# Mobius Transformations and Inversions

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Complex Analysis

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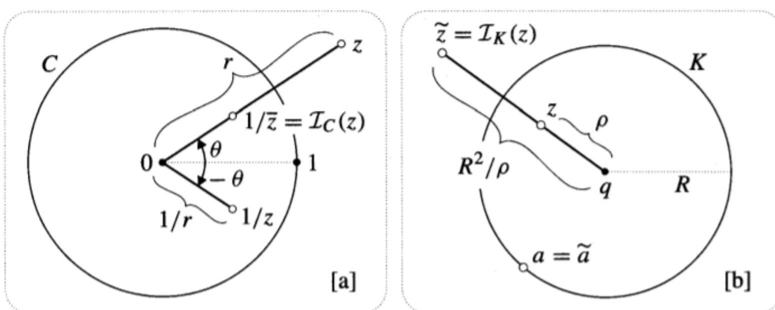
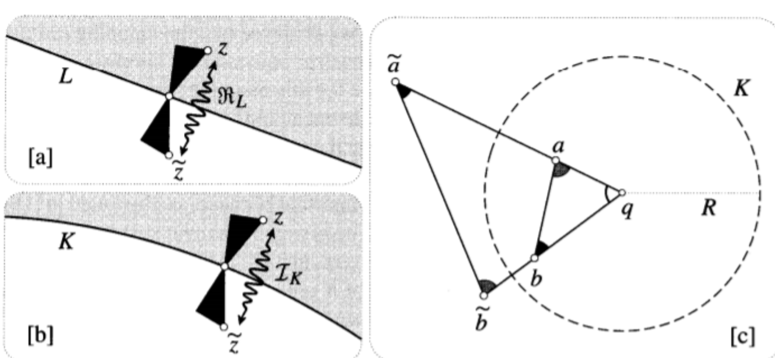
1. $M(z) = (az+b)/(cz+d)$; $M(z) = -(ad-bc)/(c^2(z+d/c)) + a/c$; Take $z$, apply translation of $d/c$, apply complex inversion, apply dilative rotation of $-(ad-bc)/c^2$ and apply translation of $a/c$, and the resulting complex number would be M.T. of $z$.

2. Complex inversion comprise of : $z \to 1/\bar{z}$, $z \to \bar{z}$; Geometric inversion: $\mathcal{I}_C(z) = 1/\bar{z}$, where $C$ is origin centered unit circle; For geometric inversion in general circle $K$, $\tilde{z} = \mathcal{I}_K(z)$ is obtained by: $(\tilde{z}-q)(\bar{z}-\bar{q}) = R^2$, $\tilde{z} = (R^2 - |q|^2 + \bar{z}q)/(\bar{z}-\bar{q})$.

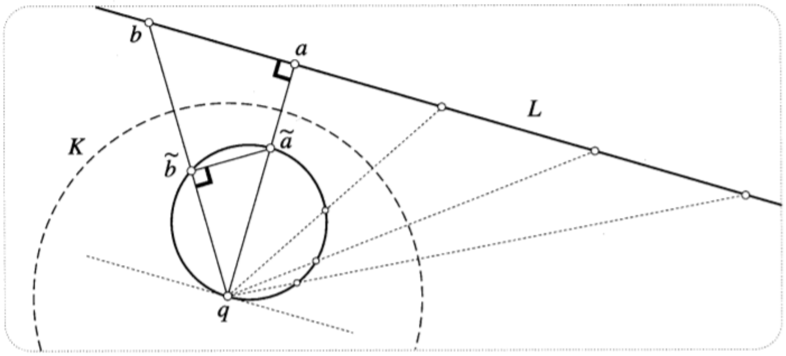



3. Line $L$ divides plane into two parts, $\mathcal{R}_L$ (reflection in $L$) interchanges those parts, $\mathcal{R}_L(L) = L$, $\mathcal{R}_L(\mathcal{R}_L(z)) = z$; $\mathcal{I}_K$ shares all three properties. As $K$ gets larger or the point to be inverted/reflected comes closer to $K$, $\mathcal{I}_K$ behaves as $\mathcal{R}_K$.

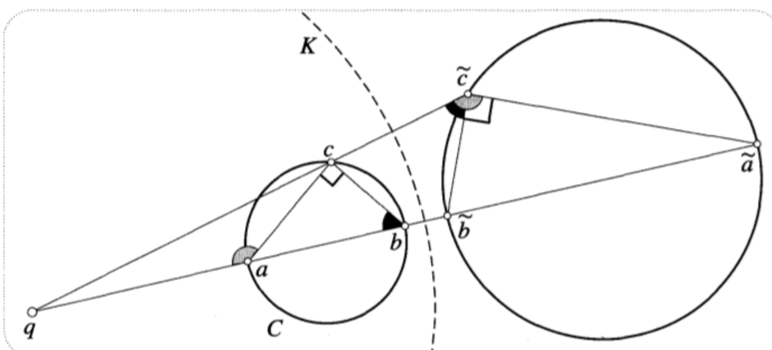



4. $\tilde{a} = \mathcal{I}_K(a)$, $\tilde{b} = \mathcal{I}_K(b)$, $[q\tilde{a}][qa] = R^2 = [qb][q\tilde{b}]$, $aqb \sim \tilde{b}q\tilde{a}$, $[\tilde{a}\tilde{b}]/[ab] = [q\tilde{a}]/[qb]$, $[\tilde{a}\tilde{b}] = ([ab]R^2)/([qa][qb])$.

5. $L$ pass through $q$ then $\mathcal{I}_K(L) = L$; $L$ does not pass through $q$ then $\mathcal{I}_K(L) = K'$ where $K'$ is a circle through $q$, tangent to which at $q$ is parallel to $L$. $K_1$,$K_2$ with centre $q$ then $[q\tilde{z}_2]/[q\tilde{z}_1] = R_1^2/R_2^2 = k$, so, $\mathcal{I}_{K_2} = \mathcal{D}_q^k \circ \mathcal{I}_{K_1}$.

