CSE 417T
Introduction to Machine Learning

Lecture 19
Instructor: Chien-Ju (CJ) Ho

Logistics

- Homework 5 is due Apr 19 (Tuesday)

- Exam 2 will be on April 28 (Thursday)
  - Will focus on the topics in the second half of the semester
  - Format / logistics will be similar to Exam 1
  - More details to come

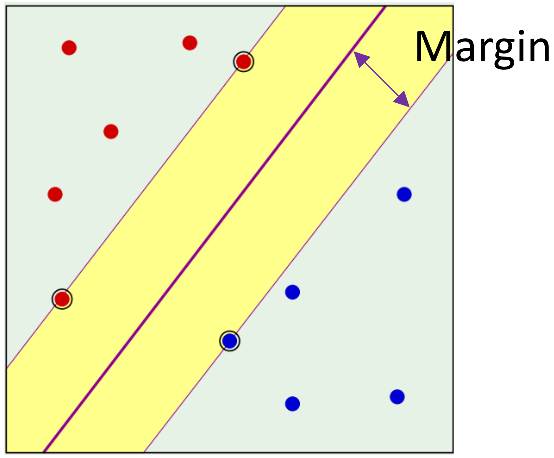
Recap

Support Vector Machines

- Goal: Find the max-margin linear separator
- If the data is linearly separable
  - Hard-Margin SVM (Assume data is linearly separable)

$$\begin{aligned} &\text{minimize}_{\vec{w},b} \quad \frac{1}{2}\vec{w}^T\vec{w} \\ &\text{subject to} \quad y_n(\vec{w}^T\vec{x}_n + b) \geq 1, \forall n \end{aligned}$$

    - $g(\vec{x}) = sign({\vec{w}^*}^T\vec{x} + b^*)$

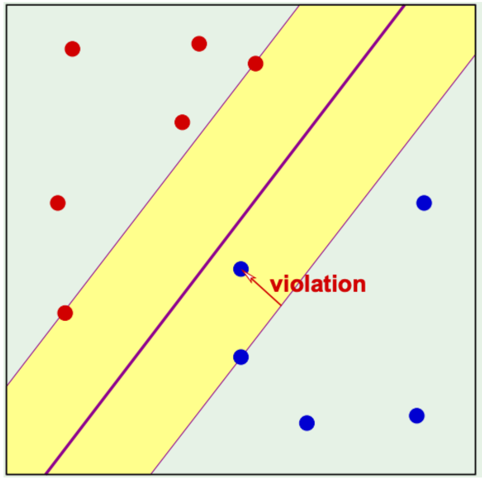


- If the data is not linearly separable
  - Soft-margin SVM
  - Nonlinear transformation – Dual Formulation and Kernel Tricks

Soft-Margin SVM

- For each point $(\vec{x}_n, y_n)$, we allow some violation $\xi_n \geq 0$
  - The constraint becomes: $y_n(\vec{w}^T \vec{x}_n + b) \geq 1 - \xi_n$
  - We add a penalty for each violation: Total penalty $C \sum_{n=1}^{N} \xi_n$

$$\begin{aligned} \text{minimize}_{\vec{w},b,\vec{\xi}} \quad & \frac{1}{2}\vec{w}^T\vec{w} + C\sum_{n=1}^{N}\xi_n \\ \text{subject to} \quad & y_n(\vec{w}^T\vec{x}_n + b) \geq 1 - \xi_n, \forall n \\ & \xi_n \geq 0, \forall n \end{aligned}$$



Remarks:

- C is a hyper-parameter we can choose, e.g., using validation
  - Larger C => less tolerable to noise => smaller margin
- Soft-margin SVM is still a Quadratic Program, with efficient solvers
- ξ_n^* indicates where $\vec{x}_n$ is with respect to the separator and the margin

Primal-Dual Formulations of Hard-Margin SVM

- Primal

$$\begin{aligned} &\text{minimize}_{\vec{w},b} && \frac{1}{2}\vec{w}^T\vec{w} \\ &\text{subject to} && y_n(\vec{w}^T\vec{x}_n + b) \geq 1, \forall n \end{aligned}$$

- Dual

$$\begin{aligned} &\text{maximize}_{\vec{\alpha}} && \textstyle\sum_{n=1}^N \alpha_n - \frac{1}{2}\sum_{n=1}^N \sum_{m=1}^N \alpha_n \alpha_m y_n y_m \vec{x}_n^T \vec{x}_m \\ &\text{subject to} && \textstyle\sum_{n=1}^N \alpha_n y_n = 0 \\ &&& \alpha_n \geq 0, \forall n \end{aligned}$$

- Key messages:
  - Both can be efficiently solved using QP solver
  - We can infer the solution from one to the other

Reminders of definitions in general convex program:

$$\begin{aligned} &\text{minimize}_{\vec{w}} && f(\vec{w}) && \\ &\text{subject to} && g_i(\vec{w}) \leq 0, && i = 1, \dots, k \\ &&& h_j(\vec{w}) = 0, && j = 1, \dots, \ell \end{aligned}$$

$$L(\vec{w}, \vec{\alpha}, \vec{\beta}) = f(\vec{w}) + \textstyle\sum_{i=1}^k \alpha_i g_i(\vec{w}) + \sum_{j=1}^{\ell} \beta_j h_j(\vec{w})$$

Primal: $\min_{\vec{w}} \max_{\vec{\alpha},\vec{\beta};\alpha_i \geq 0} L(\vec{w}, \vec{\alpha}, \vec{\beta})$

Dual: $\max_{\vec{\alpha},\vec{\beta};\alpha_i \geq 0} \min_{\vec{w}} L(\vec{w}, \vec{\alpha}, \vec{\beta})$

Given optimal $\vec{\alpha}^*$:

- $\vec{w}^* = \sum_{\alpha_n^* > 0} \alpha_n^* y_n \vec{x}_n$
- Find a $\alpha_n^* > 0$, $b^* = y_n - \vec{w}^{*T} \vec{x}_n$

Kernel Functions

- Define kernel function $K_{\Phi}(\vec{x}, \vec{x}') = \Phi(\vec{x})^T \Phi(\vec{x}')$ $(= \vec{z}^T \vec{z}')$
  - The similarity of two vectors in the projected space
- Goal: Compute $K_{\Phi}(\vec{x}, \vec{x}')$ without transforming $\vec{x}$ and $\vec{x}'$
- Why? This enables us to operate in higher dimensional spaces without really worrying about the computational overhead.

Kernel Trick: Utilize Dual and Kernel Functions

- The dual with nonlinear transform

$$\begin{aligned} &\text{maximize}_{\vec{\alpha}} \sum_{n=1}^{N} \alpha_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n \alpha_m y_n y_m \vec{z}_n^T \vec{z}_m \\ &\text{subject to} \quad \sum_{n=1}^{N} \alpha_n y_n = 0 \\ &\qquad\qquad\quad\ \alpha_n \geq 0, \forall n \end{aligned}$$

- Plug in the kernel function $K_\Phi(\vec{x}, \vec{x}') = \Phi(\vec{x})^T \Phi(\vec{x}')$

$$\begin{aligned} &\text{maximize}_{\vec{\alpha}} \sum_{n=1}^{N} \alpha_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n \alpha_m y_n y_m K_\Phi(\vec{x}_n, \vec{x}_m) \\ &\text{subject to} \quad \sum_{n=1}^{N} \alpha_n y_n = 0 \\ &\qquad\qquad\quad\ \alpha_n \geq 0, \forall n \end{aligned}$$

  - If the kernel can be computed efficiently, we can solve $\vec{\alpha}^*$ efficiently.
  - With kernel tricks, we can avoid the dependency on the dimension of $\vec{z}$

Recover $(\vec{w}^*, b^*)$ from $\vec{\alpha}^*$ with Kernel Tricks

- Note that $\vec{\alpha}^*$ is solved in the $\vec{z}$ space
  - $\vec{w}^* = \sum_{\alpha_n^*>0} \alpha_n^* y_n \Phi(\vec{x}_n)$
  - Find a $\alpha_n^* > 0,\ b^* = y_n - {\vec{w}^*}^T \Phi(\vec{x}_n)$
  - We want to avoid the transformation!

Instead of storing $(\vec{w}^*, b^*)$, we can store "support vectors" (points with $\alpha_n^* > 0$) and make predictions accordingly.

