Winding Numbers and Topology - Notes

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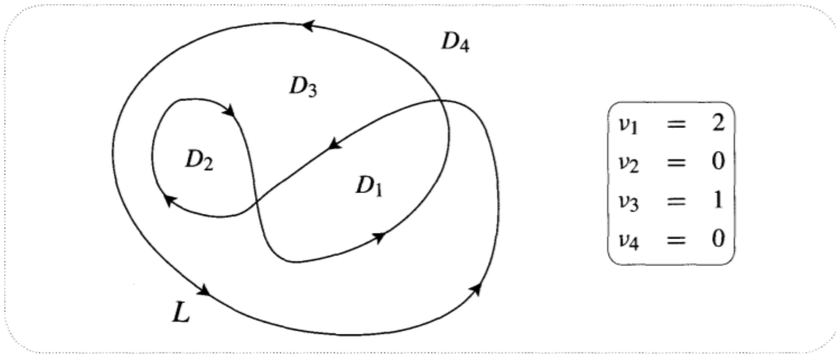
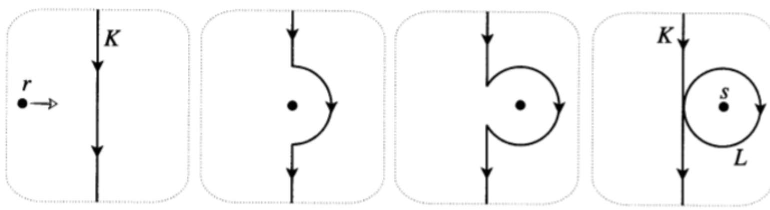
1 Winding Number

- Winding number $\nu(L, p)$ of a closed loop L about point p is the net number of revolutions of the direction of z as it traces out L once in its given sense.
- A naive way to compute $\nu(L, p)$ is to start with a random point on L and trace L with your finger - starting with 0, add 1 for each positive (anti-clockwise) rotation of the arrow from p to your finger and subtract 1 for each negative (clockwise) rotation of the arrow from p to your finger. When you have reached the starting point, the final count is the required winding number of the loop L about p.
- A simple loop - one which does not intersect itself - divides plan into two just two sets, inside and outside (Proof is not easy). For a non-simple loop, this is not the case.
- A general loop L divides the plane into a number of sets D_j. $\nu(L, p)$ is same for all p in the same set. Proof: Consider a small segment of L. As z traverses it, the rotation of $z - p$ will depend continuously on p unless p crosses L i.e. if we move p by a small amount then the rotation angle about the new point will differ from the original rotation by a small amount. Since winding number of loop about p is nothing but the sum of rotations about p due to all of its segments, it turns out that a small change in p to $\tilde{p}$ will lead to small change in $[\nu(L, p) - \nu(L, \tilde{p})]$. But this must be an integer, so, it must be 0.
- By above argument, we can assign a winding number ν_j to each set D_j. The "*inside*" can now be defined as those sets with $\nu_j \neq 0$.
- A quick way to compute winding numbers,

  Crossing rule: If L is moving from our left to our right (our right to our left) as we cross it, its winding number around us increases (or decreases) by one. (1)

- As a consequence of crossing rule, a ray emanating from a point p such that it does not pass through self intersection point of loop and it is not tangent to loop, intersects the loop $|n| + 2m$ times where $m \in \mathbb{N}$.





2 Hopf's Degree Theorem

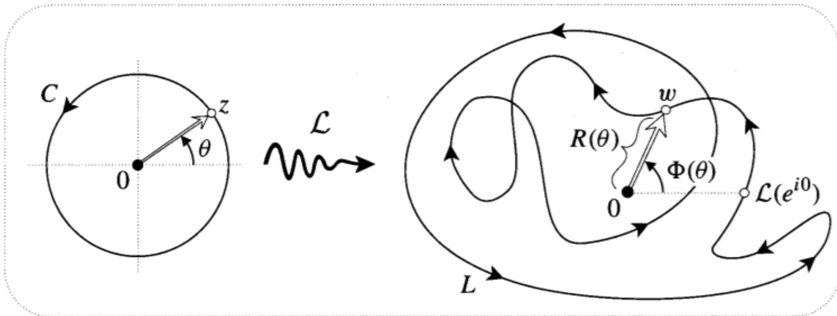
- For a fixed loop and a continuously moving point, the winding number changes only when the point crosses the loop. The same is true for a fixed point and continuously moving loop. The winding number changes by ± 1 if the continuously moving loop crosses the point. Thus if a loop L can be continuously deformed into another loop K without crossing point p then $\nu(L, p) = \nu(K, p)$. It turns out that the converse is also true.