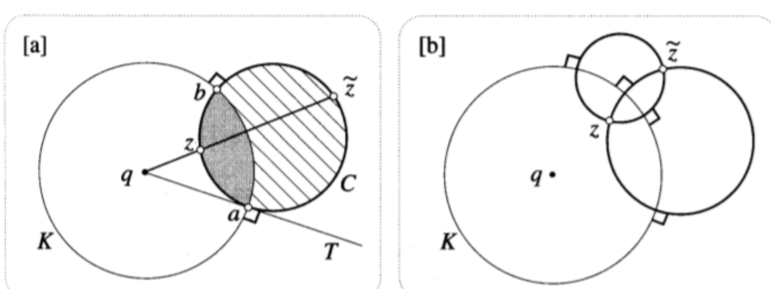



6. $K'$ passing through $q$ then $\mathcal{I}_K(K') = L$ where $L$ is parallel to tangent to $K'$ at $q$; $K'$ doesn't pass through $q$ then $\mathcal{I}_K(K') = K''$ where $K''$ doesn't pass through $q$.

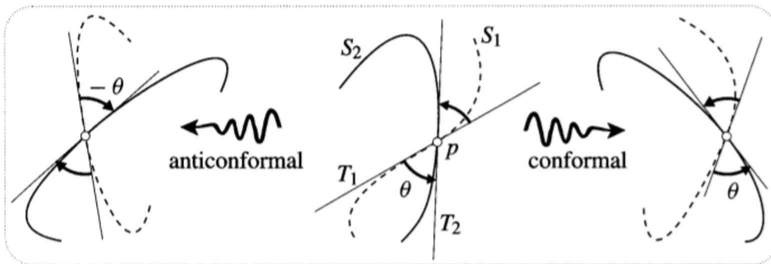



7. $K' \perp K$ then $\mathcal{I}_K(K') = K'$. $K' \perp K, K'' \perp K$ s.t. $K'$ and $K''$ intersect at $z_1, z_2$, then $I_K(z_1) = z_2$; $\tilde{z} = I_K(z)$ is the second intersection point of any two circles passing through $z$ and $\perp$ to $K$.

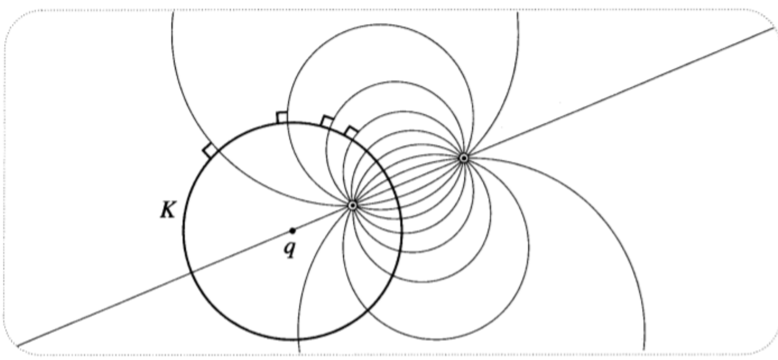
8. As $R \to \infty$, $I_K(z) = (iR\bar{z})/(\bar{z} + iR) = \bar{z}/(1 - (i\bar{z}/R)) \to \bar{z} = \mathcal{R}_L(z)$; As $z$ gets closer to a point $p$ (say 0) on circle ($R$ is fixed) and $|z| < R$, $\mathcal{I}_K(z) = \bar{z} + i\bar{z}^2/R + \ldots \to \bar{z}$.

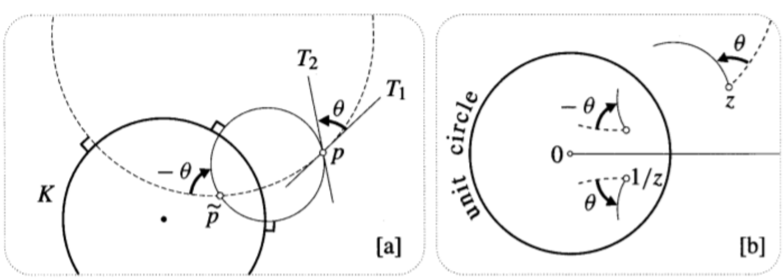


9. Conformal at $p$: When transformation preserves sign and magnitude of the angle between any two curves sufficiently smooth at $p$; Anticonformal at $p$: Magnitude preserved, sign reversed; Conformal map: Conformal for all $p$; Anticonformal map: Anti conformal for all $p$; Isogonal map: Magnitude preserved for all $p$, can't say anything about sign.




10. Geometric inversion is anticonformal (draw $\perp$ circle to $K$ passing through $z$ at a specific angle). Complex inversion is conformal. Even number of reflections (in lines or circles) is conformal, odd is anticonformal.