- Let's look at the hypothesis
  - $g(\vec{x}) = sign\left({\vec{w}^*}^T \Phi(\vec{x}) + b^*\right)$

$$
\begin{aligned}
{\vec{w}^*}^T \Phi(\vec{x}) &= \left(\sum_{\alpha_n^*>0} \alpha_n^* y_n \Phi(\vec{x}_n)\right)^T \Phi(\vec{x}) \\
&= \sum_{\alpha_n^*>0} \alpha_n^* y_n \Phi(\vec{x}_n)^{\mathrm{T}} \Phi(\vec{x}) \\
&= \sum_{\alpha_n^*>0} \alpha_n^* y_n K(\vec{x}_n, \vec{x})
\end{aligned}
$$

$$
\begin{aligned}
b^* &= y_n - {\vec{w}^*}^T \Phi(\vec{x}_n) \\
&= y_n - \left(\sum_{\alpha_m^*>0} \alpha_m^* y_m \Phi(\vec{x}_m)\right)^{\mathrm{T}} \Phi(\vec{x}_n) \\
&= y_n - \sum_{\alpha_m^*>0} \alpha_m^* y_m K(\vec{x}_m, \vec{x}_n)
\end{aligned}
$$

- Still can be computed in the $\vec{x}$ space!

Today’s Lecture

The notes are not intended to be comprehensive. They should be accompanied by lectures and/or textbook.

Let me know if you spot errors.

Kernel Functions

$K_\Phi(\vec{x}, \vec{x}')$: Inner products of two points $\Phi(\vec{x})^T\Phi(\vec{x}')$ in the transformed space

Similarity of two points $\Phi(\vec{x})$ and $\Phi(\vec{x}')$ in the transformed space

Polynomial Kernel

Kernel $K(\vec{x}, \vec{x}') = \Phi(\vec{x})^T \Phi(\vec{x}')$

- Example in the last lecture: 2nd order polynomial for 2-d $\vec{x}$
  - $\vec{x} = (x_1, x_2)$
  - $\vec{z} = \Phi_2(\vec{x}) = (1, \sqrt{2}x_1, \sqrt{2}x_2, \sqrt{2}\, x_1 x_2, x_1^2, x_2^2)$
  - $\vec{z}' = \Phi_2(\vec{x}') = (1, \sqrt{2}x_1', \sqrt{2}x_2', \sqrt{2}\, x_1' x_2', {x'}_1^2, {x'}_2^2)$
  - $$\begin{aligned}\vec{z}^T \vec{z}' &= 1 + 2x_1x_1' + 2x_2x_2' + 2x_1x_1'x_2x_2' + (x_1x_1')^2 + (x_2x_2')^2 \\ &= (1 + x_1x_1' + x_2x_2')^2 \\ &= (1 + \vec{x}^T \vec{x}')^2\end{aligned}$$
- General 2nd order polynomial
  - $\vec{x} = (x_1, x_2, \dots, x_d)$
  - $$\begin{aligned}K_{\Phi_2}(\vec{x}, \vec{x}') &= (1 + \vec{x}^T \vec{x}')^2 \\ &= (1 + x_1x_1' + x_2x_2' + \cdots + x_dx_d')^2\end{aligned}$$

Polynomial Kernel

General form of polynomial kernel:
$K(\vec{x}, \vec{x}') = (a\vec{x}^T\vec{x}' + b)^Q$

- $\vec{x} = (x_1, x_2, \ldots, x_d)$

- 2^{nd} order polynomial kernel $K_{\Phi_2}(\vec{x}, \vec{x}') = (1 + \vec{x}^T\vec{x}')^2$
- Q-th order Polynomial kernel $K_{\Phi_Q}(\vec{x}, \vec{x}') = (1 + \vec{x}^T\vec{x}')^Q$

$$= (1 + x_1 x_1' + \cdots + x_d x_d')^Q$$

- Computational complexity
  - Dimension of $\Phi_Q(\vec{x})$: $\binom{Q+d}{Q}$
  - Direct computation of $\Phi_Q(\vec{x})^T \Phi_Q(\vec{x}')$: $O\left(\binom{Q+d}{Q}\right)$
  - Computation through kernel $K_{\Phi_Q}(\vec{x}, \vec{x}')$: $O(d)$

We Only Need $\vec{z}$ Space to Exist

- In the discussion of polynomial kernels
  - We have a target transformation in mind
  - We want to find a corresponding kernel function

- In fact, as long as $K(\vec{x}, \vec{x}')$ is an inner product in **some** $\vec{z}$ space, we are good
  - Just plug in the kernel in the dual formulation
  - We obtain a linear separator in the corresponding $\vec{z}$ space

$$\begin{aligned} &\text{maximize}_{\vec{\alpha}} \textstyle\sum_{n=1}^{N} \alpha_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} \alpha_n\alpha_m y_n y_m K_\Phi(\vec{x}_n, \vec{x}_m) \\ &\text{subject to } \textstyle\sum_{n=1}^{N} \alpha_n y_n = 0 \\ &\qquad\qquad\quad \alpha_n \geq 0, \forall n \end{aligned}$$

Gaussian RBF Kernel

- $K(\vec{x}, \vec{x}') = e^{-\gamma\|\vec{x}-\vec{x}'\|^2}$
- What's the corresponding $\vec{z}$ space? (What is Φ such that $\Phi(\vec{x})^T\Phi(\vec{x}') = e^{-\gamma\|\vec{x}-\vec{x}'\|^2}$)
  - For illustrative purpose, make $\vec{x} = x$ be 1 dimensional and $\gamma = 1$
    - $K(\vec{x}, \vec{x}') = e^{-(x-x')^2}$

$$
\begin{aligned}
&= e^{-x^2+2xx'-x'^2} \\
&= e^{-x^2}e^{-x'^2}e^{2xx'} \\
&= e^{-x^2}e^{-x'^2}\sum_{k=0}^{\infty}\frac{(2xx')^k}{k!} \\
&= \sum_{k=0}^{\infty} e^{-x^2}\sqrt{\frac{2^k}{k!}}x^k e^{-x'^2}\sqrt{\frac{2^k}{k!}}x'^k
\end{aligned}
$$

Taylor expansion: $e^{2xx'} = \sum_{k=0}^{\infty}\frac{(2xx')^k}{k!}$

  - The corresponding $\Phi(x) = e^{-x^2}\left(1, \sqrt{\frac{2}{1}}x, \sqrt{\frac{2^2}{2!}}x^2, \ldots\right)$

Gaussian RBF Kernel

- $K(\vec{x}, \vec{x}') = e^{-\gamma\|\vec{x}-\vec{x}'\|^2}$
- The corresponding transform in 1-dim input $\vec{x} = x$
  - $\Phi(x) = e^{-x^2}\left(1, \sqrt{\frac{2}{1}}x, \sqrt{\frac{2^2}{2!}}x^2, \dots\right)$
- $K(\vec{x}, \vec{x}')$ is the inner product of two vectors in an infinite dimensional space!
- When we plug in $K(\vec{x}, \vec{x}')$ in dual SVM
  - We are finding the max-margin separator in an infinite dimensional space
  - Seems to introduce infinite generalization error?
    - Maximizing margin help mitigate this issue
    - The number of support vectors provides indicators on the generalization

Design Your Own Kernel? [Safe to Skip]

- Say we design a kernel function, how do we know whether it is valid, i.e., whether there is a corresponding $\vec{z}$ space?

