' &= f_3(t, y_1, y_2, \cdots, y_n) \\
&\vdots \\
y_n' &= f_n(t, y_1, y_2, \cdots, y_n)
\end{aligned}
\qquad (1)
$$

in matrix form $\mathbf{y}' = \mathbf{f}\,(\mathrm{t}, \mathbf{y})$

$$
\text{where } \mathbf{y} \equiv \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}; \qquad \mathbf{f} \equiv \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}
$$

initial conditions: $y_1(t_0) = K_1, y_2(t_0) = K_2, \cdots, y_n(t_0) = K_n$ (2)

(1)&(2): initial value problem

Theorem: Let $f_1, \cdots, f_n$ in (1) be continuous functions having continuous partial derivatives $\frac{\partial f_1}{\partial y_1}, \cdots, \frac{\partial f_1}{\partial y_n}, \cdots, \frac{\partial f_n}{\partial y_n}$ in some domain R of $ty_1y_2 \cdots y_n$ -space containing the point $(t_0, K_1, K_2 \cdots, K_n)$. Then (1) has a solution on some interval $t_0 - \alpha < t < t_0 + \alpha$ satisfying (2), and this solution is unique.

※ **Linear systems:**

extending (1)

$$
\begin{aligned}
y_1' &= a_{11}(t)y_1 + a_{12}(t)\, y_2 + \cdots + a_{1n}(t)\, y_n + g_1(t) \\
y_2' &= a_{21}(t)y_1 + a_{22}(t)\, y_2 + \cdots + a_{2n}(t)\, y_n + g_2(t) \\
&\vdots \\
y_n' &= a_{n1}(t)y_1 + a_{n2}(t)\, y_2 + \cdots + a_{nn}(t)\, y_n + g_n(t)
\end{aligned}
$$

in matrix form: $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$ (3)

where $\mathbf{A} \equiv \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ \cdots & .. & .. & .. \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$; $\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ .. \\ g_n \end{bmatrix}$

$\mathbf{g} = \mathbf{0}$ homogeneous equation $\mathbf{y}' = \mathbf{A}\mathbf{y}$ (4)

$\mathbf{g} \neq \mathbf{0}$ non-homogeneous equation $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}$

Theorem: If $\mathbf{y}^{(1)} and\ \mathbf{y}^{(2)}$ are solutions of the homogeneous linear system (4) on some interval, so is any linear combination $\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}$

※ **General solution:**

If $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \cdots, \mathbf{y}^{(n)}$ are independent solutions of (4) $\rightarrow$ **basis**

In matrix form $\mathbf{Y} = [\mathbf{y}^{(1)}\ \ \mathbf{y}^{(2)}\ \cdots\ \mathbf{y}^{(n)}]$:**fundamental matrix**

Linear combination of basis

$\Rightarrow \mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\ \mathbf{y}^{(2)} + \cdots + c_n\mathbf{y}^{(n)}$ **general solution** of (4)

determinant of **Y** is called **Wronskian** of $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \cdots, \mathbf{y}^{(n)}$

$$W(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \cdots, \mathbf{y}^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ \vdots & \vdots & \cdots & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{vmatrix}$$

4.3_0 Homogeneous linear systems with constant Coefficients

Consider the homogeneous system:

$$\left.\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots\cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots\cdots + a_{2n}y_n \\ &\cdots\cdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots\cdots + a_{nn}y_n \end{aligned}\right\} ------(1)$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} ------(1)$$

Assume the solution of (1) has the form:

$$\mathbf{y} = \mathbf{x}e^{\lambda t} ------(2)$$

(2) into (1) $\Rightarrow$ $\mathbf{A}\mathbf{x} = \lambda\mathbf{x} ------(3)$

Thus the solutions of (1) are given by (2) , in which λ and $\mathbf{x}$ are the eigenvalues and the corresponding eigenvectors of (3).

(I) Distinct real eigenvalues

When $\lambda_1, \lambda_2, \cdots\cdots \lambda_n$ are n distinct real eigenvalues of $\mathbf{A}$, and $\mathbf{x}_1, \mathbf{x}_2, \cdots\cdots \mathbf{x}_n$ are their corresponding eigenvectors .

then the general solution of (1) is:

$$\boxed{\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots\cdots + c_n\mathbf{x}_n e^{\lambda_n t}}$$

Where $c_1, c_2, \cdots, c_n$ are arbitrary constants

Ex. $\left.\begin{array}{l} y_1' = 2y_1 + 3y_2 \\ y_2' = 2y_1 + y_2 \end{array}\right\}$

$\because$ $\mathbf{A} = \begin{Bmatrix} 2 & 3 \\ 2 & 1 \end{Bmatrix} \to \left.\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = 4 \end{array}\right\} \to \mathbf{x}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}; \mathbf{x}_2 = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$

$\therefore$ $\mathbf{y} == c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{-t} + c_2 \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} e^{4t}$

(II) Complex eigenvalues

When $\mathbf{A}$ is real, suppose $\lambda_1 = \alpha + i\beta$; $\mathbf{x}_1$ is the complex

eigenvalue and the corresponding eigenvector of $\mathbf{A}$.

Then $\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t}$; $\mathbf{y}_2 = \bar{\mathbf{x}}_1 e^{\bar{\lambda}_1 t}$ are solutions of (1) where

$\bar{\lambda}_1, \bar{\mathbf{x}}_1$ is the complex conjugate of λ_1 and $\mathbf{x}_1$.

Let $\mathbf{x}_1 = \mathbf{u} + i\mathbf{v}$; $\quad$ *i.e.* $\mathbf{u} = \mathrm{Re}(\mathbf{x}_1)$; $\mathbf{v} = \mathrm{Im}(\mathbf{x}_1)$

then

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t} = (\mathbf{u} + i\mathbf{v}) e^{(\alpha + i\beta)t} = e^{\alpha t} (\mathbf{u} + i\mathbf{v})(\cos\beta t + i\sin\beta t)$$

$$= e^{\alpha t} \{(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) + i(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)\}$$

also

$$\mathbf{y}_2 = e^{\alpha t} \{(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) - i(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)\}$$

Since $\frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2)$; $\frac{1}{2i}(\mathbf{y}_1 - \mathbf{y}_2)$ are also solutions of the homogeneous equation $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

The real form solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ corresponding to $\lambda_1 = \alpha \pm i\beta$

are: $\boxed{\begin{aligned} \mathbf{y}_1 &= e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \\ \mathbf{y}_2 &= e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t) \end{aligned}}$

