

Lecture 5

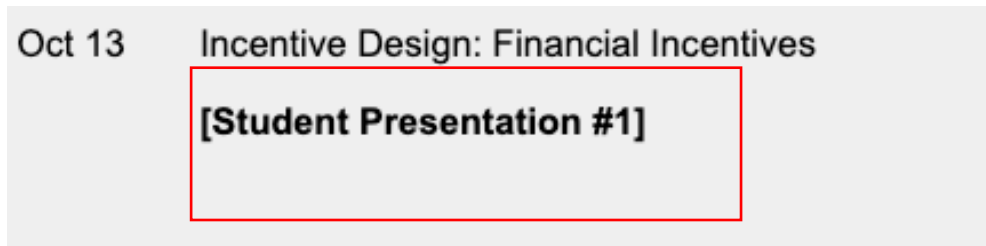
Label Aggregation: Matrix-Based Methods

Questions: <https://sli.do> #4218

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Logistics: Bidding for Presentations

- Check out the course schedule for the presentation slots:

















Oct 13	Incentive Design: Financial Incentives
	[Student Presentation #1]

- Provide around 3~5 bids **by the end of today** (hard deadline).
 - <https://doodle.com/poll/qnsgb6xqvedz7dqu>
 - You might want to glance over the papers of your bidding.
 - Bidding more than 5 bids is okay
 - It might help in decreasing the chance you get assigned to slots outside of your bids.

Logistics: Bidding for Presentations

- Bidding interface
 - Enter the **names of all members** in your group

	#1 Oct 13	#2 Oct 15	#3 Oct 20	#4 Oct 22	#5 Oct 27	#6 Oct 29	#7 Nov 3	#8 Nov 5	#9 Nov 12	#10 Nov 17	#11 Dec 1	#12 Dec 3	#13 Dec 10
 Enter your name													

Make sure you can make it before bidding these two.

- I'll announce the assignment by this Wed.
 - Manually solve the max-cover problem.
 - I'll try to accommodate your interests, but no guarantee on that.
 - Random assignments will be used if there is no feasible solutions.
 - I'll fill in the slots if there are fewer groups than slots.

Logistics: Project Proposal

- Due: October 9 (next Friday)
- [Example/past projects](#) are posted on the course website
- Requirements:
 - Title, team members
 - 1~2 paragraphs describing what you want to do
 - At least one relevant paper
- Submission:
 - Submit on Gradescope.
 - One submission per group.
 - Need to include all teammates using the Gradescope interface.

Logistics: Assignment 2

- Posted on the course website
- Due: October 16 (Friday)
- Programming assignments
 - Implement and compare the performance of majority voting, EM, and SVD
 - You can use any programming language you like
 - You will be **graded based on the report**
 - You need to submit your codes
 - Used for plagiarism tests
 - Might check the codes if we have confusions/doubts on the reported results

Logistics: Assignment 2

- Dataset:
 - <https://sites.google.com/site/nlpannotations>. (rte.standardized.tsv)
 - Recognizing Textual Entailment (RTE) task

Text:

- Many experts think that there is likely to be another terrorist attack on American soil within the next five years.

Hypothesis:

- There will be another terrorist attack on American soil within the next five years.

Answer: NO

Task: Whether the first sentence implies the second hypothesis

- 800 tasks; 164 workers; 10 labels per task

You might want to convert the labels/gold to {+1,-1}

Logistics: Assignment 2

Ignore this column		Worker ID	Task ID	Worker Label	Ground Truth (only used to evaluate the aggregation algorithm)
!amt_annotation_ids	!amt_worker_ids	orig_id	response	gold	
89KZPYXSTGTJ0CZY2Y1ZB28YQ9GBT88Z2W1KDYZT	A19IBSKBTABMR3	266	1	1	
89KZPYXSTGTJ0CZY2Y1ZYAJC56Z6FBPGXJYVPXM0	AEX5NCH03LWSG	266	1	1	
89KZPYXSTGTJ0CZY2Y1ZFWHATWX49Y3ZTPX4FYH0	A17RPF5ZMO75GW	266	1	1	
89KZPYXSTGTJ0CZY2Y1ZV89Z3WRZ6R8ZM4ZZZ070	A15L6WGIK3VU7N	266	0	1	
89KZPYXSTGTJ0CZY2Y1ZWZHYZCCYYVYPDZVNRAZ	A3U7T47F498T1P	266	1	1	
89KZPYXSTGTJ0CZY2Y1Z09PZYS137RPZT6SY4A20	AXBQF8RALCIGV	266	1	1	
89KZPYXSTGTJ0CZY2Y1ZQ30CJXY2EB96XJS543YZ	A1DCEOFAUIDY58	266	1	1	
89KZPYXSTGTJ0CZY2Y1ZXZ3ZNY7VZKZSCY0B94Z	A1Q4VUJBM78YR	266	0	1	
89KZPYXSTGTJ0CZY2Y1ZDZGGWVY8XDZTKYC9XKZ	A18941IO2ZZWW6	266	1	1	
89KZPYXSTGTJ0CZY2Y1Z3Z7ZWY9J4WFMX60VRVXZ	A11GX90QFWDLMM	266	1	1	
89KZPYXSTGTJ0CZY2Y1ZB28YQ9GBT88Z2W1KDYZT	A19IBSKBTABMR3	934	0	0	
89KZPYXSTGTJ0CZY2Y1ZYAJC56Z6FBPGXJYVPXM0	AEX5NCH03LWSG	934	0	0	
89KZPYXSTGTJ0CZY2Y1ZFWHATWX49Y3ZTPX4FYH0	A17RPF5ZMO75GW	934	0	0	
89KZPYXSTGTJ0CZY2Y1ZV89Z3WRZ6R8ZM4ZZZ070	A15L6WGIK3VU7N	934	0	0	
89KZPYXSTGTJ0CZY2Y1ZWZHYZCCYYVYPDZVNRAZ	A3U7T47F498T1P	934	0	0	
89KZPYXSTGTJ0CZY2Y1Z09PZYS137RPZT6SY4A20	AXBQF8RALCIGV	934	1	0	
89KZPYXSTGTJ0CZY2Y1ZQ30CJXY2EB96XJS543YZ	A1DCEOFAUIDY58	934	0	0	
89KZPYXSTGTJ0CZY2Y1ZXZ3ZNY7VZKZSCY0B94Z	A1Q4VUJBM78YR	934	0	0	