  A loop K may be continuously deformed to another loop L, without ever crossing the point p, if and only if K and L have the same winding number round p. (2)

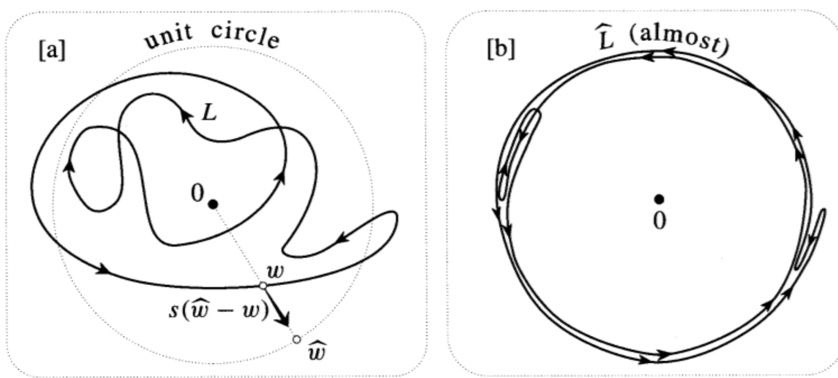
- Note that in three dimensions, a sphere encloses points inside it just once, like a circle in two dimensions enclosing points inside it just once. Hopf's theorem says that one closed surface may be continuously deformed into another closed surface, without ever crossing p, if and only if they enclose p the same number of times. Indeed, the same is true for n-dimensional surfaces in $(n + 1)$-dimensional space.

- Proof of converse of Hopf's theorem in 2 dimensions is as follows -

  - Consider a unit circular rubber band C around origin which is continuously deformed into an arbitrary loop L around origin (i.e. rubber band may cross the origin while deforming).
  - The new loop L can be represented as a mapping from the unit circle C to L given by $w = \mathcal{L}(z) = \mathcal{L}(e^{i\theta}) = R(\theta)e^{i\Phi(\theta)}$ where R and Φ are continuous functions of θ, so that $\mathcal{L}(C) = L$.



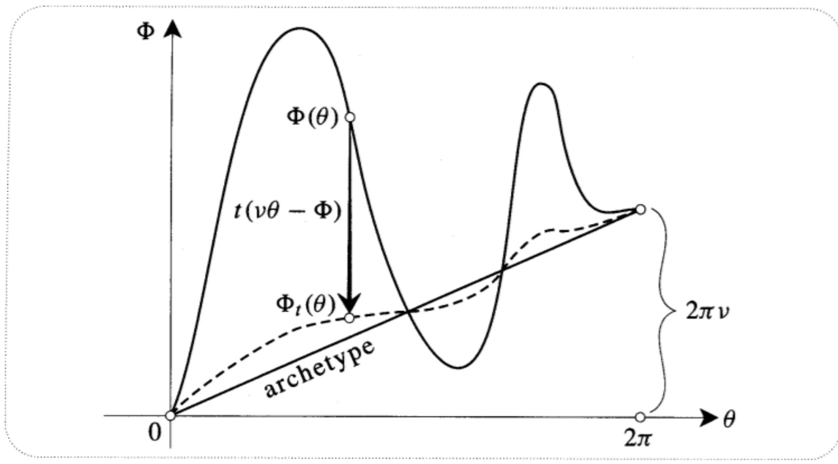
- We can always rotate L so that $\mathcal{L}(0) = 0$. As θ goes from 0 to 2π, z goes around C once and w goes around L once. The net rotation of w after it has returned to its initial point is given by $\Phi(2\pi) = 2\pi\nu$.
- To remove the distraction of the varying length w, we pull each point w radially onto $w/|w|$ on the unit circle obtaining a standardized version of L i.e. $\hat{L}$. The deformation from L to $\hat{L}$ can be represented as $\mathcal{L}_s(z) = w + s(\hat{w} - w)$ where $s \in [0, 1]$. As s varies from 0 to 1, $\mathcal{L}_s(C)$ changes from L to $\hat{L}$.
- Note that in this process origin is never crossed.



