Contents

4 Motion in Two and Three Dimensions

4.1 Position and Displacement

One general way of locating a particle is with a position vector $\vec{r}$, which is a vector that extends from a reference point (usually the origin of a coordinate system) to the particle. $\vec{r}$ may be written as

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

where the coefficients x, y, and z give the particle's location along the coordinate axes and relative to the origin.

As a particle moves, its position vector changes in such a way that the vector always extends to the particle from the origin. If the position vector changes—say, from $\vec{r}_1$ to $\vec{r}_2$ during a certain time interval—then the particle's displacement $\Delta\vec{r}$ during that time interval is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

or as

$$\begin{aligned}\Delta\vec{r} &= (x_2 - x_1)\,\hat{\imath} + (y_2 - y_1)\,\hat{\jmath} + (z_2 - z_1)\,\hat{k} \\ &= \Delta x\hat{\imath} + \Delta y\hat{\jmath} + \Delta z\hat{k}\end{aligned}$$

where coordinates (x_1, y_1, z_1) correspond to position vector $\vec{r}_1$, coordinates (x_2, y_2, z_2) correspond to position vector $\vec{r}_2$ and $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta z = z_2 - z_1$.

4.2 Average Velocity and Instantaneous Velocity

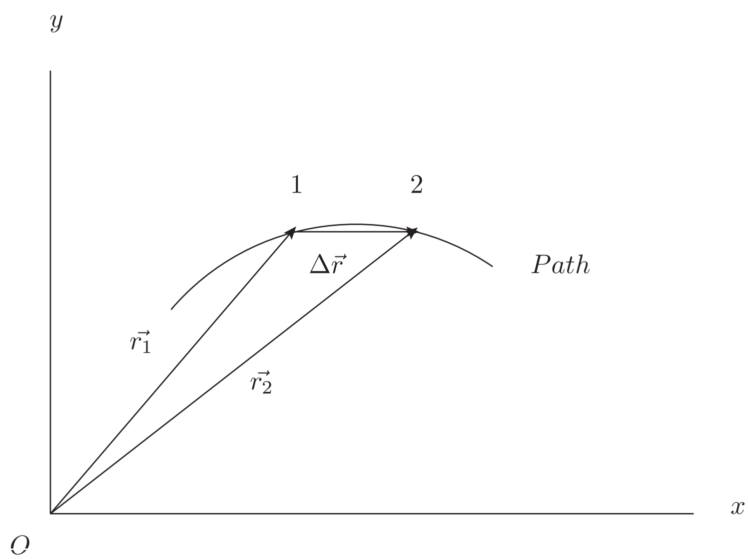
Average velocity is

$$\begin{aligned}\vec{v}_{avg} &= \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t} \\ &= \frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k}\end{aligned}$$

Instantaneous velocity is

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

$$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}, v_z = \frac{dz}{dt}$$



The direction of the instantaneous velocity $\vec{v}$ of a particle is always tangent to the particle's path at the particle's position.

4.3 Average Acceleration and Instantaneous Acceleration

Average acceleration is

$$
\begin{aligned}
\vec{a}_{avg} &= \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta \vec{v}}{\Delta t} \\
&= \frac{\Delta v_x}{\Delta t}\hat{\imath} + \frac{\Delta v_y}{\Delta t}\hat{\jmath} + \frac{\Delta v_z}{\Delta t}\hat{k}
\end{aligned}
$$

Instantaneous acceleration is

$$
\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} + \frac{dv_z}{dt}\hat{k}
$$

$$
a_x = \frac{dv_x}{dt}, a_y = \frac{dv_y}{dt}, a_z = \frac{dv_z}{dt}
$$

4.4 Projectile Motion

A particle moves in a vertical plane with some initial velocity $\vec{v}_0$ but its acceleration is always the free-fall acceleration $\vec{g}$, which is downward. Such a particle is called a projectile.

Assume

$$
\vec{v}_0 = v_{0x}\hat{\imath} + v_{0y}\hat{\jmath} = v_0 \cos\theta_0 \hat{\imath} + v_0 \sin\theta_0 \hat{\jmath}
$$

and

$$
\frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = \vec{g} = -g\hat{\jmath}
$$

So

$$
\begin{aligned}
\frac{dv_x}{dt} = 0, \frac{dv_y}{dt} &= -g \\
v_x = v_{x0}, v_y &= v_{y0} - gt \\
x(t) &= x_0 + v_{x0}t \qquad (1) \\
y(t) &= y_0 + v_{y0}t - \frac{1}{2}gt^2 \qquad (2)
\end{aligned}
$$

(1) and (2) express the coordinates x and y in terms of the time t.