Ex. $\left.\begin{aligned} y_1' &= y_1 + 2y_2 \\ y_2' &= -0.5y_1 + y_2 \end{aligned}\right\} ------(i)$

$$\because \mathbf{A} = \begin{Bmatrix} 1 & 2 \\ -0.5 & 1 \end{Bmatrix} \to \lambda_{1,2} = 1 \pm i \to \mathbf{x}_1 = \begin{Bmatrix} 2 \\ i \end{Bmatrix}$$

$$i.e.\ \ \alpha = 1;\ \ \beta = 1;\ \ \mathbf{u} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix};\ \ \mathbf{v} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$\Rightarrow$

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) = e^t(\begin{Bmatrix} 2 \\ 0 \end{Bmatrix}\cos t - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\sin t) = e^t\begin{Bmatrix} 2\cos t \\ -\sin t \end{Bmatrix}$$

$$\mathbf{y}_2 = e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t) = e^t(\begin{Bmatrix} 2 \\ 0 \end{Bmatrix}\sin t + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\cos t) = e^t\begin{Bmatrix} 2\sin t \\ \cos t \end{Bmatrix}$$

The general solution of Eq.(i) is :

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = \left[c_1\begin{Bmatrix} 2\cos t \\ -\sin t \end{Bmatrix} + c_2\begin{Bmatrix} 2\sin t \\ \cos t \end{Bmatrix}\right]e^t$$

(III) Repeated eigenvalues

Suppose λ_1 is a repeated eigenvalues (of multiplicity m) of $\mathbf{A}$:

(i) If we can find m independent corresponding eigenvectors

$\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_m$, the linearly independent solutions corresponding to λ_1 are:

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t};\ \mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_1 t};\ \cdots\cdots;\ \mathbf{y}_m = \mathbf{x}_m e^{\lambda_1 t}$$

(ii) If there is only one eigenvector, i.e. $\mathbf{x}_1$ corresponding to the repeated eigenvalue λ_1, the linearly independent solutions are :

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t};$$

$$\mathbf{y}_2 = \left\{ t \cdot \mathbf{x}_1 + \mathbf{x}_2 \right\} e^{\lambda_1 t};$$

$$\mathbf{y}_3 = \left\{ \frac{1}{2!} t^2 \cdot \mathbf{x}_1 + t \cdot \mathbf{x}_2 + \mathbf{x}_3 \right\} e^{\lambda_1 t};$$

$$\cdots\cdots;$$

$$\mathbf{y}_m = \left\{ \frac{1}{(m-1)!} t^{m-1} \cdot \mathbf{x}_1 + \cdots + t \cdot \mathbf{x}_{m-1} + \mathbf{x}_m \right\} e^{\lambda_1 t}$$

Where $\mathbf{x}_2, \mathbf{x}_3, \cdots \mathbf{x}_m$ can be obtained by substituting $\mathbf{y}_i$ $(i = 2 \sim m)$ into the equation: $\mathbf{y}' = \mathbf{A}\mathbf{y}$

Ex. $$\left.\begin{aligned} y_1' &= y_1 - 2y_2 + 2y_3 \\ y_2' &= -2y_1 + y_2 - 2y_3 \\ y_3' &= 2y_1 - 2y_2 + y_3 \end{aligned}\right\}$$

$$\because \mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \rightarrow \lambda_{1,2} = -1 \text{ (multiplicity 2)};\ \lambda_3 = 5$$

$\rightarrow \mathbf{x}_1 = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$; $\mathbf{x}_2 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$ (two indep. eig. vectors.), $\mathbf{x}_3 = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$

The general solution is: $\mathbf{y} = c_1 \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} e^{-t} + c_2 \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} e^{-t} + c_3 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} e^{5t}$

Ex. $\left.\begin{aligned} y_1' &= 4y_1 + y_2 \\ y_2' &= -y_1 + 2y_2 \end{aligned}\right\} ------(i)$

$\because \mathbf{A} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$

$\rightarrow \lambda_{1,2} = 3$ (multiplicity 2); $\mathbf{x}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$; (only one eigenvector)

$\rightarrow \mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{3t}$

let $\mathbf{y}_2 = (t \cdot \mathbf{x}_1 + \mathbf{x}_2) e^{\lambda_1 t} -----(ii)$

Eq.(ii) into Eq.(I)→

$$\mathbf{x}_1 e^{\lambda_1 t} + \lambda_1 (t \cdot \mathbf{x}_1 + \mathbf{x}_2) e^{\lambda_1 t} = \mathbf{A}(t \cdot \mathbf{x}_1 + \mathbf{x}_2) e^{\lambda_1 t}$$

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x}_2 = \mathbf{x}_1$$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{x}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ → we may take $\mathbf{x}_2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

the general solution of Eq.(i) is :

$$\mathbf{y} = c_1\mathbf{y} + c_2\mathbf{y}_2 = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2(t\cdot\mathbf{x}_1 + \mathbf{x}_2)e^{\lambda_1 t}$$

$$= c_1\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{3t} + c_2(t\cdot\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix})e^{3t}$$

Note: in this example : $\because \lambda = 3$ has multiplicity 2, but only one

eigenvector, we may set: $\left.\begin{array}{l} y_1 = c_1 e^{3t} + c_2 t\, e^{3t} \\ y_2 = c_3 e^{3t} + c_4 t\, e^{3t} \end{array}\right\} --- (iii)$

substitute (iii) into (i), then solve c_3, c_4 in terms of c_1, c_2.

4.3 Constant-coefficient systems. Phase plane method

for n=2, $\mathbf{y}' = \mathbf{A}\mathbf{y} \quad \Rightarrow \quad \begin{matrix} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2 \end{matrix}$ (6)

solution is $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ (7)

plot (7) as a single curve in the $y_1\, y_2$-plane $\rightarrow$ **trajectory**

$y_1\, y_2$-plane : **phase plane**

phase plane + trajectory $\Longrightarrow$ **phase portrait**

※ **Critical points**

from (6) $\Rightarrow$ $\dfrac{d\,y_2}{d\,y_1} = \dfrac{dy_2/dt}{dy_1/dt} = \dfrac{y_2'}{y_1'} = \dfrac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$ (8)

This associates with every point $P:(y_1, y_2)$ a unique tangent direction $\dfrac{d\,y_2}{d\,y_1}$ of the trajectory passing through P, except for the point $P:(0,0)$, where the right side of (8) becomes $\dfrac{0}{0}$.

