Chapter 9: Differential Analysis of Fluid Flow

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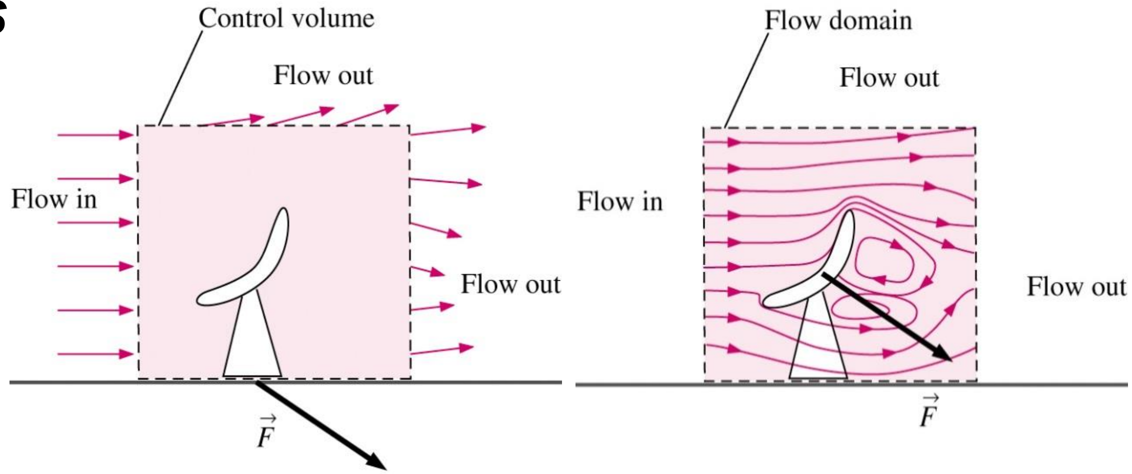
Objectives

1. Understand how the differential equations of mass and momentum conservation are derived.
2. Calculate the stream function and pressure field, and plot streamlines for a known velocity field.
3. Obtain analytical solutions of the equations of motion for simple flows.

Introduction

- Recall
  - Chap 5: Control volume (CV) versions of the laws of conservation of mass and energy
  - Chap 6: CV version of the conservation of momentum
- CV, or integral, forms of equations are useful for determining overall effects
- However, we cannot obtain detailed knowledge about the flow field ***inside*** the CV $\Rightarrow$ motivation for ***differential analysis***



Introduction

- Example: incompressible Navier-Stokes equations

$$\nabla \cdot \vec{V} = 0$$

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho \left(\vec{V} \cdot \nabla \right) \vec{V} = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{g}$$

- We will learn:
  - Physical meaning of each term
  - How to derive
  - How to solve

Introduction

- For example, how to solve?

Step	Analytical Fluid Dynamics (Chapter 9)	Computational Fluid Dynamics (Chapter 15)
1	Setup Problem and geometry, identify all dimensions and parameters	
2	List all assumptions, approximations, simplifications, boundary conditions	
3	Simplify PDE's	Build grid / discretize PDE's
4	Integrate equations	Solve algebraic system of equations including I.C.'s and B.C's
5	Apply I.C.'s and B.C.'s to solve for constants of integration	
6	Verify and plot results	Verify and plot results

Conservation of Mass

- Recall CV form (Chap 5) from Reynolds Transport Theorem (RTT)

$$0 = \int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \int_{CS} \rho \left(\vec{V} \cdot \vec{n} \right) \, dA$$

- We'll examine two methods to derive differential form of conservation of mass
  - Divergence (Gauss's) Theorem
  - Differential CV and Taylor series expansions

Conservation of Mass
Divergence Theorem

- Divergence theorem allows us to transform a volume integral of the divergence of a vector into an area integral over the surface that defines the volume.

$$\int_{\mathcal{V}} \nabla \cdot \vec{G}\, d\mathcal{V} = \oint_{A} \vec{G} \cdot \vec{n}\, dA$$

Conservation of Mass
Divergence Theorem

- Rewrite conservation of mass

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}\mathcal{V} + \oint_A \rho \left(\vec{V} \cdot \vec{n} \right) \, \mathrm{d}A = 0$$

- Using divergence theorem, replace area integral with volume integral and collect terms

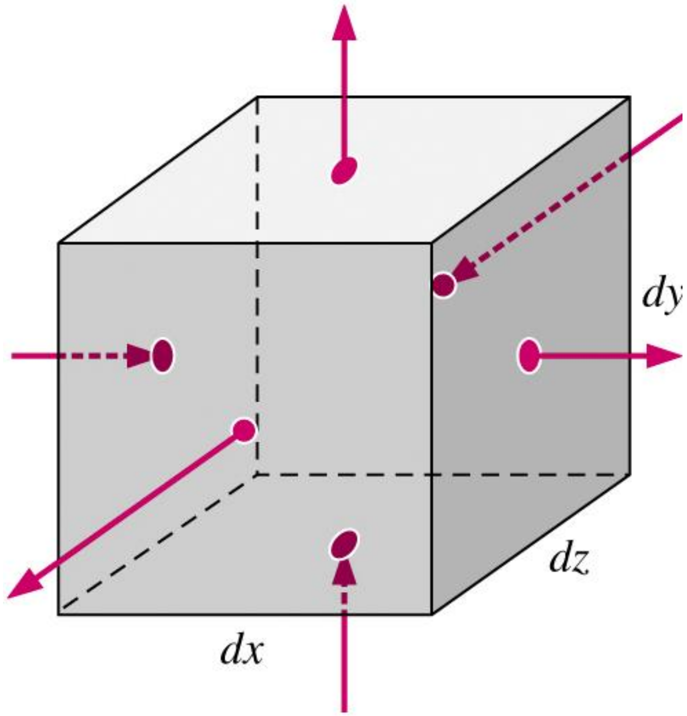
$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \nabla \cdot \rho \vec{V} \, \mathrm{d}\mathcal{V} = 0 \quad \Longrightarrow \quad \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) \right] \mathrm{d}\mathcal{V} = 0$$

- Integral holds for ***ANY*** CV, therefore:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) = 0}$$

Conservation of Mass
Differential CV and Taylor series

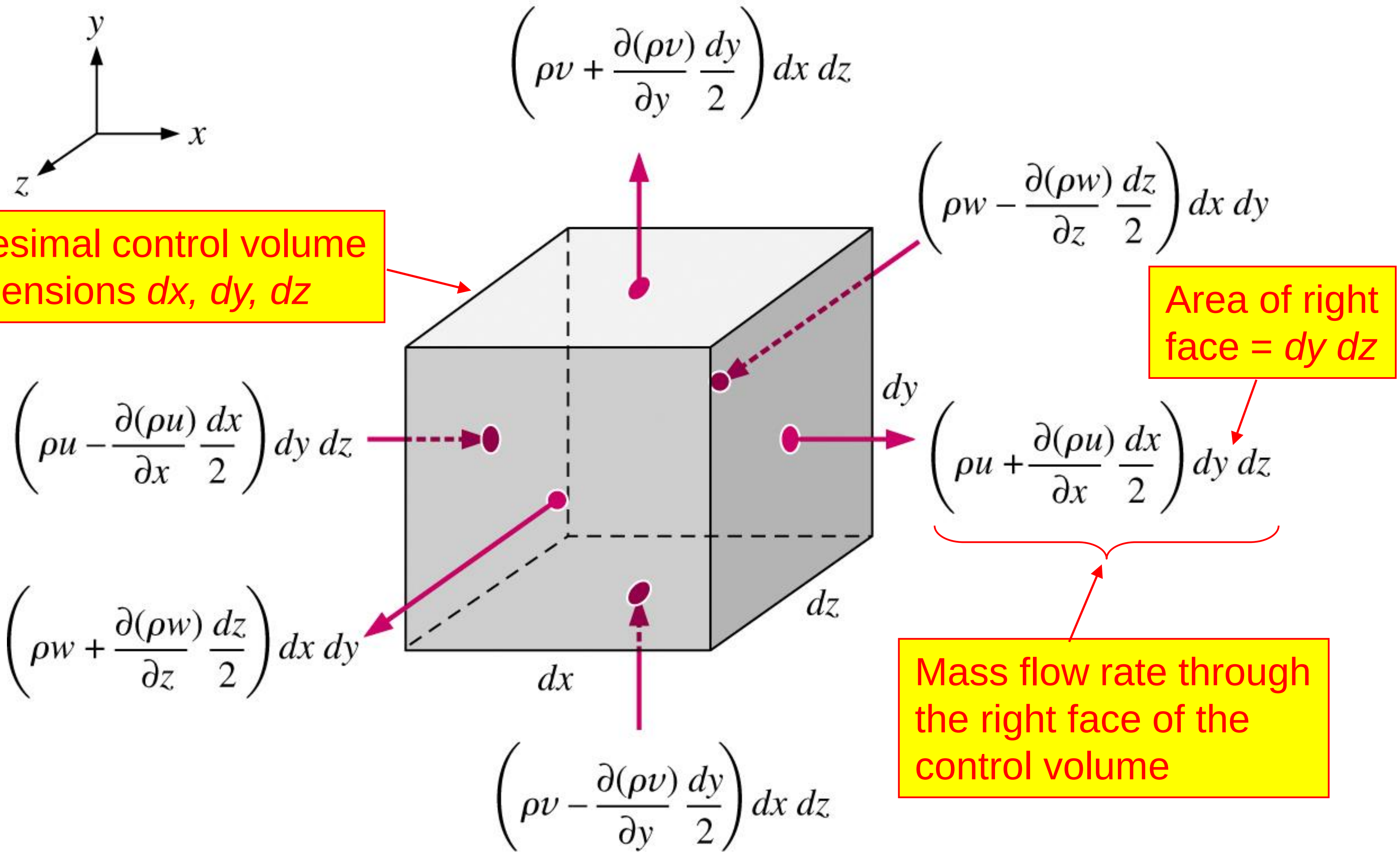
- First, define an infinitesimal control volume *dx* x *dy* x *dz*
- Next, we approximate the mass flow rate into or out of each of the 6 faces using Taylor series expansions around the center point, e.g., at the right face



Ignore terms higher than order *dx*

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \cancel{\frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2} + \ldots$$

Conservation of Mass
Differential CV and Taylor series



Conservation of Mass
Differential CV and Taylor series

- Now, sum up the mass flow rates into and out of the 6 faces of the CV

Net mass flow rate into CV:

$$\sum_{in} \dot{m} \approx \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\,dz + \left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\,dz + \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dx}{2}\right) dx\,dy$$

Net mass flow rate out of CV:

$$\sum_{out} \dot{m} \approx \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\,dz + \left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\,dz + \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dx}{2}\right) dx\,dy$$

- Plug into integral conservation of mass equation

$$\int_{CV} \frac{\partial \rho}{\partial t}\, \mathrm{d}\mathcal{V} = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

Conservation of Mass
Differential CV and Taylor series

- After substitution,

$$\frac{\partial \rho}{\partial t}\,dx\,dy\,dz = -\frac{\partial(\rho u)}{\partial x}\,dx\,dy\,dz - \frac{\partial(\rho v)}{\partial y}\,dx\,dy\,dz - \frac{\partial(\rho w)}{\partial z}\,dx\,dy\,dz$$

