

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

Our discussion of the heat equation

$$(1) \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in the last section extends to bars of infinite length, which are good models of very long bars or wires (such as a wire of length, say, 300 ft). Then the role of Fourier series in the solution process will be taken by **Fourier integrals** (Sec. 11.7).

Let us illustrate the method by solving (1) for a bar that extends to infinity on both sides (and is laterally insulated as before). Then we do not have boundary conditions, but only the **initial condition**

$$(2) \quad u(x, 0) = f(x) \quad (-\infty < x < \infty)$$

where $f(x)$ is the given initial temperature of the bar.

To solve this problem, we start as in the last section, substituting $u(x, t) = F(x)G(t)$ into (1). This gives the two ODEs

$$(3) \quad F'' + p^2 F = 0 \quad [\text{see (5), Sec. 12.6}]$$

and

$$(4) \quad \dot{G} + c^2 p^2 G = 0 \quad [\text{see (6), Sec. 12.6}].$$

Solutions are

$$F(x) = A \cos px + B \sin px \quad \text{and} \quad G(t) = e^{-c^2 p^2 t},$$

respectively, where A and B are any constants. Hence a solution of (1) is

$$(5) \quad u(x, t; p) = FG = (A \cos px + B \sin px) e^{-c^2 p^2 t}.$$

Here we had to choose the separation constant k negative, $k = -p^2$, because positive values of k would lead to an increasing exponential function in (5), which has no physical meaning.

Use of Fourier Integrals

Any series of functions (5), found in the usual manner by taking p as multiples of a fixed number, would lead to a function that is periodic in x when $t = 0$. However, since $f(x)$

in (2) is not assumed to be periodic, it is natural to use **Fourier integrals** instead of Fourier series. Also, A and B in (5) are arbitrary and we may regard them as functions of p , writing $A = A(p)$ and $B = B(p)$. Now, since the heat equation (1) is linear and homogeneous, the function

$$(6) \quad u(x, t) = \int_0^\infty u(x, t; p) dp = \int_0^\infty [A(p) \cos px + B(p) \sin px] e^{-c^2 p^2 t} dp$$

is then a solution of (1), provided this integral exists and can be differentiated twice with respect to x and once with respect to t .

Determination of $A(p)$ and $B(p)$ from the Initial Condition. From (6) and (2) we get

$$(7) \quad u(x, 0) = \int_0^\infty [A(p) \cos px + B(p) \sin px] dp = f(x).$$

This gives $A(p)$ and $B(p)$ in terms of $f(x)$; indeed, from (4) in Sec. 11.7 we have

$$(8) \quad A(p) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos pv dv, \quad B(p) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin pv dv.$$

According to (1*), Sec. 11.9, our Fourier integral (7) with these $A(p)$ and $B(p)$ can be written

$$u(x, 0) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) \cos (px - pv) dv \right] dp.$$

Similarly, (6) in this section becomes

$$u(x, t) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) \cos (px - pv) e^{-c^2 p^2 t} dv \right] dp.$$

Assuming that we may reverse the order of integration, we obtain

$$(9) \quad u(x, t) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \left[\int_0^\infty e^{-c^2 p^2 t} \cos (px - pv) dp \right] dv.$$

Then we can evaluate the inner integral by using the formula

$$(10) \quad \int_0^\infty e^{-s^2} \cos 2bs ds = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$

[A derivation of (10) is given in Problem Set 16.4 (Team Project 24).] This takes the form of our inner integral if we choose $p = s/(c\sqrt{t})$ as a new variable of integration and set

$$b = \frac{x - v}{2c\sqrt{t}}.$$

Then $2bs = (x - v)p$ and $ds = c\sqrt{t} dp$, so that (10) becomes

$$\int_0^\infty e^{-c^2 p^2 t} \cos(px - pv) dp = \frac{\sqrt{\pi}}{2c\sqrt{t}} \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\}.$$

By inserting this result into (9) we obtain the representation

$$(11) \quad u(x, t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv.$$

Taking $z = (v - x)/(2c\sqrt{t})$ as a variable of integration, we get the alternative form

$$(12) \quad u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2cz\sqrt{t}) e^{-z^2} dz.$$

If $f(x)$ is bounded for all values of x and integrable in every finite interval, it can be shown (see Ref. [C10]) that the function (11) or (12) satisfies (1) and (2). Hence this function is the required solution in the present case.