Logistics: Assignment 2

- Requirements

- Create subsampled datasets:
 - Original: 800 tasks; 164 workers; 10 labels per task
 - Randomly sub-sample the labels, such that each task has k labels
 - $k=1, 2, 3, \dots, 10$
- Implement majority voting, EM (simple version as in the discussion session), SVD
- Calculate the error (ratio of tasks the algorithms make wrong predictions)
- Compare the performance of algorithms
 - Generate a figure with x-axis being k , y-axis being error
 - Plot 3 curves, each corresponding to an algorithm
- Offer brief discussion
- "Expand" the dataset to see the performance of SVD

Quick Recap

EM-Based Approach

- Notations
 - $D = \{d_1, \dots, d_n\}$: Observations
 - θ : latent variables
- Concepts
 - Likelihood: $\Pr(D|\theta)$
- Steps for MLE approach
 - Define label generation model $\Pr(d_i|\theta)$
 - θ contains the true labels and other latent factors in your models
 - Optimization: Find $\theta^* = \operatorname{argmax}_{\theta} \sum_{i=1}^n \log \Pr(d_i|\theta)$
 - In last lecture, there are only two possible values for θ . So we brute-force find it.

EM-Based Approach

- Connection to supervised learning
 - Model of the labeling process: Hypothesis set / Loss Function
 - EM: an algorithm to find a hypothesis within the set that minimizes the error
- Pros
 - **Empirically performs well**
 - A generic framework
 - There is a HUGE amount of papers along this line, with different models of label generation
- Cons
 - EM only attempts to find the local optimal of the objective function
 - **Lack of theoretical guarantees** on the final performance
 - Are we just getting lucky

Today's Lecture

Matrix Representation of Workers' Answers

	Task 1	Task 2	Task 3	Task 4	...
Worker 1	1	-1	1	1	
Worker 2	1	-1	-1	-1	
Worker 3	-1	1	-1	1	
Worker 4	1	-1	1	1	
...					
	?	?	?	?	

Goal: Infer the true label of each task

Let's Look at Another Similar Problem

- Movie recommendation

	Movie 1	Movie 2	Movie 3	Movie 4	Movie 5	Movie 6
Alice	5	4		1		
Bob	4			2	5	
Charlie	1		4		2	
David		3	2			4
...						

Warmup Discussion:

- Which movie will you recommend to Alice? Why?

Collaborative Filtering

	Movie 1	Movie 2	Movie 3	Movie 4	Movie 5	Movie 6
Alice	5	4		1		
Bob	4			2	5	
Charlie	1		4		2	
David		3	2			4
...						

Bob is probably most similar to Alice

- User-based collaborative filtering
 - Examine users' rating *vector*
 - Alice and Bob seem to have similar tastes
 - Bob likes Movie 5
 - Alice probably also likes Movie 5
 - We can also calculate **similarities** among users, and **weight** their opinions accordingly

Collaborative Filtering

	Movie 1	Movie 2	Movie 3	Movie 4	Movie 5	Movie 6
Alice	5	4		1		
Bob	4			2	5	
Charlie	1		4		2	
David		3	2			4
...						

- Item-based collaborative filtering
 - Examine items' rating *vectors*
 - People who like/hate Movie 1 seem to like/hate Movie 5 as well
 - Since Alice likes Movie 1, she might also like Movie 5



Discussions and Intuitions

- Pros and Cons
 - Simple and interpretable
 - Cold-start and data sparsity problem (won't discuss much in this lecture)
- Key intuitions for collaborative filtering to work
 - A big number of ratings are controlled by a small number of parameters
 - You probably can see why this is related to crowdsourcing already
- Low rank matrix approximation
 - A principled method to utilize the above intuition

	Movie 1	Movie 2	Movie 3	Movie 4	Movie 5	Movie 6
Alice	5	4		1		
Bob	4			2	5	
Charlie	1		4		2	
David		3	2			4
...						

A very short intro to

Low rank matrix approximation

Rank of a Matrix

- Matrix Rank
 - # linearly independent row (or column) vectors in a matrix

- Example

$$\begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 4 & 8 & 0 \end{bmatrix} \quad \text{Rank: 2}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \\ 4 & 5 & 9 \end{bmatrix} \quad \text{Rank: 2}$$

- What does low rank matrix imply?

Singular Value Decomposition (SVD)

$$A_{[m \times n]} \approx U_{[m \times r]} \Sigma_{[r \times r]} (V_{[n \times r]})^T$$

- A : Input matrix
 - $m \times n$ matrix (m users, n movies; m workers, n tasks)
- U : Left singular matrix
 - $m \times r$ matrix (m users, r latent concepts)
- Σ : Singular values
 - $r \times r$ diagonal matrix (strength of each latent concept)
- V : Right singular matrix
 - $n \times r$ matrix (n movies, r latent concepts)

(Technically, the SVD definition here is slightly different from standard definition, but we can get this one with some discussion on matrix ranks.)

(See the [lecture notes](#) by Tim Roughgarden for more details.)

