Branch and Bound Algorithms

Satisfiability problem

Given a logic formula $f(x_1, x_2, \ldots, x_n)$ of n binary variables $x_1, x_2, \ldots, x_n$, is there a truth assignment for which the formula is **T**rue.

$f(x_1, x_2, x_3, x_4)$
$= (x_1 \vee \sim x_2 \vee \sim x_4) \wedge (x_2 \vee \sim x_3) \wedge (x_1 \vee x_3 \vee x_4)$

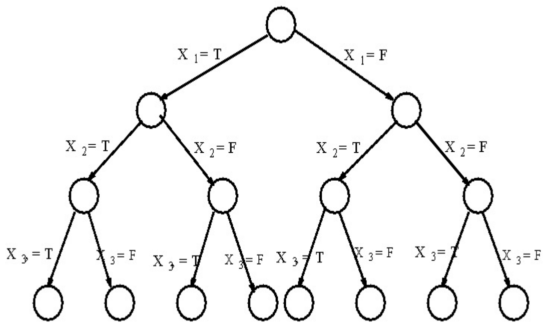
$f(\mathbf{F}, \mathbf{F}, \mathbf{F}, \mathbf{F}) = (\mathbf{T}) \wedge (\mathbf{T}) \wedge (\mathbf{F}) = \mathbf{F}$

$f(\mathbf{T}, \mathbf{T}, \mathbf{T}, \mathbf{T}) = (\mathbf{T}) \wedge (\mathbf{T}) \wedge (\mathbf{T}) = \mathbf{T}$

If there are n variables $x_1, x_2, \ldots, x_n$, then there are 2^n possible assignments.

Satisfiability problem

x_1	x_2	x_3
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T



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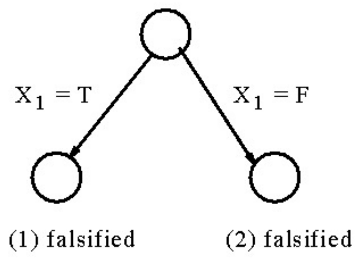
- An instance:
  - $-x_1$..................(1)
  - x_1..................(2)
  - $x_2 \vee x_5$..........(3)
  - x_3..................(4)
  - $-x_2$..................(5)



A partial tree for the satisfiability problem.

- We may not need to examine all possible assignments.

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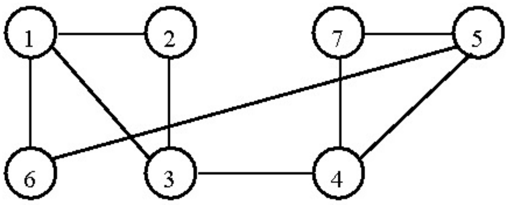
Hamiltonian circuit problem



HC: 1-2-3-4-7-5-6-1

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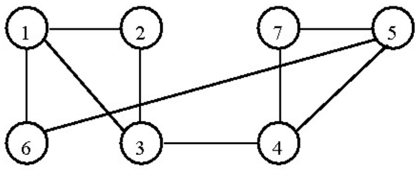
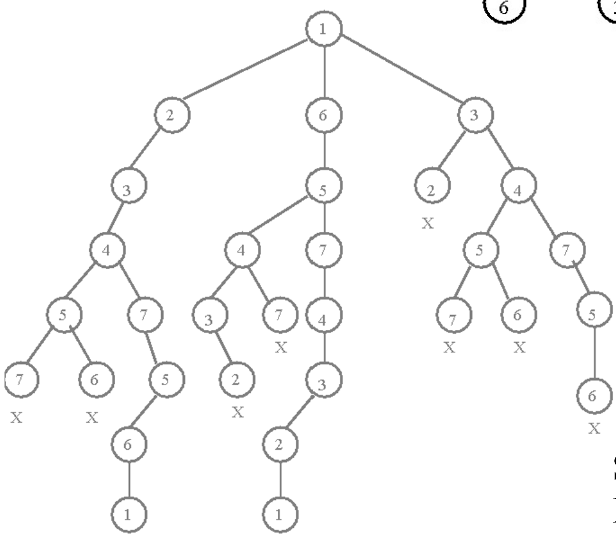
Tree representation of the Hamiltonian circuit problem.





Search tree/
Enumeration tree

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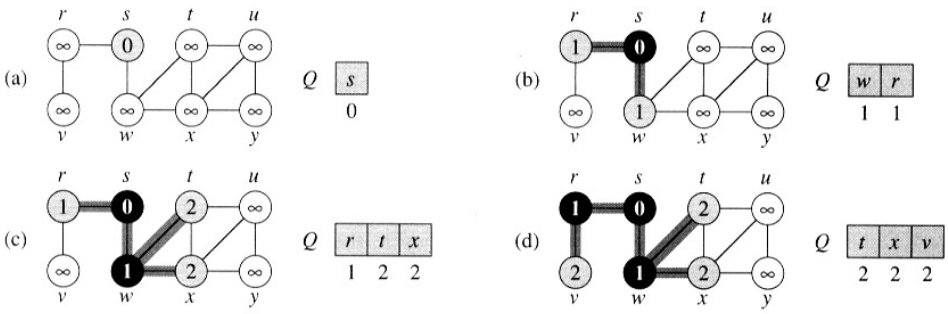
P.7

Operations of Breadth First Search in a graph



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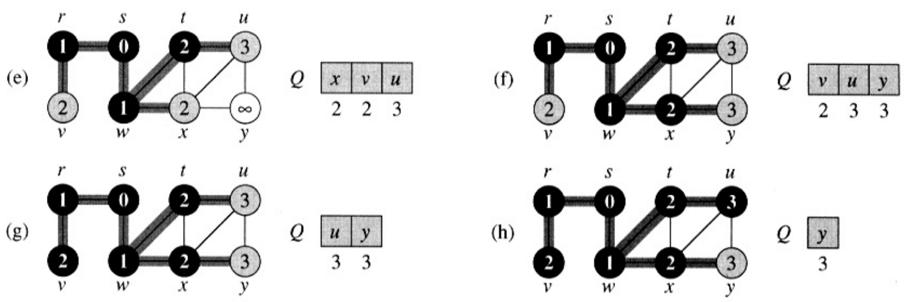
P.8

Operations of BFS



Chapter 22

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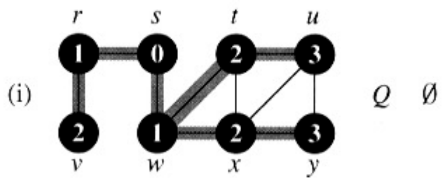
Operations of BFS