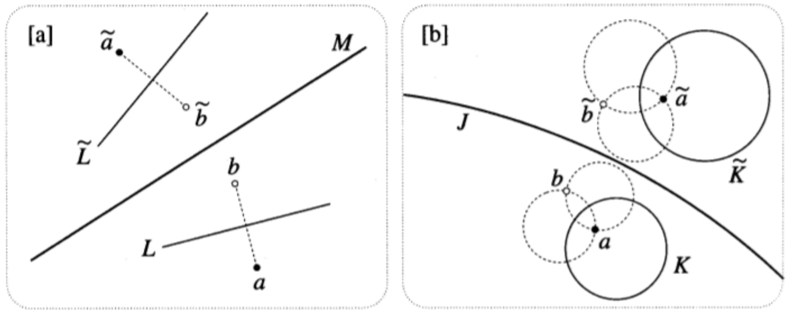


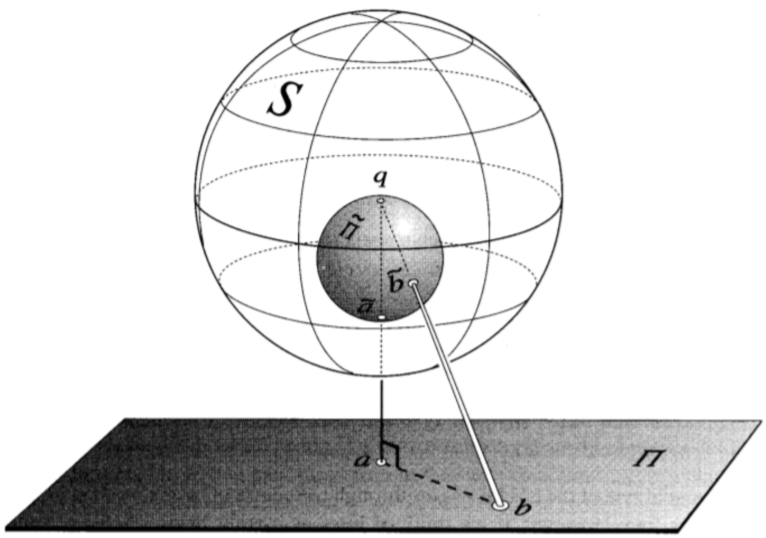



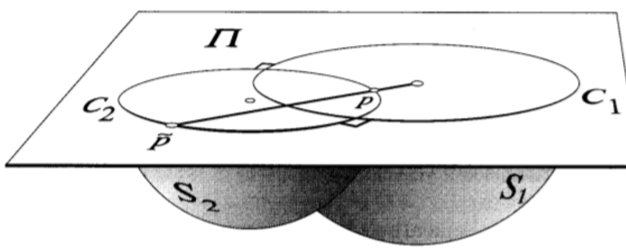

11. Inversion maps any pair of $\perp$ circles to another pair of $\perp$ circles; If $a$ and $b$ are symmetric wrt $K$ then $\tilde{a}, \tilde{b}, \tilde{K} = \mathcal{I}_J(a, b, K)$, $\tilde{a}$ and $\tilde{b}$ are symmetric wrt $\tilde{K}$.

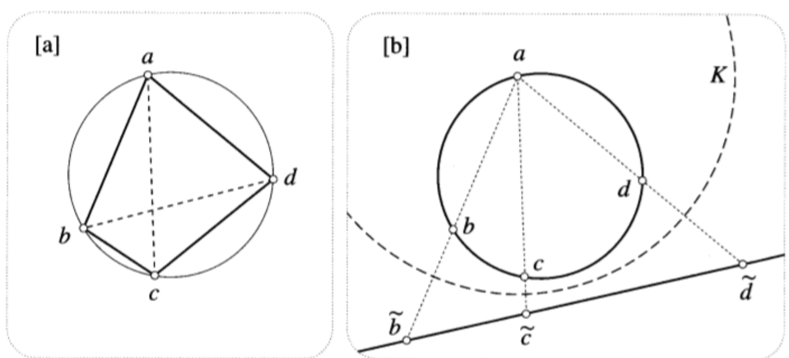
12. Analogous results for inversion in a sphere; Let $S_1$ and $S_2$ be intersecting spheres, and let $C_1$ and $C_2$ be the great circles in which these spheres intersect a plane $\prod$ passing through their centres. Then $S_1 \perp S_2 \iff C_1 \perp C_2$.

13. Plotemy's theorem $[ab][cd] + [ad][bc] = [ac][bd]$ where $a, b, c, d$ lie on a circle.



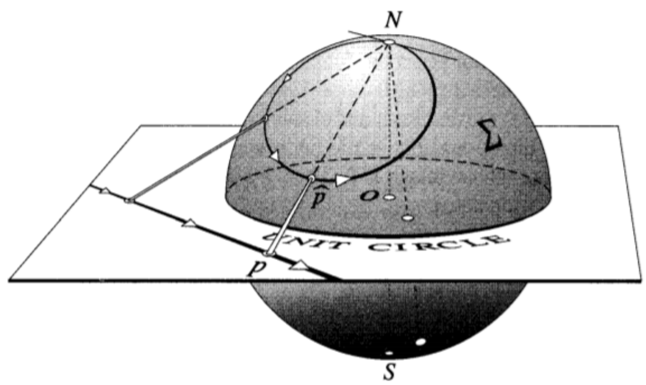



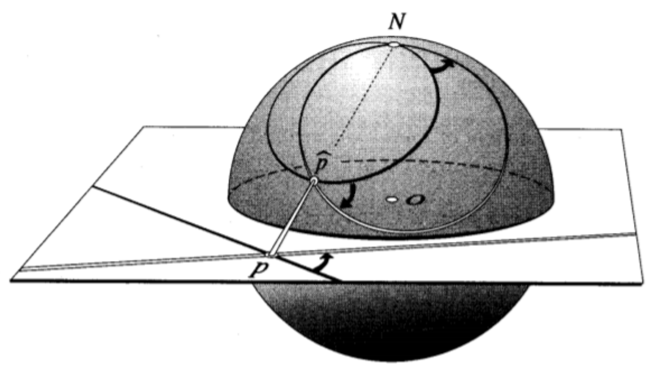



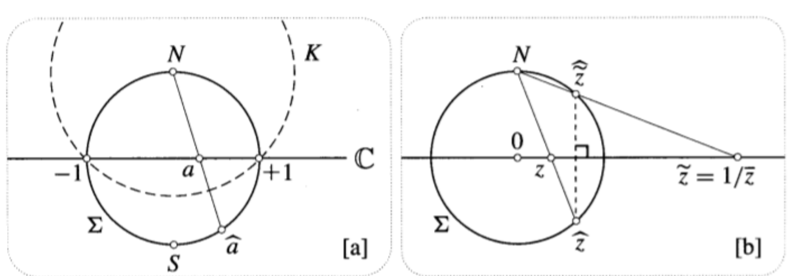

14. Extended complex plane = complex plan with a point $\infty$; Stereographic projection: Angle preserving (conformal (if sense of angle on $\Sigma$ by observer inside it)) mapping

from extended complex plane to unit sphere $\Sigma$. Stereographic image of a line in the plane is a circle on $\Sigma$ passing through $N = \infty$. Circles on plane are mapped to circles on $\Sigma$.