Singular Value Decomposition (SVD)

- It is always possible to make such decomposition exactly equal (if we don't put any restrictions on the **rank r**)

$$A_{[m \times n]} = U_{[m \times r]} \Sigma_{[r \times r]} (V_{[n \times r]})^T$$

- Low rank matrix decomposition
 - Can we approximate A with this decomposition with **small rank r**

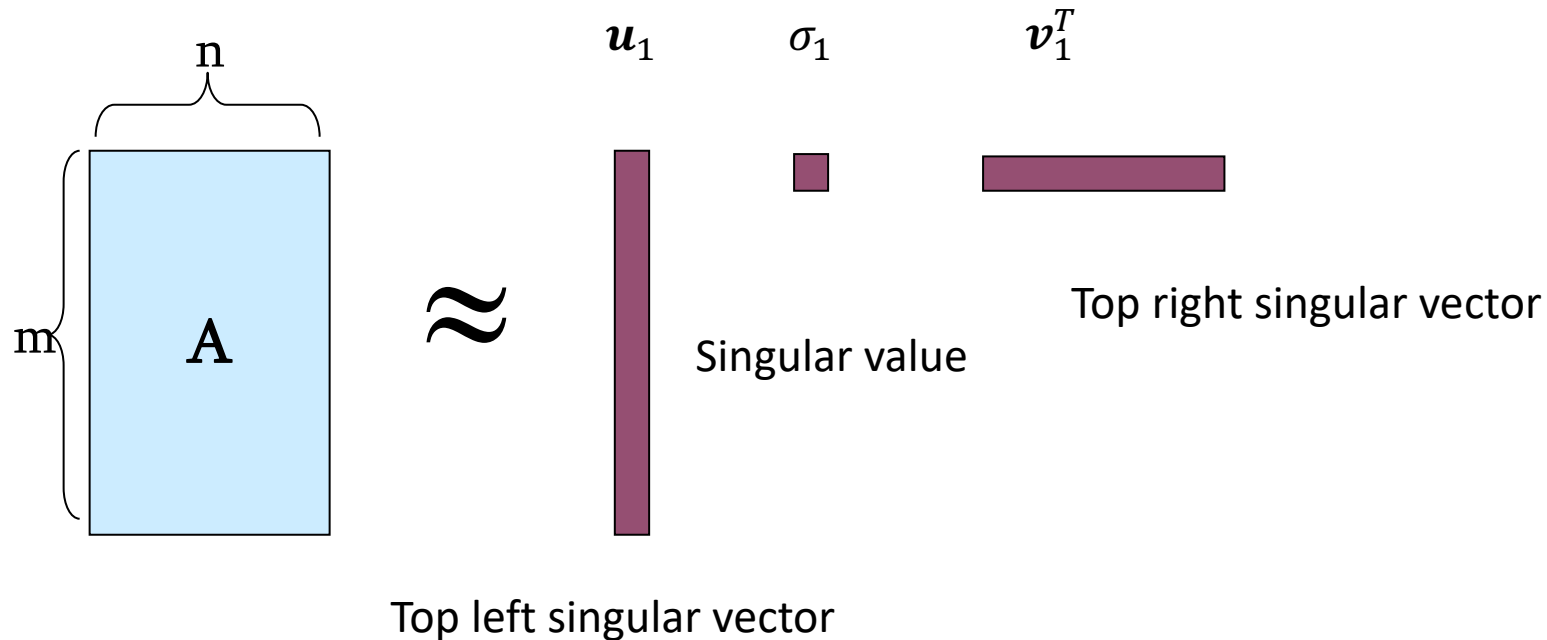
$$A_{[m \times n]} \approx U_{[m \times r]} \Sigma_{[r \times r]} (V_{[n \times r]})^T$$

A dimension reduction technique;

Reduce the number of parameters (and therefore requires less data to learn)

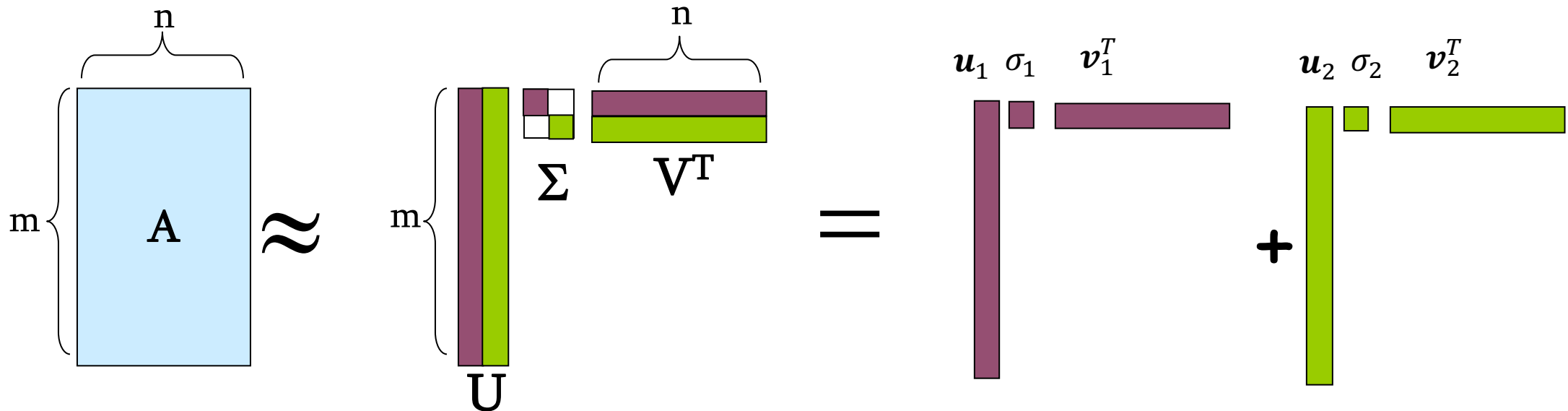
Rank 1 Approximation

$$A_{[m \times n]} \approx U_{[m \times 1]} \Sigma_{[1 \times 1]} (V_{[n \times 1]})^T$$
$$= \mathbf{u}_1 \sigma_1 \mathbf{v}_1^T$$



