Determinants

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Outline

Motivation

- How to test invertibility of a matrix?
- How to compute volume of a box in n- dimensions?
- Any explicit formula for the solution of $Ax = b$?
- Any explicit formula for pivots of A?
- What is the dependence of $A^{-1}b$ on each element of b?

Introduction

1. Determinant is defined only for square matrices.
2. $detA = 0 \iff A$ is singular.
3. $detA =$volume of a box in n-dimensional space.
4. $detA = \pm$(product of pivots) where the sign depends on number of row exchanges in elimination. Even number of exchanges implies positive sign.

- **The simple things about the determinant are not the explicit formulas, but the properties it possesses.**

Properties of the Determinant

1. $detI = 1$.
2. Determinant changes sign when two rows are exchanged because determinant of a permutation matrix P is ± 1. If the number of row exchanges required to bring P to I is even then $detP = 1$ else -1.
3. Determinant depends linearly on a row. Proof by determinant computing determinant along that row.

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Properties of the Determinant

4. If two rows of A are equal then $detA = 0$. Proof: use 2.
5. Subtracting a multiple of one row from another leaves the same determinant. Proof: use 3 and 4.
6. If A has a zero row, then $detA = 0$. Proof: use 5 and 4.
7. If A is triangular then $detA =$ product of diagonal entries. Proof: use 5 to derive diagonal matrix, then use 3 and 1.
8. $detA = \pm(\text{product of pivots}), detA = 0 \iff A$ is singular. Proof: elimination leads to U which has pivots on the diagonal. For singular matrices one of the row will be zero. Then use 7.
9. $detAB = detAdetB$. Proof: $A = P_1^T L_1 U_1, B = P_2^T L_2 U_2$.
10. $detA = detA^T$. Proof: $A = P^T LU$, $A^T = U^T L^T P$ and $detP^T P = detI = 1$. This means - we can exchange rows by columns in above results.[1]

[1] Singular case separately for 7,8,9,10

Formulas for the Determinant

1. If A is invertible then $PA = LDU$, $detP = \pm 1$ and product rule gives $detA = \pm detLdetDdetU = \pm$(product of pivots)
2. Suppose $A_{n \times n}$ is split into n^n terms by applying property 3 to each row in the following way -
$$\begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$
Among n^n terms only $n!$ terms will be non-zero when the non-zero terms are in different columns otherwise there will be atleast one column of 0s making determinant 0. The $n!$ terms correspond to $n!$ permutations of $(1, \ldots, n)$ which gives another formula for determinant:
$$detA = \sum_{\text{all } P\text{'s}} a_{1\alpha} a_{2\beta} \ldots a_{n\gamma} detP$$

Formulas for the Determinant

- Consider the terms involving $a_1 1$. This means $\alpha = 1$. This leaves some permutation $(\beta, \ldots, \gamma)$ of resulting columns $(2, \ldots, n)$. We collect all those terms as C_{11} which is the determinant of the submatrix formed by deleting row 1 and column 1.

$$C_{11} = \sum_{\text{all } P\text{'s s.t. } P_{11}=1} a_{2\beta} \ldots a_{n\gamma} detP$$

$$detA = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

M_{ij} is called a minor (smaller determinant) which is obtained by computing the determinant of the matrix when ith row and jth column are deleted.

Applications of Determinant

Result 1 - Computation of inverse A^{-1}

$A^{-1} = C^T/detA \implies AC^T = detA\ I$

Proof - Hints:

$$(AC^T)_{ij} = \sum_{k=1}^{n} A_{ik}C_{jk} = detA\ \mathbb{I}(i=j)$$

Note that when $i \neq j$, $(AC^T)_{ij}$ represents determinant of the matrix A with ith row copied into jth row (2 rows are equal).

Result 2 - Solution of $Ax = b$

$x_j = detB_j/detA$ where B_j is A with b in jth column.

Proof - Hints:

$$(A^{-1}b)_j = \left(\frac{C^T}{detA}b\right)_j = \frac{\sum_{k=1}^{n} C_{kj}b_k}{detA} = \frac{detB_j}{detA}$$

Applications of Determinant contd.

Result 3 - Volume of a box

Volume of a box whose edges are in rows of A equals $detA$.
Proof - Hints: If edges are $\perp$ and are of length $l_1, l_2, \ldots, l_n$,

$$AA^T = diag(l_i^2) \implies det(AA^T) = det(A)^2 = \prod_{i=1}^{n} l_i^2$$

Sign of $detA$ will indicate whether the edges form a RH-set of coord. $x-y-z$ or a LH-set $y-x-z$. If the edges are $\not\perp$ then with row ops. it can be made $\perp$ by reducing matrix to RREF. Det. is invariant to row ops, so vol. stays same.

Result 4 - Formula of pivots

$d_k = detA_k/detA_{k-1}$, A_k is left submatrix of A of order k.
$A_k = L_k D_k U_k \implies detA_k = \prod_{i=1}^{k} d_i \implies d_k = \frac{detA_k}{detA_{k-1}}$

Applications of Determinant contd.

Result 5

Elemination can be completed without row exchanges i.e. $P = I$ and $A = LU$ if and only if the leading submatrices $A_1, A_2, \ldots, A_n$ are all non-singular.

Proof - Hints: Follows from result 4.

Bibliography

Gilbert Strang. *Linear algebra and its applications*. Belmont, CA: Thomson, Brooks/Cole, 2006.