The point $\dfrac{d\,y_2}{d\,y_1}$ becomes undetermined is called **critical point**.

Critical point: **improper node, proper node, saddle point, center, spiral point**

Example 1: (**improper nodes**) $\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y}$

thus $\begin{matrix} y_1' = -3y_1 + y_2 \\ y_2' = y_1 - 3y_2 \end{matrix}$

Solve: set $\mathbf{y} = \mathbf{x}\, e^{\lambda t}$ substitute into the equation

$\Rightarrow$ $\mathbf{A}\mathbf{x} = \lambda\ \mathbf{x}$, the characteristic equation is

$$\det(\mathbf{A} - \lambda\, \mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

$\Rightarrow \lambda_1 = -2$ & $\lambda_2 = -4$, the eigenvectors are

$$\lambda_1 = -2 \quad \Rightarrow \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \lambda_2 = -4 \quad \Rightarrow \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{solution} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

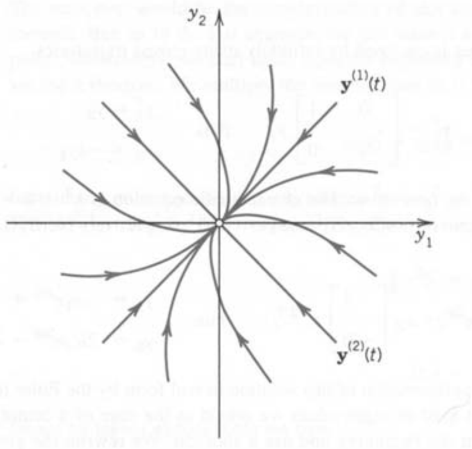


Fig. 78. Trajectories of the system (8)
(Improper node)

An **improper node** is a critical point P_0 at which all the trajectories, except for two of them, have the same limiting direction of the tangent. The two exceptional trajectories also have a limiting direction of the tangent at P_0 which, however, is different.

Example 2:(**proper nodes) :** A proper node is a critical point P_0 at which every trajectory has a definite limiting direction and for any given direction *d* at P_0 there is a trajectory having *d* as its limiting direction.

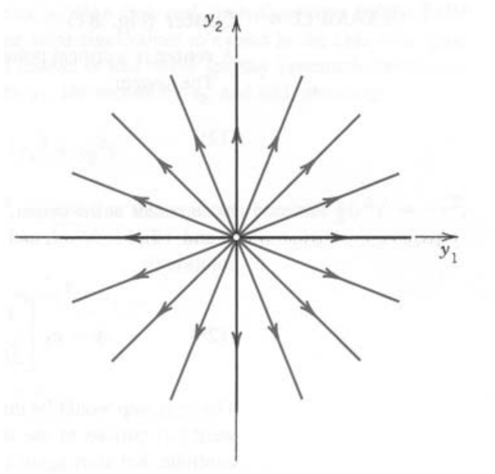


Fig. 79. Trajectories of the system (10)
(Proper node)

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y} \quad \text{thus} \quad \begin{array}{l} y_1' = y_1 \\ y_2' = y_2 \end{array}$$ has a proper node at the origin because the general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \quad or \quad \begin{array}{l} y_1 = c_1\, e^t \\ y_2 = c_2\, e^t \end{array} \quad or \quad c_1 y_2 = c_2 y_1$$

Example 3:**(saddle point) :** A saddle point is a critical point P_0 at which there are two incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 by pass P_0 .

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y} \quad \text{thus} \quad \begin{array}{l} y_1' = y_1 \\ y_2' = -y_2 \end{array}$$ has a saddle point at the origin because the general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \quad or \quad \begin{array}{l} y_1 = c_1\, e^t \\ y_2 = c_2\, e^{-t} \end{array} \quad or \quad y_1 y_2 = const.$$

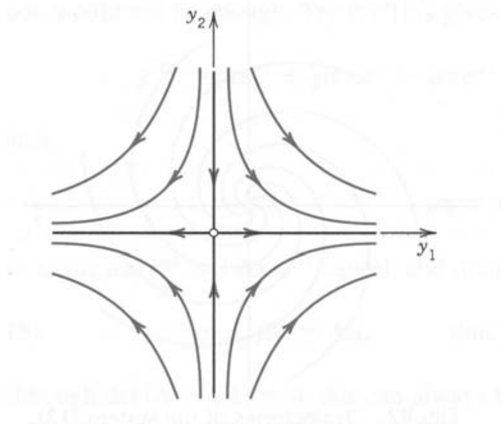


Fig. 80. Trajectories of the system (11)
(Saddle point)

Example 4:**(center) :** A center is a critical point that enclosed by infinitely many closed trajectories.

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\mathbf{y}, \text{ thus } \begin{matrix} y_1' = y_2 \\ y_2' = -4y_1 \end{matrix} \quad \text{has a center at the origin.}$$

Eigenvalues are $\Rightarrow \lambda_1 = 2i$ & $\lambda_2 = -2i$, the eigenvectors are

$$\lambda_1 = 2i \quad \Rightarrow \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}, \lambda_2 = -2i \quad \Rightarrow \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

the general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it} \quad or \quad \begin{matrix} y_1 = c_1 e^{2it} + c_2 e^{-2it} \\ y_2 = 2c_1 e^{2it} - 2ic_2 e^{-2it} \end{matrix}$$

(complex).

Need to transform to real. (section 2.3)

Another method: rewrite the eq. as $y_1' = y_2$, $4y_1 = -y_2'$, then product the two equations $4y_1 y_1' = -y_2 y_2'$ integration $2y_1^2 + \frac{1}{2}y_2^2 = const.$ Center at the origin.

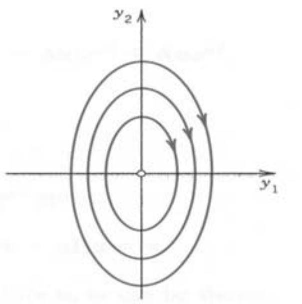


Fig. 81. Trajectories of the system (12)
(Center)

Example 5:**(spiral point) :** A center is a critical point P_0 about which the trajectories spiral, approaching P_0 as $t \to \infty$(or tracing these spiral in the opposite sense, away from P_0.)

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y} \quad \text{thus} \quad \begin{aligned} y_1' &= -y_1 + y_2 \qquad (a) \\ y_2' &= -y_1 - y_2 \qquad (b) \end{aligned}$$ has a spiral point at the origin.

