Problem Set 2

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1 Problem 1

Consider a version of the Real Business Cycle model presented in class in which there is 100% capital depreciation and utility is log-linear in consumption and leisure.

1.1

Discuss why the competitive equilibrium of this economy is equivalent to the following:

$$\max_{\{c_t, l_t\}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t (\ln c_t - \chi l_t) \tag{1}$$

subject to:

$$y_t = c_t + k_t, \tag{2}$$
$$y_t = A_t k_{t-1}^{\alpha} l_t^{1-\alpha}, \tag{3}$$
$$\ln A_t = \rho \ln A_{t-1} + \epsilon_t. \tag{4}$$

Derive the intertemporal Euler equations for consumption and labour supply, and the intratemporal Euler labour-consumption equation for this economy.

1.1.1 Discuss why the competitive equilibrium of the economy

The maximisation of the utility function gives the competitive equilibrium of this economy through the choice of two variables: Consumption c_t and labour l_t. The utility function is the expected value (because we are in a stochastic environment) of the discounted sum of log consumption minus the disutility of labour times labour. Labour has a negative sign because it is a cost in terms of utility for the household. The higher is χ, the more the household values leisure (or disvalues labour).
Constraint (2) is the budget constraint for the household, which states that the income must be equal to consumption and capital.
Constraint (3) states that the income must be equal to the firm production because of market clearing. α is the distribution of income between capital and labour, which are the two factors of production.
Constraint (4) states that productivity is equal to the sum of the previous period's productivity plus a random shock ϵ_t. If there is memory between periods ($\rho > 0$), the productivity follows a random walk path. Both current and previous period productivity are expressed in ln because productivity cannot be negative.

1.1.2 Intratemporal Euler labour-consumption equation

We know that we can simplify the problem by considering only two consecutive periods (for example 0 and 1):

$$\max_{\{c_0, l_0, k_0\}} \ln c_0 - \chi l_0 + \beta(\ln c_1 - \chi l_1) \tag{5}$$

Let's set up the Lagrangian:

$$\mathcal{L} = (\ln c_0 - \chi l_0) + \lambda_0(c_0 + k_0 - A_0 k_{-1}^{\alpha} l_0^{1-\alpha}) + \beta \mathbb{E}(\ln c_1 - \chi l_1 + \lambda_1(c_1 + k_1 - A_1 k_0^{\alpha} l_1^{1-\alpha})) \tag{6}$$

Let's derive the FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_0} = \frac{1}{c_0} + \lambda_0 = 0, \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial k_0} = \lambda_0 - \beta \mathbb{E}(\lambda_1 A_1 \alpha k_0^{\alpha-1} l_1^{1-\alpha})), \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial l_0} = -\chi - \lambda_0 (1-\alpha)(A_0 k_{-1}^{\alpha} l_0^{-\alpha}) = 0 \tag{9}$$

Taking into account (3) and the first-order conditions (7) and (9), we have:

$$\lambda_0 = -\frac{1}{c_0}, \tag{10}$$

$$-\chi = \lambda_0 \frac{y_0}{l_0}(1-\alpha) \tag{11}$$

Solving we get:

$$\chi = \frac{1}{c_0}\frac{y_0}{l_0}(1-\alpha) \tag{12}$$

$$l_0 = \frac{(1-\alpha)y_0}{c_0 \chi} \tag{13}$$

where (13) is the Intratemporal Euler labour-consumption equation.

1.1.3 Intertemporal Euler Equation for Consumption

Note that:

$$\lambda_0 = u'(c_0), \tag{14}$$

$$\lambda_1 = u'(c_1) \tag{15}$$

So from (8) :

$$\frac{1}{c_0} = \beta \mathbb{E}(\frac{1}{c_1}\alpha \frac{y_1}{k_0}) \tag{16}$$

where (16) is the Intertemporal Euler Equation for Consumption.

1.2 Bonus question

Define the output to consumption ratio $\vartheta_t = \frac{y_t}{c_t}$. Show that the Euler equation for consumption can then be written as a first-order ordinary difference equation in terms of ϑ_t and $E_t\vartheta_{t+1}$, and then calculate an expression for the steady-state value of ϑ_t. Plot a graph of the ordinary difference equation with ϑ_t on the horizontal axis and $E_t\vartheta_{t+1}$ on the vertical axis. Does your graph suggest that the ordinary difference equation for ϑ_t is stable?

(a) If you think the ordinary difference equation is stable then comment on what factors affect the speed with which ϑ_t converges to its steady-state value.

(b) If you think the ordinary difference equation is unstable then comment on whether this is a sensible model.

(c) How does the Blanchard-Kahn condition matter for whether the ordinary difference equation is stable or unstable? Is the transversality condition respected or violated in this model?

