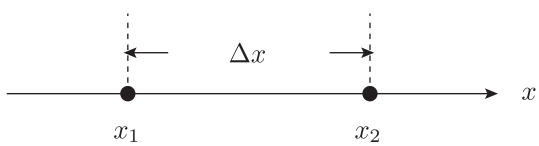
Contents

2 Motion Along a Straight Line

2.1 Position and Displacement

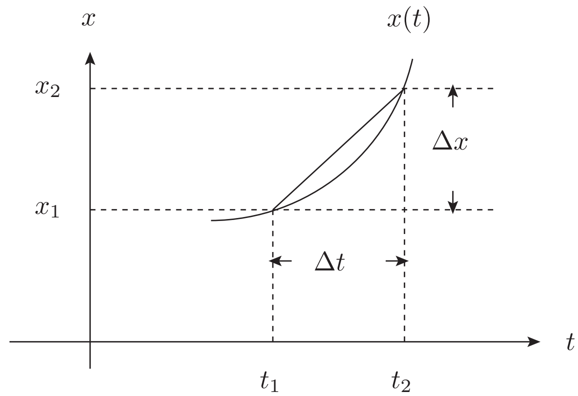


A change from one position x_1 to another position x_2 is called a displacement Δx, where

$$\Delta x = x_2 - x_1$$

2.2 Average Velocity and Average Speed

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$



2.3 Instantaneous Velocity and Speed

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

2.4 Acceleration

Average Acceleration

$$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$$

Instantaneous Acceleration

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

2.5 Constant Acceleration

When the acceleration is constant, we have

$$a = a_{avg} = \frac{v\left(t\right) - v\left(0\right)}{t - 0}$$

$$v\left(t\right) = v\left(0\right) + at$$

which means $v\left(t\right)$ increases linearly with time t. The displacement Δx is equal to the average velocity time the time span. Thus

$$x\left(t\right) - x\left(0\right) = v_{avg}\left(t - 0\right)$$

Now the velocity changes linearly with respect to time, so we have

$$v_{avg} = \frac{1}{2}\left(v\left(t\right) + v\left(0\right)\right) = v\left(\frac{t}{2}\right)$$

Consequently,

$$x\left(t\right) = x\left(0\right) + tv\left(\frac{t}{2}\right) = x\left(0\right) + t\left(v\left(0\right) + \frac{a}{2}t\right) = x\left(0\right) + v\left(0\right)t + \frac{a}{2}t^2$$

2.6 Differentiation Rules

a, b are constants. n is an integer. From

$$\frac{af\left(t+\Delta t\right) + bg\left(t+\Delta t\right) - af\left(t\right) - bg\left(t\right)}{\Delta t} = a\frac{f\left(t+\Delta t\right) - f\left(t\right)}{\Delta t} + b\frac{g\left(t+\Delta t\right) - bg\left(t\right)}{\Delta t},$$

we get

$$\frac{d}{dt}\left(af\left(t\right) + bg\left(t\right)\right) = a\frac{df\left(t\right)}{dt} + b\frac{dg\left(t\right)}{dt}$$

Since

$$\frac{f\left(t+\Delta t\right)g\left(t+\Delta t\right) - f\left(t\right)g\left(t\right)}{\Delta t} = \frac{f\left(t+\Delta t\right) - f\left(t\right)}{\Delta t}g\left(t+\Delta t\right) + f\left(t\right)\frac{g\left(t+\Delta t\right) - g\left(t\right)}{\Delta t},$$

we have

$$\frac{d}{dt}\left(f\left(t\right)g\left(t\right)\right) = \frac{df\left(t\right)}{dt}g\left(t\right) + f\left(t\right)\frac{dg\left(t\right)}{dt}$$

which leads to

$$\begin{aligned}&\frac{d}{dt}\left(f_1\left(t\right)f_2\left(t\right)..f_n\left(t\right)\right)\\ &= \frac{df_1\left(t\right)}{dt}f_2\left(t\right)..f_n\left(t\right) + f_1\left(t\right)\frac{df_2\left(t\right)}{dt}f_3\left(t\right)..f_n\left(t\right) + .. + f_1\left(t\right)f_2\left(t\right)..f_{n-1}\left(t\right)\frac{df_n\left(t\right)}{dt}\end{aligned}$$

$$\frac{d}{dt}\left(f\left(t\right)^n\right) = nf\left(t\right)^{n-1}\frac{df\left(t\right)}{dt}$$