Eigenvalues are $\Rightarrow \lambda_1 = -1 + i \quad \& \quad \lambda_2 = -1 - i$, the eigenvectors are

$$\lambda_1 = -1 + i \quad \Rightarrow \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad , \lambda_2 = -1 - i \quad \Rightarrow \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

the general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)\,t} \qquad \text{complex.}$$

Need to transform to real. (section 2.3)

Another method: $(a) \times y_1 + (b) \times y_2 \Rightarrow y_1 y_1' + y_2 y_2' = -(y_1^2 + y_2^2)$ (c)

Introducing polar coordinates *r*, *t* where $r^2 = y_1^2 + y_2^2$

$$\text{(c)} \;\Rightarrow\; \frac{1}{2}(r^2)' = -r^2 \quad \Rightarrow \quad rr' = -r^2 \quad \Rightarrow \quad r' = -r$$

$$\Rightarrow \ln r = -t + \tilde{c} \quad \Rightarrow r = c\, e^{-t} \quad \text{a spiral.}$$

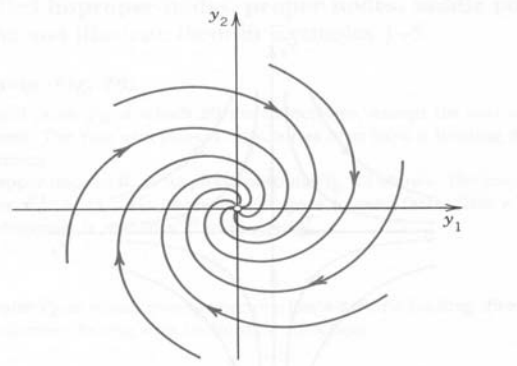


Fig. 82. Trajectories of the system (13)
(Spiral point)

4.4 Criteria for critical points. Stability

- Stability analysis of linear system

$$\frac{d\,y_2}{d\,y_1}=\frac{dy_2/dt}{dy_1/dt}=\frac{y_2'}{y_1'}=\frac{a_{21}y_1+a_{22}y_2}{a_{11}y_1+a_{12}y_2}=\frac{0}{0}$$: critical point

Consider a linear system:

$$\mathbf{x}'=\mathbf{A}\mathbf{x}------(1)$$

$$\text{where}\quad \mathbf{A}=\begin{Bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Bmatrix};\quad \mathbf{x}=\begin{Bmatrix} x \\ y \end{Bmatrix}$$

Assume $\det\mathbf{A}\neq 0 \;\Rightarrow\; p_0=(0,0)$ is the only critical of (1)

since the eigenvalues of $\mathbf{A}$ is determined by :

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}=0$$

i.e.

$$\lambda^2-(a_{11}+a_{22})\lambda+\det\mathbf{A}=0------(2)$$

let

$$\begin{cases} p\equiv a_{11}+a_{22} & \text{called "the trace of } \mathbf{A}\text{"} \\ q\equiv\det\mathbf{A} \end{cases} ------(3)$$

then (2)→

$$\lambda^2-p\lambda+q=0------(4)$$

If λ_1,λ_2 are two eigenvalues of $\mathbf{A}$,we have

$(\lambda-\lambda_1)(\lambda-\lambda_2)=0$

i.e.

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \cdot \lambda_2 = 0 - - - - - (5)$$

(4) and(5)→

$$\begin{cases} p = (\lambda_1 + \lambda_2) \\ q = \lambda_1 \cdot \lambda_2 \end{cases} \quad - - - - - (6)$$

(I) When $p^2 - 4q > 0 \Rightarrow \lambda_1; \lambda_2$ are real, the general solution of (1) has the form:

$$\mathbf{x} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}$$

$$\text{If} \begin{cases} \lambda_1 < 0 \ \& \lambda_2 < 0 \rightarrow \text{ Stable node } (p < 0 \ \ \& \ q > 0) \\ \lambda_1 > 0 \ \& \lambda_2 > 0 \rightarrow \text{ Unstable node } (p > 0 \ \ \& \ q > 0) \end{cases}$$

$$\text{If} \left\{ \begin{matrix} \lambda_1 < 0 \ \& \lambda_2 > 0 \\ \lambda_1 > 0 \ \& \lambda_2 < 0 \end{matrix} \right\} \rightarrow \text{Saddle point } (q < 0)$$

(II) When $p^2 - 4q = 0 \Rightarrow \lambda_1 = \lambda_2$ repeated real eigenvalues, the general solution of (1) has the form:

$$\mathbf{x} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 (t \cdot \mathbf{x}_1 + \mathbf{x}_2) e^{\lambda_1 t}$$

If

$$\begin{cases} \lambda_1 < 0 \ \rightarrow \text{ Called the "degenerated stable node"} (p < 0 \ \ \& \ q > 0) \\ \lambda_1 > 0 \ \rightarrow \text{ Called the "degenerated unstable node"} (p > 0 \ \ \& \ q > 0) \end{cases}$$

(III) When $p^2 - 4q < 0 \Rightarrow \lambda_{1,2} = \alpha \pm i\beta$ complex conjugate , the

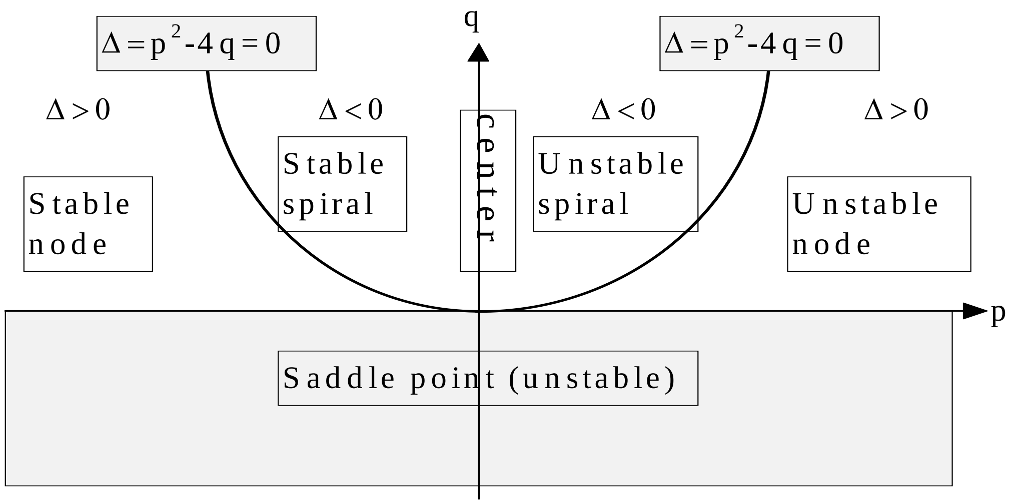
general solution of (1) has the form:

$$\mathbf{x} = e^{\alpha \cdot t}\left(c_1 \mathbf{x}_1 e^{i\beta \cdot t} + c_2 \bar{\mathbf{x}}_1 e^{-i\beta \cdot t}\right)$$

$$\text{If } \begin{cases} \alpha < 0 \;\rightarrow\; \text{Stable spiral point } (p < 0 \;\;\&\;\; q > 0) \\ \alpha > 0 \;\rightarrow\; \text{Unstable spiral point } (p > 0 \;\;\&\;\; q > 0) \\ \alpha = 0 \;\rightarrow\; \text{Center } (p = 0 \;\;\&\;\; q > 0) \end{cases}$$

Summary:

$$p \equiv \text{trace of } \mathbf{A}; \quad q \equiv \det \mathbf{A}; \quad \Delta \equiv p^2 - 4q;$$



Ex. $\mathbf{x}' = \mathbf{A}\mathbf{x}$; $\quad \mathbf{A} = \begin{Bmatrix} -1 & 1 \\ -1 & -1 \end{Bmatrix}$

since $\det \mathbf{A} \neq 0$

$\Rightarrow$ only $p_o = (0,0)$ is a critical point

$\because p = -2 < 0; \quad q = \det \mathbf{A} = 2 > 0; \quad \Delta = p^2 - 4q = 4 - 8 = -4 < 0$

$\rightarrow p_o = (0,0)$ is a stable spiral point

4.5 Qualitative methods for nonlinear system

Linearization and local stability

consider the nonlinear aut0nomous system

$$\left.\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y) \end{aligned}\right\} ------(1)$$

If $P_o = (x_o, y_o)$ is an isolated critical point of (1) i.e.

$$F(x_o,y_o)=0; \quad G(x_o,y_o)=0------(2)$$

then, the Taylor's expansion of $F(x,y)$ and $G(x,y)$ at (x_o,y_o) becomes:

$$\left.\begin{aligned} F(x,y) &= \underbrace{F(x_o,y_o)}_{0} + F_x(x_o,y_o)(x-x_o) + F_y(x_o,y_o)(y-y_o) + \cdots\cdots \\ G(x,y) &= \underbrace{G(x_o,y_o)}_{0} + G_x(x_o,y_o)(x-x_o) + G_y(x_o,y_o)(y-y_o) + \cdots\cdots \end{aligned}\right\} ---(3)$$

where

$$F_x(x_o,y_o) \equiv \frac{\partial F}{\partial x}(x_o,y_o) \qquad F_y(x_o,y_o) \equiv \frac{\partial F}{\partial y}(x_o,y_o)$$
$$G_x(x_o,y_o) \equiv \frac{\partial G}{\partial x}(x_o,y_o) \qquad G_y(x_o,y_o) \equiv \frac{\partial G}{\partial y}(x_o,y_o)$$

let

$$\left.\begin{aligned} a_{11} &\equiv F_x(x_o,y_o) \qquad a_{12} \equiv F_y(x_o,y_o) \\ a_{21} &\equiv G_x(x_o,y_o) \qquad a_{22} \equiv G_x(x_o,y_o) \end{aligned}\right\} ------(4)$$

Using (2),(3), we can linearize (1) :

$$\left.\begin{array}{l} x' = a_{11}(x-x_o)+a_{12}(y-y_o)+P(x,y) \\ y' = a_{21}(x-x_o)+a_{22}(y-y_o)+Q(x,y) \end{array}\right\} ------(5)$$

Where $P(x,y)$, $Q(x,y)$ are function of second order terms of $(x-x_o)$ and/or $(y-y_o)$.

If we define a new coordinate :

$$\left.\begin{array}{l} \bar{x} = (x-x_o) \\ \bar{y} = (y-y_o) \end{array}\right\} -------(6)$$

(5) can be linearized as:

$$\left.\begin{array}{l} \bar{x}' = a_{11}\bar{x}+a_{12}\bar{y} \\ \bar{y}' = a_{21}\bar{x}+a_{22}\bar{y} \end{array}\right\} ------(7)$$

It can be shown that the stability of the nonlinear system (1) at the critical point $P_o = (x_o, y_o)$ is almost the same as the stability of the linearized system (7)at $(\bar{x},\bar{y}) = (0,0)$, provided that:

$$P(\bar{x},\bar{y}) \to 0 \quad \text{and} \quad Q(\bar{x},\bar{y}) \to 0 \quad \text{as limit} \left\{\begin{array}{l} \bar{x} \to 0 \\ \bar{y} \to 0 \end{array}\right. ---(8)$$

Therefore, the stability analysis of the critical point of a non-linear system can be carried out as following:

$$\text{Given } x' = \left\{\begin{array}{c} F(x,y) \\ G(x,y) \end{array}\right\}$$

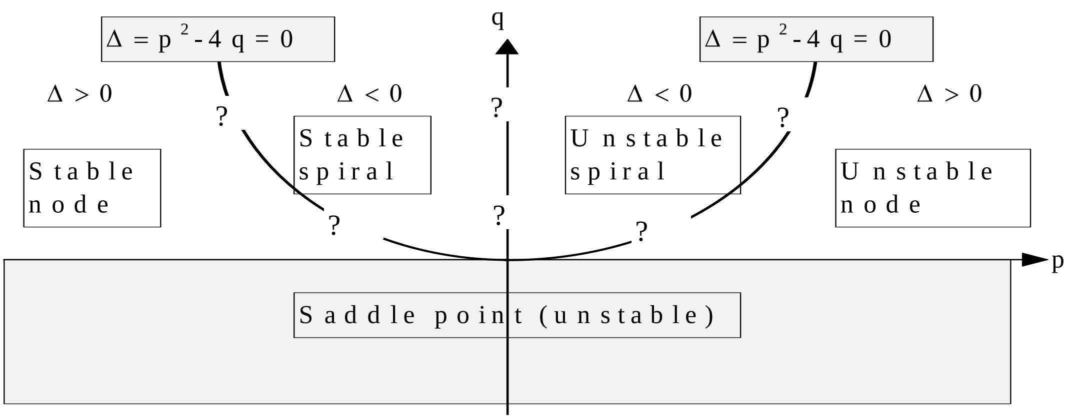
(i) Solve $\left.\begin{array}{l} F(x,y) = 0 \\ G(x,y) = 0 \end{array}\right\} \to P_o = (x_o, y_o)$ the critical point

