CSE 417T
Introduction to Machine Learning

Lecture 11

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Logistics

- Homework 2: due on **Mar 8, Monday**

- Exam 1: **Mar 23 (Tuesday)**
  - More details will be posted on Piazza today or tomorrow
  - Topics: LFD Chapters 1 to 5

- Homework 3 will be posted early next week
  - Expected due: **Mar 19 (Friday)**
  - A bit shorter amount of time than usual, but the goal is for you to read the topics that Exam 1 covers

Recap

VC Dimension of d-dimension Perceptron

- Claim:
  - The VC Dimension of d-dim perceptron is $d + 1$

- How to prove it?
  1. Show that the VC dimension of d-dim perceptron $\geq d + 1$
  2. Show that the VC dimension of d-dim perceptron $\leq d + 1$

- To prove $d_{vc}(H) \geq d+1$, what do we need to prove?
  There is a set of $d+1$ points that can be shattered by H

Proof Sketch:

1. Let's construct a dataset of $d+1$ points: $X = \begin{bmatrix} \vec{x}_1^T \\ \vdots \\ \vec{x}_{d+1}^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 0 & \dots & 0 & 0 \end{bmatrix}$; It's easy to check that X^{-1} exist

2. For any possible dichotomy $\vec{y}$, there exists a $\vec{w}$ such that $X\vec{w} = \vec{y}$, i.e., $\vec{w} = X^{-1}\vec{y}$
3. Therefore, d-dim perceptron can shatter X

- To prove $d_{vc}(H) \leq d+1$, what do we need to prove?
  Every set of $d+2$ points cannot be shattered by H

Proof Sketch:

1. For every set of $d+2$ points (in d+1 dimensions), there exists a point that can be written as linear combinations of the others.
2. Denote the point $\vec{x}_{d+2}$, we have $\vec{x}_{d+2} = \sum_{i=1}^{d+1} a_i \vec{x}_i$
3. Consider the dichotomy $(y_1, \dots, y_{d+2}) = (\mathrm{sign}(\mathrm{a}_1), \dots, \mathrm{sign}(\mathrm{a}_{d+1}), -1)$, we can show that no linear separator can generate this dichotomy (think about why).
4. Therefore, for every set of $d+2$ points, there exist a dichotomy that H cannot shatter.

VC "Dimension"

- Degrees of freedom for your hypothesis in H
- (effective) # of parameters that control the hypothesis

- Examples:
  - d-dim perceptron: h is represented by $(w_0, \dots, w_d)$; $d_{vc} = d + 1$
  - Positive rays: h is represented by a threshold; $d_{vc} = 1$
  - Positive or negative rays: h is represented by a threshold and a direction; $d_{vc} = 2$
  - Positive intervals: h is represented by two thresholds; $d_{vc} = 2$
  - Positive or negative intervals: h is represented by two thresholds and a direction; $d_{vc} = 3$

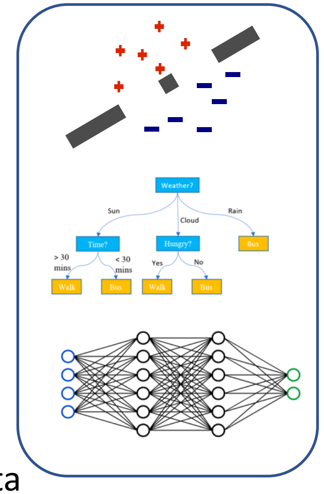
- Effective # parameters: An "approximation" for VC dimension

What We Have Taught So Far

- The explanation of ”machine learning” from the first lecture

Data (Historical Customer Data)

Hypothesis / Model



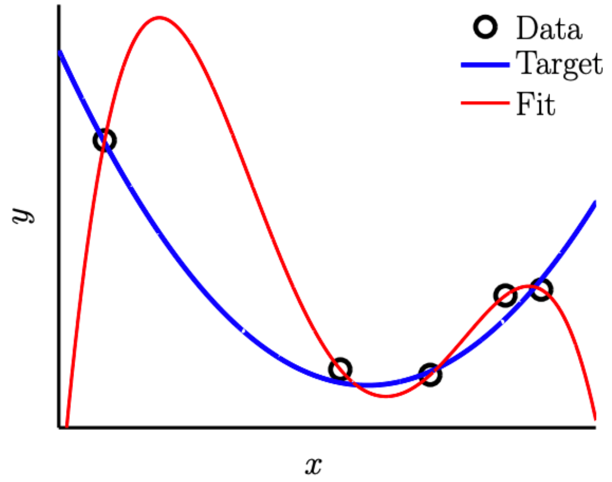
Find a hypothesis that “fits” the data
(The process requires a lot of computation)

