Explaining Systematic Departures from Gibrat's Law

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Abstract. One of the most well-known "laws" in economics is Gibrat's Law of proportionate effect. Concerning firm dynamics, Gibrat's law predicts that firm growth will be independent of its size. While Gibrat's Law is frequently used as a benchmark in models of industry dynamics, the majority of recent empirical literature demonstrates a systematic departure from Gibrat's Law: small firms exhibit higher volatility than larger ones. This paper aims to provide an explanation for this systematic deviation. Utilizing a hierarchical production network, we demonstrate that upstream firms, which tend to be smaller, display more volatility than downstream firms. The model also offers an explanation for the observation that the firm size distribution follows a power law.

Keywords: Production Network, Power Law, Gilbrat's Law, Uniqueness

1. Introduction

One of the most debated statistical regularities in industrial dynamics is Gibrat's Law, which predicts that the growth rate of a firm is independent of its size (Gibrat, 1931, Sutton, 1997). Gibrat's Law allows a firm's growth to be viewed as a stochastic process, which brings in heterogeneity. The law also explains the skewed distribution for firm size successfully. These properties make Gibrat's Law become a prevalent assumption or a stylized fact in theoretical work (Ijiri and Simon, 1964, Lucas, 1978). For instance, Gibrat's Law plays a crucial role in explaining the power law of firm sizes and the tent-shaped distribution of firm growth (Simon and Bonini, 1958, Gabaix,

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2009, Fu et al., 2005). Gabaix and Landier (2008) model CEO pay by assuming an exogenous distribution of firm sizes from a random growth process.

However, the empirical tests for Gibrat's Law have generated controversy. It has been observed that small firms exhibit faster growth rates or greater volatility than their larger counterparts, leading to the rejection of Gibrat's Law, especially for downstream firms.[1] For instance, in a study of Italian firms, Grazzi and Moschella (2018) find a negative size-growth relationship, even when accounting for firm age. Nevertheless, it is worth noting that the breakdown in adherence to Gibrat's Law appears to occur systematically, suggesting that it may still hold for large firms.

Recently, economists have discovered the breakdown of Gibrat's Law for production networks. Guilmi and Fujiwara (2020) and Hiromitsu and Wataru (2020) both observe in the Japanese production networks that small firms experience higher fluctuation. Hiromitsu and Wataru (2020) find that the upstream firms tend to be smaller than those firms in the other portions of the network. Since upstream firms tend to be small, Guilmi and Fujiwara (2020) argue that the upstream firms also have higher volatility than the downstream firms. Further, Hiromitsu and Wataru (2020) observe that firms that follow a power law are usually not at the upstream level of production chains.[2]Since Gibrat's Law explains the power-law distribution, it implies that upstream firms depart from Gibrat's Law.

Overall, their observation suggests that a small firm's size and growth may be affected by its relative position or the hierarchical structure of a production chain. Motivated by these studies, this paper attempts to explain the breakdown of Gibrat's Law for upstream firms and the power-law distribution of firm sizes for downstream firms by a production network model.

[1]Hall (1987), Evans (1987a),Evans (1987b),, Dunne and Hughes (1994), Almus (2000), Calvo (2006), Falk (2008), Daunfeldt and Elert (2013), Oke (2018) and Aydogan and Donduran (2019) find that firm growth and firm size are negatively correlated, considering small firm or aggregated data. Hall (1987) and Dunne and Hughes (1994) document that the growth of small firms is more volatile growths than that of large firms.

[2]Guilmi and Fujiwara (2020) and Hiromitsu and Wataru (2020) also demonstrate that the distribution of large firms' sizes is Pareto, while that of small firms' sizes is not. This breakdown of Pareto distribution among small firms is a well-known observation (See e.g., (Saito et al., 2007, Fujiwara and Aoyama, 2010, Almsafir et al., 2015)).

This paper establishes a balanced production chain and shows both the existence and uniqueness of equilibrium. We discover that the hub-like network exhibits robust features such that the prices are increasing in transaction costs, and subcontractors' sizes grow as they progress to downstream levels. Additionally, simulations reveal that the production chain length decreases in transaction costs but increases with the degree of decreasing return to management. These properties enable us to study the impact of cost shocks on both price and the structure of the production network.

Furthermore, our findings indicate that when assembly costs exponentially increase with the task, the firm sizes of the subcontractors follow a power law in each production chain. In addition, if home producers experience higher volatility in technology compared to subcontractors, Gibrat's Law breaks down. In other words, when assembly technology is more stable in the short run, the home producers at the most upstream level exhibit greater volatility in firm sizes than downstream subcontractors. If we further examine the economy across many production chains, it appears that the upstream or small firms experience more fluctuations than downstream or large firms.

Moreover, we show that it is possible to identify a locally optimal number of suppliers, although it may be unrealistic to be fixed. Finally, this paper extends a balanced production network to a more realistic model that allows firms to choose an optimal number of suppliers when they subcontract. We show that the extended network also has a unique equilibrium and exhibits robust comparative statics.

This paper modifies the framework with multiple upstream partners from Kikuchi et al. (2018), Yu and Zhang (2019) and Kikuchi et al. (2021). They consider a perfectly competitive economy where all firms are ex-ante identical.[3] Facing prices, all firms minimize the home producing costs and outsourcing costs, which are subject to transaction costs and input prices. There exists a trade-off between intra-firm coordinate costs and inter-firm transaction costs. The idea is proposed by Coase (1937) and Williamson (1979). They address that the external transaction cost encourages firm growth while the intra-firm coordination cost discourages it. Hence, the point

[3]Some features of a multi-sourcing strategy are that it could mitigate the risk of suppliers' failure, allow firms to obtain broader supplier capabilities, and encourage competition among suppliers (see, e.g., Jin and Ryan (2012)). Considering the competition, our model also assumes that the intermediate market is perfectly competitive as Fally and Hillberry (2018) and firms are ex-ante identical.

where the cost of managing an additional task is equal to the cost of acquiring a similar input or service from the market pins down the firm's boundary. Kikuchi et al. (2018) construct the pricing function capturing the above concept and compute the equilibrium prices for the whole chain by the recursive method. They also show the existence of equilibrium but do not provide its uniqueness.

The model of Kikuchi et al. (2018) has the advantage that the equilibrium exists and is tractable by dynamic programming technique, so it pins down the firm sizes in each production stage endogenously. In their model, the firm's size increases as it goes downstream levels. Therefore, their framework can capture some realistic features. Moreover, they generate heterogeneity by the endogenous network but with ex-ante identical agents. This feature helps us to understand how the network structure affects firm-size distribution.

Building on Kikuchi et al. (2018), we construct the hub-like production network. Two reasons make us concentrate on the hubs: they render the network even more tractable, and some literature suggests that the hub-like firms are the key to forming a power-law distribution.[4] Unlike Kikuchi et al. (2018), we let each firm make a binary decision on outsourcing, rather than decide the range of home-producing tasks, to focus on the hub-like production network. In this case, it can be shown that the equilibrium not only exists but also is unique. Moreover, it successfully explains the power law for firm size.

In our model, the structure of the network is more tractable than Kikuchi et al. (2018). In detail, subcontractors divide the allocated task into a fixed number of multiple pieces of sub-module and outsource each sub-module to upstream suppliers. Subcontractors have to pay the assembly cost and transaction cost. Then, the minimum number of suppliers is two since the transaction and assembly costs prevent firms from outsourcing to only one supplier. Therefore, only hubs exist in the production network except for the upstream level. In other words, home producers only exist in the most upstream.

[4]For instance, Bernard et al. (2019) suggests that the number of suppliers for a firm is proportional to its firm size. The hub also plays an important role in transmitting shocks. Acemoglu et al. (2012) imply that super large suppliers with multiple customers can have a significant impact on aggregate shocks. Carvalho (2014) underscores that hub-like units provide shortcuts or act as influential conductors such that they help shocks propagate throughout the economy.

Related Literature. Guilmi and Fujiwara (2020) offer an explanation for the breakdown of Gibrat's Law among upstream firms, attributing it to amplified demand shocks along a production chain. However, both the production network and the prices are exogenous in their model, meaning the network is randomly given, and the price is integrated into stochastic shocks. Hence, their model is not able to study the impact of the shape of networks and cost shocks. Furthermore, Guilmi and Fujiwara (2020) do not explain explicitly why the sizes of the downstream or large firm follow a power law that Gibrat's Law holds for downstream firms in their model.

Nevertheless, the higher volatility among small firms documented in Guilmi and Fujiwara (2020) does not necessarily imply that Gibrat's Law holds for large firms and breaks down for small firms. It could be the case that the breakdown occurs for both small and large firms. Therefore, if Gibrat's Law is rejected, it would be better to explain the power-law distribution without it. In response, this paper provides one origin of the power-law distribution of firm sizes without Gibrat's Law and ex-ante heterogeneity. To simplify the model, we focus on a balanced and hub-like production network, in the sense that firms have a fixed number of multiple suppliers if they choose to outsource. We find that the hub-like network can generate a power-law distribution of firm size for subcontractors and illustrate the departure for upstream firms.

Another reason for the interest in the breakdown of Gibrat's Law is its connection to the development policy. If small firms exhibit higher growth rates, policymakers may be motivated to invest in young and innovative small firms for economic development.[5]. While small firms may have to do with economic development, large firms could impact the business cycle. The power law of the firm size distribution is influential in the research of aggregate shock. For example, Gabaix (2011) shows that firm-level idiosyncratic shocks can translate into aggregate fluctuations when the firm size distribution has a heavy tail, and the largest firms contribute disproportionately to aggregate output.[6] Moreover, Carvalho and Grassi (2019) show that the dispersion

[5]For example, a policymaker would support young small firms with high R&D intensity by relaxing financial constraints, since they can achieve significantly higher innovative sales (Schneider and Veugelers (2010)) Also, small firms have higher job creation rates than large firms (Schreyer (2000)).

[6]Gabaix (2011) shows that when the company size has infinite variance, the aggregate volatility decays at a rate slower than $1/\sqrt{N}$ such that the aggregate fluctuation is substantial even if N is large, where N denotes the number of firms in the economy. The fat tail breaks the central limit theorem, and the idiosyncratic shocks remain on aggregate output.

of firm sizes affects the variance of aggregate output in a Hopenhayn economy, which implies that large firms' dynamics induce significant macro-level fluctuations.

It also has been proved that the shape of the production network affects aggregate shock. Acemoglu et al. (2012) show how weighted out-degree and second-order interconnections of the inter-sectoral network affect aggregate volatility, where the aggregate output is a linear combination of sectoral shocks.[7] If the network is asymmetric, then the aggregate volatility can not be averaged out.[8] Carvalho (2014) stresses that the shape of the network influences the propagation of sectoral shocks, where the idiosyncratic shock in a single sector propagates along the chain and then generate aggregate shocks.

Similarly, Tahbaz-Salehi et al. (2016) illustrate that the rate at which aggregate volatility decays is determined by the structure of the input-output network as the economy becomes more disaggregated. They demonstrate how network interactions can propagate and amplify microeconomic shocks. Baqaee (2018) shows that the network influences the amplification and pattern of shocks. In the study by Bigio and La'O (2016), a network model with financial constraints reveals that shocks propagate idiosyncratic shocks and manifest as aggregate shocks through two channels: a fall in total factor productivity and an aggregate labor wedge distortion. Consequently, if our concern is economic growth and stability policy, it becomes imperative to understand the existence of a power-law distribution of firm sizes and the reason behind the breakdown of Gibrat's Law.

The real-world production network is intricate and challenging to analyze comprehensively. However, there are two types of networks that serve as approximations to real-world complexities. Baldwin and Venables (2013) describe two highly simplified models: snakes and spiders. A snake represents a sequence of production stages, with value added from the upstream stage to the downstream stage. Conversely, a spider is a process where numerous components (limbs) converge from the upstream stage to be assembled into a final output. The spider model results from the unbundling

[7] The weighted out-degree is the output share of one sector in the production network which can be computed from the input-output matrix. The weighted out-degree is higher when the supplier is large. The second-order interconnection is higher when there is a clustering of significant sectors in the network.

[8] In the sense that the corresponding weighted out-degree sequence or second-order degree sequence have power tails.

process, which breaks down production tasks into various sub-modules All production networks can be viewed as a combination of snakes and spiders.

Some research employs snakes as a model to investigate production behavior, such as integration and specialization. For example, Antràs and Chor (2013) show that the incentive to integrate suppliers depends on the relative position in a sequential chain. They predict that the average demand elasticity faced by final-good producers affects integration choices. Costinot et al. (2012) study wage inequality and vertical specialization within countries with a sequential feature.[9]

In reality, firms may engage with multiple partners. This paper specifically concentrates on spiders or hub-like networks. Acemoglu et al. (2012) and Barrot and Sauvagnat (2016) present a network of U.S. firms, wherein the majority of firms and sectors adopt the "star network" structure. Moreover, the result of a sequential chain may not hold if the assumption of a sequential structure is dropped. For example, Fattorini et al. (2017) find that demand elasticity is not a significant determinant for integration choices, contradicting the prediction of Antràs and Chor (2013) when the assumption of a unique linear sequence of production stages is removed. They argue that an input participates in multiple stages of production, but not only a single stage as a sequential chain. Therefore, this paper focuses on the more realistic spider networks.

Our model endogenously determines the equilibrium price and can analyze cost shocks under a multi-sourcing strategy. The model exhibits the price co-movement through the input-output network. Input-output linkages generate price co-movement and inflation synchronization, where the local cost shocks could be translated into global inflation (Antoun de Almeida, 2016, Auer et al., 2017, Bilgin and Yilmaz, 2018, Auer et al., 2019, Kamber and Wong, 2020).

This paper also explores how transaction costs impact the network. As transaction costs influence firm boundaries, the structure of the production network depends on both transaction costs and production costs. Aral et al. (2018) indicate that

[9] The other examples are: Costinot et al. (2013) study the sequential chain subject to failure and the connection between the degree of specialization and the stage of the supply chain. They show that countries with lower probabilities of making mistakes specialize in later stages of production, with a continuum of sequential tasks produced in countries that are themselves sequentially ordered in equilibrium. Levine (2012) studies that the trade-off between specialization and failure determines the optimal chain length, where a longer chain is more fragile.

coordination information technology reducing coordination and communication cost, which is one of transaction cost, is correlated with the number of suppliers, while the vendor-specific I.T. is associated with fewer suppliers. Boehm and Oberfield (2020) imply that the cost of contract enforcement affects the firms' outsourcing decision and then distorts the economy. Recent economic shocks have significant impacts on transaction costs. For instance, the trade war between China and the U.S. (Amiti et al., 2019, Fajgelbaum et al., 2020). Another example is the COVID-19 pandemic. Barua (2020) and Das (2020) point out that the causes the logistic problem and increases transportation costs. In our framework, the transaction cost is completely passed through to the price of the final good, which is compatible with Amiti et al. (2019).

Other related literature investigates whether the positions of firms affect their performance. For example, Mahy et al. (2019) and Gagliardi et al. (2019) find that Belgian firms' productivity increases on average as they go upstream levels and workers in more upstream firms obtain higher wages in the Belgian manufacturing industry, respectively. Chen (2017) find that the upstream industries have more severe wage inequality than downstream industries in China manufacturing data, where the downstream industries tend to do processing and assembly work. Szymczak et al. (2019) observe that workers in Central and Eastern European countries earn more when their industries are at either the beginning or the end of the production chain than in the middle. These papers demonstrate the importance of the position and structure of the network.

Section 2 establishes a model of .a balanced production chain. It also presents the proofs for existence and uniqueness and discusses some robust properties, including comparative statics. Section 3 shows the distribution of firm sizes of subcontractors and the departure of Gibrat's Law for upstream firms. Section 4 provides an extended model that allows firms to choose an arbitrary number of suppliers. The code can be found at github.com/chien-y/Departure_Gibrat_law.

2. Model

A supply chain produces a single unit of the final good in a perfectly competitive market. Positioned at the most downstream, there exists one and only one firm that sells this final good to consumers. Suppose that all firms are ex-ante identical, indicating that they share the same cost functions for both in-house production and assembly work.

Within the production chain, the firms collaborate to execute a continuum of tasks $[0, 1]$ to produce the final good. The interval $[0, 1]$ serves as a normalized measure of tasks required to produce one unit of good. To elaborate, the measure of tasks is denoted as $\ell \in [0, 1]$, representing the allocation of tasks in an outsourcing contract.

Moreover, each firm in the supply chain has a total of $\kappa > 1$ upstream partners. Upon receiving an allocation of tasks ℓ, each firm makes a strategic decision between in-house production and subcontracting the tasks ℓ to κ suppliers. In the case of in-house production, the firm incurs a production cost $c(\ell)$. Conversely, if the firm ops to be a subcontractor, it divides the tasks into κ portions and outsources each portion, equivalent to ℓ/κ units, to its upstream partners. Subsequently, the subcontractor is responsible for the assembly work, combining the κ pieces of components, incurring an assembly cost denoted as α.