EXAMPLE 1 Temperature in an Infinite Bar

Find the temperature in the infinite bar if the initial temperature is (Fig. 298)

$$f(x) = \begin{cases} U_0 = \text{const} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

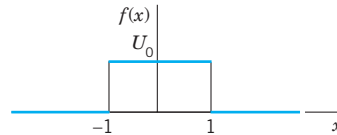


Fig. 298. Initial temperature in Example 1

Solution. From (11) we have

$$u(x, t) = \frac{U_0}{2c\sqrt{\pi t}} \int_{-1}^1 \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv.$$

If we introduce the above variable of integration z , then the integration over v from -1 to 1 corresponds to the integration over z from $(-1 - x)/(2c\sqrt{t})$ to $(1 - x)/(2c\sqrt{t})$, and

$$(13) \quad u(x, t) = \frac{U_0}{\sqrt{\pi}} \int_{-(1+x)/(2c\sqrt{t})}^{(1-x)/(2c\sqrt{t})} e^{-z^2} dz \quad (t > 0).$$

We mention that this integral is not an elementary function, but can be expressed in terms of the error function, whose values have been tabulated. (Table A4 in App. 5 contains a few values; larger tables are listed in Ref. [GenRef1] in App. 1. See also CAS Project 1, p. 574.) Figure 299 shows $u(x, t)$ for $U_0 = 100^\circ\text{C}$, $c^2 = 1 \text{ cm}^2/\text{sec}$, and several values of t . ■

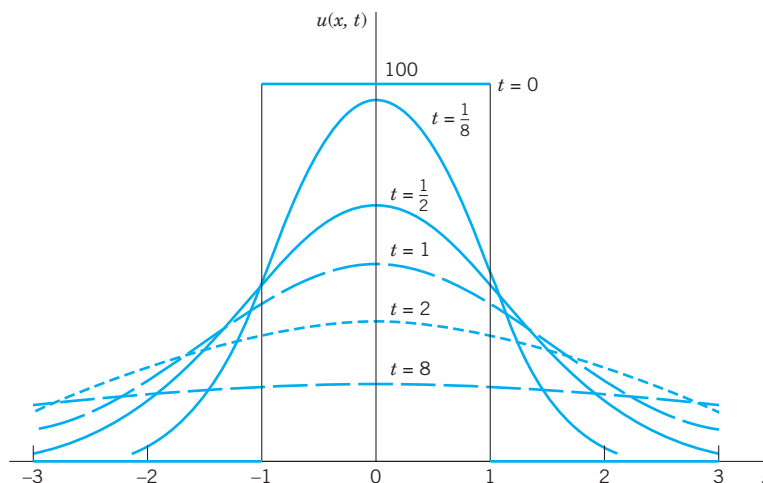


Fig. 299. Solution $u(x, t)$ in Example 1 for $U_0 = 100^\circ\text{C}$, $c^2 = 1 \text{ cm}^2/\text{sec}$, and several values of t

Use of Fourier Transforms

The Fourier transform is closely related to the Fourier integral, from which we obtained the transform in Sec. 11.9. And the transition to the Fourier cosine and sine transform in Sec. 11.8 was even simpler. (You may perhaps wish to review this before going on.) Hence it should not surprise you that we can use these transforms for solving our present or similar problems. The Fourier transform applies to problems concerning the entire axis, and the Fourier cosine and sine transforms to problems involving the positive half-axis. Let us explain these transform methods by typical applications that fit our present discussion.

EXAMPLE 2 Temperature in the Infinite Bar in Example 1

Solve Example 1 using the Fourier transform.

Solution. The problem consists of the heat equation (1) and the initial condition (2), which in this example is

$$f(x) = U_0 = \text{const} \quad \text{if } |x| < 1 \quad \text{and } 0 \text{ otherwise.}$$

Our strategy is to take the Fourier transform with respect to x and then to solve the resulting *ordinary* DE in t . The details are as follows.

Let $\hat{u} = \mathcal{F}(u)$ denote the Fourier transform of u , *regarded as a function of x* . From (10) in Sec. 11.9 we see that the heat equation (1) gives

$$\mathcal{F}(u_t) = c^2 \mathcal{F}(u_{xx}) = c^2(-w^2) \mathcal{F}(u) = -c^2 w^2 \hat{u}.$$

On the left, assuming that we may interchange the order of differentiation and integration, we have

$$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u e^{-iwx} dx = \frac{\partial \hat{u}}{\partial t}.$$

