Chapter 3: Pressure and Fluid Statics

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Pressure

- **Pressure** is defined as a *normal force exerted by a fluid per unit area*.
- Units of pressure are N/m^2, which is called a **pascal** (Pa).
- Since the unit Pa is too small for pressures encountered in practice, *kilopascal* (1 kPa $= 10^3$ Pa) and *megapascal* (1 MPa $= 10^6$ Pa) are commonly used.
- Other units include *bar*, *atm, kgf/cm²,* *lbf/in²=psi*.

Pressure

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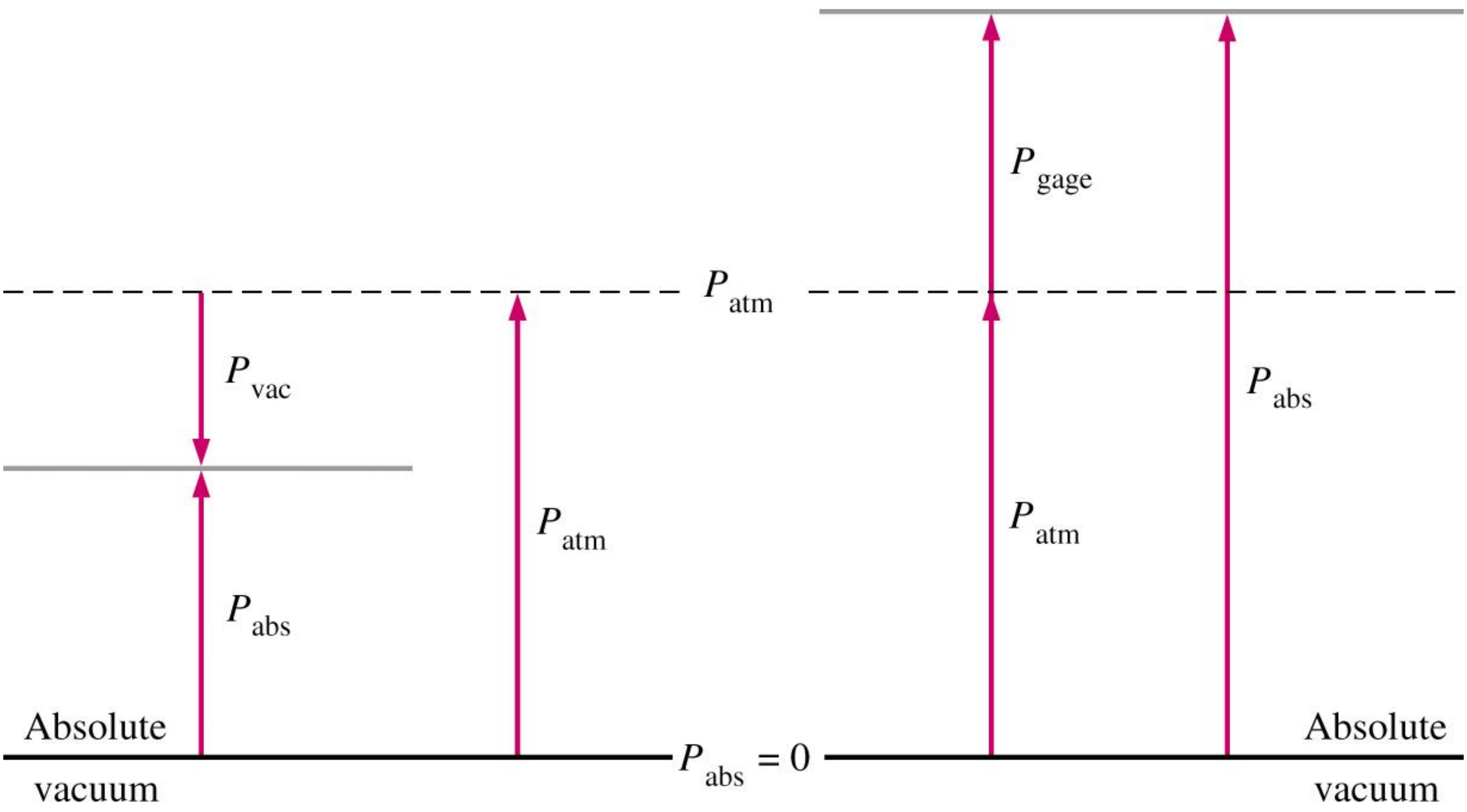
- 1 bar = 10^5 Pa = 0.1 MPa = 100 kPa
- 1 atm = 101,325 Pa = 101.325 kPa = 1.01325 bars
- 1 kgf/cm^2 = 9.807 N/cm^2 = 9.807×10^4 N/m^2 = 9.807×10^4 Pa = 0.9807 bar = 0.9679 atm
- 1 atm = 14.696 psi.
- 1 kgf/cm^2 = 14.223 psi.

Absolute, gage, and vacuum pressures

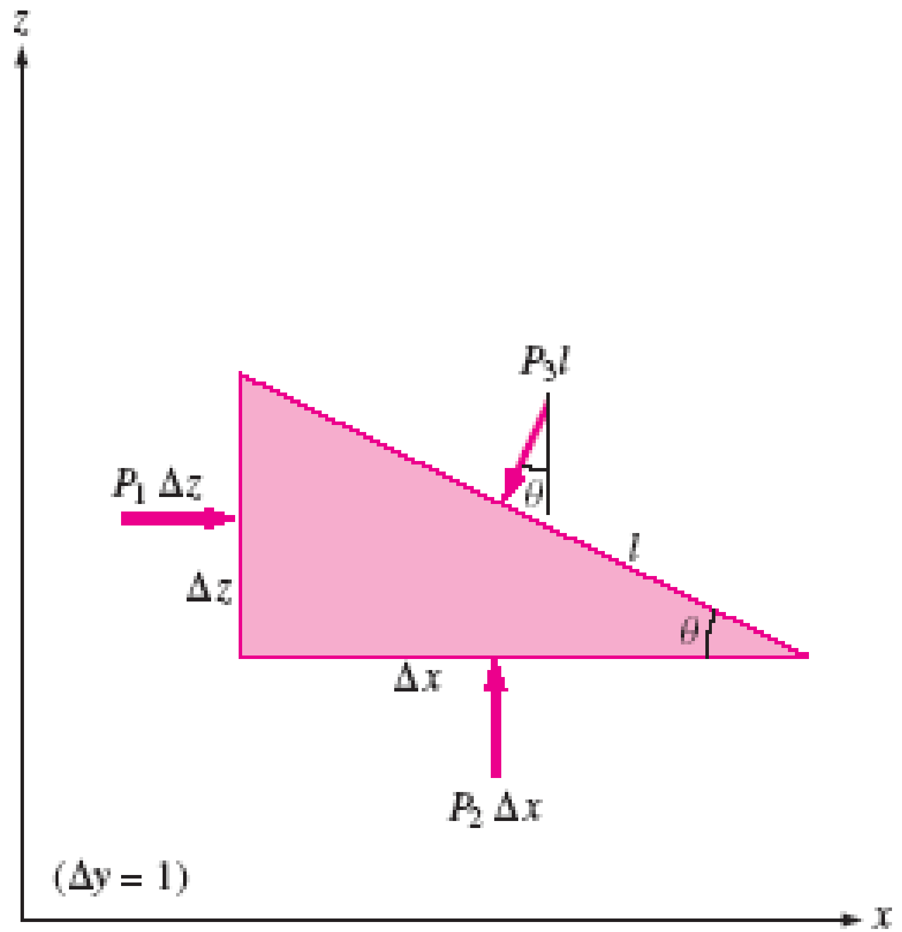
- Actual pressure at a given point is called the **absolute pressure**.
- Most pressure-measuring devices are calibrated to read zero in the atmosphere, and therefore indicate **gage pressure**, $P_{gage} = P_{abs} - P_{atm}$.
- Pressure below atmospheric pressure are called **vacuum pressure**, $P_{vac} = P_{atm} - P_{abs}$.

Absolute, gage, and vacuum pressures

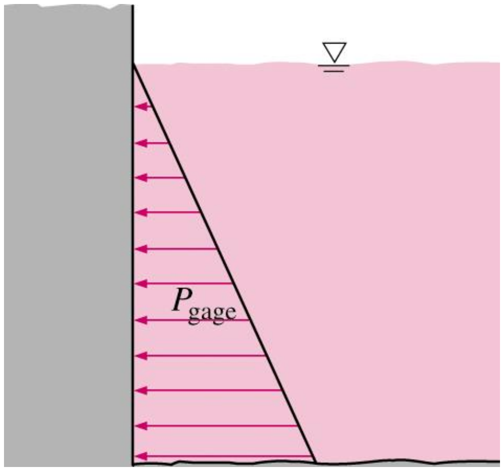


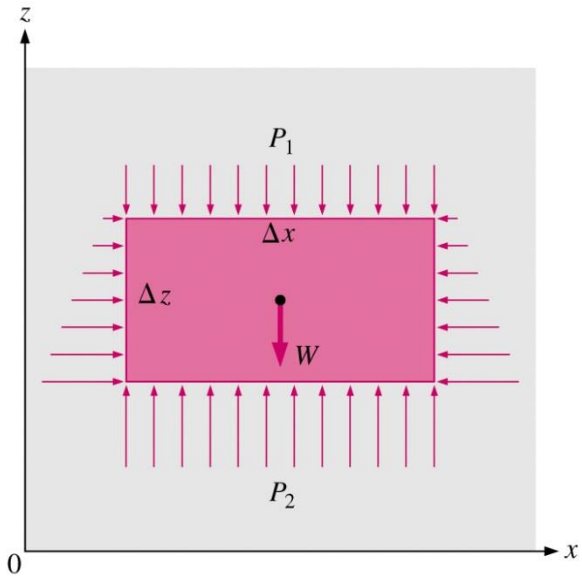
Pressure at a Point

- Pressure at any point in a fluid is the same in all directions.
- Pressure has a magnitude, but not a specific direction, and thus it is a scalar quantity.
- Proof on the Blackboard



Variation of Pressure with Depth





- In the presence of a gravitational field, pressure increases with depth because more fluid rests on deeper layers.
- To obtain a relation for the variation of pressure with depth, consider rectangular element
  - Force balance in *z*-direction gives

$$\sum F_z = ma_z = 0$$

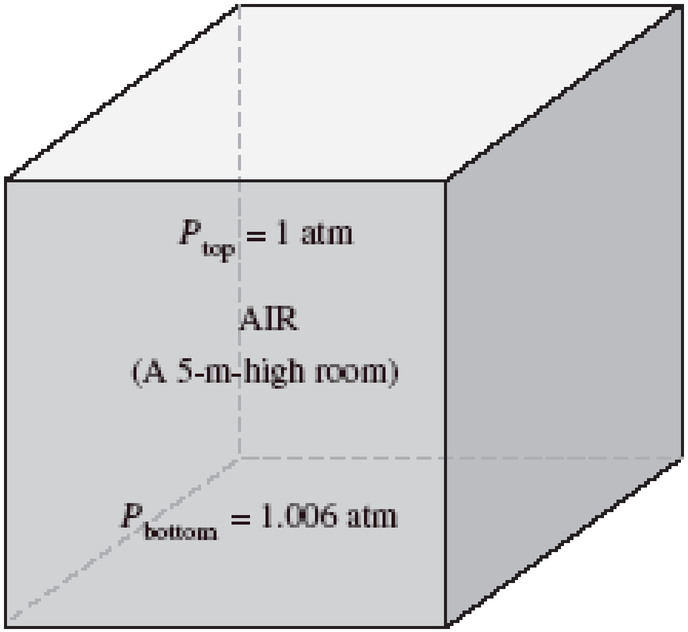
$$P_2 \Delta x - P_1 \Delta x - \rho g \Delta x \Delta z = 0$$