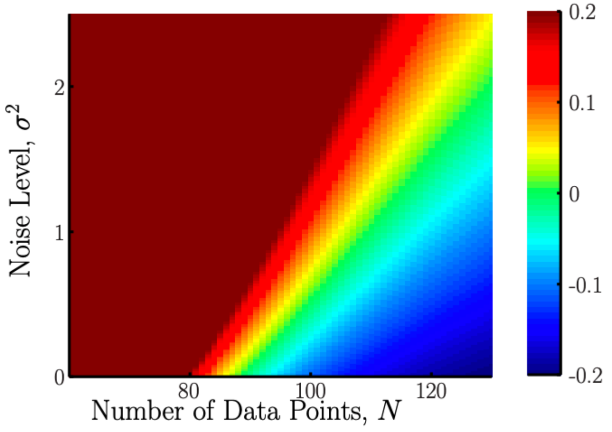
Our progress

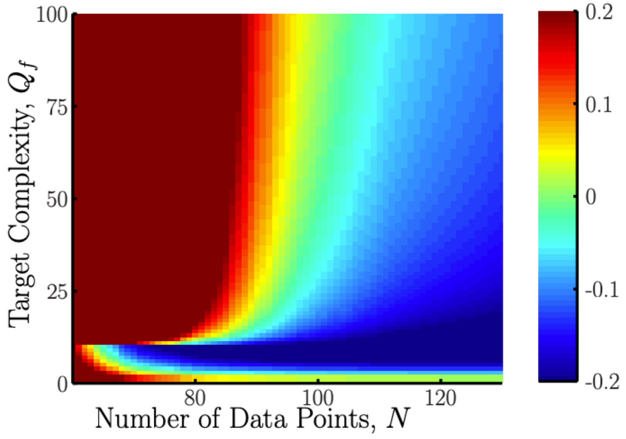
- Generalization of learning
  - What to say about $E_{out}(g)$ from $E_{in}(g)$
- How to find g
  - Focus on $g = argmin_{h \in H} E_{in}(g)$

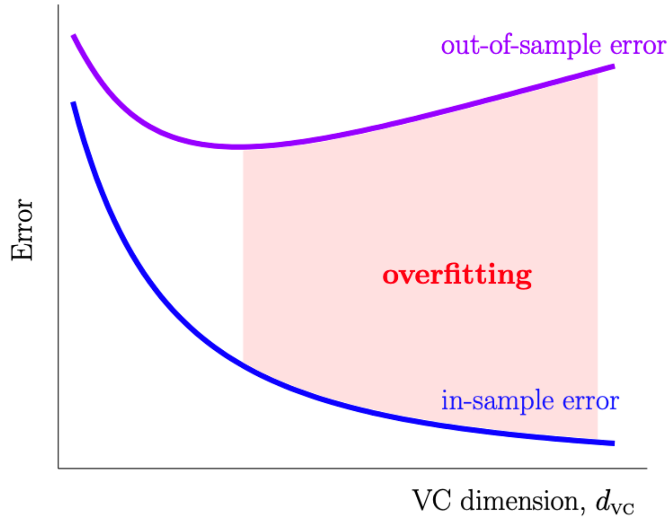
Seems to make sense, but…

Overfitting









Number of data points $\uparrow$	Overfitting $\downarrow$
Noise $\uparrow$	Overfitting $\uparrow$
Target complexity $\uparrow$	Overfitting $\uparrow$

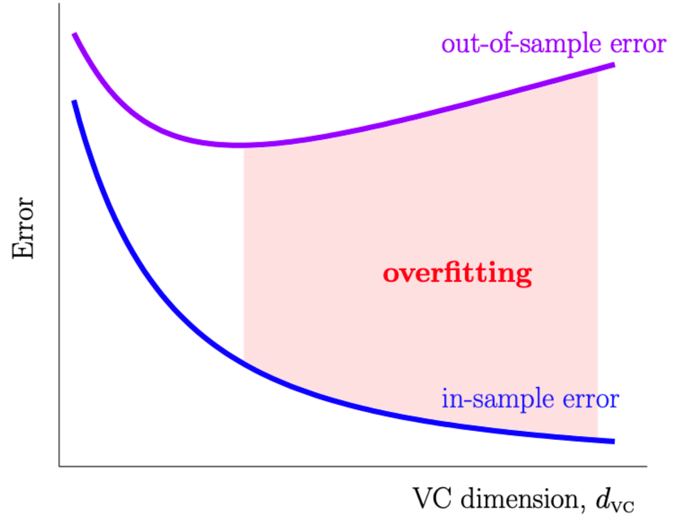
Today’s Lecture

The notes are not intended to be comprehensive. They should be accompanied by lectures and/or textbook.