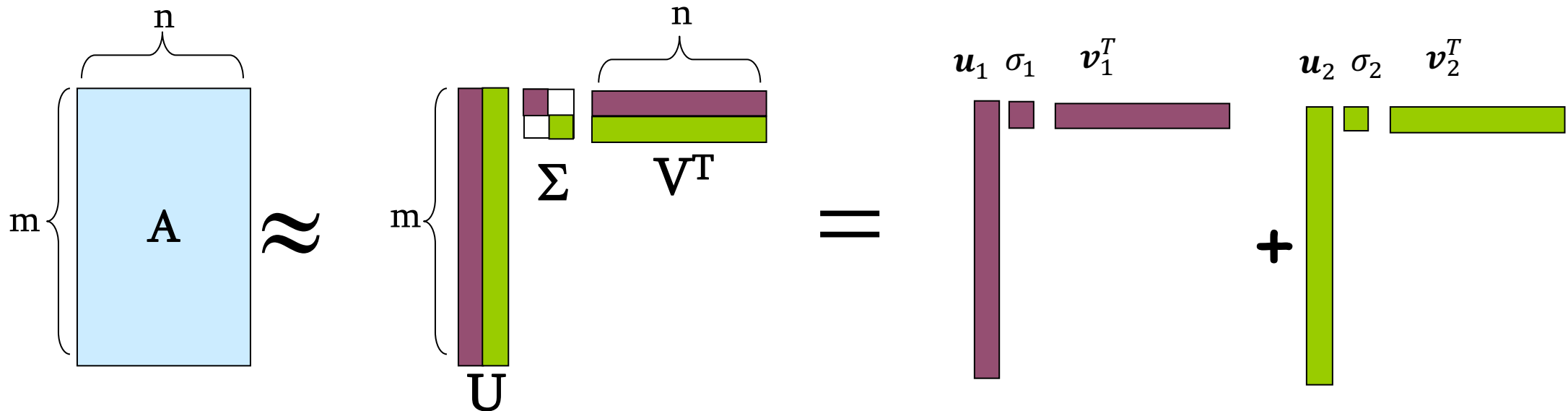
Rank k Approximation

$$A_{[m \times n]} \approx U_{[m \times k]} \Sigma_{[k \times k]} (V_{[n \times k]})^T$$



Rank k Approximation

$$A_{[m \times n]} \approx U_{[m \times k]} \Sigma_{[k \times k]} (V_{[n \times k]})^T = \sum_i u_i \sigma_i v_i^T$$



