Problem Set 1

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1 Problem 1

An economy is composed of identical individuals. Each individual lives for 2 periods (you may imagine them as adulthood and old age). Individuals may work during the first period of their life for a proportion L of the day, for an income equal to wL. In the second period they retire and consume their remaining lifetime savings. Their lifetime utility is given by:

$$U = \ln C_1 + \ln C_2 + \ln (1 - L) \tag{1}$$

where C_i is consumption in period i and L is the fraction of time the individual spends working in period 1.

1.1

If the rate of interest on savings is R, write down the individual's budget constraints for both periods, and then combine them in a lifetime (intertemporal) budget constraint

1.1.1 Solution

The individual's budget constraint for the first period is the following:

$$C_1 + S = wL \tag{2}$$

Where C_1 is the consumption for adulthood, S refers to savings and wL is income.
The individual's budget constraint for the second period is the following:

$$C_2 = S(1 + R) \tag{3}$$

Where C_2 is the consumption for old age, S refers to savings and R is the interest rate.
In order to obtain the lifetime (intertemporal) budget constraint we derive S from (3):

$$S = \frac{C_2}{1 + R} \tag{4}$$

The next step is to insert S into (2):

$$C_1 + \frac{C_2}{1 + R} = wL \tag{5}$$

1.2

Solve for optimal consumption each period and the optimal work effort. Comment on what you find.

1.2.1 Solution

To obtain the optimal consumption for each period, it is necessary to implement the Lagrangian optimization method using the utility function and intertemporal budget constraint

$$\mathcal{L} = \ln C_1 + \ln C_2 + \ln(1-L) - \lambda(C_1 + \frac{C_2}{1+R} - wL) \tag{6}$$

$$\frac{d\mathcal{L}}{dC_1} = \frac{1}{C_1} - \lambda \tag{7}$$

$$\frac{d\mathcal{L}}{dC_2} = \frac{1}{C_2} - \frac{\lambda}{1+R} \tag{8}$$

$$\frac{d\mathcal{L}}{\lambda} = -C_1 - \frac{C_2}{1+R} + wL \tag{9}$$

$$\frac{d\mathcal{L}}{L} = -\frac{1}{1-L} + \lambda w \tag{10}$$

The equations (7) to (10) must be equal to 0. Given that the 4 equations contain unknowns, we need to solve for C_1, C_2 and L. From (7) we derive

$$C_1 = \frac{1}{\lambda} \tag{11}$$

From (11) we derive λ and insert it into (8) therefore resulting in

$$\frac{C_2}{C_1} = 1 + R \tag{12}$$

Then, deriving $C_2 = C_1(1+R)$ from (12) and inserting it into (9) we get

$$-C_1 - \frac{C_1(1+R)}{1+R} + wL = 0 \implies -2C_1 = -wL \implies C_1 = \frac{wL}{2} \tag{13}$$

Now we can go back to (12), substitute C_1 into the equation and solve for C_2 which gives back

$$C_2 = \frac{wL}{2}(1+R) \tag{14}$$

From (11) we derive $\lambda = \frac{1}{C_1}$. Then we insert it into (10) and get

$$\frac{1}{1-L} = \frac{w}{C_1} \implies C_1 = w(1-L) \tag{15}$$

This result together with number (13) is going to convey to:

$$\frac{wL}{2} = w(1-L) \implies \frac{wL}{2} + wL = w \implies L(\frac{w}{2} + w) = w \tag{16}$$

$$L = \frac{2}{3} \tag{17}$$

Now that labor is found, we put it into (13) and obtain

$$C_1 = \frac{w}{2}\frac{2}{3} = \frac{w}{3} \tag{18}$$

This result is substituted into (12), getting back C_2:

$$C_2 = \frac{w}{2}\frac{2}{3}(1+R) = \frac{w}{3}(1+R) \tag{19}$$

The derived optimal fraction of time spent working in the first period is $\frac{2}{3}$ and it indicates that an individual allocates two-thirds of the available time to labor in adulthood, thereby maximizing their utility derived from both consumption and leisure, where the latter is equal to $\frac{1}{3}$. The optimal consumption in the first period is equal to $1 - L = \frac{w}{3}$, suggesting that half of their income is being used in the first period (since whole income is $w\frac{2}{3}$). The second half of the income is saved in order to have sufficient resources to sustain consumption during retirement in the second period. In fact, the optimal consumption for the second period is equal to $\frac{w}{3}(1 + R)$, since it needs to be evaluated with respect to the gross interest rate. By consuming an equal amount in both periods, individuals avoid the diminishing marginal utility that would come up from consuming significantly more in one period over the other. This smoothing ensures that individuals derive consistent utility from consumption and leisure throughout their lifetime.