Breadth-first search

u.d: the distance from *s* to node *u*
u.π: the predecessor of node *u*

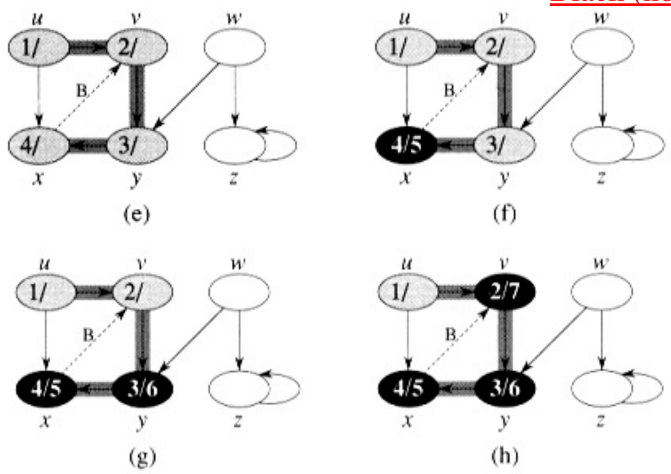
```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Time analysis: $O(V+E)$

P.11

Operations of Depth First Search

White (untouched)
Gray (explored)
Black (no more expansion)



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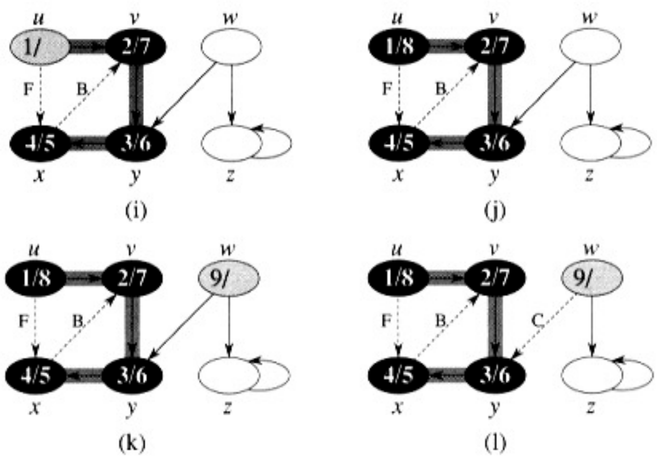
P.12

DFS

White (untouched)
Gray (explored)
Black (no more expansion)



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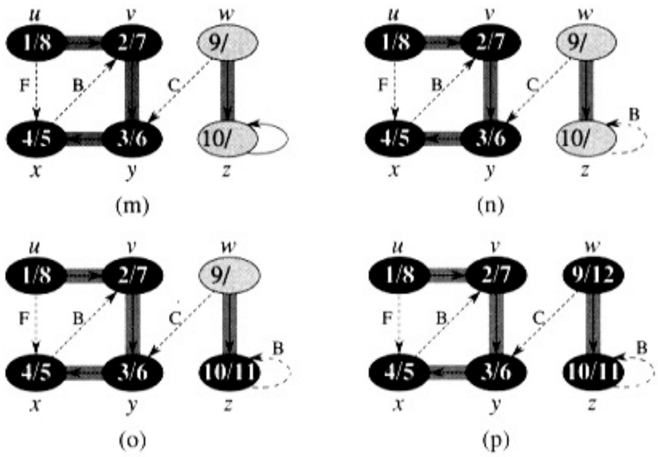
P.13

DFS

White (untouched)
Gray (explored)
Black (no more expansion)



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P.14

Depth-First Search

DFS(G)
for each vertex $u \in V[G]$
    $color[u] \leftarrow white$
    $\pi[u] \leftarrow NIL$
$time \leftarrow 0$
for each vertex $u \in V[G]$
    if $color[u] = white$
    then DFS-VISIT(u)

DFS-VISIT(u)
$color[u] = gray$
$d[u] \leftarrow time \leftarrow time + 1$
for each $v \in adj[u]$
    if $color[u] = white$
    then $\pi[v] \leftarrow u$
        DFS-VISIT(v)
$color[u] = black$
$f[u] \leftarrow time \leftarrow time + 1$

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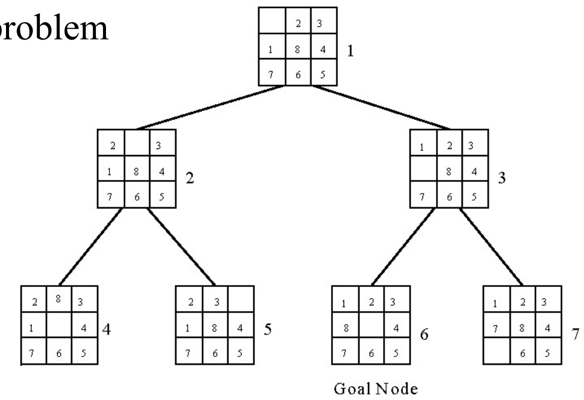
Breadth-First Search (BFS)

- 8-puzzle problem



- Breadth-first search uses a queue to hold all expanded nodes.

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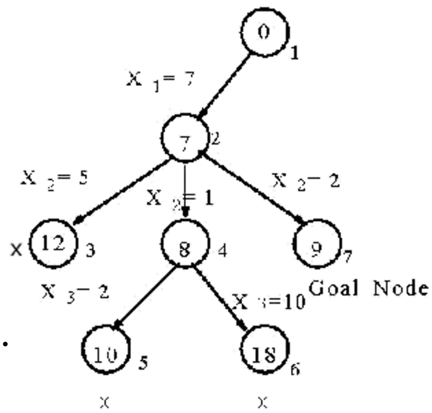
Depth-First Search (DFS)

- Sum of subset problem
  Given S={7, 5, 1, 2, 10},
  $\exists S' \subseteq S$ such that the sum of elements in S' = 9?



- A stack can be used to guide the depth-first search.

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Minimizing the Total Weighted Late Work

- There are n jobs to be processed on a machine
- Each job j is associated three parameters:
  - Processing length p_j
  - Due date d_j
  - Weight w_j
- In a schedule S, the completion time of job j is denoted by $C_j\,(S)$
- The late work of job j is given by $Y_j = \max\{p_j, C_j - d_j\}$
- We want to find a permutation of the jobs such that the total weighted late work $\Sigma w_j Y_j$ is minimized

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Hill Climbing, Steepest Descent

- A variant of depth-first search

  The method selects the locally optimal node to expand.
- e.g. 8-puzzle problem

  evaluation function $f(n) = d(n) + w(n)$

  where $d(n)$ is the depth of node n

  $w(n)$ is the number of misplaced tiles in node n.

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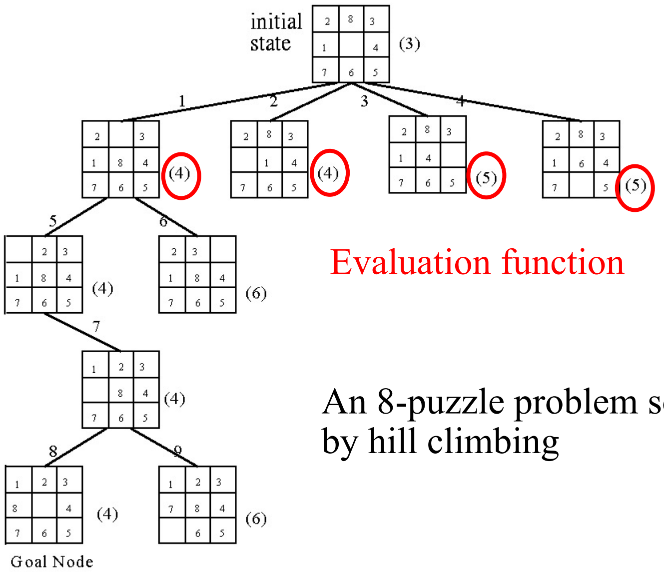
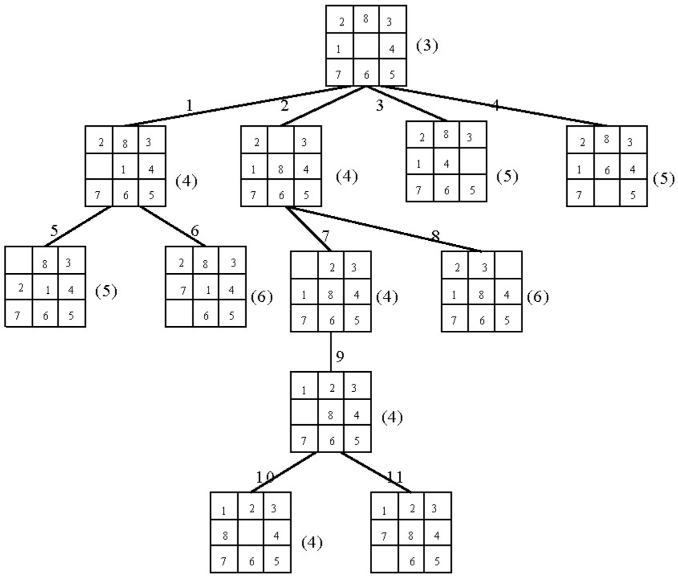


Evaluation function

An 8-puzzle problem solved by hill climbing

Best-First Search (Beam Search)

- Combing depth-first search and breadth-first search.
- Selecting the node with the best estimated cost among all active nodes.
- This method has a global view.



An 8-puzzle problem solved by best-first search

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Best-First Search Scheme

Step1: Form a one-element list consisting of the root node.

Step2: Remove the first element from the list. Expand the selected element. If one of the descendants of the first element is a goal node, then stop; otherwise, add the descendants into the list.

Step3: Sort the entire list by the values of some estimation function.

Step4: If the list is empty, then failure. Otherwise, go to Step 2.

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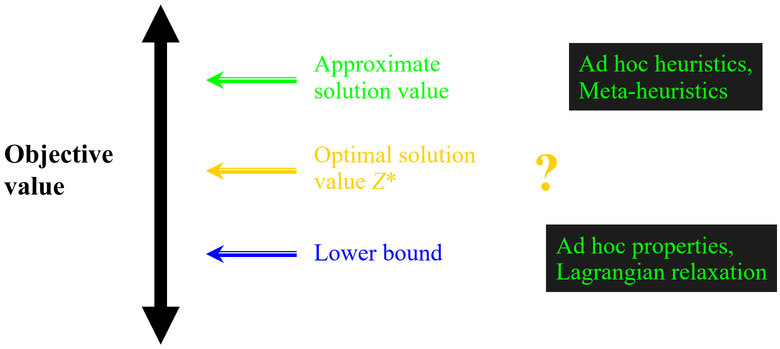
Why Lower Bounds

- Consider some *minimization* problem that is computationally challenging



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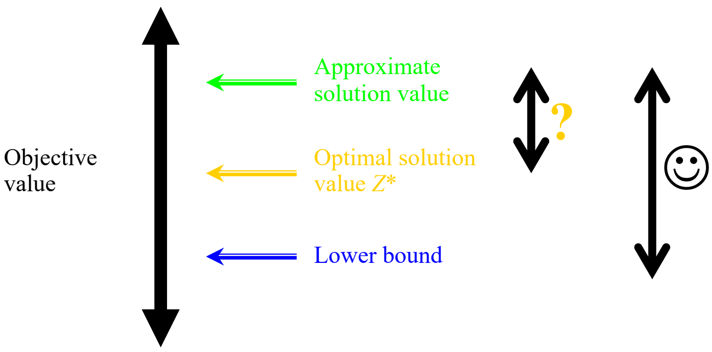
Why Lower Bounds

- Consider some *minimization* problem that is computationally challenging



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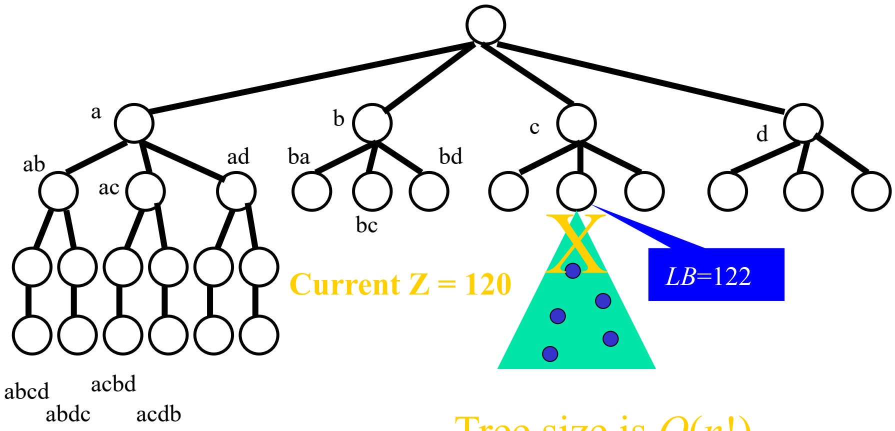
Why Lower Bounds

4 (*a*, *b*, *c*, *d*) orders to be assigned to 4 positions



Tree size is $O(n!)$

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Lower bounds Vs. Upper bounds

- Minimization problems
  - lower bound <= optimal value
  - larger lower bounds are preferred
  - approximate solutions are upper bounds
- Maximization problems
  - upper bound >= optimal value
  - smaller upper bounds are preferred
  - approximate solutions are lower bounds

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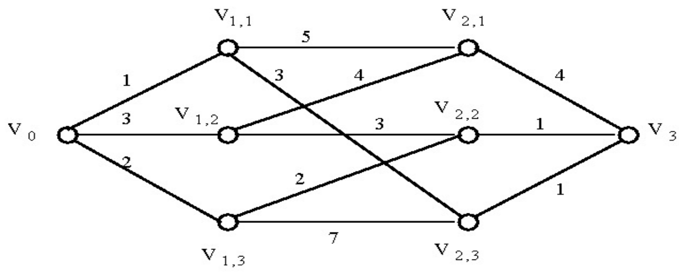
Branch-and-bound strategy

- This strategy can be used to efficiently solve optimization problems.