4.4.1 The equation of the Path

We may eliminate t from (1) and (2) to arrive at a relationship between x and y. (1) gives

$$t = \frac{x - x_0}{v_{x0}}$$

which can be substituted into (2) to yield the orbital equation

$$y - y_0 = v_{y0}\frac{x - x_0}{v_{x0}} - \frac{1}{2}g\left(\frac{x - x_0}{v_{x0}}\right)^2$$
$$y - y_0 = (x - x_0)\tan\theta_0 - \frac{g}{2v_0^2\cos^2\theta_0}(x - x_0)^2$$

This is the equation of a parabola and the path is parabolic.

4.4.2 The Horizontal Range

The horizontal range $R = x - x_0$ occurs when $y - y_0 = 0$. So

$$0 = \tan\theta_0 - \frac{g}{2v_0^2\cos^2\theta_0}R$$

$$R = \frac{v_0^2}{g}2\cos^2\theta_0\tan\theta_0 = \frac{v_0^2}{g}\sin 2\theta_0$$

which has the maximum at $\sin 2\theta_0 = 1$ or $\theta_0 = \frac{\pi}{4}$.

4.5 Circular Motion

If a particle moves on a circle of radius R centered at the origin, then

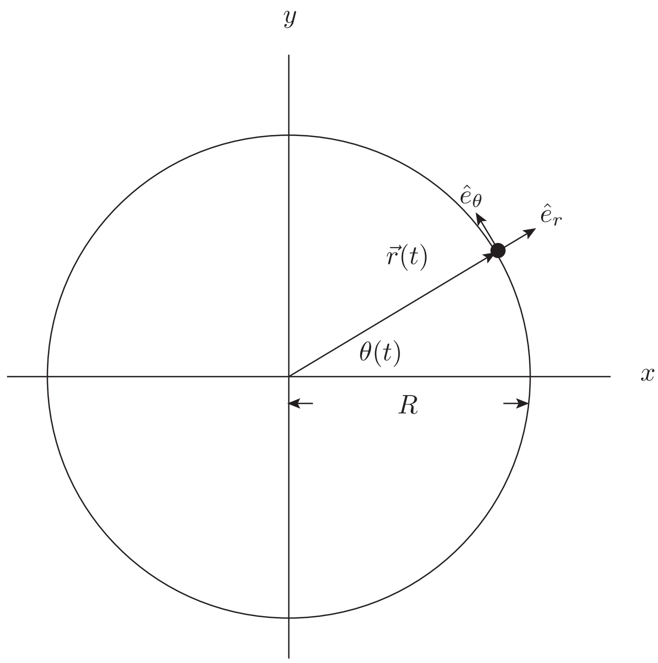
$$|\vec{r}(t)| = R \text{ or } \vec{r}\cdot\vec{r} = R^2$$

Thus

$$0 = \frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d\vec{r}}{dt}\cdot\vec{r} + \vec{r}\cdot\frac{d\vec{r}}{dt} = 2\vec{r}\cdot\frac{d\vec{r}}{dt} = 2\vec{r}\cdot\vec{v}$$

For a particle moving on a circle, the velocity is perpendicular to the position vector extending from the center of the circle to the position of the particle.

Let θ be the angle between $\vec{r}$ and $\hat{\imath}$ which is the direction of the x-axis.



Then,

$$\vec{r}\,(t) = x\,(t)\,\hat{\imath} + y\,(t)\,\hat{\jmath} = R\,(\cos\theta\,(t)\,\hat{\imath} + \sin\theta\,(t)\,\hat{\jmath})$$

The vector $\cos\theta\,(t)\,\hat{\imath} + \sin\theta\,(t)\,\hat{\jmath} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{R}$ is a unit vector denoted by

$$\hat{e}_r = \cos\theta\,(t)\,\hat{\imath} + \sin\theta\,(t)\,\hat{\jmath}$$

$\hat{e}_r$ points to the direction of the particle in motion and changes in time through the dependence of $\theta\,(t)$. It has the derivative

$$\begin{aligned}\frac{d\hat{e}_r\,(t)}{dt} &= \frac{d\theta\,(t)}{dt}\frac{d\hat{e}_r}{d\theta\,(t)} = \frac{d\theta\,(t)}{dt}\frac{d\,(\cos\theta\,(t)\,\hat{\imath} + \sin\theta\,(t)\,\hat{\jmath})}{d\theta\,(t)}\\ &= \frac{d\theta\,(t)}{dt}\,(-\sin\theta\,(t)\,\hat{\imath} + \cos\theta\,(t)\,\hat{\jmath})\end{aligned}$$