1.2.1 Defining the expression and computing the steady state

Given the output to consumption ratio in t and t+1

$$\vartheta_t = \frac{y_t}{c_t} \tag{17}$$

$$\vartheta_{t+1} = \frac{y_{t+1}}{c_{t+1}} \tag{18}$$

and the following Euler equation

$$\frac{1}{c_t} = \beta \mathbb{E}\left(\frac{1}{c_{t+1}} \alpha \frac{y_{t+1}}{K_t}\right) \tag{19}$$

we perform a substitution into the latter and obtain

$$\frac{1}{c_t} = \frac{\alpha\beta}{K_t} \mathbb{E}(\vartheta_{t+1}) \tag{20}$$

where K_t is not subject to random fluctuations or uncertainty. Therefore

$$\frac{K_t}{c_t} = \alpha\beta \mathbb{E}(\vartheta_{t+1}) \tag{21}$$

Given that, previously

$$y_t = c_t + K_t \implies K_t = y_t - c_t \tag{22}$$

we have

$$\frac{y_t}{c_t} - \frac{c_t}{c_t} = \alpha\beta \mathbb{E}(\vartheta_{t+1}) \tag{23}$$

$$\frac{y_t}{c_t} = \alpha\beta \mathbb{E}(\vartheta_{t+1}) + 1 \tag{24}$$

So, in the steady state

$$\vartheta_t = \vartheta_{t+1} = \overline{\vartheta} \tag{25}$$

$$\overline{\vartheta} = \alpha\beta\overline{\vartheta} + 1 \tag{26}$$

$$\overline{\vartheta}(1 - \alpha\beta) = 1 \tag{27}$$

and ultimately

$$\overline{\vartheta} = \frac{1}{1 - \alpha\beta} \tag{28}$$

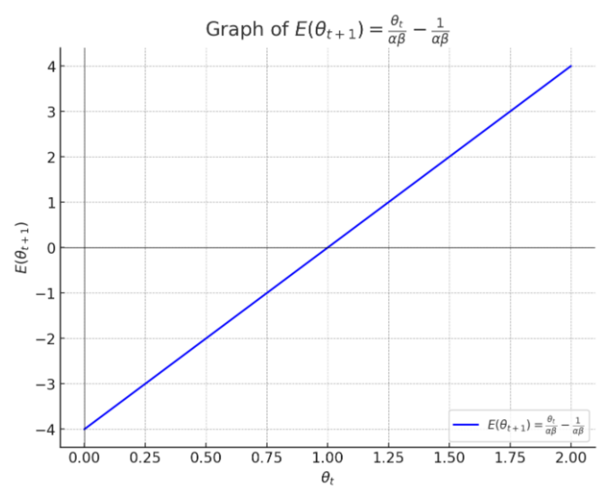

Figure 1: It's unstable because ϑ is diverging from $E[\vartheta]$

No, the model is not sensible because it's not converging (or we made some mistake).
The transversality condition is violated, because the behaviour of ϑ explodes over time.

1.3

Assume that ϑ_t is equal to its steady state value, i.e., $\vartheta_t = \vartheta = \frac{y}{c}$, $\forall t$. Show that in this case the solution of the model is given by:

$$l_t = \frac{1-\alpha}{\chi(1-\beta\alpha)} \tag{29}$$

$$y_t = A_t k_{t-1}^{\alpha} \left[\frac{1-\alpha}{X(1-\alpha\beta)}\right]^{1-\alpha} \tag{30}$$

$$c_t = (1-\alpha\beta)y_t \tag{31}$$

$$k_t = y_t - c_t \tag{32}$$

$$lnA_t = \rho \ln A_t + \epsilon_t \tag{33}$$

A calibration of the model is $\alpha = 0.4$, $\chi = 2$, $\beta = 0.99$, $\rho = 0.95$, and $\sigma_a = 0.01$. Derive the numerical equations corresponding to the second equation of y_t and calculate the steady state values of l, y, c, k, and A. Simulate the economy for 40 periods and calculate the standard deviation of simulated output, consumption, and labour supply. Use impulse response functions to show how the endogenous variables in the economy react to a one standard deviation technology.

1.3.1 Solution

From (16) we can derive

$$\frac{1}{c_t} = \frac{\beta\alpha}{k_t}\vartheta \tag{34}$$

Also we omit expectation as $\frac{y}{c}$ is constant

$$k_t = c_t \vartheta \beta \alpha \tag{35}$$

$$k_t = y_t - c_t \tag{36}$$

By equating (35) and (36) we get:

$$y_t - c_t = c_t \vartheta \beta \alpha \tag{37}$$

$$\frac{y_t}{c_t} - 1 = \vartheta \beta \alpha \tag{38}$$

$$\vartheta - 1 = \vartheta \beta \alpha \tag{39}$$

$$\vartheta = \frac{1}{1-\beta\alpha} \tag{40}$$

$$\vartheta = \frac{y_t}{c_t} = \frac{1}{1-\beta\alpha} \tag{41}$$

$$c_t = (1-\alpha\beta)y_t \tag{42}$$

$$l_t = \frac{1-\alpha}{\chi(1-\beta\alpha)} \tag{43}$$

And the rest of the equations can easily be derived to be as per the problem setup.