As an example,

$$\frac{dt^n}{dt} = nt^{n-1}\frac{dt}{dt} = nt^{n-1}$$

Since

$$\frac{df\left(t\right)}{dt} = \frac{d}{dt}\left(f\left(t\right)^{\frac{1}{n}}\right)^n = n\left(f\left(t\right)^{\frac{1}{n}}\right)^{n-1}\frac{d}{dt}\left(f\left(t\right)^{\frac{1}{n}}\right)$$

we have
$$\frac{d}{dt}\left(f(t)^{\frac{1}{n}}\right)=\frac{1}{n\left(f(t)^{\frac{1}{n}}\right)^{n-1}}\frac{df(t)}{dt}=\frac{1}{n}f^{\frac{1}{n}-1}\frac{df}{dt}$$
In general, we can show that, for rational $p=\frac{n}{m}$ where n,m are integers,
$$\frac{d}{dt}f(t)^{p}=p\frac{df(t)}{dt}f(t)^{p-1}$$
In fact, by continuity, the above identity is valid for arbitrary p.

2.6.1 Taylor Series Expansion

By definition
$$f'(a)=\frac{df(x)}{dx}|_{x=a}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$
If $x-a$ is a small non-zero number, the above identity is approximate and can be rewritten as
$$f(x)\simeq f(a)+f'(a)(x-a)$$
This is called the Taylor series expansion to first order. Let us define
$$g(x)=f(x)-f(a)-f'(a)(x-a)$$
Then
$$g(a)=0$$
$$\begin{aligned}g'(x)&=\frac{dg(x)}{dx}=f'(x)-f'(a),\\ g'(a)&=0\end{aligned}$$
and
$$g''(x)=\frac{d^2g(x)}{dx^2}=f''(x)$$
For $x\sim a$
$$\frac{dg(x)}{dx}=g'(x)\simeq g'(a)+g''(a)(x-a)=f''(a)(x-a)=\frac{d}{dx}\left(\frac{f''(a)}{2}(x-a)^2\right)$$

Thus $g(x)$ and $\frac{f''(a)}{2}(x-a)^2$ have approximately the same derivative with respect to x, and their difference must be a constant which is zero because they both vanish at $x = a$.

$$g(x) \simeq \frac{f''(a)}{2}(x-a)^2$$

$$f(x) \simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

This is the Taylor series expansion up to second order. In fact, it can be shown and will likely be proven in your Calculus class that

$$\begin{aligned} f(x) &\simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

where $f^{(0)}(a) = f(a)$ and for $n > 1$, $f^{(n)}(x)$ is the nth derivative of $f(x)$. The above series is called the Taylor series for $f(x)$ expanded around $x = a$.

2.6.2 Exponential and Logarithmic Functions

By definition, the derivative of $f_a(x) = a^x$ is

$$f_a'(x) = \frac{df_a(x)}{dx} = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{\left(a^{\Delta x} - 1\right)}{\Delta x} = a^x f_a'(0)$$

$f_a'(0)$ depends on a. In particular, the constant e is the value of a that gives $f_a'(0) = 1$. The constant e can be shown to be an irrational number. The function $e^x = \exp(x)$ is called the exponential function which is equal to its own derivative

$$\frac{d}{dx}(e^x) = e^x \tag{1}$$

Since the derivative of exponential function e^x remains equal to e^x, the Taylor series for e^x expanded around $x = 0$ gives us

$$e^x = \sum_{n=0}^{\infty} \frac{e^0}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... \tag{2}$$

Taking the derivative with respect to the above, we get

$$\frac{d}{dx}\left(e^x\right) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ..\right)$$
$$= 0 + 1 + x + \frac{x^2}{2} + ... = e^x$$

which is the same as (1). Set $x = 1$ in (2).