Furthermore, when firms engage in the sell of intermediates or final goods, they are obligated to pay the transaction costs τ.[10] The transaction costs and the cost function satisfy the following assumptions.

Assumption 2.1. The transaction cost τ satisfies $0 < \tau < 1$.

Assumption 2.2. The cost function $c : [0, 1] \to \mathbb{R}$ is convex, $c(0) = 0$ and $c'(0) > 0$.

Under Assumption 2.2, given convexity of the cost function and the derivative $c'(0)$ as $\lim_{h \downarrow 0} c(h)/h$, the cost function is strictly increasing and non-negative.

Additionally, we assume that the assembly cost increases with the allocated tasks ℓ in Assumption 2.2. Two intuitions support this assumption. Firstly, if ℓ not only measures the quantity of tasks but also the diversity of tasks, a higher ℓ implies increased variety, leading to heightened complexity.[11] This increased complexity, in turn, results in higher assembly costs. Secondly, even though firms consistently divide

[10]It matters little whether the transaction is paid by the buyer or the seller in the model. The transaction costs include search and information costs, bargaining and decision costs, and contract enforcement costs (Dahlman, 1979).

[11]This concept is similar to ElMaraghy et al. (2013), Hu et al. (2008) and Hu et al. (2011). They show that the product variety complicates the design and operation of the assembly system and decreases the efficiency and quality of production.

tasks into κ pieces and subcontract κ/ℓ tasks to suppliers, a greater variety of tasks introduces more possibilities for task combinations. Firms need to strategically plan the task division and assembly line decision, incurring costs that rise with the task variety ℓ. Consequently, the assembly costs are proportional to the variety or quantity of tasks, making the assumption reasonable.

Specifically, we further assume that the assembly cost takes the form $b\ell^q$, where $q \geqslant 0$, b represents the efficiency for assembly work and q captures the return to scale of management.

Assumption 2.3. The assembly cost is $\alpha(\ell) \coloneqq b\ell^q$ for some $q \geqslant 0$ and $b > 0$.

In Assumption 2.3, it is permissible for the assembly cost function to be concave. The idea stems from the concept that labor can specialize in the assembly line, resulting in an increasing return to scale for assembly work. Aizcorbe (1992) illustrates this with an example that an automobile assembly plant experiences an increasing return to labor in the short run, attributed to the specialization of tasks. Thus, the assumption allows for the assembly cost to exhibit concavity, in contrast to the convex nature of the production cost.

Given these assumptions, each firm in the chain receives a contract assigning it ℓ tasks to complete. The firm then minimizes costs by deciding between home production and subcontracting. If the firm opts for home production, the cost is $c(\ell)$. Conversely, if the firm chooses subcontracting, the cost becomes $\kappa p(\ell/\kappa)+\alpha(\ell)$ for purchasing inputs from κ suppliers and conducting the assembly work, where $\kappa = 2, 3, \ldots$, and each supplier completes ℓ/κ fraction of tasks. Subsequently, accounting for the transaction costs $\tau \in (0, 1)$ associated with selling the intermediate inputs, the profit is given by

$$(1-\tau)p(\ell) - \min\{\kappa p(\ell/\kappa) + \alpha(\ell),\, c(\ell)\}$$

for each $\ell \in [0, 1]$ and $\kappa \in 2, 3, \ldots$. Note that if $\kappa = 1$, firms do not have the incentive to subcontract (choose the first term inside the minimum function), since all firms have identical costs and the transaction costs are larger than zero. This implies that it must be $\kappa > 1$.

Let $\delta = 1/(1-\tau)$. The parameter δ represents the transaction cost if buyers instead of sellers bear it. Since the market is perfectly competitive, firms have zero profit in equilibrium. Hence, the pricing equation solves

$$p(\ell) \coloneqq \delta \min\left\{\kappa p\left(\frac{\ell}{\kappa}\right) + \alpha(\ell), c(\ell)\right\}. \tag{1}$$

2.1. **Equilibrium of Production Chain.** Under the equilibrium price (1), the structure of the supply chain is as follows. At the most downstream level, indexed level 1, there exists a single firm that receives a task allocation ($\ell = 1$). If this firm chooses to subcontract, then there are κ firms in the next upstream level, indexed level 2, since the number of suppliers is fixed at κ in the model. All firms at level 2 receive $\ell = 1/\kappa$ units of task allocations. Again, if the firms at level 2 choose to subcontract, then there are κ^2 firms at level 3 and each supplier at level 3 implements $1/\kappa^2$ units of tasks. Therefore, by induction, there are κ^{n-1} number of firms at level $n \geqslant 1$, and each firm at level n implements κ^{1-n} units of tasks. See Figure 1 for the visual structure.

Moreover, if $m < \infty$ is the level index for the most upstream level of the chain, there are $1 + \kappa + \cdots + \kappa^{m-1} = (\kappa^m - 1)/(\kappa - 1)$ firms in the chain.

An allocation of tasks $\boldsymbol{\ell} \coloneqq \{\ell_i\}$ is called *feasible* if there is $m \in \mathbb{N}$ such that $i = 1, \cdots, (\kappa^m - 1)/(\kappa - 1)$ and $\ell_i = \kappa^{1-n}$ for $(\kappa^{n-1} - 1)/(\kappa - 1) < i \leqslant (\kappa^n - 1)/(\kappa - 1)$ for some integer $1 \leqslant n \leqslant m$. Here, the integer m defines the length of the production chain. If firm i at level $n \leqslant m$ receives a feasible allocation, its allocated measure of tasks is κ^{1-n}. The length of the chain m and the length of sequence $\{\ell_i\}$ are finite for a feasible allocation.

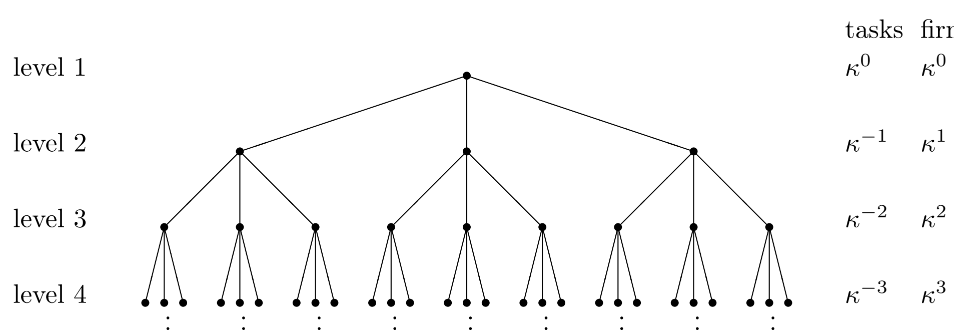

FIGURE 1. The structure of a balanced production chain. Take $\kappa = 3$ for example.

Given a feasible ℓ with length $m \in \mathbb{N}$, the profit to firm i with the outsourcing or home production costs (1) is

$$\pi_i := (1-\tau)p(\ell_i) - \min\{\kappa p(\ell_i/\kappa) + \alpha(\ell_i),\, c(\ell_i)\}, \tag{2}$$

for $\ell_i \in \boldsymbol{\ell}$, where the revenue is $(1-\tau)p(\ell_i)$ with transaction cost $\tau \in (0,1)$. Define the set of all possible tasks by $\mathcal{D} := \{\kappa^{-n} : n \in \mathbb{N} \cup \{0\}\}$.

Definition 2.1. Given a price function $p\colon [0,1] \to [0,\infty)$, a feasible allocation of tasks $\boldsymbol{\ell} = \{\ell_i\}$ with length $m \in \mathbb{N}$ and the corresponding profits $\{\pi_i\}$ defined by (2), a tuple $(p,\, \boldsymbol{\ell})$ is called an *equilibrium* if

(1) $p(0) = 0$,
(2) $(1-\tau)p(\ell) - \min\{\kappa p(\ell/\kappa) + \alpha(\ell),\, c(\ell)\} \leqslant 0$, for all $\ell \in \mathcal{D}$, and
(3) $\pi_i = 0$ for all i.

Condition (1) in definition 2.1 excludes the possibility of positive profits for suppliers providing initial inputs, considering $c(0) = 0$. Moreover, condition (2) ensures that no firm in the supply chain has an incentive to deviate from its decision, and inactive firms are unable to extract positive profits. Furthermore, given the assumption of perfect competition in the market, all firms in the chain are expected to have zero profits, thus establishing condition (3).

2.2. **Existence and Uniqueness of Equilibrium.** In this section, we show both the existence and uniqueness of equilibrium. A computational method is also provided. The proofs in the following sections are all presented in Appendix A.

Define the set of price functions by $\mathcal{P} := \{p : \mathcal{D} \to \mathbb{R}_+ : \delta c'(0)\ell \leqslant p(\ell) \leqslant \delta c(\ell), \forall \ell \in \mathcal{D}\}$. We will show that there is a unique equilibrium price in $\mathcal{P}$. Define operator T by

$$Tp(\ell) := \delta \min\left\{\kappa p\left(\frac{\ell}{\kappa}\right) + \alpha(\ell), c(\ell)\right\}. \tag{3}$$

where $\ell \in \mathcal{D}$ and $p : \mathcal{D} \to \mathbb{R}_+$. The operator T is analogous to a Bellman operator with respect to price function (1). If T has a fixed point, that fixed point is also a solution to price function (1) by its definition. Define $\bar{x} := \sup\{x \in (0,1] : c'(x) \leqslant \delta c'(0)\}$ for the following proposition. The next proposition shows both the existence and uniqueness of the fixed point of T.

Proposition 2.1. *If Assumption 2.1, 2.2 and 2.3 hold, the following statements are true.*

(a) T is a self-map on $\mathcal{P}$.

(b) T has a unique fixed point $p^* \in \mathcal{P}$.

(c) p^* is a unique solution to (1).

(d) p^* can be computed by finite iterations: $p^* = T^n p$ for all $n \geqslant 1 - \ln \bar{x} / \ln \kappa$ and any $p \in \mathcal{P}$

(e) Under p^*, firms always choose to produce in-house if their task allocations are smaller or equal to $\bar{x}$, i.e. $\ell \leqslant \bar{x}$.

(f) p^* is strictly increasing.

Proposition 2.1 asserts the existence and uniqueness of equilibrium price p^* in $\mathcal{P}$, which can be computed by iteration with any initial guess from $\mathcal{P}$. Moreover, since c is strictly increasing, Proposition 2.1 shows that p^* is strictly increasing.

As indicated by Proposition 2.1, the bounds for $\mathcal{P}$ carry substantial economic significance, capturing all equilibrium price. The upper bound $\delta c(\ell)$ for $\mathcal{P}$ suggests that the equilibrium price is less than or equal to the costs of home production. This upper limit aligns with the rationale that in a perfect competition market, where the price equals the total cost, outsourcing becomes economically favorable due to reduced costs, even after factoring in transaction costs.

As for the lower bound $\delta c'(0)\ell$ for $\mathcal{P}$, it represents the costs when the decreasing return to management is eliminated through outsourcing. Image a scenario where both transaction costs and assembly costs are absent; under such conditions, firms would invariably opt for outsourcing. Consequently, this leads to an infinite number of levels in the production network. The "infinite outsourcing" results in the production technology approximating a linear function when we view across the entire chain. This elimination of decreasing return of management translate to a production technology with a slope of $c'(0)$. Therefore, the lower bound not only leverages the convexity of cost function for the proof, but also reflects the cost or price when the return to management is constant.

Proposition 2.1 also shows that the equilibrium price can be determined through a finite number of iterations, enhancing computational feasibility. Moreover, a firm invariably becomes a home producer if its task allocation is below the critical value $\bar{x}$, which implies that the length of the production chain is always finite under equilibrium price. Note that the length of the production chain is also the number of iteration for computation $1 - \ln(\bar{x} - \kappa)$, provided in Proposition 2.1. As discussed in Section 2.1, if we ascertain the equilibrium length of the production, we can determine the task allocations. Therefore, there exists a feasible allocation $\boldsymbol{\ell}^*$ such that $(p^*, \boldsymbol{\ell}^*)$ constitutes an equilibrium for the production chain, in line with Definition 2.1.

Let p^* be the fixed point of operator T. Then, p^* is the solution of price equation (1). Let m^* denote the number of total levels corresponding to p^*. According to Proposition 2.1, there exists a maximal possible length $\bar{m} \in N$ such that $m^* \leqslant \bar{m}$. Thus, m^* is finite, and the corresponding allocations $\boldsymbol{\ell}^*$ are well defined. The following proposition shows that the solution $(p^*, \boldsymbol{\ell}^*)$ constitutes a unique equilibrium.

Proposition 2.2. *Suppose that 2.1, 2.2 and 2.3 hold. If p^* is a fixed point of T, then there exists a feasible task allocations $\boldsymbol{\ell}^*$ such that $(p^*, \boldsymbol{\ell}^*)$ is a unique equilibrium for the production chain.*

Since the number of suppliers is fixed and then the network structure and firm sizes are tractable, Proposition 2.2 shows the uniqueness of equilibrium, although we lose some degree of freedom for network structure. Note that the proof employs the convention that firms will engage in in-house production when the costs for home production and subcontracting are equal. Proposition 2.2 also implies that the unique equilibrium of the production chain can be identified, after we compute the equilibrium price from Proposition 2.1. Consequently, these results show that the upper bound and lower bound for the pricing function are deemed reasonable.

2.3. **Properties of Equilibrium.** This section attempts to characterize various properties of equilibrium $(p^*, \boldsymbol{\ell}^*)$. It includes comparative statics analysis using numerical methods and examples to reinforce the validity of the model.

2.3.1. *Length of Production Chain, Price, and Allocations.* We presents some observations and intuitions about the equilibrium in this section. To begin with, we characterize the maximal attainable length and task allocations within the production chain in equilibrium. Note that the range of tasks is $\kappa^{-(m-1)}$ at level m, and the largest attainable length $\bar{m}$ is the larges integer satisfying $\kappa^{-(m-1)} \leqslant \bar{x}$ by Proposition 2.1. Then, the equilibrium number of levels m^* in the production chain is at most

$\bar{m} := \lceil 1 - \ln(\bar{x})/\ln(\kappa) \rceil$, where $\lceil r \rceil$ denotes the smallest integer greater than $r \in \mathbb{R}$. Since $\bar{x} := \sup\{x \in (0,1] : c'(x) \leqslant \delta c'(0)\}$, it is evident that the transaction costs, convexity of cost function and the number of suppliers influence the length of the chain.

The maximum possible number of levels $\bar{m}$ decreases with both the number of upstream partners κ and transaction costs τ, which results in an increase in $\bar{x}$.[12] This indicates that transaction cost discourage the subcontract and limit the length of the chain. Hence, we can characterize some properties of the equilibrium price, by $\kappa, \delta, \alpha(.)$ and $c(.)$ functions, before computing it explicitly.

Here, we presents a method to compute the optimal length without explicitly computing the equilibrium prices. Define the function $f(\ell) := \kappa\delta c(\ell/k) + \alpha(\ell) - c(\ell)$ for $\ell \in [0,1]$. This function represents the additional costs of subcontracting compared to in-house production, assuming that the subcontracting partners are home producers. Observe that $f(0) = 0$ by our assumptions, and $f(\ell) < 0$ indicates that the outsourcing costs are less than in-house production coasts. Suppose that f has a root $\hat{\ell}$ in $(0,1]$. In the appendix, we demonstrate that if f is strictly concave, then a firm chooses in-house production when $\ell \leqslant \hat{\ell}$, and opts for outsourcing when $\ell > \hat{\ell}$.[13] This also implies that if $\hat{\ell} < 1$ then the equilibrium length exceeds 1.

Regarding the equilibrium prices, we are aware that p^* is strictly increasing in ℓ, as shown by Proposition 2.1. In addition, we can gain insight into the price function by the following iteration. If the total number of levels is $m \in \mathbb{N}$, the price of final

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[12]Since $-\ln \bar{x} > 0$, $\bar{m}$ is decreasing in κ.