- e.g.



A multi-stage graph searching problem.

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Branch-and-Bound Algorithm

- An implicit enumeration approach to dealing with hard optimization problems
- Ingredients of a BaB algorithm
  - Enumeration Tree – The way in which the solution space is constructed and explored
  - Bounding Function – An estimation function of the objective value of a partial solution
  - Dominance/Elimination Rule – Conditions under which some part of the solution space can be skipped without sacrificing the optimality

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Efficiency of an Example BaB Algorithm

n	Crude enumeration CE		Dominance Rule DR		Lower Bound LB		DR & LB	
	Avg_Node	Avg_Time	llAvg_Node	Avg_Time	llAvg_Node	Avg_Time	llAvg_Node	Avg_Time
6	1956	0.00	624	0.00	52	0.00	33	0.00
8	109,600	0.02	15,477	0.01	154	0.00	67	0.00
10	9,864,100	1.49	711,574	0.24	428	0.00	117	0.01
12	1,302,061,344	205.73	47,312,610	15.98	834	0.00	151	0.00
14	2.370E+11	—	511,829,995	907.02	2472	0.01	265	0.00
15	3.555E+12	—	—	—	2423	0.01	278	0.00
20	6.613E+18	—	—	—	43,074	0.09	690	0.00
25	4.216E+25	—	—	—	584,526	1.27	1595	0.01
30	7.210E+32	—	—	—	760,590	1.94	2361	0.02
35	2.809E+40	—	—	—	9,096,072	23.78	3857	0.03
40	2.218E+48	—	—	—	—	—	5786	0.05
50	8.267E+64	—	—	—	—	—	42,507	0.24
60	2.619E+82	—	—	—	—	—	74,606	0.64
70	3.256E+100	—	—	—	—	—	281,947	1.64
80	1.945E+119	—	—	—	—	—	540,570	3.68
90	4.039E+138	—	—	—	—	—	3,247,508	18.22
100	2.537E+158	—	—	—	—	—	2,572,461	28.86

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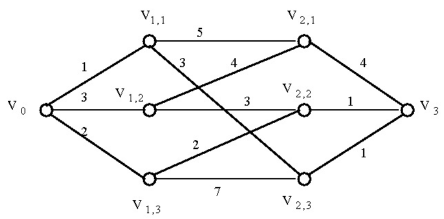
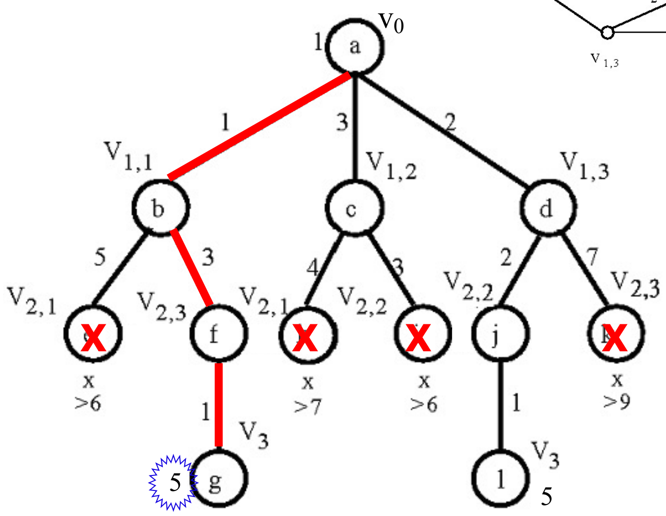
- Solved by branch-and-bound





Cost = 5 is an upper bound

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Personnel assignment problem

- A linearly ordered set of persons $P=\{P_1, P_2, \ldots, P_n\}$ where $P_1 < P_2 < \ldots < P_n$
- A partially ordered set of jobs $J=\{J_1, J_2, \ldots, J_n\}$. Suppose that P_i and P_j are assigned to jobs $f(P_i)$ and $f(P_j)$ respectively. If $f(P_i) \le f(P_j)$, then $P_i \le P_j$. Cost C_{ij} is the cost of assigning P_i to J_j. We want to find a feasible assignment with the minimum cost. i.e.

  $X_{ij} = 1$ if P_i is assigned to J_j

  $X_{ij} = 0$ otherwise.
- Minimize $\sum_{i,j} C_{ij}X_{ij}$

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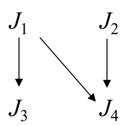
- e.g. A partial ordering of jobs



- After topological sorting, one of the following 5 feasible sequences will be generated:

  J_1, J_2, J_3, J_4

  J_1, J_2, J_4, J_3

  J_1, J_3, J_2, J_4

  J_2, J_1, J_3, J_4

  J_2, J_1, J_4, J_3

- One of the feasible assignments:
  $(P_1, J_1), (P_2, J_2), (P_3, J_3), (P_4, J_4)$

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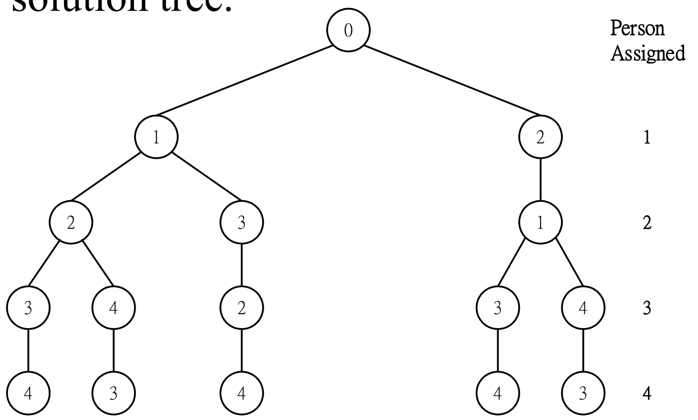
Solution Tree

- All possible solutions can be represented by a solution tree.



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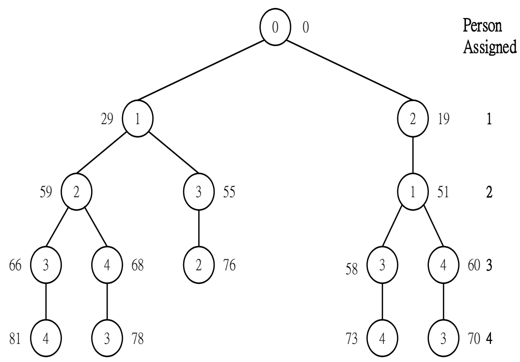
Cost matrix

- Cost matrix

Jobs / Persons	1	2	3	4
1	29	19	17	12
2	32	30	26	28
3	3	21	7	9
4	18	13	10	15

- Best-first search scheme:



Only one node is pruned away.

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Reduced cost matrix

- Cost matrix

Persons \ Jobs	1	2	3	4
1	29	19	17	12
2	32	30	26	28
3	3	21	7	9
4	18	13	10	15

- Reduced cost matrix

Persons \ Jobs	1	2	3	4	
1	17	4	5	0	(-12)
2	6	1	0	2	(-26)
3	0	15	4	6	(-3)
4	8	0	0	5	(-10)
		(-3)			

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- A reduced cost matrix can be obtained:

  subtract a constant from each row and each column respectively such that each row and each column contains at least one zero.

- Total cost subtracted: 12+26+3+10+3 = 54
- This is a *lower bound* of our solution.

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Traveling salesperson problem

- Cost matrix

i \ j	1	2	3	4	5	6	7
1	∞	3	93	13	33	9	57
2	4	∞	77	42	21	16	34
3	45	17	∞	36	16	28	25
4	39	90	80	∞	56	7	91
5	28	46	88	33	∞	25	57
6	3	88	18	46	92	∞	7
7	44	26	33	27	84	39	∞

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- Reduced cost matrix

i \ j	1	2	3	4	5	6	7	
1	∞	0	90	10	30	6	54	(-3)
2	0	∞	73	38	17	12	30	(-4)
3	29	1	∞	20	0	12	9	(-16)
4	32	83	73	∞	49	0	84	(-7)
5	3	21	63	8	∞	0	32	(-25)
6	0	85	15	43	89	∞	4	(-3)
7	18	0	7	1	58	13	∞	(-26)

Reduced: 84

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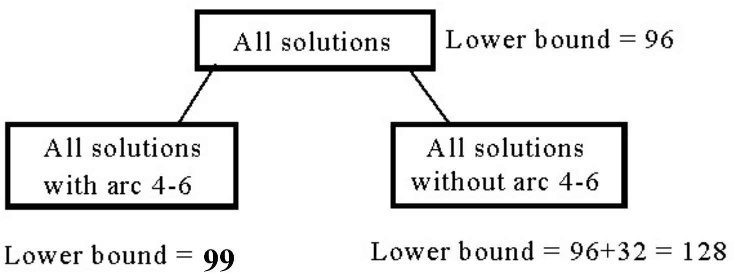
- Further reduced

i \ j	1	2	3	4	5	6	7
1	∞	0	83	9	30	6	50
2	0	∞	66	37	17	12	26
3	29	1	∞	19	0	12	5
4	32	83	66	∞	49	0	80
5	3	21	56	7	∞	0	28