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... = 2.7182818283\ldots$$

The expression (2) can be generalized to define $\exp\left(M\right)$

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} = 1 + M + \frac{M^2}{2} + \frac{M^3}{3!} + ...$$

in which M can be a complex number or a matrix. The inverse of the exponential function is called the (natural) logarithmic function ln. Thus,

$$y = e^x$$

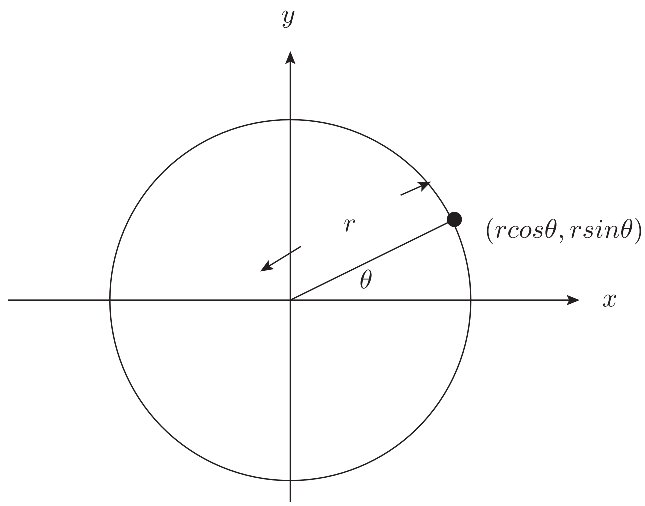
and

$$x = \ln y$$

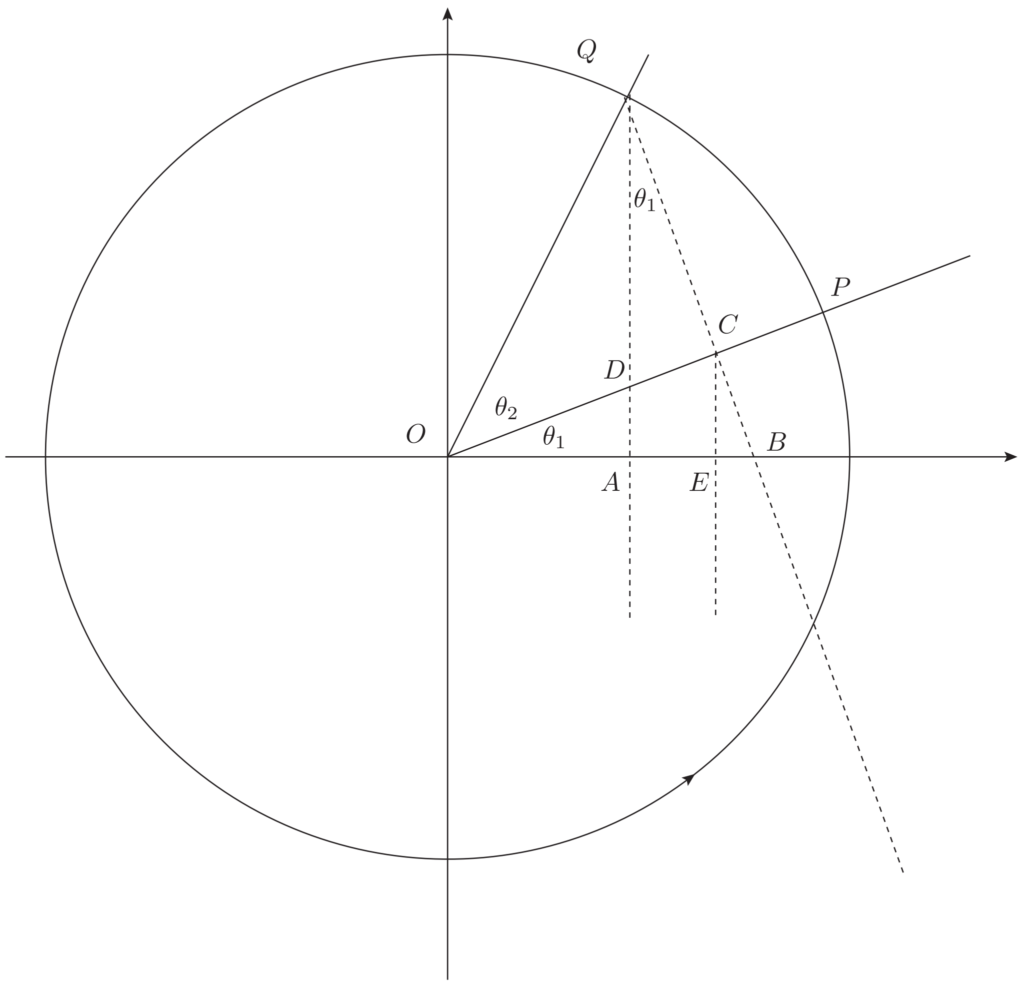
are equivalent just as $y = a^x$ and $x = \log_a b$ are equivalent statements mathematically.