Let me know if you spot errors.

Overfitting and Its Cures

- Overfitting
  - Fitting the data more than is warranted
  - Fitting the noise instead of the pattern of the data
  - Decreasing E_{in} but getting larger E_{out}
  - When H is too strong, but N is not large enough

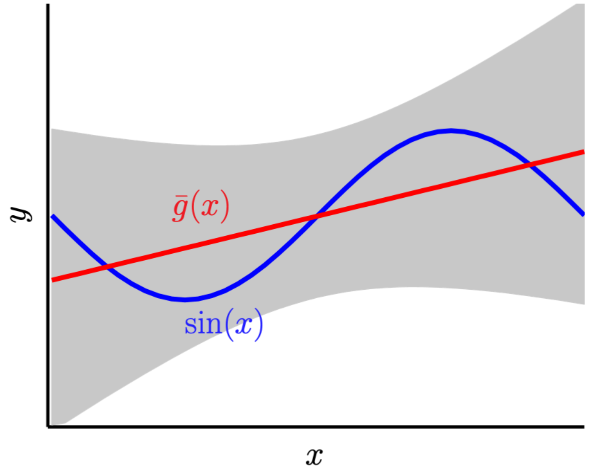


- Regularization
  - Intuition: Constraining H to make overfitting less likely to happen
  - (Topic of this lecture)

- Validation
  - Intuition: Reserve data to estimate E_{out}
  - (Focus of next lecture)

Regularization (Constraining H)

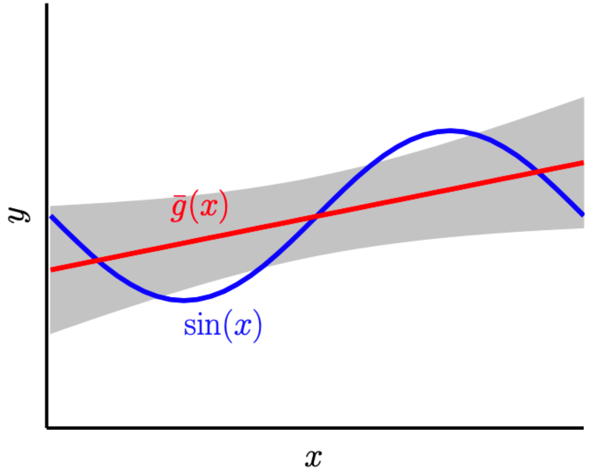
- Informal example:
  - Regression; $f = \sin(\pi x)$; $H = \{h(x) = ax + b\}$; $N = 2$



no regularization

bias $= 0.21$
var $= 1.69$

Regularization:
Constrain the hypothesis set to avoid large a and b



regularization

bias $= 0.23$
var $= 0.33$

How to do this in a principled way?

Hard Constraints

- We have seen hard constraints already

$$H_2 = \{h(x) = w_0 + w_1 x + w_2 x^2\}$$

$$H_{10} = \{h(x) = w_0 + w_1 x + w_2 x^2 + \cdots + w_{10} x^{10}\}$$

- H_2 can be written as constrained H_{10}

$$H_2 = \{h \in H_{10} \text{ and } w_3 = w_4 = \cdots = w_{10} = 0\}$$

Constraints

Soft-Order Constraints

Hard constraints

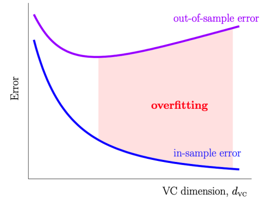
$H_2 = \{h \in H_{10} \text{ and } w_3 = w_4 = \cdots = w_{10} = 0\}$

- Instead of setting the weights to 0

$$H(C) = \left\{h \in H_Q \text{ and } \sum_{q=0}^{Q} w_q^2 \le C\right\}$$

$$= \left\{h \in H_Q \text{ and } \vec{w}^T \vec{w} \le C\right\}$$

- Observations
  - When $C \to \infty$, $H(C) = H_Q$
  - When $C_1 \le C_2$, $H(C_1) \subseteq H(C_2)$ and therefore $d_{vc}\big(H(C_1)\big) \le d_{vc}(H(C_2))$
  - A smoother way to tune the complexity of hypothesis set