- Dividing through by volume *dxdydz*

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

Or, if we apply the definition of the divergence of a vector

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0}$$

Conservation of Mass

Alternative form

- Use product rule on divergence term

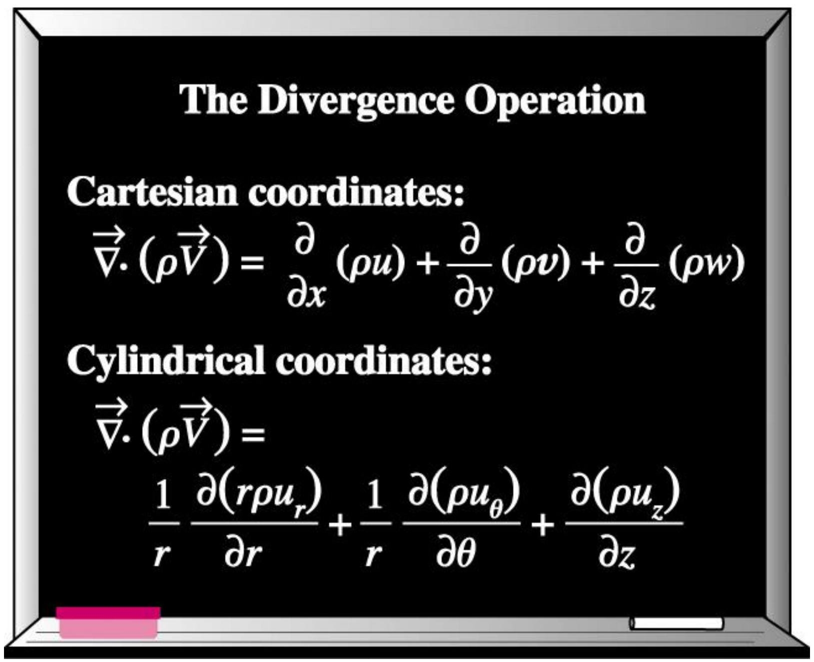
$$\frac{\partial \rho}{\partial t} + \boxed{\vec{\nabla} \cdot \left(\rho \vec{V}\right)} = \frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0$$

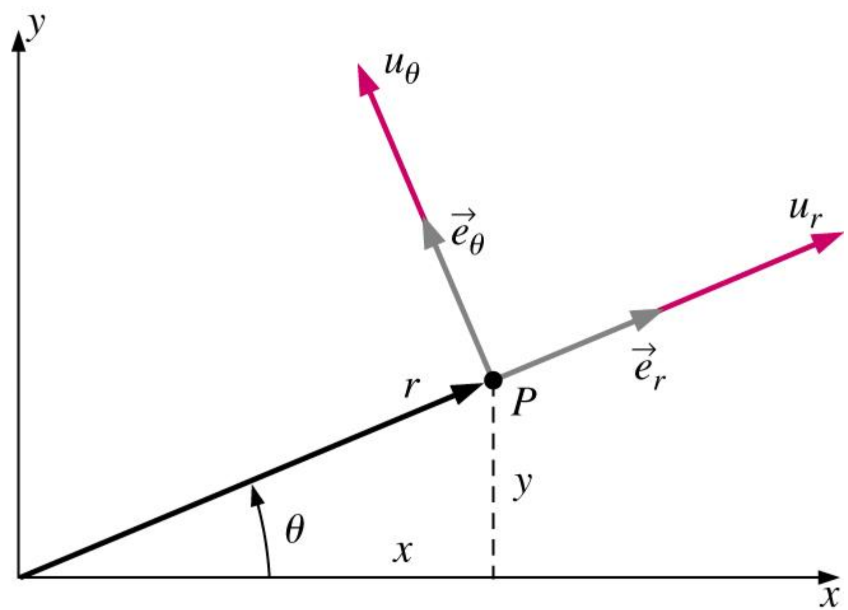
$$\boxed{\frac{D\rho}{Dt}} + \rho \nabla \cdot \vec{V} = 0$$

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{V} = 0}$$

Conservation of Mass
Cylindrical coordinates

- There are many problems which are simpler to solve if the equations are written in cylindrical-polar coordinates
- Easiest way to convert from Cartesian is to use vector form and definition of divergence operator in cylindrical coordinates





Conservation of Mass
Cylindrical coordinates

$$\vec{\nabla} = \frac{1}{r}\frac{\partial(r)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$

$$\vec{V} = U_r\hat{e}_r + U_\theta\hat{e}_\theta + U_z\hat{e}_z$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho U_\theta)}{\partial \theta} + \frac{\partial(\rho U_z)}{\partial z} = 0}$$

Conservation of Mass
Special Cases

- Steady compressible flow

$$\cancel{\frac{\partial \rho}{\partial t}} + \vec{\nabla} \cdot \left(\rho \vec{V} \right) = 0$$

$$\boxed{\vec{\nabla} \cdot \left(\rho \vec{V} \right) = 0}$$

Cartesian

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Cylindrical

$$\frac{1}{r}\frac{\partial(r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho U_\theta)}{\partial \theta} + \frac{\partial(\rho U_z)}{\partial z} = 0$$

Conservation of Mass
Special Cases

- Incompressible flow

$$\frac{\partial \rho}{\partial t} = 0 \text{ and } \rho = \text{constant}$$

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

Cartesian

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical

$$\frac{1}{r}\frac{\partial (rU_r)}{\partial r} + \frac{1}{r}\frac{\partial (U_\theta)}{\partial \theta} + \frac{\partial (U_z)}{\partial z} = 0$$

Conservation of Mass

- In general, continuity equation cannot be used by itself to solve for flow field, however it can be used to
  1. Determine if velocity field is incompressible
  2. Find missing velocity component

EXAMPLE 9–4: Finding a Missing Velocity Component

- Two velocity components of a steady, incompressible, three-dimensional flow field are known, namely, $u = ax^2 + by^2 + cz^2$ and $w = axz + byz^2$, where *a*, *b*, and *c* are constants. The *y* velocity component is missing. Generate an expression for *v* as a function of *x*, *y*, and *z*.
- Solution:

Condition for incompressibility:

$$\frac{\partial v}{\partial y} = -\underbrace{\frac{\partial u}{\partial x}}_{2ax} - \underbrace{\frac{\partial w}{\partial z}}_{ax + 2byz} \quad \rightarrow \quad \frac{\partial v}{\partial y} = -3ax - 2byz$$

- Therefore,

$$v = -3axy - by^2z + f(x, z)$$

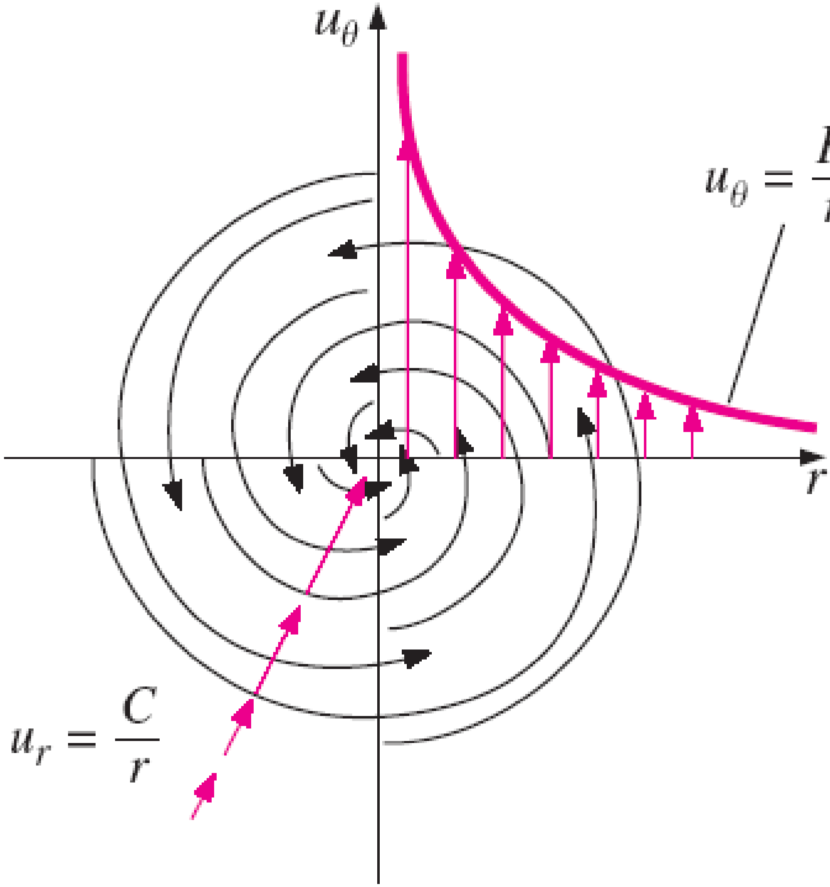
EXAMPLE 9–5: Two-Dimensional, Incompressible, Vortical Flow

- Consider a two-dimensional, incompressible flow in cylindrical coordinates; the tangential velocity component is $u_\theta = K/r$, where K is a constant. This represents a class of vortical flows. Generate an expression for the other velocity component, u_r.
- Solution: The incompressible continuity equation for this two dimensional case simplifies to

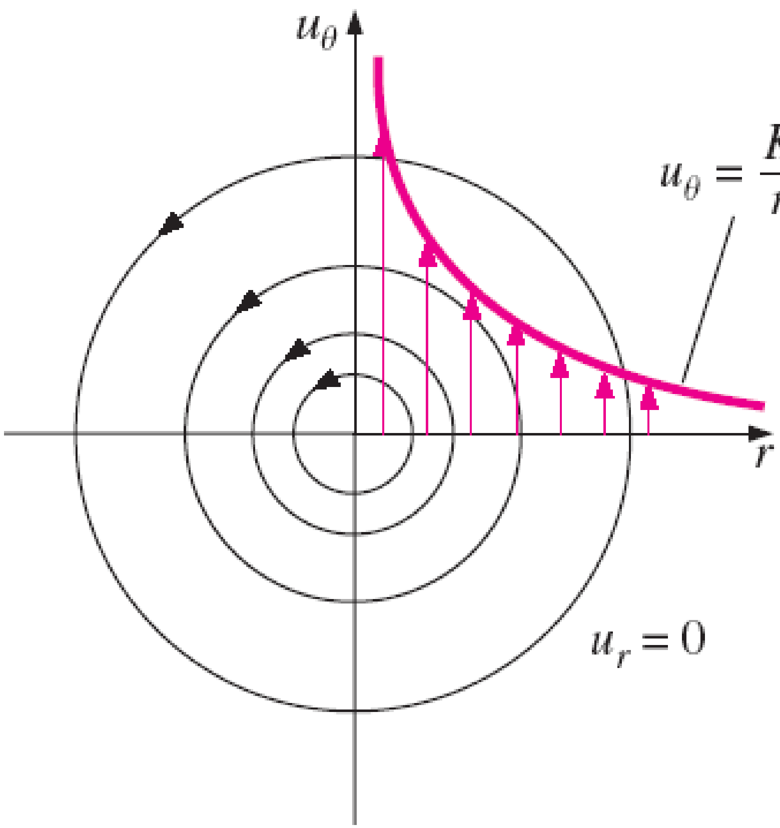
$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \underbrace{\cancel{\frac{\partial u_z}{\partial z}}}_{0\ (2\text{-D})} = 0 \quad \rightarrow \quad \frac{\partial(ru_r)}{\partial r} = -\frac{\partial u_\theta}{\partial \theta}$$

$$\frac{\partial(ru_r)}{\partial r} = 0 \quad \rightarrow \quad ru_r = f(\theta, t) \quad \Rightarrow \quad \boldsymbol{u_r = \frac{f(\theta, t)}{r}}$$

EXAMPLE 9–5: Two-Dimensional, Incompressible, Vortical Flow



A spiraling line vortex/sink flow



Line Vortex

The Stream Function

- Consider the continuity equation for an incompressible 2D flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- Substituting the clever transformation