Thus

$$\frac{\partial \hat{u}}{\partial t} = -c^2 w^2 \hat{u}.$$

Since this equation involves only a derivative with respect to t but none with respect to w , this is a first-order *ordinary DE*, with t as the independent variable and w as a parameter. By separating variables (Sec. 1.3) we get the general solution

$$\hat{u}(w, t) = C(w) e^{-c^2 w^2 t}$$

with the arbitrary “constant” $C(w)$ depending on the parameter w . The initial condition (2) yields the relationship $\hat{u}(w, 0) = C(w) = \hat{f}(w) = \mathcal{F}(f)$. Our intermediate result is

$$\hat{u}(w, t) = \hat{f}(w)e^{-c^2w^2t}.$$

The inversion formula (7), Sec. 11.9, now gives the solution

$$(14) \quad u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-c^2w^2t} e^{iwx} dw.$$

In this solution we may insert the Fourier transform

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{i vw} dv.$$

Assuming that we may invert the order of integration, we then obtain

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) \left[\int_{-\infty}^{\infty} e^{-c^2w^2t} e^{i(wx - vw)} dw \right] dv.$$

By the Euler formula (3), Sec. 11.9, the integrand of the inner integral equals

$$e^{-c^2w^2t} \cos(wx - vw) + ie^{-c^2w^2t} \sin(wx - vw).$$

We see that its imaginary part is an odd function of w , so that its integral is 0. (More precisely, this is the principal part of the integral; see Sec. 16.4.) The real part is an even function of w , so that its integral from $-\infty$ to ∞ equals twice the integral from 0 to ∞ :

$$u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\int_0^{\infty} e^{-c^2w^2t} \cos(wx - vw) dw \right] dv.$$

This agrees with (9) (with $p = w$) and leads to the further formulas (11) and (13). ■

EXAMPLE 3 Solution in Example 1 by the Method of Convolution

Solve the heat problem in Example 1 by the method of convolution.

Solution. The beginning is as in Example 2 and leads to (14), that is,

$$(15) \quad u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-c^2w^2t} e^{iwx} dw.$$

Now comes the crucial idea. We recognize that this is of the form (13) in Sec. 11.9, that is,

$$(16) \quad u(x, t) = (f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{iwx} dw$$

where

$$(17) \quad \hat{g}(w) = \frac{1}{\sqrt{2\pi}} e^{-c^2w^2t}.$$

Since, by the definition of convolution [(11), Sec. 11.9],

$$(18) \quad (f * g)(x) = \int_{-\infty}^{\infty} f(p) g(x - p) dp,$$

as our next and last step we must determine the inverse Fourier transform g of \hat{g} . For this we can use formula 9 in Table III of Sec. 11.10,

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$$

with a suitable a . With $c^2 t = 1/(4a)$ or $a = 1/(4c^2 t)$, using (17) we obtain

$$\mathcal{F}(e^{-x^2/(4c^2 t)}) = \sqrt{2c^2 t} e^{-c^2 w^2 t} = \sqrt{2c^2 t} \sqrt{2\pi} \hat{g}(w).$$

Hence \hat{g} has the inverse

$$\frac{1}{\sqrt{2c^2 t} \sqrt{2\pi}} e^{-x^2/(4c^2 t)}.$$

Replacing x with $x - p$ and substituting this into (18) we finally have

$$(19) \quad u(x, t) = (f * g)(x) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p) \exp\left\{-\frac{(x-p)^2}{4c^2 t}\right\} dp.$$

This solution formula of our problem agrees with (11). We wrote $(f * g)(x)$, without indicating the parameter t with respect to which we did not integrate. ■

EXAMPLE 4 Fourier Sine Transform Applied to the Heat Equation

If a laterally insulated bar extends from $x = 0$ to infinity, we can use the Fourier sine transform. We let the initial temperature be $u(x, 0) = f(x)$ and impose the boundary condition $u(0, t) = 0$. Then from the heat equation and (9b) in Sec. 11.8, since $f(0) = u(0, 0) = 0$, we obtain

$$\mathcal{F}_s(u_t) = \frac{\partial \hat{u}_s}{\partial t} = c^2 \mathcal{F}_s(u_{xx}) = -c^2 w^2 \mathcal{F}_s(u) = -c^2 w^2 \hat{u}_s(w, t).$$

This is a first-order ODE $\partial \hat{u}_s / \partial t + c^2 w^2 \hat{u}_s = 0$. Its solution is

$$\hat{u}_s(w, t) = C(w) e^{-c^2 w^2 t}.$$

From the initial condition $u(x, 0) = f(x)$ we have $\hat{u}_s(w, 0) = \hat{f}_s(w) = C(w)$. Hence

$$\hat{u}_s(w, t) = \hat{f}_s(w) e^{-c^2 w^2 t}.$$

Taking the inverse Fourier sine transform and substituting

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(p) \sin wp \, dp$$

on the right, we obtain the solution formula

$$(20) \quad u(x, t) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(p) \sin wp \, e^{-c^2 w^2 t} \sin wx \, dp \, dw.$$