For deriving numerical values in the steady state, we need to note that $A_t = 1$, and all other endogenous variables follow the rule $k_{t-1} = k_t = k$. So, as per problem setup, we have:

$$k = y - (1-\alpha\beta)y \implies y = \frac{k}{\alpha\beta} \tag{44}$$

Then, by equating equations for y, we have:

$$\frac{k}{\alpha\beta} = k^{\alpha} A \left(\frac{1-\alpha}{\chi(1-\beta\alpha)}\right)^{1-\alpha} \tag{45}$$

$$k^{1-\alpha} = \frac{1}{\alpha\beta A(\frac{1-\alpha}{\chi(1-\beta\alpha)})} \tag{46}$$

$$k = (\frac{1}{\alpha\beta A(\frac{1-\alpha}{\chi(1-\beta\alpha)})})^{\frac{1}{1-\alpha}} \tag{47}$$

So,

$$y = (\frac{1}{\alpha\beta A(\frac{1-\alpha}{\chi(1-\beta\alpha)})})^{\frac{\alpha}{1-\alpha}} A(\frac{1-\alpha}{\chi(1-\beta\alpha)}) \tag{48}$$

The Dynare part can be found on the GitHub.
The standard deviations of the requested variables are the following:

$$Y = 0.016182, \tag{49}$$
$$C = 0.0097738, \tag{50}$$
$$L = 0 \tag{51}$$

Labour is a constant, so it has a 0 standard deviation.

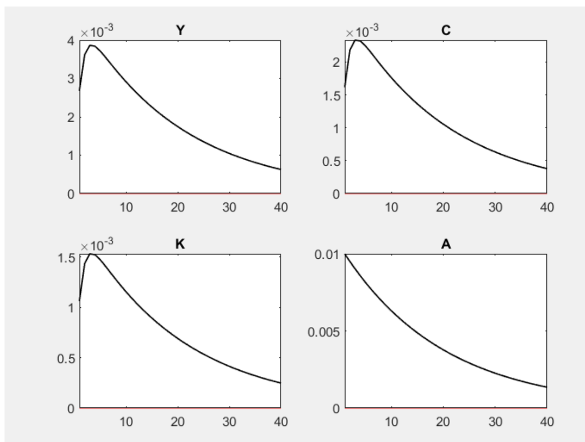

Figure 2: Dynare Simulation $\sigma = 0.01$

2 Problem 2

Consider a basic RBC model, where the social planner wants to maximize

$$\mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) - V(L_{t+i})) \right] \tag{52}$$

where C_t is consumption, L_t is hours worked, and β is the representative household's rate of time preference (discount factor). The economy faces constraints described by:

$$Y_t = C_t + I_t, \tag{53}$$
$$Y_t = F(K_t, L_t), \tag{54}$$
$$K_{t+1} = I_t + (1-\delta)K_t \tag{55}$$

where $F(K_t, L_t)$ is the production technology of output Y_t with constant returns to scale, I_t is investment, and δ is the rate of depreciation of capital. We can simplify the problem by combining the constraints into one equation:

$$F(K_t, L_t) = C_t + K_{t+1} - (1-\delta)K_t. \tag{56}$$

2.1

Does the real business cycle (RBC) model predict that real wages should be procyclical or countercyclical? How about employment? Why?

2.1.1 Wages

According to the RBC model, real wages are procyclical. This happens because a positive productivity shock increases the marginal product of labour, which we assume is equal to wages. Intuitively, if productivity increases the firm will want to hire more because it can produce more with a single unit of labour. This will increase labour demand and consequently wages.

2.1.2 Labour

On the other hand, employment has an ambiguous response to an income shock. This is because when income increases, as a consequence of the productivity shock, also consumption rises. This creates a decrease in labour supply because there is a trade-off between labour and consumption.

2.2

What does the empirical evidence say about the direction and magnitude of the fluctuations in these variables in comparison with the model's predictions?

2.2.1 Labour and Wages

Check GitHub for the code. The empirical evidence shows that wages and labour are both procyclical. As we learned in class, the RBC model overestimates the correlation of labour and wages with output.

2.3

What are the implications of labour market developments for interpreting the validity of the RBC framework?

2.3.1

In the real world, there are several reasons that make labour and wages less flexible.
For example: contracts fix wages until their expiry.
In the model, we also have perfect information and rational agents, and we know that this is possible only in theory.
In addition, the model doesn't consider important factors such as monetary and fiscal policy.
In conclusion: when we interpret the results of the RBC model, we have to check that the variables that are not taken in account by the model didn't played a role in the years we are analysing.