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

- Gives

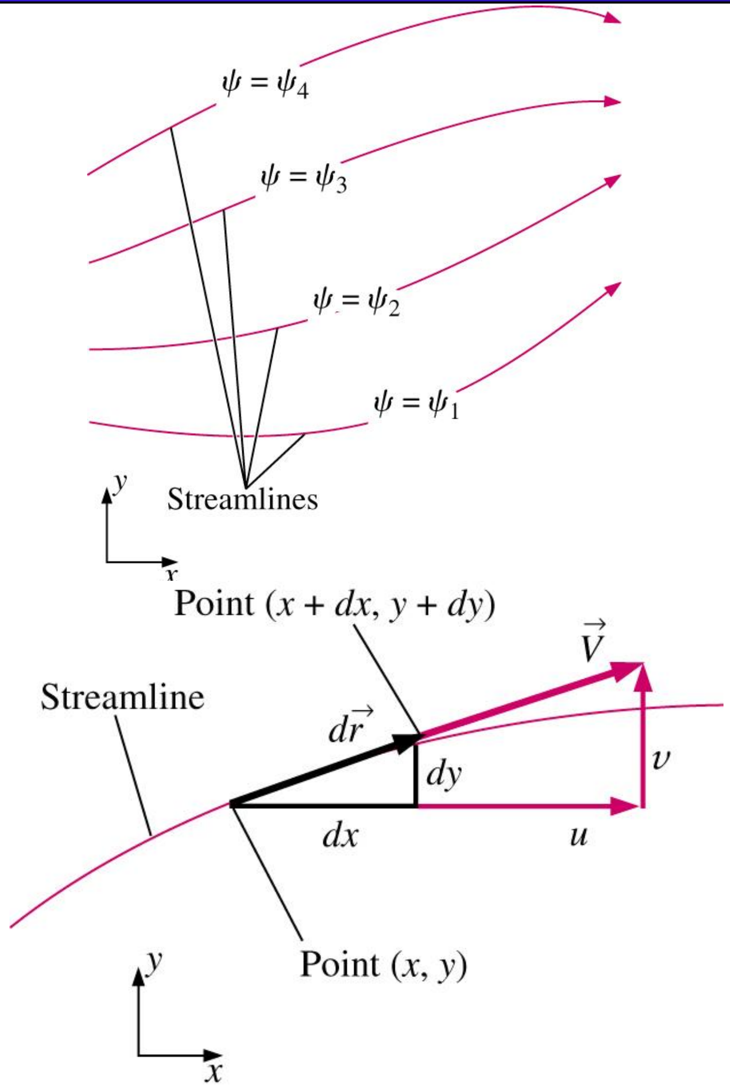
$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \equiv 0$$

This is true for any smooth function $\psi(x,y)$

The Stream Function

- Why do this?
  - Single variable ψ replaces (u,v). Once ψ is known, (u,v) can be computed.
  - Physical significance
    1. Curves of constant ψ are streamlines of the flow
    2. Difference in ψ between streamlines is equal to volume flow rate between streamlines
    3. The value of ψ increases to the left of the direction of flow in the *xy*-plane, "left-side convention."

The Stream Function
Physical Significance



Recall from Chap. 4 that along a streamline

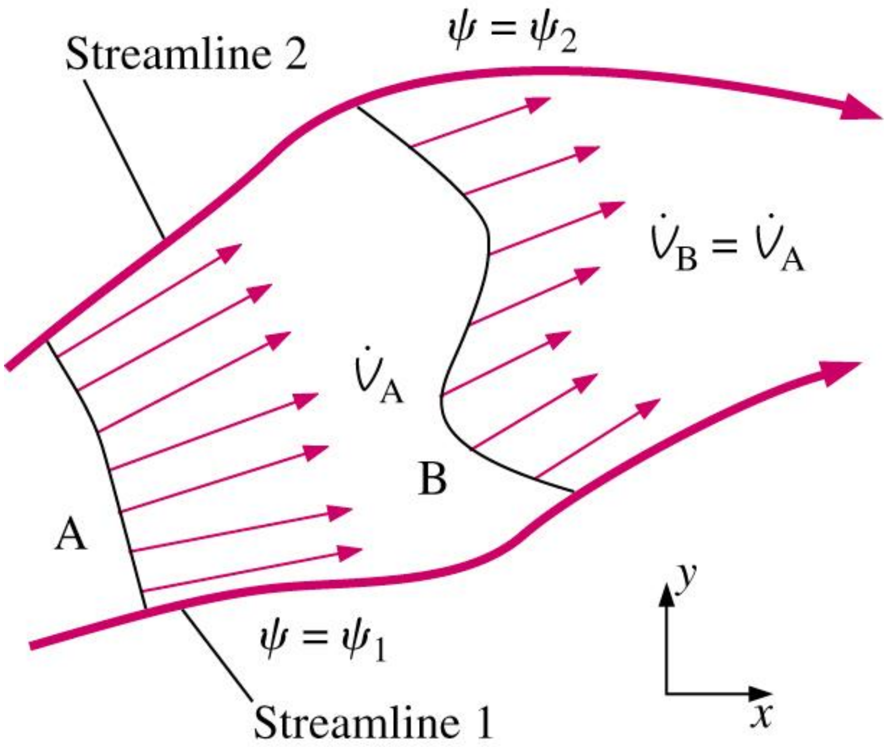
$$\frac{dy}{dx} = \frac{v}{u} \;\Rightarrow\; -v\,dx + u\,dy = 0$$

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$$

$$d\psi = 0$$

$\therefore$ Change in ψ along streamline is zero

The Stream Function
Physical Significance



Difference in ψ between streamlines is equal to volume flow rate between streamlines (Proof on black board)

$$\boxed{\dot{V}_A = \dot{V}_B = \psi_2 - \psi_1}$$

The Stream Function in Cylindrical Coordinates

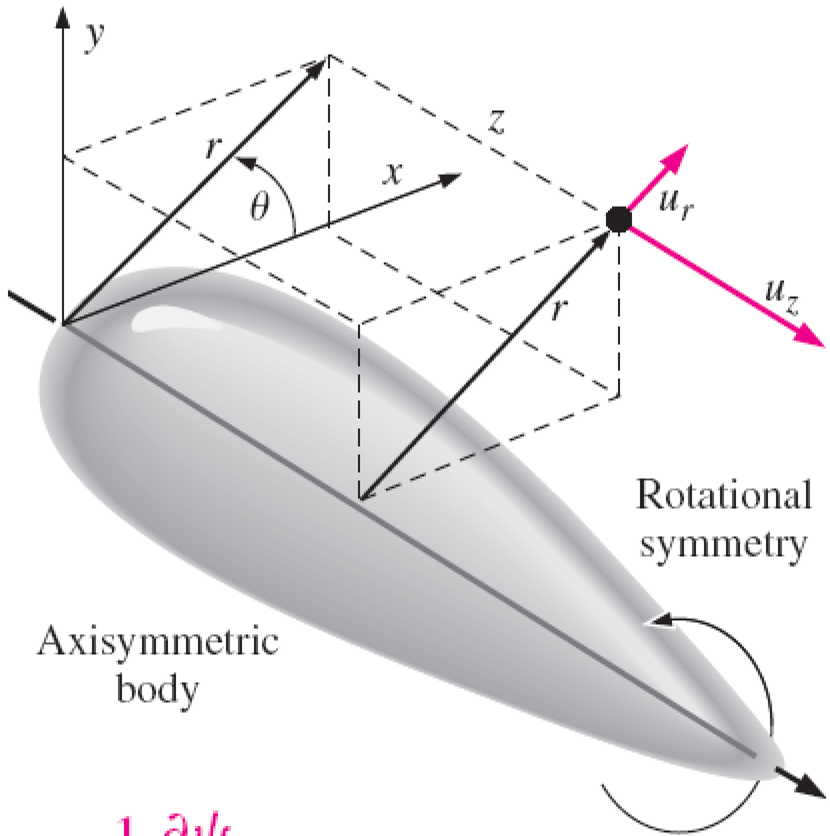
- *Incompressible, planar stream function in cylindrical coordinates*:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

- For incompressible axisymmetric flow, the continuity equation is

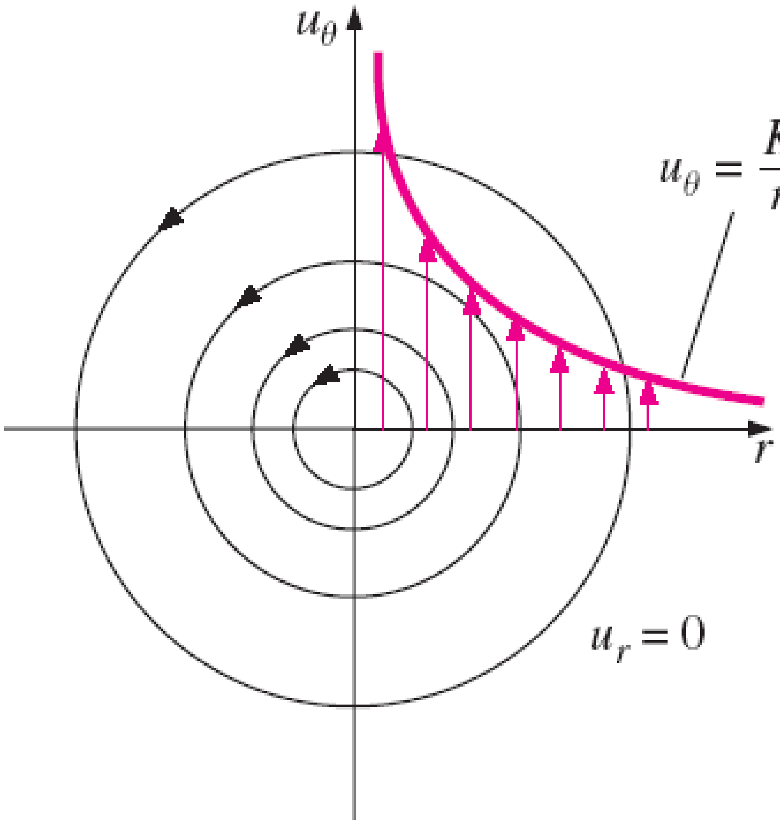
$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial (u_z)}{\partial z} = 0$$

$$\Rightarrow \quad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad \text{and} \qquad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$



EXAMPLE 9–12 Stream Function in Cylindrical Coordinates

- Consider a line vortex, defined as steady, planar, incompressible flow in which the velocity components are $u_r = 0$ and $u_\theta = K/r$, where K is a constant. Derive an expression for the stream function $\psi(r, \theta)$, and prove that the streamlines are circles.



Line Vortex

EXAMPLE 9–12 Stream Function in Cylindrical Coordinates

- Solution:

$$\frac{\partial \psi}{\partial r} = -u_\theta = -\frac{K}{r} \quad \rightarrow$$

$$\psi = -K \ln r + f(\theta)$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r} f'(\theta)$$

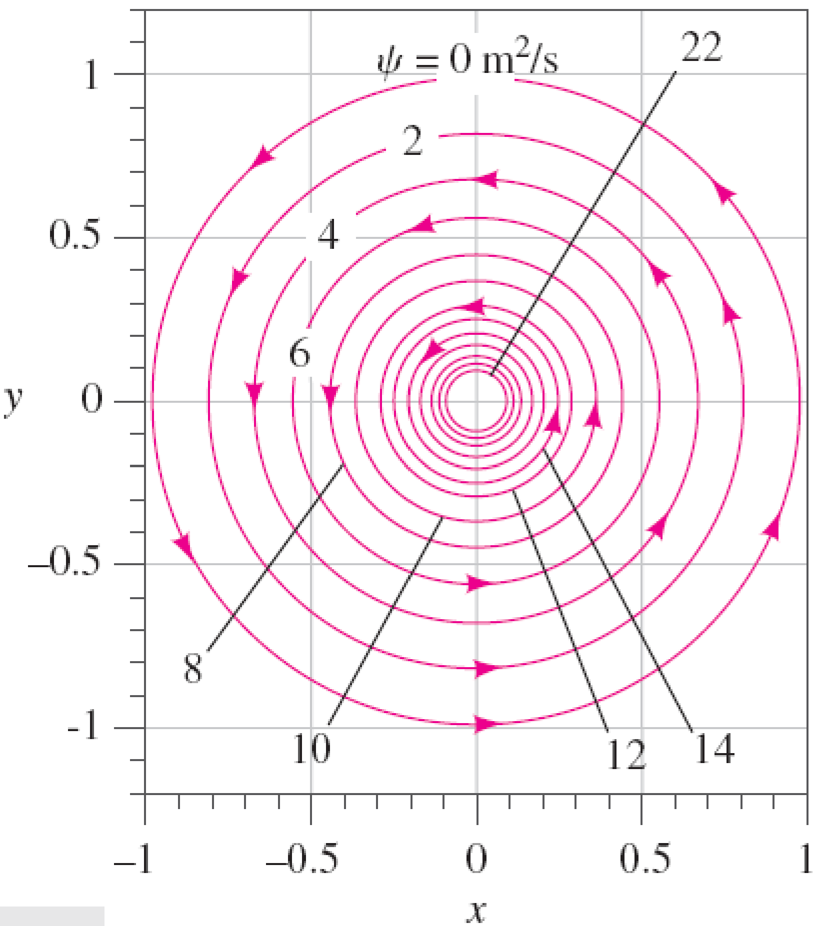
$$f'(\theta) = 0 \quad \rightarrow$$

$$f(\theta) = C$$

$$\Rightarrow \quad \psi = -K \ln r + C$$

Equation for streamlines:

$$r = e^{-(\psi - C)/K}$$



Conservation of Linear Momentum

- Recall CV form from Chap. 6

$$\sum \vec{F} = \underbrace{\int_{CV} \rho g \, \mathrm{d}\mathcal{V}}_{\text{Body Force}} + \underbrace{\boxed{\int_{CS} \sigma_{ij} \cdot \vec{n} \, \mathrm{d}A}}_{\text{Surface Force}} = \int_{CV} \frac{\partial}{\partial t} \left(\rho \vec{V} \right) \, \mathrm{d}\mathcal{V} + \boxed{\int_{CS} \left(\rho \vec{V} \right) \vec{V} \cdot \vec{n} \, \mathrm{d}A}$$