[13]Since $f''(\ell) = \delta/\kappa c''(\ell/\kappa) - c''(\ell) + \alpha''(\ell)$, we can check that $f'' < 0$ if $\delta/\kappa < 1$, $c'' > 0$ and $\alpha'' \leqslant 0$.

product $p(1,m)$ follows

$$\begin{aligned}
p(1,m) &= \delta\kappa p(\kappa^{-1}) + \delta b \\
&= \delta\kappa\left[\delta\kappa p(\kappa^{-2}) + \delta b\kappa^{-q}\right] + \delta b \\
&= (\delta\kappa)^2 p\left(\kappa^{-2}\right) + \left(\delta\kappa^{1-q}+1\right)\delta b \\
&= \dots \\
&= (\delta\kappa)^{m-1} p\left(\kappa^{1-m}\right) + \left[(\delta\kappa^{1-q})^{m-2} + \cdots + \delta\kappa^{1-q} + 1\right]\delta b \\
&= (\delta\kappa)^{m-1}\delta c\left(\kappa^{1-m}\right) + \left[\left(\delta\kappa^{1-q}\right)^{m-2} + \cdots + \delta\kappa^{1-q} + 1\right]\delta b \\
&= (\delta\kappa)^{m-1}\delta c\left(\kappa^{1-m}\right) + \frac{\left(\delta\kappa^{1-q}\right)^{m-1} - 1}{\delta\kappa^{1-q} - 1}\delta b.
\end{aligned} \tag{4}$$

In (4), the first term of the right-hand side represents the in-house production costs with tasks κ^{1-m}, and the second term reflects the assembly costs, accounting for transaction costs. Clearly, if m is fixed, equation (4) shows that the price of finished goods increases with transaction costs δ, assembly cost parameter b, and the costs of in-house production c. However, note that the optimal length of the chain m^* will change if we alter these parameters. Consequently, the impact of these parameters on the equilibrium price is ambiguous from this perspective, as changes in these parameters may influence the optimal length of the chain, thus affecting the overall equilibrium price.

Recall that the firms become home producers if they are at level $\bar{m}$. To this end, we can also define the equilibrium price of the final good $p^*(1)$ as

$$p^*(1) \coloneqq \min_{m=1,\dots,\bar{m}} \left\{(\delta\kappa)^{m-1}\delta c\left(\kappa^{1-m}\right) + \frac{\left(\delta\kappa^{1-q}\right)^{m-1} - 1}{\delta\kappa^{1-q} - 1}\delta b\right\}. \tag{5}$$

This equation can be interpreted as that the organizer of the supply chain selects an optimal length, given that the number of suppliers is fixed. However, since the choice variable m is discrete, it is challenging to characterize m^* by (5).

2.3.2. *Firm Sizes.* Define the value-added as $c(\ell)$ for home producers and as $\alpha(\ell)$ for subcontractors. Given that the assembly cost is increasing in tasks, the value-added for each subcontractor increases as it progresses downstream levels. If we consider firm size as the value-added, the model exhibits the characteristic that downstream subcontracting firms are larger than upstream subcontracting firms. This result aligns with studies such as Kikuchi et al. (2018) and Fally and Hillberry (2018), which observe a negative correlation between "upstreamness" and the value-added.[14]

On the other hand, the firm sizes for the most upstream firms or home producers are indeterminate. Home producers may be smaller than downstream firms since they receive the least amount of tasks in the production chain. However, they could also be large under some cost functions and parameters. This feature reflects the complexity of the real world, where empirical studies show that countries or industries at the upstream and downstream extremities of the chain often have higher shares of value-added than those in the middle. The most upstream firms may have higher value-added such that the curve of value-added along the chain is like a "smile" curve (Ito and Vézina, 2016, Aggarwal, 2017, Rungi and Del Prete, 2018, Stöllinger, 2019).

2.3.3. *Comparative Statics.* In a perfect competitive economy, prices reflects total costs, leading to an intuitive expectation that equilibrium prices would increase with any costs, including transaction costs, assembly costs, or production costs. The following proposition demonstrates that the shift in equilibrium price is monotonic with respect to these costs.

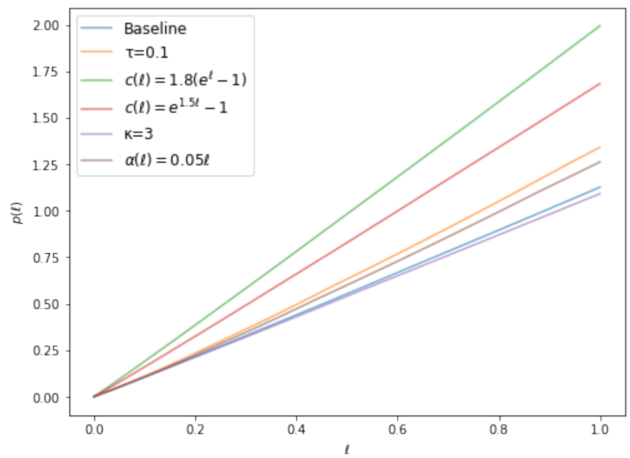
Proposition 2.3. *Suppose that 2.1, 2.2 and 2.3 hold. The equilibrium price is increasing in transaction cost, assembly cost, and home production cost.*

The proof relies on the monotonicity of operator T. It can also be shown that the price of the final good is strictly increasing in transaction costs, assembly costs, and home production costs, where the proof leverages the fact that the equilibrium price can be characterized using the optimal length, as expressed in (4).

Example 2.1. This numerical example employs the function $c(\ell) = A(e^{\theta\ell} - 1)$ as an illustration. The change in price with respect to parameters $\tau, c(\ell), \kappa, \alpha(\ell)$ are depicted in Figure 2. The result align with Proposition 2.3. It also indicates that the prices are decreasing in the number of suppliers κ. The intuition for decreasing prices with κ lies in the fact that the as κ increases, tasks are split into smaller pieces, reaching home production criterion $\ell \leqslant \bar{x}$ in a shorter length.

[14] The upstreamness is the relative position in the chain. The firm is in a more upstream position if the measure of upstreamness is higher. See Antràs et al. (2012), Antràs and Chor (2018) and Wang et al. (2017).

Moreover, the length is observed to increase in cost function $c(\ell)$, and decrease with transaction costs, the number of suppliers, and assembly costs. In this example, the total assembly costs decline as the length become shorter, but it may not be the case if $q > 1$. □



(A)

Model	Length
Baseline	5
$\tau = 0.1$	4
$c(\ell) = 1.8(e^{\ell} - 1)$	6
$c(\ell) = e^{1.5\ell} - 1$	6
$\kappa = 3$	4
$\alpha(\ell) = 0.05\ell$	4

(B)

FIGURE 2. The baseline model is with $\tau = 0.01, \alpha(\ell) = 0.01\ell, \kappa = 2, c(\ell) = e^{\ell} - 1$. We first plot the baseline price function and then plot the case for increasing $\tau = 0.1$, $c(\ell) = 1.8(e^{\ell} - 1)$, $c(\ell) = e^{1.5\ell} - 1$, $\kappa = 3$, $\alpha(\ell) = 0.05\ell$, respectively.

2.3.4. *An Example of Assembly Cost and Cost Function.* This section presents an example of a closed-form solution for the cost function and assembly cost derived from the firm's cost minimization problem. The motivation for this example arises from the question: Does the share of intermediate inputs affect the shape of the production chain? Intuitively, if the production process involves a higher utilization of inputs, the production entity would tend to outsource more tasks, potentially resulting in a longer chain. This section discusses how the cost share of intermediate inputs influences the structure of the production network.

Moreover, it is known that the transaction cost can impact the share of the intermediate. For example, Boehm and Oberfield (2020) point out that court congestion makes the contract enforcement slow, leading plants in industries reliant on relationship-specific inputs and in states with more congested courts d to have lower shares of intermediate inputs. Given that the cost of contract enforcement is a component of transaction costs, we would expect that it influences the structure of a production network both directly through the effect of transaction costs and indirectly through the share of intermediate inputs.

Suppose that firms have the Cobb-Douglas production function as

$$q = \begin{cases} A_s[(k^a n^{1-a})^{1-\sigma} x^\sigma]^\theta & \text{if subcontract,} \\ A_h(k^a n^{1-a})^\theta & \text{if produce in-house,} \end{cases} \tag{6}$$

where q denotes output, k denotes capital, n denotes labor, $x$denotes intermediate input, and σ, a and θ are all in $(0,1)$ representing intermediate share, capital share and degree of decreasing return, respectively. Given task ℓ, wage rate w and rental rate r, if firms choose to subcontract, firms using κ pieces of intermediate ℓ/n and minimize the costs to the output ℓ

$$\min_{k,n} rk + wn + \kappa p(\ell/\kappa) \qquad \text{s.t. } \ell = A_s \left[(k^a n^{1-a})^{1-\sigma} \left(\kappa \frac{\ell}{\kappa} \right)^\sigma \right]^\theta.$$

Let the assembly cost function be the resulting minimum cost function of labor and capital as

$$\alpha(\ell) \coloneqq \min_{k,n}\{rk + wn\} = rk^* + wn^* = C_a \ell^{(1-\sigma\theta)/(\theta(1-\sigma))}, \tag{7}$$

for a positive constant C_s.[15] Since $\theta < 1$, we have $(1-\sigma\theta)/(\theta(1-\sigma)) > 1$.

On the other side, if the firm is a home producer, assume that the cost is $c(\ell) = \min_{k,n}\{rk + wn\} + h\ell$, where h is a constant. Given ℓ, a home producer solves

$$\min_{k,n} rk + wn + h\ell \qquad \text{s.t. } \ell = A_h(k^a n^{1-a})^\theta. \tag{8}$$

Then, the resulting cost is

$$c(\ell) = C_h \ell^{1/\theta} + f\ell$$

satisfying $c'(0) > 0$, where C_h is some constant.[16] It is known that the capital share is around 1/3 and the intermediate share is around 1/2. Suppose that $\theta = 9/10$, for example, then $\alpha(\ell) = C_h \ell^{11/9}$ and $c(\ell) = C_h \ell^{10/9} + h\ell$.

This simple model also sheds light on how production efficiency affects the outsourcing decision. If assembly is significantly more efficient than home production, i.e., A_s is

[15] $C_s = (1/A_s)^{1/(1-\sigma)\theta} \left[(r/w)^a \left((1-a)/a\right)^a + (w/r)^{1-a} \left(a/(1-a)\right)^{1-a} \right]$.

[16] $C_h = (1/A_h)^{1/\theta} \left[(r/w)^a \left((1-a)/a\right)^a + (w/r)^{1-a} \left(a/(1-a)\right)^{1-a} \right]$

much higher than A_h, then C_s is much smaller than C_h. Then, outsourcing will occur if the transaction cost is also small enough.

Observe that the parameter q of assembly costs is $(1-\sigma\theta)(\theta(1-\sigma))$, which is increasing in σ. Since $\ell \leqslant 1$, the assembly cost $\alpha(\ell)$ is decreasing in the share of intermediate σ. Thus, if the share of intermediate is lower, the higher assembly costs make subcontracting less attractive, yielding a shorter the production chain.

According to Boehm and Oberfield (2020), when courts are more congested, transaction cost τ is higher and the share of intermediate σ is lower. Together with both changes, the model predicts that the chain is shorter, prices are higher and firms at the most upstream level will have larger vertical span of production. This aligns with the findings in Boehm and Oberfield (2020), showing that plants tend to have large vertical span of production if they are confronting higher-congested courts and relying on relationship-specific inputs.

3. An Economy of Balanced Production Chains

Guilmi and Fujiwara (2020) and Hiromitsu and Wataru (2020) suggests that a firm's position in a production network is related with its size, which causes the departure of Gibrat's law for upstream firms. Moreover, Gibrat's law typically breaks down for small firms due to their higher growth rates driven by innovation.[17] This phenomenon is particularly pronounced for small firms situated at the upstream levels of a production chain, as noted by Hiromitsu and Wataru (2020). Guilmi and Fujiwara (2020) further show that upstream firms tend to be more volatile than the downstream firms. Hence, Gibrat's law breaks down for those smaller upstream firms.

This section aims to illustrate this departure within a production network. Specifically, we construct an economy based on the balanced trees of a production network. This economy has the features that the firm sizes of assemblers or subcontractors follow a power-law distribution, and home producers are more fluctuated than subcontractors, assuming the assembly technology remains unchanged in the short run. Consequently, when examining the upstream firms across all balanced chains, the upstream firms are more volatile.

[17]e.g. Lotti et al. (2003), Calvo (2006), Daunfeldt and Elert (2013), Tang (2015) and Aydogan and Donduran (2019))

3.1. **Distribution of Firm Size.** We first show that when in a balanced supply chain, where the number of suppliers is fixed for every subcontractor, the sizes of firms follow a power-law distribution. This section is motivated by the fact that the Pareto distribution of firm size holds for mid-stream and downstream firms (Hiromitsu and Wataru, 2020).

Let m^* be the optimal length of the chain in equilibrium. Recall the network structure that there are κ^{m-1} number of firms at level m and each firm of level m implements κ^{1-m} of tasks for all $m \leqslant m^*$.

Define the firm size by the value-added v_j for firm j in the chain. If firm j is a home producer, then its value-added is defined as the cost of in-house production, $v_j = c(\ell_j)$ for $\ell_j \in \mathcal{D}$. On the other hand, if firm j is a subcontractor, then the value-added is the assembly cost, $v_j = \alpha(\ell_j)$.[18] If the subcontractor j is at level m, its value-added or firm size is $v_j = b\ell_j^q = b(\kappa^{1-m})^q = b\kappa^{(1-m)q}$.

If we also let $q = 0$ for assembly cost function, then the distribution of firm size is flat since the value-added for all subcontractors is a constant $\alpha(\ell) \equiv b$. Alternatively, if $q > 0$, then the firm sizes, $b\kappa^{(1-m)q}$, decline exponentially as we go upstream levels, while at the same time the number of firms κ^{1-m} at each level grow exponentially. Hence, we have the following lemma.

Let the firm size of subcontractor be s and the fraction of firms with sizes greater than or equaled to s be $F(s)$.

Proposition 3.1. *Suppose that Assumption 2.1 and 2.2 hold, assembly cost is $b(\ell)^q$ with $b, q > 0$, and the price function is*

$$p(\ell) = \delta \min\{kp(\ell/\kappa) + b\ell^q,\, c(\ell)\} \qquad (\ell \in \mathcal{D}).$$

Then, there exist $s_{min>0}$ and $\gamma > 0$ such that $F(s) = Cs^{-\gamma} - D$ for all $\bar{s} \geqslant y \geqslant s_{min}$, where C and D are some positive constants, and $\bar{s}$ is the size of the largest subcontractor. In particular, $\gamma = 1/q$.

Proof. Let all the stated assumptions hold. Suppose that there are m^* levels in total in equilibrium. Since there is no subcontractor if $m^* = 1$, assume that $m^* > 1$. By the discussion in section 3.1, there are κ^{i-1} firms at level i and the subcontractors'

[18] A subcontractor buys intermediate inputs form suppliers and do the assembly work, so its value-added is the assembly cost.

sizes are $b\kappa^{(1-i)q}$ at the level i of chain in equilibrium. Let $s_{min} = b\kappa^{(2-m^*)q}$, which is the sizes of the smallest subcontractors at level $m^* - 1$.[19] Let s be the firm size of one of subcontractors. Then, there exists $m \leqslant m^*$ such that $s = b\kappa^{(1-m)q}$. Clearly, $b \geqslant s \geqslant s_{min}$.

Moreover, there are in total $1 + \kappa + \cdots + \kappa^{m^*-1}$ firms in the chain and there are $1 + \kappa + \cdots + \kappa^{m-1}$ firms at level $1, \ldots, m$.[20] In this setting, $F(s)$ is the fraction of subcontractors at level $1, \ldots, m$. Then, by definition of f and changing variables of $s = b\kappa^{(1-m)q}$, we have

$$F(s) = \frac{1 + \kappa + \cdots + \kappa^{m-1}}{1 + \kappa + \cdots + \kappa^{m^*-1}} = \frac{\kappa^m - 1}{\kappa^{m^*} - 1} = \frac{\kappa b^{1/q} s^{-1/q} - 1}{\kappa^{m^*} - 1}.$$

Let $\gamma \coloneqq 1/q$, $C \coloneqq \kappa b^{1/q}/(\kappa^{m^*} - 1)$ and $D \coloneqq 1/(\kappa^{m^*} - 1)$. Hence, $F(s) = Cs^{-\gamma} - D$ for $b \geqslant s \geqslant s_{min}$. Generally, $F(s) \propto s^{-\gamma}$. $\square$

Proposition 3.1 implies that the firm sizes for subcontractors are Pareto distributed in equilibrium. In addition, when the in-house producers have the smallest size in equilibrium, then the upper tail, excluding the smallest firms, is Pareto distributed. Furthermore, if $q = 1$, then the above distributions follow a Zipf's law.[21] If the "span of control" parameter of assembly cost q is less than one, the assembly cost is strictly concave and the Pareto exponent $1/q$ is greater than one by Proposition 3.1.

Example 3.1. This part presents the computation results given the proportional assembly cost $b\ell^q$. With the parameters $\tau = 0.01, b = 0.001, \kappa = 2, q = 0.92$ and $c(\ell) = 0.01(e^{25x} - 1)$, the production network is shown in Figure 3a. There are 8 levels in equilibrium. Next, we plot the log-rank plot of the distribution of firm sizes for subcontractors in this equilibrium, see Figure 3b. Note that the ranks are the same if the firms have the same size and there are κ^{m-1} firms at level $m = 1, ..., m^*$. Define the rank as κ^{m-1} if the firm is at level m.[22] Thus, the ranks for, say, $\kappa = 2$ are $1, 2, 2, 4, 4, 4, 4, 8, \ldots$. There are 8 levels in equilibrium, so there are 7 points in Figure 3b, where κ^{m-1} points of subcontractors collapse into one point in every level.

