CSE 518A: Assignment 2 Solution

Note: These are not intended to be comprehensive, just to help you see what the answers should be.

1. **Cooperation and Repeated Prisoner's Dilemma.**

   (a) By definition, a subgame perfect equilibrium is a Nash equilibrium for every subgame.

   Below we use backward induction to prove that both players choosing to defect at every time step is a subgame perfect equilibrium.

   - Consider the subgame starting at time T. Since this is the standard prisoner's dilemma, we know that both players choosing to defect is the only equilibrium.
   - Consider the subgame starting at time $t' < T$. Assume both players choosing to defect for every time step $t \geq t'$ is the only Nash. Now for the subgame starting at time $t' - 1$. Condition on both players always choosing to defect in time t' to T, the players' reasoning reduces to standard one-shot prisoner's dilemma again (by adding a constant payoff, obtained by always choosing to defect from time t' to T, for the payoff of every possible strategy combinations). Therefore, both players choosing to defect at time $t' - 1$ is the only Nash, which means both players choosing to defect at time $t' - 1$ to T is the only Nash.
   - Using backward induction, both players choosing to defect at time 1 to T is the only Nash.

   (b) Using one-deviation property, we only need to show that no player can obtain higher payoff by deviating from the tit-for-tat strategy for one of the time round (while assuming another player follows the tit-for-tat strategy). Below we discuss the case that player 2 always follow the tit-for-tat strategy and see if player 1 can benefit by deviating in one of the time round.

   - Case 1: Player 1 follows tit-for-tat except at time t'.
     Before time t', both players choose to cooperate. When player 1 deviates and chooses to defect at time t', player 2 will choose to defect while player 1 will choose to cooperate at time $t' + 1$. Then at $t' + 2$, player 1 will defect and player 2 will cooperate. The two players' strategy sequence would look like the following (we use C for cooperate and D for defect):
     * before time t': (player 1: C, player 2: C); player 1 obtains payoff 2 in each step before t'
     * at time t': (player 1: D, player 2: C); player 1 obtains payoff 3 in this step
     * at time $t' + 1$: (player 1: C, player 2: D); player 1 obtains payoff 0 in this step
     * at time $t' + 2$: (player 1: D, player 2: C); player 1 obtains payoff 3 in this step
     * at time $t' + 3$: (player 1: C, player 2: D); player 1 obtains payoff 0 in this step
     * ...

     Therefore, the total time-discounted payoff for player 1 would be

$$
\begin{aligned}
U_1 &= \sum_{t=1}^{t'-1} 2\delta^t + 3\delta^{t'} + 0 + 3\delta^{t'+2} + 0 + 3\delta^{t'+4} + 0 \ldots \\
&= \sum_{t=1}^{t'-1} 2\delta^t + 3\delta^{t'} \sum_{t=0}^{\infty} \delta^{2t} \\
&= \sum_{t=1}^{t'-1} 2\delta^t + 3\delta^{t'} \frac{1}{1-\delta^2}
\end{aligned}
$$

– Case 2: Player 1 also follows the tit-for-tat strategy.
  In this case, both players always choose to cooperate. Therefore, the total time-discounted payoff for player 1 would be

$$U_2 = 2 + 2\delta + 2\delta^2 + ... = \sum_{t=1}^{t'-1} 2\delta^t + 2\delta^{t'} \frac{1}{1-\delta}$$

Using one-deviation property, as long as $U_2 \geq U_1$, tit-for-tat is a subgame perfect equilibrium.

$$\begin{aligned} & U_2 \geq U_1 \\ \Rightarrow & \frac{2}{1-\delta} \geq \frac{3}{1-\delta^2} \\ \Rightarrow & 2\delta^2 - 3\delta + 1 \leq 0 \\ \Rightarrow & (2\delta - 1)(\delta - 1) \leq 0 \\ \Rightarrow & 1/2 \leq \delta \leq 1 \end{aligned}$$

Therefore, tit-for-tat is a subgame perfect equilibrium if $\delta \geq 1/2$ (recall $\delta < 1$ in our definition).

