Question:
The Gaussian probability density function of a one-dimensional random variable x is:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Derive the variance of this distribution by brute-force integration.

Solution:
The definition of variance is

$$var\{x\} = E\{(x - E\{x\})^2\} \tag{1}$$

As we have derived from exercise 1, the expected value of $E\{(x\}$ is μ.
eq. (1) can be written as

$$var\{x\} = E\{(x-\mu)^2\} \tag{2}$$

With probability density function p(x), $var\{(x\}$ can be written as

$$\begin{aligned} var\{(x\} &= \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\, dx \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)\, dx \end{aligned} \tag{3}$$

Let

$$\begin{aligned} y &= \frac{x-\mu}{\sqrt{2}\sigma} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{2}\sigma} \\ dx &= \sqrt{2}\sigma dy \end{aligned}$$

Substitute it to eq. (3), then we get

$$\begin{aligned} var\{x\} &= \int_{-\infty}^{\infty} 2\sigma^2 y^2 \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-y^2\right)\, \sqrt{2}\sigma dy \\ &= \int_{-\infty}^{\infty} \frac{2}{\sqrt{\pi}}\sigma^2 y^2 exp\left(-y^2\right) dy \\ &= \frac{2}{\sqrt{\pi}}\sigma^2 \int_{-\infty}^{\infty} y^2 exp\left(-y^2\right) dy \\ &= \frac{2}{\sqrt{\pi}}\sigma^2 \int_{-\infty}^{\infty} -\frac{1}{2} y\ d exp\left(-y^2\right) \\ &= \frac{2}{\sqrt{\pi}}\sigma^2 \cdot -\frac{1}{2}\left[y \cdot exp(-y^2)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} exp(-y^2) dy \right] \\ &= \frac{2}{\sqrt{\pi}}\sigma^2 \cdot -\frac{1}{2}\left[0 - \int_{-\infty}^{\infty} exp(-y^2) dy \right] \\ &= \frac{2}{\sqrt{\pi}}\sigma^2 \cdot \frac{1}{2} \cdot \sqrt{\pi} \quad = \sigma^2 \end{aligned} \tag{4}$$