Soft-Order Constraints

$$H(C) = \{h \in H_Q \text{ and } \vec{w}^T\vec{w} \le C\}$$

- Two main questions
  - How do we choose C
    - Model selection: The same question as selecting H
    - The focus of the next lecture
  - How do we perform learning, i.e., find a $g \in H(C)$ such that $g \approx f$
    - Solve the following constrained optimization problem

minimize $E_{in}(\vec{w})$

subject to $\vec{w}^T\vec{w} \le C$

Constrained to Unconstrained Optimization

- Constrained optimization ⇔ Unconstrained optimization

minimize $E_{in}(\vec{w})$

subject to $\vec{w}^T\vec{w} \leq C$

equivalent

minimize $E_{in}(\vec{w}) + \frac{\lambda_C}{N}\vec{w}^T\vec{w}$

- Why the above is true?
  - Will talk about how to utilize Lagrangian relaxation to get this in the 2nd half of the semester
  - For now, let's think about it graphically

minimize $E_{in}(\vec{w})$ subject to $\vec{w}^T\vec{w} \leq C$

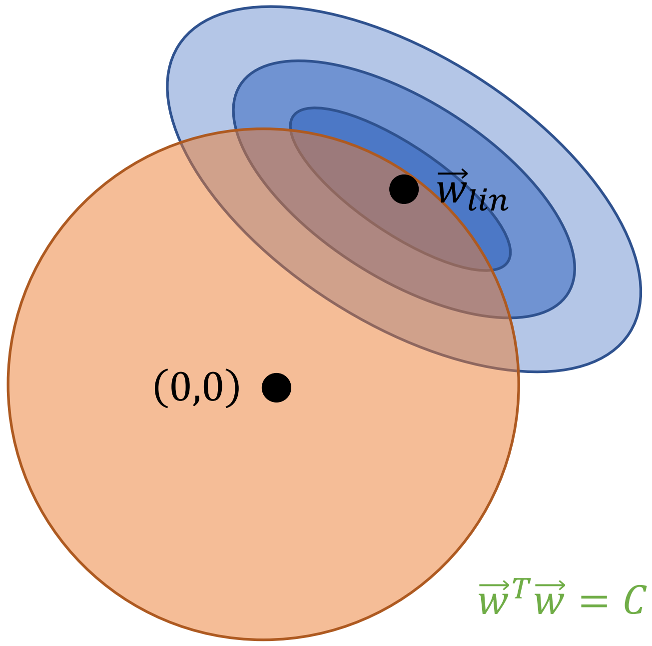
- Notations
  - $\vec{w}_{lin}$: the solution for $\min E_{in}(\vec{w})$
  - $\vec{w}_{reg}$:the solution for $\min E_{in}(\vec{w})$ subject to $\vec{w}^T\vec{w} \leq C$

minimize $E_{in}(\vec{w})$ subject to $\vec{w}^T\vec{w} \leq C$

- Notations
  - $\vec{w}_{lin}$: the solution for $\min E_{in}(\vec{w})$
  - $\vec{w}_{reg}$:the solution for $\min E_{in}(\vec{w})$ subject to $\vec{w}^T\vec{w} \leq C$

- When C is large enough, i.e., $\vec{w}_{lin}^T\vec{w}_{lin} \leq C$
  - $\vec{w}_{reg} = \vec{w}_{lin}$

- When C is not large enough, i.e., $\vec{w}_{lin}^T\vec{w}_{lin} > C$
  - $\vec{w}_{reg}^T\vec{w}_{reg} = C$



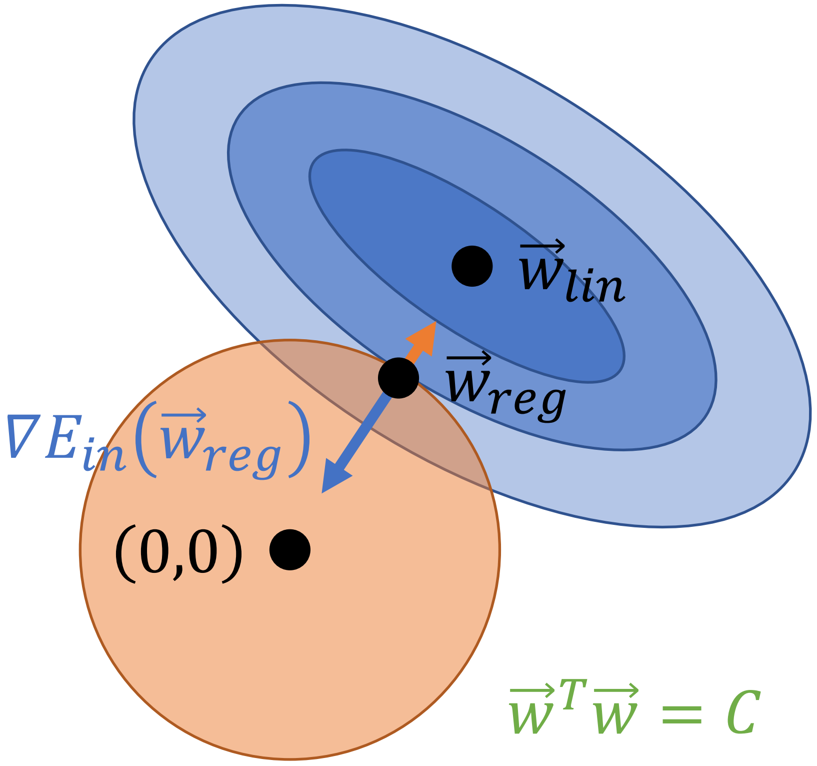
minimize $E_{in}(\vec{w})$ subject to $\vec{w}^T\vec{w} \le C$