2.6.3 Trigonometric Functions

The definitions for the trigonometric functions sin and cos are shown below.



$\sin(\theta_2 + \theta_1)$ can be expanded according to the following figure:



in which the radius for the circle is unity.

$$\overline{OC} = \cos\theta_2,$$
$$\overline{QC} = \sin\theta_2,$$
$$\overline{CE} = \cos\theta_2 \sin\theta_1,$$
$$\overline{CB} = \frac{\overline{CE}}{\cos\theta_1} = \frac{\cos\theta_2 \sin\theta_1}{\cos\theta_1}$$

$$
\begin{aligned}
\sin(\theta_2+\theta_1) = \overline{QA} = \overline{CE}\frac{\overline{QC}+\overline{CB}}{\overline{CB}} &= \cos\theta_2\sin\theta_1\left(\frac{\sin\theta_2}{\frac{\cos\theta_2\sin\theta_1}{\cos\theta_1}}+1\right) \\
&= \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\cos(\theta_2+\theta_1) &= \sin\left(\frac{\pi}{2}-\theta_2-\theta_1\right) \\
&= \sin\left(\frac{\pi}{2}-\theta_2\right)\cos(-\theta_1) + \cos\left(\frac{\pi}{2}-\theta_2\right)\sin(-\theta_1) \\
&= \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1
\end{aligned}
$$

When $\epsilon \ll 1$,

$$
\begin{aligned}
\sin\epsilon &\sim \epsilon \\
\cos\epsilon = \sqrt{1-\sin^2\epsilon} &\sim \sqrt{1-\epsilon^2} \sim 1-\frac{\epsilon^2}{2}
\end{aligned}
$$

The above two expressions enable us to derive the derivative of $\sin x$:

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\epsilon\to 0}\frac{\sin(\epsilon+x)-\sin x}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\sin\epsilon\cos x + \cos\epsilon\sin x - \sin x}{\epsilon} \\
&= \cos x\lim_{\epsilon\to 0}\frac{\sin\epsilon}{\epsilon} + \sin x\lim_{\epsilon\to 0}\frac{\cos\epsilon - 1}{\epsilon} = \cos x
\end{aligned}
$$

The derivative of $\cos x$ can the derived via

$$
\begin{aligned}
\frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right) &= \frac{d\left(\frac{\pi}{2}-x\right)}{dx}\frac{d}{d\left(\frac{\pi}{2}-x\right)}\sin\left(\frac{\pi}{2}-x\right) \\
&= -\cos\left(\frac{\pi}{2}-x\right) = -\sin x
\end{aligned}
$$

The two conditions: $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$ for real x, may be combined as

$$
\frac{d}{dx}(\cos x + i\sin x) = i(\cos x + i\sin x) \tag{3}
$$

The function e^{ix} and its derivative $\frac{d}{dx}e^{ix}$ are related by a similar equation

$$
\begin{aligned}
\frac{d}{dx}e^{ix} = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(ix)^n}{n!} = \sum_{n=0}^{\infty}\frac{i^n}{n!}\frac{dx^n}{dx} &= \sum_{n=0}^{\infty}\frac{i^n}{n!}nx^{n-1} \\
&= i\sum_{n=1}^{\infty}\frac{i^{n-1}x^{n-1}}{(n-1)!} = i\sum_{n=0}^{\infty}\frac{i^n x^n}{n!} = ie^{ix}
\end{aligned} \tag{4}
$$

e^{ix} and $(\cos x + i\sin x)$ are both equal at $x = 0$:

$$e^{i0} = (\cos 0 + i\sin 0) = 1$$

(3) and (4) also show their derivatives are equal to $x = 0$:

$$\frac{d}{dx}e^{ix}|_{x=0} = \frac{d}{dx}\left(\cos x + i\sin x\right)|_{x=0} = i$$