(ii) Calculate $\mathbf{A} \equiv \begin{Bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Bmatrix} \equiv \begin{Bmatrix} F_x & F_y \\ G_x & G_y \end{Bmatrix}_{(x_o, y_o)}$ the Jacobin matrix

(iii) Calculate :

$p = a_{11} + a_{12}$ trace of $\mathbf{A}$; $\quad q = \det \mathbf{A}$; $\quad \Delta = p^2 - 4q$;

(iv) Determine the stability and the type of the critical point P_o by the chart:



Note:

(i) on $\Delta \equiv p^2 - 4q = 0$, P_o may be stable $(p < 0)$ or unstable $(p > 0)$ spiral, node, or degenerated node.

(ii) on $p = 0$ and $q > 0$; P_o may be stable or unstable spiral, or stable center.

Ex. Find the critical points and determine the type of the

$$\left.\begin{aligned} x' &= y \\ y' &= x^3 - x \end{aligned}\right\} ------(1)$$

Sol

$$\because \left.\begin{matrix} F(x,y)=0 \\ G(x,y)=0 \end{matrix}\right\} \to \left.\begin{matrix} y=0 \\ x^3-x=0 \end{matrix}\right\} \to \left.\begin{matrix} y=0 \\ x(x^2-1)=0 \end{matrix}\right\}$$

the critical points of (1) are: $(0,0);\ (1,0);\ (-1,0);$

$$\because \text{ the Jacobin matrix } \mathbf{A} \equiv \begin{Bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Bmatrix} \equiv \begin{Bmatrix} F_x & F_y \\ G_x & G_y \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ 3x^2-1 & 0 \end{Bmatrix}$$

(i) at critical point $(0,0)$:

$$\mathbf{A} = \begin{Bmatrix} 0 & 1 \\ -1 & 0 \end{Bmatrix} \qquad \Rightarrow p=0;\ q=1;$$

$\therefore (0,0)$ may be a stable spiral, center or unstable spiral.

(ii) at critical point $(1,0)$

$$\mathbf{A} = \begin{Bmatrix} 0 & 1 \\ 2 & 0 \end{Bmatrix} \qquad \Rightarrow p=0;\ q=-2;$$

$\therefore (1,0)$ is a saddle point (unstable).

(iii) at critical point $(-1,0)$

$$\mathbf{A} = \begin{Bmatrix} 0 & 1 \\ 2 & 0 \end{Bmatrix}$$

$\therefore (-1,0)$ is also a saddle point.

- Phase-plane method

the linearization method, when successful, can provide information on the local behavior of the solution near a critical point. If it is not the case, or if we wish to obtain the global view of the solution, the nonlinear system may be analyzed by the phase-plane method:

Ex. Determine the nature of the solution of the following system near critical point $(0,0)$.

$$\left.\begin{aligned} x' &= y \\ y' &= x^3 - x \end{aligned}\right\} \text{------}(1)$$

Sol.: $(1) \Rightarrow \dfrac{dy}{dx} = \dfrac{x^3 - x}{y} \Rightarrow y \cdot dy = (x^3 - x) \cdot dx$

$$\Rightarrow \frac{y^2}{2} = \frac{x^4}{4} - \frac{x^2}{2} + c$$

or $\boxed{y^2 = \frac{1}{2}(x^2 - 1)^2 + c}$ ------(2) c is arbitrary constant

If a point $(x_o,0)$, assume $0 < x_o < 1$

substitute this point into (2), we have:

$$c = -\frac{1}{2}(x_o{}^2 - 1)^2$$

the behavior of the solution near $(x_o,0)$ is:

$$y^2 = \frac{1}{2}(x^2 - 1)^2 - \frac{1}{2}(x_o{}^2 - 1)^2 \text{------}(3)$$

(i) when $x = \pm x_o$ → $y = 0$

(ii) when $|x| < x_o$ → y has two real values for every x.

therefore the solution of (1) is periodic near the point $(0,0)$,

→ $(0,0)$ is a center.

Ex. Lotka-Volterra Predator-Prey model

let $x \equiv$ the population of the Predator (say lynx 山貓)

$y \equiv$ the population of the Prey (say hare 野兔)

the first predator-prey model was constructed independently by A. Lotka (1925) and V. Volterra (1926):

$$\left.\begin{aligned} x' &= -a \cdot x + b \cdot xy \\ y' &= -c \cdot xy + d \cdot y \end{aligned}\right\} \text{------}(1)$$

Where a, b, c, d are positive constants

Note:

(i) when $y = 0 \to x' < 0$ i.e. $x = e^{-at} \to$ extinct

(ii) When $x = 0 \to y' > 0$ i.e. $y = e^{dt} \to$ Grows exponentially

(iii) $b \cdot xy; -c \cdot xy \to$ time rate change of x and y due to encounters.

Set $\left.\begin{aligned} -a \cdot x + b \cdot xy = 0 \\ -c \cdot xy + d \cdot y = 0 \end{aligned}\right\} \to \left.\begin{aligned} x(-a + b \cdot y) = 0 \\ y(-c \cdot x + d) = 0 \end{aligned}\right\}$

$\therefore$ The critical points of (1) are $(0,0)$ and $(\frac{d}{c}, \frac{a}{b})$

The Jacobian matrix of (1) is $\mathbf{A} = \begin{Bmatrix} -a + by & bx \\ -cy & -cx + d \end{Bmatrix}$

At $(0,0)$;

$$\mathbf{A}=\begin{Bmatrix}-a & 0\\ 0 & d\end{Bmatrix}\Rightarrow p=-a+d;\ \ q=-ad<0\ \Rightarrow (0,0) \text{ is a saddle point}$$

At $(\frac{d}{c},\frac{a}{b})$

$$\mathbf{A}=\begin{Bmatrix}0 & {bd}/{c}\\ -{ac}/{b} & 0\end{Bmatrix}\Rightarrow p=0;\ \ q=ad>0\ \Rightarrow (\frac{d}{c},\frac{a}{b}) \text{ may be stable or}$$

unstable spiral, or stable center.