15. If $K$ is sphere of radius $\sqrt{2}$ centred at $N$ then stereographic projection of plane is nothing but its inversion in $K$; This is another reason why steregraphic projection preserves circles.

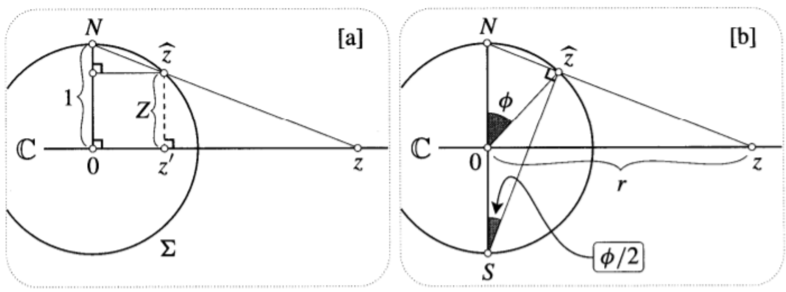



16. Transformation on stereographic image of complex plane corresponding to: complex conjugation is reflection of $\Sigma$ in vertical plane through real axis; geometric inversion is reflection of $\Sigma$ in its equitorial plane; complex inversion is rotation of $\Sigma$ with $\pi$ about real axis.

17. Transferring a function from complex plane to $\Sigma$ can tell about its behaviour at $\infty \equiv N$; Complex inversion is conformal everywhere; $z \to\ z^2$ is conformal everywhere except

at 0 and $N = \infty$; Such points where conformality of an otherwise conformal map breaks down are called critical points; To investigate conformality of $f(z)$ at $\infty$, take $F(z) = f(1/z)$ and check its conformality at $O$.

18. Stereographic formulae: Given Cartesian coordinates of $z$ as $x + iy$, and Cartesian coordinates of its stereographic image on $\Sigma$ as $(X, Y, Z)$, we have, $x + iy = (X + iY)/(1 - Z)$ and $X + iY = 2z/(1 + |z|^2)$ where $Z = (|z|^2 - 1)/(|z|^2 + 1)$; Also, if polar coordinates of stereographic image are $(\phi, \theta)$ where $\theta$ measures angle around $Z$-axis and $\phi$ is the angle subtended at the centre of $\Sigma$ by points $N$ and $\hat{z}$, then, $z = e^{i\theta} \cot(\phi/2)$; $\hat{p}$ and $\hat{q}$ are antipodal on $\Sigma$ then $q = -1/\bar{p}$.




19. M.T. map circles to circles, are conformal, if two points are symmetric wrt a circle then their images are symmetric wrt image circle, maps an oriented circle $C$ to an oriented circle $\tilde{C}$ s.t. region to the left of $C$ is mapped to the region to the left of $\tilde{C}$.

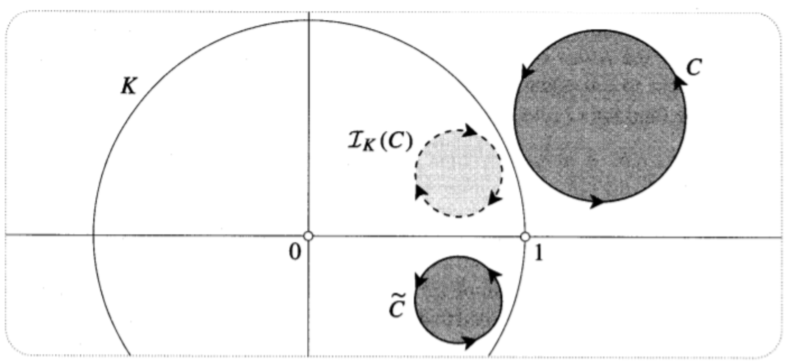
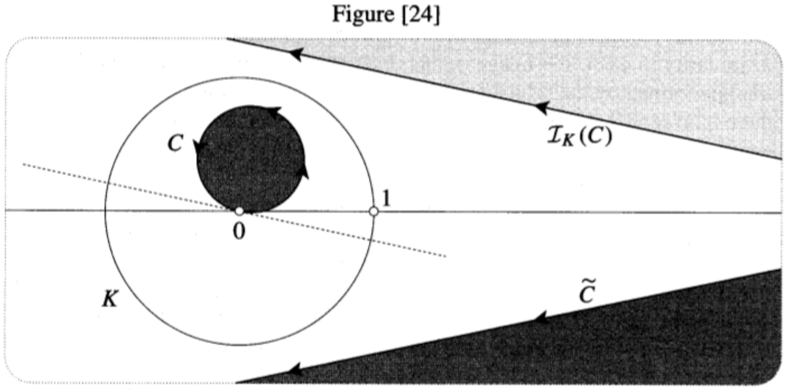
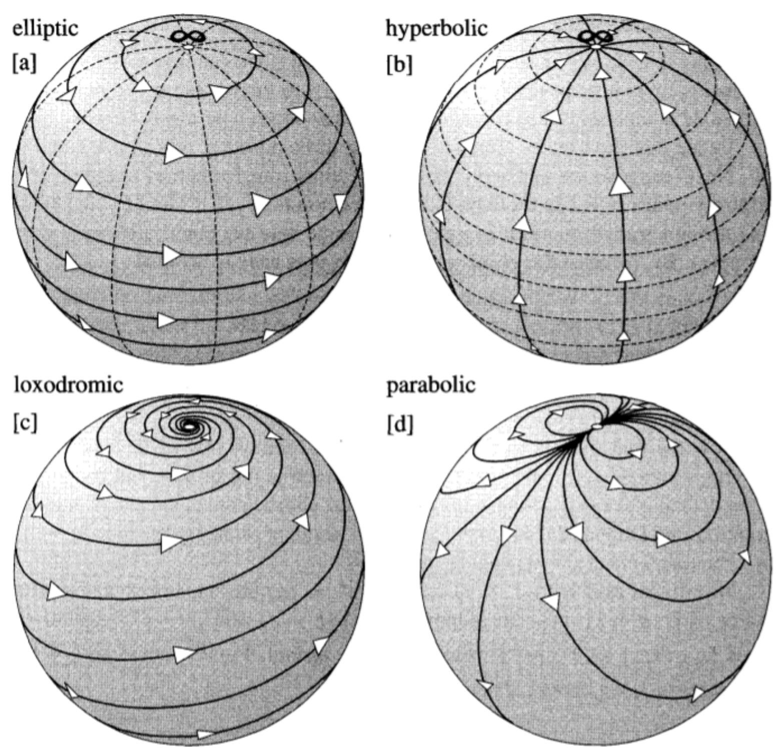