σ_{ij} = stress tensor

- Using the divergence theorem to convert area integrals

$$\int_{CS} \sigma_{ij} \cdot \vec{n} \, \mathrm{d}A = \int_{CV} \nabla \cdot \sigma_{ij} \, \mathrm{d}\mathcal{V}$$

$$\int_{CS} \left(\rho \vec{V} \right) \vec{V} \cdot \vec{n} \, \mathrm{d}A = \int_{CV} \nabla \cdot \left(\rho \vec{V} \vec{V} \right) \, \mathrm{d}\mathcal{V}$$

Conservation of Linear Momentum

- Substituting volume integrals gives,

$$\int_{CV} \left[\frac{\partial}{\partial t}\left(\rho \vec{V}\right) + \nabla \cdot \left(\rho \vec{V}\vec{V}\right) - \rho \vec{g} - \nabla \cdot \sigma_{ij} \right] d\mathcal{V} = 0$$

- Recognizing that this holds for <u>**any**</u> CV, the integral may be dropped

$$\boxed{\frac{\partial}{\partial t}\left(\rho \vec{V}\right) + \nabla \cdot \left(\rho \vec{V}\vec{V}\right) = \rho \vec{g} + \nabla \cdot \sigma_{ij}}$$

<u>*This is Cauchy's Equation*</u>

Can also be derived using infinitesimal CV and Newton's 2nd Law (see text)

Conservation of Linear Momentum

- Alternate form of the Cauchy Equation can be derived by introducing

$$\frac{\partial\left(\rho\vec{V}\right)}{\partial t}=\rho\frac{\partial\vec{V}}{\partial t}+\vec{V}\frac{\partial\rho}{\partial t}$$ (Chain Rule)

$$\nabla\cdot\left(\rho\vec{V}\vec{V}\right)=\vec{V}\nabla\cdot\left(\rho\vec{V}\right)+\rho\left(\vec{V}\cdot\nabla\right)\vec{V}$$

- Inserting these into Cauchy Equation and rearranging gives

$$\rho\left[\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\nabla\right)\vec{V}\right]=\rho\vec{g}+\nabla\cdot\sigma_{ij}$$

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}+\nabla\cdot\sigma_{ij}$$

Conservation of Linear Momentum

- Unfortunately, this equation is not very useful
  - 10 unknowns
    - Stress tensor, σ_{ij} : 6 independent components
    - Density ρ
    - Velocity, $\vec{V}$: 3 independent components
  - 4 equations (continuity + momentum)
  - 6 more equations required to close problem!

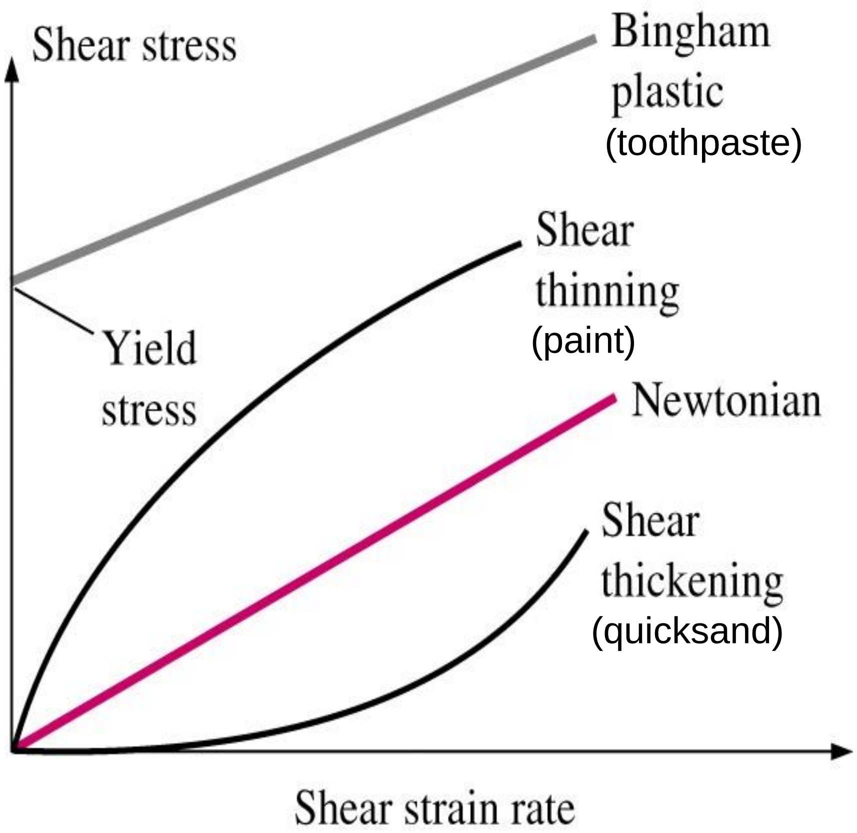
Navier-Stokes Equation

- First step is to separate σ_{ij} into pressure and viscous stresses

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \underbrace{\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}}_{\text{Viscous (Deviatoric) Stress Tensor}}$$

- Situation not yet improved
  - 6 unknowns in σ_{ij} $\Rightarrow$ 6 unknowns in τ_{ij} + 1 in *P*, *which means that we've added 1!*

Navier-Stokes Equation



Newtonian fluid includes most common fluids: air, other gases, water, gasoline

- Reduction in the number of variables is achieved by relating shear stress to strain-rate tensor.
- For Newtonian fluid with constant properties

$$\tau_{ij} = 2\mu\epsilon_{ij}$$

Newtonian closure is analogous to Hooke's Law for elastic solids

Navier-Stokes Equation

- Substituting Newtonian closure into stress tensor gives

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij}$$

- Using the definition of ε_{ij} (Chapter 4)

$$\sigma_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} 2\mu\frac{\partial U}{\partial x} & \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) & \mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) \\ \mu\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) & 2\mu\frac{\partial V}{\partial y} & \mu\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right) \\ \mu\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right) & \mu\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) & 2\mu\frac{\partial W}{\partial z} \end{pmatrix}$$

Navier-Stokes Equation

- Substituting σ_{ij} into Cauchy's equation gives the Navier-Stokes equations

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

$$\nabla \cdot \vec{V} = 0$$

Incompressible NSE written in vector form

- This results in a *closed* system of equations!
  - 4 equations (continuity and momentum equations)
  - 4 unknowns (U, V, W, p)

Navier-Stokes Equation

- In addition to vector form, incompressible N-S equation can be written in several other forms
  - Cartesian coordinates
  - Cylindrical coordinates
  - Tensor notation

Navier-Stokes Equation
Cartesian Coordinates

Continuity
$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0$$

X-momentum
$$\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}\right)=-\frac{\partial P}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)$$

Y-momentum
$$\rho\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}\right)=-\frac{\partial P}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}\right)$$

Z-momentum
$$\rho\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}\right)$$

See page 431 for equations in cylindrical coordinates

Navier-Stokes Equation
Tensor and Vector Notation

Tensor and Vector notation offer a more compact form of the equations.

Continuity

Tensor notation

$$\boxed{\frac{\partial U_i}{\partial x_i} = 0}$$

Vector notation

$$\boxed{\nabla \cdot \vec{V} = 0}$$

Conservation of Momentum

Tensor notation

$$\boxed{\rho \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \rho g_{x_i} + \mu \left(\frac{\partial^2 U_i}{\partial x_j \partial x_j} \right)}$$

Vector notation

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}}$$

Repeated indices are summed over j
($x_1 = x$, $x_2 = y$, $x_3 = z$, $U_1 = U$, $U_2 = V$, $U_3 = W$)

Differential Analysis of Fluid Flow Problems

- Now that we have a set of governing partial differential equations, there are 2 problems we can solve

1. Calculate pressure (*P*) for a known velocity field
2. Calculate velocity (*U, V, W*) and pressure (*P*) for known geometry, boundary conditions (BC), and initial conditions (IC)

EXAMPLE 9–13 Calculating the Pressure Field in Cartesian Coordinates

- Consider the steady, two-dimensional, incompressible velocity field, namely, $\vec{V} = (u, v) = (ax + b)\vec{i} + (-ay + cx)\vec{j}$ Calculate the pressure as a function of *x* and *y*.
- Solution: Check continuity equation,

$$\underbrace{\frac{\partial u}{\partial x}}_{a} + \underbrace{\frac{\partial v}{\partial y}}_{-a} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0\text{ (2-D)}} = a - a = 0$$

Consider the *y*-component of the Navier–Stokes equation:

$$\rho\left(\underbrace{\cancel{\frac{\partial v}{\partial t}}}_{0\text{ (steady)}} + \underbrace{u\frac{\partial v}{\partial x}}_{(ax+b)c} + \underbrace{v\frac{\partial v}{\partial y}}_{(-ay+cx)(-a)} + \underbrace{\cancel{w\frac{\partial v}{\partial z}}}_{0\text{ (2-D)}}\right) = -\frac{\partial P}{\partial y} + \underbrace{\cancel{\rho g_y}}_{0} + \mu\left(\underbrace{\cancel{\frac{\partial^2 v}{\partial x^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 v}{\partial y^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 v}{\partial z^2}}}_{0\text{ (2-D)}}\right)$$

EXAMPLE 9–13 Calculating the Pressure Field in Cartesian Coordinates

The *y*-momentum equation reduces to

$$\frac{\partial P}{\partial y} = \rho(-acx - bc - a^2y + acx) = \rho(-bc - a^2y)$$

In similar fashion, the *x*-momentum equation reduces to

$$\frac{\partial P}{\partial x} = \rho(-a^2x - ab)$$

Pressure field from y-momentum:

$$P(x, y) = \rho\left(-bcy - \frac{a^2y^2}{2}\right) + g(x)$$

$$\Rightarrow \quad \frac{\partial P}{\partial x} = g'(x) = \rho(-a^2x - ab)$$

EXAMPLE 9–13 Calculating the Pressure Field in Cartesian Coordinates

Then we can get

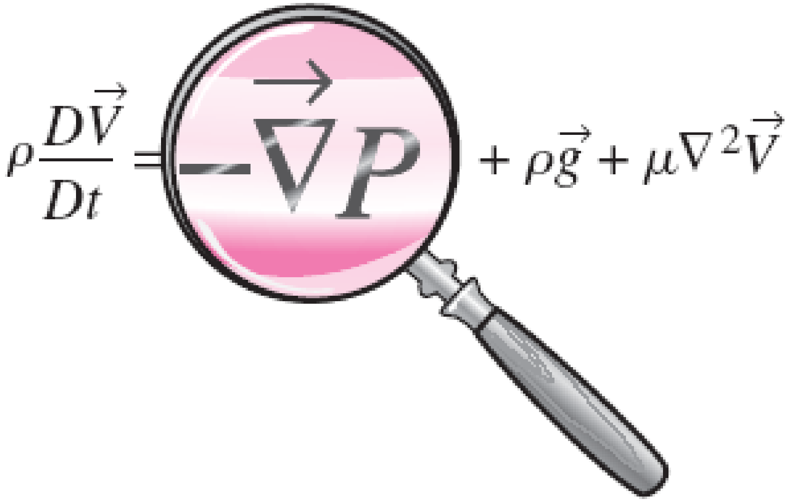
$$g(x) = \rho\left(-\frac{a^2x^2}{2} - abx\right) + C_1$$