It can be shown that $(\frac{d}{c},\frac{a}{b})$ is a center.

Ex. Van der Pol equation

$$y''-\mu(1-y^2)y'+y=0 \quad (\mu>0\ \text{ constant})------(1)$$

Note:

When $\mu=0$; (1)$\rightarrow$ $y''+y=0$ $\rightarrow$ harmonic oscillation.

When $\mu>0$; $-\mu(1-y^2)$ becomes “negative damping” when $y<1$ (i.e. small oscillation) and becomes “positive damping” when $y>1$ (i.e. larger oscillation).

let $V\equiv y'$ then (1)$\rightarrow$

$$\left.\begin{aligned} y'&=V\\ V'&=\mu(1-y^2)V-y\end{aligned}\right\}------(2)$$

$(y,V)=(0,0)$ is the only critical point of (2)

The Jacobian matrix of (2) is $\mathbf{A} = \begin{Bmatrix} 0 & 1 \\ -2\mu y V - 1 & \mu(1 - y^2) \end{Bmatrix}$

At $(y, V) = (0,0)$,

$$\mathbf{A} = \begin{Bmatrix} 0 & 1 \\ -1 & \mu \end{Bmatrix} \rightarrow p = \mu > 0;\ q = 1 > 0;\ \Delta = p^2 - 4q = \mu^2 - 4$$

When $\mu = 0$ the critical point $(0,0)$ is a center

When $\mu > 2$ the critical point $(0,0)$ is an unstable node

When $\mu < 2$ the critical point $(0,0)$ is an unstable spiral

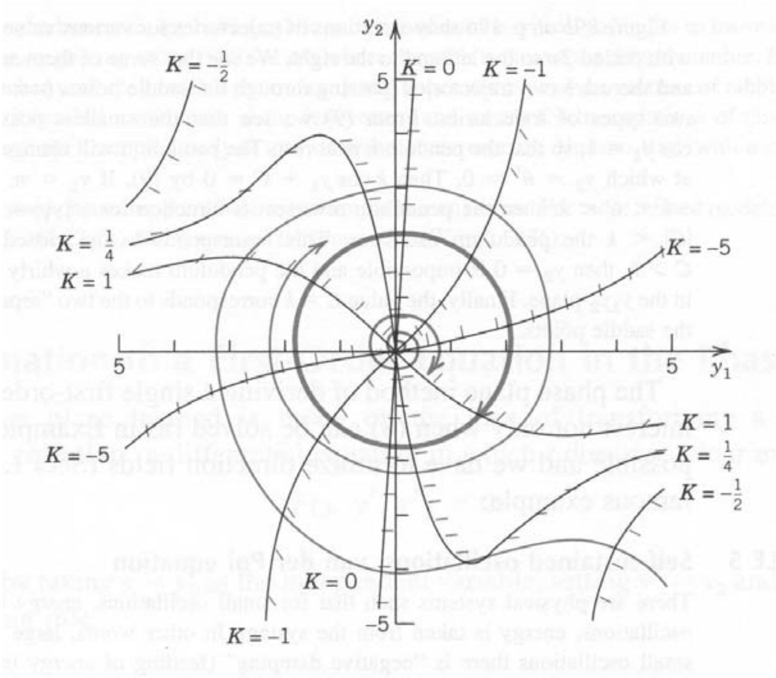


Fig. 92. Lineal element diagram for the van der Pol equation with $\mu = 0.1$ in the phase plane, showing also the limit cycle and two trajectories

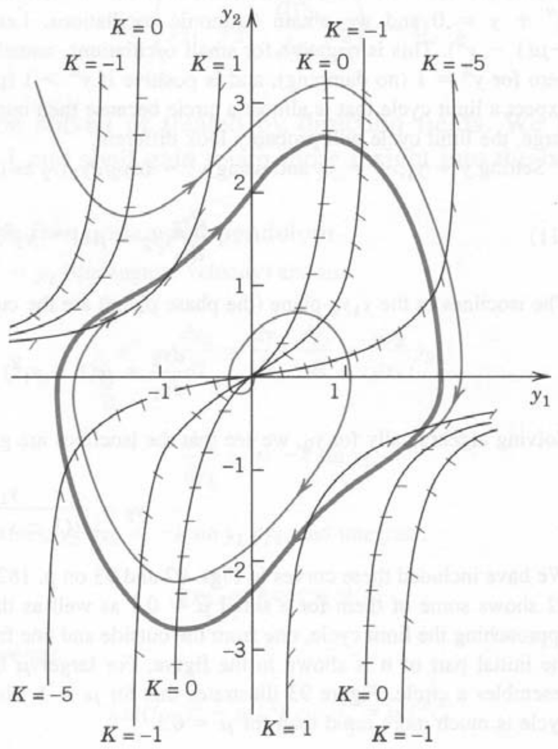


Fig. 93. Lineal element diagram for the van der Pol equation with $\mu = 1$ in the phase plane, showing also the limit cycle and two trajectories approaching it

4.7 non-homogeneous linear systems

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \tag{1}$$

$\mathbf{y}^{(h)}(t)$ is the general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and

$\mathbf{y}^{(p)}(t)$ is the particular solution of (1)

the **general solution** of (1) is $\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)}$

※ method of undetermined coefficients

Example: $$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix} \tag{2}$$

solve: (a) homogeneous solution:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\mathbf{y} \quad , \quad \text{try} \quad \mathbf{y} = \mathbf{x}\, e^{\lambda t}$$

$$\Rightarrow \mathbf{A}\mathbf{x} = \lambda\,\mathbf{x} \quad \Rightarrow \det(\mathbf{A} - \lambda\,\mathbf{I}) = \begin{vmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^{2} + \lambda - 2 = 0 \quad \Rightarrow \quad \lambda = -2,\, \lambda = 1$$

eigenvectors are $\lambda = 1 \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\quad \lambda = -2 \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\mathbf{y}^{(h)}(t) = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