2. **Incentivizing Effort with Scoring Rules.**

   (a) Using Bayes' theorem,

$$Pr(Rainy|R) = \frac{Pr(R|Rainy)Pr(Rainy)}{Pr(R)} = \frac{0.8 * 0.5}{Pr(R)} = \frac{0.4}{Pr(R)}$$

   Similarly we can get $Pr(Not\ Rainy|R) = 0.15/Pr(R)$. Note that since $Pr(Not\ Rainy|R) + Pr(Rainy|R) = 1$, we can solve $Pr(R) = 0.4 + 0.15 = 0.55$. Therefore,

$$Pr(Rainy|R) = \frac{0.4}{0.55} = \frac{8}{11}$$

   (b) Again, using Bayes' theorem,

$$Pr(Rainy|RN) = \frac{Pr(RN|Rainy)Pr(Rainy)}{Pr(RN)} = \frac{2Pr(R|Rainy)P(N|Rainy)Pr(Rainy)}{Pr(RN)} = \frac{0.16}{Pr(RN)}$$

   The additional factor of 2 is the binomial factor (remember, the probability of seeing one head and one tail in two coin flips is $2*0.5*0.5 = 0.5$).[1] Similarly, we can calculate $Pr(Not\ Rainy|R\&N) = 0.21/Pr(R\&N)$ and solve $Pr(R\&N) = 0.37$. Therefore,

$$Pr(Rainy|R\&N) = \frac{0.16}{0.37} = \frac{16}{37}$$

   (c) Using the same idea in (b), we can build a useful table as in Table 1. The value in each cell represents the conditional probability of Rainly / Not Rainy given the signals.

	Rainy	Not Rainy
R	$\frac{8}{11}$	$\frac{3}{11}$
N	$\frac{2}{9}$	$\frac{7}{9}$
RR	$\frac{64}{73}$	$\frac{9}{73}$
RN	$\frac{16}{37}$	$\frac{21}{37}$
NN	$\frac{4}{53}$	$\frac{49}{53}$

Table 1: Each cell in the table is the conditional probability of Rainy (Not Rainy) given the signals

   If the expert initially gets the R signal, her belief on rainy and not rainy would be $8/11$ and $3/11$. Therefore, her expected payoff if paid using the scoring rule would be

$$U_0 = \frac{8}{11}(\log\frac{8}{11} + b) + \frac{3}{11}(\log\frac{3}{11} + b) = b + \frac{8}{11}\log\frac{8}{11} + \frac{3}{11}\log\frac{3}{11}$$

   If the expert acquires an additional signal, we can separately discuss the cases on whether the true state is Rainy or Not Rainy, calculate the payoff the expert expects to receive in each state, and weighted them according to the expert's beliefs. I'm giving the full arguments below. The key intuition is just to list all cases and calculate the payoff generated by each case.

   – with probability $8/11$, she believes the true state is Rainy
     * given the state is Rainy, she draws signal R with probability 80%
       · given she draws another R signal, her beliefs on Rainy is $Pr(Rainy|RR)$, which is $64/73$ according to Table 1.

[1] In fact, not including the factor shouldn't change your answer, as the same factor will also appear in the denominator and will be cancelled out.

- the expert's expected payoff contributed by this part would be $8/11*(0.8*\log(64/73))$. Note that in this part, we are under the condition that the true state is Rainy, so we only need to consider the payoff contributed by Rainy.
    - given the state is Rainy, she draws signal N with probability 20%
      - given she draws another N signal, her beliefs on Rainy is $Pr(Rainy|RN)$, which is $16/37$ according to Table 1.
      - the expert's expected payoff contributed by this part would be $8/11*(0.2*\log(16/37))$
  - with probability $3/11$, she believes the true state is Not Rainy, and we can repeat the above process.
    - given the state is Not Rainy, she draws signal R with probability 30%
      - given she draws another R signal, her beliefs on Not Rainy is $Pr(Not\ Rainy|RR)$, which is $9/73$ according to Table 1.
      - the expert's expected payoff contributed by this part would be $3/11*(0.3*\log(9/73))$. Note that in this part, we are under the condition that the true state is Not Rainy, so we only need to consider the payoff contributed by Not Rainy.
    - given the state is Not Rainy, she draws signal N with probability 70%
      - given she draws another N signal, her beliefs on Not Rainy is $Pr(Not\ Rainy|RN)$, which is $21/37$ according to Table 1.
      - the expert's expected payoff contributed by this part would be $3/11*(0.7*\log(16/37))$