6	0	85	8	42	89	∞	0
7	18	0	0	0	58	13	∞
			(-7)	(-1)			(-4)

Total cost reduced: 84+7+1+4 = 96 (lower bound)

- The highest level of a decision tree:



- A reduced cost matrix if arc (4,6) is included in the solution.

j / i	1	2	3	4	5	7
1	∞	0	83	9	30	50
2	0	∞	66	37	17	26
3	29	1	∞	19	0	5
5	3	21	56	7	∞	28
6	0	85	8	∞	89	0
7	18	0	0	0	58	∞

Arc (6,4) is changed to be infinity since it can not be included in the solution.

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- The reduced cost matrix for all solutions with arc 4-6

j / *i*	1	2	3	4	5	7	
1	∞	0	83	9	30	50	
2	0	∞	66	37	17	26	
3	29	1	∞	19	0	5	
5	0	18	53	4	∞	25	(-3)
6	0	85	8	∞	89	0	
7	18	0	0	0	58	∞	

- Total cost reduced: 96+3 = 99 (new lower bound)

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Arc (4, 6) is excluded

i \ j	1	2	3	4	5	6	7
1	∞	0	83	9	30	6	50
2	0	∞	66	37	17	12	26
3	29	1	∞	19	0	12	5
4	32	83	66	∞	49	∞	80
5	3	21	56	7	∞	0	28
6	0	85	8	42	89	∞	0
7	18	0	0	0	58	13	∞

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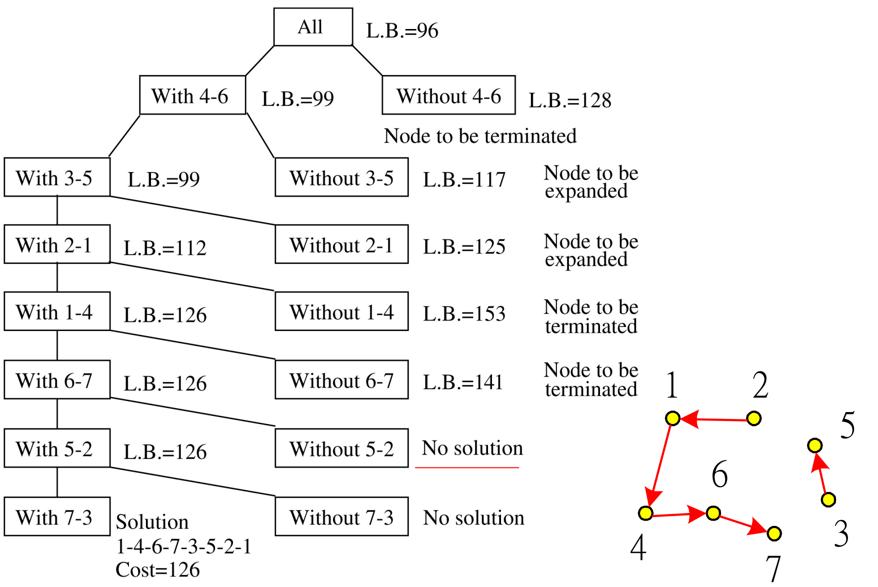

A branch-and-bound solution of a traveling salesperson problem.

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When will the expansion of a node be terminated?

- Current node is infeasible
- The lower bound value of the current node is greater than or equal to the best solution value ever encountered
- Upper bound and lower bound values of the current node are the same

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The 0/1 knapsack problem

- Positive integer $P_1, P_2, \ldots, P_n$ (profit)
  $W_1, W_2, \ldots, W_n$ (weight)
  M (capacity)

maximize $\sum_{i=1}^{n} P_i X_i$

subject to $\sum_{i=1}^{n} W_i X_i \leq M \quad X_i = 0 \text{ or } 1, i = 1, \ldots, n.$

The problem is modified:

minimize $-\sum_{i=1}^{n} P_i X_i$

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- e.g. $n = 6$, $M = 34$

i	1	2	3	4	5	6
P_i	6	10	4	5	6	4
W_i	10	19	8	10	12	8

$(P_i/W_i \geq P_{i+1}/W_{i+1})$

- A feasible solution: $X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 0, X_5 = 0, X_6 = 0$
  $-(P_1+P_2) = -16$ (upper bound)
  Any solution higher than -16 cannot be optimal.

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Relax the restriction

- Relax our restriction from $X_i = 0$ or 1 to $0 \leq X_i \leq 1$ (knapsack problem)

Let $-\sum_{i=1}^{n} P_i X_i$ be an optimal solution for 0/1 knapsack problem and $-\sum_{i=1}^{n} P_i X_i'$ be an optimal solution for knapsack problem. Let $Y = -\sum_{i=1}^{n} P_i X_i$,

$Y' = -\sum_{i=1}^{n} P_i X_i'$.

$\Rightarrow Y' \leq Y$

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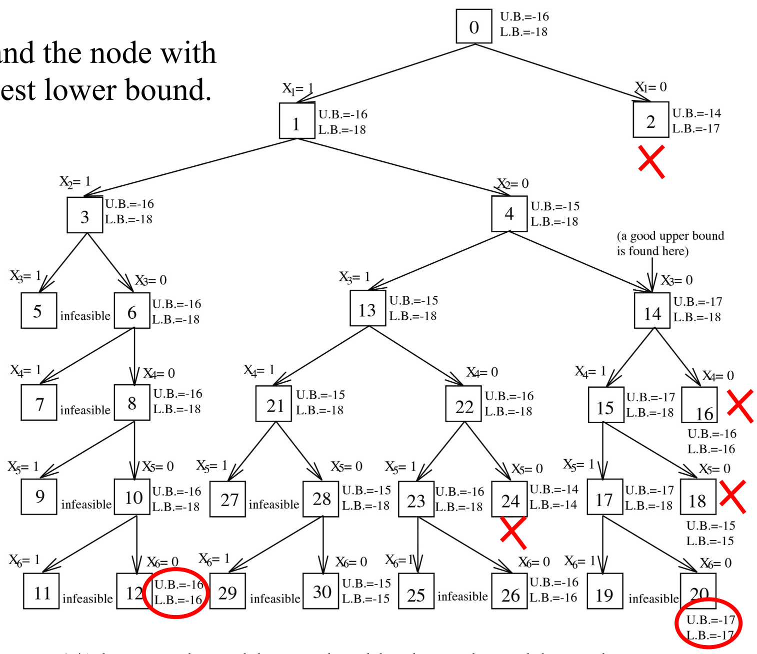
Upper bound and lower bound

- We can use the greedy method to find an optimal solution for knapsack problem:

  $X_1 = 1, X_2 =1, X_3 = 5/8, X_4 = 0, X_5 = 0, X_6 =0$
  $-(P_1+P_2+5/8P_3) = -18.5$ (lower bound)
  -18 is our lower bound. (only consider integers)

$\Rightarrow$ $-18 \leq$ optimal solution ≤ -16
optimal solution: $X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 1, X_5 = 1, X_6 = 0$
$-(P_1+P_4+P_5) = -17$

Expand the node with the best lower bound.



0/1 knapsack problem solved by branch-and-bound strategy.

Dominance Rule

- If there exist items i and j with $p_i \geq p_j$ and $w_i \leq w_j$, then there is an optimal solution in which
  - (1) if item j is included, then item i is also included;
  - (2) if item i is excluded, then item j is also excluded;

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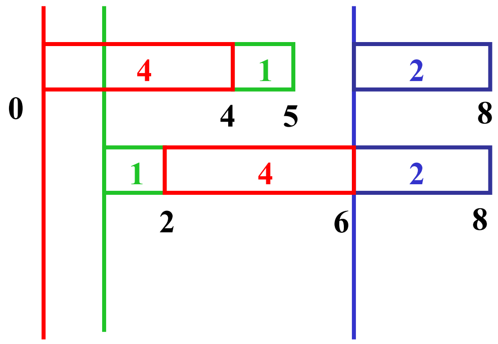
Minimizing the Total Completion Time on a Single Machine with Release Dates

- Each job has a processing time p_i and a release date r_i.
- We want to find a feasible schedule such that the Total completion time ΣC_i is minimized.



NP-hard

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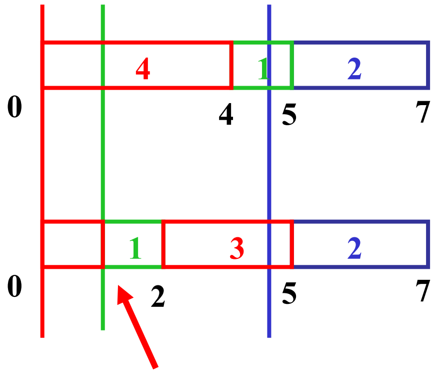
Find a Lower Bound

- Consider the situation where *preemption* is allowed



Consider the objective $\Sigma w_i C_i$

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Total completion time minimization

- The total completion time minimization problem is strongly NP-hard
- The preemptive case is polynomially solvable
- Dominance Rule: If there exist jobs i and j with $p_i \le p_j$ and $r_i \le r_j$, then job i precedes job j in some optimal solution.
- Consider the objective $\Sigma w_i C_i$:
  - NP-hard even if preemption is allowed

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Heuristics and lower bounds for the bin packing problem with conflicts