- Mercer's condition (See discussion in LFD 8.3.2)
  - Kernel matrix

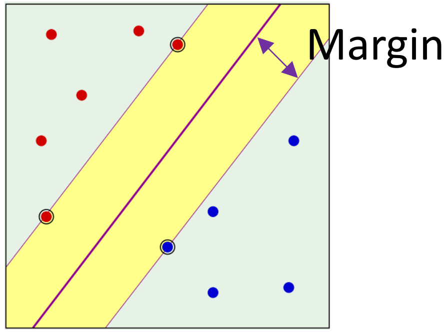
$$\begin{bmatrix} K(\mathbf{x}_1,\mathbf{x}_1) & K(\mathbf{x}_1,\mathbf{x}_2) & \dots & K(\mathbf{x}_1,\mathbf{x}_N) \\ K(\mathbf{x}_2,\mathbf{x}_1) & K(\mathbf{x}_2,\mathbf{x}_2) & \dots & K(\mathbf{x}_2,\mathbf{x}_N) \\ \dots & \dots & \dots & \dots \\ K(\mathbf{x}_N,\mathbf{x}_1) & K(\mathbf{x}_N,\mathbf{x}_2) & \dots & K(\mathbf{x}_N,\mathbf{x}_N) \end{bmatrix}$$

  - $K(\vec{x}, \vec{x}')$ is a valid kernel if and only if the kernel matrix is always symmetric positive semi-definite for any $\vec{x}_1, \dots, \vec{x}^N$

Summary of What We Talked About So Far

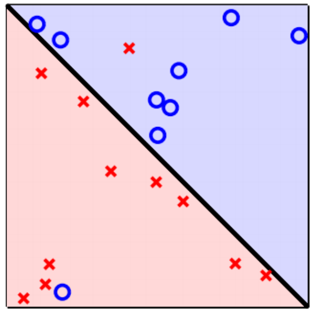
Hard-Margin SVM (Separable Data)

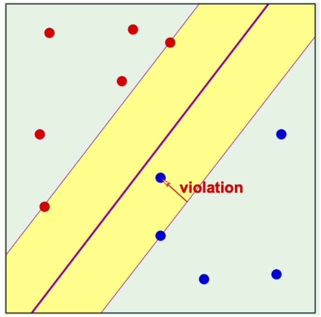
$$\text{minimize}_{\vec{w},b} \quad \frac{1}{2}\vec{w}^T\vec{w}$$
$$\text{subject to} \quad y_n(\vec{w}^T\vec{x}_n + b) \geq 1, \forall n$$



Soft-Margin SVM (Tolerate Noise)

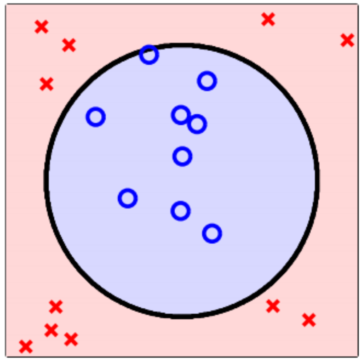
$$\text{minimize}_{\vec{w},b,\vec{\xi}} \quad \frac{1}{2}\vec{w}^T\vec{w} + C\sum_{n=1}^{N}\xi_n$$
$$\text{subject to} \quad y_n(\vec{w}^T\vec{x}_n + b) \geq 1 - \xi_n, \forall n$$
$$\xi_n \geq 0, \forall n$$





Kernel Formulation of Hard-Margin SVM

$$\text{maximize}_{\vec{\alpha}} \sum_{n=1}^{N}\alpha_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\alpha_m y_n y_m K_\Phi(\vec{x}_n, \vec{x}_m)$$
$$\text{subject to} \quad \sum_{n=1}^{N}\alpha_n y_n = 0$$
$$\alpha_n \geq 0, \forall n$$



Kernel Version of Soft-Margin SVM

- Soft-Margin SVM

$$
\begin{aligned}
&\text{minimize}_{\vec{w},b,\vec{\xi}} && \frac{1}{2}\vec{w}^T\vec{w} + C\textstyle\sum_{n=1}^{N}\xi_n \\
&\text{subject to} && y_n(\vec{w}^T\vec{x}_n + b) \geq 1 - \xi_n, \forall n \\
& && \xi_n \geq 0, \forall n
\end{aligned}
$$

- Kernel Version of Soft-Margin SVM

$$
\begin{aligned}
&\text{maximize}_{\vec{\alpha}} \textstyle\sum_{n=1}^{N}\alpha_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\alpha_m y_n y_m K_\Phi(\vec{x}_n, \vec{x}_m) \\
&\text{subject to } \textstyle\sum_{n=1}^{N}\alpha_n y_n = 0 \\
&\qquad\qquad\quad 0 \leq \alpha_n \leq C, \forall n
\end{aligned}
$$

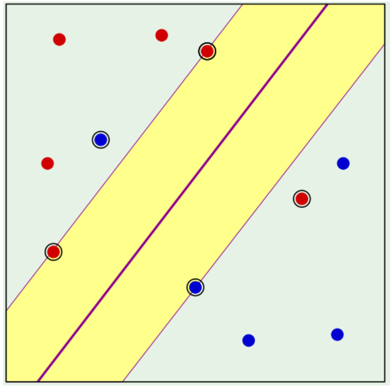
  - It can be obtained by similar procedure as hard-margin version
  - We can obtain the same relationship between $\vec{\alpha}^*$ and $(\vec{w}^*, b^*)$

Interpretation of Support Vectors

- $\alpha_n^* > 0$ => $(\vec{x}_n, y_n)$ is a support vector
  - $y_n\left(\vec{w}^{*T}\vec{x}_n + b^*\right) = 1 - \xi_n$
- Utilizing complementary slackness
  - When $0 < \alpha_n^* < C$
    - $\xi_n = 0$
    - $y_n\left(\vec{w}^{*T}\vec{x}_n + b^*\right) = 1$
    - $(\vec{x}_n, y_n)$ is a "margin" support vector
  - When $\alpha_n^* = C$
    - $\xi_n > 0$
    - $y_n\left(\vec{w}^{*T}\vec{x}_n + b^*\right) < 1$
    - $(\vec{x}_n, y_n)$ is a "non-margin" support vector

$$\begin{aligned} &\text{minimize}_{\vec{w},b,\vec{\xi}} && \frac{1}{2}\vec{w}^T\vec{w} + C\sum_{n=1}^{N}\xi_n \\ &\text{subject to} && y_n(\vec{w}^T\vec{x}_n + b) \geq 1 - \xi_n, \forall n \\ &&& \xi_n \geq 0, \forall n \end{aligned}$$

$$\begin{aligned} &\text{maximize}_{\vec{\alpha}} \sum_{n=1}^{N}\alpha_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\alpha_m y_n y_m K_\Phi(\vec{x}_n, \vec{x}_m) \\ &\text{subject to } \sum_{n=1}^{N}\alpha_n y_n = 0 \\ &\qquad 0 \leq \alpha_n \leq C, \forall n \end{aligned}$$



Another Look at Primal vs. Dual SVM

- Primal

$$\begin{aligned} &\text{minimize}_{\vec{w},b} && \frac{1}{2}\vec{w}^T\vec{w} \\ &\text{subject to} && y_n(\vec{w}^T\vec{z}_n + b) \geq 1, \forall n \end{aligned}$$

  - Learned hypothesis
    - $g(\vec{x}) = sign\left(\vec{w}^{*^T}\Phi(\vec{x}) + b^*\right)$

- Dual

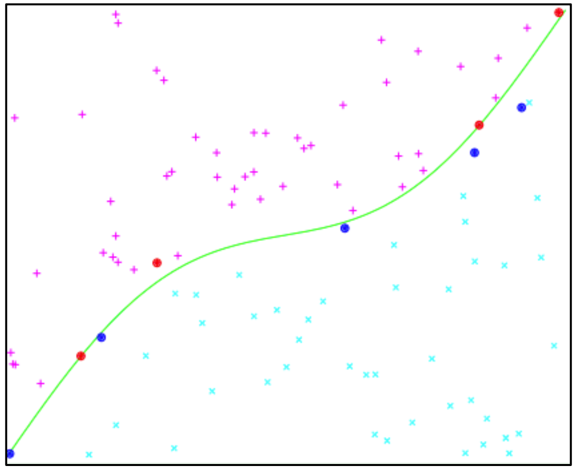
$$\begin{aligned} &\text{maximize}_{\vec{\alpha}} && \textstyle\sum_{n=1}^{N}\alpha_n - \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\alpha_n\alpha_m y_n y_m \vec{z}_n^T\vec{z}_m \\ &\text{subject to} && \textstyle\sum_{n=1}^{N}\alpha_n y_n = 0 \\ &&& \alpha_n \geq 0, \forall n \end{aligned}$$

  - Learned hypothesis
    - $g(\vec{x}) = sign(\sum_{\alpha_n^*>0}\alpha_n^* y_n K(\vec{x}_n, \vec{x}) + b^*))$
    - ($\alpha_n^* > 0 \Rightarrow \vec{x}_n$ is a support vector)