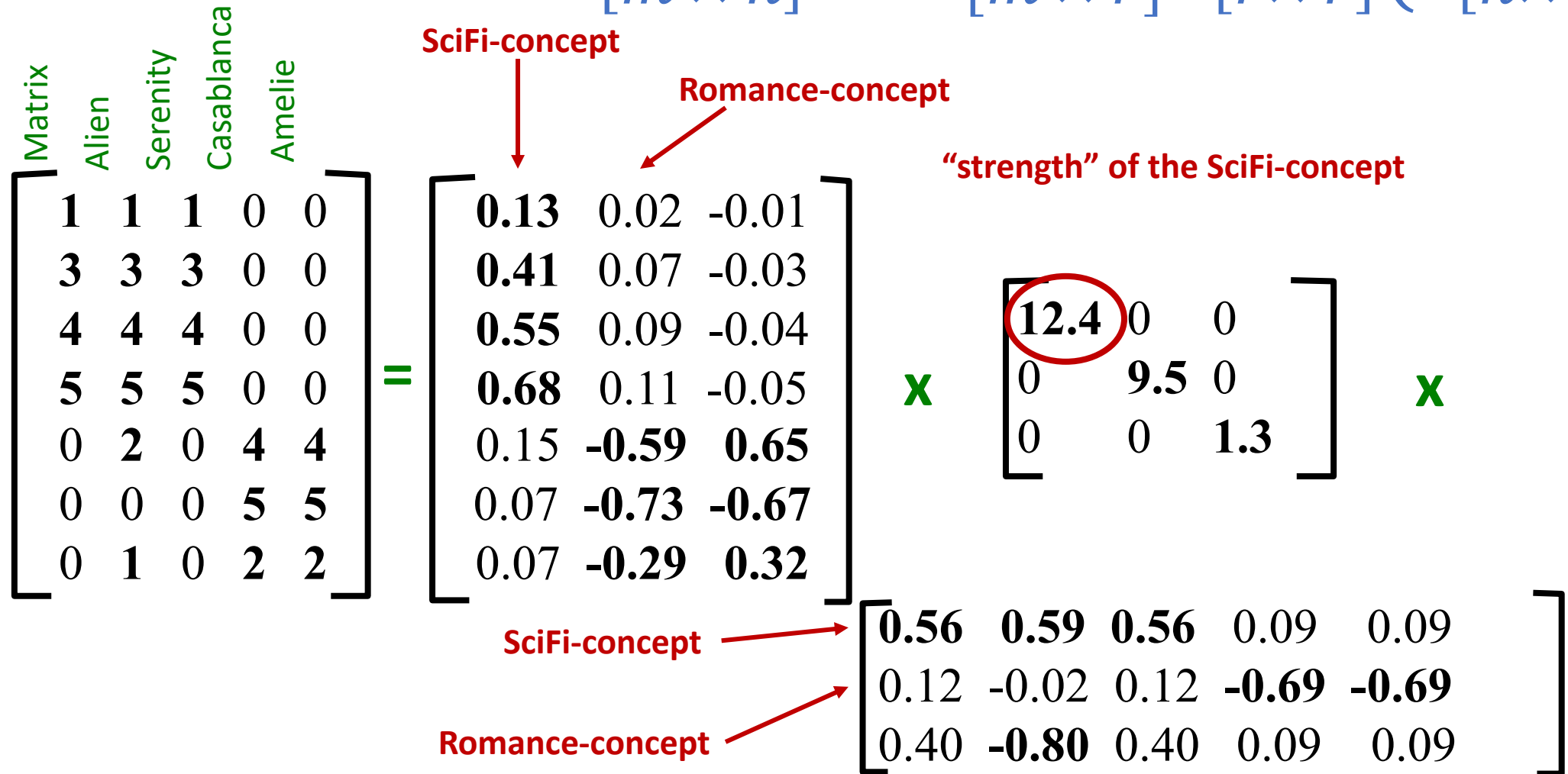
Movie Recommendation Example

$$A_{[m \times n]} \approx U_{[m \times r]} \Sigma_{[r \times r]} (V_{[n \times r]})^T$$

Matrix	Alien	Serenity	Casablanca	Amelie														
<div style="display: flex; flex-direction: column; align-items: center;"> <div>1</div> <div>3</div> <div>4</div> <div>5</div> <div>0</div> <div>0</div> <div>0</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>1</div> <div>3</div> <div>4</div> <div>5</div> <div>2</div> <div>0</div> <div>1</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>1</div> <div>3</div> <div>4</div> <div>5</div> <div>0</div> <div>0</div> <div>0</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0</div> <div>0</div> <div>0</div> <div>0</div> <div>4</div> <div>5</div> <div>2</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0</div> <div>0</div> <div>0</div> <div>0</div> <div>4</div> <div>5</div> <div>2</div> </div>	=	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0.13</div> <div>0.41</div> <div>0.55</div> <div>0.68</div> <div>0.15</div> <div>0.07</div> <div>0.07</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0.02</div> <div>0.07</div> <div>0.09</div> <div>0.11</div> <div>-0.59</div> <div>-0.73</div> <div>-0.29</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>-0.01</div> <div>-0.03</div> <div>-0.04</div> <div>-0.05</div> <div>0.65</div> <div>-0.67</div> <div>0.32</div> </div>	x	<div style="display: flex; flex-direction: column; align-items: center;"> <div>12.4</div> <div>0</div> <div>0</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0</div> <div>9.5</div> <div>0</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0</div> <div>0</div> <div>1.3</div> </div>	x	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0.56</div> <div>0.12</div> <div>0.40</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0.59</div> <div>-0.02</div> <div>-0.80</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0.56</div> <div>0.12</div> <div>0.40</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0.09</div> <div>-0.69</div> <div>0.09</div> </div>	<div style="display: flex; flex-direction: column; align-items: center;"> <div>0.09</div> <div>-0.69</div> <div>0.09</div> </div>

Movie Recommendation Example

$$A_{[m \times n]} \approx U_{[m \times r]} \Sigma_{[r \times r]} (V_{[n \times r]})^T$$



Netflix Challenge

- \$1 million award for people beating their algorithm by 10%
- Simply implementing SVD already beats the algorithm Netflix was using...
- The winning team uses an ensemble of many methods
 - SVD is one major component

How to Perform SVD

- Should be covered in linear algebra class....
- Most likely, you will just use an existing library

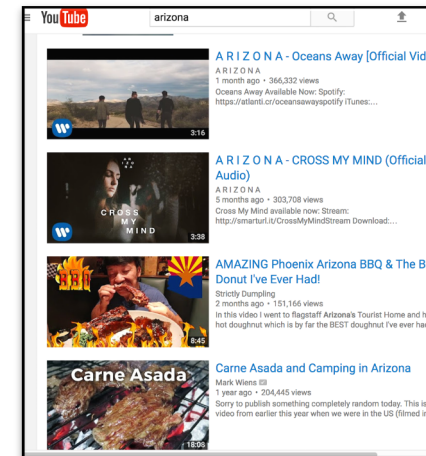
Let's (finally) get back to label aggregation

Who moderates the moderators? Crowdsourcing abuse detection in user-generated content. Ghosh, Kale, and McAfee. EC 2011.

User Generated Content

- Common practice to use user ratings to determine whether a content is good or not

- When a content receives a bad rating
 - is the content bad, or
 - is the rating bad?



1,504,905 views

42K 1K

12.2k Views · 119 Upvotes

- Given users ratings, how to decide content quality.
- This is a crowdsourcing label aggregation problem. With each rating being a label provided by a worker.

Model

NOTE:

1. The notations here are different from the previous slides about SVD.
2. The matrix also uses a different representation compared with the movie rating examples. In movie rating, each row is a user's ratings. Here, each column is a rater's ratings.

- Basic components

- n raters, $i = 1, \dots, n$
- T contributions, $t = 1, \dots, T$
- $u_{t,i} \in \{-1, 1\}$ is the rating rater i gives to contribution t

	Rater 1	Rater 2	Rater 3	Rater 4	...
Contribution 1	1	-1	-1	1	
Contribution 2	-1	1	1	-1	
Contribution 3	1	1	-1	1	
...					

U

- Label generation process

- Each contribution t has a true quality $q_t \in \{-1, 1\}$
- Each rater i gives *correct* rating with probability ψ_i

- Goal: Infer q_t for all t from rating matrix U (both q_t and ψ_i are unknown)

The Goal Seems Different from Movie Recommendation

- In movie recommendation
 - Goal: Fill in the empty ratings (and recommend the movie with highest one)
- In this paper
 - Assumption: all ratings are given
 - Goal: Infer the latent variable (true quality)

	Rater 1	Rater 2	Rater 3	Rater 4	...
Contribution 1	1	-1	-1	1	
Contribution 2	-1	1	1	-1	
Contribution 3	1	1	-1	1	
...					