[19] This s_{min} captures the tail of all subcontractors. We can also choose larger sizes of subcontractors.

[20] Including the firm of size s itself.

[21] For example, Axtell (2001) finds that the Pareto exponent is 1.059 , using employees as sizes, or 0.994, using receipts as sizes, for U.S. firms.

[22] Under this definition, the slope of log-rank plot reveals the Pareto exponent for convenience.

The slope of log-rank plot is -0.92 which is equal to the parameter $-q$ of assembly function $b\ell^q$ as predicted by Proposition 3.1.[23] In this case, the Pareto exponent is $1/q = 1.087$.

In addition, let κ increase from 2 to 5 and fix other parameters. Then, the computation indicates that the total number of levels of the chain declines to 4 in equilibrium, see Figure 4. This result is reasonable since the maximal number of levels $\bar{m}$ is decreasing in κ. On the other hand, under these particular specifications, the firms at the most upstream level have the smallest sizes. As the previous example, the slope is $-q$ in Figure 4b and the Pareto exponent is unchanged. The comparative statics implies that the Pareto exponent is not affected by the structure of the network if the curvature of assembly cost is identical. $\square$

3.2. **Breakdown of Gibrat's Law.** Given a equilibrium production chain $(p^*, \boldsymbol{\ell}^*)$, Proposition 3.1 implies that large firms at the downstream exhibit power-law distributed sizes. In addition, it indicates that the sizes of firms at the most upstream do not necessarily follow a power-law distribution. To this end, the sizes of small firms within the production chain generally deviate from a power law. It suggests that there exists a departure of Gibrat's Law; otherwise, the sizes of small firms would also exhibit a power-law distribution.

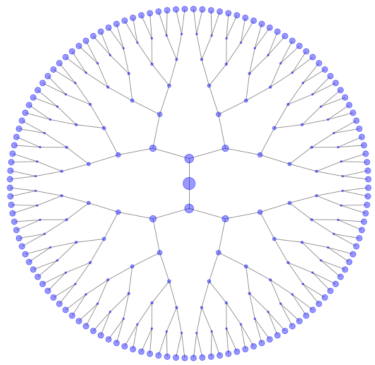
We restrict the time frame to the short run since we are interested in the short-term volatility. This assumption is inspired by Tang (2015), who demonstrates that there is a steady state in the firm's expansion and that Gibrat's law breaks down in the short run but holds in the long run.[24]

Suppose that firms solve cost minimization problems outlined in Section 2.3.4. Let the cost function be (8) and assembly cost function be (7). Suppose that the efficiency of in-house production grows over time in the short run, to the extend that the structure of the network undergoes minimum change.[25] That is, let $A_{h,t+1} = \zeta_{t+1} A_{h,t}$ with growth rate $\zeta_t > 0$ and let the other parameters are fixed, indicating that C_h in

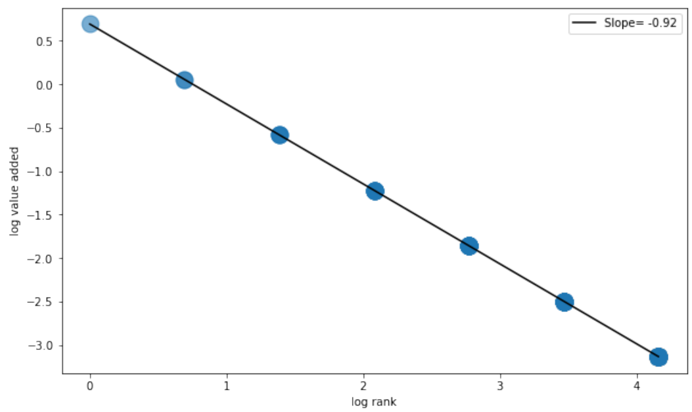
[23] Pareto law can also be formulated as $n = Ax^{-\gamma}$, where n is the number of people having wealth $\geqslant x$. Thus, we can write $x(n) = Cn^{-1/\gamma}$ and then $\log x(n) = \log C - (1/\gamma)\log n$. The slope of log-rank is $-1/\gamma$, which is $-q$ of Proposition 3.1.

[24] Tang (2015) studies firm-level data in Swedish energy market.

[25] Hiromitsu and Wataru (2020) also show that the structure of the production network is stable in the short run.



(A) Production network.

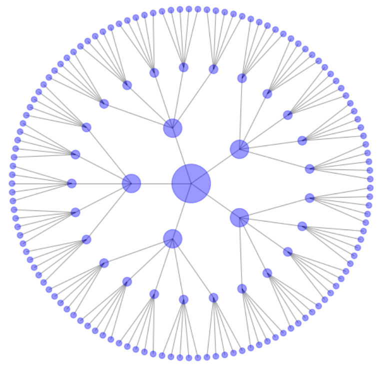


(B) The log-rank plot for the distribution of firm sizes.

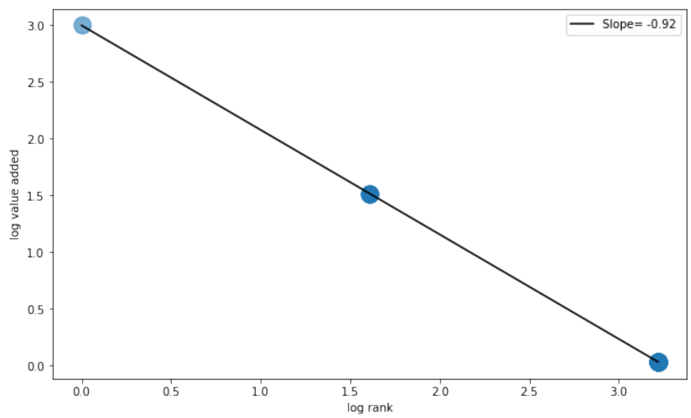
FIGURE 3. The production chain with $\tau = 0.01, b = 0.001, \kappa = 2, q = 0.92$ and $c(\ell) = 0.01(e^{25x} - 1)$.

(8) varies over time. Suppose that the growth ζ_{t+1} is a random process such that the optimal length of the production chain remains unchanged in the short run. There is no innovation in assembly technology in the short run. Consequently, the sizes of home producers change over time, while the sizes of subcontractors are stable over time. As a result, smaller firms at the upstream have higher volatility, while large firms at downstream maintain more stable sizes. Furthermore, if the assembly efficiency C_s in (7) or assembly productivity A_s is volatile in the short run, the most upstream firms also experience greater fluctuations. This is because firms may alter their outsourcing decisions or exit the production chain.

We expand the model from a chain to an industry, considering N heterogeneous goods in a industry. For each good $i = 1, \cdots, N$, there is a production chain dedicated to



(A) The network of production.



(B) The log-rank plot of the firm-size distribution for the subcontractors.

FIGURE 4. Production chain with $\tau = 0.01, b = 0.001, \kappa = 5, q = 0.92$ and $c(\ell) = 0.01(e^{25x} - 1)$.

produce good i. The pricing function (1) determines the price for each good, based on a fixed number of suppliers κ_i so that the chain is balanced.[26]

Production chain i is characterized by production cost c_i, assembly cost α_i and transaction cost τ_i. Specifically, we assume that $c_i(\ell) = C_{h,i}\ell^{\theta_i} + \varepsilon_i \ell$, where ε_i is a constant, $\alpha_i(\ell) = C_{a,i}\ell^{q_i}$ and parameter $\delta_i = 1/(1 - \tau_i)$. Here, the parameters $C_{h,i}, \varepsilon_i$ and $C_{a,i}$ are selected such that home-producing firms tend to be smaller in size compared to most outsourcing firms. Suppose that Assumption 2.1 , 2.2 and 2.3 hold. Then, Proposition 2.2 confirms the existence of a unique equilibrium for each production

[26] In Section 4, we can introduce a method to determine κ_i by the local optimality.

chain. In the equilibrium, each production line has the optimal length m_i^* and price p_i^*.

Assume that each firm produces all N final goods. Each firm j is randomly at level m_j in all chains and then has a random size S_j. The cost or value added of subcontractor firm j producing good i is X_i, which follows a power law with parameter γ_i by Proposition 3.1. From Gabaix (2009), we know that if Y_1 and Y_2 are power-law variables, then $Y_1 + Y_2$ is also power-law variable.[27] Then, a subcontractor's size $S_j = X_1 + \cdots + X_N$ follows a power law.

Contrary to home producers, the firm sizes of home producers do not necessarily follow a power law. If further the home production technology $C_{h,i}$ is more volatile than assembly technology $C_{a,i}$, then the sizes of home producers exhibit higher volatility. Examining the entire industry, the upstream firms in the economy tend to be more volatile than downstream firms in the short run. Given that home producers typically have smaller firm sizes, smaller firms inherently possess higher volatility.

In summary, it appears that the growth rates of firms are contingent on their sizes, leading to a breakdown of Gibrat's Law for upstream firms. Simultaneously, downstream subcontractors conform to a power-law size distribution across all production chains. The model elucidates that small firms and upstream firms deviate from Gibrat's law, while large firms and downstream firms adhere to a power-law distribution of firm sizes.

4. Extension: An Endogenous Number of Suppliers

The assumption that the number of suppliers is fixed may be unrealistic. However, we can approximate the real world by choosing a reasonable fixed number of suppliers for our model. This section first shows that it is able to find the local optimum for the supplier number by the numerical computation.

Moreover, we establish an alternative model relaxing the assumption of the fixed number of suppliers, so that subcontractors can choose an arbitrary number of suppliers. We show the existence and uniqueness of equilibrium for the extended models.

[27] If X and Y are power-law variables with exponent ζ_x and ζ_y satisfying $\zeta_y \geqslant \zeta_x$, then $X + Y$ is a power-law variable with exponent ζ_x.

4.1. **Local Optimality for Number of Suppliers.** Throughout this section, we denote the number of suppliers as $n \in \mathbb{N} \setminus \{1\}$. To simplify the problem, suppose that the assembly cost is fixed at a constant α. As indicated by $\bar{m} = \lceil 1 - \ln \bar{x} / \ln n \rceil$ from Section 2, we express the maximal possible number of levels as a function of n, denoted as $\bar{m}(n)$. Then, (5) implies that the price of final good is a function of n,

$$\begin{aligned} p^*(1,n) &:= \min_{m=1,\ldots,\bar{m}(n)} \left\{ (\delta n)^{m-1} \delta c \left(n^{1-m} \right) + \frac{(\delta n)^{m-1} - 1}{\delta n - 1} \delta \alpha \right\} \\ &= \min_{m=1,\ldots,\bar{m}(n)} \{ p(1,n,m) \}. \end{aligned} \tag{9}$$

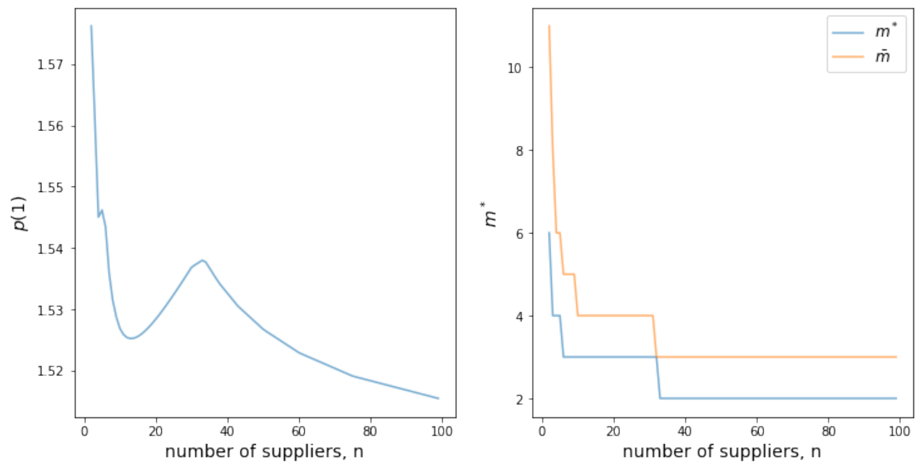
In words, the planner of the production chain strategically determines the optimal number of levels m, given the number of suppliers n in each level. This decision-making process is illustrated in Figure 5b, where the gray line represents the minimum cost, i.e. the prices of final good $p^*(1,n)$, among all possible m. Notably, for each $m = 2, \ldots, 6$, the pricing functions for the final good exhibit convexity, featuring local minimum prices. Therefore, if there are frictions to increase supplier numbers, such as contract costs and the expenses associated with searching for new suppliers, a firm may opt to remain at these locally minimized points to minimize the total cost effectively.

Moreover, Figure 5a reveals that the number of optimal levels m^* is less than $\bar{m}$ and converges to 2, and the final price is generally decreasing in n. This suggests that, in the absence of costs related to expanding suppliers, the most-downstream firm can achieve cost reduction by increasing suppliers to an arbitrary number.

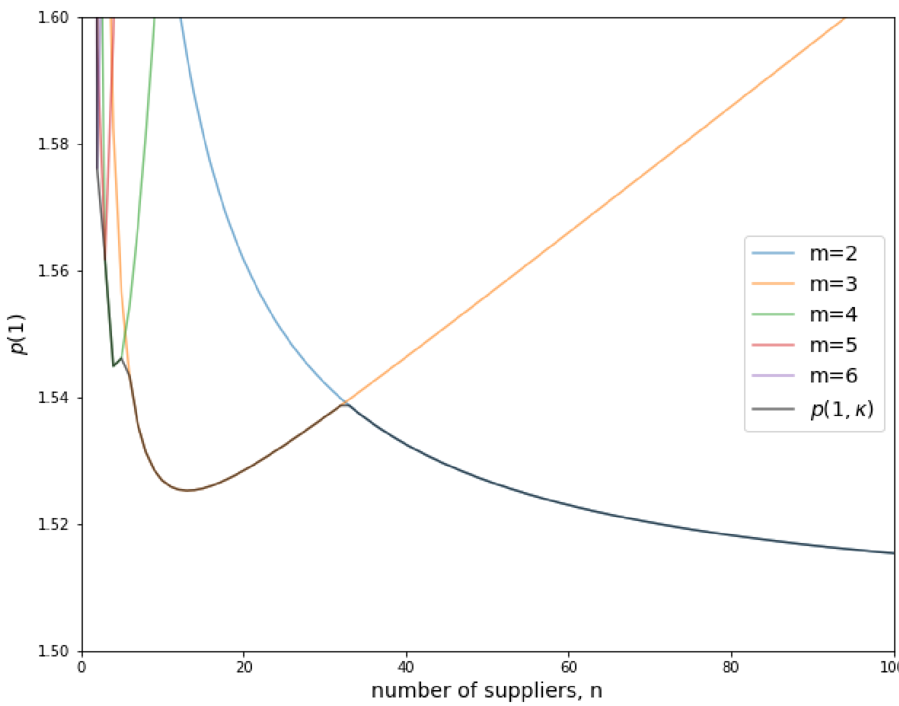
In general, the planner overseeing the chain has an incentive to increase the number of suppliers when there is no friction affecting the search of supplier. Since $\bar{m} = \lceil 1 - \ln \bar{x} / \ln n \rceil$, we have $\bar{m} \to 2$ as $n \to \infty$.[28] Moreover, given $m^* \leqslant \bar{m}$, we know that the optimal number of levels is $m^* \leqslant 2$ if the number of suppliers n is sufficiently large.

If the firm at the most downstream level has the incentive to subcontract and then there are exactly two levels in the chain $m^* = 2$, the pricing function implies $np(1/n) + \alpha = n\delta c(1/n) + \alpha < c(1)$. Since the cost function c is convex, strictly increasing, and $c(0) = 0$, the function $nc(1/n)$ is strictly decreasing in n and bounded below by zero. Hence, $nc(1/n)$ converges to some $t \geqslant 0$ as $n \to \infty$ by the monotone convergence

[28] $-\ln \bar{x} > 0$.



(A) Prices of final good and the corresponding number of levels



(B) Prices of final good, fixing number of levels m.

FIGURE 5. Prices of final good, given that $\tau = 0.001, \alpha(\ell) = 0.001, n = 2, c(\ell) = e^{1.5\ell} - 1$.

theorem of sequence. Thus, if $2\delta c(1/2) + \alpha < c(1)$, there will be two levels for all $n \geqslant 2$, and the firm at the most downstream level will subcontract, regardless of the number of suppliers.

Note that the condition $\delta 2c(1/2) + \alpha < c(1)$ is necessary for having at least one subcontractor in equilibrium. In this case, the firm can minimize the outsourcing costs by subcontracting the tasks to as many suppliers as possible, iven that the cost $\delta nc(1/n) + \alpha \downarrow \delta t + \alpha$. Therefore, if there is at least one subcontractor and no cost for expanding upstream partners, the planner has an incentive to raise the number of suppliers unlimitedly, resulting in only two levels in the chain.

