(a)
Compute least squares estimators for c and d in

$$y_n = cx_n + d + r_n \tag{1}$$

by computing the minimum of $J(c,d) = \sum_{n=1}^{N}(y_n - cx_n - d)^2$.

Solution:
We differentiate J with c

$$\frac{\partial J}{\partial c} = \sum_{n=1}^{N} 2(y_n - cx_n - d)(-x_n) = 2\sum_{n=1}^{N}(-x_n y_n + cx_n^2 n + dx_n) = 0$$

$$-\sum_{n=1}^{N} x_n y_n + c\sum_{n=1}^{N} x_n^2 + \sum_{n=1}^{N} dx_n = 0 \tag{2}$$

$$\hat{c} = \frac{\sum_{n=1}^{N} x_n y_n - \sum_{n=1}^{N} dx_n}{\sum_{n=1}^{N} x_n^2} \tag{3}$$

Then, we differentiate J with d

$$\frac{\partial J}{\partial d} = \sum_{n=1}^{N} 2(y_n - cx_n - d) = 0 \tag{4}$$

$$\hat{d} = \frac{1}{N}(\sum_{n=1}^{N} y_n - \sum_{n=1}^{N} \hat{c}x_n) = \bar{y}_n - \hat{c}\bar{x}_n \tag{5}$$

Substitute $\hat{d}$ from eq (5) into eq (3)

$$\hat{c} = \frac{\sum_{n=1}^{N} x_n y_n - (\bar{y}_n - \hat{c}\bar{x}_n)\sum_{n=1}^{N} x_n}{\sum_{n=1}^{N} x_n^2} \tag{6}$$

Reorganize eq (6)

$$\begin{aligned}\hat{c}(\sum_{n=1}^{N} x_n^2) &= \sum_{n=1}^{N} x_n y_n - (\bar{y}_n - \hat{c}\bar{x}_n)\sum_{n=1}^{N} x_n \\ \hat{c}(\sum_{n=1}^{N} x_n^2 - \bar{x}_n\sum_{n=1}^{N} x_n) &= \sum_{n=1}^{N} x_n y_n - \bar{y}_n\sum_{n=1}^{N} x_n\end{aligned} \tag{7}$$

The estimator $\hat{c}$ is

$$\hat{c} = \frac{\sum_{n=1}^{N} x_n y_n - \bar{y}_n\sum_{n=1}^{N} x_n}{\sum_{n=1}^{N} x_n^2 - \bar{x}_n\sum_{n=1}^{N} x_n} = \frac{\sum_{n=1}^{N} x_n y_n - N\bar{x}_n\bar{y}_n}{\sum_{n=1}^{N} x_n^2 - N\bar{x}_n^2} \tag{8}$$

The estimator $\hat{d}$ is

$$\hat{d} = \bar{y}_n - \hat{c}\bar{x}_n \tag{9}$$

(b)
Convert the above problem into a problem of the form $J(\mathbf{x}) = (\mathbf{y} - \mathbf{G}\ \mathbf{x})^T(\mathbf{y} - \mathbf{G}\ \mathbf{x})$, minimize it with the given matrix expressions in the course book, and show that the result is the same as in (a). Hint: put $x = (c, d)$.

Solution:
Let $\mathbf{x} = [c, d]^T$

$$y_n = \begin{bmatrix} x_n & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} + r_n, \mathbf{G} = \begin{bmatrix} x_n & 1 \end{bmatrix} \tag{10}$$

$$\begin{aligned} \hat{\mathbf{X}}_{LS} &= (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{y} \\ &= \left(\begin{bmatrix} \mathbf{X}^T \\ \mathbf{1}_N \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{1}_N \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{X} \\ \mathbf{1}_N \end{bmatrix} \mathbf{y} \\ &= \left(\begin{bmatrix} \mathbf{X}^T\mathbf{X} & \mathbf{X}^T \\ \mathbf{X} & N \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{X} \\ \mathbf{1}_N \end{bmatrix} \mathbf{y} \end{aligned} \tag{11}$$

Let

$$A = \begin{bmatrix} \mathbf{X}^T\mathbf{X} & \mathbf{X}^T \\ \mathbf{X} & N \end{bmatrix}$$

Then,

$$\begin{aligned} |\mathbf{A}| &= N\mathbf{X}^T\mathbf{X} + \mathbf{X}^T\mathbf{X} \\ &= N\sum_{n=1}^{N} x_n^2 - (\sum_{n=1}^{N} x_n)^2 \\ &= N\sum_{n=1}^{N} x_n^2 - N^2\bar{x}^2 \end{aligned} \tag{12}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} N & -\mathbf{X} \\ -\mathbf{X}^T & \mathbf{X}^T\mathbf{X} \end{bmatrix} \tag{13}$$

Substitute eq (13) back to (11),

$$\begin{aligned} \begin{bmatrix} \hat{c} \\ \hat{b} \end{bmatrix} &= \begin{bmatrix} \frac{1}{|A|}\sum_{n=1}^{N} x_n y_n - \frac{1}{|A|}\sum_{n=1}^{N} x_n \sum_{n=1}^{N} y_n \\ -\frac{1}{|A|}\sum_{n=1}^{N} x_n \sum_{n=1}^{N} x_n y_n + \frac{1}{|A|}\sum_{n=1}^{N} x_n^2 \sum_{n=1}^{N} y_n \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sum_{n=1}^{N} x_n y_n - N\bar{x}\bar{y}}{\sum_{n=1}^{N} x_n^2 - N\bar{x}^2} \\ \bar{y} - \hat{c}\bar{x} \end{bmatrix} \end{aligned} \tag{14}$$