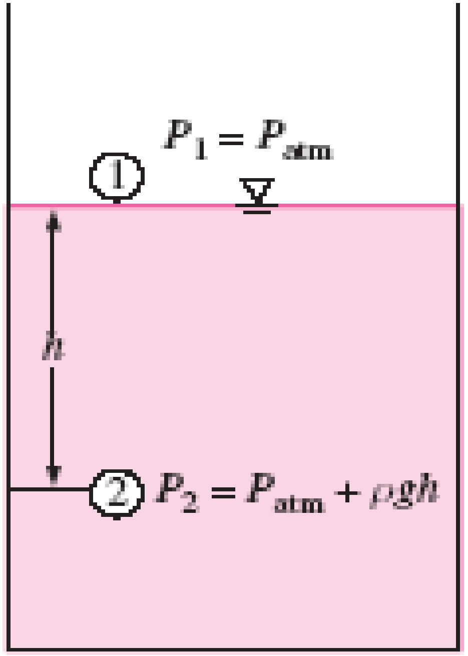
  - Dividing by *Δx* and rearranging gives

$$\boxed{\Delta P = P_2 - P_1 = \rho g \Delta z = \gamma_s \Delta z}$$

- *Δz* is called the *pressure head*

Variation of Pressure with Depth

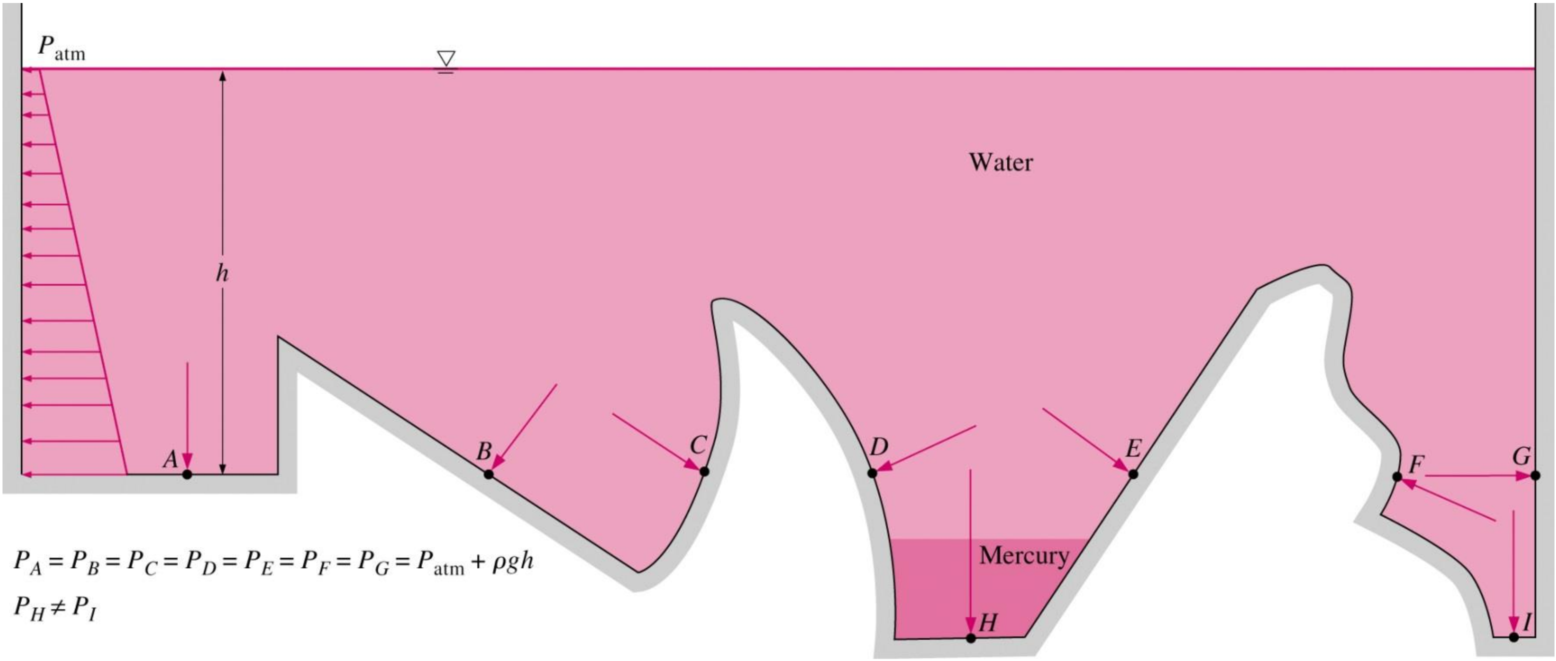




$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g \, dz$$

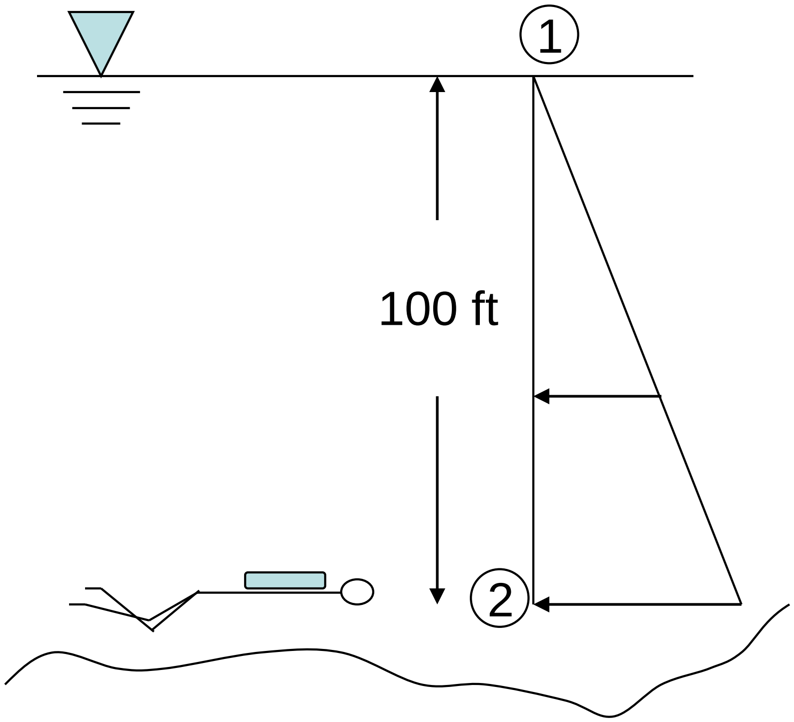
Variation of Pressure with Depth

- Pressure in a fluid at rest is independent of the shape of the container.
- Pressure is the same at all points on a horizontal plane in a given fluid.



Scuba Diving and Hydrostatic Pressure

Scuba Diving and Hydrostatic Pressure



- Pressure on diver at 100 ft?

$$P_{gage,2} = \rho g z = \left(998\frac{kg}{m^3}\right)\left(9.81\frac{m}{s^2}\right)(100\,ft)\left(\frac{1m}{3.28\,ft}\right)$$

$$= 298.5kPa\left(\frac{1atm}{101.325kPa}\right) = 2.95atm$$

$$P_{abs,2} = P_{gage,2} + P_{atm} = 2.95atm + 1atm = \boxed{3.95atm}$$

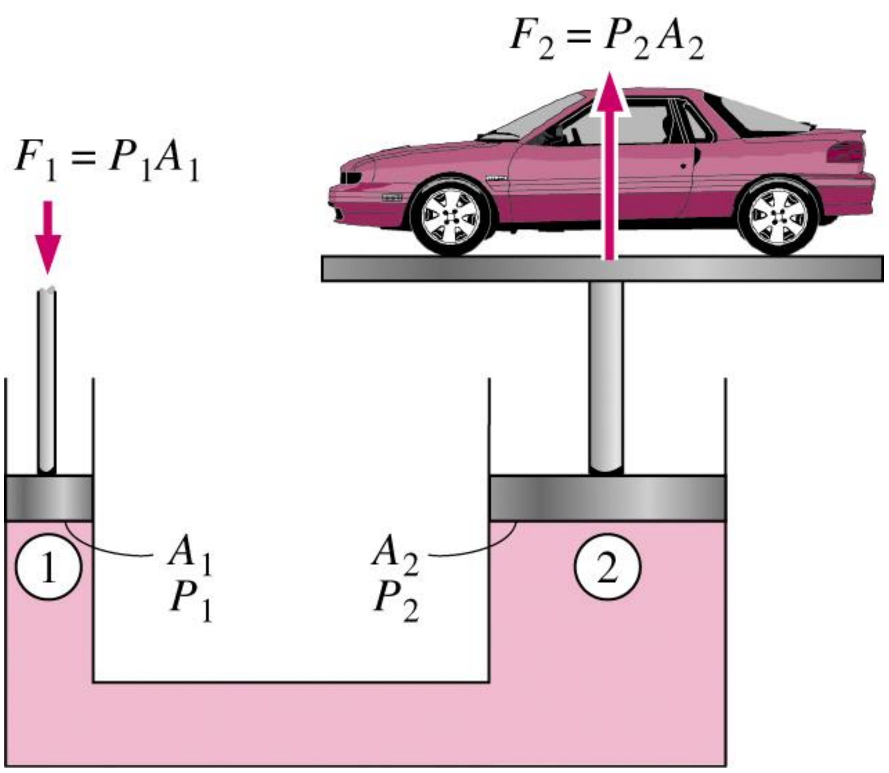
- Danger of emergency ascent?

$$P_1V_1 = P_2V_2 \qquad \text{Boyle's law}$$

$$\frac{V_1}{V_2} = \frac{P_2}{P_1} = \frac{3.95atm}{1atm} \approx 4$$

If you hold your breath on ascent, your lung volume would increase by a factor of 4, which would result in embolism and/or death.

Pascal's Law

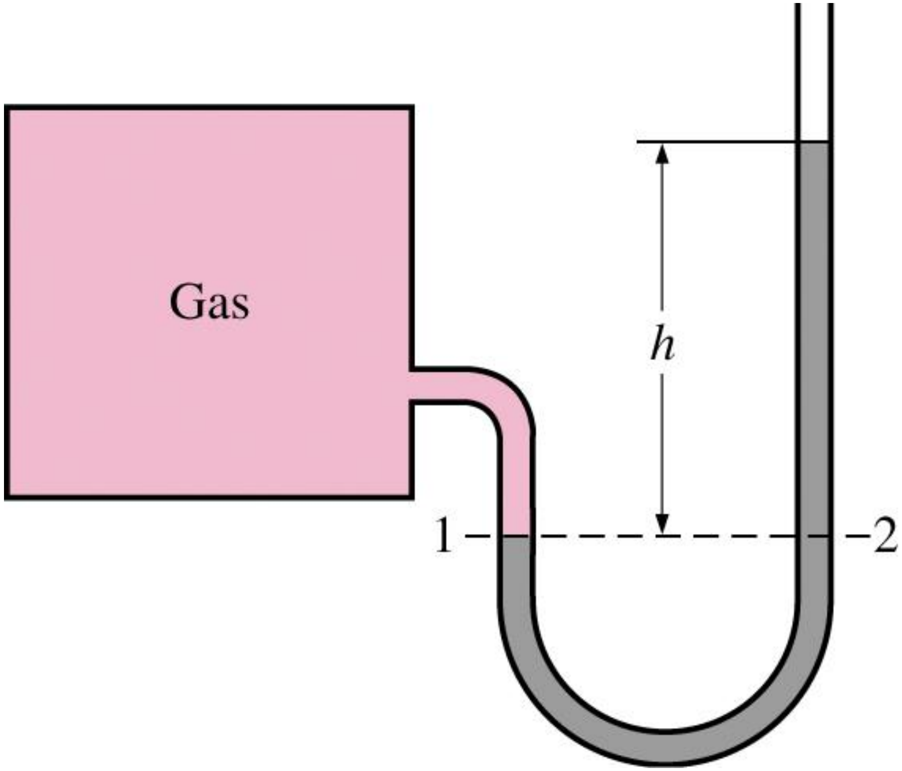


- Two points at the same elevation in a continuous fluid at rest are at the same pressure, called *Pascal's law,*
- Pressure applied to a confined fluid increases the pressure throughout by the same amount.
- In picture, pistons are at same height:

$$P_1 = P_2 \rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2} \rightarrow \frac{F_2}{F_1} = \frac{A_2}{A_1}$$