The additional possibility to look at the results is that in the utility function agent values equally consumption in the first period C_1, consumption in the second period $C2$, and leisure $1 - L$. That's why we have a coefficient $\frac{1}{3}$ near each of these terms in the final result.

1.3

The government introduces a fixed pension paid to individuals in the second period of their lives, funded by a "lump sum tax" paid by those who work. Re-write the intertemporal budget constraint. What will be the impact of the pension on consumption and labour supply decisions of the young? What about the old when the pension is introduced? Give intuition for your answers.

1.3.1 Solution

We assume that the pension paid to retired individuals (P) is equal to the tax paid by those who work (T), i.e., $T = P$. For convenience, only T is used in the further calculations. Now that a pension is introduced and a tax has to be paid by those who work, the budget constraint for the first period is:

$$C_1 + S + T = wL \tag{20}$$

The budget constraint for the second period:

$$C_2 = S(1 + R) + T \tag{21}$$

By combining the new budget constraints, we obtain once again the intertemporal budget constraint through the following passages:

$$S = \frac{C_2 - T}{1 + R} \tag{22}$$

Now we substitute S into equation (20)

$$C_1 = wL - \frac{C_2}{1 + R} + \frac{T}{1 + R} - T \tag{23}$$

$$C_1 + \frac{C_2}{1 + R} + T = wL + \frac{T}{1 + R} \tag{24}$$

Next we set up the Langrangian and solve the partial derivatives

$$\mathcal{L} = \ln C_1 + \ln C_2 + \ln (1 - L) - \lambda(C_1 + \frac{C_2}{1 + r} - wL + T - \frac{T}{1 + R}) \tag{25}$$

$$\frac{d\mathcal{L}}{dC_1} = \frac{1}{C_1} - \lambda \tag{26}$$

$$\frac{d\mathcal{L}}{dC_2} = \frac{1}{C_2} - \frac{\lambda}{1 + R} \tag{27}$$

$$\frac{d\mathcal{L}}{dL} = -\frac{1}{1 - L} + \lambda L \tag{28}$$

$$\frac{d\mathcal{L}}{d\lambda} = -C_1 - \frac{C_2}{1 + R} + wL - T + \frac{T}{1 + R} \tag{29}$$

Now we need to set up derivatives to zero and solve the system for C_1, C_2, L. Here derivatives by consumption and labor remain unchanged since previous task, so we can define relations between C_1 and λ (11), between C_1 and L (15) as well as C_2 and C_1 (12). Plugging C_2 from (12) into (29) we get:

$$-C_1 = \frac{-C_1(1+R)}{1+R} + wL - T + \frac{T}{1+R} \implies -2C_1 = T - \frac{T}{1+R} - wL \tag{30}$$

$$C_1 = \frac{wL}{2} - \frac{T}{2}(1 - \frac{1}{1+R}) \tag{31}$$

Then, making this value of C_1 equal to the value derived in (15), we get

$$\frac{wL}{2} - \frac{T}{2}(1 - \frac{1}{1+R}) = w(1-L) = w - wL \implies \frac{wL}{2} + wL = W + \frac{T}{2}(1 - \frac{1}{1+R}) \tag{32}$$

Rearranging terms and dividing everything by w implies

$$\frac{L}{2} + L = 1 + \frac{T}{2w}(1 - \frac{1}{1+R}) \tag{33}$$

So,

$$L = \frac{2}{3} + \frac{T}{3w}(1 - \frac{1}{1+R}) \tag{34}$$

By putting this value of L into (31) we get:

$$C_1 = \frac{w}{2}(\frac{2}{3} + \frac{T}{3w}(1 - \frac{1}{1+R})) - \frac{T}{2}(1 - \frac{1}{1+R}) = \frac{w}{3} + \frac{T}{6}(1 - \frac{1}{1+R}) - \frac{T}{2}(1 - \frac{1}{1+R}) \tag{35}$$