Figure [24]



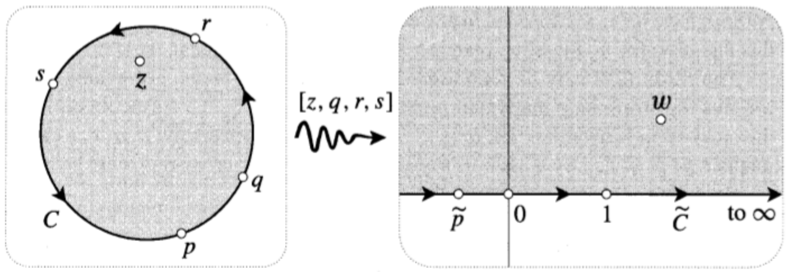
20. There exist a unique M.T. sending any three points to any other three points; $ad-bc=1$ then M.T. is normalized; set of M.T. forms a group under composition; $z = M(z)$ is a quadratic in $z$, so, with exception of identity mapping, a $M.T.$ has atmost two fixed points, this fact is used to prove uniqueness part; if $c \neq 0$ then both fixed points lie on a finite plane; if $c = 0$ then $M(z) = Az + B$, which is a similarity, has a fixed points at $\infty$;

21. Transfer $M(z) = e^{i\theta}z$ on $\Sigma$ to see that origin centered circles are invariant curves, origin originating rays map to another such ray, and fixed points are 0 and $\infty$: Such M.T. is called elliptic M.T.; transfer $M(z) = \rho z$ on $\Sigma$ to see that origin centered circles map to another such circle, origin originating rays are invariant curves, and fixed points are 0 and $\infty$: Such M.T. is called hyperbolic M.T.; Transfer $M(z) = \rho e^{i\theta}z$ to see the combined effect, 0 and $\infty$ are fixed points: Such M.T. is called loxodromic M.T.; $M(z) = z + b$ has lines parallel to $b$ as invariant curves and only $\infty$ as its fixed point: Such M.T. is called parabolic M.T.; A M.T. has a fixed point at $\infty \iff$ it is a similarity $M(z) = az + b$; $\infty$ is the sole fixed point $\iff$ it is a translation $M(z) = z + b$.




22. M.T. taking $q, r, s$ to $q', r', s'$ is given by $M'(z) = M^{-1}_{q',r',s'} \circ M_{q,r,s}(z)$ where $M_{q,r,s} = [z, q, r, s] = ((z-q)(r-s))/((z-s)(r-q))$ is a M.T. mapping $q \to 0$, $r \to 1$, $s \to \infty$; $[z, q, r, s]$ is called cross-ratio; A point $p$ lies on the circle through $q, r, s$ $\iff Im[p, q, r, s] = 0$; if $q, r, s$ induce positive orientation to circle then $z$ lies outside $\iff Im[p, q, r, s] < 0$ and inside if $Im[p, q, r, s] > 0$.

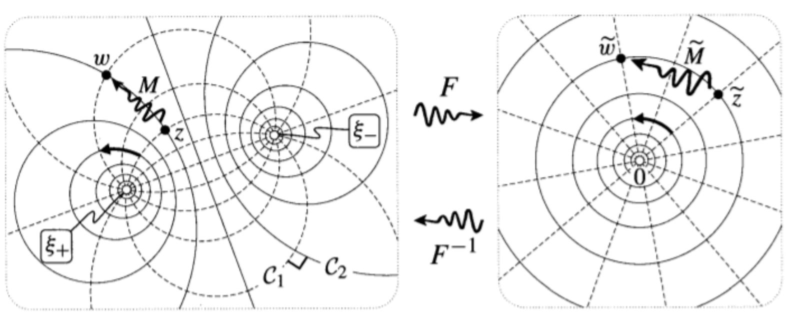
23. M.T. in matrix form $[a, b; c, d]$ which is non-unique; if M.T. is normalized then matrix




form is unique upto sign; check following in matrix form: Identity M.T., normalization coefficient as determinant, normalized $\times$ normalized $=$ normalized, composition of M.T., inverse of M.T.; M.T. are linear transformations, only they act on homogeneous coordinates in $\mathbb{C}^2$; $z = \zeta_1/\zeta_2$ is a fixed point of $M(z) \iff [\zeta_1, \zeta_2]^T$ is evec of $[M]$; Check with matrix form that if $\infty$ is a fixed point then $c = 0$; Suppose $M(z)$ is normalized then $det([M]) = 1$, so, $det([M] - \lambda I) = \lambda^2 - (a+d)\lambda + 1 = 0$; $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 = a + d$.

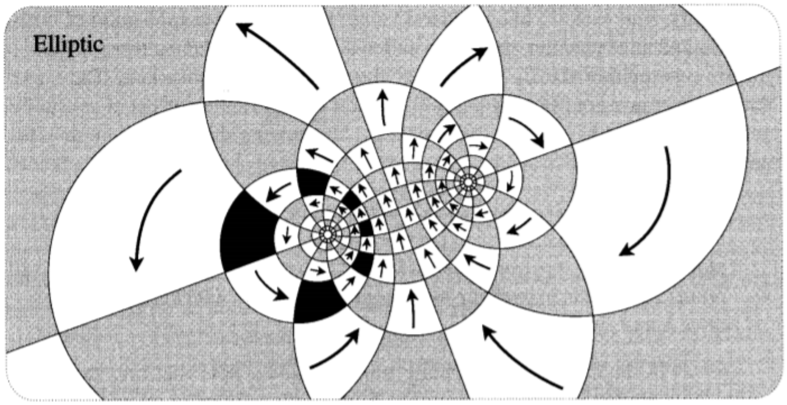
24. Two vectors in $\mathbb{C}^2$ are $\perp \iff$ they are homogeneous coordinates of antipodal points on $\Sigma$; A linear transformation $[R]$ analogous to a rotation must preserve inner product: $< [R]p, [R]q >=< p, q >$, so, $[R]^*[R] = I$; We get form of $[R] = [a, b; -\bar{b}, \bar{a}]$ and therefore the most general rotation of $\Sigma$ can be expressed as $R(z) = (az+b)/(-\bar{b}z+\bar{a})$.