4.2. **Global Optimality of Suppliers.** To curb the endogenously unbounded number of suppliers, we consider assembly costs that are intuitively proportional to the number of suppliers. The underlying rationale stems from that as subcontractors further divide tasks into smaller components, the process of assembling inputs necessitates additional effort. Precisely, let the assembly cost function be $\alpha(n)$ where n is the number of upstream suppliers and $\alpha(n)$ is strictly increasing in n.

Assumption 4.1. The assembly cost $\alpha : \mathbb{N} \setminus \{1\} \to \mathbb{R}$ is strictly increasing and $\alpha(2) > 0$.

Then, the profits for firms in the chain are

$$(1 - \tau)p(\ell) - \min\{np(\ell/n) + \alpha(n), c(\ell)\} \tag{10}$$

The corresponding Bellman equation of pricing function is

$$p(\ell) = \delta \min\{np(\ell/n) + \alpha(n), c(\ell)\} \tag{11}$$

Following the proofs in section 2.2, we can also show the existence and uniqueness of equilibrium price and allocation under the profits (10). Similar to Proposition 2.1 and 2.2, the unique equilibrium can be computed by iteration, i.e. $p^* = T^k p$ for any $p \in \mathcal{P}$ and large enough k.

Lemma 4.1. *If Assumption 2.1, 2.2 and 4.1 hold, then*

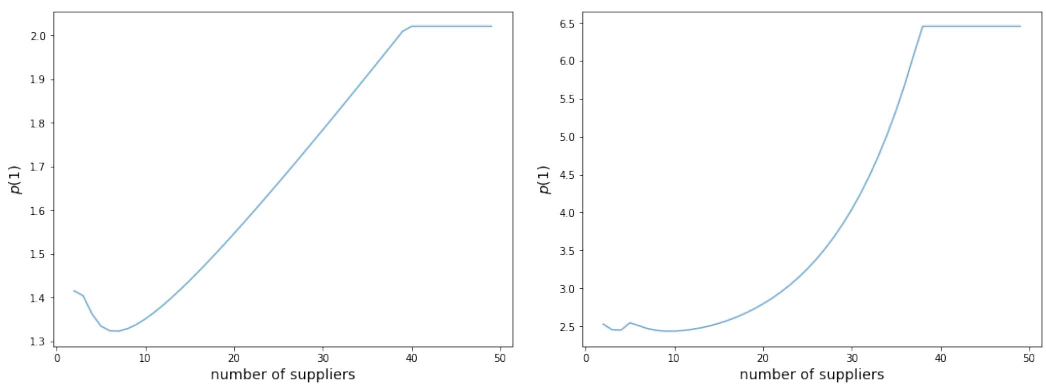
(a) the solution to the Bellman equation (11) is an equilibrium of the production chain given the profits (10),

(b) the equilibrium price p^* and allocations $\boldsymbol{\ell}^*$ are unique, and

(c) $p^* = T^k p$ for any $p \in \mathcal{P}$ and sufficiently large k.

Since $\alpha(n)$ is strictly increasing by Assumption 4.1, there exists $\bar{n}$ such that there is only one firm or level in the chain when $n \geqslant \bar{n}$. For $n \geqslant \bar{n}$, the firm at the most downstream level does not have any incentive to subcontract, so $\bar{n}$ can be characterized by $\delta nc(1/n) + d\alpha(n) \geqslant c(1)$. The interpretation is that, if there are many pieces of modules to assemble, the sum of assembly costs and the transaction costs may be greater than the costs of production at home so that the firm does not subcontract at all.

Moreover, the global minimum exists given that $\alpha(n)$ is increasing. The intuition is that if the assembly cost $\alpha(n)$ is increasing fast enough in the number of supplier n but still possible to subcontract, then the global minimum could be at $n = 2$. These properties are illustrated in the examples presented in Figure 6, which also depicts the upper bound for the price of final goods.[29]



(A) (B)

FIGURE 6. Examples of global minimum of prices. (a) Prices of final good are plotted given that $\tau = 0.01, \alpha(n) = 0.01(n+n^{1.1}), c(\ell) = \ell+\ell^2$. The global minimal price is at $n = 7$. (b) Prices of final good are plotted given that $\tau = 0.01, \alpha(n) = 0.1(e^{0.1n} - 1), c(\ell) = e^{2\ell} - 1$. The global minimal price is at $n = 10$.

4.3. **Choice over Number of Suppliers.** This section extends the assumption that the number of suppliers is fixed along the production chain. Firms have flexibility to select the number of suppliers when opting to subcontract their tasks. In particular, we introduce the pricing function

$$p(\ell) = \delta \min \left\{ \min_{n=2,3,4,\ldots} \{np(\ell/n) + \alpha(\ell, n)\},\ c(\ell) \right\}. \tag{12}$$

[29]Since the assembly is too high for large n, firms all choose to home production and the bound is $\delta c(1)$.

To provide a more comprehensive framework, suppose that the assembly cost depend on both tasks and the number of upstream partners. To prevent an unbounded partner number n^*, we have the below assumption for assembly cost.

Assumption 4.2. Assume that $\alpha : [0,1] \times \{2,3,\dots\} \to (0,\infty)$ is strictly increasing in both ℓ and n.

Assumption 4.2 makes the optimal number of suppliers n finite if there is outsourcing. Analogous to Section 2.2, let the set of tasks be $\mathcal{D} \coloneqq \{k^{-i} : k \in \mathbb{N} \setminus \{1\}, i \in \mathbb{N} \cup \{0\}\}$, and $\mathcal{P} \coloneqq \{p : \mathcal{D} \to \mathbb{R} : \delta c'(0)\ell \leqslant p(\ell) \leqslant \delta c(\ell)\}$.[30]

Under this pricing function, we say that $\boldsymbol{\ell} = \{\ell_i\}$ is a *feasible* allocation of tasks if there are $m \in \mathbb{N}$ and a finite sequence $\{n_t\}_{t=1}^m$ with $n_1 = 1$ and $n_t \geqslant 2$ for all $t = 2, \dots, m$ such that $i = 1, 2, \dots, n_1 n_2 \cdots n_m$ and $\ell_i = (n_1 n_2 \cdots n_t)^{-1}$ for $n_1 + n_1 n_2 + \cdots + n_1 n_2 \cdots n_{t-1} < i \leqslant n_1 + n_1 n_2 + \cdots + n_1 n_2 \cdots n_t$. In words, the length of the chain, m, is finite and there are $n_1 n_2 \dots n_t$ number of firms with allocation $(n_1 n_2 \dots n_t)^{-1}$ at level t. Each firm at level $t-1$ has n_t number of subcontractors if it chooses home production. At the most upstream level m, there are $n_1 \cdots n_m$ number of firms with $(n_1 \cdots n_m)^{-1}$ measure of allocated tasks.

Moreover, the profit of firm i is

$$\pi_i = (1-\tau)p(\ell_i) - \min\left\{ \min_{n=2,3,\dots} \{np(\ell_i/n) + \alpha(\ell_i, n)\},\ c(\ell_i) \right\} \tag{13}$$

We can define the equilibrium under price (12) as definition 2.1 using the above feasibility definition and profit (13).

Definition 4.1. Given a price function p, a feasible allocation $\boldsymbol{\ell} = \{\ell_i\}$ and the corresponding profit defined by (13), $(p, \boldsymbol{\ell})$ is an equilibrium if

(1) $p(0) = 0$,
(2) for all $\ell \in \mathcal{D}$,

$$(1-\tau)p(\ell) - \min\left\{ \min_{n=2,3,\dots} \{np(\ell/n) + \alpha(\ell, n)\},\ c(\ell) \right\} \leqslant 0,$$

(3) $\pi_i = 0$ for all i.

[30] It can also be defined as $D = [0,1]$ and the results hold.

Then, we can prove the existence of equilibrium. For the following proposition, define the operator T as

$$Tp(\ell) \coloneqq \delta \min \left\{ \min_{n=2,3,4,\dots} \{np(\ell/n) + \alpha(\ell, n)\},\, c(\ell) \right\}.$$

Suppose that firms choose the minimum number of suppliers when there are multiple number of suppliers that minimize the cost. That is, suppose firms choose $\min\{\arg\min_{n\geqslant 2}\{np(\ell/n)+\alpha(\ell,n)\}\}$ as their number of suppliers if $np(\ell/n)+\alpha(\ell,n) < c(\ell)$. The proof of unique equilibrium uses this assumption to pin down the feasible task allocation.

Proposition 4.1. *If 2.2, 2.1 and 4.2 hold, and the price is defined by (12), the following statements are true.*

(a) The operator T is a self-map on $\mathcal{P}$ and has a unique fixed point p^* in $\mathcal{P}$
(b) The fixed point can be computed by $T^k p = p^*$ for all $k \geqslant 1 - \ln \bar{x} / \ln 2$ and all $p \in \mathcal{P}$.
(c) The production chain has a unique equilibrium $(p^*, \boldsymbol{\ell}^*)$, where p^* is the fixed point of T, and $\boldsymbol{\ell}^*$ is the corresponding task allocation under p^*.

Hence, we can compute the unique equilibrium in a manner consistent with the basic model. The comparative statics also yield similar results as previously observed.

Proposition 4.2. *If 2.2, 2.1 and 4.2 hold, and the price is defined by (12), then the equilibrium price is increasing in transaction cost, assembly cost and home production cost.*

As in Section 2.3, we can gain insights into the equilibrium price by the properties of cost function before computing it. The following two lemmas illustrate the equilibrium price's characteristics, demonstrating that, under certain conditions, the equilibrium price is strictly convex almost everywhere.

Suppose that the assembly function $\alpha(n)$ only depends on the number of suppliers and Assumption 2.3 holds. Moreover, assume that $c, \alpha \in \mathrm{C}^2$ are twice differentiable. Define $t(n,\ell) \coloneqq \delta n c(\ell/n) - c(\ell)$. Then, the second derivative $t_{22}(n,\ell) = (\delta/n)c''(\ell/n) - c''(\ell)$ implies that $t(n, .)$ is strictly concave when $0 < \tau < 0.5$ and c is strictly convex, with $n \geqslant 2$. In addition, define $f(\ell) \coloneqq \min_{n=2,3,\dots}\{t(n,\ell) + \alpha(n)\}$. The next lemma helps us to determine the boundary of subcontracting and length of the chain.

Lemma 4.2. *Suppose that $t(n, .)$ is strictly concave. Then, $f(\ell)$ has a root $\hat{\ell}$ in $(0, 1]$ if and only if $p(\ell) = \delta c(\ell)$ for all $1 \leqslant \ell \leqslant \hat{\ell}$ and $p(\ell) < \delta c(\ell)$ for all $\hat{\ell} < \ell \leqslant 1$.*[31]

This lemma shows that when the transaction cost is not excessively large ($\tau < 0.5$) and the cost is strictly convex, the boundary allocation for subcontracting can be characterized by computing the root of $f(\ell)$, which is determined by the home-producing cost and assembly cost. This allows us to calculate the boundary task of of subcontracting (the root $\hat{\ell}$) before knowing the equilibrium price.

Lemma 4.2 also shows that if the root of $f(\ell)$ is less than 1, then the length of chain is greater than or equal to 2. Consequently, there must exist subcontractors in the chain, as the firms with $\ell \in (\hat{\ell}, 1]$ always choose to outsource. Furthermore, if the assembly is relatively economical, meaning that firms can choose numerous partners and divide the tasks in numerous modules with little cost, then there would be only two levels of length, as $1/n < \hat{\ell}$ for large n.

Lemma 4.3. *Let p be the equilibrium price of (12). Suppose that the assumptions in Lemma 4.2 hold and further assume that c is strictly convex and $\alpha(n)$ is convex. Then p is twice differentiable with $p' > 0$ and $p'' > 0$ almost everywhere.*

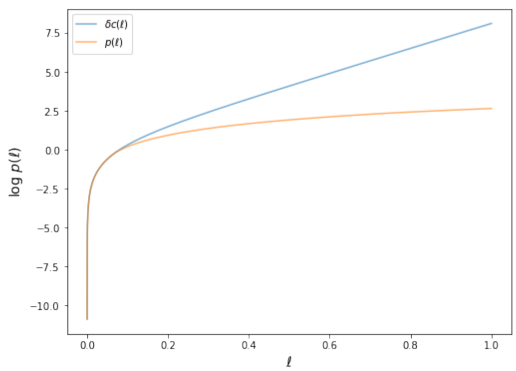
Figure 7 gives an example of a production network under pricing function (12), with assembly cost $\alpha(n)$ depending only on n. The scale of pricing function is in log. This example also demonstrates a fat-tailed distribution of the number of buyer-seller links.[32] Moreover, numerical simulations indicate that the number of suppliers may decrease as the firm goes to upstream levels, when the assembly cost only contigent on the supplier count.

5. Conclusion

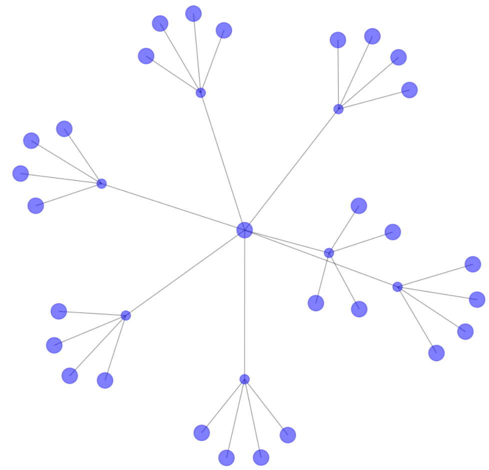
This paper demonstrates the influence of a hierarchical production chain. The production network with hub-like firms can generate a power-law distribution in firm size without any ex-ante heterogeneity or adhering to Gibrat's law. The production network coordinates the positions and task allocations for ex-ante identical firms in

[31] This lemma also holds if the assembly cost is $\alpha(\ell, n)$ and $\alpha(., n)$ is concave.

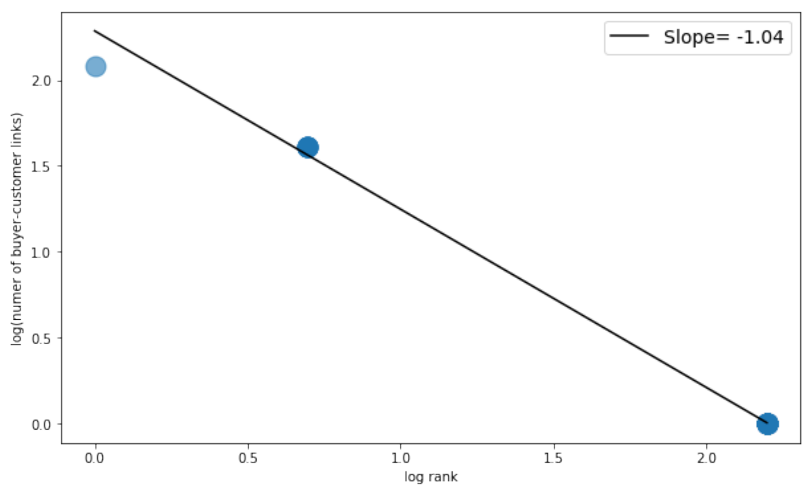
[32] See Bernard et al. (2019) for an empirical example.



(A) The log-price function.



(B) The network.



(C) The distribution of number of seller-buyer links.

FIGURE 7. An example of production chain under (12) with $\tau = 0.1, \alpha(n) = 0.01n^{1.8}, c(\ell) = e^{8\ell} - 1$. This chain has length 3 and $(n_1, n_2, n_3) = (1, 7, 4)$. Also, $\hat{\ell} = 0.0711$.

a way that a large firm tends to be a hub and be at a downstream level. The framework in this paper endeavors to capture this phenomenon through the minimization of costs. Our model provides insights into why firms following a power law often occupy downstream positions. Moreover, it explains the departure of Gibrat's law for upstream firms, which are more volatile than the downstream counterparts. The model underscores the significance of a spider network, akin to the concept introduced by Baldwin and Venables (2013).

Appendix A. Appendix

Proofs in Section 2.2. Throughout this section, we assume that Assumption 2.1, 2.2 and 2.3 hold.