By simplifying terms, we get

$$C_1 = \frac{w}{3} - \frac{T}{3}(1 - \frac{1}{1+R}) \tag{36}$$

Putting this value of C_1 into (12) we get

$$C_2 = (\frac{w}{3} - \frac{T}{3}(1 - \frac{1}{1+R}))(1+R) = \frac{w}{3}(1+R) - \frac{1}{3}TR \tag{37}$$

As can be seen from (32), when the tax is introduced young will work more if $R > 0$ than they would in the economy without taxes, which we have shown in (17). Consumption of the young will decrease if $R > 0$, as can be seen in (36), as well as consumption of the old (37). In case when $R = 0$, no behavior will change compared to no tax economy.

This behavior can be explained by the fact that in our model pension is given to the old without interest which they would gain if they saved this amount instead of giving it as a tax, as can be seen from (20), (21), and (24). Also, the young need to consume less compared to the economy without taxes since part of their income is being paid as tax, as well as they need to work more.

1.4

The pension is now funded by an income tax, i.e., a tax equal to τwL where τ is the tax rate. Will the behaviour of the young change in the case where R = 0? Interpret this result.

1.4.1 Solution

As in the previous examples, we define the individual's budget constraints for both periods, and then combine them in a lifetime (intertemporal) budget constraint:

$$C_1 = (1-\tau)wL - S = wL - \tau wL - S \tag{38}$$

$$C_2 = (1+R)S + \tau wL \tag{39}$$

$$S = \frac{C_2 - \tau w L}{1+R} \tag{40}$$

$$C_1 + \frac{C_2}{1+R} + \tau w L = \frac{\tau(wL)}{1+R} + wL = wL(1 + \frac{\tau}{1+R}) \tag{41}$$

The intertemporal budget constraint just found contains all key economic indexes showing a complete economic background. In fact, it represents the total resources available to the individual in both periods, accounting for income, taxes, and interest rates. It shows how individuals allocate their resources between current and future consumption while considering the impact of taxes and interest rates.

Now as done in the previous points, we set up the new Lagrangian and the partial derivatives:

$$\mathcal{L} = \ln C_1 + \ln C_2 + \ln(1-L) - \lambda(C_1 + \frac{C_2}{1+r} + \tau w L - \frac{\tau w L}{1+R} - wL) \tag{42}$$

$$\frac{d\mathcal{L}}{dC_1} = \frac{1}{C_1} - \lambda = 0 \tag{43}$$

$$\frac{d\mathcal{L}}{dC_2} = \frac{1}{C_2} - \frac{\lambda}{1+R} = 0 \tag{44}$$

$$\frac{d\mathcal{L}}{dL} = -\frac{1}{1-L} - \lambda w L + \frac{\lambda w L}{1+R} + \lambda w = 0 \tag{45}$$

$$\frac{d\mathcal{L}}{d\lambda} = -C_1 - \frac{C_2}{1+R} - \tau w L + \frac{\tau w L}{1+R} + wL = 0 \tag{46}$$

Since $C_2 = C_1(1+R)$, we can plug it into (46) and get

$$-C_1 - \frac{C_1(1+R)}{1+R} - \tau w L(1 - \frac{1}{1+R}) + wL = 0 \implies 2C_1 = wL - \tau w L(1 - \frac{1}{1+R}) \tag{47}$$

Therefore,

$$C_1 = \frac{wL}{2} - \frac{\tau w L}{2}(1 - \frac{1}{1+R}) \tag{48}$$

Then, by plugging $\lambda = \frac{1}{C_1}$ into (45) we get

$$\frac{1}{1-L} = \frac{1}{C_1}(w - \tau w + \frac{\tau w}{1+R}) \tag{49}$$

$$C_1 = (1-L)(w - \tau w(1 - \frac{1}{1+R})) \tag{50}$$

By combining (50) and (48) we have

$$(1-L)(w - \tau w(1 - \frac{1}{1+R})) = \frac{wL}{2} - \frac{\tau w L}{2}(1 - \frac{1}{1+R}) \tag{51}$$