Such that

$$P(x, y) = \rho\left(-\frac{a^2x^2}{2} - \frac{a^2y^2}{2} - abx - bcy\right) + C_1$$

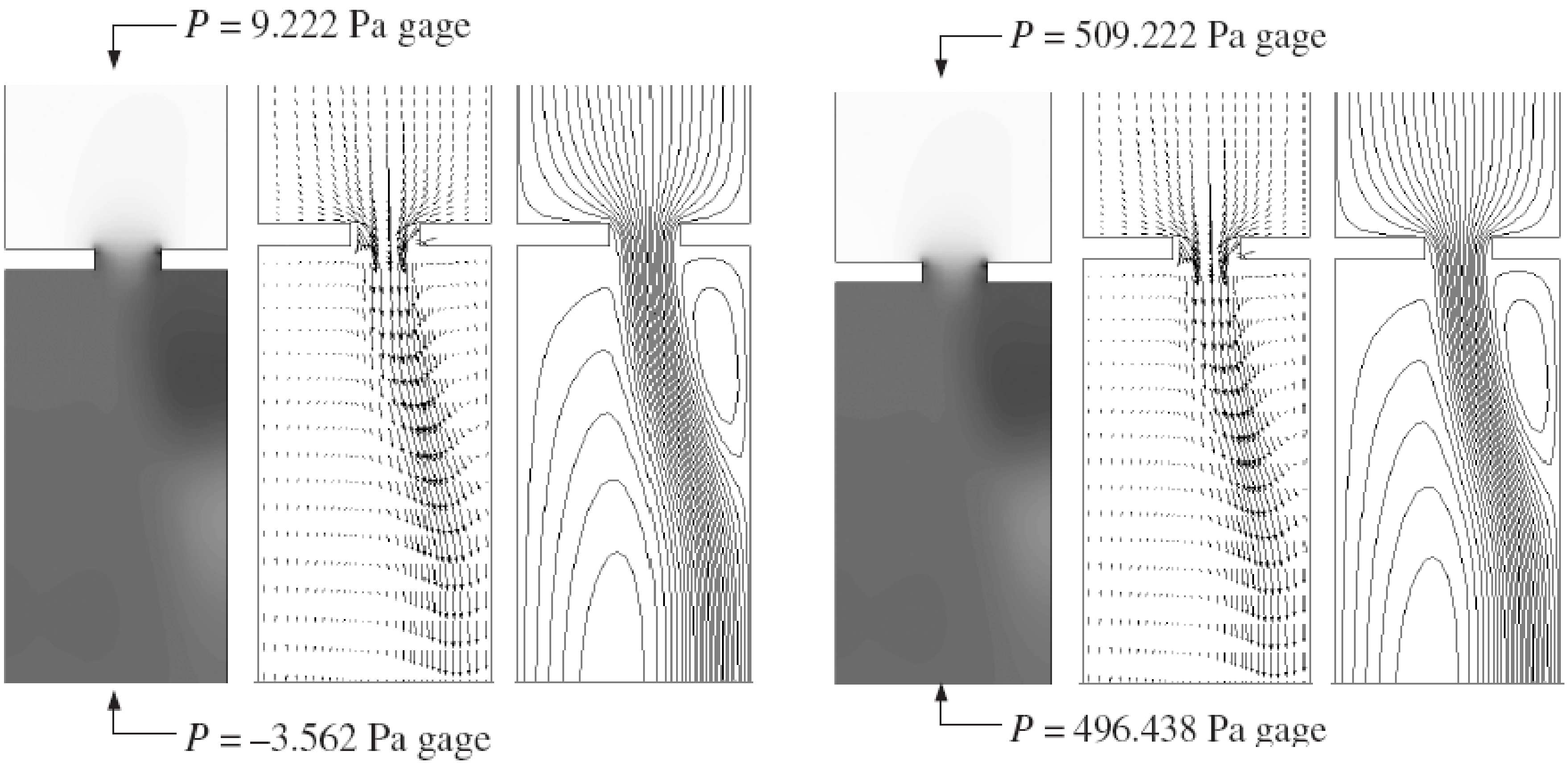
- Will the C_1 in the equation affect the velocity field? No. The velocity field in an incompressible flow is not affected by the absolute magnitude of pressure, but only by pressure differences.

EXAMPLE 9–13 Calculating the Pressure Field in Cartesian Coordinates



- From the Navier-Stokes equation, we know the velocity field is affected by pressure gradient.
- In order to determine that constant (C_1 in Example 9–13), we must measure (or otherwise obtain) *P* somewhere in the flow field. In other words, we require a pressure boundary condition. Please see the CFD results on the next page.

EXAMPLE 9–13 Calculating the Pressure Field in Cartesian Coordinates



Two cases are identical except for the pressure condition. The results of the velocity fields and streamline patterns confirm that the velocity field is affected by pressure gradient.

Exact Solutions of the NSE

- There are about 80 known exact solutions to the NSE
- The can be classified as:
  - Linear solutions where the convective $\left(\vec{V}\cdot\nabla\right)\vec{V}$ term is zero
  - Nonlinear solutions where convective term is not zero

- Solutions can also be classified by type or geometry
  1. Couette shear flows
  2. Steady duct/pipe flows
  3. Unsteady duct/pipe flows
  4. Flows with moving boundaries
  5. Similarity solutions
  6. Asymptotic suction flows
  7. Wind-driven Ekman flows

Exact Solutions of the NSE

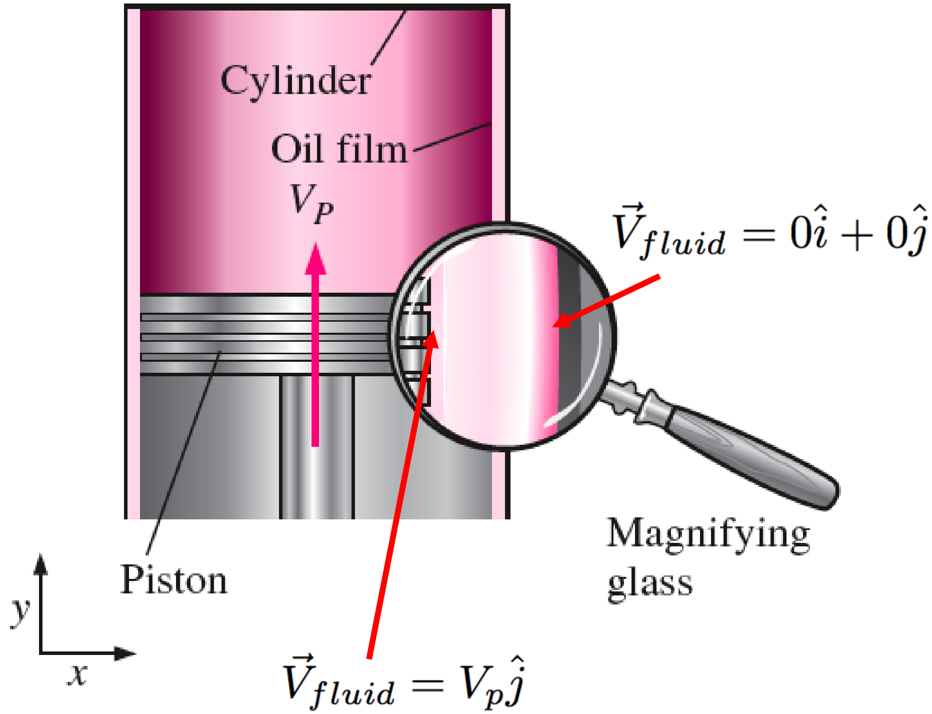
Procedure for solving continuity and NSE

1. Set up the problem and geometry, identifying all relevant dimensions and parameters
2. List all appropriate assumptions, approximations, simplifications, and boundary conditions
3. Simplify the differential equations as much as possible
4. Integrate the equations
5. Apply BC to solve for constants of integration
6. Verify results

Boundary conditions

- Boundary conditions are critical to exact, approximate, and computational solutions.
- Discussed in Chapters 9 & 15
  - BC's used in analytical solutions are discussed here
    - No-slip boundary condition
    - Interface boundary condition
  - These are used in CFD as well, plus there are some BC's which arise due to specific issues in CFD modeling. These will be presented in Chap. 15.
    - Inflow and outflow boundary conditions
    - Symmetry and periodic boundary conditions

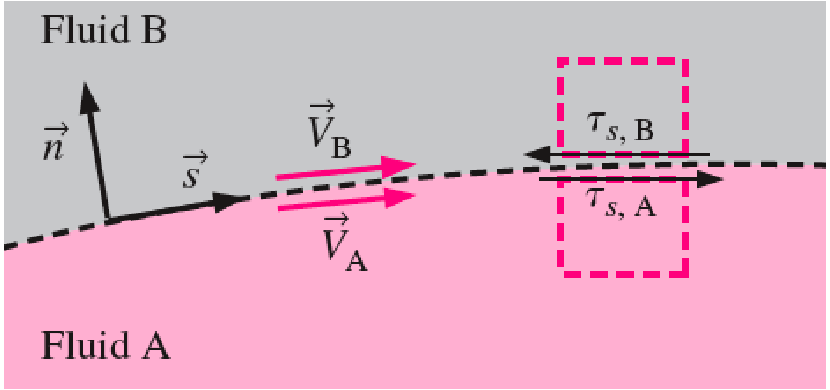
No-slip boundary condition



- For a fluid in contact with a solid wall, the velocity of the fluid must equal that of the wall

$$\vec{V}_{fluid} = \vec{V}_{wall}$$

Interface boundary condition



- When two fluids meet at an interface, the velocity and shear stress must be the same on both sides

$$\vec{V}_A = \vec{V}_B \qquad \tau_{s,A} = \tau_{s,B}$$

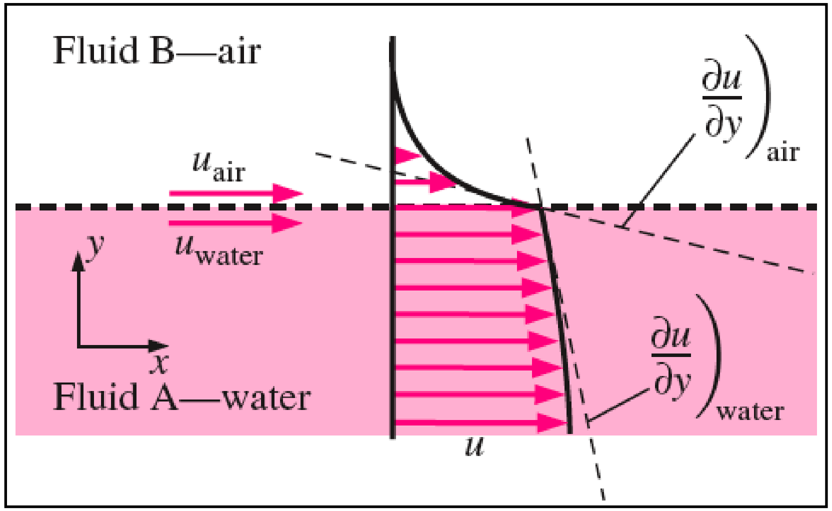
- If surface tension effects are negligible and the surface is nearly flat

$$P_A = P_B$$

Interface boundary condition

- Degenerate case of the interface BC occurs at the free surface of a liquid.
- Same conditions hold $u_{air} = u_{water}$

$$\tau_{s,water} = \mu_{water}\left(\frac{\partial u}{\partial y}\right)_{water} = \tau_{s,air} = \mu_{air}\left(\frac{\partial u}{\partial y}\right)_{air}$$