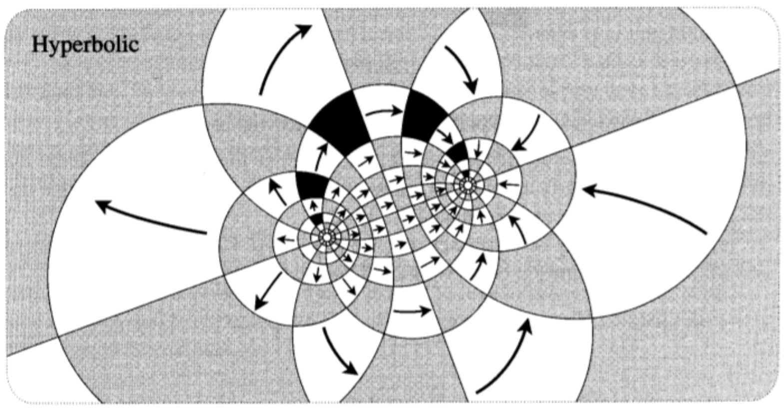
25. M.T. $M(z)$ with two fixed points $\xi_+$ and $\xi_i$; Consider family of circles through them as $C_1$ and family of circles s.t. each circle is $\perp$ all circles of $C_1$ as $C_2$; Note that $\xi_+$ and $\xi_i$ are symmetric wrt to a circle in $C_2$; $F(z) = (z - \xi_+)/(z - \xi_-)$ sends $\xi_+ \to 0$ and $\xi_- \to \infty$; $C_1$ circles become straight lines from origin and $C_2$ circles become concentric origin centred circles; $w = M(z)$ in $z$-plane and in $w$-plane $\tilde{w} = \tilde{M}(\tilde{z})$ so that $\tilde{M}(\tilde{z}) = F(M(F^{-1}(\tilde{z})))$; Two fixed points, so, $\tilde{M}(z) = mz$ where $m = \rho w^{i\alpha}$; $m$ is the multiplier of $M(z)$; $M(z)$ is elliptic if $m = e^{i\alpha}$ - $C_1$ circles permutate among themselves and $C_2$ circles are invariant, $\alpha = (m/n)2\pi$ then period of $M$ is $n$.

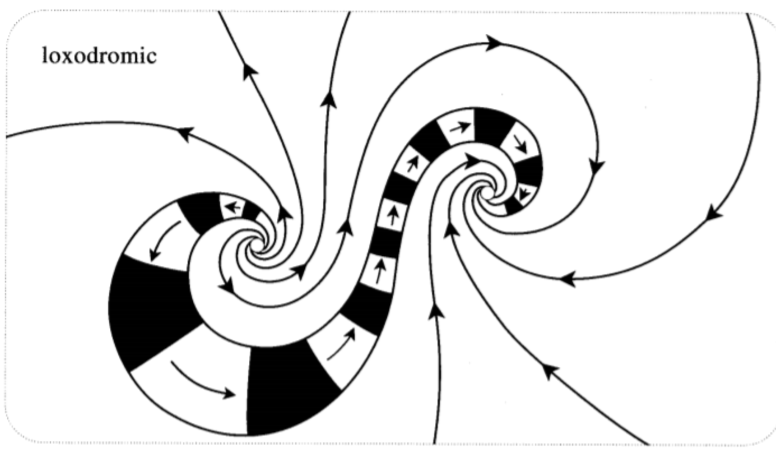



26. $M(z)$ is hyperbolic if $m = \rho$: $C_1$ circles are invariant and $C_2$ circles permutate, $\rho < 1$ is contraction and movement is from $\xi_-$ to $\xi_+$ and analogously with $\rho > 1$.

27. $M(z)$ is loxodromic if $m = \rho e^{i\alpha}$; $m$ is the multiplier associated with $\xi_+$ and $1/m$ is the multiplier associated with $\xi_-$; Locally, i.e. near $\xi_+$ the effect of $M(z)$ is just dilative rotation $m$ and near $\xi_-$ the effect of $M(z)$ is again dilative rotation $1/m$.

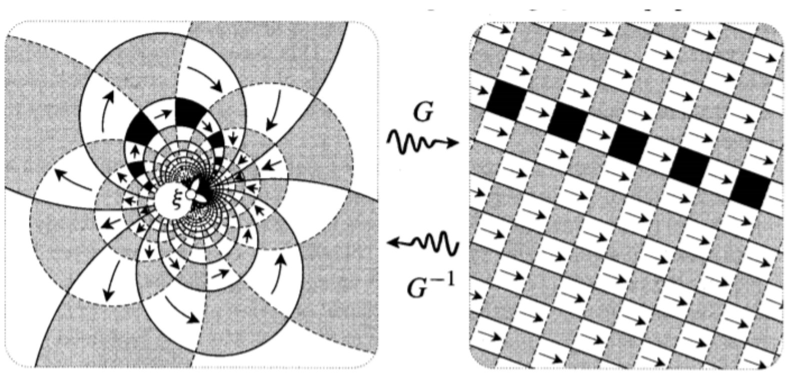



28. M.T. $M(z)$ with one fixed point $\xi$; $C_1$ and $C_2$ be two $\perp$ families of circle passing through $\xi$, so, $\perp$ at the second point of intersections too; $G(z) = 1/(z - \xi)$ sends $\xi \to \infty$; $C_1$ and $C_2$ are mapped to two $\perp$ families of lines; $\widetilde{M}(\tilde{z}) = G(M(G^{-1}(\tilde{z})))$; $\infty$ is only fixed point, so, $\widetilde{M}(\tilde{z}) = \tilde{z} + T$; $C_1$ circles permutate and $C_2$ circles permutate; $M(z)$ is said to be parabolic in such case; Also, normalized $M(z)$ is parabolic $\iff$ $(a + d) = \pm 2$ so $\xi = (a - d)/2c$ and we get $T = \pm c$.