The rate of change $\frac{d\theta(t)}{dt}$ for the angle between the position vector and the x-axis is called the angular velocity ω:

$$\omega\,(t) = \frac{d\theta\,(t)}{dt}$$

Note θ is sometimes called angular displacement. The derivative of $\omega(t)$ with respect to time is called the angular acceleration $\alpha(t)$:

$$\alpha(t) = \frac{d\omega(t)}{dt}$$

The vector $-\sin(t)\,\hat{\imath} + \cos\theta(t)\,\hat{\jmath}$ is of unit length and is denoted by

$$\hat{e}_\theta = -\sin\theta(t)\,\hat{\imath} + \cos\theta(t)\,\hat{\jmath}$$

Note that $\hat{e}_\theta \cdot \hat{e}_r = 0$ and $\hat{e}_\theta$ is perpendicular to $\hat{e}_r$ or to the position vector. $\hat{e}_\theta$ is thus tangent to the circle as shown in the above figure. $\hat{e}_\theta$ also changes in time and has the derivative

$$\begin{aligned}\frac{d\hat{e}_\theta(t)}{dt} &= \frac{d\theta(t)}{dt}\frac{d\hat{e}_\theta(t)}{d\theta(t)} = \omega\frac{d}{d\theta(t)}\left(-\sin\theta(t)\,\hat{\imath} + \cos\theta(t)\,\hat{\jmath}\right) \\ &= -\omega\left(\cos\theta(t)\,\hat{\imath} + \sin\theta(t)\,\hat{\jmath}\right) = -\omega\hat{e}_r\end{aligned}$$

To summarize, we have

$$\frac{d\hat{e}_r}{dt} = \omega\hat{e}_\theta, \frac{d\hat{e}_\theta}{dt} = -\omega\hat{e}_r \tag{3}$$

Since position vector $\vec{r} = R\hat{e}_r$, the velocity $\vec{v}$ can be written as

$$\vec{v} = \frac{d\vec{r}}{dt} = R\frac{d\hat{e}_r}{dt} = R\omega\hat{e}_\theta$$

which shows that the speed for the particle in circular motion is

$$v = |\vec{v}| = |R\omega\hat{e}_\theta| = R|\omega|$$

4.5.1 Uniform Circular Motion

If there is no angular acceleration, $\alpha = 0$, the particle's angular velocity $\omega = \omega_0$ is constant and its angular displacement changes linearly with respect to time

$$\theta(t) = \omega_0 t + \theta_0$$

To complete a revolving motion, the angular displacement changes by an increment of 2π. So the period of the motion T must satisfy

$$2\pi = \theta(t+T) - \theta(t) = \omega_0 T$$

or

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi R}{\omega_0 R} = \frac{2\pi R}{v}$$

The frequency

$$f = \frac{1}{T} = \frac{\omega_0}{2\pi}$$

is the number of revolutions per second. The angular velocity (sometimes called angular frequency) ω_0 is related to the frequency f by a factor 2π:

$$\omega_0 = 2\pi f$$

For a particle in a uniform circular motion with $\omega(t) = \omega_0$, thanks to ,

$$\vec{r} = R\hat{e}_r,$$

$$\vec{v} = \frac{d\vec{r}}{dt} = R\frac{d\hat{e}_r}{dt} = R\omega_0\hat{e}_\theta,$$

and

$$\vec{a} = \frac{d\vec{v}}{dt} = R\omega_0\frac{d\hat{e}_\theta}{dt} = -R\omega_0^2\hat{e}_r = -\omega_0^2\vec{r}$$