- Since $\mu_{air} << \mu_{water}$,

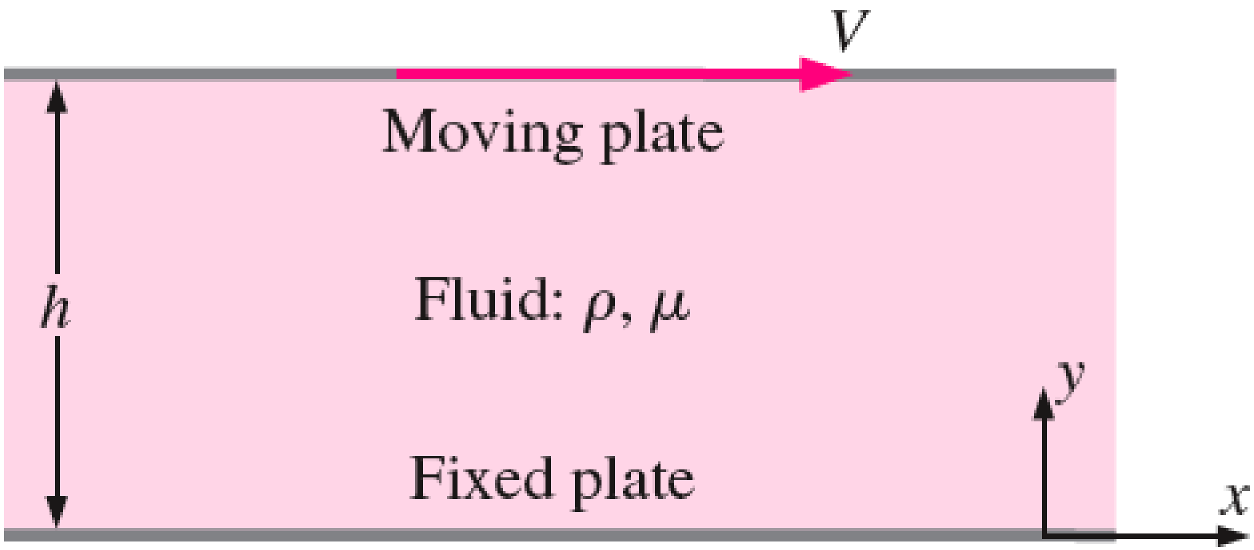
$$\left(\frac{\partial u}{\partial y}\right)_{water} \approx 0$$

- As with general interfaces, if surface tension effects are negligible and the surface is nearly flat $P_{water} = P_{air}$

Example exact solution (Ex. 9-15)
Fully Developed Couette Flow

- For the given geometry and BC's, calculate the velocity and pressure fields, and estimate the shear force per unit area acting on the bottom plate

- Step 1: Geometry, dimensions, and properties



Example exact solution (Ex. 9-15)
Fully Developed Couette Flow

- Step 2: Assumptions and BC's
  - Assumptions
    1. Plates are infinite in x and z
    2. Flow is steady, $\partial/\partial t = 0$
    3. Parallel flow, V=0
    4. Incompressible, Newtonian, laminar, constant properties
    5. No pressure gradient
    6. 2D, W=0, $\partial/\partial z = 0$
    7. Gravity acts in the -z direction, $\vec{g} = -g\vec{k}, g_z = -g$
  - Boundary conditions
    1. Bottom plate (y=0) : u=0, v=0, w=0
    2. Top plate (y=h) : u=V, v=0, w=0

Example exact solution (Ex. 9-15)
Fully Developed Couette Flow

- Step 3: Simplify

Note: these numbers refer to the assumptions on the previous slide

<u>Continuity</u>

$$\frac{\partial U}{\partial x} + \underbrace{\cancel{\frac{\partial V}{\partial y}}}_{3} + \underbrace{\cancel{\frac{\partial W}{\partial z}}}_{6} = 0$$

$$\boxed{\frac{\partial U}{\partial x} = 0}$$

This means the flow is "*fully developed*" or not changing in the direction of flow

<u>X-momentum</u>

$$\rho\left(\underbrace{\cancel{\frac{\partial U}{\partial t}}}_{2} + \underbrace{\cancel{U\frac{\partial U}{\partial x}}}_{\text{Cont.}} + \underbrace{\cancel{V\frac{\partial U}{\partial y}}}_{3} + \underbrace{\cancel{W\frac{\partial U}{\partial z}}}_{6}\right) = \underbrace{\cancel{-\frac{\partial P}{\partial x}}}_{5} + \underbrace{\cancel{\rho g_x}}_{7} + \mu\left(\underbrace{\cancel{\frac{\partial^2 U}{\partial x^2}}}_{\text{Cont.}} + \frac{\partial^2 U}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 U}{\partial z^2}}}_{6}\right)$$

$$\boxed{\frac{d^2 u}{dy^2} = 0}$$

Example exact solution (Ex. 9-15)
Fully Developed Couette Flow

- Step 3: Simplify, cont.

<u>Y-momentum</u>

$$\rho\left(\overset{2,3}{\cancel{\frac{\partial V}{\partial t}}}+\overset{3}{\cancel{U\frac{\partial V}{\partial x}}}+\overset{3}{\cancel{V\frac{\partial V}{\partial y}}}+\overset{3,6}{\cancel{W\frac{\partial V}{\partial z}}}\right)=-\frac{\partial P}{\partial y}+\overset{7}{\cancel{\rho g_y}}+\mu\left(\overset{3}{\cancel{\frac{\partial^2 V}{\partial x^2}}}+\overset{3}{\cancel{\frac{\partial^2 V}{\partial y^2}}}+\overset{3}{\cancel{\frac{\partial^2 V}{\partial z^2}}}\right)$$

$$\frac{\partial p}{\partial y}=0 \Rightarrow \boxed{p=p(z)}$$

<u>Z-momentum</u>

$$\rho\left(\overset{2,6}{\cancel{\frac{\partial W}{\partial t}}}+\overset{6}{\cancel{U\frac{\partial W}{\partial x}}}+\overset{6}{\cancel{V\frac{\partial W}{\partial y}}}+\overset{6}{\cancel{W\frac{\partial W}{\partial z}}}\right)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\overset{6}{\cancel{\frac{\partial^2 W}{\partial x^2}}}+\overset{6}{\cancel{\frac{\partial^2 W}{\partial y^2}}}+\overset{6}{\cancel{\frac{\partial^2 W}{\partial z^2}}}\right)$$

$$\frac{\partial p}{\partial z}=\rho g_z \Rightarrow \boxed{\frac{dp}{dz}=-\rho g}$$

Example exact solution (Ex. 9-15)
Fully Developed Couette Flow

- Step 4: Integrate

X-momentum

$$\frac{d^2u}{dy^2} = 0 \xrightarrow{\text{integrate}} \frac{du}{dy} = C_1 \xrightarrow{\text{integrate}} \boxed{u(y) = C_1 y + C_2}$$

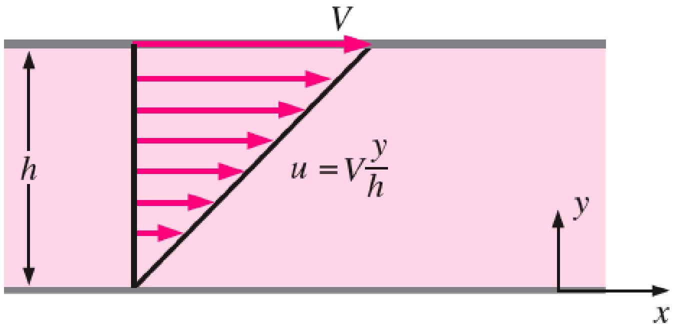
Z-momentum

$$\frac{dp}{dz} = -\rho g \xrightarrow{\text{integrate}} \boxed{p = -\rho g z + C_3}$$

Example exact solution (Ex. 9-15)
Fully Developed Couette Flow

- Step 5: Apply BC's
  - y=0, $u=0=C_1(0) + C_2 \Rightarrow \underline{C_2 = 0}$
  - y=h, $u=V=C_1h \Rightarrow \underline{C_1 = V/h}$
  - This gives

$$\boxed{u(y) = V\frac{y}{h}}$$



  - For pressure, no explicit BC, therefore C_3 can remain an arbitrary constant (recall only ∇P appears in NSE).
    - Let $p = p_0$ at $z = 0$ (C_3 renamed p_0)

$$\boxed{p(z) = p_0 - \rho g z}$$

1. Hydrostatic pressure
2. Pressure acts independently of flow

Example exact solution (Ex. 9-15)
Fully Developed Couette Flow

- Step 6: Verify solution by back-substituting into differential equations
  - Given the solution (u,v,w)=(Vy/h, 0, 0)

$$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0, \frac{\partial w}{\partial z} = 0$$

  - Continuity is satisfied

$$0 + 0 + 0 = 0$$

  - X-momentum is satisfied

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

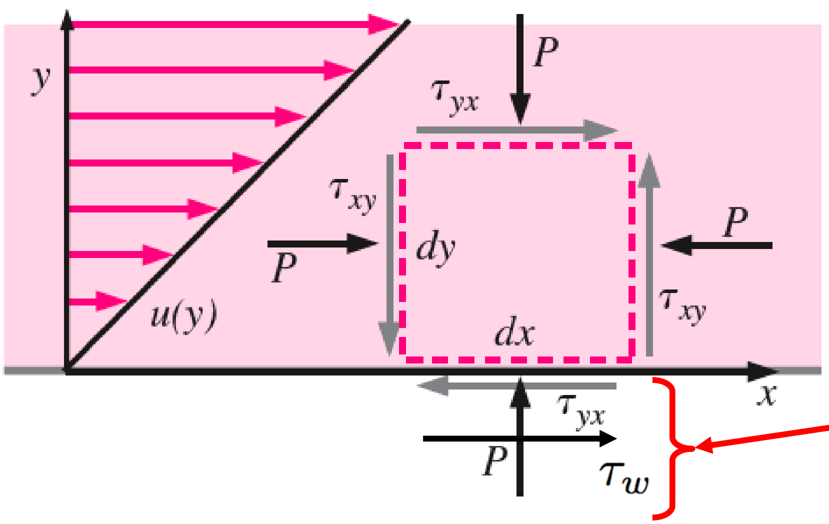
$$\rho\left(0 + V\frac{y}{h}\cdot 0 + 0\cdot V/h + 0\cdot 0\right) = -0 + \rho\cdot 0 + \mu\left(0+0+0\right)$$

$$0 = 0$$

Example exact solution (Ex. 9-15)
Fully Developed Couette Flow

- Finally, calculate shear force on bottom plate

$$\tau_{ij} = \begin{pmatrix} 2\mu\frac{\partial U}{\partial x} & \mu\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right) & \mu\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right) \\ \mu\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right) & 2\mu\frac{\partial V}{\partial y} & \mu\left(\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\right) \\ \mu\left(\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\right) & \mu\left(\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z}\right) & 2\mu\frac{\partial W}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & \mu\frac{V}{h} & 0 \\ \mu\frac{V}{h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



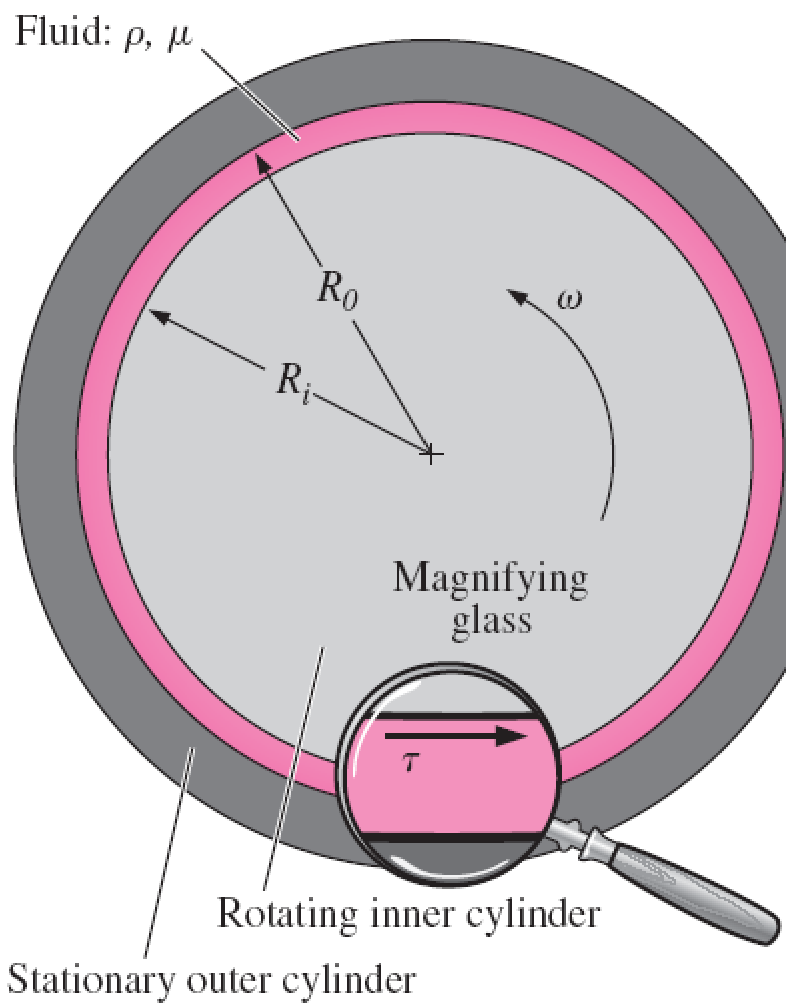
Shear force per unit area acting on the wall

$$\boxed{\frac{\vec{F}}{A} = \tau_w = \mu\frac{V}{h}\hat{i}}$$

Note that τ_w is equal and opposite to the shear stress acting on the fluid τ_{yx} (Newton's third law).