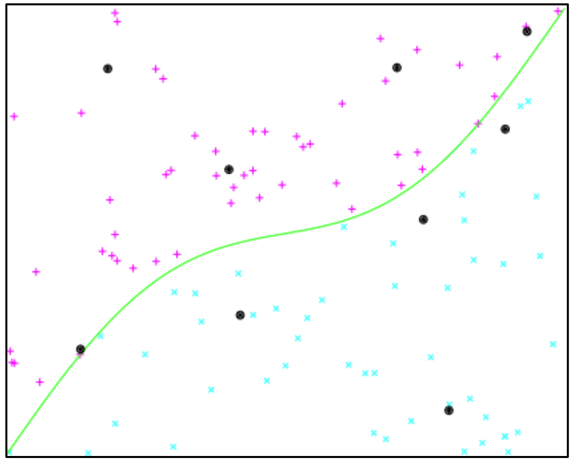
- Primal view of SVM (parametric)
  - We are learning the weights for SVM, i.e., $(\vec{w}^*, b^*)$
  - When using RBF Kernel, there are infinite number of parameters
- Dual kernel view of SVM (nonparametric)
  - We are learning the support vectors, and use those for prediction

Kernel SVM and Radial Basis Functions

- Kernel SVM
  - $g(\vec{x}) = sign\left(\sum_{\alpha_n^*>0} \alpha_n^* y_n K(\vec{x}_n, \vec{x}) + b^*\right)$
  - Use support vectors to characterize a hypothesis



- Radial Basis Functions
  - $h(\vec{x}) = sign\left(\sum_{k=1}^{K} w_k \, \phi\left(\frac{\|\vec{x}-\vec{\mu}_k\|}{r}\right)\right)$
  - Use cluster centers to characterize a hypothesis



Neural Networks

Perceptron

- What is a hypothesis in Perceptron

$$h(\vec{x}) = sign(\vec{w}^T \vec{x})$$

- Note that we have reverted back to our original notations

  - $\vec{x} = (x_0, x_1, \dots, x_d)$
  - $\vec{w} = (w_0, w_1, \dots, w_d)$
  - Linear separator

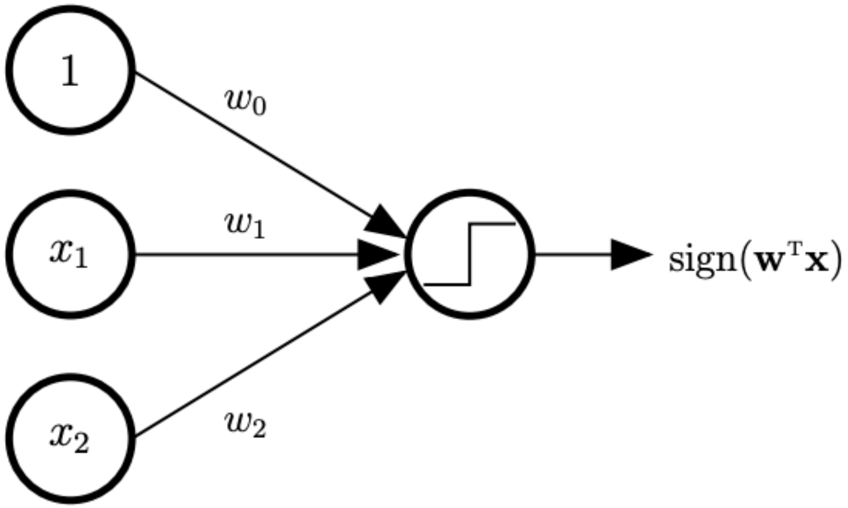
  $$h(\vec{x}) = sign(\vec{w}^T \vec{x})$$

Perceptron

- What is a hypothesis in Perceptron

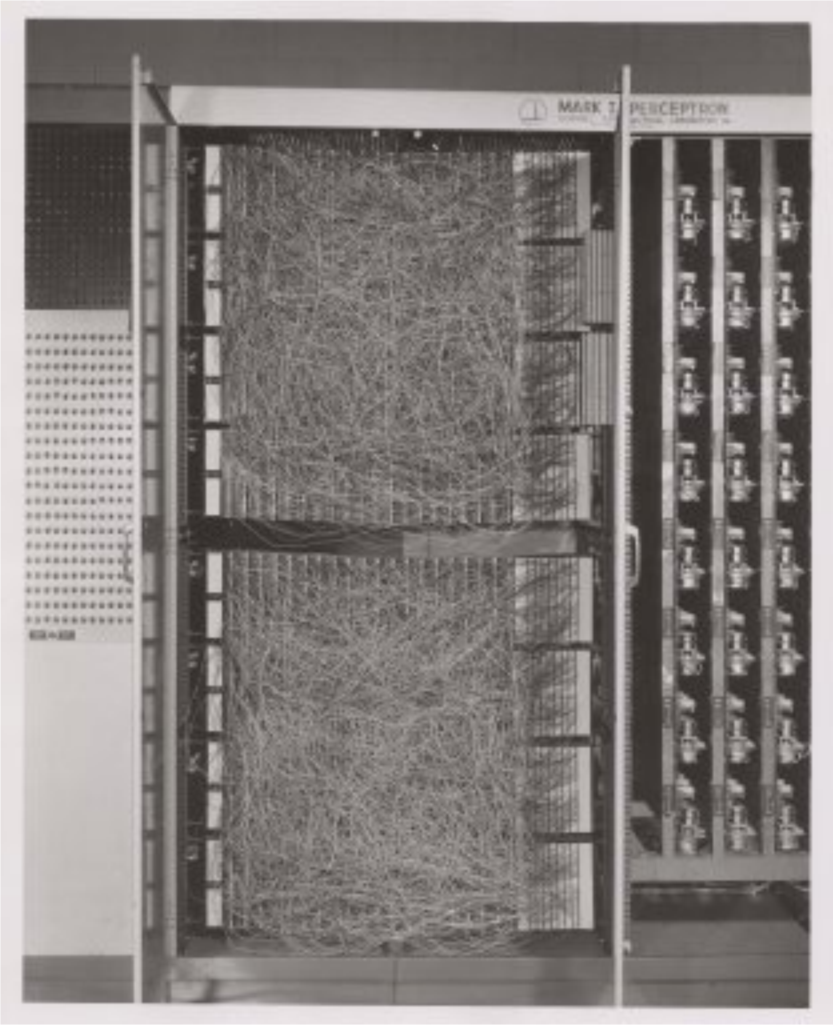
$$h(\vec{x}) = sign(\vec{w}^T\vec{x})$$

- Graphical representation of Perceptron



Inspired by neurons:
The output signal is triggered when the weighted combination of the inputs is larger than some threshold

The First Perceptron Machine



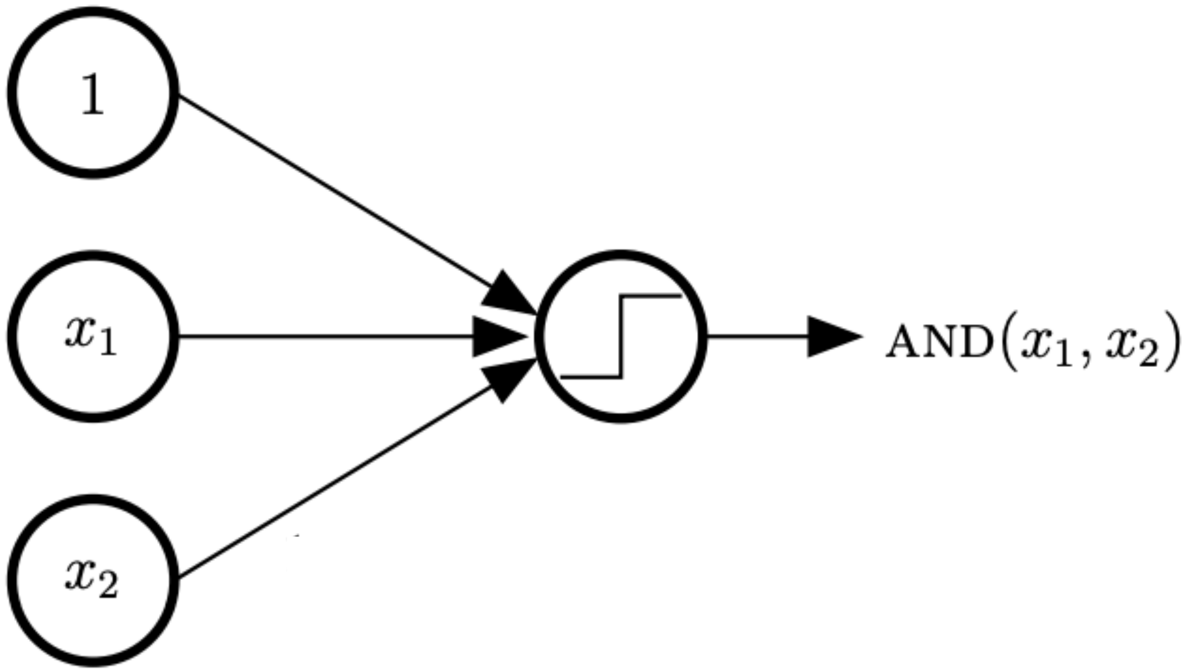
Mark I Perceptron machine, the first implementation of the perceptron algorithm. (From Wikipedia)