Lemma A.1. *The operator T defined by (3) is a self-map on $\mathcal{P}$ and has a fixed point in $\mathcal{P}$. Moreover, $Tp_1 \leqslant Tp_2$ if $p_1 \leqslant p_2$ for $p_1, p_2 \in \mathcal{P}$.*

Proof. The proof is similar to Kikuchi et al. (2021). Let T be the operator of (3). We want to show that $T : \mathcal{P} \to \mathcal{P}$, and T preserves orders so that we can apply Knaster-Tarski fixed point theorem, with the fact that $\mathcal{P}$ is a complete lattice.[33]

Fix $p \in \mathcal{P}$. Clearly, the definition of T implies $Tp(\ell) \leqslant \delta c(\ell)$ for all ℓ. Next, we check that $\delta c'(0)\ell \leqslant Tp(\ell)$ for all ℓ. Fix $\ell \in \mathcal{D}$. Since $p \in \mathcal{P}$, $b\ell^q \geqslant 0$ and c is convex, we have

$$
\begin{aligned}
Tp(\ell) &= \delta \min\{\kappa p(\ell/\kappa) + b\ell^q,\, c(\ell)\} \\
&\geqslant \delta \min\{\kappa \delta c'(0)\ell/\kappa + b\ell^q,\, c(\ell)\} \\
&\geqslant \delta \min\{c'(0)\ell,\, c(\ell)\} \\
&= \delta c'(0)\ell.
\end{aligned}
$$

Then, $Tp \in \mathcal{P}$. In addition, suppose that $p_1, p_2 \in \mathcal{P}$ with $p_1(\ell) \leqslant p_2(\ell)$ for all $\ell \in \mathcal{D}$. Thus, we have

$$
\min\{\kappa p_1(\ell/\kappa) + \alpha(\ell),\, c(\ell)\} \leqslant \min\{\kappa p_2(\ell/\kappa) + \alpha(\ell),\, c(\ell)\},
$$

for all ℓ. Hence, we have $Tp_1 \leqslant Tp_2$ and T preserves orders. By Knaster-Tarski fixed point, the set of fixed points of T in $\mathcal{P}$ is also a complete lattice. Since a complete lattice is nonempty, the fixed points exist. $\square$

Next, we characterize the price function in equilibrium and show its uniqueness. The subsequent lemma suggests that there exists a maximum number of levels. In other words, the firms will choose to home production when their task allocations are sufficiently small. Recall that $\bar{x} \coloneqq \sup\{x \in (0,1] : c'(x) \leqslant \delta c'(0)\}$.

[33] A partially ordered set is a *complete lattice* if every subset has both an infimum and a supremum.

Lemma A.2. *Suppose that $p \in \mathcal{P}$ is a solution to price function (1). Then, given $\ell \in \mathcal{D}$, if $\ell \leqslant \bar{x}$, then $p(\ell) = \delta c(\ell)$. That is, firms choose to home production if $\ell \leqslant \bar{x}$.*

Proof. Suppose that $p \in \mathcal{P}$ is a solution to price function (1). Since $\delta = 1/(1-\tau) > 1$ with $\tau \in (0,1)$ and Assumption 2.2 holds, there is $x > 0$ such that $c'(x) < \delta c'(0)$. Thus, $\bar{x}$ is well-defined. Toward contradiction, suppose that there is $\ell \in \mathcal{D}$ with $\ell \leqslant \bar{x}$ such that $\kappa p(\ell/\kappa) + \alpha(\ell) < c(\ell)$. Since $p \in \mathcal{P}$, we have

$$c(\ell) > \kappa p(\ell/\kappa) + \alpha(\ell) \geqslant \kappa \delta c'(0)\frac{\ell}{\kappa} + \alpha(\ell) = \delta c'(0)\ell + \alpha(\ell) \geqslant \delta c'(0)\ell.$$

Hence, we have $c(\ell)/\ell > \delta c'(0)$. Since c is convex and $\ell \leqslant \bar{x}$, this contradicts the fact that $c(\ell)/\ell \leqslant c'(\ell) \leqslant \delta c'(0)$. Thus, it must be $\kappa p(\ell/\kappa) + \alpha(\ell) \geqslant c(\ell)$ and then $p(\ell) = \delta c(\ell)$, where we follow the convention that $\min\{\kappa p(\ell/\kappa) + \alpha(\ell), c(\ell)\} = c(\ell)$ if equality holds.[34] □

Lemma A.3. *If $p \in \mathcal{P}$, then $Tp(\ell) = \delta c(\ell)$ for all $\ell \leqslant \bar{x}$.*

Proof. The statement follows from Lemma A.2. □

Lemma A.4. *If $p, q \in \mathcal{P}$, then $T^n p = T^n q$ for $n \geqslant 1 - \ln(\bar{x} - \kappa)$.*

Proof. Let $p, q \in \mathcal{P}$. The proof is by induction. From the proof of Lemma A.2, we can show that Tp and Tq are agree on $[0, \bar{x}]$, where $\bar{x} \coloneqq \sup\{x \in (0,1] : c'(x) \leqslant \delta c'(0)\}$. Next, we prove that $T^n p = T^n q$ on $[0, \kappa^{n-1}\bar{x}]$ for all $n \in N$. Suppose that the claim is true for some $k \in \mathbb{N}$. Fix $\ell \in [0, \kappa^k \bar{x}]$. Then, since $\ell/\kappa \in [0, \kappa^{k-1}\bar{x}]$, we have

$$\begin{aligned}
T^{k+1}p(\ell) &= T(T^k p)(\ell) \\
&= \delta \min\left\{\kappa T^k p\left(\frac{\ell}{\kappa}\right) + \alpha(\ell), c(\ell)\right\} \\
&= \delta \min\left\{\kappa T^k q\left(\frac{\ell}{\kappa}\right) + \alpha(\ell), c(\ell)\right\} \\
&= T^{k+1}q(\ell)
\end{aligned}$$

Therefore, $T^n p = T^n q$ on $[0, \kappa^{n-1}\bar{x}]$ is true for all $n \in \mathbb{N}$. Since $\kappa^{n-1}\bar{x} \geqslant 1$ for $n \geqslant 1 - (\ln \bar{x})/(\ln \kappa)$, we have $T^n p = T^n q$ on $[0,1]$ for $n \geqslant 1 - (\ln \bar{x})/(\ln \kappa)$. □

[34] That is, firms choose to produce in-house if the costs for home production and subcontracting are identical.

Lemma A.5. *T has a unique fixed point in $\mathcal{P}$ and thus the solution to the pricing function (1) is also unique.*

Proof. Let p, q be the fixed points of T. By Lemma A.4, $p = T^n p = T^n q = q$ for large enough n. Moreover, by the definition of T, p^* is the fixed point of T if and only if it is the solution to pricing equation (1). Thus, the solution exists and is unique. $\square$

Lemma A.6. *If $p \in \mathcal{P}$ is (resp. strictly) increasing, then Tp is also (resp. strictly) increasing. Moreover, if p^* is the fixed point of T, then p^* is strictly increasing.*

Proof. Since c is strictly increasing in $\ell \in [0.1]$, Tp is (strictly) increasing if p is (strictly) increasing. It follows from Lemma A.4 that $p^* = T^n c$ for large enough n, which implies that p^* is strictly increasing. $\square$

Lemma A.7. *If p^* is a solution to the price equation (1), then there is a feasible $\boldsymbol{\ell}^*$ such that $(p^*, \boldsymbol{\ell}^*)$ is an equilibrium for the production chain.*

Proof. Let p^* be a solution to the price equation (1). By Lemma A.2, there exists m such that $p(\ell) = \delta c(\ell)$ with $\ell = \kappa^{1-m} \leqslant \bar{x}$, where $\bar{x} = \sup\{x \in (0, 1] : c'(x) \leqslant \delta c'(0)\}$. Let $\bar{m} := \lceil 1 - \ln \bar{x} / \ln \kappa \rceil$, which is the smallest integer satisfying $\kappa^{1-\bar{m}} \leqslant \bar{x}$ and the largest possible length of chain. Therefore, the corresponding maximal level $m^* \in \mathbb{N}$ for price p^* satisfies $m^* \leqslant \bar{m}$. The corresponding allocations are $\boldsymbol{\ell}^* = \{\ell_i^*\}$ are $\ell_i^* = \kappa^{1-n}$ for $i = 1, \ldots, (\kappa^m - 1)/(\kappa - 1)$ and $(\kappa^{n-1} - 1)/(\kappa - 1) < i \leqslant (\kappa^n - 1)/(\kappa - 1)$ for some $1 \leqslant n \leqslant m^*$. Therefore, $\boldsymbol{\ell}^*$ is feasible.

Since $p^* \in \mathcal{P}$, we have $\delta c'(0)\ell \leqslant p^*(\ell) \leqslant c(\ell)$ for all ℓ in $\mathcal{D}$. It implies $p^*(0) = 0$. Moreover, since the definition of price equation implies that for all $\ell \in \mathcal{D}$ we have

$$p^*(\ell) - \delta \min\{\kappa p^*(\ell/\kappa) + \alpha(\ell), c(\ell)\} = 0,$$

condition (2) and (3) of Definition 2.1 are clearly satisfied. Therefore, $(p^*, \boldsymbol{\ell}^*)$ is an equilibrium. $\square$

In Lemma A.7, we construct an equilibrium $(p^*, \boldsymbol{\ell}^*)$ if p^* is a unique solution to price function. We next show that $(p^*, \boldsymbol{\ell}^*)$ is the unique equilibrium. The proof in the following lemma uses the convention that the firms choose to produce in-house when the costs for home production and subcontract are indifferent.

Lemma A.8. *If the equilibrium prices and allocations exist, then they are unique.*

Proof. Suppose that there are two arbitrary equilibria $(p_1, \boldsymbol{\ell}^1)$ and $(p_2, \boldsymbol{\ell}^2)$ following Definition 2.1. Since $(p_1, \boldsymbol{\ell}^1)$ and $(p_2, \boldsymbol{\ell}^2)$ are feasible, there exist $m_1, m_2 > 0$ such that the allocations $\boldsymbol{\ell}^1$ and $\boldsymbol{\ell}^2$ have m_1 and m_2 length of chain or maximal levels, respectively. Clearly, $\boldsymbol{\ell}^1 \neq \boldsymbol{\ell}^2$ if and only if $m_1 \neq m_2$, by the definition of feasibility. Without loss of generality, assume that $m_1 \geqslant m_2$. We first show that $p_1(\ell) = p_2(\ell)$ for all $\ell = 1, \kappa^{-1}, \kappa^{-2}, \dots, \kappa^{1-m_2}$ and then extend to $\ell = 1, \kappa^{-1}, \dots, \kappa^{1-m_2}, \dots, \kappa^{1-m_1}$.

We claim that, for $\ell = \kappa^{-1}, \kappa^{-2}, \dots, \kappa^{1-m_2}$, $p_1(\ell) = p_2(\ell)$ implies $p_1(\kappa\ell) = p_2(\kappa\ell)$. Let $p_1(\ell) = p_2(\ell)$ for all $\ell = \kappa^{-1}, \kappa^{-2}, \dots, \kappa^{1-m_2}$. Since the firms are subcontractors at level $1, \dots, m_2 - 1$ for both equilibrium, condition (3) of Definition 2.1 implies that

$$(1-\tau)p_1(\kappa\ell) = \kappa p_1(\ell) + \alpha(\kappa\ell) = \kappa p_2(\ell) + \alpha(\kappa\ell) = (1-\tau)p_2(\kappa\ell).$$

Therefore, we have $p_1(\kappa\ell) = p_2(\kappa\ell)$. In words, if the prices are equal at level i for $i = 2, \dots, m_2$, then the prices are equaled at its downstream level $i-1$. To this end, if we can show that $p_1(\kappa^{1-m_2}) = p_2(\kappa^{1-m_2})$ at level m_2, then it implies that $p_1(\ell) = p_2(\ell)$ for all $\ell = 1, \kappa^{-1}, \dots, \kappa^{1-m_2}$.

If $m_1 = m_2$, then firms are home producers at level m_1 in both equilibrium so that $p_1(\kappa^{1-m_1}) = \delta c(\kappa^{1-m_1}) = p_2(\kappa^{1-m_1})$. Then, $p_1(\ell) = p_2(\ell)$ for all $\ell = 1, \kappa^{-1}, \dots \kappa^{1-m_1}$ and two equilibrium prices are the same.

Suppose that $m_1 > m_2$. Let $\delta = 1/(1-\tau)$ and $t = \kappa^{1-m_2}$ to simplify notations. Then, $p_1(t) \neq p_2(t)$. At level m_2, the firms in equilibrium $(p_1, \boldsymbol{\ell}^1)$ are subcontractors, while the firms in equilibrium $(p_2, \boldsymbol{\ell}^2)$ are in-house producers. Hence, condition (3) of

Definition 2.1 implies $p_2(t) = \delta c(t)$ and the following iteration

$$\begin{aligned}
p_1(t) &= \delta\kappa p_1(t/\kappa) + \delta\alpha(t) \\
&= \delta\kappa[\delta\kappa p_1(t/\kappa^2) + \delta\alpha(t/\kappa)] + \delta\alpha(t) \\
&= (\delta\kappa)^2 p_1(t/\kappa^2) + \delta^2\kappa b(t/\kappa)^q + \delta b(t)^q \\
&= (\delta\kappa)^2 p_1(t/\kappa^2) + \delta b t^q[\delta\kappa^{1-q} + 1] \\
&= (\delta\kappa)^2[\delta\kappa p_1(t/\kappa^3) + \delta\alpha(t/\kappa^2)] + \delta b t^q[\delta\kappa^{1-q} + 1] \\
&= (\delta\kappa)^3 p_1(t/\kappa^3) + \delta b t^q[(\delta\kappa^{1-q})^2 + \delta\kappa^{1-q} + 1] \\
&= \dots \\
&= (\delta\kappa)^{m_1-m_2} p_1(t/\kappa^{m_1-m_2}) \\
&\qquad + \delta b t^q[(\delta\kappa^{1-q})^{m_1-m_2-1} + \dots + \delta\kappa^{1-q} + 1] \\
&= (\delta\kappa)^{m_1-m_2}\delta c(\kappa^{1-m_1}) + \delta b t^q \frac{(\delta\kappa^{1-q})^{m_1-m_2} - 1}{\delta\kappa^{1-q} - 1}
\end{aligned} \tag{14}$$

Similarly, condition (2) of equilibrium implies $p_1(t) \leqslant \delta c(t)$ and the iteration

$$\begin{aligned}
p_2(t) &\leqslant \delta\kappa p_2(t/\kappa) + \delta\alpha(t) \\
&= (\delta\kappa)^2 p_2(t/\kappa^2) + \delta b t^q[(\delta\kappa^{1-q}) + 1] \\
&\leqslant (\delta\kappa)^2[\delta\kappa p_2(t/\kappa^3) + \delta\alpha(t/\kappa^2)] + \delta b t^q[\delta\kappa^{1-q} + 1] \\
&= (\delta\kappa)^3 p_2(t/\kappa^3) + \delta b t^q[(\delta\kappa^{1-q})^2 + \delta\kappa^{1-q} + 1] \\
&\leqslant \dots \\
&\leqslant (\delta\kappa)^{m_1-m_2} p_2(t/\kappa^{m_1-m_2}) + \delta b t^q[(\delta\kappa^{1-q})^{m_1-m_2-1} + \dots + 1] \\
&\leqslant (\delta\kappa)^{m_1-m_2}\delta c(\kappa^{1-m_1}) + \delta b t^q \frac{(\delta\kappa^{1-q})^{m_1-m_2} - 1}{\delta\kappa^{1-q} - 1} \\
&= p_1(t)
\end{aligned} \tag{15}$$

where the last inequality follows from $p_2 \in \mathcal{P}$. Therefore, $\delta c(t) = p_2(t) \leqslant p_1(t) \leqslant \delta c(t)$. We have $p_1(t) = p_2(t)$ and then $p_1(\ell) = p_2(\ell)$ for all $\ell = 1, \kappa^{-1}, \dots, \kappa^{1-m_2}$ at lower levels by the previous claim. It implies that firms at level m_2 in equilibrium $(p_1, \boldsymbol{\ell}_1)$ have the same costs as the home production.

Since firms choose to produce in-house when the home production costs and subcontracting costs are the same, the firms at level m_2 under equilibrium p_1 will choose to home production since $p_1(\kappa^{1-m_2}) = \delta c(\kappa^{1-m_2}) = \delta\kappa p_1(\kappa^{-m_2}) + \delta\alpha(\kappa^{1-m_2})$. That is, the firms have no incentive to subcontract and make longer lengths than m_2. Then, it must be $m_1 = m_2$. This implies that $\boldsymbol{\ell}^1 = \boldsymbol{\ell}^2$ since the tasks in each level are identical for their feasibility. $\square$

Proof of Proposition 2.1. The statements follow from Part (a) follows from Lemma A.1, Part (b) and (c) follow from Lemma A.5, Part (d) follows from Lemma A.4, Part (e) follows from A.2, and Part (f) follows from Lemma A.6. $\square$

Proof of Proposition 2.2. The statements follow from Lemma A.7 and Lemma A.8. $\square$

Proofs in Section 2.3. Suppose that 2.1, 2.2, and 2.3 hold throughout this section. Let $f(\ell) \coloneqq \delta\kappa c(\ell/k) + \alpha(\ell) - c(\ell)$ for all $\ell \in [0,1]$.