- Ratio A_2/A_1 is called *ideal mechanical advantage*

The Manometer

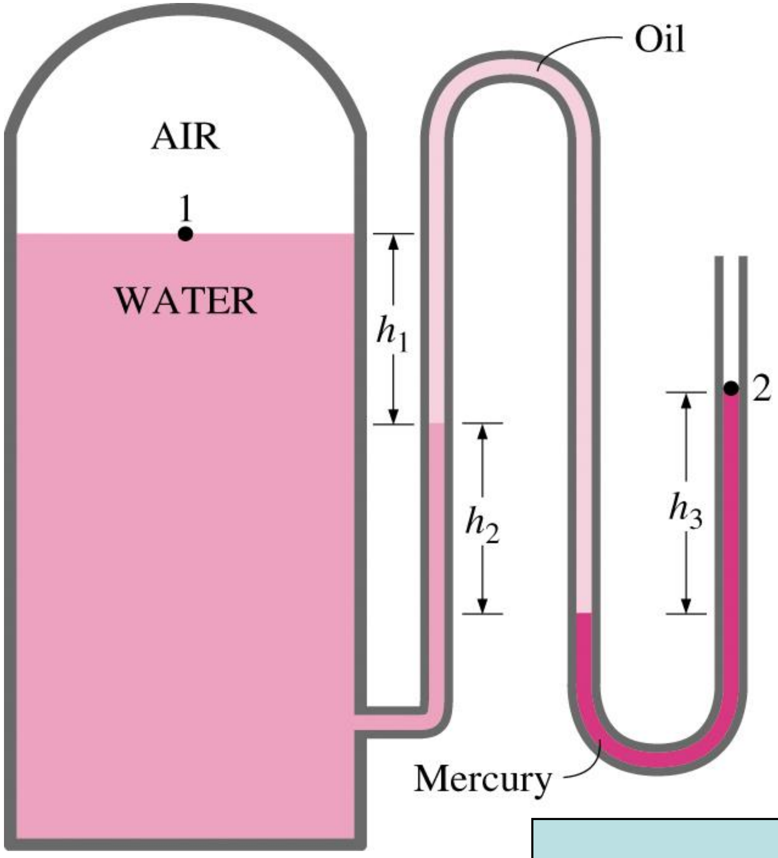


$$P_1 = P_2$$

$$P_2 = P_{atm} + \rho g h$$

- An elevation change of Δz in a fluid at rest corresponds to $\Delta P/\rho g$.
- A device based on this is called a **manometer.**
- A manometer consists of a U-tube containing one or more fluids such as mercury, water, alcohol, or oil.
- Heavy fluids such as mercury are used if large pressure differences are anticipated.

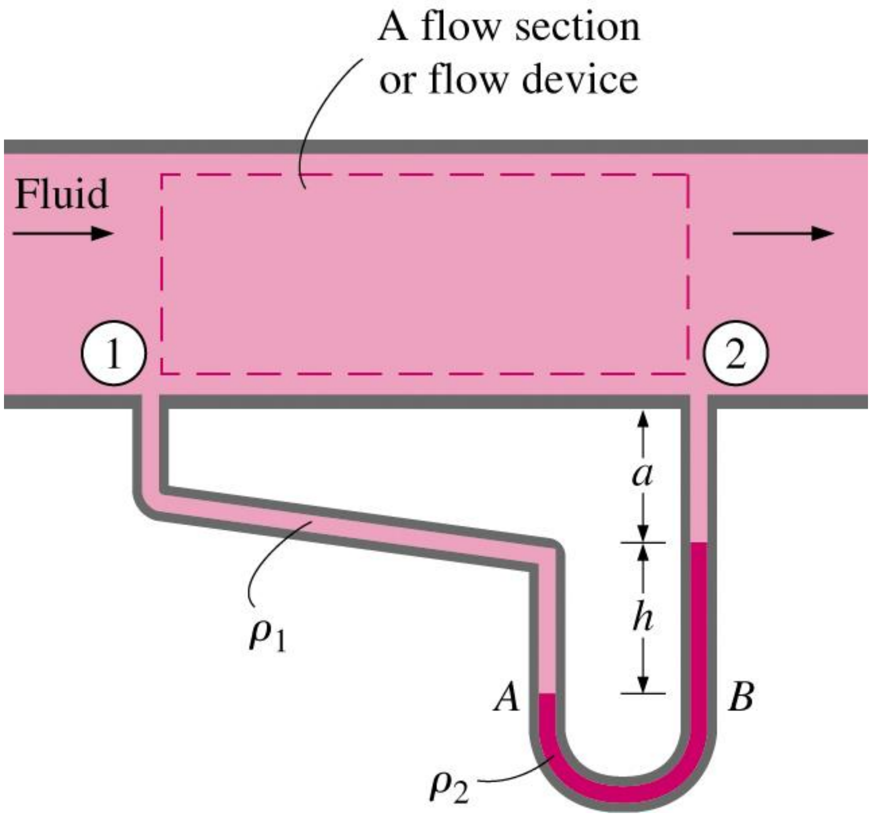
Mutlifluid Manometer



- For multi-fluid systems
- Pressure change across a fluid column of height h is $\Delta P = \rho g h$.
- Pressure increases downward, and decreases upward.
- Two points at the same elevation in a continuous fluid are at the same pressure.
- Pressure can be determined by adding and subtracting $\rho g h$ terms.

$$P_1 = P_{atm} - \rho_w g h_1 - \rho_o g h_2 + \rho_m g h_3$$

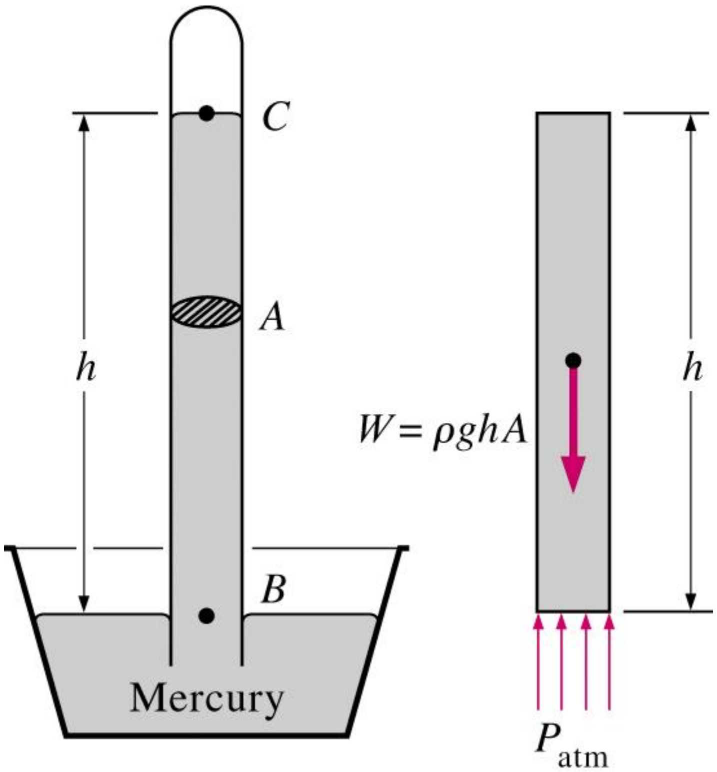
Measuring Pressure Drops



$$P_1 + \rho_1 g(a + h) - \rho_2 gh - \rho_1 ga = P_2$$

- Manometers are well-- suited to measure pressure drops across valves, pipes, heat exchangers, etc.
- Relation for pressure drop P_1-P_2 is obtained by starting at point 1 and adding or subtracting ρgh terms until we reach point 2.
- If fluid in pipe is a gas, $\rho_2 >> \rho_1$ and P_1-$P_2 \cong \rho gh$ (Mistyped on page 73)

The Barometer



$$P_C + \rho g h = P_{atm}$$

$$P_{atm} = \rho g h$$

- Atmospheric pressure is measured by a device called a **barometer**; thus, atmospheric pressure is often referred to as the *barometric pressure*.
- P_C can be taken to be zero since there is only Hg vapor above point C, and it is very low relative to P_{atm}.
- Change in atmospheric pressure due to elevation has many effects: Cooking, nose bleeds, engine performance, aircraft performance.

The Barometer

- *Standard atmosphere* is defined as the pressure produced by a column of mercury 760 mm (29.92 inH_g or of water about 10.3 m) in height at 0°C (ρ_{Hg} = 13,595 kg/m^3) under standard gravitational acceleration (g = 9.807 m/s^2).

- 1 atm = 760 torr and 1 torr = 133.3 Pa

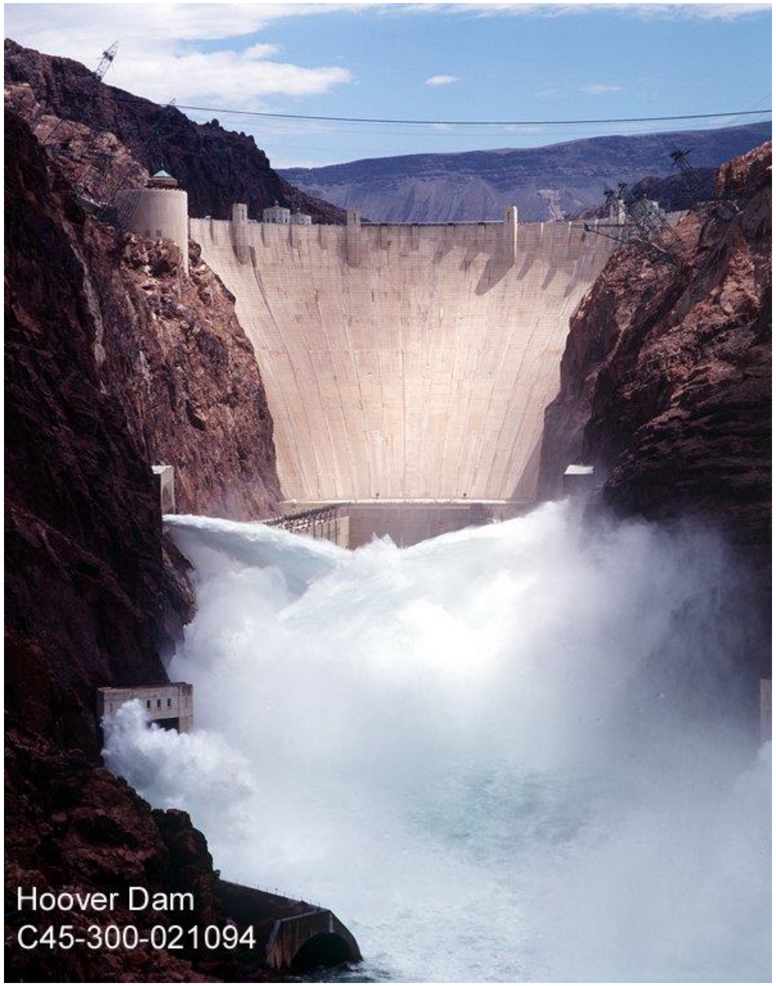
Fluid Statics

- **Fluid Statics** deals with problems associated with fluids at rest.
- In fluid statics, there is no relative motion between adjacent fluid layers.
- Therefore, there is no shear stress in the fluid trying to deform it.
- The only stress in fluid statics is *normal stress*
  - Normal stress is due to pressure
  - Variation of pressure is due only to the weight of the fluid $\rightarrow$ fluid statics is only relevant in presence of gravity fields.
- Applications: Floating or submerged bodies, water dams and gates, liquid storage tanks, etc.