- When C is not large enough

- $\vec{w}_{lin}$: the solution for $\min E_{in}(\vec{w})$
- $\vec{w}_{reg}$:the solution for $\min E_{in}(\vec{w})$ subject to $\vec{w}^T\vec{w} \le C$

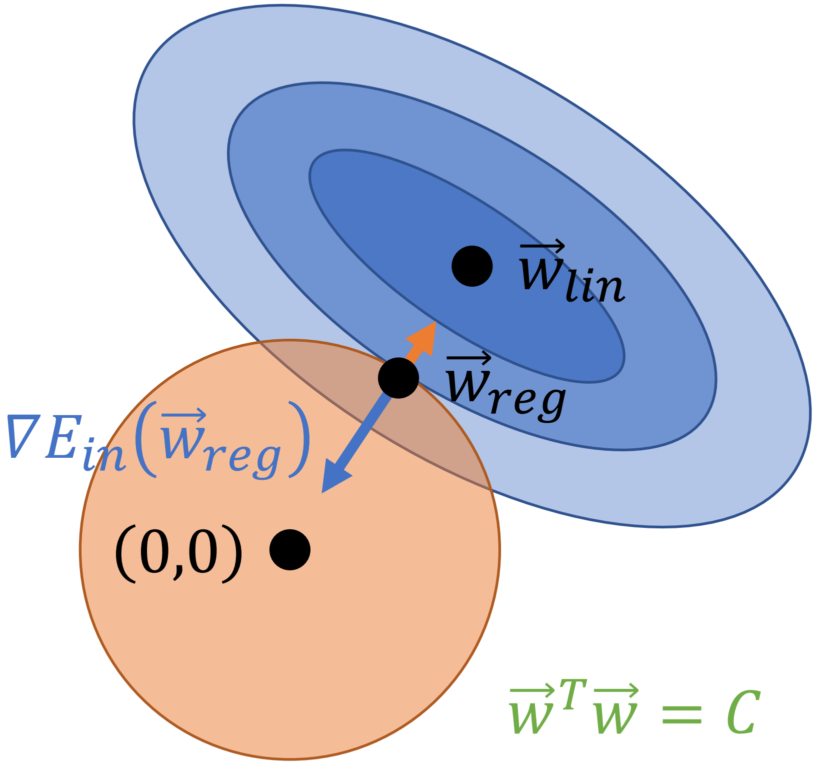
minimize $E_{in}(\vec{w})$ subject to $\vec{w}^T\vec{w} \leq C$

- When C is not large enough
  - Using graphical arguments
    - $\vec{w}_{reg} \propto -\nabla_{\vec{w}} E_{in}(\vec{w}_{reg})$



minimize $E_{in}(\vec{w})$ subject to $\vec{w}^T\vec{w} \leq C$

- When C is not large enough
  - Using graphical arguments
    - $\vec{w}_{reg} \propto -\nabla_{\vec{w}} E_{in}(\vec{w}_{reg})$
  - That is, we can find some constant $\lambda_C \geq 0$ such that
    - $\nabla_{\vec{w}} E_{in}(\vec{w}_{reg}) = -\frac{2\lambda_C}{N}\vec{w}_{reg}$
  - Therefore,
    - $\nabla_{\vec{w}}\left(E_{in}(\vec{w}_{reg}) + \frac{\lambda_C}{N}\vec{w}_{reg}^T\vec{w}_{reg}\right) = 0$
  - This implies, $\vec{w}_{reg}$ is the solution for
    - minimize $E_{in}(\vec{w}) + \frac{\lambda_C}{N}\vec{w}^T\vec{w}$



Constrained to Unconstrained Optimization

- Constrained optimization ⇔ Unconstrained optimization

minimize $E_{in}(\vec{w})$

subject to $\vec{w}^T\vec{w} \leq C$

equivalent

minimize $E_{in}(\vec{w}) + \frac{\lambda_C}{N}\vec{w}^T\vec{w}$

augmented error

- Interpretations of regularization
  - Constraining H (by adding constraints)
  - Adding penalty to complex hypothesis in augmented errors