Lemma A.9. *Suppose that p is the equilibrium price and f is strictly concave. Then, $\hat{\ell} \in (0,1]$ is a root of f if and only if $\hat{\ell}$ satisfies $p(\ell) = \delta c(\ell)$ for all $0 \leqslant \ell \leqslant \hat{\ell}$ and $p(\ell) < \delta c(\ell)$ for all $\hat{\ell} < \ell \leqslant 1$.*

Proof of Lemma A.9. Let the stated assumptions hold. Let $P = \{p : [0,1] \to \mathbb{R} : \delta c'(0)\ell \leqslant p(\ell) \leqslant \delta c(\ell), \forall \ell \in [0,1]\}$. The same arguments in Lemma A.1, Lemma A.2 and A.4 show that $T\colon P \to P$ has a unique fixed point in P. Let p be the fixed point of P, which is also a equilibrium price.

Since $f(0) = 0$ by assumption 2.1, 2.2 and 2.3, and f is strictly concave, the root of f on $[0,1]$ is unique. Suppose that $\hat{\ell} \in (0,1]$ satisfies $f(\hat{\ell}) = 0$. Fix $\ell \in [0,\hat{\ell}]$. We have $f(\ell) \geqslant 0$ and $\delta\kappa c(\ell/\kappa) + \alpha(\ell) \geqslant c(\ell)$, so

$$T\delta c(\ell) = \delta \min\{\kappa\delta c(\ell/\kappa) + \alpha(\ell), c(\ell)\} = \delta c(\ell).$$

Then, $\delta c(\ell)$ is a fixed point of T if ℓ is restricted in $[0,\hat{\ell}]$. Since T only has one fixed point, we have $p(\ell) = \delta c(\ell)$.

Next, let $\ell \in (\hat{\ell}, \kappa\hat{\ell}]$. since $\ell/\kappa \leqslant \hat{\ell}$ and $f(\ell) < 0$, we have

$$\begin{aligned} p(\ell) &= \delta \min\{\kappa p(\ell/\kappa) + \alpha(\ell), c(\ell)\} \\ &= \delta \min\{\kappa\delta c(\ell/\kappa) + \alpha(\ell), c(\ell)\} \\ &= \delta^2\kappa c(\ell/\kappa) + \delta\alpha(\ell) < \delta c(\ell), \end{aligned}$$

where the last inequality follows from $f(\ell) < 0$. Hence, firms subcontract when $\ell \in (\hat{\ell}, \kappa\hat{\ell}]$. Similarly, fixing $\ell \in (\kappa\hat{\ell}, \kappa^2\hat{\ell}]$, we have $p(\ell/\kappa) < \delta c(\ell/\kappa)$ and then

$$\begin{aligned} p(\ell) &= \delta \min\{\kappa p(\ell/\kappa) + \alpha(\ell), c(\ell)\} \\ &\leqslant \delta \min\{\kappa \delta c(\ell/\kappa) + \alpha(\ell), c(\ell)\} \\ &= \delta^2 \kappa c(\ell/\kappa) + \delta\alpha(\ell) < \delta c(\ell), \end{aligned}$$

where the last inequality follows from $f(\ell) < 0$. Iteration implies $p(\ell) < \delta c(\ell)$ for $\ell \in (\hat{\ell}, \kappa^n\hat{\ell}]$ for $n \geqslant 2$. Therefore, we have $p(\ell) < \delta c(\ell)$ for $\ell > \hat{\ell}$. For the sufficiency, suppose that $\hat{\ell}$ satisfies $p(\ell) = \delta c(\ell)$ for $\ell \leqslant \hat{\ell}$ and $p(\ell) < \delta c(\ell)$ for $\ell > \hat{\ell}$. Then, by the definition of equilibrium price p, we have $f(\ell) \geqslant 0$ for $\ell \leqslant \hat{\ell}$ and $f(\ell) < 0$ for $\ell > \hat{\ell}$. Since the root of f is unique in $(0, 1]$, $f(\hat{\ell}) = 0$. □

Lemma A.10. *Following Lemma A.9, if $0 < \hat{\ell} < 1$, then the equilibrium length has at least two levels.*

Lemma A.11. *Suppose that Assumptions 2.2 and 2.3 hold, and c is twice differentiable. If $f(\ell)$ is strictly concave and has a unique root in $(0, 1]$, then the equilibrium price p is twice differentiable except for a finite number of points with $p'(\ell) > 0$. If further $\alpha'' \geqslant 0$, then $p''(\ell) \geqslant 0$.*

Proof of Lemma A.11. Let all the stated assumptions hold. By Lemma A.9, there is $\hat{\ell}$ such that $p(\ell) = \delta c(\ell)$ for $\ell \in [0, \hat{\ell}]$. Fixing $\ell \in (\hat{\ell}, \kappa\hat{\ell})$, Lemma A.9 implies that $p(\ell) = \delta[\kappa\delta c(\ell/\kappa) + \alpha(\ell)] = \delta^2\kappa c(\ell/\kappa) + \delta\alpha(\ell)$. Since c and α are twice differentiable, $p'(\ell) = \delta^2 c'(\ell/\kappa) + \delta\alpha'(\ell)$ and $p''(\ell) = \delta^2/\kappa c''(\ell/\kappa) + \delta\alpha''(\ell)$. Since c and α are strictly increasing, $p'(\ell) > 0$. If a is convex, then $p''(\ell) \geqslant 0$.

Fix $\ell \in (\kappa\hat{\ell}, \kappa^2\hat{\ell})$. We have

$$\begin{aligned} p(\ell) &= \delta[\kappa p(\ell/\kappa) + \alpha(\ell)] \\ &= \delta[\kappa[\delta^2\kappa c(\ell/\kappa^2) + \delta\alpha(\ell/\kappa)] + \alpha(\ell)] \\ &= \delta^3\kappa^2 c(\ell/\kappa^2) + \delta^2\kappa\alpha(\ell/\kappa) + \delta\alpha(\ell) \end{aligned}$$

Then, we can compute $p'(\ell) = \delta^3 c'(\ell/\kappa^2) + \delta^2\alpha'(\ell/\kappa) + \delta\alpha'(\ell)$ and $p''(\ell) = \delta^3/\kappa^2 c''(\ell/\kappa^2) + \delta^2/\kappa\alpha''(\ell/\kappa) + \delta\alpha''(\ell)$ and confirm the statement for ℓ in $(\kappa\hat{\ell}, \kappa^2\hat{\ell})$.

By iteration, we conclude that p is differentiable, $p'(\ell) \geqslant 0$ and $p''(\ell) \geqslant 0$ in $[0, 1]$ except for the points $\{\hat{\ell}, \kappa\hat{\ell}, \ldots, \kappa^n\hat{\ell}\}$, where n is the greatest integer such that $\kappa^n\hat{\ell} \leqslant 1$. □

Lemma A.12. *If $\delta_1 \leqslant \delta_2$ and $p^*_{\delta_1}, p^*_{\delta_2}$ are the corresponding equilibrium prices, then $p^*_{\delta_1} \leqslant p^*_{\delta_2}$.*

Proof of Lemma A.12. Suppose that $\delta_1 \leqslant \delta_2$. Let T_1 and T_2 be the corresponding operator with respect to δ_1 and δ_2, respectively. Let $p \in \mathcal{P}$. Clearly, $T_1 p \leqslant T_2 p$. Since operator T is order preserving, we have

$$T_1(T_1 p) \leqslant T_1(T_2 p) \leqslant T_2(T_2 p).$$

By iteration, $p^*_{\delta_1} = T_1^n p \leqslant T_2^n p = p^*_{\delta_2}$ for $n \geqslant 1 - \frac{\ln \bar{x}}{\ln \kappa}$ □

Proof of Proposition 2.3. Let all the stated assumptions hold. For the transaction cost, Lemma A.12 and $\delta = 1/(1-\tau)$ shows that the equilibrium price is increasing in τ. For home production costs, suppose that $c_1, c_2 \colon \mathcal{D} \to \mathbb{R}_+$ satisfying $c_1(\ell) \leqslant c_2(\ell)$ for all $\ell \in \mathcal{D}$. Let T_1 and T_2 be the corresponding operator, respectively. Clearly, $T_1 p \leqslant T_2 p$ for all $p \in \mathcal{P}$, since we have

$$\min\{\kappa p(\ell/\kappa) + \alpha(\ell), c_1(\ell)\} \leqslant \min\{\kappa p(\ell/\kappa) + \alpha(\ell), c_2(\ell)\}.$$

Thus, $p^*_1 \leqslant p^*_2$ by the iteration as Lemma A.12, where p^*_i is the equilibrium price with respect to c_i, $i = 1, 2$. Similarly, it can be shown that the price is increasing in assembly cost.

Next, we show that the price of the final good is strictly increasing in transaction cost, assembly cost, and home production cost. Let the price of final good $p(\alpha, c, m, \delta, \kappa)$ be defined as equation (4). In particular,

$$\begin{aligned} p(\alpha, c, m, \delta, \kappa) &\coloneqq (\delta\kappa)^{m-1}\delta c(\kappa^{1-m}) \\ &\quad + \delta\alpha(1) + \delta^2\kappa\alpha(\kappa^{-1}) + \cdot + \delta^{m-1}\kappa^{m-2}\alpha(\kappa^{2-m}) \end{aligned} \tag{16}$$

Suppose that $\alpha_1(\ell) < \alpha_2(\ell)$ for all ℓ and let p^*_1 and p^*_2 be the corresponding equilibrium price. Also let m^*_1 and m^*_2 be the corresponding optimal length in equilibrium, $m^*_1 \geqslant m^*_2$. Then, by the optimality of p^*, we have $p^*_1(1) = p(\alpha_1, m^*_1; \delta, \kappa, c) \leqslant p(\alpha_1, m^*_2; \delta, \kappa, c) < p(\alpha_2, m^*_2; \delta, \kappa, c) = p^*_2(1)$. For the other task allocation $t = 1, \kappa^{-1}, \cdots, \kappa^{1-m^*_2}$, we can also characterize the price $p^*(t)$ as equation (14) and show that $p^*_1(t) < p^*_2(t)$.

For the production cost, suppose that $c_1(\ell) < c_2(\ell)$ for all ℓ and let p^*_1 and p^*_2 be the corresponding equilibrium price. Also let m^*_1 and m^*_2 be the corresponding optimal length in equilibrium, $m^*_1 \leqslant m^*_2$. Then, by the optimality of p^*, we have $p^*_1 =$

$p(\alpha, c_1, m_1^*, \delta, \kappa) \leqslant p(\alpha, c_1, m_2^*, \delta, \kappa) < p(\alpha, c_2, m_2^*, \delta, \kappa) = p_2^*$. Similarly, we can compute $p^*(t)$ and show that $p_1^*(t) < p_2^*(t)$ by the same argument for $t = 1, \kappa^{-1}, \cdots, \kappa^{1-m_1^*}$. We can also prove the statement for transaction costs following the same procedures. □

Proofs in Section 4. Throughout this section, suppose that 2.2, 2.1 and 4.2 hold, and the price is defined by (12).

Proof of Lemma 4.1. Consider the operator $T : \mathcal{P} \to \mathcal{P}$ as

$$Tp(\ell) = \delta \min\{np(\ell/n) + \alpha(n), c(\ell)\}.$$

The similar proofs as in Lemma A.1, Lemma A.2 and Lemma A.4 show that the fixed point uniquely exists. The statements then follow from Lemma A.7 and A.8. □

Lemma A.13. *There is a unique solution to the pricing function (12), and the equilibrium price of Definition 4.1 exists.*

Proof. Consider the operator $T : \mathcal{P} \to \mathcal{P}$ as

$$Tp(\ell) = \delta \min\left\{ \min_{n=2,3,4,\ldots} \{np(\ell/n) + \alpha(\ell, n)\},\, c(\ell) \right\},$$

for all $\ell \in [0, 1]$. We first show that T has a unique fixed point, which is an equilibrium price. To the existence, following the argument of Lemma A.1, it suffices to show that T is a self-map on $\mathcal{P}$ and preserves orders. Fix $\ell \in [0, 1]$. Clearly, we have $Tp(\ell) \leqslant \delta c(\ell)$. Since $p \in \mathcal{P}$ and c is convex, we have

$$\begin{aligned} Tp(\ell) &\geqslant \delta \min\left\{ \min_{n=2,3,\ldots} \{n\delta c'(0)\ell/n + \alpha(\ell, n)\},\, c(\ell) \right\} \\ &= \delta \min\left\{ \delta c'(0)\ell + \min_n\{\alpha(\ell, n)\},\, c(\ell) \right\} \\ &\geqslant \delta \min\{c'(0)\ell, c(\ell)\} = \delta c'(0)\ell \end{aligned}$$

Then, $Tp(\ell) \geqslant \delta c'(0)\ell$ and $T : \mathcal{P} \to \mathcal{P}$. To see that T preserves orders, assume $p_1(\ell) \leqslant p_2(\ell)$ for all $\ell \in [0, 1]$. Fix $\ell \in [0, 1]$ and $n \geqslant 2$, we have $np_1(\ell/n) + \alpha(\ell, n) \leqslant np_2(\ell/n) + \alpha(\ell, n)$. Hence, we have $\min_n\{np_1(\ell/n) + \alpha(\ell, n)\} \leqslant np_2(\ell/n) + \alpha(\ell, n)$ for all $n \geqslant 2$, implying $\min_n\{np_1(\ell/n) + \alpha(\ell, n)\} \leqslant \min_n\{np_2(\ell/n) + \alpha(\ell, n)\}$. This implies that $Tp_1(\ell) \leqslant Tp_2(\ell)$ for all $\ell \in [0, 1]$. Therefore, Knaster-Tarski Fixed Point Theorem shows that T has at least one fixed point.

Similar to Lemma A.2 and A.4, we next show that $T^k p = T^k q$ for $n \geqslant 1 - \ln \bar{x}/\ln 2$ for arbitrary $p, q \in \mathcal{P}$, where $\bar{x} := \sup\{x \in [0,1] : c'(x) \leqslant \delta c'(0)\}$. Let $p, q \in \mathcal{P}$. We claim that $Tp(\ell) = Tq(\ell) = \delta c(\ell)$ for $\ell \leqslant \bar{x}$. Suppose that $Tp(\ell) \neq \delta c(\ell)$ for $\ell \leqslant \bar{x}$. Then, firms choose to subcontract, so it must be $\min_n\{np(\ell/n) + \alpha(\ell, n)\} < c(\ell)$[35]. Since $p(\ell/n) \geqslant \delta c'(0)\ell/n$ for $p \in \mathcal{P}$,

$$c(\ell) > \min_n\{np(\ell/n) + \alpha(\ell, n)\} \geqslant \delta c'(0)\ell$$

Therefore, $c(\ell)/\ell > \delta c'(0)$. This contradicts that $\ell \leqslant \bar{x}$ and convexity of c. Hence, we have $Tp(\ell) = Tq(\ell)$ for $\ell \leqslant \bar{x}$. Next, suppose that $T^k p(\ell) = T^k q(\ell)$ for $\ell \in [0, 2^{k-1}\bar{x}]$. Let $\ell \in [0, 2^k \bar{x}]$. Since $nT^k p(\ell/n) + \alpha(\ell, n) = nT^k q(\ell/n) + \alpha(\ell, n)$ for all $n \geqslant 2$, we can prove that $T^{k+1}p(\ell) = T^{k+1}q(\ell)$ for $\ell \in [0, 2^k \bar{x}]$. By induction, $T^k p(\ell) = T^k q(\ell)$ for all $\ell \in [0,1]$ and $k \geqslant 1 - \ln \bar{x}/\ln 2$.

By definition of T and pricing function (12), p is a fixed point of T if and only if it is a solution to (12). Suppose that there are two fixed points or solutions of (12) p, q and $p \neq q$. Then, by the above induction, $p = T^k p = T^k q = q$ for large enough k. Hence, T has only one fixed point and the solution of (12) is also unique.

Let $\delta = 1/(1-\tau)$ and $p^* \in \mathcal{P}$ be the solution to (12). The above induction implies that there is a maximal possible length $\bar{m}$, and the equilibrium length m^* is finite. Therefore, we can construct a feasible $\boldsymbol{\ell}$ by iteration, starting from $\ell_1 = 1$, under the price p^*. Moreover, since $p^* \in \mathcal{P}$ and $c(0) = 0$, we have $p^*(0) = 0$. Condition (1) of Definition (2.1) is satisfied. By definition of (12) and the profit (13), for all $\ell \in \mathcal{D}$, we have[36]

$$p^*(\ell) - \delta \min\left\{\min_{n=2,3,4,\ldots}\{np^*(\ell/n) + \alpha(\ell, n)\},\, c(\ell)\right\} = 0.$$

Therefore, Condition (2) and (3) are satisfied, so the solution to (12) is the equilibrium price. □

Lemma A.14. *Given the equilibrium price, the equilibrium allocation is uniquely determined.*

Proof. Let $(p, \boldsymbol{\ell}^{\mathbf{1}})$ and $(p, \boldsymbol{\ell}^{\mathbf{2}})$ be two equilibrium with the same price p and different allocations. Since $\boldsymbol{\ell}^1 = \{\ell_i^1\}$ and $\boldsymbol{\ell}^2 = \{\ell_i^2\}$ are feasible allocations, there exist $m_1^*, m_2^* \geqslant 2$ and two integer sequence $\{n_t^1\}_{t=1}^{m_1^*}$ and $\{n_t^2\}_{t=1}^{m_2^*}$ such that $\ell_i^j = (n_1^j n_2^j \cdots n_t^j)^{-1}$ for

[35]If " = " holds, then firms choose to home production by assumption.