Rotational viscometer

- An instrument used to measure viscosity. is constructed of two concentric circular cylinders of length *L*— a solid, rotating inner cylinder of radius R_i and a hollow, stationary outer cylinder of radius R_o.
- The gap is small, i.e. $(R_o - R_i) << R_o$.
- Find the viscosity of the fluid in between the cylinders.



Rotational viscometer

- The viscous shear stress acting on a fluid element adjacent to the inner cylinder is approximately equal to

$$\tau = \tau_{yx} \cong \mu \frac{V}{R_o - R_i} = \mu \frac{\omega R_i}{R_o - R_i}$$

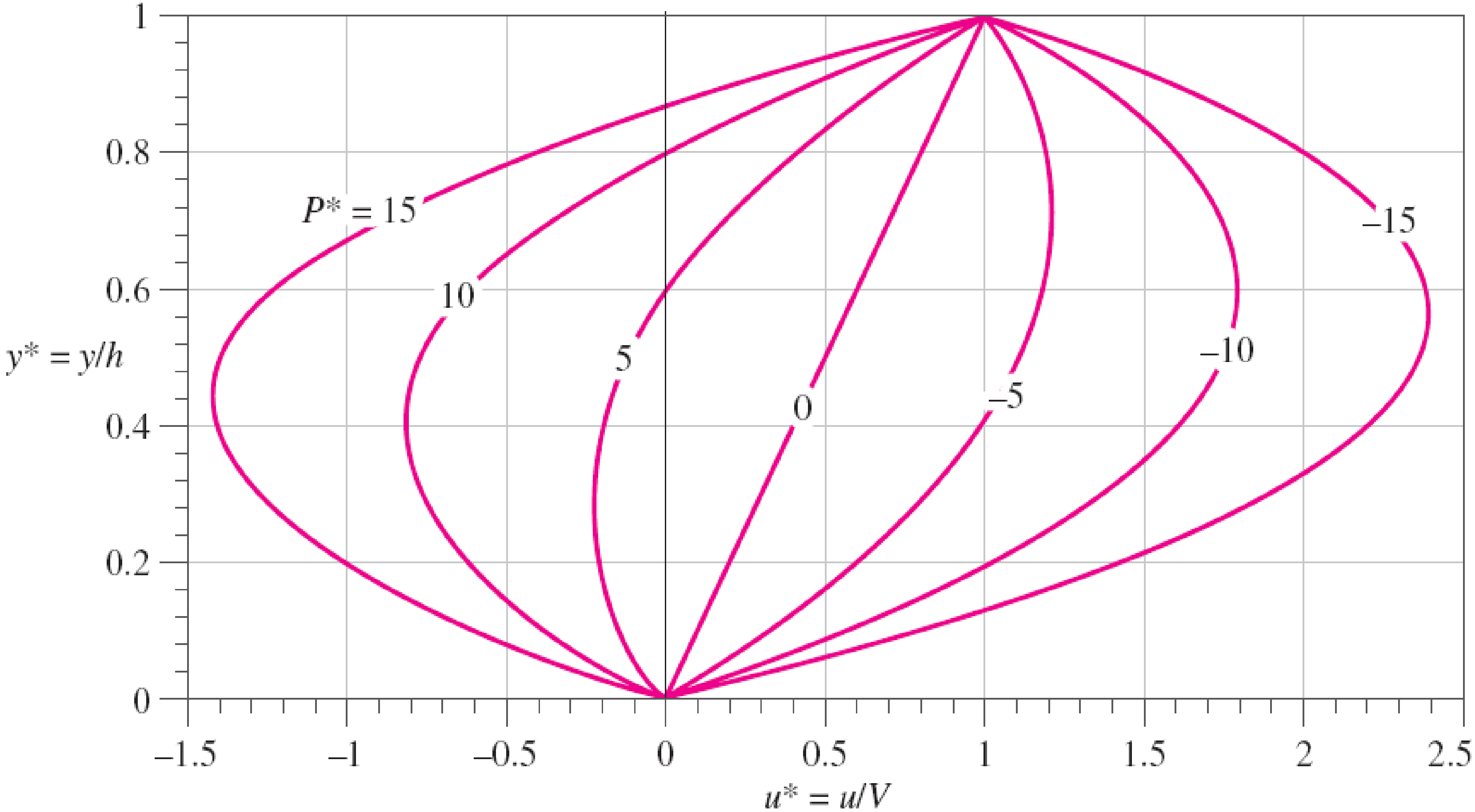
- The total *clockwise* torque acting on the inner cylinder wall due to fluid viscosity is

$$\mathrm{T}_{\text{viscous}} = \tau A R_i \cong \mu \frac{\omega R_i}{R_o - R_i} \left(2\pi R_i L \right) R_i$$

- Under steady conditions, the clockwise torque $\mathrm{T}_{\text{viscous}}$ is balanced by the applied counterclockwise torque $\mathrm{T}_{\text{applied}}$.
- Therefore, *viscosity of the fluid*:

$$\mu = \mathrm{T}_{\text{applied}} \frac{(R_o - R_i)}{2\pi \omega R_i^3 L}$$

EXAMPLE 9–16 Couette Flow with an Applied Pressure Gradient

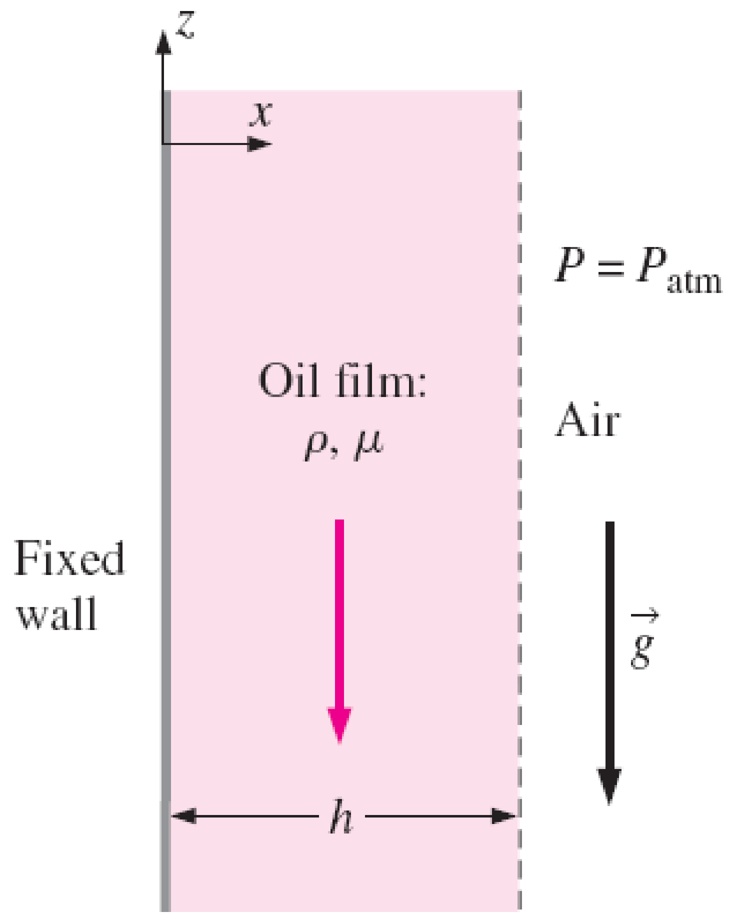


- The detailed derivation is referred to pages 443 -446 in the text.

$$P^* = \frac{h^2}{\mu V}\frac{\partial P}{\partial x}$$

EXAMPLE 9–17 Oil Film Flowing Down a Vertical Wall by Gravity

- Consider steady, incompressible, parallel, laminar flow of a film of oil falling slowly down an infinite vertical wall. The oil film thickness is *h*, and gravity acts in the negative *z*-direction. There is no applied (forced) pressure driving the flow—the oil falls by gravity alone. Calculate the velocity and pressure fields in the oil film and sketch the normalized velocity profile. You may neglect changes in the hydrostatic pressure of the surrounding air.



EXAMPLE 9–17 Oil Film Flowing Down a Vertical Wall by Gravity

- Solution:
  - Assumptions
    1. Plates are infinite in y and z
    2. Flow is steady, $\partial/\partial t = 0$
    3. Parallel flow, u=0
    4. Incompressible, Newtonian, laminar, constant properties
    5. $P=P_{atm}$ = constant at free surface and no pressure gradient
    6. 2D, v=0, $\partial/\partial y = 0$
    7. Gravity acts in the -z direction,
  - Boundary conditions
    1. No slip at wall (x=0) : u=0, v=0, w=0
    2. At the free surface ($x = h$), there is negligible shear, means $\partial w/\partial x = 0$ at $x = h$.

EXAMPLE 9–17 Oil Film Flowing Down a Vertical Wall by Gravity

- **Step 3:** *Write out and simplify the differential equations.*

$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{\text{assumption 6}} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial w}{\partial z} = 0$$

- Therefore, $w = w(x)$ only

- Since $u = v = 0$ everywhere, and gravity does not act in the *x*- or *y*-directions, the *x*- and *y*-momentum equations are satisfied exactly (in fact all terms are zero in both equations). The *z*-momentum equation reduces to

EXAMPLE 9–17 Oil Film Flowing Down a Vertical Wall by Gravity

$$\rho\left(\underbrace{\cancel{\frac{\partial w}{\partial t}}}_{\text{assumption 2}} + \underbrace{u\cancel{\frac{\partial w}{\partial x}}}_{\text{assumption 3}} + \underbrace{v\cancel{\frac{\partial w}{\partial y}}}_{\text{assumption 6}} + \underbrace{w\cancel{\frac{\partial w}{\partial z}}}_{\text{continuity}}\right) = \underbrace{-\cancel{\frac{\partial P}{\partial z}}}_{\text{assumption 5}} + \underbrace{\rho g_z}_{-\rho g}$$

$$+\mu\left(\frac{\partial^2 w}{\partial x^2} + \underbrace{\cancel{\frac{\partial^2 w}{\partial y^2}}}_{\text{assumption 6}} + \underbrace{\cancel{\frac{\partial^2 w}{\partial z^2}}}_{\text{continuity}}\right) \quad \rightarrow \quad \frac{d^2 w}{dx^2} = \frac{\rho g}{\mu}$$

- **Step 4** *Solve the differential equations.* (Integrating twice)

$$w = \frac{\rho g}{2\mu}x^2 + C_1 x + C_2$$

EXAMPLE 9–17 Oil Film Flowing Down a Vertical Wall by Gravity

- **Step 5** *Apply boundary conditions.*

$$\text{Boundary condition (1):} \qquad w = 0 + 0 + C_2 = 0 \qquad C_2 = 0$$

Boundary condition (2):

$$\left.\frac{dw}{dx}\right)_{x=h} = \frac{\rho g}{\mu} h + C_1 = 0 \quad \rightarrow \quad C_1 = -\frac{\rho g h}{\mu}$$