Augmented Error

$\vec{w}^T\vec{w}$: weight decay

- Define augmented error
  - $E_{aug}(\vec{w}) = E_{in}(\vec{w}) + \frac{\lambda_C}{N}\vec{w}^T\vec{w}$
  - Algorithm: Find $\vec{w}^* = argmin\ E_{aug}(\vec{w})$

- A bit more discussion
  - When $C \to \infty$, $\lambda_C = 0$
  - Smaller C (stronger constraints)
    - => larger λ_C
    - => smaller H
    - => stronger regularization
  - Use λ_C to tune the level of regularization

Side note:
You will see people/us interchangeably use λ_C and $\frac{\lambda_C}{N}$ to be the constant, depending on whether the dependency on N is emphasized.

General Form of Regularization

$$E_{aug}(h, \lambda, \Omega) = E_{in}(\vec{w}) + \frac{\lambda}{N}\Omega(h)$$

- Key parameters
  - Ω: Regularizer
  - λ: Amount of regularization

- Does the form look familiar: VC Theory
  - $E_{out}(g) \leq E_{in}(g) + O\left(\sqrt{d_{vc}\frac{\ln N}{N}}\right)$

- If we pick the right Ω, E_{aug} can be a better proxy for E_{out}

How to Pick the Right Ω

- No definite answer, but generally
  - We like to pick Ω that leads to “smoother” hypothesis
    - Overfitting is due to noise
    - Informally, noise is usually ”high frequency”
  - We prefer Ω that makes the optimization easier (e.g., convex/differentiable)
    - Similar to picking the error measure
  - We might have some other objective in mind
    - Ex: L-1 regularizer leads to weight vectors with more 0s
      - $E_{aug}(\vec{w}) = E_{in}(\vec{w}) + \lambda \|\vec{w}\|_1 = E_{in}(\vec{w}) + \lambda \sum_i |w_i|$
- What if we pick the wrong Ω (Think about weight growth)
  - We might still fix it by picking the right λ – using validation in the next lecture

More Discussion on Regularization

Why $\vec{w}^T\vec{w}$ is Called Weight Decay

- Run gradient descent on $E_{aug}(\vec{w}) = E_{in}(\vec{w}) + \lambda_C \vec{w}^T\vec{w}$

- The update rule would be

$$\vec{w}(t+1) \leftarrow \vec{w}(t) - \eta \nabla_{\vec{\mathrm{w}}} E_{aug}(\vec{w}(t))$$

$$\Rightarrow \vec{w}(t+1) \leftarrow (1 - 2\eta\lambda_C)\vec{w}(t) - \eta \nabla_{\vec{\mathrm{w}}} E_{in}(\vec{w}(t))$$

We are decaying the weights first, then do the update

Linear Regression with Weight Decay

- $E_{aug}(\vec{w}) = E_{in}(w) + \frac{\lambda_C}{N}\vec{w}^T\vec{w} = \frac{1}{N}\|X\vec{w} - \vec{y}\|^2 + \frac{\lambda_C}{N}\vec{w}^T\vec{w}$

- Solve $\nabla_{\vec{w}} E_{aug}(\vec{w})|_{\vec{w}=\vec{w}_{reg}} = 0$, we get
  - $\frac{2}{N}\left(X^T X\vec{w}_{reg} - X^T\vec{y} + \lambda_C\vec{w}_{reg}\right) = 0$
  - $(X^T X + \lambda_C I)\vec{w}_{reg} = X^T\vec{y}$
  - $\vec{w}_{reg} = (X^T X + \lambda_C I)^{-1} X^T\vec{y}$

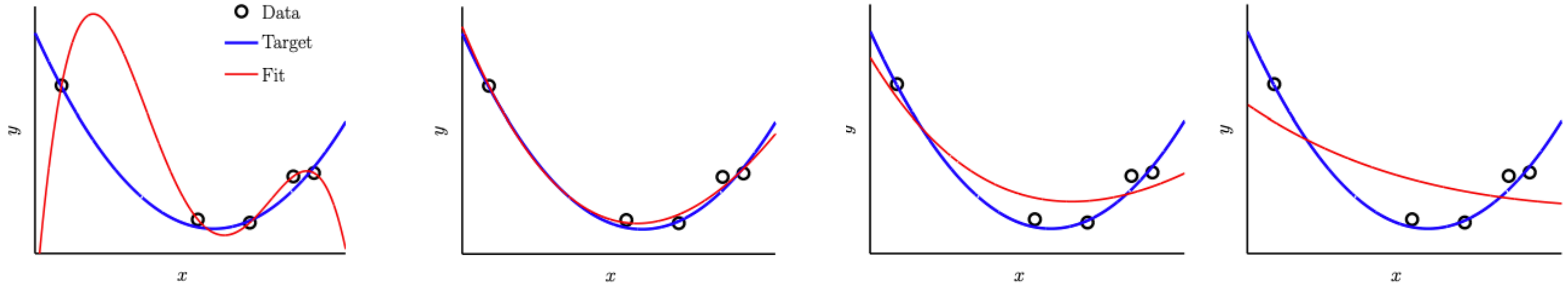
Notation: I is an identity matrix: only the elements in the diagonals are 1, and all others are 0.