Hoover Dam

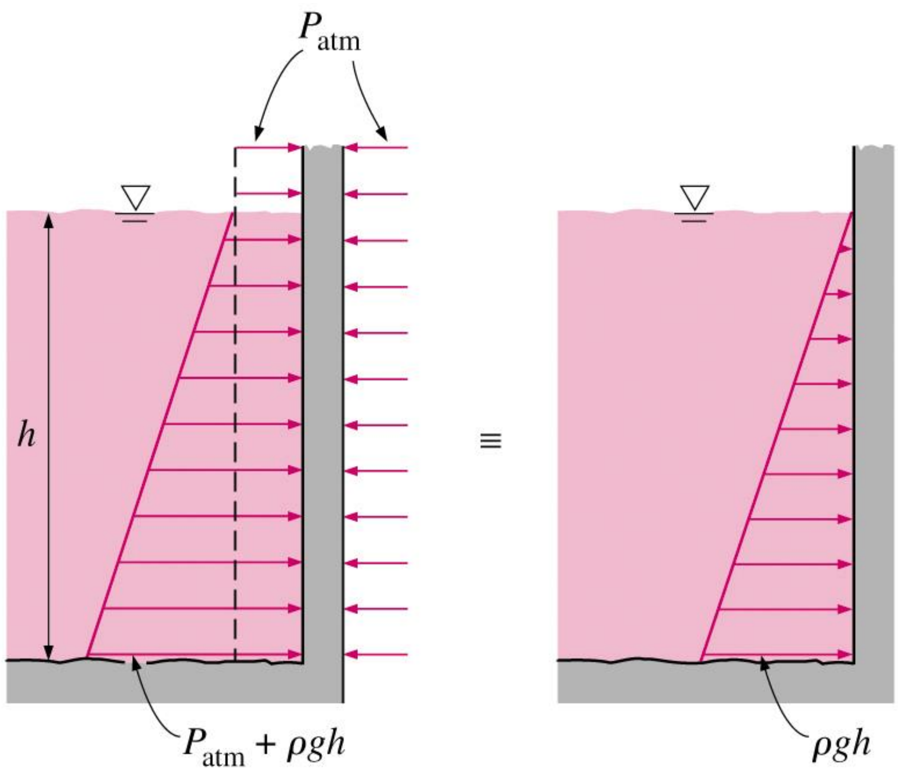
Hoover Dam

Hoover Dam



- Example of elevation head *z* converted to velocity head $V^2/2g$. We'll discuss this in more detail in Chapter 5 (Bernoulli equation).

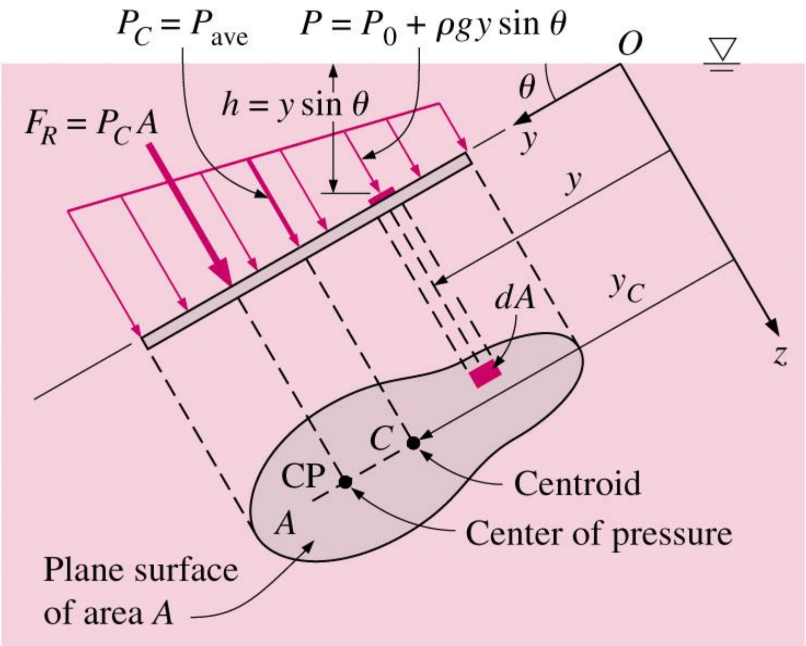
Hydrostatic Forces on Plane Surfaces

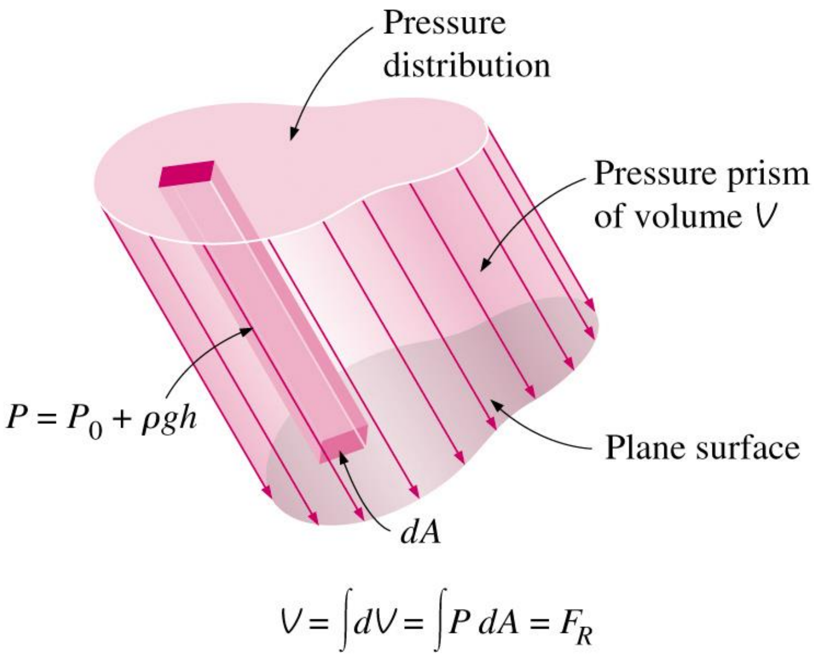


(*a*) P_{atm} considered (*b*) P_{atm} subtracted

- On a *plane* surface, the hydrostatic forces form a system of parallel forces
- For many applications, magnitude and location of application, which is called **center of pressure**, must be determined.
- Atmospheric pressure P_{atm} can be neglected when it acts on both sides of the surface.

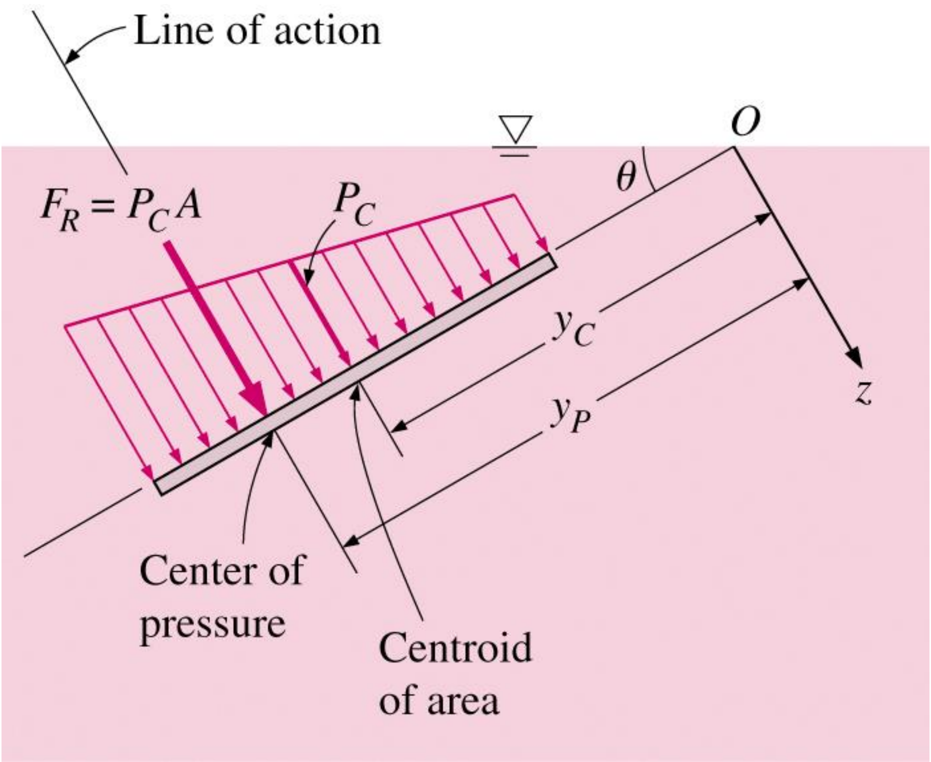
Resultant Force





The magnitude of F_R acting on a plane surface of a completely submerged plate in a homogenous fluid is equal to the product of the pressure P_C at the centroid of the surface and the area A of the surface

Center of Pressure

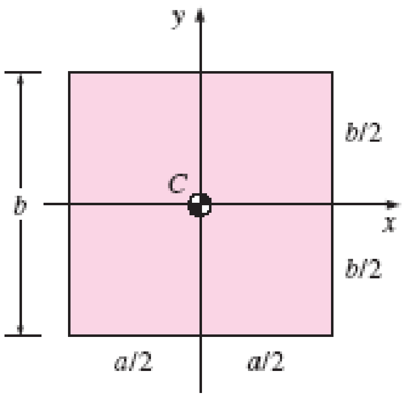


- Line of action of resultant force $F_R=P_CA$ does not pass through the centroid of the surface. In general, it lies underneath where the pressure is higher.
- Vertical location of **Center of Pressure** is determined by equation the moment of the resultant force to the moment of the distributed pressure force.

$$y_p = y_C + \frac{I_{xx,C}}{y_c A}$$

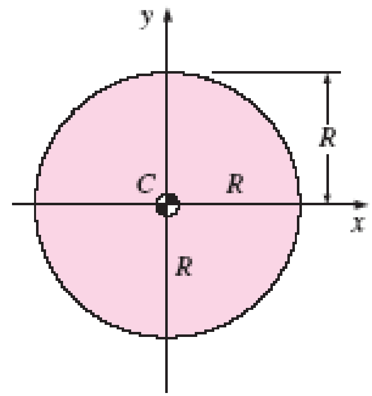
- $I_{xx,C}$ is tabulated for simple geometries.
- Derivation of F_R and examples on blackboard