Multiplying everything by $\frac{2}{L}$, we have

$$2(\frac{1-L}{L})(w - \tau w(1 - \frac{1}{1+R})) = (w - \tau w(1 - \frac{1}{1+R})) \implies 2(\frac{1-L}{L}) = 1 \tag{52}$$

Therefore,

$$L = \frac{2}{3} \tag{53}$$

Plugging it into (48) gives:

$$C_1 = \frac{w}{2}\frac{2}{3} - \frac{2}{3}\frac{w}{2} - \frac{\tau w}{2}(1 - \frac{1}{1+R}) = \frac{w}{3} - \frac{\tau w}{3}(1 - \frac{1}{1+R}) \tag{54}$$

In conclusion, the labor supply of the young is not dependent on R and is $\frac{2}{3}$, as in no-tax economy, described in a) and b). Consumption of the young, however, depends on R. But in case the latter is equal to 0, consumption is equal to $\frac{w}{3}$, which is the same as in the no-tax economy. Therefore, the behaviour of the young will not change with the introduction of the tax in the case where $R = 0$.

The reason for this is that since the interest rate is 0, individuals don't have incentives to save while they are young. Therefore, in this economy, individuals are indifferent between savings and paying taxes since the latter are returned as pensions in the second period. So, all savings from the no-tax economy are turned into taxes in this economy.

2 Problem 2

The link to the GitHub repository can be found here.

2.1 Replicating the tables

We used the FRED API to retrieve economic data for analysis. Using the Python script automated the fetching and preprocessing of multiple economic time series from 1950 to 2023, focusing on indicators like GDP, consumption, employment, and average hours worked. Since quarterly data was not available for all the variables, the data retrieved is annual.

2.1.1 Replicating Table 1

We calculated the variability and cross-correlation between GDP and other economic variables across different time lags.

	-4	-3	-2	-1	0	1	2	3	4	SD%
GDP	0.996157	0.996902	0.998006	0.999012	1.0	0.999012	0.998006	0.996902	0.996157	0.456230
CND	0.989164	0.990058	0.989764	0.987779	0.986553	0.986209	0.98596	0.98637	0.988491	0.750827
CD	0.991568	0.993298	0.989355	0.985254	0.981415	0.978577	0.976664	0.974612	0.974694	0.781369
H	0.956995	0.951185	0.950265	0.951792	0.953955	0.948412	0.940684	0.933495	0.926917	0.206383
L	0.975264	0.971531	0.969728	0.970067	0.971627	0.9699	0.966303	0.962447	0.958619	0.255275
AveW	0.995252	0.995148	0.994857	0.994162	0.993219	0.993012	0.992738	0.992388	0.993354	0.583056
GDP/L	0.988295	0.990481	0.99224	0.99373	0.99417	0.994016	0.995472	0.996699	0.997268	0.226308
AveH	-0.881602	-0.883717	-0.879279	-0.875903	-0.876424	-0.889888	-0.908078	-0.924259	-0.930663	0.065301

The correlation coefficients are very high, suggesting that the Economist might have used some de-trending techniques that were not specified in the Problem Set.

2.1.2 Replicating Table 2

We set up API request parameters to fetch data for specific economic indicators from the FRED API. After retrieving and merging the data into a unified data frame, we cleaned the dataset and restrict our analysis to the years 1982 to 2007. By applying the Hodrick-Prescott filter, we isolated the cyclical components of each time series to understand their economic fluctuations better. Finally, we calculated and compiled key statistics such as standard deviation, relative standard deviation, and first-order autocorrelation into a summary table.

	SD	SD%	p	corr_Y
Y	0.034149	1.000000	0.764941	1.000000
C	0.035240	1.031949	0.827277	0.978454
I	0.031741	0.929493	0.717496	0.518569
w	0.019735	0.577904	0.712322	-0.155843
r	1.802591	52.786217	0.516510	0.142384
A	0.011091	0.324782	0.558993	0.154666
N	0.012061	0.353200	0.643295	0.558470
Y/N	0.029182	0.854545	0.789934	0.939387