- *Velocity field*:

$$w = \frac{\rho g}{2\mu} x^2 - \frac{\rho g}{\mu} h x = \frac{\rho g x}{2\mu}(x - 2h)$$

- Since $x < h$ in the film, w is negative everywhere, as expected (flow is downward). The pressure field is trivial; namely, $\boldsymbol{P = P_{atm}}$ **everywhere.**

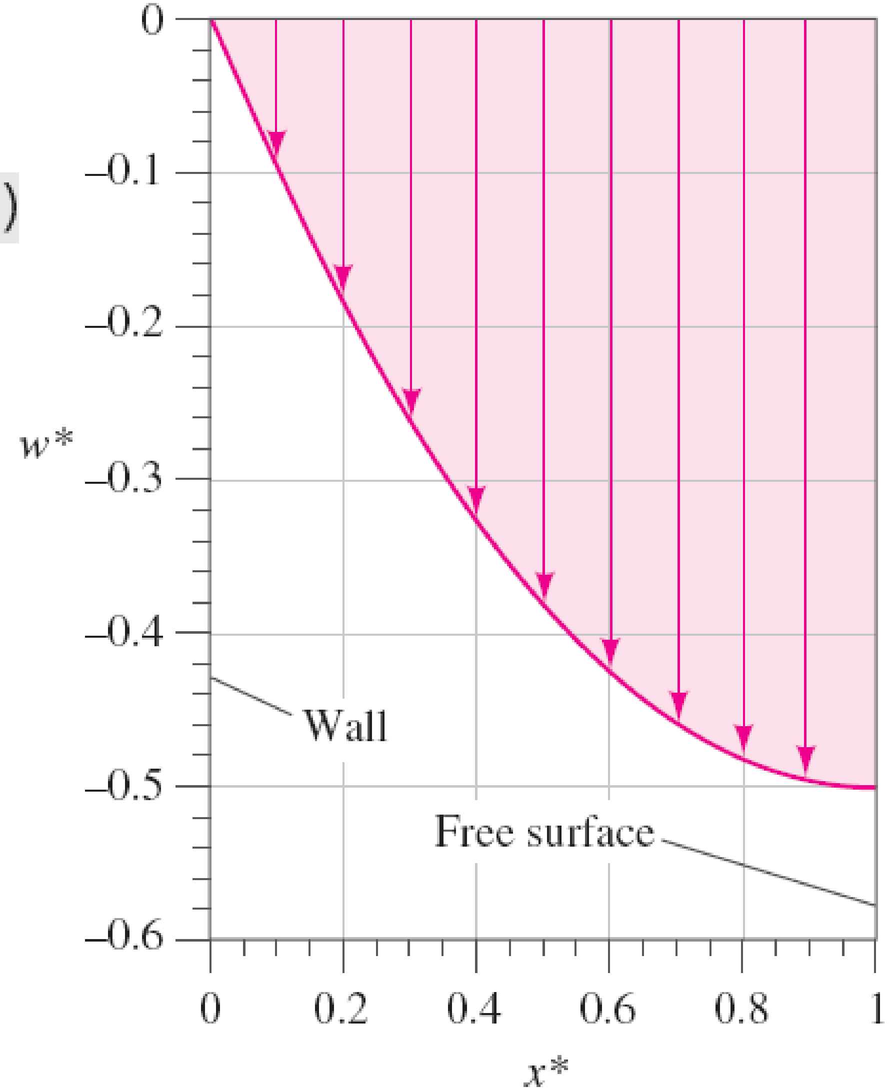
EXAMPLE 9–17 Oil Film Flowing Down a Vertical Wall by Gravity

- **Step 6** *Verify the results.*

let $x^* = x/h$ and $w^* = w\mu/(\rho g h^2)$

Normalized velocity profile:

$$w^* = \frac{x^*}{2}(x^* - 2)$$

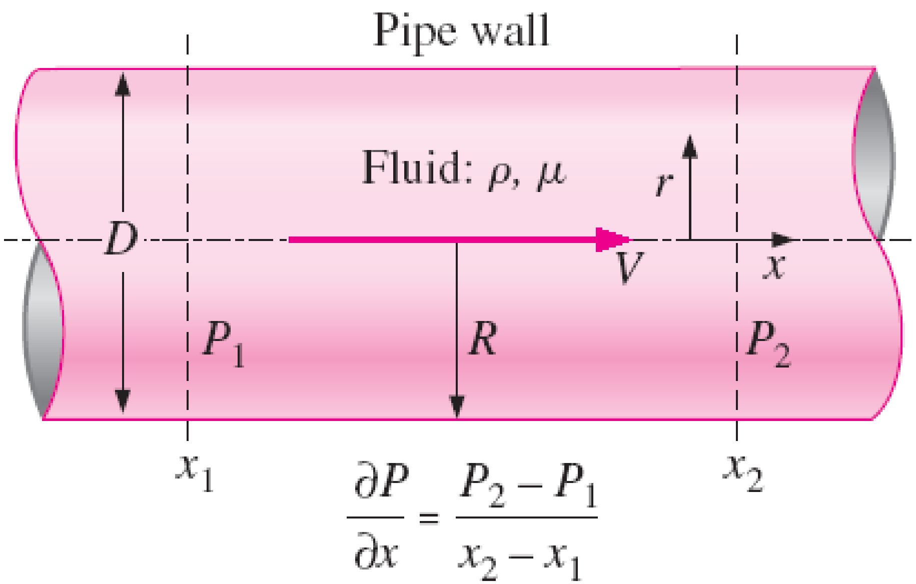


EXAMPLE 9–18 Fully Developed Flow in a Round Pipe — Poiseuille Flow

- Consider steady, incompressible, laminar flow of a Newtonian fluid in an infinitely long round pipe of radius $R = D/2$. We ignore the effects of gravity. A constant pressure gradient P/x is applied in the x-direction,

$$\frac{\partial P}{\partial x} = \frac{P_2 - P_1}{x_2 - x_1} = \text{constant}$$

where x_1 and x_2 are two arbitrary locations along the x-axis, and P_1 and P_2 are the pressures at those two locations.



EXAMPLE 9–18 Fully Developed Flow in a Round Pipe — Poiseuille Flow

- Derive an expression for the velocity field inside the pipe and estimate the viscous shear force per unit surface area acting on the pipe wall.

Solution:

Assumptions

1. The pipe is infinitely long in the *x*-direction.
2. Flow is steady, $\partial/\partial t = 0$
3. Parallel flow, u_r = zero.
4. Incompressible, Newtonian, laminar, constant properties
5. A constant-pressure gradient is applied in the *x*-direction
6. The velocity field is axisymmetric with no swirl, implying that $u_\theta = 0$ and all partial derivatives with respect to θ are zero.
7. ignore the effects of gravity.

EXAMPLE 9–18 Fully Developed Flow in a Round Pipe — Poiseuille Flow

Solution:

Step 2 *List boundary conditions.*

(1)at $r = R$, $\vec{V} = 0.$

(2) at $r = 0$, $du/dr = 0$.

Step 3 *Write out and simplify the differential equations.*

$$\underbrace{\cancel{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{1}{r}\frac{\partial(u_\theta)}{\partial \theta}}}_{\text{assumption 6}} + \frac{\partial u}{\partial x} = 0 \qquad \rightarrow \qquad \frac{\partial u}{\partial x} = 0$$

EXAMPLE 9–18 Fully Developed Flow in a Round Pipe — Poiseuille Flow

Solution:

Result of continuity: $u = u(r)$ only

We now simplify the axial momentum equation

$$\rho\left(\underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{assumption 2}} + \underbrace{\cancel{u_r \frac{\partial u}{\partial r}}}_{\text{assumption 3}} + \underbrace{\cancel{\frac{u_\theta}{r}\frac{\partial u}{\partial \theta}}}_{\text{assumption 6}} + \underbrace{\cancel{u\frac{\partial u}{\partial x}}}_{\text{continuity}}\right)$$

$$= -\frac{\partial P}{\partial x} + \underbrace{\cancel{\rho g_x}}_{\text{assumption 7}} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}}_{\text{assumption 6}} + \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{continuity}}\right)$$

Or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x} \qquad \textbf{(4)}$$

EXAMPLE 9–18 Fully Developed Flow in a Round Pipe — Poiseuille Flow

Solution:

In similar fashion, every term in the *r*-momentum equation

$$\textit{r-momentum:} \qquad \frac{\partial P}{\partial r} = 0$$

$$\textit{Result of r-momentum:} \qquad P = P(x) \text{ only}$$

Finally, all terms of the θ-component of the Navier–Stokes equation go to zero.

Step 4 *Solve the differential equations*.

After multiplying both sides of Equation (4) by *r*, we integrate once to obtain

$$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dP}{dx} + C_1 \qquad \textbf{(7)}$$

EXAMPLE 9–18 Fully Developed Flow in a Round Pipe — Poiseuille Flow

Solution:

Dividing both sides of Eq. 7 by *r*, we integrate again to get

$$u = \frac{r^2}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2 \qquad \textbf{(8)}$$

Step 5 *Apply boundary conditions*.

Boundary condition (2): $0 = 0 + C_1 \quad \rightarrow \quad C_1 = 0$

Boundary condition (1): $u = \frac{R^2}{4\mu}\frac{dP}{dx} + 0 + C_2 = 0$

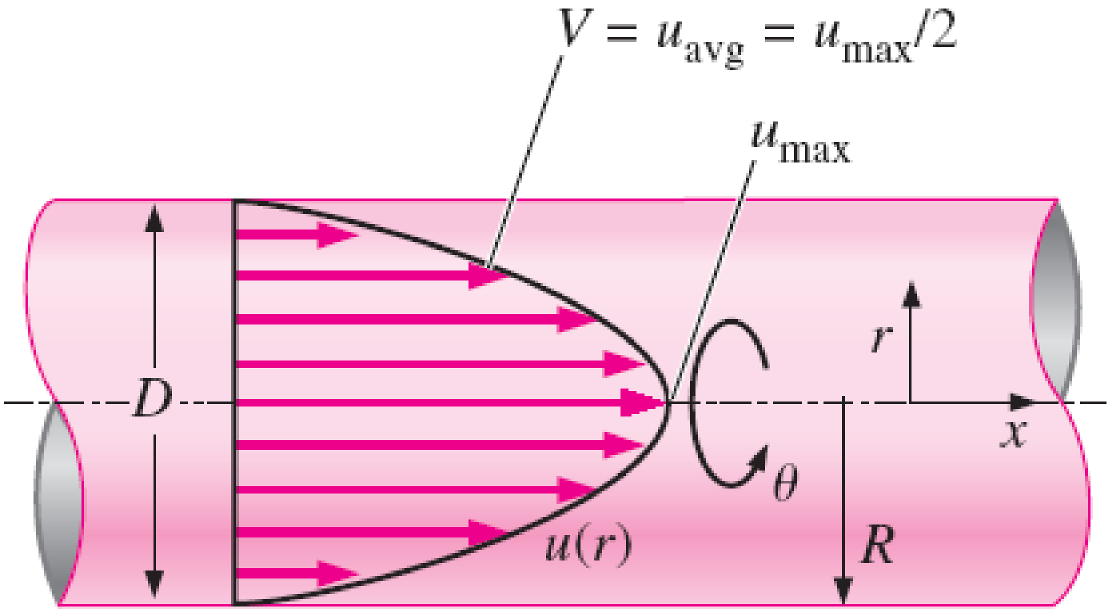
$$\rightarrow \quad C_2 = -\frac{R^2}{4\mu}\frac{dP}{dx}$$

EXAMPLE 9–18 Fully Developed Flow in a Round Pipe — Poiseuille Flow

Solution:

Finally, the result becomes

$$u = \frac{1}{4\mu}\frac{dP}{dx}(r^2 - R^2)$$



Step 6 *Verify the results*. You can verify that all the differential equations and boundary conditions are satisfied.