PROBLEM SET 3.1

1–6 BASES: TYPICAL EXAMPLES

To get a feel for higher order ODEs, show that the given functions are solutions and form a basis on any interval. Use Wronskians. In Prob. 6, $x > 0$,

1. $1, x, x^2, x^3, \quad y^{\text{iv}} = 0$
2. $e^x, e^{-x}, e^{x/2}, \quad 2y''' - y'' - 2y' + y = 0$
3. $\cos x, \sin x, x\cos x, x\sin x, \quad y^{\text{iv}} + 2y'' + y = 0$
4. $e^{-4x}, xe^{-4x}, x^2e^{-4x}, \quad y''' + 12y'' + 48y' + 64y = 0$
5. $1, e^{-2x}\cos x, e^{-2x}\sin x, \quad y''' + 4y'' + 5y' = 0$
6. $1, x^2, x^4, \quad x^2y''' - 3xy'' + 3y' = 0$
7. **TEAM PROJECT. General Properties of Solutions of Linear ODEs.** These properties are important in obtaining new solutions from given ones. Therefore extend Team Project 38 in Sec. 2.2 to nth-order ODEs. Explore statements on sums and multiples of solutions of (1) and (2) systematically and with proofs. Recognize clearly that no new ideas are needed in this extension from $n = 2$ to general n.

8–15 LINEAR INDEPENDENCE

Are the given functions linearly independent or dependent on the half-axis $x \geq 0$? Give reason.

8. $x^2, 1/x^2, 0$
9. $\tan x, \cot x, \frac{\pi}{4}$
10. $e^{2x}, xe^{2x}, x^2e^{2x}$
11. $e^{-x}\cos x, e^{-x}\sin x, e^{-x}$
12. $\sin^2 x, \cos^2 x, \cos 2x$
13. $\sin x, \cos x, \sin 2x$
14. $\cos^2 x, \sin^2 x, 2\pi$
15. $\cosh x, \sinh x, e^x$
16. **TEAM PROJECT. Linear Independence and Dependence.** (a) Investigate the given question about a set S of functions on an interval I. Give an example. Prove your answer.

(1) If S contains the zero function, can S be linearly independent?

(2) If S is linearly independent on a subinterval J of I, is it linearly independent on I?

(3) If S is linearly dependent on a subinterval J of I, is it linearly dependent on I?

(4) If S is linearly independent on I, is it linearly independent on a subinterval J?

(5) If S is linearly dependent on I, is it linearly independent on a subinterval J?

(6) If S is linearly dependent on I, and if T contains S, is T linearly dependent on I?

(b) In what cases can you use the Wronskian for testing linear independence? By what other means can you perform such a test?

PROBLEM SET 3.2

1–6 GENERAL SOLUTION

Solve the given ODE. Show the details of your work.

1. $y''' + 9y' = 0$
2. $y^{\text{iv}} + 2y'' + y = 0$
3. $y^{\text{iv}} + 16y'' = 0$
4. $(D^3 - D^2 - D + I)y = 0$
5. $(D^4 + 10D^2 + 9I)y = 0$
6. $(D^5 + 2D^3 + D)y = 0$

7–13 INITIAL VALUE PROBLEM

Solve the IVP by a CAS, giving a general solution and the particular solution and its graph.

7. $y''' + 3y'' + 4.81y' = 0, \quad y(0) = 6, \quad y'(0) = -3.15, \quad y''(0) = -12.195$
8. $y''' + 7.5y'' + 14.25y' - 9.125y = 0, \quad y(0) = 10.05, \quad y'(0) = -54.975, \quad y''(0) = 257.5125$
9. $4y''' + 8y'' + 41y' + 37y = 0, \quad y(0) = 9, \quad y'(0) = -6.5, \quad y''(0) = -39.75$
10. $y^{\text{iv}} + 4y = 0, \quad y(0) = \frac{1}{2}, \quad y'(0) = -\frac{3}{2}, \quad y''(0) = \frac{5}{2}, \quad y'''(0) = -\frac{7}{2}$
11. $y^{\text{iv}} - 9y'' - 400y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 41, \quad y'''(0) = 0$
12. $y^{\text{v}} - 5y''' + 4y' = 0, \quad y(0) = 3, \quad y'(0) = -5, \quad y''(0) = 11, \quad y'''(0) = -23, \quad y^{\text{iv}}(0) = 47$
13. $y^{\text{iv}} + 0.35y''' + 3.85y'' + 1.4y' - 0.6y = 0, \quad y(0) = 7.4125, \quad y'(0) = 0.51, \quad y''(0) = 0.849, \quad y'''(0) = -2.3831$
14. **PROJECT. Reduction of Order.** This is of practical interest since a single solution of an ODE can often be guessed. For second order, see Example 7 in Sec. 2.1.