"the embryo of an electronic computer that [the Navy] expects will be able to walk, talk, see, write, reproduce itself and be conscious of its existence." [1958]

Implement Logic Gates with Perceptron

- AND(x_1, x_2)
  - Use $+1$ to denote "true" and -1 to denote "false"

x_1	x_2	AND(x_1, x_2)
+1	+1	+1
+1	-1	-1
-1	+1	-1
-1	-1	-1



Implement Logic Gates with Perceptron

- AND(x_1, x_2)
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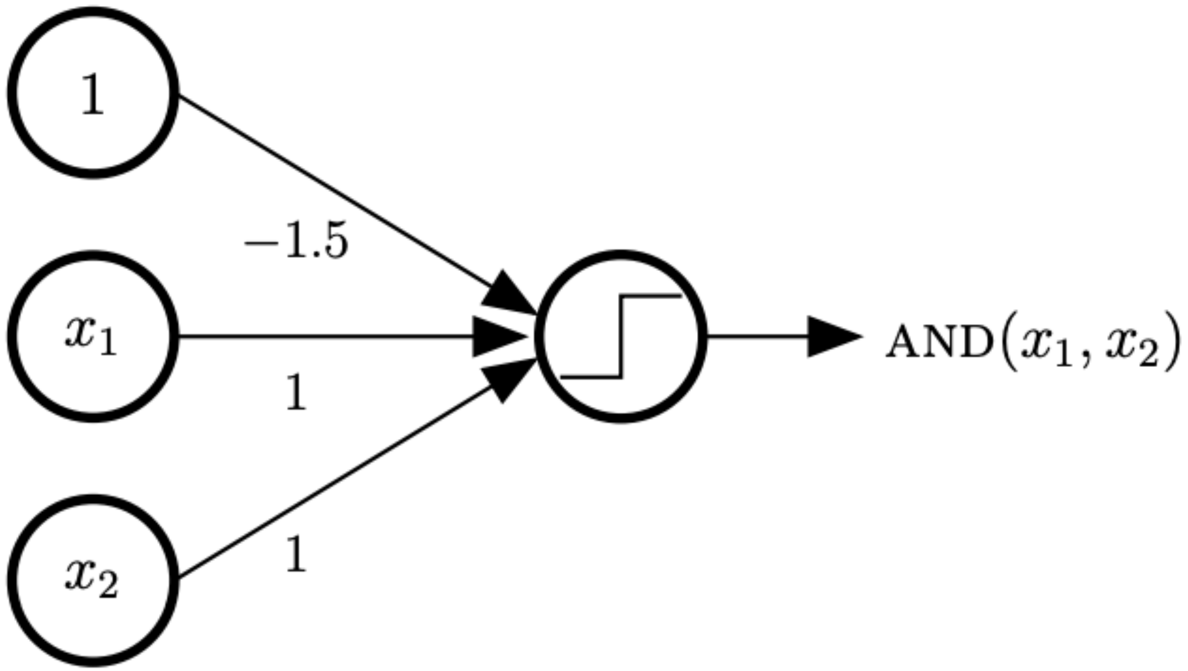
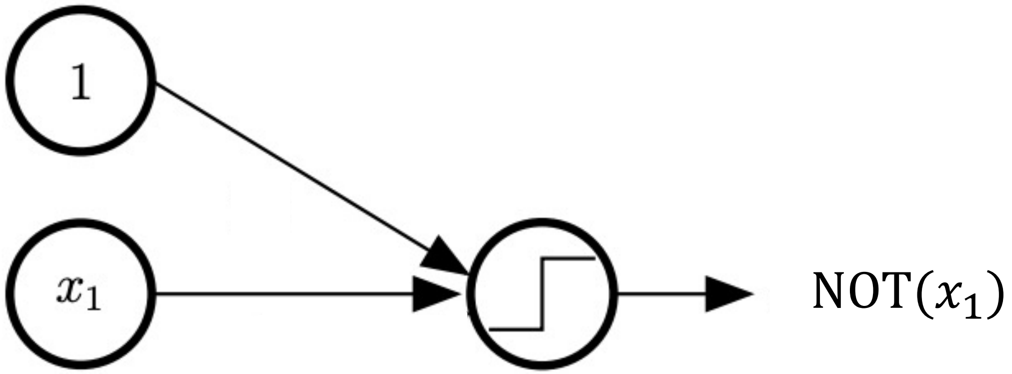
x_1	x_2	AND(x_1, x_2)
+1	+1	+1
+1	-1	-1
-1	+1	-1
-1	-1	-1



Implement Logic Gates with Perceptron

- NOT(x_1)
  - Use $+1$ to denote "true" and -1 to denote "false"

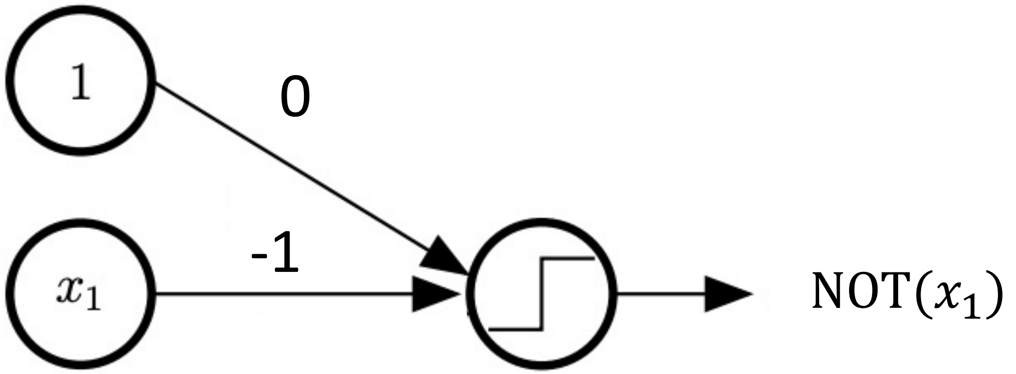
x_1	NOT(x)
+1	-1
-1	+1



Implement Logic Gates with Perceptron

- NOT(x_1)
  - Use $+1$ to denote “true” and -1 to denote “false”

x_1	OR(x)
+1	-1
-1	+1



Practice: How to Implement OR and XOR?

- Use $+1$ to denote "true" and -1 to denote "false"

- $\text{OR}(x_1, x_2)$

x_1	x_2	$\text{OR}(x_1, x_2)$
+1	+1	+1
+1	-1	+1
-1	+1	+1
-1	-1	-1

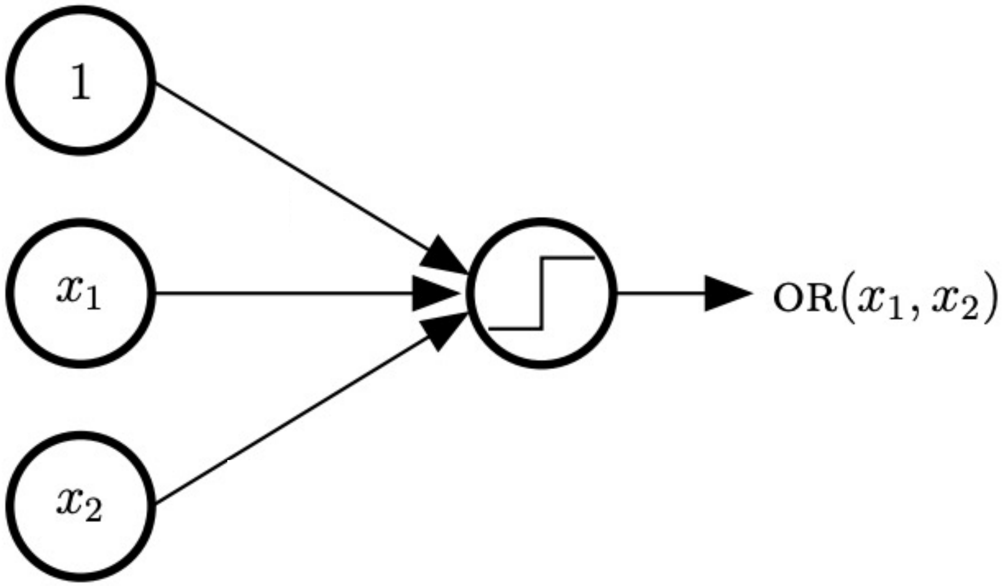
- $\text{XOR}(x_1, x_2)$

x_1	x_2	$\text{XOR}(x_1, x_2)$
+1	+1	-1
+1	-1	+1
-1	+1	+1
-1	-1	-1

Implement Logic Gates with Perceptron

- OR(x_1, x_2)
  - Use $+1$ to denote "true" and -1 to denote "false"