In fact, it is easy to see that their nth derivative are equal

$$\frac{d^n}{dx^n}e^{ix}|_{x=0} = \frac{d^n}{dx^n}\left(\cos x + i\sin x\right)|_{x=0} = i^n$$

Thus e^{ix} and $\frac{d^n}{dx^n}\left(\cos x + i\sin x\right)$ have the same Taylor series expansion around $x = 0$ and they should be equal at all x. *i.e.*,

$$e^{ix} = \cos x + i\sin x \tag{5}$$

This shows that exponential functions and trigonometric functions are not much different. In fact, it is often conveniently to express the latter in terms of the former to facilitate practical calculations.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

As an example, knowing that the law of exponential gives us

$$\left(e^{ix}\right)^3 = e^{i3x}$$

we readily derive the trigonometric identity

$$(\cos x + i\sin x)^3 = \cos 3x + i\sin 3x$$

The real part of the above identity gives

$$\cos^3 x - 3\cos x\sin^2 x = \cos 3x$$

while the imaginary part yields

$$3\sin x\cos^2 x - \sin^3 x = \sin 3x$$

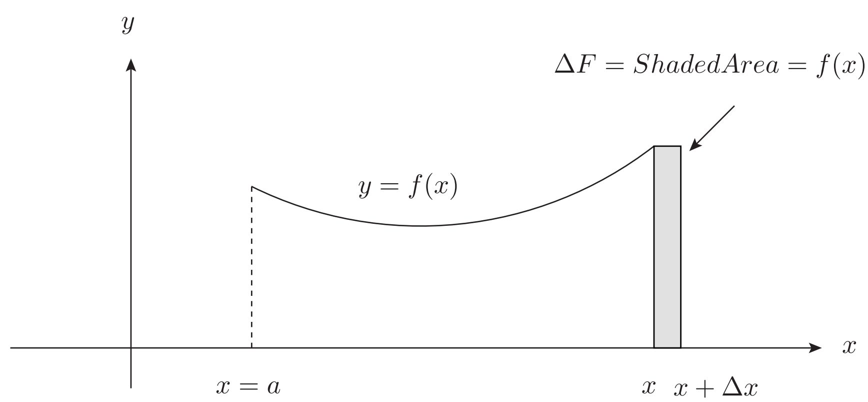
The identity (5) will be utilized when dealing with oscillations in chapter 15. (5) will also be involved when we discuss the circuit with alternating current next semester.

2.7 Integration

If

$$f(x) = \frac{dF(x)}{dx},$$

$F(x)$ is then called an antiderivative of $f(x)$. The area under the curve $y = f(x)$ and above the x-axis and between the vertical lines $x = a$ and $x = b$ is equal to $F(b) - F(a)$ as can be seen from the following figure:



In the above, the shaded area is equal to $f(x)\Delta x$. To first order of Δx, we have

$$F(x + \Delta x) - F(x) \simeq \frac{dF(x)}{dx}\Delta x = f(x)\Delta x$$

Let us call $A(b)$ the area under the curve $y = f(x)$ and above the x-axis in the interval (a, b). $A(b)$ is also denoted by the definite integral

$$A(b) = \int_a^b f(x)\,dx$$

where $f(x)$ is called the integrand, x is called the integration variable and a (b) is called the lower(upper) limit. Note the integration variable x is dummy can be replace by other symbol which has not been defined yet. Thus $A(b)$ can also be written as

$$A(b) = \int_a^b f(u)\,du = \int_a^b f(t)\,dt$$

in which u is the dummy integration variable for the first integral whereas t is the dummy variable for the second. The shaded area is equal to

$$A(x+\Delta x) - A(x) \simeq f(x)\,\Delta x \simeq F(x+\Delta x) - F(x)$$

In the limit $\Delta x \to 0$, we have

$$\frac{dA(x)}{dx} = \lim_{\Delta x\to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{dF(x)}{dx} = f(x)$$