(a) How could you reduce the order of a linear constant-coefficient ODE if a solution is known?

(b) Extend the method to a variable-coefficient ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0.$$

Assuming a solution y_1 to be known, show that another solution is $y_2(x) = u(x)y_1(x)$ with $u(x) = \int z(x)\,dx$ and z obtained by solving

$$y_1z'' + (3y_1' + p_2y_1)z' + (3y_1'' + 2p_2y_1' + p_1y_1)z = 0.$$

(c) Reduce

$$x^3y''' - 3x^2y'' + (6 - x^2)xy' - (6 - x^2)y = 0,$$

using $y_1 = x$ (perhaps obtainable by inspection).

15. **CAS EXPERIMENT. Reduction of Order.** Starting with a basis, find third-order linear ODEs with variable coefficients for which the reduction to second order turns out to be relatively simple.

PROBLEM SET 3.3

1–7 GENERAL SOLUTION

Solve the following ODEs, showing the details of your work.

1. $y''' - 3y'' + 3y' - y = e^x - x - 1$
2. $y''' + 2y'' - y' - 2y = 1 - 4x^3$
3. $(D^4 + 5D^2 + 4I)y = 3.5\sinh 2x$
4. $(D^3 + 3D^2 - 5D - 39I)y = -300\cos x$
5. $(x^3D^3 + 2x^2D^2 - xD + I)y = x^{-2}$
6. $(D^3 + 4D)y = \sin x$
7. $(D^3 - 3D^2 + 3D - I)y = 4\cos x$

8–13 INITIAL VALUE PROBLEM

Solve the given IVP, showing the details of your work.

8. $y^{\text{iv}} - 5y'' + 4y = 10e^{-3x}, \quad y(0) = 1, y'(0) = 0, \quad y''(0) = 0, y'''(0) = 0$
9. $y^{\text{iv}} + 5y'' + 4y = 90\sin 4x, \quad y(0) = 1, y'(0) = 2, \quad y''(0) = -1, y'''(0) = -32$
10. $x^3y''' + xy' - y = x^2, \quad y(1) = 1, y'(1) = 3, \quad y''(1) = 14$
11. $(D^3 + D^2 - 2D)y = 4e^{-2x}/\cos x, \quad y(0) = 0.4, \quad y'(0) = -0.4, y''(0) = -0.4$
12. $(D^3 - 2D^2 - 9D + 18I)y = e^{2x}, \quad y(0) = 4.5, \quad y'(0) = 8.8, y''(0) = 17.2$
13. $(D^3 - 4D)y = 10\cos x + 5\sin x, \quad y(0) = 3, \quad y'(0) = -2, y''(0) = -1$
14. **CAS EXPERIMENT. Undetermined Coefficients.** Since variation of parameters is generally complicated, it seems worthwhile to try to extend the other method. Find out experimentally for what ODEs this is possible and for what not. *Hint:* Work backward, solving ODEs with a CAS and then looking whether the solution could be obtained by undetermined coefficients. For example, consider

$$y''' - 3y'' + 3y' - y = x^{1/2}e^x$$

and

$$x^3y''' + x^2y'' - 2xy' + 2y = x^3\ln x.$$

15. **WRITING REPORT. Comparison of Methods.** Write a report on the method of undetermined coefficients and the method of variation of parameters, discussing and comparing the advantages and disadvantages of each method. Illustrate your findings with typical examples. Try to show that the method of undetermined coefficients, say, for a third-order ODE with constant coefficients and an exponential function on the right, can be derived from the method of variation of parameters.