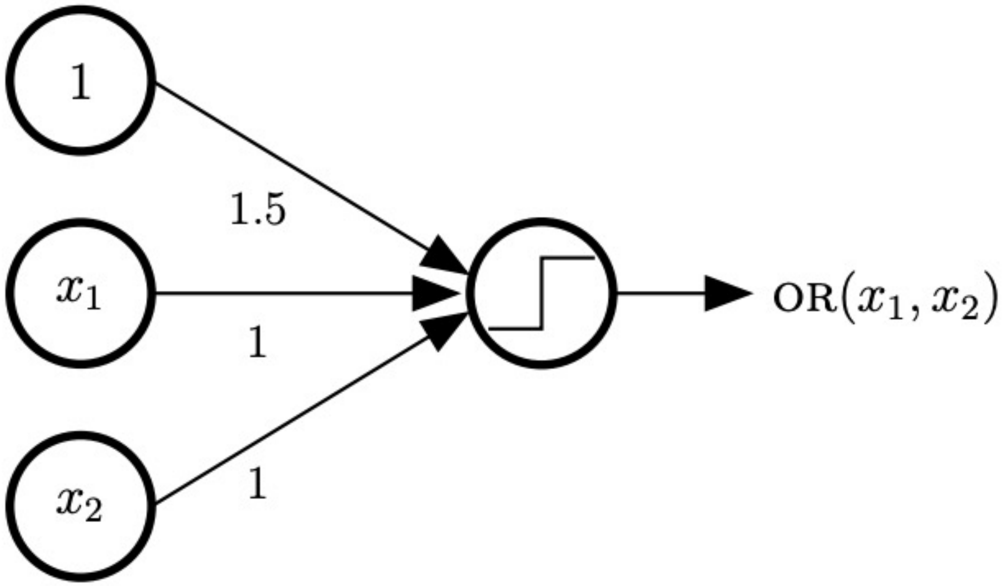
x_1	x_2	OR(x_1, x_2)
+1	+1	+1
+1	-1	+1
-1	+1	+1
-1	-1	-1



Implement Logic Gates with Perceptron

- OR(x_1, x_2)
  - Use $+1$ to denote "true" and -1 to denote "false"

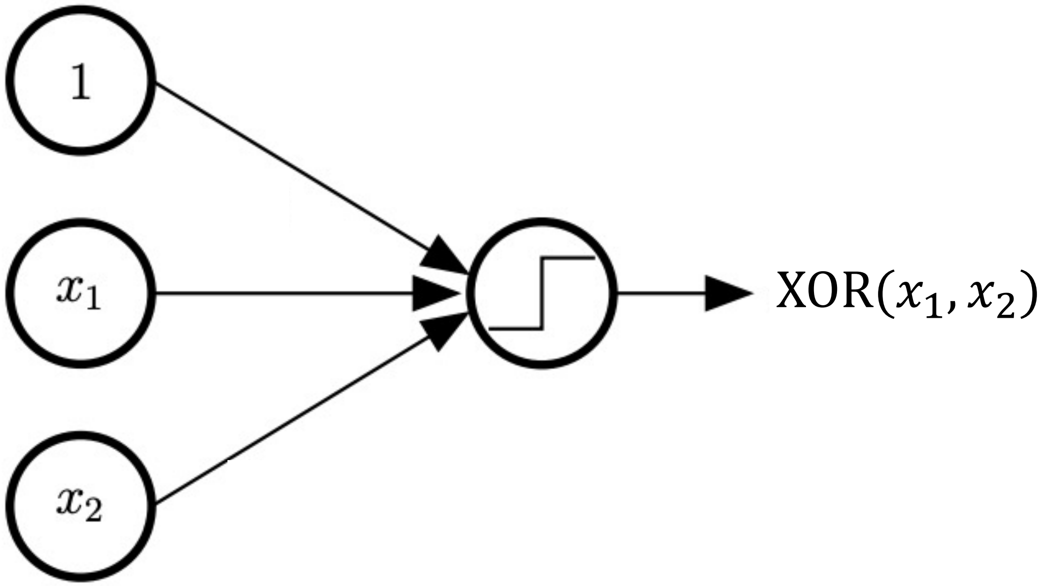
x_1	x_2	OR(x_1, x_2)
+1	+1	+1
+1	-1	+1
-1	+1	+1
-1	-1	-1



Implement Logic Gates with Perceptron

- XOR(x_1, x_2)
  - Use $+1$ to denote "true" and -1 to denote "false"

x_1	x_2	XOR(x_1, x_2)
+1	+1	-1
+1	-1	+1
-1	+1	+1
-1	-1	-1



Implement Logic Gates with Perceptron

- XOR(x_1, x_2)
  - Use $+1$ to denote "true" and -1 to denote "false"

x_1	x_2	XOR(x_1, x_2)
+1	+1	-1
+1	-1	+1
-1	+1	+1
-1	-1	-1

It is impossible to implement XOR
using a single perceptron
(draw the points in the 2-D space,
you will see they are not linearly separable)

Stronger version:
It is impossible to implement XOR
using a single layer of perceptrons

Multi-Layer Perceptron

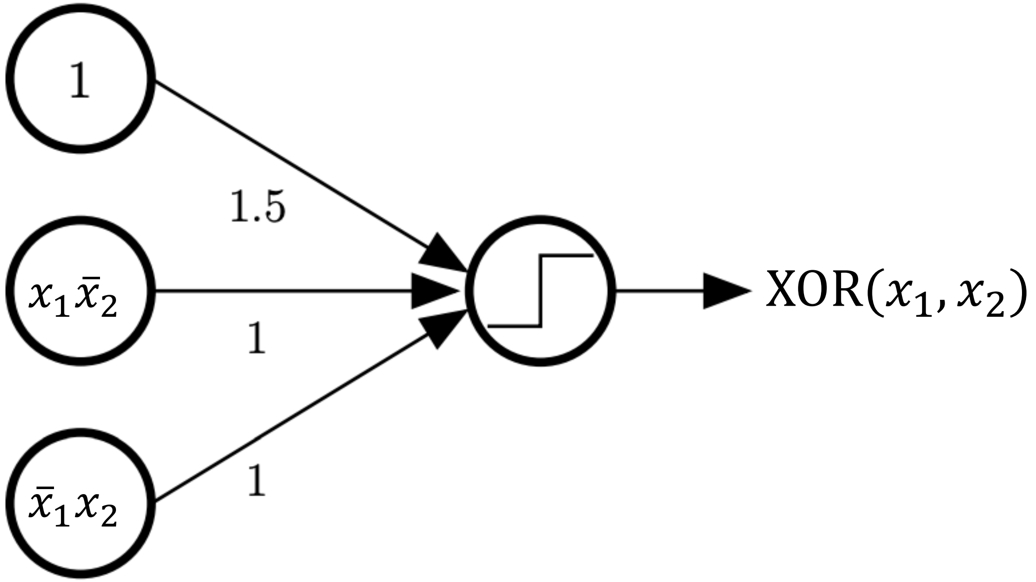
Representing Boolean Operations

- $\text{AND}(x_1, x_2) \rightarrow x_1 x_2$
- $\text{OR}(\mathrm{x}_1, x_2) \rightarrow x_1 + x_2$
- $\text{NOT}(x_1) \rightarrow \bar{x}_1$