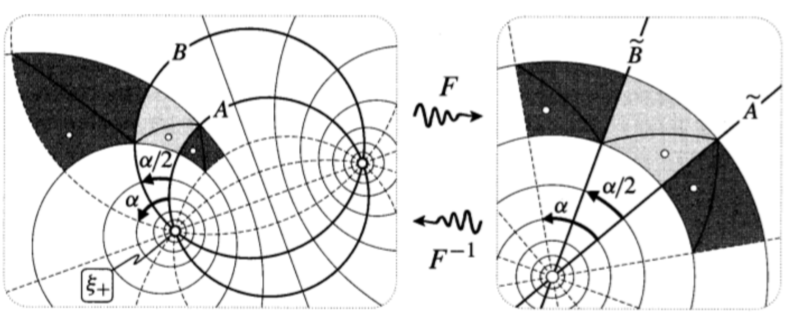
29. Since $M(z)$ maps $z = \infty$ to $w = a/c$ the multiplier can be computed by $(a/c - \xi_+)/(a/c - \xi_-) = m(z - \xi_+)/(z - \xi_-) = m$; Using $det([\widetilde{M}]) = det([F][M][F^{-1}]) =$



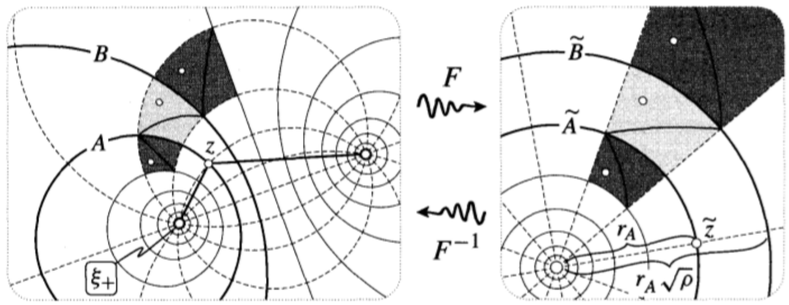

$det([M])$, $\tilde{M}(\tilde{z}) = m\tilde{z}$ and normalized form of $[\tilde{M}] = [\sqrt{m}, 0; 0, 1/\sqrt{m}]$ we get $tr([\tilde{M}]) = tr([F][M][F^{-1}]) = tr([M]) = a + d$, so, $\sqrt{m} + 1/\sqrt{m} = a + d$; $M(z)$ is elliptic $\iff$ $a+d$ is real and $|a+d| < 2$; $M(z)$ is parabolic $\iff$ $(a+d) = \pm 2$; $M(z)$ is hyperbolic $\iff$ $a = d$ is real and $|a+d| > 2$; $M(z)$ is loxodromic $\iff$ $a+d$ is complex.

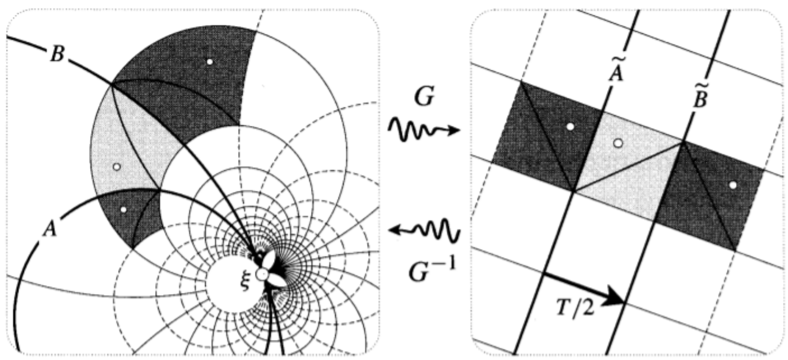
30. If a fixed points of $M(z)$ is represented as an evec with eval $\lambda$ of a normalized matrix $[M]$ then the multiplier associated with the fixed point is given by $m = 1/\lambda^2$. The two reciprocal values of $m$ equal to the two reciprocal vallues of $\lambda^2$. Easy exercise to show that $m_+ = 1/\lambda_+^2$ where $\lambda_+$ is the eval corresponsing to evec/fixed point $\xi_+$.

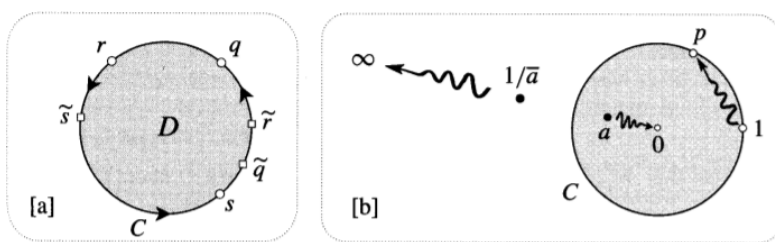
31. Composition of any two reflections is a M.T.; Composition of 2 reflections is a non-loxodromic M.T. and composition of 4 reflections is a loxodromic M.T.; Elliptic case - $m = e^{i\alpha}$ then $M(z) = \mathcal{I}_B(\mathcal{I}_A(z))$ where $A$ and $B$ are two circles from $C_1$ s.t. angle from $A$ to $B$ is $\alpha/2$.; Hyperbolic case - $m = \rho$ then $M(z) = \mathcal{I}_B(\mathcal{I}_A(z))$ where $A$ and $B$ are two circles of Appolonius with limit points $\xi_\pm$ s.t. $r_B/r_A = \sqrt{\rho}$ - if a point moves s.t. ratio of its distance from $\xi_+$ and $\xi_-$ is constant then the point moves on a circle, $r_A = |\tilde{z}| = |F(z)| = |(z - \xi_+)/(z - \xi_-)|$; Parabolic case - $M(z) = \mathcal{I}_B(\mathcal{I}_A(z))$ where $A$ and $B$ are circles that touch each other at $\xi$ s.t. the distance between parallel line $G(A)$ and $G(B)$ is $T/2$.