This is called “Ridge Regression” in statistics.

Effect of Regularization (Different λ)

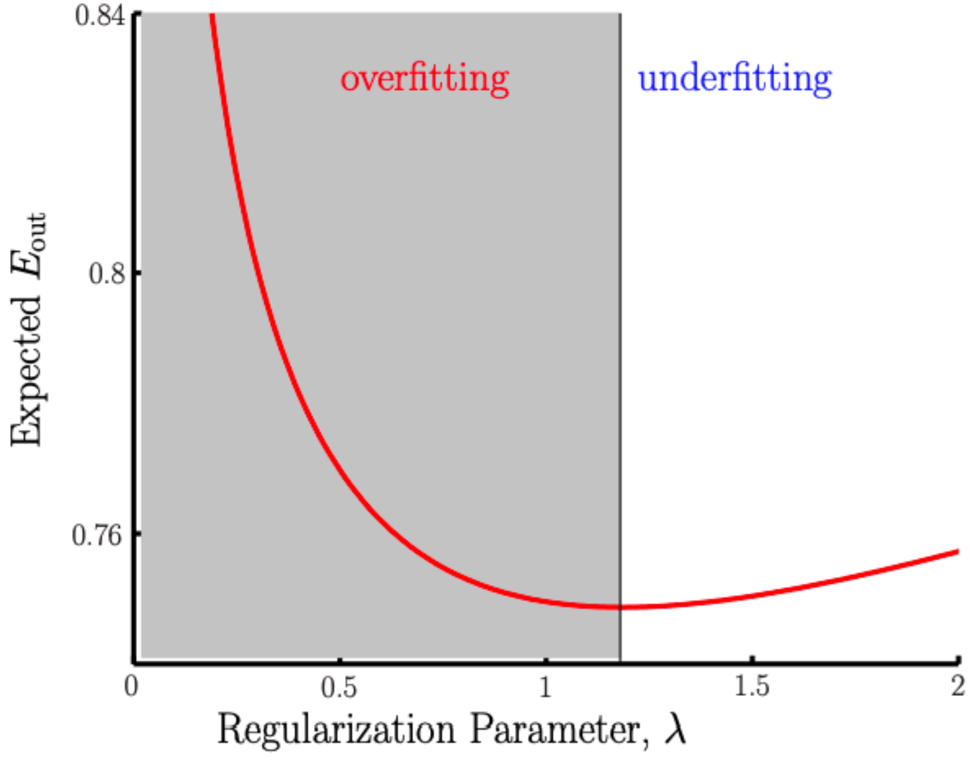
- Minimizing $E_{in}(\vec{w}) + \frac{\lambda}{N}\vec{w}^T\vec{w}$ with different λ

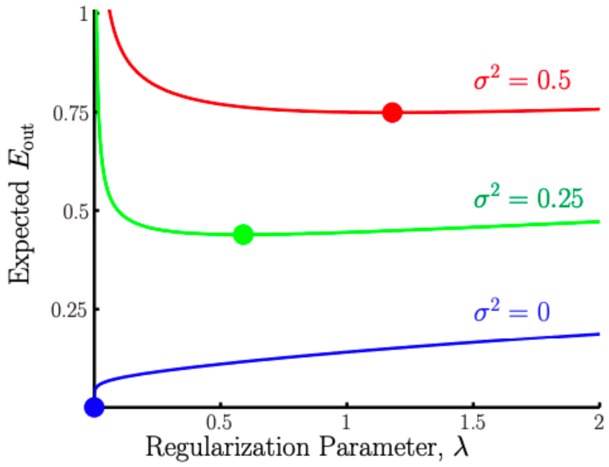
$\lambda = 0$ $\quad$ $\lambda = 0.0001$ $\quad$ $\lambda = 0.01$ $\quad$ $\lambda = 1$