- $\text{XOR}(x_1, x_2) \rightarrow x_1 \bar{x}_2 + \bar{x}_1 x_2$

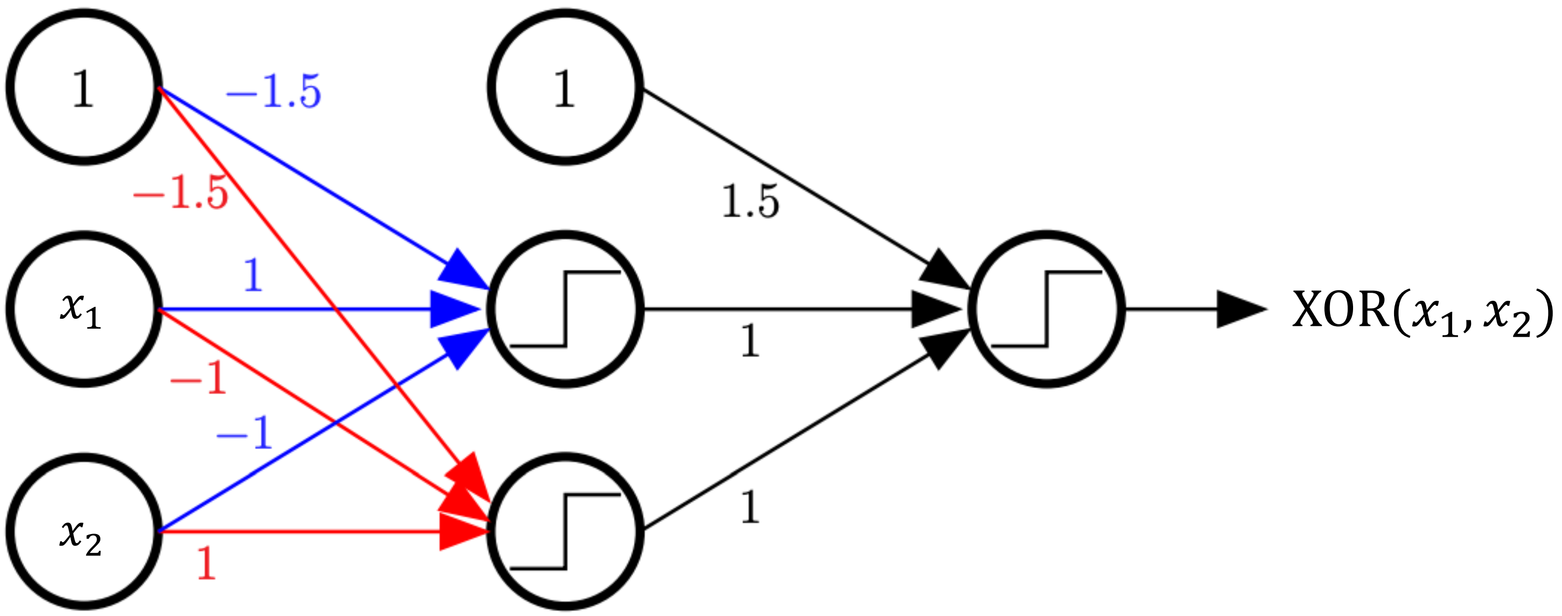
Implementing XOR

- $\text{XOR}(x_1, x_2) \rightarrow x_1\bar{x}_2 + \bar{x}_1x_2$

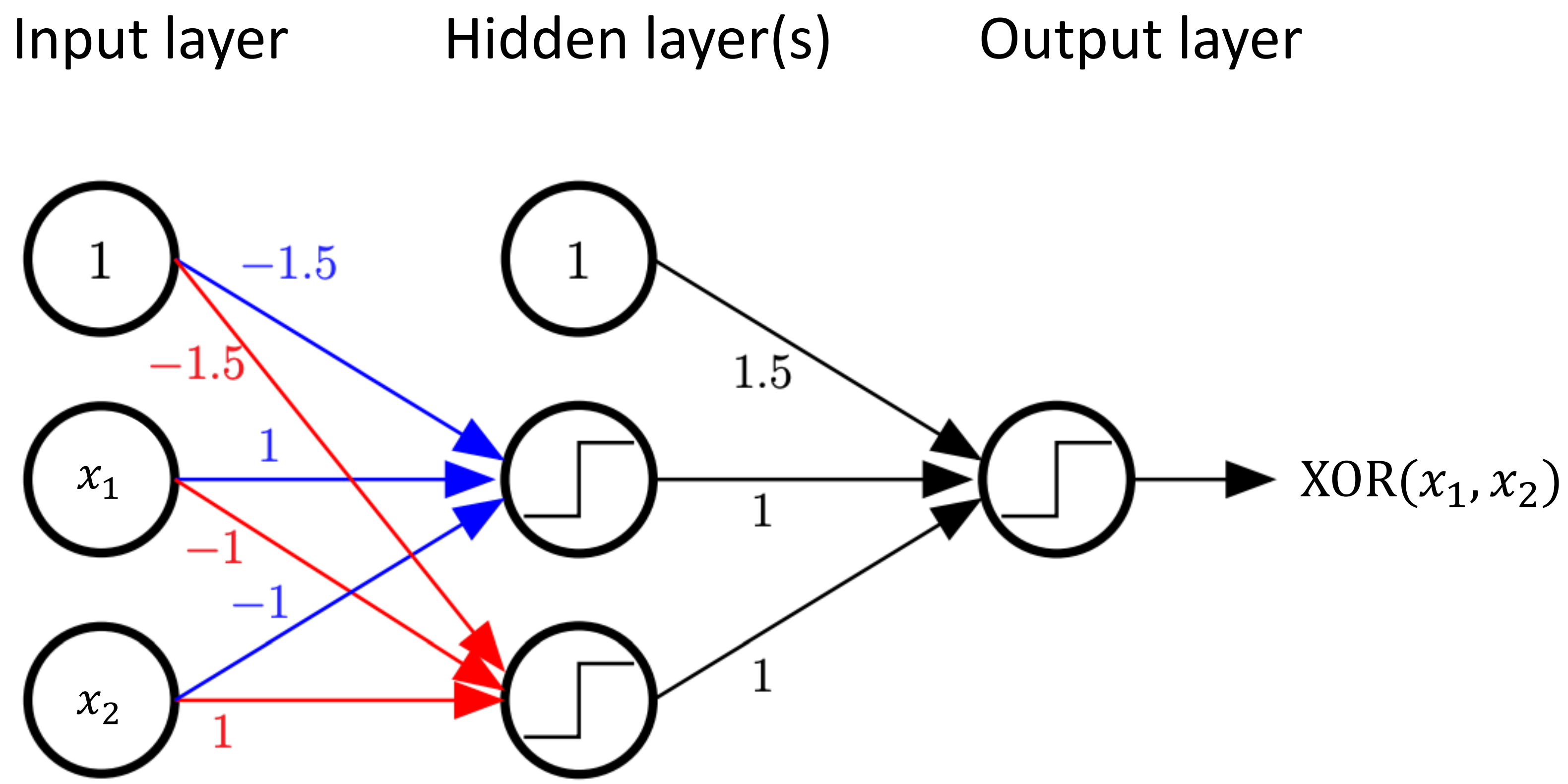


Implementing XOR

- $\text{XOR}(x_1, x_2) \rightarrow x_1\bar{x}_2 + \bar{x}_1 x_2$



Multi-Layer Perceptron (MLP)



Feed-forward network

The Power of Multi-Layer Perceptron (MLP)

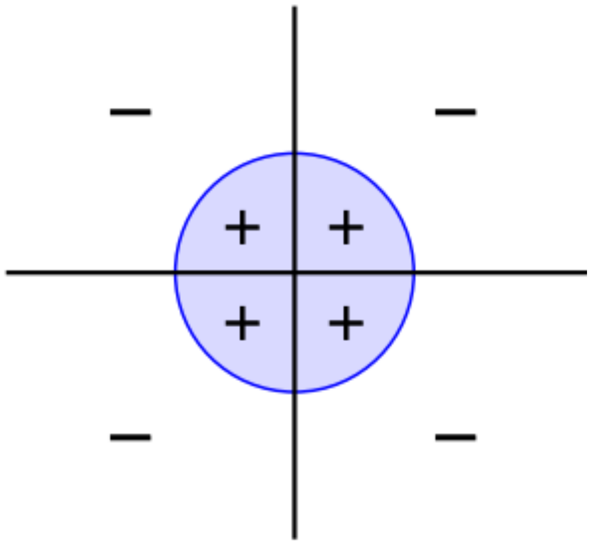
- We now know that we can implement XOR by introducing the hidden layer in MLP. But generally how powerful is MLP?