Figure 300 shows (20) with $c = 1$ for $f(x) = 1$ if $0 \leq x \leq 1$ and 0 otherwise, graphed over the xt -plane for $0 \leq x \leq 2$, $0.01 \leq t \leq 1.5$. Note that the curves of $u(x, t)$ for constant t resemble those in Fig. 299. ■

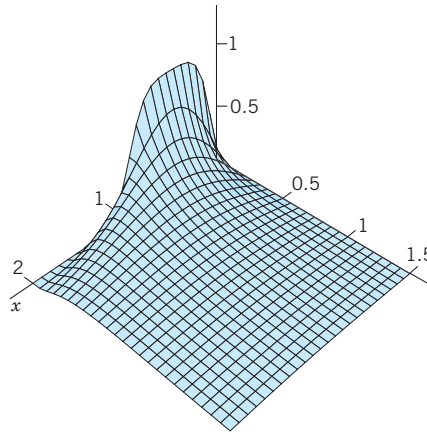


Fig. 300. Solution (20) in Example 4

PROBLEM SET 12.7

1. **CAS PROJECT. Heat Flow.** (a) Graph the basic Fig. 299.
 (b) In (a) apply animation to “see” the heat flow in terms of the decrease of temperature.
 (c) Graph $u(x, t)$ with $c = 1$ as a surface over a rectangle of the form $-a < x < a$, $0 < y < b$.

2-8 SOLUTION IN INTEGRAL FORM

Using (6), obtain the solution of (1) in integral form satisfying the initial condition $u(x, 0) = f(x)$, where

2. $f(x) = 1$ if $|x| < a$ and 0 otherwise
3. $f(x) = 1/(1 + x^2)$.
4. $f(x) = e^{-|x|}$
5. $f(x) = |x|$ if $|x| < 1$ and 0 otherwise
6. $f(x) = x$ if $|x| < 1$ and 0 otherwise
7. $f(x) = (\sin x)/x$.

Hint. Use Prob. 4 in Sec. 11.7.

8. Verify that u in the solution of Prob. 7 satisfies the initial condition.

9-12 CAS PROJECT. Error Function.

$$(21) \quad \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2} dw$$

This function is important in applied mathematics and physics (probability theory and statistics, thermodynamics, etc.) and fits our present discussion. Regarding it as a typical case of a special function defined by an integral that cannot be evaluated as in elementary calculus, do the following.

9. Graph the **bell-shaped curve** [the curve of the integrand in (21)]. Show that $\operatorname{erf} x$ is odd. Show that

$$\int_a^b e^{-w^2} dw = \frac{\sqrt{\pi}}{2} (\operatorname{erf} b - \operatorname{erf} a).$$

$$\int_{-b}^b e^{-w^2} dw = \sqrt{\pi} \operatorname{erf} b.$$

10. Obtain the Maclaurin series of $\operatorname{erf} x$ from that of the integrand. Use that series to compute a table of $\operatorname{erf} x$ for $x = 0(0.01)3$ (meaning $x = 0, 0.01, 0.02, \dots, 3$).
11. Obtain the values required in Prob. 10 by an integration command of your CAS. Compare accuracy.
12. It can be shown that $\operatorname{erf}(\infty) = 1$. Confirm this experimentally by computing $\operatorname{erf} x$ for large x .
13. Let $f(x) = 1$ when $x > 0$ and 0 when $x < 0$. Using $\operatorname{erf}(\infty) = 1$, show that (12) then gives

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{\pi}} \int_{-x/(2c\sqrt{t})}^{\infty} e^{-z^2} dz \\ &= \frac{1}{2} - \frac{1}{2} \operatorname{erf} \left(-\frac{x}{2c\sqrt{t}} \right) \quad (t > 0). \end{aligned}$$

14. Express the temperature (13) in terms of the error function.

$$\begin{aligned} 15. \text{ Show that } \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-s^2/2} ds \\ &= \frac{1}{2} + \frac{1}{2} \operatorname{erf} \left(\frac{x}{\sqrt{2}} \right). \end{aligned}$$

Here, the integral is the definition of the “distribution function of the normal probability distribution” to be discussed in Sec. 24.8.