The centroidal moments of inertia for some common geometries



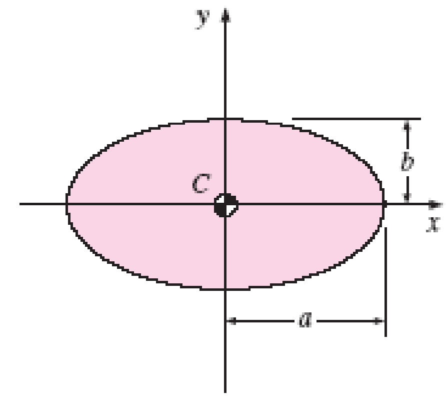
$A = ab,\ I_{xx,\,C} = ab^3/12$

(*a*) Rectangle



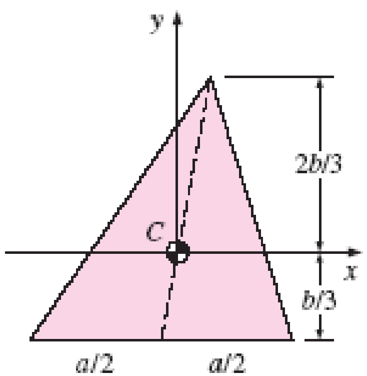
$A = \pi R^2,\ I_{xx,\,C} = \pi R^4/4$

(*b*) Circle



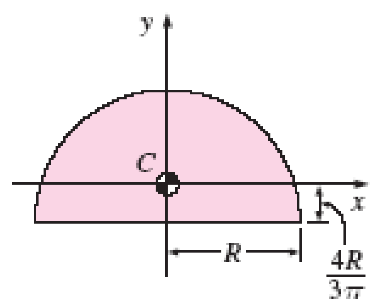
$A = \pi ab,\ I_{xx,\,C} = \pi ab^3/4$

(*c*) Ellipse



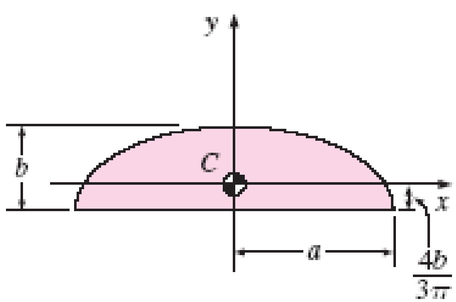
$A = ab/2,\ I_{xx,\,C} = ab^3/36$

(*d*) Triangle



$A = \pi R^2/2,\ I_{xx,\,C} = 0.109757R^4$

(*e*) Semicircle

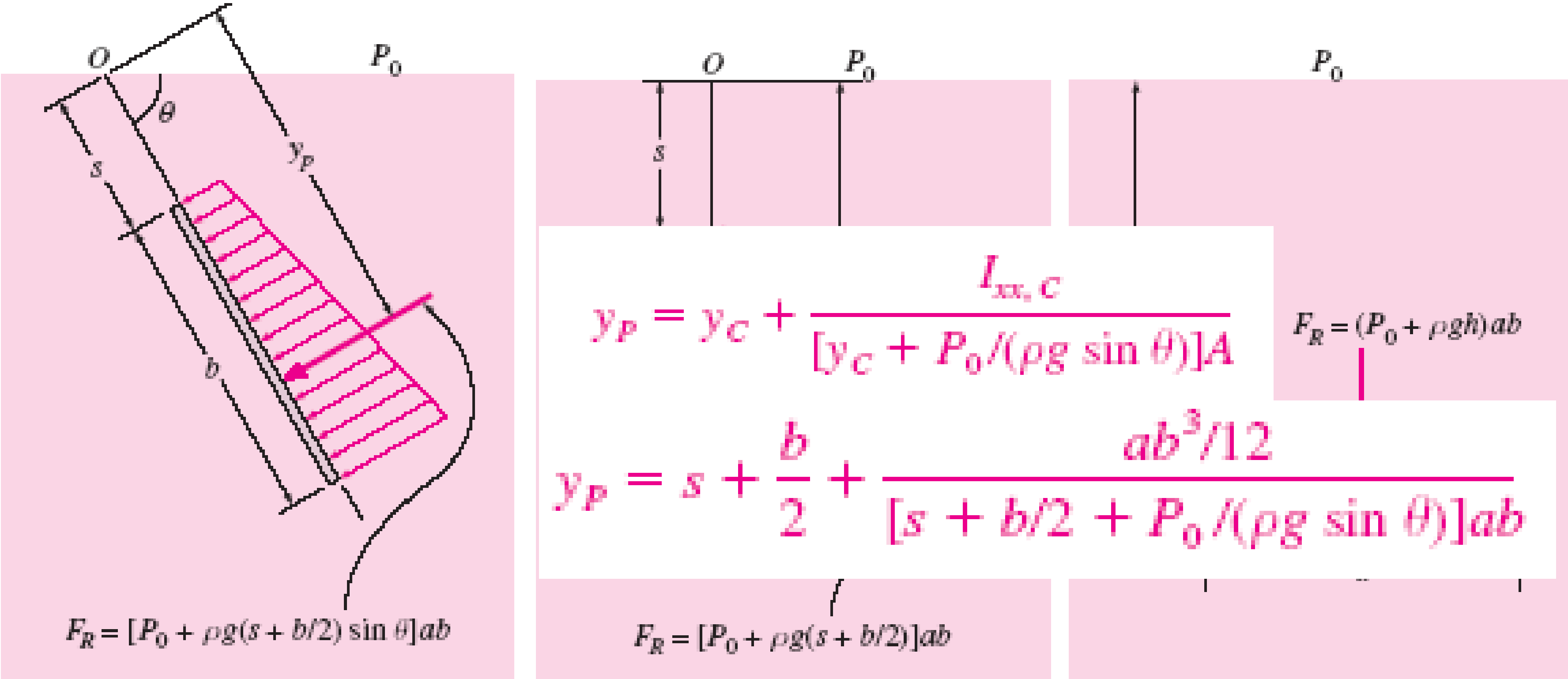


$A = \pi ab/2,\ I_{xx,\,C} = 0.109757ab^3$

(*f*) Semiellipse

Submerged Rectangular Plate

What is the y_p for case (a)?



(a) Tilted plate

(b) Vertical plate

(c) Horizontal plate

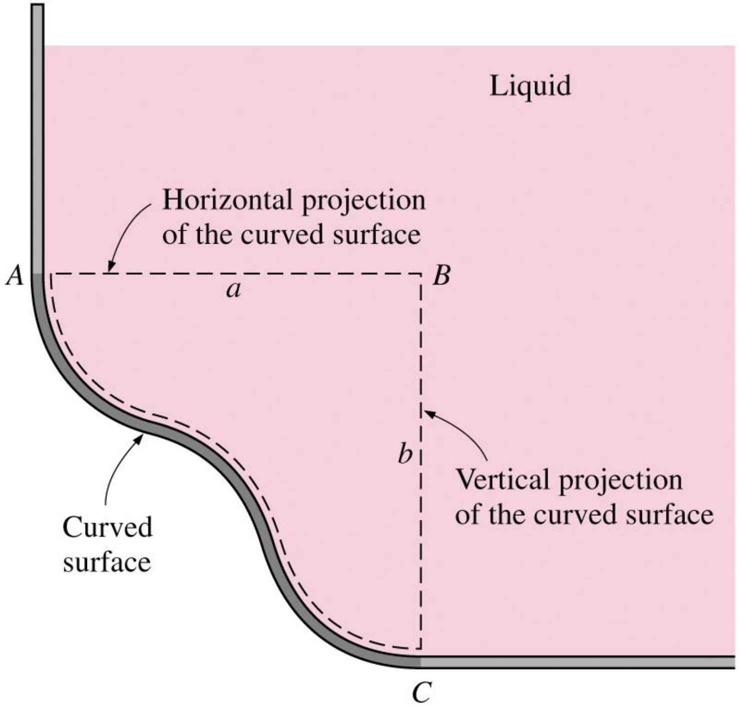
$$y_P = y_C + \frac{I_{xx,\,C}}{[y_C + P_0/(\rho g \sin\theta)]A}$$

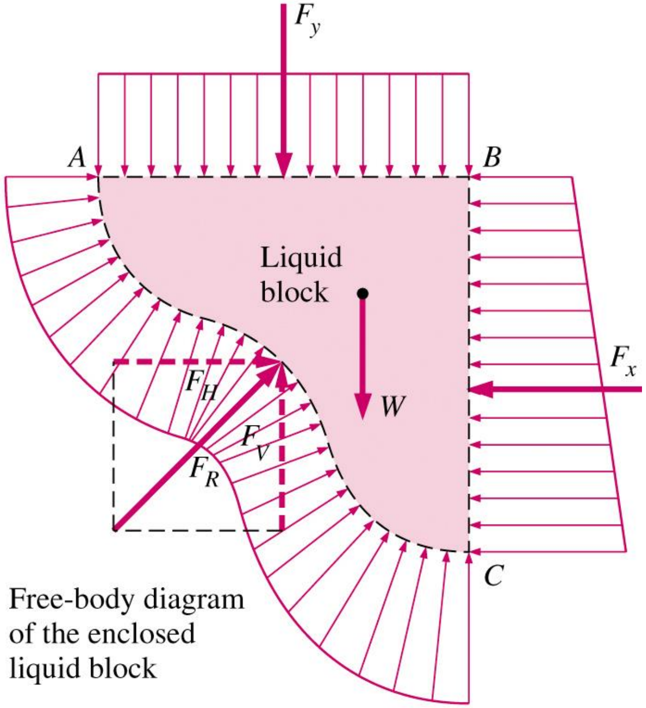
$$y_P = s + \frac{b}{2} + \frac{ab^3/12}{[s + b/2 + P_0/(\rho g \sin\theta)]ab}$$

Example: **Hydrostatic Force Acting on the Door of a Submerged Car**

Discussion A strong person can lift 100 kg, whose weight is 981 N or about 1 kN. Also, the person can apply the force at a point farthest from the hinges (1 m farther) for maximum effect and generate a moment of 1 kN · m. The resultant hydrostatic force acts under the midpoint of the door, and thus a distance of 0.5 m from the hinges. This generates a moment of 50.6 kN · m, which is about 50 times the moment the driver can possibly generate. Therefore, it is impossible for the driver to open the door of the car. The driver's best bet is to let some water in (by rolling the window down a little, for example) and to keep his or her head close to the ceiling. The driver should be able to open the door shortly before the car is filled with water since at that point the pressures on both sides of the door are nearly the same and opening the door in water is almost as easy as opening it in air.

Hydrostatic Forces on Curved Surfaces





- F_R on a curved surface is more involved since it requires integration of the pressure forces that change direction along the surface.
- Easiest approach: determine horizontal and vertical components F_H and F_V separately.

Hydrostatic Forces on Curved Surfaces

- Horizontal force component on curved surface: $F_H = F_x$. Line of action on vertical plane gives y coordinate of center of pressure on curved surface.
- Vertical force component on curved surface: $F_V = F_y + W$, where W is the weight of the liquid in the enclosed block $W = \rho g V$. x coordinate of the center of pressure is a combination of line of action on horizontal plane (centroid of area) and line of action through volume (centroid of volume).
- Magnitude of force $F_R = (F_H^2 + F_V^2)^{1/2}$
- Angle of force is $\alpha = tan^{-1}(F_V/F_H)$

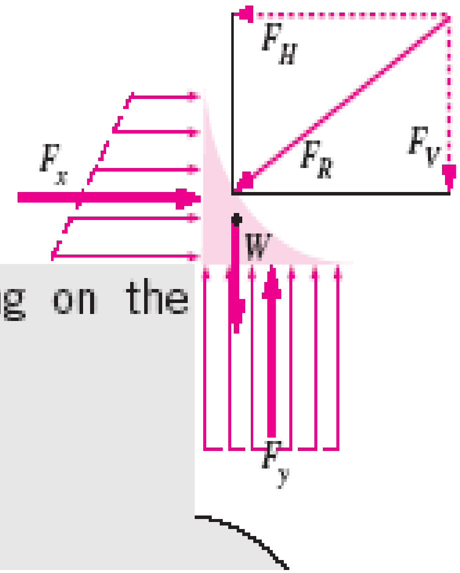
Example: **A Gravity-Controlled Cylindrical Gate**