- Connection: Low rank approximation of the rating matrix

Look at the “Expected” Rating Matrix $E[U]$

$U =$

	Rater 1	Rater 2	Rater 3	Rater 4	...
Contribution 1	1	-1	-1	1	
Contribution 2	-1	1	1	-1	
Contribution 3	1	1	-1	1	
...					

$E[U] =$

	Rater 1	Rater 2	Rater 3	Rater 4	...
Contribution 1	$q_1(2\psi_1 - 1)$	$q_1(2\psi_2 - 1)$	$q_1(2\psi_3 - 1)$	$q_1(2\psi_4 - 1)$	
Contribution 2	$q_2(2\psi_1 - 1)$	$q_2(2\psi_2 - 1)$	$q_2(2\psi_3 - 1)$	$q_2(2\psi_4 - 1)$	
Contribution 3	$q_3(2\psi_1 - 1)$	$q_3(2\psi_2 - 1)$	$q_3(2\psi_3 - 1)$	$q_3(2\psi_4 - 1)$	
...					

q_t : true label of contribution t
 ψ_i : prob of rater i to give correct rating

Look at the “Expected” Rating Matrix $E[U]$

$$E[U] =$$

	Rater 1	Rater 2	Rater 3	Rater 4	...
Contribution 1	$q_1(2\psi_1 - 1)$	$q_1(2\psi_2 - 1)$	$q_1(2\psi_3 - 1)$	$q_1(2\psi_4 - 1)$	
Contribution 2	$q_2(2\psi_1 - 1)$	$q_2(2\psi_2 - 1)$	$q_2(2\psi_3 - 1)$	$q_2(2\psi_4 - 1)$	
Contribution 3	$q_3(2\psi_1 - 1)$	$q_3(2\psi_2 - 1)$	$q_3(2\psi_3 - 1)$	$q_3(2\psi_4 - 1)$	
...					

$$= \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \dots \end{bmatrix} [(2\psi_1 - 1) \quad (2\psi_2 - 1) \quad (2\psi_3 - 1) \quad \dots]$$

$$= \mathbf{q} [\mathbf{1}] (2\boldsymbol{\psi} - \mathbf{1})^T$$

This is the exact singular value decomposition with **rank 1**

Look at the “Expected” Rating Matrix $E[U]$

$$E[U] = \mathbf{q} [\mathbf{1}] (2\boldsymbol{\psi} - \mathbf{1})^T$$

- The decomposition is not unique:
 - Multiply all elements in \mathbf{q} by -1 and all elements in $(2\boldsymbol{\psi} - \mathbf{1})$ by -1
 - What does this mean intuitively?
- Additional assumption:
 - $\psi_1 > 0.5$
 - Use this assumption to determine the sign

From $E[U]$ to U

- Exact rank 1 matrix decomposition

$$E[U] = \mathbf{q} [1] (2\boldsymbol{\psi} - \mathbf{1})^T$$

- Rank 1 matrix approximation

$$U \approx \mathbf{q}' [1] (2\boldsymbol{\psi}' - \mathbf{1})^T$$

Take the sign of \mathbf{q}' as the prediction of \mathbf{q}

More Details

- They don't directly do singular value decomposition
- The top left singular vector of U = Top eigenvector of UU^T
- In the algorithm, they perform eigenvalue decomposition of UU^T
 - They do it for efficiency reasons
 - In assignment 2, you can also directly do SVD of U
- Proposed algorithm: Spectral-Rating
 - Calculate the top eigenvector \mathbf{v} of UU^T
 - Let $\mathbf{s} = \text{sign}(\mathbf{v})$
 - Correct sign
 - If the majority of the sign is the same as the prediction of user 1, do nothing
 - Else, $\mathbf{s} \leftarrow -\mathbf{s}$
 - \mathbf{s} is the final prediction

Theoretical Guarantee

THEOREM 3.1. *There is a constant c such that if $T > \frac{2}{\gamma^2} \log(4/\eta)$ and $\frac{n}{\log(n)} > \frac{128}{c\bar{\kappa}^2}$, then for any $\eta \in (0, 1)$, with probability at least $1 - \eta$, we have*

$$\frac{1}{T} |\{t : q'_t \neq q_t\}| \leq \frac{8}{\bar{\kappa}} \sqrt{\frac{\log(n)}{cnT} \log\left(\frac{4}{\eta}\right)}.$$

n : # raters

T : # contributions

- Utilizing the matrix form of Hoeffding's inequality
- Average prediction error = $O\left(\frac{1}{2\bar{\psi}-1} \sqrt{\frac{\log n}{nT}}\right)$
- Focus on parameters you care about
 - How does error change as $\bar{\psi}$ changes
 - How does error change as n changes
 - How does error change as T changes

How to interpret a bound like this?

Extensions

- Not every rater rates every contribution

	Rater 1	Rater 2	Rater 3	Rater 4	...
Contribution 1	1		-1	1	
Contribution 2	-1	1			
Contribution 3			-1	1	
...					

Extensions

- Not every rater rates every contribution

	Rater 1	Rater 2	Rater 3	Rater 4	...
Contribution 1	1	0	-1	1	
Contribution 2	-1	1	0	0	
Contribution 3	0	0	-1	1	
...					

- An updated label generation process
 - Each rater i has a probability p_i to rate a contribution

$$u_{ti} = \begin{cases} q_t & \text{w.p. } p_i \psi_i \\ -q_t & \text{w.p. } p_i (1 - \psi_i) \\ 0 & \text{w.p. } 1 - p_i. \end{cases}$$

Is this a reasonable model?
Why do you think we need this model?