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Bin Packing with Conflicts

- A set V of n items with weights $w_1, w_2, \ldots, w_n$
- Each bin has a capacity c
- Conflict graph representing the possible conflicts amongst the items
- Goal: Finding the minimum number of bins that can accommodate all items without violating conflict constraints

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Related Problems

- Examination scheduling
- Assignment of processes to processors
- Bounded graph coloring
- Delivery problem

Integer Programming Model

$$\text{minimize} \quad z = \sum_{k=1}^{n} y_k$$

Bin k is used or not

subject to

Item i is dispatched to bin k.

$$\sum_{i=1}^{n} w_i x_{ik} \leqslant c y_k \quad (k = 1, \ldots, n)$$

$$\sum_{k=1}^{n} x_{ik} = 1 \quad (i = 1, \ldots, n)$$

$$x_{ik} + x_{jk} \leqslant 1 \quad ((i, j) \in E, \ k = 1, \ldots, n)$$

$$y_k = 0 \text{ or } 1 \quad (k = 1, \ldots, n)$$

$$x_{ik} = 0 \text{ or } 1 \quad (i, k = 1, \ldots, n).$$

Problem-Solving Approaches

- Heuristics
  - FFD
  - Variants based upon coloring heuristics
  - Variants based upon clique heuristics
- Lower bounds
  - Relaxation of conflict constraints
  - Transportation problem

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Step 1: Determine a large clique D on $G=(V,E)$ using Johnson's (1974) heuristic, and assign each item of D to a different bin. Let r_k be the residual capacity of bin k and let $h = \text{argmin}_{i \in D}\{w_i\}$.

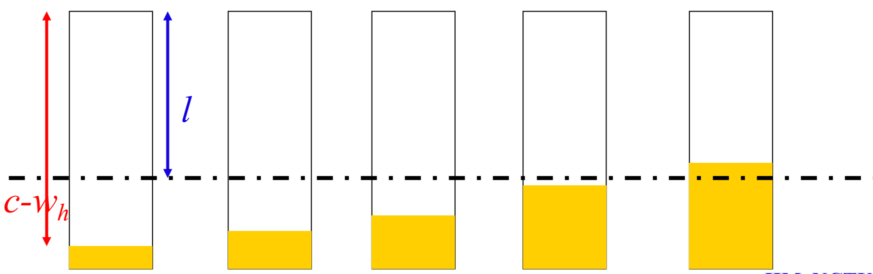
Step 2: For a given weight level $\ell \leqslant c - w_h$, define the three sets

$$I'_\ell = \{i \in V:\ w_i < \ell\},$$

$$I_\ell = \{i \in V:\ \ell \leqslant w_i \leqslant c - w_h\},$$

$$I''_\ell = \{i \in V: w_i > c - w_h\}.$$

In addition to the already created $|D|$ bins, create a dummy bin 0 of infinite residual capacity. Let K_ℓ be the set of bins k having a residual capacity $r_k \geqslant \ell$.



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Transportation Problem

$$\text{minimize} \quad \sum_{i \in I_\ell} \sum_{k \in K_\ell} c_{ik} x_{ik}$$

subject to

$$\sum_{k \in K_\ell} x_{ik} = w_i \quad (i \in I_\ell)$$

$$\sum_{i \in I_\ell} x_{ik} \leqslant r_k \quad (k \in K_\ell)$$

$$x_{ik} \text{ integer} \quad (i \in I_\ell, k \in K_\ell).$$

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BPC Lower Bound

$$b_\ell = |D| + \left\lceil \left(\sum_{i \in I_\ell} x_{i0} + \sum_{i \in I''_\ell} w_i + z_\ell \right) \Big/ c \right\rceil .$$

$$z_\ell = \max \left\{ 0, \sum_{i \in I'_\ell} w_i - \left(\sum_{k \in D} r_k - \sum_{i \in I_\ell} \sum_{k \in K_\ell \setminus \{0\}} x_{ik} \right) \right\} .$$

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Computational results for the Falkenauer uniform instances ($n = 120, c = 150$)

d (%)	MAX L_1	L_2	$\{L_1, L_2\}$	H1 UB/LB	sec.	H2 UB/LB	sec.	H3 UB/LB	sec.	H4 UB/LB	sec.	H5 UB/LB	sec.	H6 UB/LB	nb. opt.	sec.
0	48.3	48.3	48.3	**1.017**	0.1	1.279	0.5	1.049	0.5	1.261	1.9	1.302	1.3	**1.017**	2	0.2
10	48.3	48.3	48.3	1.025	0.1	1.281	0.6	1.080	0.4	1.242	2.1	1.263	1.2	**1.017**	2	0.2
20	48.3	48.3	48.3	**1.056**	0.1	1.291	0.6	1.109	0.6	1.180	2.8	1.213	1.2	**1.056**	0	0.3
30	48.3	48.3	48.3	**1.152**	0.2	1.295	0.7	1.192	0.7	1.188	4.6	1.237	1.0	1.192	0	0.3
40	48.3	49.7	49.7	1.248	0.2	1.351	1.0	1.289	1.0	1.289	13	1.349	0.8	**1.182**	0	0.4
50	48.3	59.8	59.8	1.197	0.2	1.265	1.3	1.260	1.3	1.242	15	1.284	0.6	**1.108**	0	0.5
60	48.3	72.1	72.1	1.130	0.2	1.178	1.7	1.178	1.8	1.165	13.5	1.192	0.4	**1.091**	0	0.5
70	48.3	85.8	85.8	1.081	0.3	1.100	2.0	1.100	2.0	1.091	16.4	1.112	0.3	**1.056**	0	0.7
80	48.3	96.3	96.3	1.046	0.4	1.059	2.2	1.059	2.3	1.051	15.7	1.063	0.4	**1.033**	0	0.6
90	48.3	108.3	108.3	1.017	0.4	1.029	2.4	1.029	2.4	1.017	16.1	1.023	0.3	**1.015**	4	0.6

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Table 4
Computational results for the Falkenauer uniform instances ($n = 1000, c = 150$)

d (%)	MAX L_1	L_2	$\{L_1, L_2\}$	H1 UB/LB	sec.	H2 UB/LB	sec.	H3 UB/LB	sec.	H4 UB/LB	sec.	H5 UB/LB	sec.	H6 UB/LB	nb. opt.	sec.
0	401.8	401.8	401.8	**1.012**	0.9	1.372	134.3	1.141	134.6	—	—	1.332	374.2	**1.012**	0	40.3
10	401.8	401.8	401.8	1.014	1.1	1.333	142.5	1.146	142.8	—	—	1.277	346.1	**1.012**	0	83.1
20	401.8	401.8	401.8	**1.037**	1.8	1.255	167.2	1.137	167.6	—	—	1.226	303.3	1.058	0	336.3
30	401.8	401.8	401.8	**1.095**	3.2	1.233	211.2	1.139	211.1	—	—	1.240	251.2	1.216	0	670.4
40	401.8	407.5	407.5	1.217	7.9	1.363	275.1	1.318	274.8	—	—	1.375	193.6	**1.149**	0	809
50	401.8	509.5	509.5	1.138	16.2	1.238	364.2	1.237	363.6	—	—	1.251	140.2	**1.063**	0	810.9