Weight of fluid block per m length (downward):

$$W = mg = \rho g V = \rho g(R^2 - \pi R^2/4)(1\text{ m}) = 1.3\text{ kN}$$

Therefore, the net upward vertical force is

$$F_V = F_y - W = 39.2 - 1.3 = 37.9\text{ kN}$$



Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{36.1^2 + 37.9^2} = \mathbf{52.3\text{ kN}}$$

$$\tan\theta = F_V/F_H = 37.9/36.1 = 1.05 \rightarrow \theta = 46.4°$$

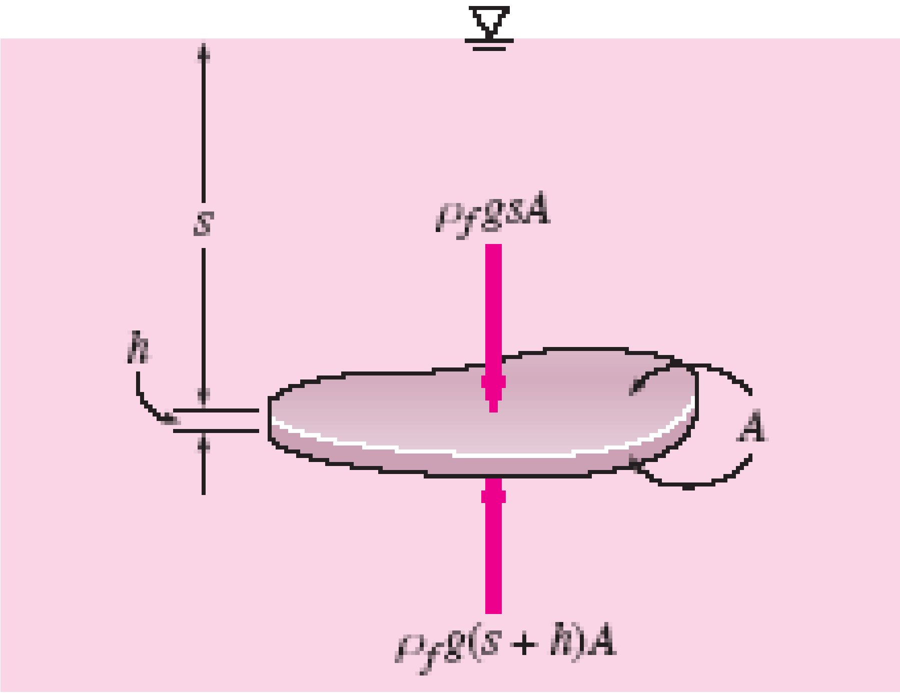
The weight of the cylinder is

$$F_R R \sin\theta - W_{cyl} R = 0 \rightarrow W_{cyl} = F_R \sin\theta = (52.3\text{ kN})\sin 46.4° = \mathbf{37.9\text{ kN}}$$

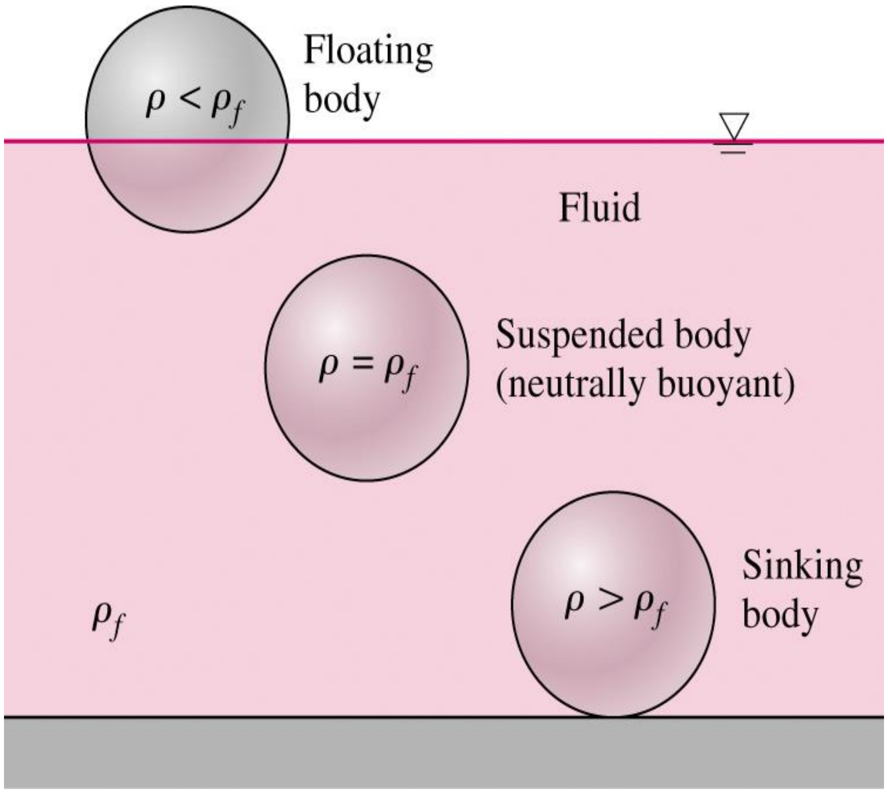
Discussion The weight of the cylinder per m length is determined to be 37.9 kN. It can be shown that this corresponds to a mass of 3863 kg per m length and to a density of 1921 kg/m³ for the material of the cylinder.

Buoyancy and Stability

- Buoyancy is due to the fluid displaced by a body. $F_B = \rho_f g V$.
- **Archimedes principal** : The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume.



Buoyancy and Stability



For *floating* bodies,

$$F_B = W \rightarrow \rho_f g V_{\text{sub}} = \rho_{\text{ave, body}} g V_{\text{total}}$$

$$\rightarrow \frac{V_{\text{sub}}}{V_{\text{total}}} = \frac{\rho_{\text{ave, body}}}{\rho_f}$$

- Buoyancy force F_B is equal only to the displaced volume $\rho_f g V_{displaced}$.
- Three scenarios possible
  1. $\rho_{body}<\rho_{fluid}$: Floating body
  2. $\rho_{body}=\rho_{fluid}$: Neutrally buoyant
  3. $\rho_{body}>\rho_{fluid}$: Sinking body

Example: Galilean Thermometer

- Galileo's thermometer is made of a sealed glass cylinder containing a clear liquid.
- Suspended in the liquid are a number of weights, which are sealed glass containers with colored liquid for an attractive effect.
- As the liquid changes temperature it changes density and the suspended weights rise and fall to stay at the position where their density is equal to that of the surrounding liquid.
- If the weights differ by a very small amount and ordered such that the least dense is at the top and most dense at the bottom they can form a temperature scale.

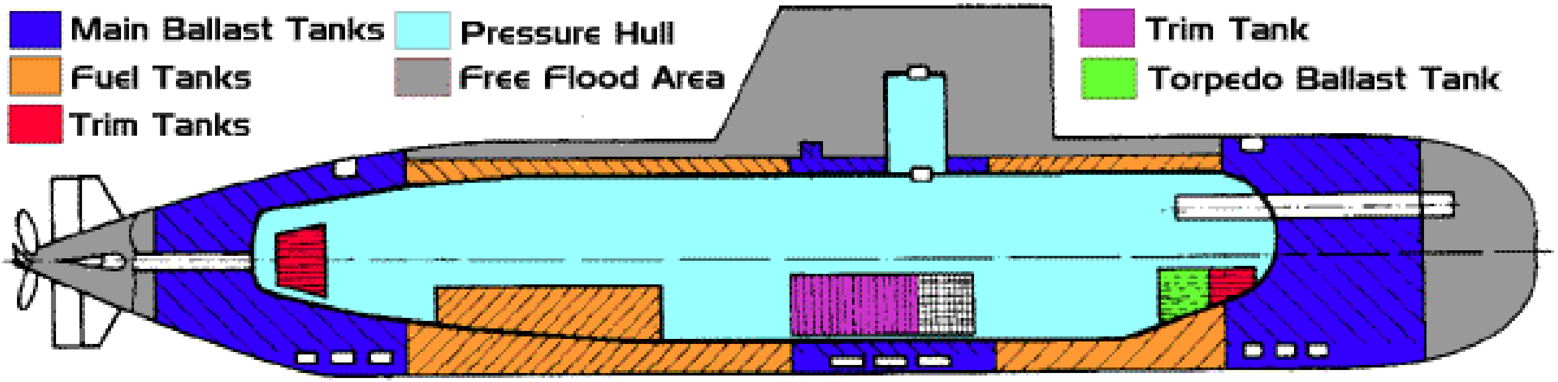
Example: Floating Drydock

Auxiliary Floating Dry Dock Resolute (AFDM-10) partially submerged

Submarine undergoing repair work on board the AFDM-10

Using buoyancy, a submarine with a displacement of 6,000 tons can be lifted!

Example: Submarine Buoyancy and Ballast



- Submarines use both static and dynamic depth control. Static control uses ballast tanks between the pressure hull and the outer hull. Dynamic control uses the bow and stern planes to generate trim forces.

Example: Submarine Buoyancy and Ballast

Normal surface trim

SSN 711 nose down after accident which damaged fore ballast tanks

Example: Submarine Buoyancy and Ballast

Damage to SSN 711 (USS San Francisco) after running aground on 8 January 2005.

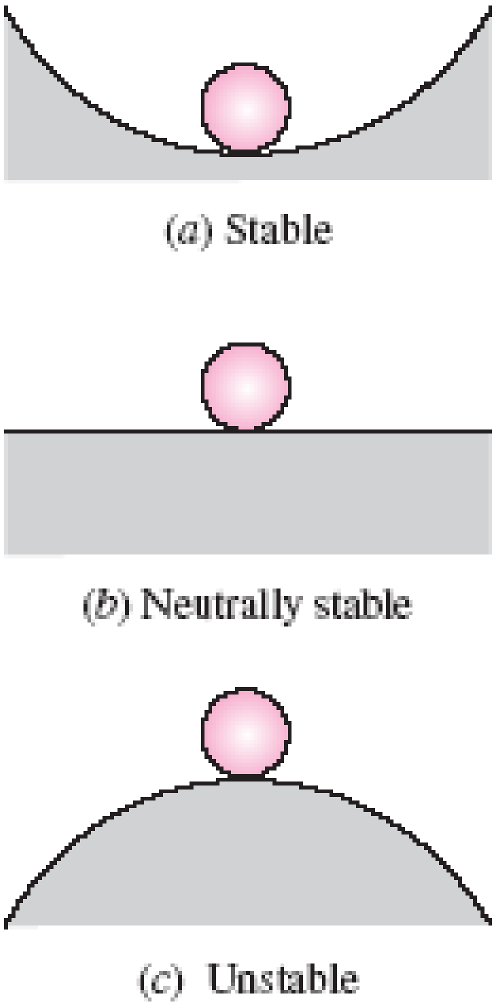
Example: Submarine Buoyancy and Ballast

Ballast Control Panel: Important station for controlling depth of submarine

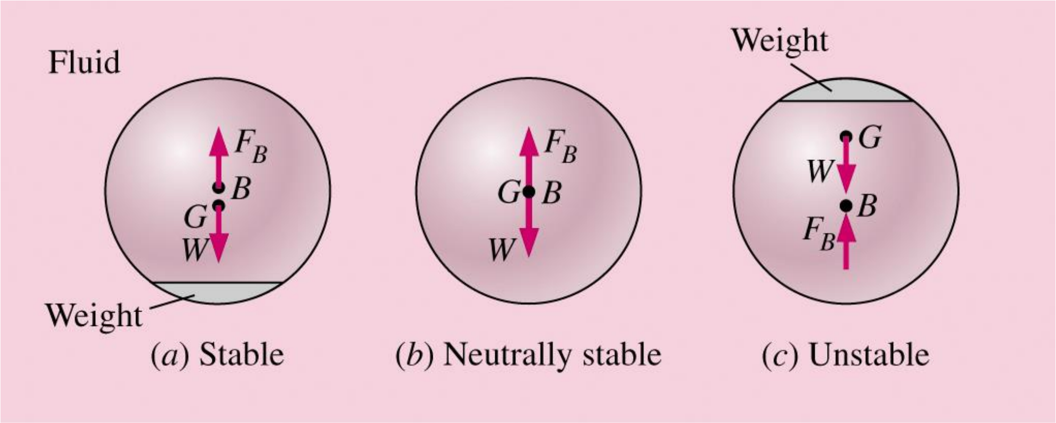
Stability

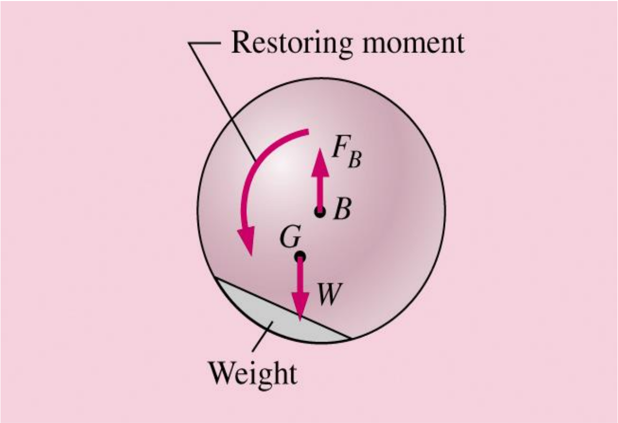
What about a case where the ball is on an *inclined* floor?