(b) particular solution:

the form of **g** suggests to assume $\mathbf{y}^{(p)}$ in the form

$$\mathbf{y}^{(p)}(t) = \mathbf{u} + \mathbf{v}\,t + \mathbf{w}\,t^{2}$$

$$(2) \quad \Rightarrow \mathbf{v}t + 2\mathbf{w}\,t = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v}\,t + \mathbf{A}\mathbf{w}\,t^{2} + \mathbf{g}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 2w_1 t \\ 2w_2 t \end{bmatrix} = \begin{bmatrix} 2u_1 - 4u_2 \\ u_1 - 3u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 - 4v_2 \\ v_1 - 3v_2 \end{bmatrix} t + \begin{bmatrix} 2w_1 - 4w_2 \\ w_1 - 3w_2 \end{bmatrix} t^{2} + \begin{bmatrix} 2t^{2} + 10t \\ t^{2} + 9t + 3 \end{bmatrix}$$

for t^2 term: $\quad \Rightarrow w_1 = -1, \quad w_2 = 0$

for t term: $\quad \Rightarrow v_1 = 0, \quad v_2 = 3$

for constant term: $\quad \Rightarrow u_1 = 0, \quad u_2 = 0$

general solution: $\mathbf{y} = \mathbf{y}^{(h)}(t) + \mathbf{y}^{(p)}(t) = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -t^{2} \\ 3t \end{bmatrix}$

※ Method of Variation of Parameters

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \qquad (3)$$

if $\mathbf{y}^{(h)}(t)$ is the general solution of the homogeneous system

$$\mathbf{y}^{(h)} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \cdots + c_n\mathbf{y}^{(n)}, \qquad \text{rewrite as}$$

$$\mathbf{y}^{(h)} = \begin{bmatrix} c_1 y_1^{(1)} + c_2 y_1^{(2)} + \cdots + c_n y_1^{(n)} \\ \vdots \\ c_1 y_n^{(1)} + c_2 y_n^{(2)} + \cdots + c_n y_n^{(n)} \end{bmatrix} = \begin{bmatrix} y_1^{(1)} & y_1^{(2)} \cdots y_1^{(n)} \\ & \vdots \\ y_n^{(1)} & y_n^{(2)} \cdots y_n^{(n)} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{Y}(t)\mathbf{c}$$

$\mathbf{Y}(t)$: fundamental matrix

If we replace the constant vector $\mathbf{c}$ by a variable vector $\mathbf{u}(t)$

$$\mathbf{y}^{(p)} = \mathbf{Y}(t)\mathbf{u}(t)$$

$$(3) \Rightarrow \quad \mathbf{Y}'\mathbf{u} + \mathbf{Y}\mathbf{u}' = \mathbf{A}\mathbf{Y}\mathbf{u} + \mathbf{g},$$

since $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \cdots, \mathbf{y}^{(n)}$ are homogeneous solutions

$$\mathbf{y}^{(1)\prime} = \mathbf{A}\mathbf{y}^{(1)},\ \mathbf{y}^{(2)\prime} = \mathbf{A}\mathbf{y}^{(2)},\ \cdots,\ \mathbf{y}^{(n)\prime} = \mathbf{A}\mathbf{y}^{(n)} \quad \Rightarrow \mathbf{Y}' = \mathbf{A}\mathbf{Y}$$

hence $\quad \mathbf{Y}'\mathbf{u} = \mathbf{A}\mathbf{Y}\mathbf{u} \quad \Rightarrow \quad \mathbf{Y}\mathbf{u}' = \mathbf{g} \quad \Rightarrow \quad \mathbf{u}' = \mathbf{Y}^{-1}\mathbf{g}$

since $\mathbf{Y}$ is fundamental matrix $\rightarrow$ $\mathbf{Y}$ is nonsingular, i.e. $\mathbf{Y}^{-1}$ exists

$$\Rightarrow \mathbf{u}(t) = \int_{t_0}^{t} \mathbf{Y}^{-1}(\tilde{t})\,\mathbf{g}(\tilde{t})\, d\,\tilde{t} + \mathbf{c}$$

for $\mathbf{c} = 0$, we get the particular solution

$$\mathbf{y}^{(\mathrm{p})} = \mathbf{Y}\mathbf{u} = \mathbf{Y}\int_{t_0}^{t} \mathbf{Y}^{-1}(\tilde{t})\,\mathbf{g}(\tilde{t})\, d\,\tilde{t}$$

Example: $\quad \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t} \qquad (4)$

$$\mathbf{y}^{(h)}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \qquad (5)$$

A. method of undetermined coefficients

since $\lambda = -2$ is a eigenvalue of **A**, we must assume

$$\mathbf{y}^{(p)} = \mathbf{u}\, t e^{-2t} + \mathbf{v} e^{-2t}$$

$$(5) \quad \Rightarrow \mathbf{u}\; e^{-2t} - 2\mathbf{u}\, t e^{-2t} - 2\mathbf{v} e^{-2t} = \mathbf{Au}\, t e^{-2t} + \mathbf{Av} e^{-2t} + \mathbf{g}$$

for te^{-2t} term: $\Rightarrow -2\mathbf{u} = \mathbf{Au} \quad \Rightarrow \quad \mathbf{u} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}, a \neq 0$

for the other term:

$$\Rightarrow \; \mathbf{u} - 2\mathbf{v} = \mathbf{Av} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \quad \Rightarrow \quad (\mathbf{A} + 2\mathbf{I})\mathbf{v} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$\begin{aligned} -v_1 + v_2 &= a + 6 \\ v_1 - v_2 &= a - 2 \end{aligned} \quad \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a+6 \\ a-2 \end{bmatrix}$$

since $\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$, for nontrivial solution of $\mathbf{v}$

$$\Rightarrow \begin{vmatrix} a+6 & 1 \\ a-2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & a+6 \\ 1 & a-2 \end{vmatrix} = 0$$

therefore a = - 2, then $\Rightarrow v_2 = v_1 + 4$, say $v_1 = k \Rightarrow v_2 = k + 4$

$$\therefore \mathbf{v} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

choose k = 0

$$\Rightarrow \mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-2t} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-2t}$$

choose k = -2

$$\Rightarrow \mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$

B. Method of Variation of Parameters

From (5) and (6)

$$\mathbf{Y} = \left[\mathbf{y}^{(1)} \quad \mathbf{y}^{(2)}\right] = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\Rightarrow \mathbf{Y^{-1}} = \frac{1}{-2e^{-6t}} \begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}$$

$$\Rightarrow \mathbf{u}' = \mathbf{Y^{-1}g} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$$\Rightarrow \mathbf{u}(t) = \int_0^t \begin{bmatrix} -2 \\ -4e^{2\tilde{t}} \end{bmatrix} d\tilde{t} = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$\Rightarrow \mathbf{Yu} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix} = \begin{bmatrix} -2t-2 \\ -2t+2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}$$

$$\Rightarrow \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$