2.2 Verify whether or not the following business cycle facts from Cooley and Prescott (1995) still hold today:

2.2.1 Consumption is smoother than output.

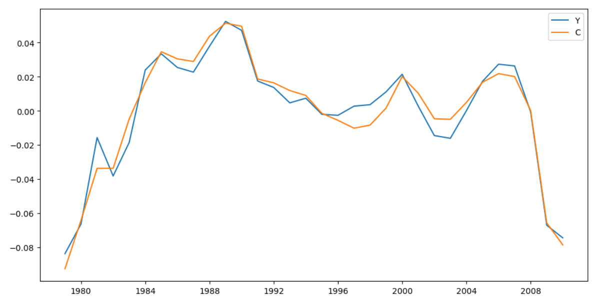

Figure 1: Line Graph of Output and Consumption

Smoothness coefficient for Y: 0.021577125401106397
Smoothness coefficient for C: 0.018898095189354272

We have calculated the smoothness of Y and C by measuring the standard deviation of the differences between various years. Consumption is smoother than output due to its lower standard deviation of differences.

2.2.2 Volatility in GDP is similar in magnitude to volatility in total hours.

GDP Relative SD: 0.45622985774816327
Total Hours Relative SD: 0.20638266428542026

They are not similar in magnitude; the relative standard deviation of GDP is more than double that of total hours.

2.2.3 Volatility in employment is greater than volatility in average hours. Therefore most labour market adjustments operate on the extensive rather than intensive margin.

Employment Relative SD: 0.25527535497308446
Average Hours Relative SD: 0.0653007267399741

Yes, the volatility in employment is much greater than the volatility in average hours.

2.2.4 Productivity is slightly pro-cyclical.

Correlation between Y and A: 0.15466648915714679

Yes, productivity is slightly pro-cyclical because the correlation is slightly positive.

2.2.5 Wages are less variable than productivity.

Wages Relative SD: 0.03201728814382639
Productivity Relative SD: 0.07198321702621593

Yes, wages are less variable than productivity.

2.2.6 There is no correlation between wages and output (nor with employment for that matter).

Correlation between Y and w: 0.7189419826488415

Yes, there is a strong positive correlation between wages and output.

2.3 Verify whether or not the following business cycle facts from King and Rebelo (1999) still hold today:

2.3.1 Consumption of non-durables is less volatile than output.

CND Relative SD: 0.7508268658127834

Output Relative SD: 0.45622985774816327

No, consumption of non-durables is more volatile than output.

2.3.2 Consumer durables are more volatile than output.

CD Relative SD: 0.781368866709871

Yes, durables consumption is more volatile than output.

2.3.3 Investment is three times more volatile than output.

Investment SD: 0.18126323777437398

Output Relative SD: 0.39756127545406267

No, investment is less volatile than output.

2.3.4 Government expenditures are less volatile than output.

We don't have data for government spending and trade balance in the dataset. We will fetch the data from FRED. Government Spending Relative SD: 0.3883564096965604

Output Relative SD: 0.45622985774816327

Yes, Government Spending is less volatile than output.

2.3.5 Total hours worked are about the same volatility as output.

Total Hours Relative SD: 0.20638266428542026

Output Relative SD: 0.45622985774816327

The volatility of the output is more than twice that of the total hours worked.

2.3.6 Capital is much less volatile than output.

We were unable to find data on real total capital, so we retrieved nominal data from FRED for both capital and GDP. Comparing real GDP with nominal capital wouldn't make sense, as the volatility of the latter is influenced by inflation.

Capital Relative SD: 0.9768640775216679

Output Relative SD: 1.0005333399020928

Capital is slightly less volatile than output.

2.3.7 Employment is as volatile as output, while hours per worker are much less volatile than output.

Employment Relative SD: 0.25527535497308446

Output Relative SD: 0.45622985774816327

No, employment is less volatile than output.

Hours per Worker SD: 0.033349727279074724

Output Relative SD: 0.39756127545406267

Yes, hours per worker is much less volatile than output.

2.3.8 Labour productivity is less volatile than output.

Productivity SD: 0.07198321702621593
Output Relative SD: 0.39756127545406267

Yes, productivity is less volatile than output.

2.3.9 The real wage is much less volatile than output.

Real Wage Relative SD: 0.03201728814382639
Output Relative SD: 0.39756127545406267

Yes, the real wage is less volatile than output.