60	401.8	602.3	602.3	1.098	25.6	1.161	498.3	1.161	459.1	—	—	1.173	80.1	**1.047**	0	575.4
70	401.8	707.4	707.4	1.063	37.0	1.103	539.2	1.103	538.5	—	—	1.108	44.4	**1.027**	0	515.3
80	401.8	799.2	799.2	1.042	49.4	1.065	598.9	1.065	597.2	—	—	1.066	24.7	**1.024**	0	512.2
90	401.8	899.2	899.2	1.020	63.0	1.028	649.9	1.028	642.2	—	—	1.031	15.4	**1.011**	0	501.0

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Film Production Problem

Film Production Problem

- Holding Days :
  - The days when an actor is not required, but still has some other filming left to complete.

- The objective is to minimize the total holding cost of the actors/actress.

Film Production Problem

actor i	Shooting day j 1 2 3 4 5 6 7 8	Holding Cost c_i
1	1 1 1 0 0 1 0 0	810
2	1 1 0 0 0 1 1 0	700
3	0 1 0 1 0 0 0 1	910
4	0 1 0 1 1 1 1 0	880
5	0 1 0 0 0 0 0 1	400
6	0 0 1 1 0 0 0 1	610

Total cost = 2*810+3*700+4*910
+880+5*400+3*610

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Film Production Problem

actor i	Shooting day j 8 2 3 4 5 6 7 1	Holding Cost c_i
1	0 1 1 0 0 1 0 1	810
2	0 1 0 0 0 1 1 1	700
3	1 1 0 1 0 0 0 0	910
4	0 1 0 1 1 1 1 0	880
5	1 1 0 0 0 0 0 0	400
6	1 0 1 1 0 0 0 0	610

Total cost = 3*810+3*700+1*910
+880+610

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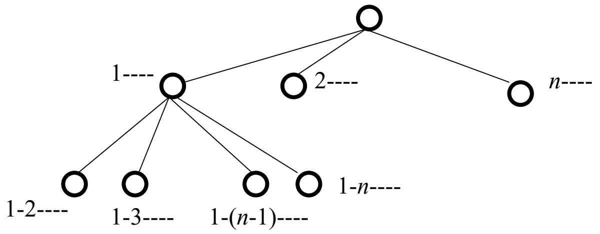
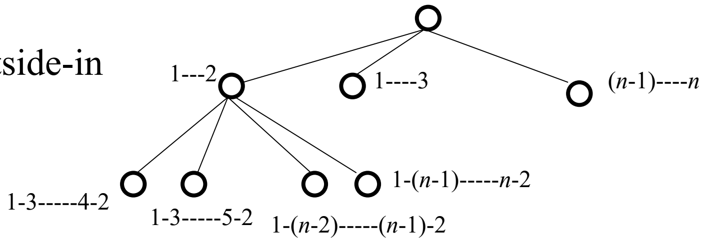
Enumeration Tree



Outside-in



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Lower Bound

Actor *i*	*T(s)* 4	2	3	1	5	8	7	6	**Holding Cost** c_i
1	0	1					0	0	$1,000
2	0	1					1	1	$200
3	1	0					0	0	$700
4	1	1					1	0	$350
5	0	1					0	0	$500
6	1	1					0	1	$840
7	0	0					1	0	$1,020
8	1	0					1	0	$980
9	0	0					1	1	$1,300
10	0	1					0	0	$450

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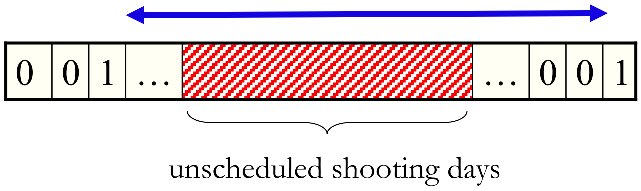
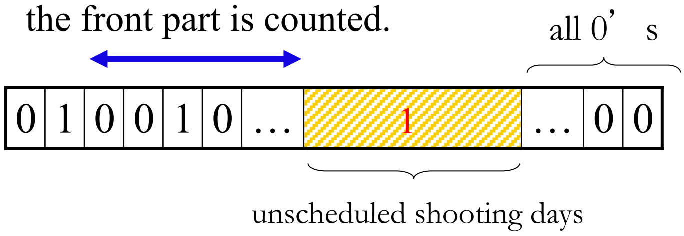
Any 0 enclosed by two 1's is counted.



Any 0 already scheduled in
the front part is counted.



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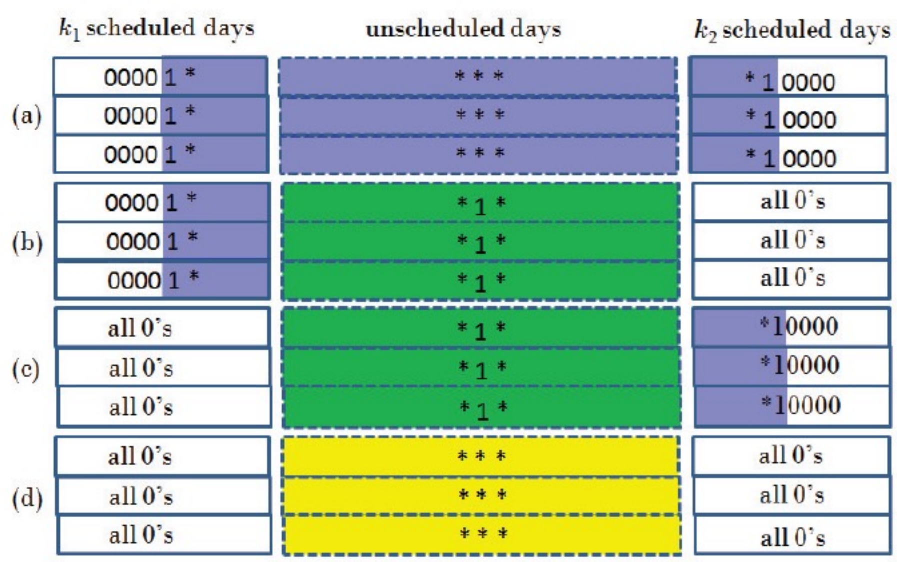

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Lower Bound

Actor i	T(s) 4 2 3 1 5 8 7 6	Holding Cost c_i
√ 1	0 1 0 0	\$1,000
√ 3	1 0 0 0	\$700
√ 5	0 1 0 0	\$500
7	0 0 1 0	\$1,020
9	0 0 1 1	\$1,300
√ 10	0 1 0 0	\$450

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- Define two subsets:

$$E_P = \{i|\varepsilon_i(k) = 1 \text{ and } \lambda_i(k) = 0\}; \text{ and}$$

$$L_P = \{i|\varepsilon_i(k) = 0 \text{ and } \lambda_i(k) = 1\}.$$

- $\alpha_p(x,y)$ denotes the number of unscheduled shooting days which actor x is required but y is not.

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- P = 38----75, E_P = {1, 3, 5, 6} and L_P = {2, 4}.

actress	Shooting day j								Holding Cost
i	3	8	4	2	1	6	7	5	c_i
1	1	0	0	1	1	1	0	0	810
2	0	0	0	1	1	1	1	0	700
3	0	1	1	1	0	0	0	0	910
4	0	0	1	1	0	1	1	1	880
5	0	1	0	1	0	0	0	0	400
6	1	1	1	0	0	0	0	0	610

$\pi_P(1,3) = 2$; $\alpha_P(3,1) = 1$; $\alpha_P(5,1) = 0$; $\alpha_P(6;1) = 1$;

$\alpha_P(1,5) = 2$; $\alpha_P(3,5) = 1$; $\alpha_P(5,3) = 0$; $\alpha_P(6,3) = 0$;

$\alpha_P(1,6) = 3$; $\alpha_P(3,6) = 1$; $\alpha_P(5,6) = 1$; $\alpha_P(6,5) = 1$;

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- Extra holding days incurred by actors x and y is is no less than $min\{\alpha_P(x,y), \alpha_P(y,x)\}$.

x 1 | 000111111100 | 0

y 1 | 111110000000 | 0

Scheduled days

x 1 | 001111111000 | 0

y 1 | 111100000100 | 0

Scheduled days

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- The extra holding cost incurred by actors x and y is is no less than $min\{c_y\alpha_P(x,y), c_x\alpha_P(y,x)\}$.

- Formulate the problem of finding the extra holding cost for the subset E_P into a maximum weighted matching problem in graph theory by letting the weight of the edge connecting nodes x and y be defined as $min\{c_y\alpha_P(x,y), c_x\alpha_P(y,x)\}$.

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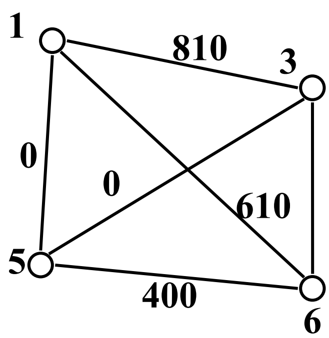
- Maximum weighted matching for E_P.
- With this graph, the maximum weighted matching of E_P is 810+400 = 1,210.



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- The situation of L_P can be similarly estimated with a value of 700.
- Therefore, we conclude the example with a lower bound of P is 810+1,210+700 = 2,720.

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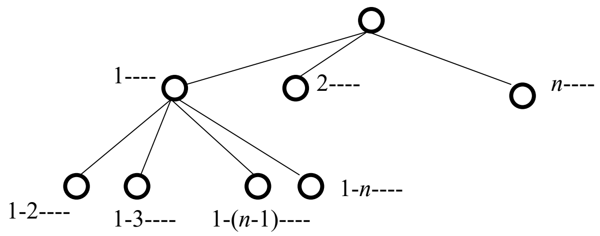
- Assume the conventional branching scheme is adopted.
- Can we deploy LB_1 or LB_2?



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$\beta_{i_1,\tilde{i}_2,i_3} \cup \beta_{i_1,i_2,\tilde{i}_3}$