Extensions

- Computation is expensive for large datasets
 - UU^T is a T by T matrix
 - T (# contributions) is often huge in practice
- Online algorithm
 - For a small subset of contributions, solve for their quality
 - Used this subset to infer rater's skills ψ
 - Use weighted majority voting for new contributions (as in our lecture 3)

$$q_t = \text{sign} \left(\sum_i \ln \frac{\psi_i}{1-\psi_i} u_{t,i} \right)$$

Extensions

- Algorithm robustness against rater manipulations
 - Change their “skills” to influence the algorithm outcome
 - Change their labeling strategy (not randomized) to influence the algorithm outcome
 - Collude with other raters to influence the outcome for a particular contribution
- The results are “robust” if the ratio of manipulations is not large
- What if manipulations are prevalent
 - Learning with the presence of strategic behavior
 - We will discuss more on this in the lecture of Dec 8

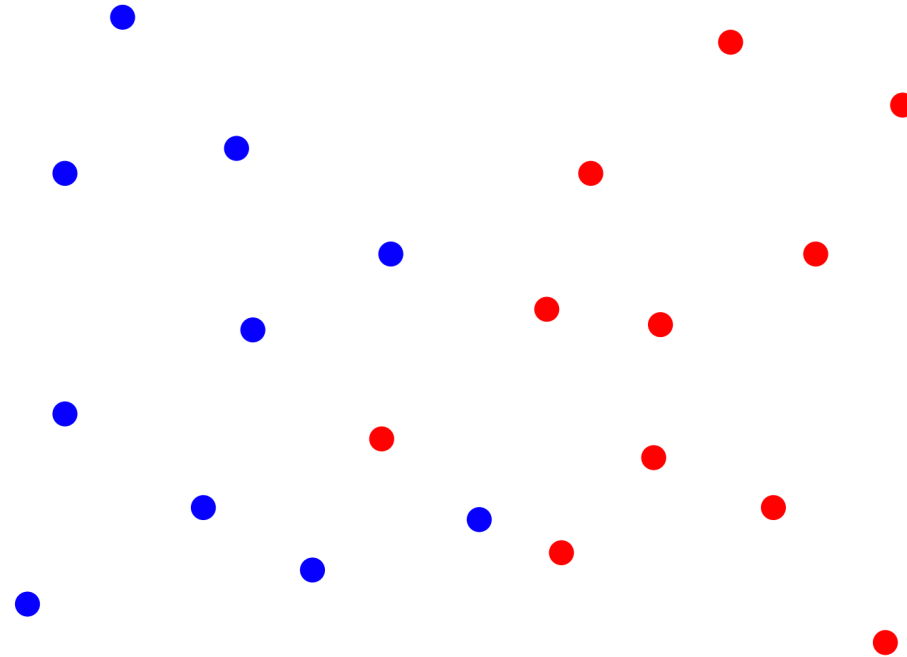
Learning with the Presence of Strategic Behavior