32. Automorphisms of unit disc. An automorphism of a region $R$ of the complex plane is a one-to-one, conformal mapping of $R$ to itself. A M.T. has six degress of freedom (need image of 3 fixed complex numbers to specify). M.A. of unit disc $D$ have three degrees of freedom (need three angles to specify images of three fixed points on the boundary of disc). Another way to fill up 3 degrees of freedom: specify which point $a$ inside $D$ is to be mapped to origin and which point $p$ on $C$ is image of the point 1.

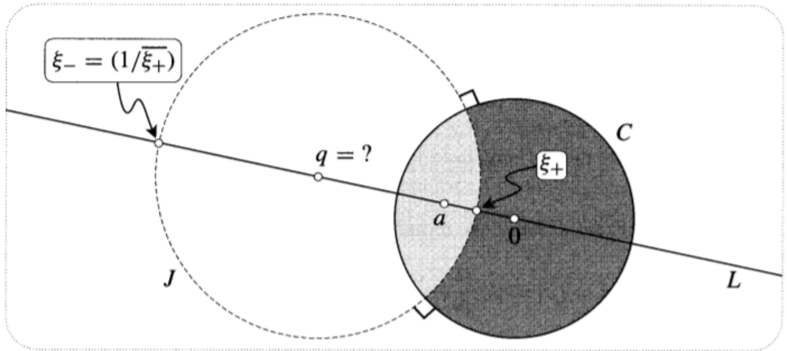


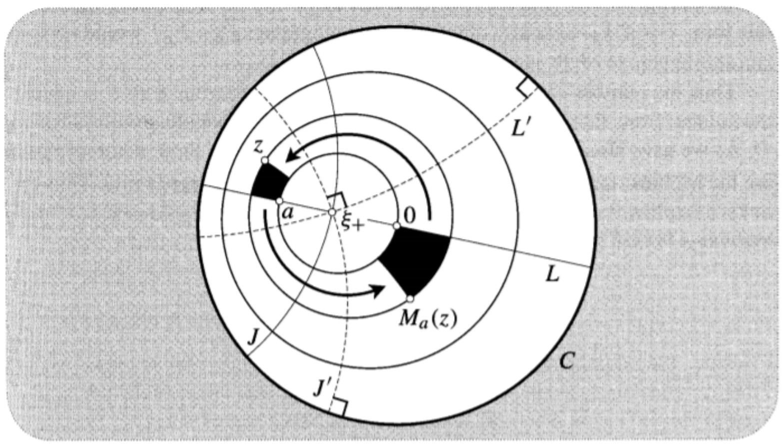




33. If two M.A. $M$ and $N$ map two interior points to the same image points, then $M = N$; Since $C$ is mapped to itself by $M$, the symmetry principle tells us that if a pair of points are symmetric wrt to $C$ then so are their images. Since $a$ is mapped to 0, $1/\bar{a}$ will map to $\infty$. Thus, form of $M$ is $M = k(z-a)/(\bar{a}z-1)$ where $k$ is a constant. Also, $p = M(1)$. So, $1 = |p| = |k|(|1-a|)/(|\bar{a}-1|) = |k|$. So $k = e^{i\phi}$. Choice of $p$ is equivalent to choice of $\phi$. $M_0^{\phi} = e^{-i\phi}$ which rotates $D$ about origin by $\pi + \phi$. $M_a^{\phi} = R_0^{\phi} \circ M_a^0$. $M_a^0 \equiv M_a$; $M_a$ swaps 0 and $a$. This is the only M.A. with this property. $M_a = R_L \circ I_J$ where $J$ is the circle orthogonal to $C$ which swaps $a$ and 0 and has $1/\bar{a}$ as centre, and $L$ is the line passing through 0 and $a$ (and so through $1/\bar{a}$). Fixed points $\xi_{\pm}$ are intersection points of $J$ and $L$, and so they are symmetric with repect to $C$. Since reflections occur in orthogonal circles through these points, $M_a$ is elliptic and $m = e^{i\pi}$ associated with both $\xi_{pm}$. So, $M_a$ is involuntary and any pair of points $z$, $M_a(z)$ is swapped by $M_a$. $M_a$ can also be expressed as $I_{L'} \circ I_{J'}$ where $J'$ and $L'$ are any two circles through $\xi_+$ that are orthogonal to $C$. If $\Phi \equiv 2\cos^{-1}|a|$, then $M_a^{\phi}$ is elliptic if $|\phi| < \Phi$, parabolic if $|\phi| = \Phi$ and hyperbolic if $|\phi| > \Phi$.

34. Riemann's Mapping Theorem: Any simply connected region $R$ (other than the entire plane) may be mapped one-to-one and conformally to any other such region $S$. It is




sufficient to establish this in case of $S$ being $D$, for if, $F_R$ is a one-to-one conformal mapping from $R$ to $D$, and $F_S$ is a one-to-one mapping of $S$ to $D$, then $F_S^{-1} \circ F_R$ is a one-to-one conformal mapping of $R$ to $S$. $\tilde{F}_R \circ F_R^{-1}$ would always be some automorphism $M$ of $D$ so that $\tilde{F}_R = M \circ F_R$. Number of one-to-one conformal mappings from $R$ to $S$ is equal to the number from $R$ to $D$, which in turn is equal to the number of automorphisms of $D$. We will show that these automorphisms are $M_a^{\phi}$ which form a 3 parameter family. This imples that there exist a 3-parameter family of one-to-one conformal mappings from $R$ to $S$.