a_{i_1}	**0**	1111 1111111	**0**
a_{i_2}	**1**	0000 1111111	**1**
a_{i_3}	**1**	1111 0000000	**1**

(a) Case x-1

$\beta_{i_1,i_2,\tilde{i}_3}$

a_{i_1}	**0**	1111111111	**1**
a_{i_2}	**1**	1111111111	**0**
a_{i_3}	**1**	0000000000	**1**

(b) Case x-4

$\beta_{i_1,\tilde{i}_2,\tilde{i}_3} \cup \beta_{i_1,i_2,i_3} \cup \beta_{i_1,\tilde{i}_2,\tilde{i}_3}$

a_{i_1}	**0**	000000111111111	**0**
a_{i_2}	**0**	111111110000000	**0**
a_{i_3}	**1**	000000110000000	**1**

(a) Case y-2

$\beta_{i_1,i_2,i_3} \cup \beta_{i_1,\tilde{i}_2,\tilde{i}_3}$

a_{i_1}	**0**	111111111111	**0**
a_{i_2}	**0**	111100000000	**1**
a_{i_3}	**1**	111100000000	**0**

or

$\beta_{i_1,i_2,i_3} \cup \beta_{i_1,\tilde{i}_2,\tilde{i}_3}$

a_{i_1}	**0**	111111111111	**0**
a_{i_2}	**0**	000000001111	**1**
a_{i_3}	**1**	000000001111	**0**

(b) Case y-5

Theorem 4.2. *Given partial schedule P,*

$$LB_3(P) = LB_2(P) + \textsc{Max-3-Grouping}(P)$$

is a lower bound on the optimal solution derived from P.

`IP-3-Grouping:`

$$\text{Maximize} \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} z_{i,j,k} Z_{i,j,k}$$

subject to

$$\sum_{j=1}^{m}\sum_{k=1}^{m} Z_{i,j,k} + \sum_{j=1}^{m}\sum_{k=1}^{m} Z_{j,i,k} + \sum_{j=1}^{m}\sum_{k=1}^{m} Z_{j,k,i} \leq 1 \qquad \text{for all } i;$$

$$Z_{i,j,k} \in \{0,1\} \qquad \text{for all } i,j,k.$$

TABLE 3. Lower bounds subject to outside-in branching scheme.

Density	lower bound	k=1 value	k=1 ratio	$k=3$ value	$k=3$ ratio	$k=5$ value	$k=5$ ratio	$k=10$ value	$k=10$ ratio
0.1	LB_1	11,413	1.00	106,365	1.00	241,830	1.00	515,424	1.00
	LB_2	22,297	1.95	129,686	1.22	265,002	1.10	530,076	1.03
	LB_3	50,023	4.38	143,619	1.35	272,854	1.13	531,030	1.03
0.2	LB_1	43,299	1.00	282,387	1.00	462,986	1.00	723,402	1.00
	LB_2	81,428	1.88	324,608	1.15	496,798	1.07	727,837	1.01
	LB_3	112,749	2.60	333,710	1.18	498,486	1.08	727,837	1.01
0.3	LB_1	102,531	1.00	393,930	1.00	564,109	1.00	693,117	1.00
	LB_2	160,744	1.57	439,086	1.11	585,516	1.04	693,716	1.00
	LB_3	188,908	1.84	442,229	1.12	585,516	1.04	693,716	1.00

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TABLE 4. Lower bounds subject to sequential branching.

Density	lower bound	k=1 value	k=1 ratio	$k=3$ value	$k=3$ ratio	$k=5$ value	$k=5$ ratio	$k=10$ value	$k=10$ ratio
0.1	LB_1	0	N/A	7,764	1.00	22,918	1.00	87,358	1.00
	LB_2	6,541	N/A	26,995	3.48	51,991	2.27	127,306	1.46
	LB_3	37,745	N/A	49,103	6.32	67,435	2.94	132,759	1.52
0.2	LB_1	0	N/A	11,737	1.00	35,605	1.00	11,7147	1.00
	LB_2	24,446	N/A	74,227	6.32	113,462	3.19	206,271	1.76
	LB_3	71,501	N/A	96,454	8.22	125,353	3.52	212,210	1.81
0.3	LB_1	0	N/A	14,769	1.00	39,563	1.00	113,626	1.00
	LB_2	44,348	N/A	119,872	8.12	275,254	6.96	236,553	2.08
	LB_3	95,967	N/A	135,891	9.20	281,540	7.12	236,553	2.08

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Dominance Rule

- Assume that k shooting days are already scheduled in P. We consider the (k-1)th and the kth days.
- If actress i is required on at least one of the first k-2 days of P, then $\varepsilon_i(k\text{-}2) = 1$; otherwise, $\varepsilon_i(k\text{-}2) = 0$.
- If $\sum_{j=1}^{n} t_{ij} - \sum_{j \in P} t_{ij} > 0$, $\lambda_i(n\text{-}k) = 1$; otherwise, let $\lambda_i(n\text{-}k) = 0$.

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Lagrangian Relaxation

- M.L. Fisher, An applications oriented guide to Lagrangian relaxation, *Interfaces*, Vol. 15, No. 2, 1985, pp. 10-21.
- M.L. Fisher, The Lagrangian relaxation method for solving integer programming problems, *Management Science*, Vol. 27, No. 1, 1981, pp. 1-18.

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u is a vector of multipliers

Max $Z = cX$
Subject to
$AX \le b$
$DZ \le e$
$X \ge 0$ and integral

Add a non-negative term into Z

Max Z
$= cX + u(b - AX)$
Subject to
$AX \le b$
$DZ \le e$
$X \ge 0$ and integral

Provides an upper bound on Z

Remove some constraints

1. Removing the constraints cannot decrease the optimal value
2. The relaxed problem is relatively easy to solve

Max $Z = cX + u(b - AX)$
Subject to
$DZ \le e$
$X \ge 0$ and integral

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$$
\begin{aligned}
&\text{Max } Z = 16x_1 + 10x_2 + 4x_4 \\
&\text{subject to} \\
&\quad \boxed{8x_1 + 2x_2 + x_3 + 4x_4 \le 10} \\
&\quad x_1 + x_2 \le 1 \\
&\quad x_3 + x_4 \le 1 \\
&\quad x_j = 0 \text{ or } 1,\ j = 1,2,3,4.
\end{aligned}
$$

$$
\begin{aligned}
&\text{Max } Z_D(u) = (16-8u)x_1 + (10-2u)x_2 + (0-u)x_3 + (4-4u)x_4 + 10u \\
&\text{subject to} \\
&\quad x_1 + x_2 \le 1 \\
&\quad x_3 + x_4 \le 1 \\
&\quad x_j = 0 \text{ or } 1,\ j = 1,2,3,4.