It is not really appropriate to discuss stability for this case since the ball is not in a state of equilibrium. In other words, it cannot be at rest and would roll down the hill even without any disturbance.



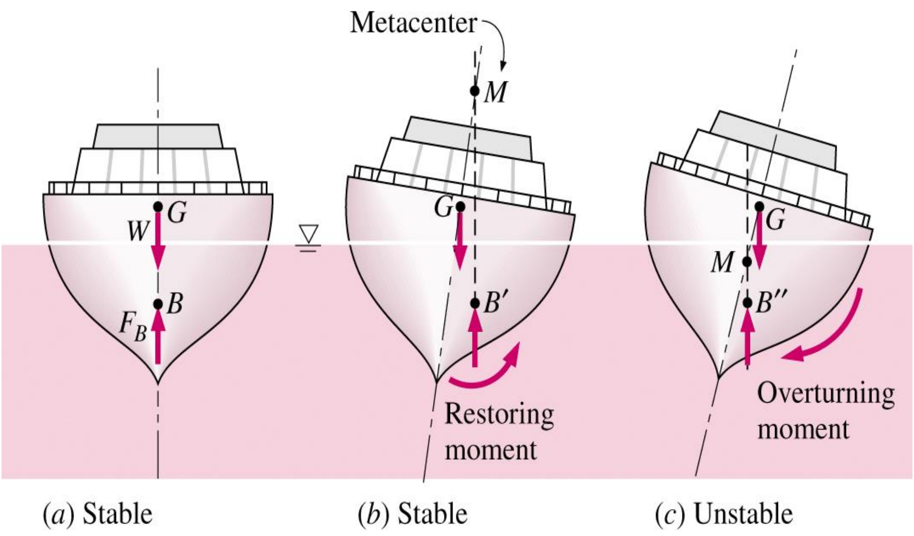
Stability of Immersed Bodies





- Rotational stability of immersed bodies depends upon relative location of *center of gravity G* and *center of buoyancy B*.
  - *G* below *B*: stable
  - *G* above *B*: unstable
  - *G* coincides with *B*: neutrally stable.

Stability of Floating Bodies

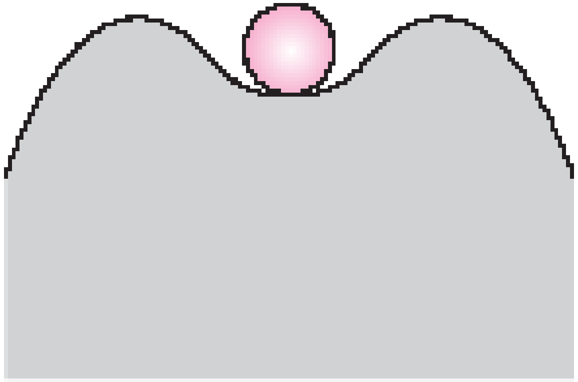


- The metacenter may be considered to be a fixed point for most hull shapes for small rolling angles up to about 20°.

- If body is bottom heavy (*G* lower than *B*), it is always stable.
- Floating bodies can be stable when *G* is higher than *B* due to shift in location of center buoyancy and creation of restoring moment.
- Measure of stability is the metacentric height G*M*. If *GM*>1, ship is stable.

Stability of Floating Bodies

A boat can tilt to some maximum angle without capsizing, but beyond that angle it overturns (and sinks). We make a final analogy between the stability of floating objects and the stability of a ball rolling along the floor. Namely, imagine the ball in a trough between two hills. The ball returns to its stable equilibrium position after being perturbed—up to a limit.



Fluids in rigid – body motion

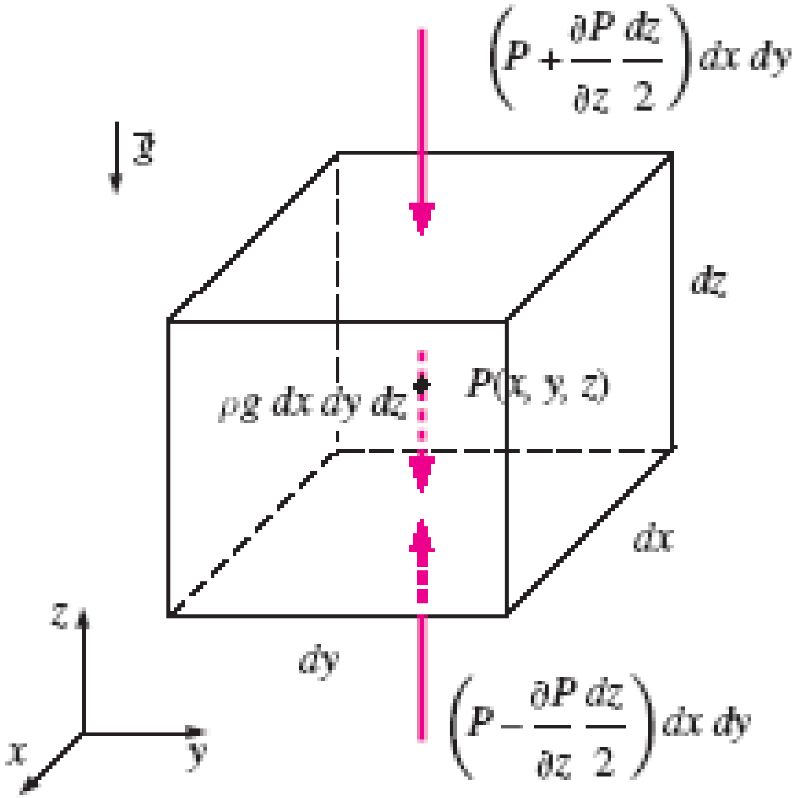
- In this section we obtain relations for the variation of pressure in fluids moving like a solid body with or without acceleration in the absence of any shear stresses (i.e., no motion between fluid layers relative to each other).

Fluids in rigid – body motion

The differential fluid element behaves like a *rigid body, Newton's second law of motion* for this element can be expressed as

$$\delta\vec{F} = \delta m \cdot \vec{a}$$

The forces acting on the fluid element consist of *body forces* such as gravity (and also electrical and magnetic forces), and *surface forces* such as the pressure forces



Rigid-body motion of fluids

$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z}\, dx\, dy\, dz$$

Similarly, $\delta F_{S,x} = -\frac{\partial P}{\partial x}\, dx\, dy\, dz$ and $\delta F_{S,y} = -\frac{\partial P}{\partial y}\, dx\, dy\, dz$

$$\delta \vec{F}_S = \delta F_{S,x}\vec{i} + \delta F_{S,y}\vec{j} + \delta F_{S,z}\vec{k}$$

$$= -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right) dx\, dy\, dz = -\vec{\nabla} P\, dx\, dy\, dz$$

$$\delta \vec{F}_{B,z} = -g\delta m\vec{k} = -\rho g\, dx\, dy\, dz\vec{k}$$

$$\delta \vec{F} = \delta \vec{F}_S + \delta \vec{F}_B = -(\vec{\nabla} P + \rho g\vec{k})\, dx\, dy\, dz$$

Equation of Motion: $\vec{\nabla} P + \rho g\vec{k} = -\rho\vec{a}$

$$\frac{\partial P}{\partial x} = -\rho a_x, \frac{\partial P}{\partial y} = -\rho a_y, \frac{\partial P}{\partial z} = -\rho\left(g + a_x\right)$$

Rigid-body motion of fluids

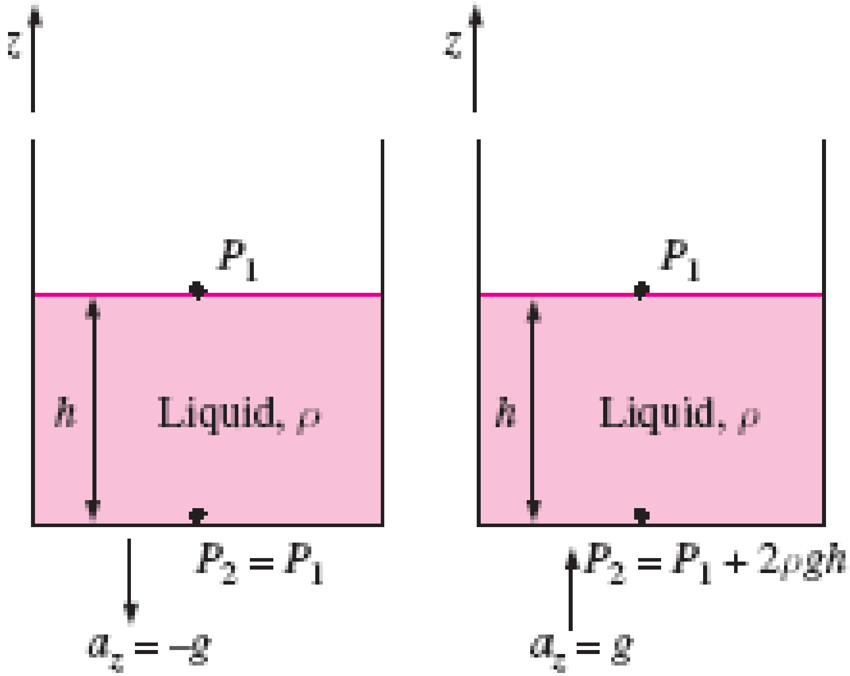
Special Case 1: Fluids at Rest

$$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{dP}{dz} = -\rho g$$

Special Case 2:
Free Fall of a Fluid Body

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$$

$$\rightarrow \qquad P = \text{constant}$$



Rigid-Body Motion

- A 2-D ($a_y = 0$) container is accelerated on a straight path with a constant acceleration.

$$\nabla P + \rho g \vec{k} = -\rho \vec{a}$$

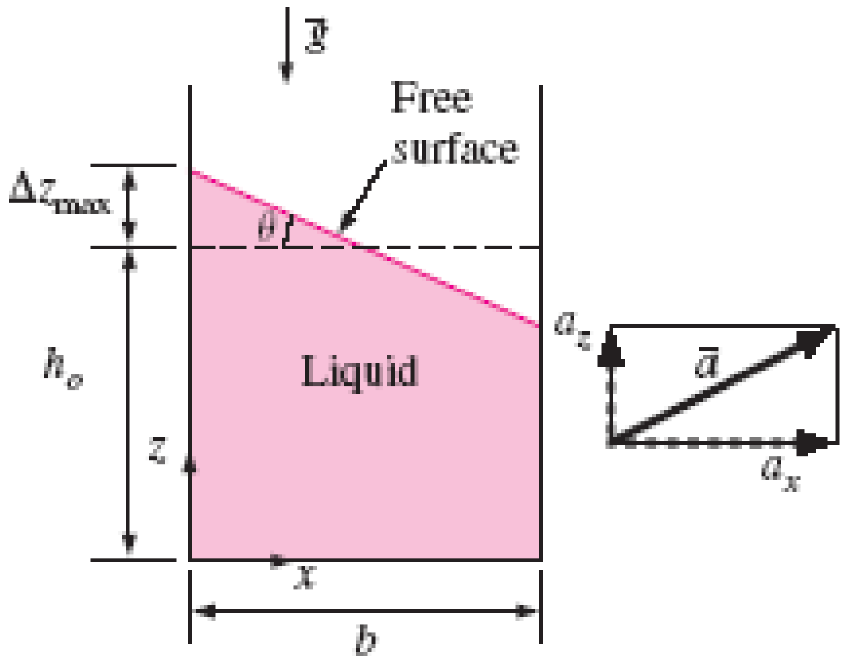
$$\frac{\partial P}{\partial x} = -\rho a_x, \frac{\partial P}{\partial y} = -\rho a_y, \frac{\partial P}{\partial z} = -\rho (g + a_x)$$