[36]It is also valid for $\ell \in [0, 1]$.

$n_1^j + \cdots + n_1^j n_2^j \cdots n_{t-1}^j < i \leqslant n_1^j + \cdots + n_1^j n_2^j \cdots n_t^j$ for $j = 1, 2$. Suppose that $m_1^* \geqslant m_2^*$ without loss of generality. That is, $\boldsymbol{\ell}^1$ has a longer chain.

If $m_2^* = 1$, then, by the assumption that firms choose home production if home production and subcontract are indifferent, it can be shown that $m_1^* = m_2^* = 1$ and $\boldsymbol{\ell}^1 = \boldsymbol{\ell}^2$ by the bellowing argument. Thus, assume that $m_2^* > 1$. At level 1, using condition (3) of Definition 4.1, we have

$$\begin{aligned}\pi &= (1-\tau)p(1) - \min\{\min_{n=2,3,\ldots}\{np(1/n) + \alpha(1,n)\}, c(1)\}\\ &= (1-\tau)p(1) - n_2^1 p(1/n_2^1) + \alpha(1, n_2^1) = 0\\ &= (1-\tau)p(1) - n_2^2 p(1/n_2^2) + \alpha(1, n_2^2) = 0\end{aligned}$$

Thus, $n_2^1 = n_2^2$ by the assumption that firms choose the minimum optimal number of suppliers. If $n_s^1 = n_s^2 = n_s$ for $s = 1, \ldots, t$ and $t < m_2^*$, for $\ell = (n_1 n_2 \cdots n_t)^{-1}$, then the profit at ℓ is

$$\begin{aligned}&(1-\tau)p(\ell) - \min\{\min_{n=2,3,\ldots}\{np(\ell/n) + \alpha(\ell,n)\}, c(\ell)\}\\ &\qquad = (1-\tau)p(\ell) - \min_{n=2,3,\ldots}\{np(\ell/n) + \alpha(\ell,n)\}\end{aligned}$$

Again, by choosing the minimal minimizer, $n_{t+1}^1 = n_{t+1}^2$. Then, by induction, we have $n_t^1 = n_t^2 = n_t$ for all $t = 1, \ldots, m_2^*$.

Suppose that $m_1^* > m_2^*$. At level m_2^*, we have tasks $\ell = (n_1 n_2 \cdots n_{m_2^*})^{-1}$. The firms of $\boldsymbol{\ell}^1$ at this level choose to outsource while the firms of $\boldsymbol{\ell}^2$ choose to home production. That is, for $t = m_2^*$,

$$\begin{aligned}&(1-\tau)p(\ell) - \min\{\min_{n=2,3,\ldots}\{np(\ell/n) + \alpha(\ell,n)\}, c(\ell)\}\\ &= (1-\tau)p(\ell) - \min\{n_{t+1}^1 p(\ell/n_{t+1}^1) + \alpha(\ell, n_{t+1}^1), c(\ell)\}\\ &= (1-\tau)p(\ell) - n_{t+1}^1 p(\ell/n_{t+1}^1) + \alpha(\ell, n_{t+1}^1)\\ &= (1-\tau)p(\ell) - c(\ell) = 0\end{aligned}$$

By the assumption that firms produce at home if home production and subcontract are indifferent, it should be $m_1^* = m_2^*$. Therefore, $\boldsymbol{\ell}^1 = \boldsymbol{\ell}^2$. □

Proof of Proposition 4.1. The statement follows from the proof of Lemma A.13 and Lemma A.14 □

Proof of Proposition 4.2. For the first statement, let $\tau_1 \leqslant \tau_2$ and δ_1, δ_2 be the corresponding parameters of transaction costs with $\delta_1 \leqslant \delta_2$. Also, let T_i be the corresponding operator for d_i, $i = 1, 2$. Clearly, $T_1 p \leqslant T_2 p$ for all p in $\mathcal{P}$. By iteration, we have $T_1 T_1 p \leqslant T_1 T_2 p \leqslant T_2 T_2 p$ or $T_1^2 p \leqslant T_2^2 p$. Repeat the iteration, we have the fixed points that $p_1^* = T_1^k \leqslant T_2^k = p_2^*$ for $k \to \infty$.

For the assembly cost, suppose that $\alpha_1(\ell) \leqslant \alpha_2(\ell)$ for all ℓ. Let T_i and p_i^* be the corresponding operator and equilibrium price with respect to α_i, $i = 1, 2$. Fix $p \in \mathcal{P}$. Then, $\min_n\{np(\ell/n) + \alpha(\ell, n)\} \leqslant np(\ell/n) + \alpha_2(\ell, n)$. Taking the minimum on the right-hand side, we have $T_1 p \leqslant T_2 p$. By the similar iteration, we also have $p_1^* \leqslant p_2^*$. Moreover, the same technique proves the statement for home production cost c.

We can also show that the price of the final good is strictly increasing in τ, c, and α. Take the home production cost, for example. Let the costs be $c_1 < c_2$. Let $(p_1^*, \boldsymbol{\ell}_1)$ and $(p_2^*, \boldsymbol{\ell}_2)$ be the corresponding equilibrium. Define the equilibrium price as $p(c, \alpha, \delta)$ under parameter c, α and δ. By the optimality of equilibrium, $p_1^* = p(c_1, \boldsymbol{\ell}_1; \delta) \leqslant p(c_1, \boldsymbol{\ell}_2; \delta) < p(c_2, \boldsymbol{\ell}_2; \delta) = p_2^*$. Thus, the price of final good is increasing in home production cost. Similarly, we can prove that the price of the final good is strictly increasing in transaction cost and assembly cost. □

Lemma A.15. *Suppose that $\alpha(\ell, n)$ is increasing in ℓ. If p is the solution of the pricing function (12), then it is strictly increasing in ℓ.*

Proof. Suppose that $p \in \mathcal{P}$ is strictly increasing in ℓ. Let $\ell_1 \leqslant \ell_2$. Then,

$$\begin{aligned}\min_n\{np(\ell_1/n) + \alpha(\ell_1, n)\} &\leqslant np(\ell_1/n) + \alpha(\ell_1, n) \\ &< np(\ell_2/n) + \alpha(\ell_2, n) \quad \text{for all n} \geqslant 2.\end{aligned}$$

Hence, $\min_n\{np(\ell_1/n)+\alpha(\ell_1, n)\} < \min_n\{np(\ell_2/n)+\alpha(\ell_2, n)\}$. Since c is also strictly increasing,

$$\begin{aligned}Tp(\ell_1) &= \delta \min\{\min_n\{np(\ell_1/n) + \alpha(\ell_1, n)\}, c(\ell_1)\} \\ &< \delta \min\{\min_n\{np(\ell_2/n) + \alpha(\ell_2, n)\}, c(\ell_2)\} \\ &= Tp(\ell_2)\end{aligned}$$

$Tp(\ell)$ is also strictly increasing. Therefore, the fixed point $T^N p(\ell)$, for large enough N, is also strictly increasing. By Lemma A.13, the solution is unique and equaled to the fixed point of T, so the solution of (12) is strictly increasing. □

Proof of Lemma 4.2. ($\Rightarrow$) Let $\ell, \ell' \in [0,1]$ with $\ell \neq \ell'$ and $\theta \in (0,1)$. Since $t(n,.)$ is strictly concave in ℓ,

$$\begin{aligned}
&\min_{n=2,3,\dots} \left\{ n\delta c\left(\frac{\theta\ell + (1-\theta)\ell'}{n}\right) + \alpha(n) - c(\theta\ell + (1-\theta)\ell') \right\} \\
&\quad = \min_{n=2,3,\dots} \{t(\theta\ell + (1-\theta)\ell') + \alpha(n)\} \\
&\quad > \min_{n=2,3,\dots} \{\theta t(\ell/n) + (1-\theta)t(\ell'/n) + \alpha(n)\} \\
&\quad \geqslant \min_{n=2,3,\dots} \{\theta(t(\ell/n) + \alpha(n)) + (1-\theta)(t(\ell'/n) + \alpha(n))\} \\
&\quad \geqslant \theta \min_{n=2,3,\dots} \{t(\ell/n) + \alpha(n)\} + (1-\theta) \min_{n=2,3,\dots} \{t(\ell'/n) + \alpha(n)\}
\end{aligned}$$

Thus, the function $f(\ell) \coloneqq \min_n\{n\delta c(\ell/n) + \alpha(n)\} - c(\ell)$ is strictly concave. Together with $f(0) > 0$, the root $\hat{\ell} \in (0,1]$ is unique. Therefore, fixing $\ell \leqslant \hat{\ell}$, we have $f(\ell) \geqslant 0$ and then

$$c(\ell) \leqslant \min_{2,3,\dots}\{n\delta c(\ell/\kappa) + \alpha(n)\}.$$

So, $T\delta c(\ell) = \delta c(\ell)$. Hence, $\delta c(\ell)$ is a fixed point of T considering only $\ell \in [0, \hat{\ell}]$.

Fixing $\ell \in (\hat{\ell}, 2\hat{\ell}]$. Observe that $p(\ell/n) = \delta c(\ell/n)$ for all $n \geqslant 2$. Then, since $f(\ell) < 0$, we have

$$\begin{aligned}
p(\ell) &= \delta \min\{\min_{n=2,3,\dots} \{np(\ell/n) + \alpha(n)\}, c(\ell)\} \\
&= \delta \min\{\min_{n=2,3,\dots} \{n\delta c(\ell/n) + \alpha(n)\}, c(\ell)\} \\
&< \delta c(\ell).
\end{aligned}$$

Fixing $\ell \in (2\hat{\ell}, 2^2\hat{\ell}]$. Observe that $p(\ell/2) < \delta c(\ell)$ for $\ell/2 \in (\hat{\ell}, 2\hat{\ell})$ and $p(\ell/n) = \delta c(\ell/n)$ for $n > 2$ and $\ell/n < \hat{\ell}$. Again, since $f(\ell) < 0$, we have

$$\begin{aligned}
p(\ell) &= \delta \min\{\min_{n=2,3,\dots} \{np(\ell/n) + \alpha(n)\}, c(\ell)\} \\
&\leqslant \delta \min\{\min_{n=2,3,\dots} \{n\delta c(\ell/n) + \alpha(n)\}, c(\ell)\} \\
&< \delta c(\ell).
\end{aligned}$$

By iteration, $p(\ell) < \delta c(\ell)$ for all $\ell \in (\hat{\ell}, 2^N\hat{\ell}]$. Hence, we obtain that $p(\ell) < \delta c(\ell)$ for all $\hat{\ell} < \ell \leqslant 1$ when N is large enough.

($\Leftarrow$) This is done by the definition of equilibrium price p and strict concavity of function f. $\square$

Proof. By Lemma 4.2, there is $\hat{\ell}$ such that $p(\ell) = \delta c(\ell)$ for all $\ell \leqslant [0, \hat{\ell}]$, so $p'(\ell) = \delta c'(\ell) > 0$ and $p''(\ell) = dc''(\ell) > 0$ for all $\ell \in [0, \hat{\ell})$. Considering $\ell \in (\hat{\ell}, 1)$. Lemma 4.2 implies that $p(\ell) = \delta \min_{n=2,3,\ldots}\{np(\ell/n) + \alpha(n)\}$ for all $\ell > \hat{\ell}$.

Considering $\ell \in (\hat{\ell}, 2\hat{\ell})$, we have $p(\ell) = \delta \min_{n=2,3,\ldots}\{n\delta c(\ell/n) + \alpha(n)\}$. Relaxing the choice set of n from $\{2, 3, \ldots\}$ to $[2, \infty)$. Define the optimal choice $n^*(\ell)$ as

$$n^*(\ell) \coloneqq \arg\min_{n\geqslant 2}\{n\delta c(\ell/n) + \alpha(n)\}.^{37}$$

The first order condition is $r(n, \ell) \coloneqq \delta c(\ell/n) - (\ell/n)\delta c'(\ell/n) + \alpha'(\ell/n) = 0$. By Implicit Function Theorem,

$$\frac{\partial n^*}{\partial \ell} = -\frac{\partial r}{\partial \ell}\Big/\frac{\partial r}{\partial n^*} = \frac{c''(\ell/n^*)\ell/n^{*2}}{c''(\ell/n^*)\ell^2/n^{*3} + \alpha''(n)}.$$

Hence, $\partial n^*(\ell)/\partial \ell > 0$ by the assumption that c is strictly convex and α is convex. Since $s(n; \ell) \coloneqq n\delta c(\ell/n) + \alpha(n)$ is strictly increasing in ℓ and $s''(n; \ell) = (\ell^2/n^3)\delta c''(\ell/n) + \alpha''(n) > 0$, the curve $s(n; \ell)$ on $s(n) - n$ plane is strictly convex and shifts upward as ℓ increases. Moreover, the minimum point of $s(n; \ell)$ shifts to right as ℓ increases since $\partial n^*(\ell)/\partial \ell > 0$. Now, restrict n back to the grids $n \geqslant 2$. Since $s(\ell; n)$ shifts upward and the minimum point shifts to right as ℓ increases, the optimal grid $n^*(\ell)$ is increasing in ℓ.[38] Therefore, given n^*, there is a neighborhood $U(n^*) \subset (\hat{\ell}, 2\hat{\ell})$ such that $n^*(\ell') = n^*$ for all $\ell' \in U(n^*)$.

Suppose that $n^*(\ell) = n^*$ for a fixed $\ell \in (\hat{\ell}, 2\hat{\ell})$. Then, ℓ must be on the neighborhood $U(n^*)$. If ℓ is not in the boundary of $U(n^*)$, then $p(\ell') = \delta^2 n^* c(\ell'/n^*) + \delta\alpha(n^*)$ for $\ell' \in U(n^*)$. Given $c \in \mathrm{C}^2$, this implies that $p'(\ell') = \delta^2 c'(\ell'/n^*) > 0$ and $p''(\ell') = \delta^2 c''(\ell'/n^*)/n^* > 0$ for $\ell' \in U(n^*)$. If ℓ is on the boundary of $U(n^*)$, we exclude it. Since the optimal $n^*(\ell)$'s are discrete and countable points, the set of boundary points of such U has measure 0. Since $\ell \in (\hat{\ell}, 2\hat{\ell})$ is arbitrary, $p'(\ell) > 0$ and $p''(\ell) > 0$ for almost all ℓ in $(\hat{\ell}, 2\hat{\ell})$.

Denote $p(\ell)$ by $q(\ell)$ for $\ell \in (\hat{\ell}, 2\hat{\ell})$. Then, $q' > 0$ and $q'' > 0$ almost everywhere. Fixing $\ell \in (2\hat{\ell}, 2^2\hat{\ell})$. Again,

$$p(\ell) = \delta \min_{n=2,3,\ldots}\{np(\ell/n) + \alpha(n)\} = \delta n^*(\ell) q(\ell/n^*(\ell)) + \alpha(n^*(\ell)).$$

[37] The maximum theorem implies that $n^*(\ell)$ is single-valued and continuous if we restrict the upper bound on n.

[38] This can be seen by drawing the plot. $n^*(\ell)$ is not "strictly" increasing since the minimum grid may be the same as ℓ increases.

Similarly, let $N \subset (2\hat{\ell}, 2^2\hat{\ell})$ be the neighborhood of ℓ such that $n^*(\ell') = n^*(\ell) = n^*$ for all $\ell' \in N$. Excluding the boundary points of such neighborhood, suppose that ℓ is not on the boundary of N. If $n^* = 2$, then $p(\ell') = \delta n^* q(\ell'/n^*) + \delta\alpha(n^*)$ so that $p'(\ell') = \delta q'(\ell'/n^*) > 0$ and $p''(\ell') = (\delta/n^*)q''(\ell'/n^*) > 0$ for $\ell' \in N$. If $n^* > 2$, then $p(\ell') = \delta n^* \delta c(\ell'/n^*) + \delta\alpha(n^*)$ so that $p'(\ell') = \delta^2 c'(\ell'/n^*) > 0$ and $p''(\ell') = \delta^2 c''(\ell')/n^* > 0$ for all $\ell' \in N$. Either case gives that $p'(\ell) > 0$ and $p''(\ell) > 0$ for $\ell \in (\hat{\ell}, 2\hat{\ell})$ almost everywhere.

Repeating the process, we can conclude that $p'(\ell) > 0$ and $p''(\ell) > 0$ for almost all $\ell \in [0, 2^k\hat{\ell}]$. The conclusion follows by large enough k. $\square$

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