\end{aligned}
$$

For a fixed u, this problem can be easily solved.

Goal: Find some u>0 for which $Z_D(u)$ is minimized

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$$\text{Max } Z_D(u) = (16-8u)x_1 + (10-2u)x_2 + (0-u)x_3 + (4-4u)x_4 + 10u$$

subject to

$$x_1 + x_2 \le 1$$

$$x_3 + x_4 \le 1$$

$$x_j = 0 \text{ or } 1,\ j = 1,2,3,4.$$

$$\text{Max } Z = 16x_1 + 10x_2 + 4x_4$$

subject to

$$8x_1 + 2x_2 + x_3 + 4x_4 \le 10$$

$$x_1 + x_2 \le 1$$

$$x_3 + x_4 \le 1$$

$$x_j = 0 \text{ or } 1,\ j = 1,2,3,4.$$

u	x_1	x_2	x_3	x_4	$Z_D(u)$	Z
0	1	0	0	1	20	-
6	0	0	0	0	60	0
3	0	1	0	0	34	10
2	0	1	0	0	26	10
1	1	0	0	0	18	16
1	1	0	0	1	18	-
1	0	1	0	0	18	10
1	0	1	0	1	18	14
1/2	1	0	0	1	19	-
3/4	1	0	0	1	18.5	-

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- Is $Z_D(u) = 18$ optimal?
- If we substitute any x into the objective function for the Lagrangian problem, we obtain a linear function of u.
- The function is given by the upper envelop of a family of linear equations corresponding to the possible x's.

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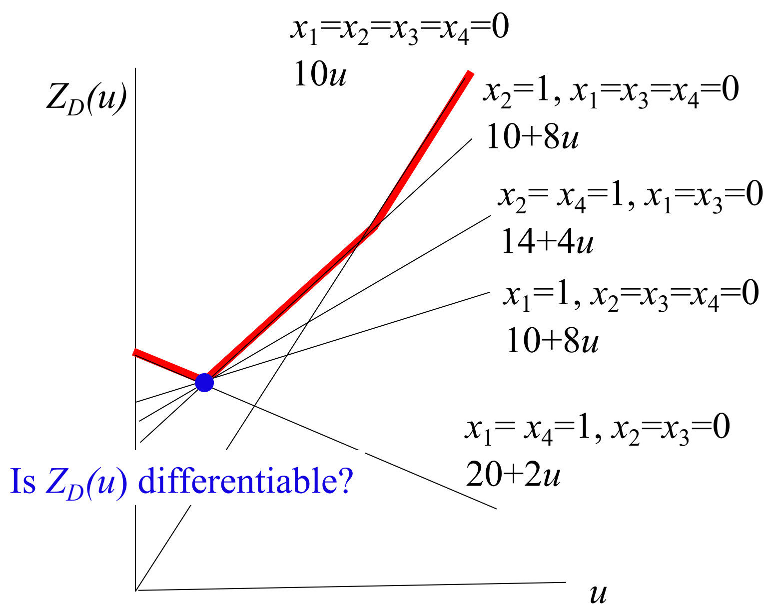

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Non-differentiable Case

$$u^{k+1} = \max\{0,\ u^k - t_k(b - Ax^k)\}. \quad (6)$$

In this formula, t_k is a scalar stepsize and x^k is an optimal solution to $(LR_u{}^k)$, the Lagrangian problem with dual variables set to u^k.

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$u^0 = 0$
$u^1 = 0 - (-2) = 2$
$u^2 = \max\{0, 2\text{-}8\} = 0$
$u^3 = 0 - (-2) = 2$
$u^4 = \max\{0, 2\text{-}8\} = 0$

Table 2: Subgradient method with $t_k = 1$ for all k.

$u^0 = 0$
$u^1 = 0 - (-2) = 2$
$u^2 = \max\{0, 2 - ½(8)\} = 0$
$u^3 = 0 - ¼(-2) = ½$
$u^4 = ½ - ⅛(-2) = ¾$
$u^5 = ¾ - 1/16(-2) = ⅞$
$u^6 = ⅞ - 1/32(-2) = 15/16$

Table 3: Subgradient method with $t_k = 1, ½, ¼, ⅛, \ldots$

$u^0 = 0$
$u^1 = 2$
$u^2 = \max\{0, 2 - 1/3(8)\} = 0$
$u^3 = 0 - 1/9(-2) = 2/9$
$u^4 = 2/9 - 1/27(-2) = .296$
$u^5 = .296 - 1/81(-2) = .321$
$u^6 = .321 - 1/243(-2) = .329$
$u^7 = .329 - 1/729(-2) = .332$

Table 4: Subgradient method with $t_k = 1, 1/3, 1/9, 1/27, 1/81, \ldots$

If $t_k \to 0$ and $\sum_{i=1}^{k} t_i \to \infty$ as $k \to \infty$

then $Z_D(u_k)$ converges to its optimal

solution value Z_D.

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An effective procedure

$$t_k = \frac{\lambda_k\left(Z_D(u^k) - Z^*\right)}{\sum_{i=1}^{m}\left(b_i - \sum_{j=1}^{n} a_{ij} x_j^k\right)^2}$$

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Relationship with LP Solutions

the LP dual of example (1) – (5) is

$$\min\ 10_u + v_1 + v_2 + w_1 + w_2 + w_3 + w_4$$

$$8u + v_1 + w_1 \geq 16$$
$$2u + v_1 + w_2 \geq 10$$
$$u + v_2 + w_3 \geq 0$$
$$4u + v_2 + w_4 \geq 4$$
$$u,\ v_1,\ v_2,\ w_1, \ldots, w_4 \geq 0.$$

The optimal solution to the primal LP is $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = ½$ and the optimal solution to the dual LP is $u = 1$, $v_1 = 8$, $v_2 = w_1 = \ldots w_4 = 0$.

$$Z_D = \min_{u \geq 0} \{ \max_{\substack{x \\ Dx \leq e \\ x \geq 0 \text{ and integral}}} (cx + u(b - Ax)) \}$$

$$\leq \min_{u \geq 0} \{ \max_{\substack{x \\ Dx \leq e \\ x \geq 0}} (cx + u(b - Ax)) \}$$

$$\text{(by } LP \text{ duality)} = \min_{u \geq 0} \{ \min_{\substack{v \geq 0 \\ vD \geq c - uA}} ub + ve \}$$

$$= \min_{\substack{u, v \geq 0 \\ uA + vD \geq c}} ub + ve$$

$$\text{(by } LP \text{ duality)} = \max_{\substack{Ax \leq b \\ Dx \leq e}} cx$$

$$= Z_{LP}$$

$Z_D < Z_{LP}$

$Z_D = Z_{LP}$

LP relaxation can improve the Lagrangian bound.

Max $Z = 16x_1 + 10x_2 + 4x_4$

subject to

$$8x_1 + 2x_2 + x_3 + 4x_4 \le 10$$
$$x_1 + x_2 \le 1$$
$$x_3 + x_4 \le 1$$
$$x_j = 0 \text{ or } 1,\ j = 1,2,3,4.$$

$$Z_D(v_1,v_2) = \max\ (16-v_1)x_1 + (10-v_1)x_2 + (0-v_2)x_3 + (4-v_2)x_4 + v_1 + v_2$$

subject to $8x_1 + 2x_2 + x_3 + 4x_4 \le 10$ (2)

$0 \le x_j \le 1,\ j=1,\ldots,4$ (5)

x_j integral, $j=1,\ldots,4$ (6)

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Lagrangian Solution

v_1	v_2	λ_k	x_1	x_2	x_3	x_4	$Z_D(v_1,v_2)$	Z^*
0	0	1	1	1	0	0	26	0
13	0	1	0	0	0	1	17	4
(feasible with $Z = 4$)								
0	0	1	1	1	0	0	26	4
11	0	1	1	0	0	0	16	16
(feasible with $Z = 16$)								

Table 5: The subgradient method applied to the improved relaxation.

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Key Issues for Lagrangian Relaxation

- Which constraint set to relax
- How to compute good multipliers *u*
- How to deduce a good, feasible solution to the original problem

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