$A(x)$ is also an antiderivative of $f(x)$ and thus

$$\frac{d}{dx}(A(x) - F(x)) = 0$$

The only function that has vanishing derivative is a constant function. Since $A(a) = 0$, we must have

$$A(x) = F(x) - F(a)$$

We have thus arrived at the theorem

Theorem 1 *The Fundamental Theorem of Calculus: The integral for $f(x)$ is given by*

$$\int_a^b f(x)\,dx = F(x)\,|_{x=a}^{x=b} = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$ with $\frac{dF(x)}{dx} = f(x)$.

Note $F(x)\,|_{x=a}^{x=b}$ is often omitted as $F(x)\,|_a^b$ if there is no confusion. For convenience, the indefinite integral $\int f(x)\,dx$ without specifying the upper and lower limits often means an antiderivative of $f(x)$ or equivalently

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

This notation of $\int f(x)\,dx$ for the indefinite integral is a little sloppy because x is dummy in $\int f(x)\,dx$. A more rigorous expression is

$$\frac{d}{dx}\int^x f(t)\,dt = f(x)$$

where the dependence on the variable x is explicitly demonstrated to be the upper limit of the integral. Mathematically, the integral $\int_a^b f(x)\,dx$ is defined as the limit of a series:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\,\Delta x_i$$

where $(b-a) = \sum_{i=1}^{N} \Delta x_i$,

each $\Delta x_i \to 0$ for $i = 1, 2, ..N$ as $N \to \infty$ and

$$x_i \in (a + \Delta x_1 + \Delta x_2 + ..\Delta x_{i-1}, a + \Delta x_1 + \Delta x_2 + ... + \Delta x_{i-1} + \Delta x_i)$$

In particular, if we choose equal spacing of $\Delta x_i = \Delta x = \frac{b-a}{N}$ and $x_i = a + \left(i - \frac{1}{2}\right)\Delta x$ for $i = 1, 2, ..N$, then

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f\left(a + \frac{\left(i-\frac{1}{2}\right)}{N}(b-a)\right)\frac{b-a}{N}$$

2.7.1 Linear Motion

The velocity $v(t)$ is the derivative of displacement $x(t)$.

$$v(t) = \frac{dx(t)}{dt}$$

$x(t)$ is an antiderivative of $v(t)$ and thus

$$x(t_1) - x(t_0) = \int_{t_0}^{t_1} v(t)\,dt$$

Likewise, from

$$a(t) = \frac{dv(t)}{dt}$$

we get

$$v(t_1) - v(t_0) = \int_{t_0}^{t_1} a(t)\, dt$$

For a motion with constant acceleration $a(t) = a_0$,

$$v(t) - v(t_0) = \int_{t_0}^{t} a_0 dt' = a_0 (t - t_0) = \frac{d}{dt}\left(\frac{1}{2} a_0 (t - t_0)^2\right)$$

$$v(t) = v_0 + a_0 (t - t_0)$$

where $v_0 = v(t_0)$.

$$\begin{aligned} x(t) - x(t_0) &= \int_{t_0}^{t} v(t')\, dt' = \int_{t_0}^{t} a_0 (t' - t_0)\, dt' + \int_{t_0}^{t} v_0 dt' \\ &= \frac{1}{2} a_0 (t' - t_0)^2 \Big|_{t'=t_0}^{t'=t} + v_0 t' \Big|_{t'=t_0}^{t'=t} \\ &= \frac{1}{2} a_0 (t - t_0)^2 + v_0 (t - t_0) \\ &= \frac{1}{2} a_0 t^2 + (v_0 - a_0 t_0)\, t + \frac{1}{2} a_0 t_0^2 - v(t_0)\, t_0 \end{aligned}$$