Pressure variation:

$$P = P_0 - \rho a_x x - \rho(g + a_z)z$$

Vertical rise of the free surface:

$$\Delta z_s = z_{s2} - z_{s1} = -\frac{a_x}{g + a_z}(x_2 - x_1)$$

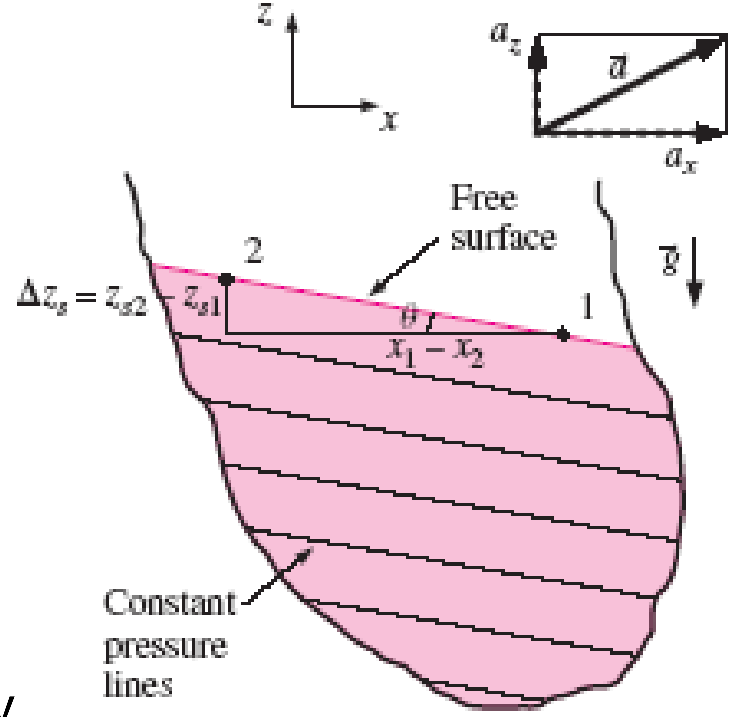


Rigid-Body Motion

Isobars: for surfaces of constant pressure
(On the same isobar, *d*P = 0)

$$\frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = \text{constant}$$

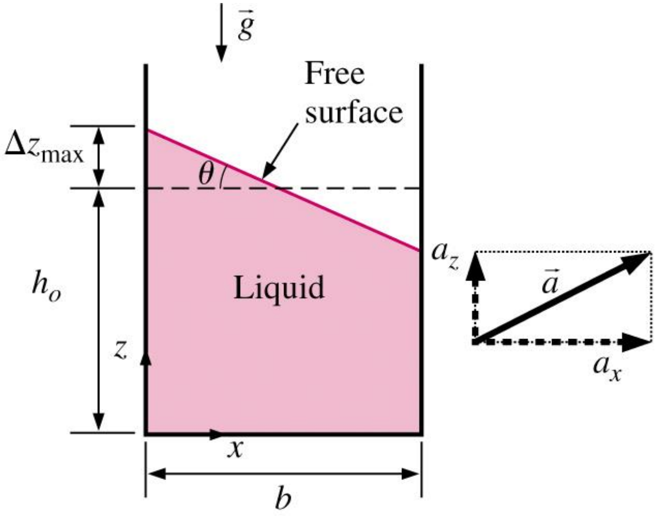
$$\text{Slope} = \frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = -\tan\theta$$



Conservation of mass implies the rise of fluid level on one side must be balanced by a drop of fluid level on the other side.

Linear Acceleration

How does the fluid behave as the car decelerates or moves with a constant speed?



Container is moving on a straight path

$$a_x \neq 0, a_y = a_z = 0$$

$$\frac{\partial P}{\partial x} = \rho a_x, \frac{\partial P}{\partial y} = 0, \frac{\partial P}{\partial z} = -\rho g$$

Total differential of P

$$dP = -\rho a_x dx - \rho g dz$$

Pressure difference between 2 points

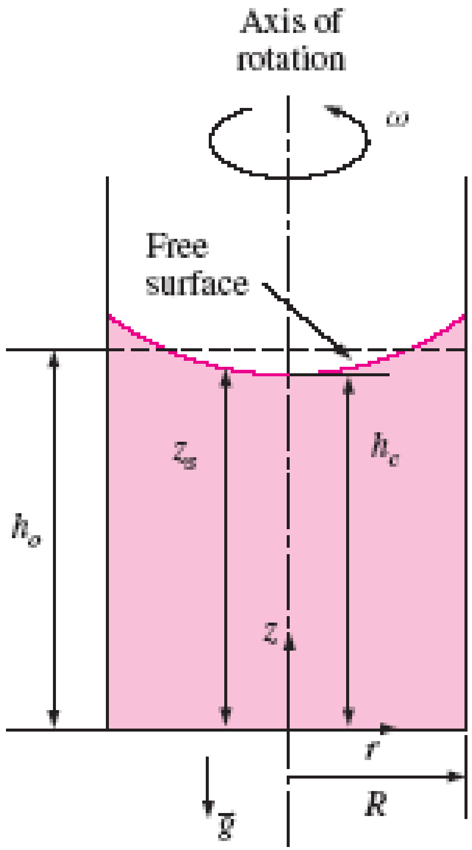
$$P_2 - P_1 = -\rho a_x \left(x_2 - x_1\right) - \rho g \left(z_2 - z_1\right)$$

Find the rise by selecting 2 points on free surface $P_2 = P_1$

$$\Delta z_s = z_{s2} - z_{s1} = -\frac{a_x}{g}\left(x_2 - x_1\right)$$

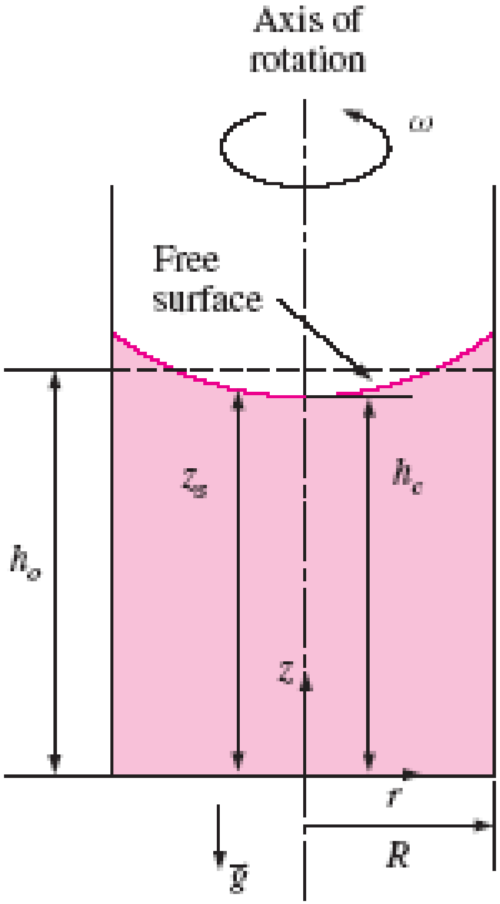
Rotation in a Cylindrical Container

A glass filled with water is rotated about its axis at a constant angular velocity of ϖ, The fluid is forced outward as a result of the so-called centrifugal force, and the free surface of the liquid becomes concave. This is known as the *forced vortex motion*.



After initial transients, the liquid will move as a rigid body together with the container. There is no deformation, and thus there can be no shear stress, and every fluid particle in the container moves with the same angular velocity.

Rotation in a Cylindrical Container



Container is rotating about the z-axis

$$a_r = -r\omega^2, a_\theta = a_z = 0$$

$$\frac{\partial P}{\partial r} = \rho r\omega^2, \frac{\partial P}{\partial \theta} = 0, \frac{\partial P}{\partial z} = -\rho g$$

Total differential of P = P(r,z)

$$dP = \rho r\omega^2 dr - \rho g dz$$

On an isobar, dP = 0

$$\frac{dz_{isobar}}{dr} = \frac{r\omega^2}{g} \rightarrow z_{isobar} = \frac{\omega^2}{2g}r^2 + C_1$$

$z_{isobar}(0) = C_1 = h_c$,

Rotation in a Cylindrical Container

The original volume of the fluid in the container

$$V = \pi R^2 h_0$$

The volume of the paraboloid formed by the free surface is

$$V = \int_{r=0}^{R} 2\pi z_s r\, dr = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^2 + h_c \right) r\, dr = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c \right)$$

From conservation of mass, we can get

$$h_c = h_0 - \frac{\omega^2 R^2}{4g}$$

The equation of the free surface

$$z_s = h_0 - \frac{\omega^2}{4g}\left(R^2 - 2r^2 \right)$$

Rotation in a Cylindrical Container

Maximum height difference: $$\Delta z_{s,\,\max} = z_s(R) - z_s(0) = \frac{\omega^2}{2g}R^2$$

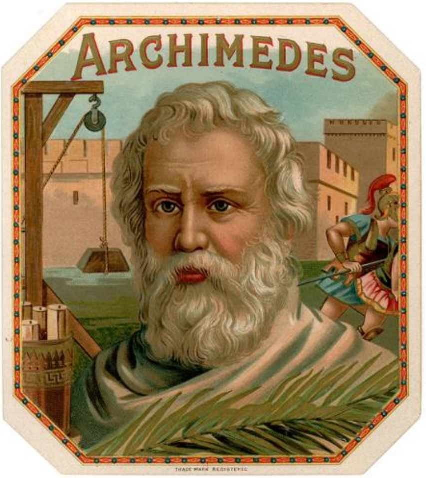
Taking point 1 to be the origin ($r = 0$, $z = 0$) where the pressure is P_0 and point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as

Pressure variation: $$P = P_0 + \frac{\rho\omega^2}{2}r^2 - \rho g z$$

In any horizontal plane, the pressure difference between the center and edge of the container of radius R is $\Delta P = \rho\varpi^2 R^2/2$.

Examples of Archimedes Principle

The Golden Crown of Hiero II, King of Syracuse



- Archimedes, 287-212 B.C.
- Hiero, 306-215 B.C.
- Hiero learned of a rumor where the goldsmith replaced some of the gold in his crown with silver. Hiero asked Archimedes to determine whether the crown was pure gold.
- Archimedes had to develop a nondestructive testing method

The Golden Crown of Hiero II, King of Syracuse

- The weight of the crown and nugget are the same in air: $W_c = \rho_c V_c = W_n = \rho_n V_n$.
- If the crown is pure gold, $\rho_c = \rho_n$ which means that the volumes must be the same, $V_c = V_n$.
- In water, the buoyancy force is $B = \rho_{H2O} V$.
- If the scale becomes unbalanced, this implies that the $V_c \neq V_n$, which in turn means that the $\rho_c \neq \rho_n$
- Goldsmith was shown to be a fraud!