Therefore, the expert's expected utility for acquiring another signal is

$$U_1 = b - c + \frac{8}{11}\left(0.8\log\frac{64}{73} + 0.2\log\frac{16}{37}\right) + \frac{3}{11}\left(0.3\log\frac{9}{73} + 0.7\log\frac{21}{37}\right)$$

The expert is only willing to acquire another signal if $U_1 \geq U_0$, which means if $c < 0.108066$, when she initially obtains a signal R.

If the expert initially obtains a signal N, we can apply the same analysis.

$$U_0 = b + \frac{2}{9}\log\frac{2}{9} + \frac{7}{9}\log\frac{7}{9}$$

and also we have

$$U_1 = b - c + \frac{2}{9}\left(0.8\log\frac{16}{37} + 0.2\log\frac{4}{53}\right) + \frac{7}{9}\left(0.3\log\frac{21}{37} + 0.7\log\frac{49}{53}\right)$$

Solving $U_1 \geq U_0$, we know that when the expert initially obtains a signal N, the expert is willing to acquire another signal if $c < 0.0909432$.

(d) When $a = 1$, we know that the expert always chooses to acquire another signal if $c < \min\{0.108066, 0.0909432\} = 0.0909432$. Since a is linear proportional to the difference of payments in acquiring an additional signal or not. We just need to set $a = c/0.0909432$.

3. **Peer Grading and Peer Prediction.**

(a) Using the law of total probability, we can get

$$Pr(G) = Pr(G|Good)Pr(Good) + Pr(G|Bad)Pr(Bad) = 0.8 * 0.8 + 0.4 * 0.2 = 0.72$$
$$Pr(B) = Pr(B|Good)Pr(Good) + Pr(B|Bad)Pr(Bad) = 0.2 * 0.8 + 0.6 * 0.2 = 0.28$$

(b) When a grader receives a signal B, she believes (using Bayes' theorem)

$$Pr(Good|B) = \frac{Pr(B|Good)Pr(Good)}{Pr(B|Good)Pr(Good) + Pr(B|Bad)Pr(Bad)} = \frac{16}{28}$$
$$Pr(Bad|B) = \frac{Pr(B|Bad)Pr(Bad)}{Pr(B|Good)Pr(Good) + Pr(B|Bad)Pr(Bad)} = \frac{12}{28}$$

Therefore, the probability she believes that her opponent receiving signal B would be

$$\frac{16}{28} * 0.2 + \frac{12}{28} * 0.6 = \frac{10.4}{28} \approx 0.3714$$

Assume her opponent truthfully report, then if she also truthfully reports, her expected payoff would be 3.714. However, if she report G even if she got signal B, her expected payoff would be 6.286. Therefore, truthfully reporting is not a best response when all others truthfully report. This means truthfully reporting signals is not the Nash equilibrium.

(c) Again, if the grader receives signal B. Assume all other grades truthfully report. When the grader reports G, her expected payoff is $0.6286 * 10/Pr(G) = 8.73$. When the grader reports B, her expected payoff would be $0.3714 * 10/Pr(B) = 13.26$. Therefore, when the grader receives signal B, truthfully report is the best response when others truthfully report. We can repeat the same argument for the case when the grader receives signal G.

(d) In both mechanisms, if all other players reporting G no matter what the signal is, when you report G, you get a positive point, and when you report B, you get 0 point. Therefore, reporting G is a best response. Therefore, everyone reporting G no matter that the signal is is a Nash equilibrium.