- Universal approximation theorem
  - a feed-forward network with a single hidden layer containing a finite number of neurons can approximate continuous functions on compact subsets of $\mathbb{R}^n$, under mild assumptions on the activation function.

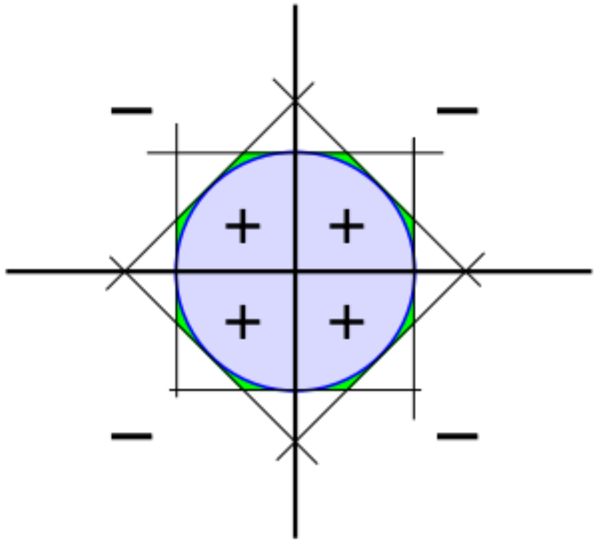
- Three-layer MLP can approximate ANY continuous target function!

Informal Intuitions of Universal Approximation

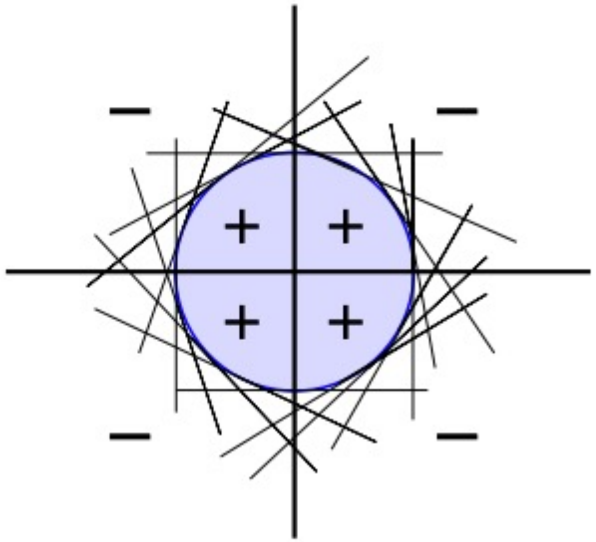
- A continuous separator can be ”decomposed” into linear separators



Target



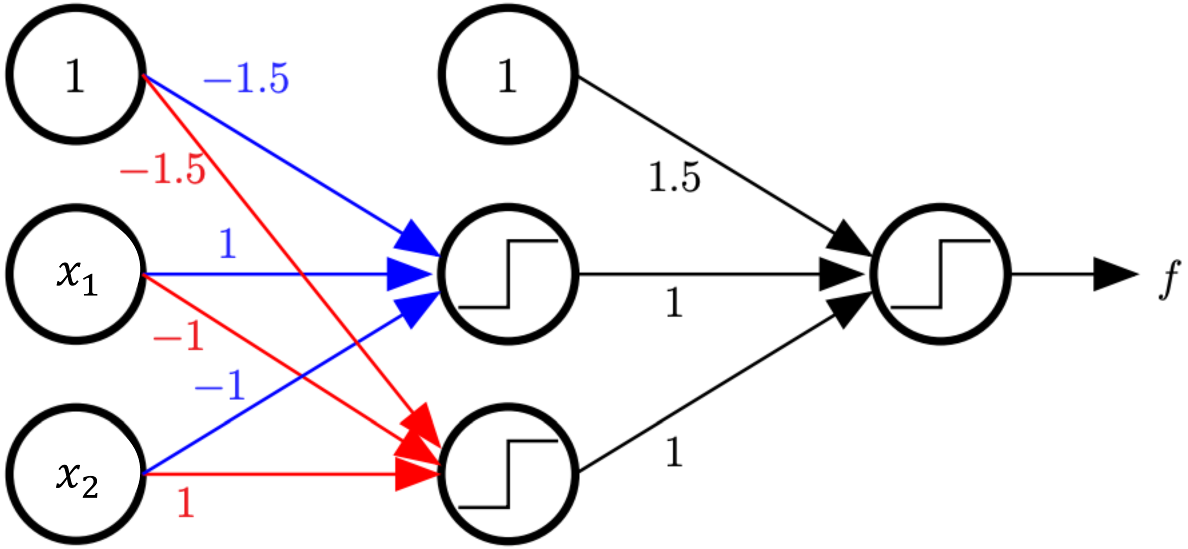
8 perceptrons

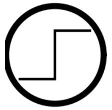


16 perceptrons

How to Learn MLP From Data?

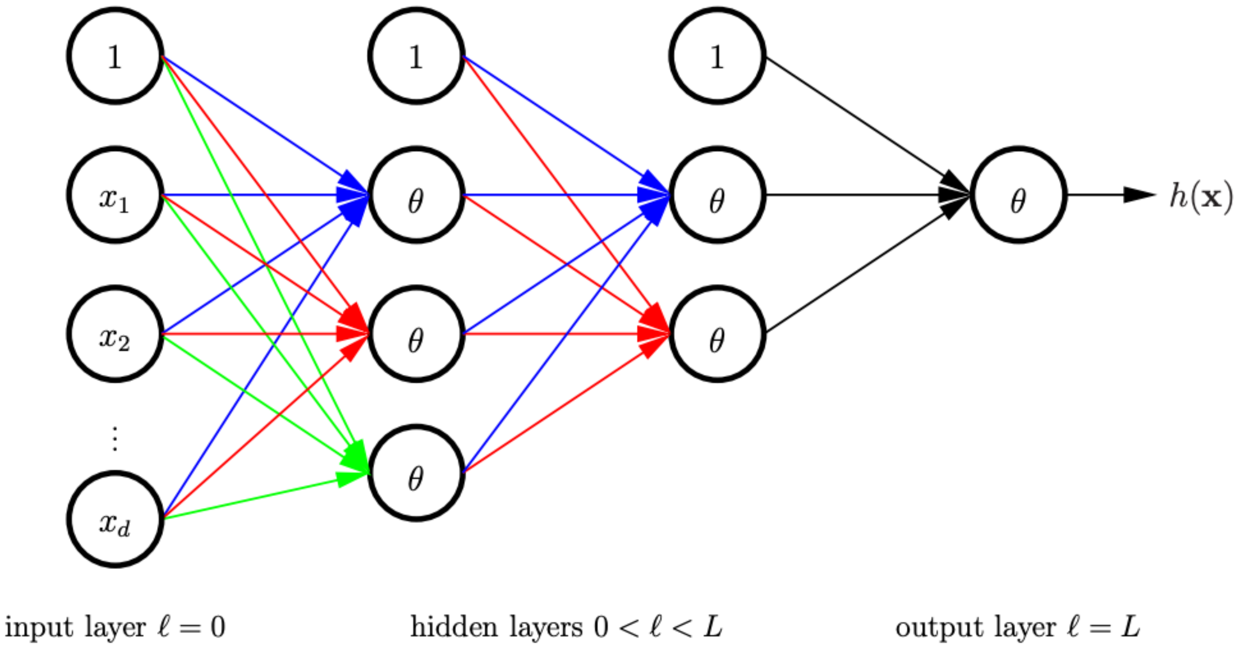
- Given D and the network structure, how to learn the “weights” (i.e., the weight vectors of every Perceptron)?



- Computationally challenging due to the ”sign” function 

Neural Networks

- A softened version of multi-layer Perceptron (MLP)



θ: activation function

(Specify the "activation" of the neuron)

Next lecture: formally introduce neural networks and how to learn it from data