Thus the acceleration is in the opposite direction of $\vec{r}$. This is called the centripetal acceleration and its magnitude is

$$|\vec{a}| = R\omega_0^2 = \frac{(R\omega_0)^2}{R} = \frac{v^2}{R}$$

4.5.2 Another Derivation of Centripetal Acceleration

There is another derivation for the centripetal acceleration in uniform motion without referring to the coordinate system. Since $|\vec{r}| = R$ and $|\vec{v}| = v$ is constant, we have

$$0 = \frac{d\vec{r}\cdot\vec{r}}{dt} = 2\vec{r}\cdot\vec{v}$$

and

$$0 = \frac{d\vec{v}\cdot\vec{v}}{dt} = 2\vec{a}\cdot\vec{v}$$

So both $\vec{r}$ and $\vec{a}$ are perpendicular to $\vec{v}$ and therefore they must be parallel or anti-parallel. We may write

$$\vec{a} = s\vec{r}$$

Now

$$0 = \frac{d^2\vec{r}\cdot\vec{r}}{dt^2} = 2\frac{d\left(\vec{r}\cdot\vec{v}\right)}{dt} = 2\left(\vec{v}\cdot\vec{v} + \vec{r}\cdot\vec{a}\right)$$

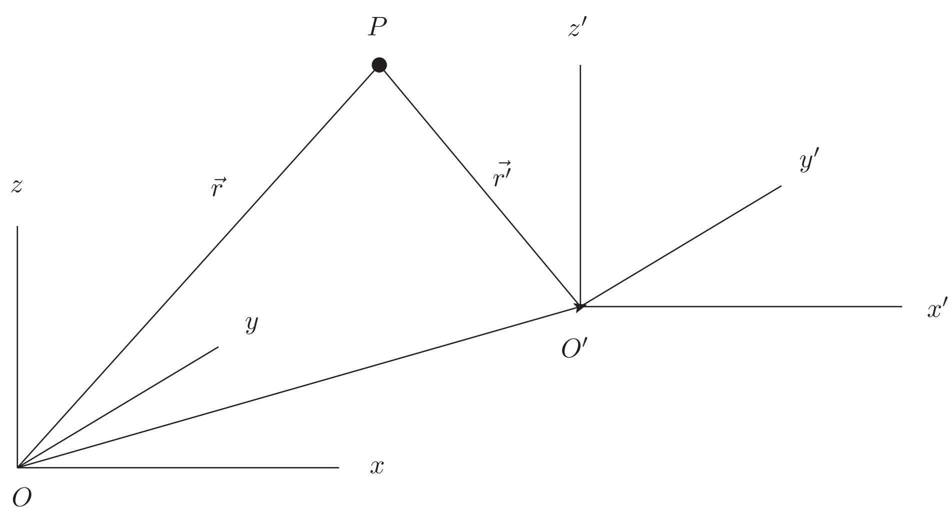
So

$$\vec{v}\cdot\vec{v} = -\vec{r}\cdot\vec{a} = -s\vec{r}\cdot\vec{r}$$

$$\vec{a} = s\vec{r} = -\frac{\vec{v}\cdot\vec{v}}{\vec{r}\cdot\vec{r}}\vec{r} = -\frac{v^2}{R}\hat{e}_r$$

4.6 Relative Motion

The position vector depends on the reference frame. Suppose we have two frames: xyz with origin O and $x'y\prime z\prime$ with origin O'. We will require the x, y, and z axes are parallel to the x', y', and z' axes respectively in this course. The case in which the two reference frames are not parallel are handled in more advanced course.



Let $\vec{r}$ be the position vector in frame O and $\vec{r\prime}$ be the position vector in frame O'. Then

$$\vec{r} = \overrightarrow{OP}, \vec{r\prime} = \overrightarrow{O'P}$$

Since

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$$

we have

$$\vec{r} = \vec{r}' + \overrightarrow{OO'}$$

The velocity in frame O is $\vec{v} = \frac{d\vec{r}}{dt}$ and the velocity in frame O is $\vec{v}' = \frac{d\vec{r}'}{dt'}$. We assume that the time is universal and $t' = t$. (This assumption is actually incorrect and will be discussed in relativity next semester. $t' = t$ does not contradicts special relativity when the particle moves at a much smaller speed in comparison to the speed of light.) Taking the derivative with respect to t for the above identity, we get

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d}{dt}\overrightarrow{OO'} = \vec{v}' + \vec{u}$$

where $\vec{u} = \frac{d}{dt}\overrightarrow{OO'}$ is the velocity of O' observed in frame O. Taking another time derivative of the above, we can relate the acceleration $\vec{a}$ and $\vec{a}'$:

$$\vec{a} = \vec{a}' + \frac{d\vec{u}}{dt}$$

If the two reference frames are moving at constant velocity relative to each other $\frac{d\vec{u}}{dt} = 0$, then

$$\vec{a} = \vec{a}'$$

and the accelerations $\vec{a}$ and $\vec{a}'$ observed are the same.