More on the lecture of Dec 8

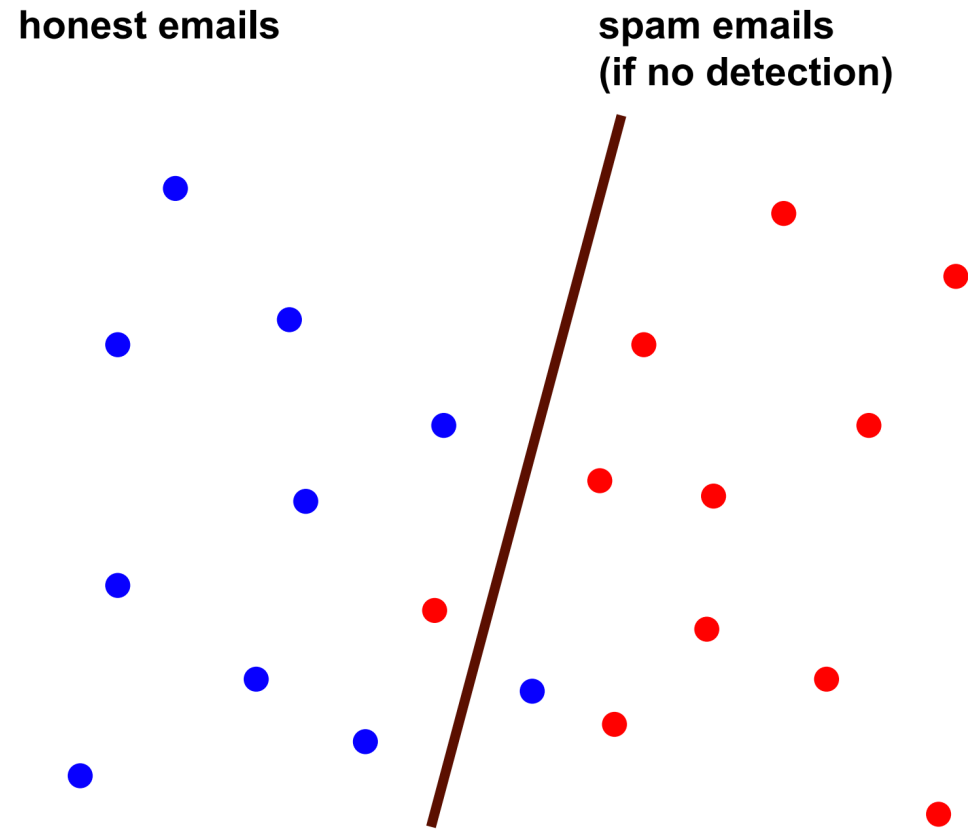
Example: Spam Classification

honest emails

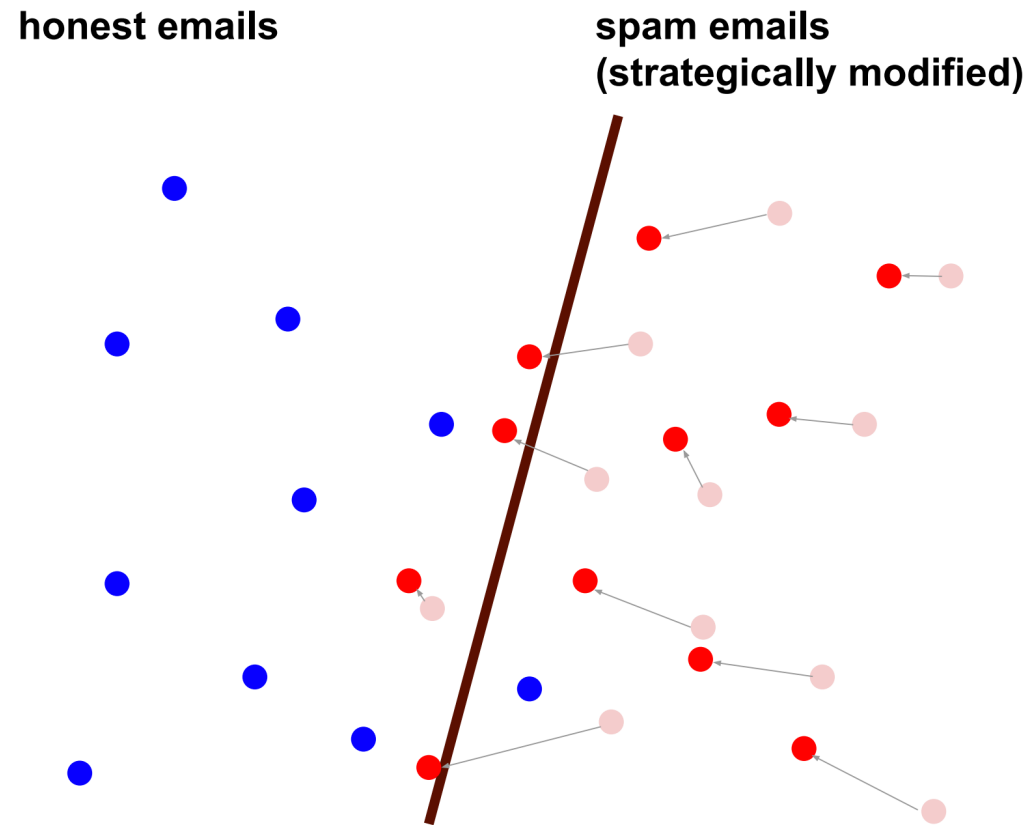
**spam emails
(if no detection)**



Example: Spam Classification



Example: Spam Classification



Goodhart's law:

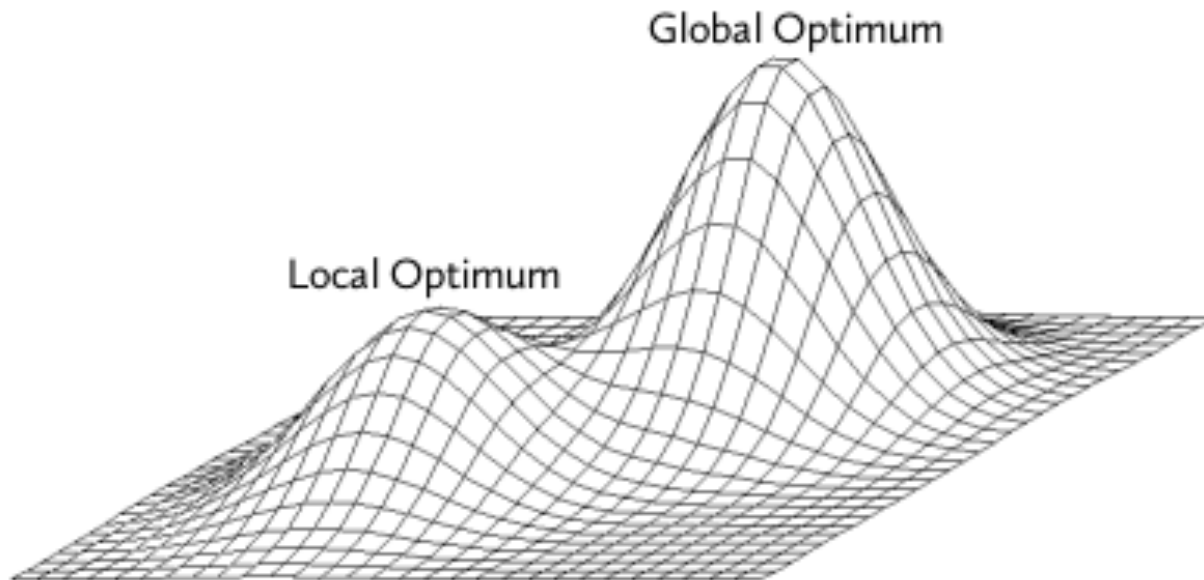
“If a measure becomes the public's goal, it is no longer a good measure.”

What We Learned So Far

- EM-based methods
 - Empirically performs well
 - Relatively computationally efficient
 - No theoretical guarantee
- Matrix-based methods
 - Comes with theoretical guarantee
 - Computationally expensive
- Can we achieve the best of both worlds?

Spectral Methods Meet EM

- Spectral Methods Meet EM: A Provably Optimal Algorithm for Crowdsourcing. Zhang et al. JMLR 2016.
- The main issue for EM: Might converge to local optimum



Spectral Methods Meet EM

- Key idea:
 - Estimate the “confusion matrix” from data
 - Using the estimation as the initial point for running the EM algorithm
- Key results
 - Given this fine-tuned starting point, with high probability, EM can achieve global optimal

Discussion

- General thoughts about this work.
- What are your thoughts on the manipulation issue? Any way to fight again it?
- What kind of research do you like? Why?
 - Empirically oriented: Design new algorithms and examine them on some datasets. No theoretical guarantees.
 - Theoretically oriented: Make assumptions on the target problems. Design algorithms and prove theoretical properties of the algorithms.