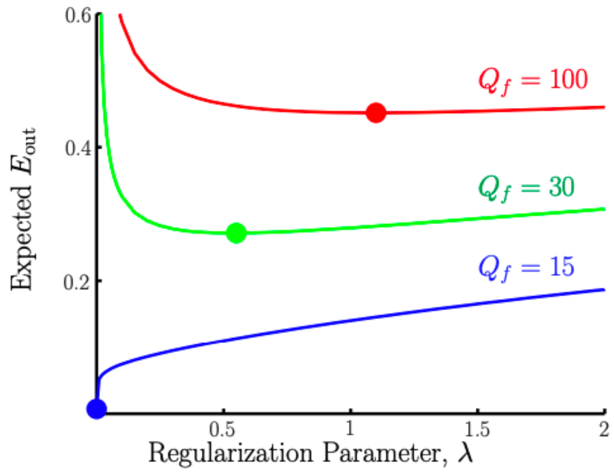


Overfitting $\longrightarrow$ $\longrightarrow$ **Underfitting**

Overfitting and Underfitting



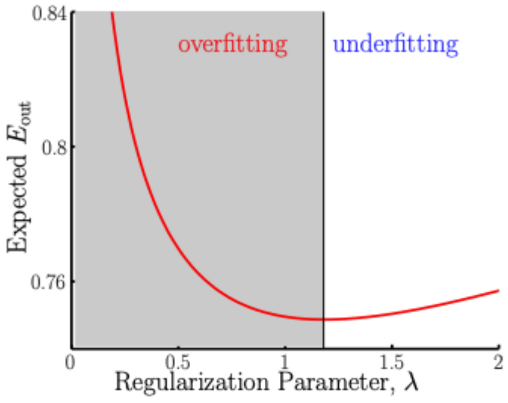




Need to pick the right λ:
Using validation: Focus of next lecture

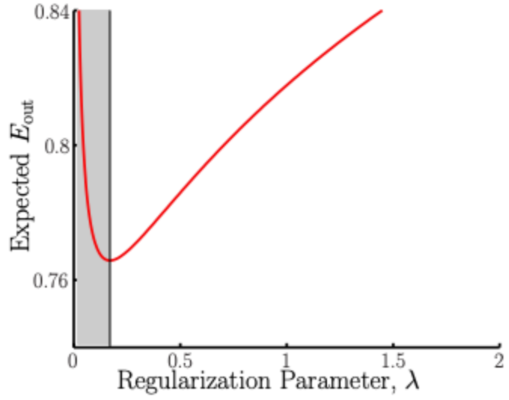
Variations on Weight Decay (Different Ω)

Uniform Weight Decay



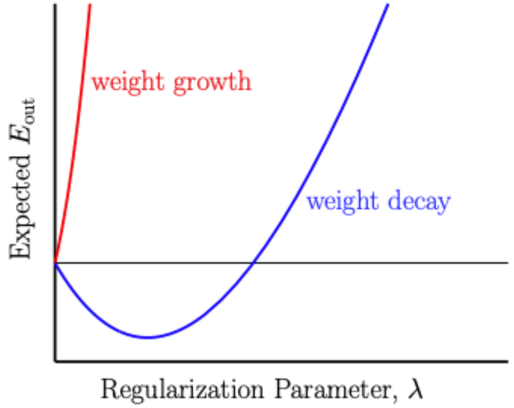
$$\sum_{q=0}^{Q} w_q^2$$

Low Order Fit



$$\sum_{q=0}^{Q} q w_q^2$$

Weight Growth!



$$\sum_{q=0}^{Q} \frac{1}{w_q^2}$$

How to Pick the Right Ω

- As discussed earlier
  - Intuition: pick Ω that leads to “smoother” hypothesis
    - Overfitting is due to noise
    - Informally, noise is generally ”high frequency”
  - Computation: prefer Ω that makes the optimization easier (e.g., convex/differentiable)
    - Similar to picking the error measure
  - We might have some other objective in mind
    - Ex: L-1 regularizer leads to weight vectors with more 0s

- What if we pick the wrong Ω (weight growth)
  - We might still fix it by picking the right λ – using validation

Summarizing Regularization

- Regularization is everywhere in machine learning

- Two main ways of thinking about regularization
  - Constraining H to make overfitting less likely to happen
    - Will discuss more regularization methods in the 2nd half of the semester
    - Pruning for decision trees, early stopping / dropout for neural networks, etc
  - Define augmented error E_{aug} to better approximate E_{out}
    - $E_{aug}(h, \lambda, \Omega) = E_{in}(\vec{w}) + \frac{\lambda}{N}\Omega(h)$

- We show the equivalence of the two for weight decay
  - The conceptual equivalence is general with Lagrangian relaxation (will cover later in the semester)