- So, $\hat{w} = \hat{\mathcal{L}}(e^{i\theta}) = e^{i\Phi(\theta)}$. Note that $\Phi(\theta)$ completely describes this mapping. As z moves around C with unit speed, $\hat{w}$ moves around $\hat{L}$ with speed $|\Phi'(\theta)|$ at θ.
- The archetypal mapping of degree ν is given by $\hat{\mathcal{J}}_\nu(z) = z^\nu$, for which $\Phi(\theta) = \nu\theta$. As z goes around C with unit speed, $\hat{w}$ travels around $\hat{J}_\nu$ with constant speed $|\nu|$, completing $|\nu|$ circuits of unit circle.
- Recall that C was made of a rubber band. Thinking of unit circle as boundary of a solid cylinder, as $\hat{L}$ is released, it contracts to $\hat{J}_\nu$. This process of taking up slack can be described in terms of the graph of Φ i.e. $\Phi_t(\theta) = \Phi(\theta) + t(\nu\theta - \Phi(\theta)$ where as t varies from 0 to 1, Φ_t from original Φ to straight line graph of archetype.

$$\textit{Any } \hat{L} \textit{ of winding number } \nu \textit{ can be continuously deformed to archetypal loop } \hat{J}_\nu \textit{ and vice versa.} \tag{3}$$

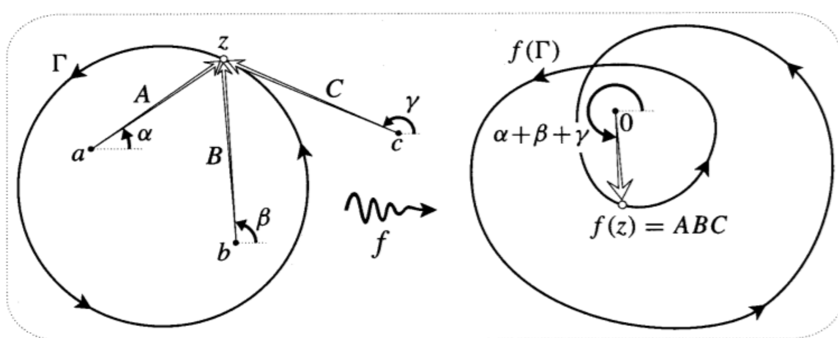
- Defining $\hat{\mathcal{L}}_t(e^{i\theta}) = e^{i\Phi_t(\theta)}$, $\hat{\mathcal{L}}_t(C)$ evolves continuously and reversibly from $\hat{L}$ to $\hat{J}_\nu$ as t varies from 0 to 1.



– Finally, if K and L have same winding number ν then K can be deformed to $\hat{J}_\nu$ which can further be deformed to L by reversing the steps of deformation of L to $\hat{J}_\nu$.

3 Polynomials and the Argument Principle

If $f(z)$ is analytic on and inside Γ, and N is the number of p-points [counted with their multiplicities] inside Γ, then $N = \nu[f(\Gamma), p]$. (4)



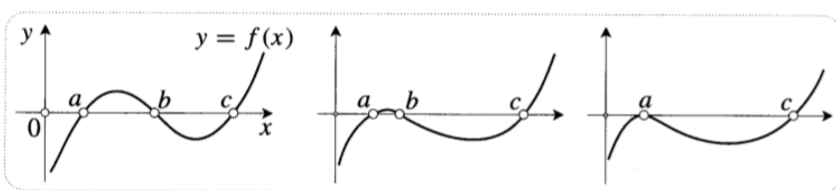
4 A Topological Argument Principle

4.1 Counting Preimages Algebraically

For an analytic function $f(z)$ s.t. $f(a) = p$ that is a is a preimage of p, algebraic multiplicity (order or valence) of a is defined as the degree of the dominant term in the Taylor series expansion of $f(z) - p$ about a. (5)

$$f(z) - p = f(z) - f(a) = \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots \tag{6}$$

- If $f'(a) \neq 0$ then algebraic multiplicity of a is 1 and a is called *simple* root of $f(z) - p$.
- The first nonzero term on right is one that dominates the local behaviour of $f(z) - p$.
- Note that the above definition solely depends on the fact that analytic functions can always be represented as a convergent Taylor series.
- Also, note that when the algebraic multiplicity of a is n s.t. $n > 1$ i.e. the dominanting term in the Taylor series of $f(z) - p$ is $f^{(n)}(z-a)^n/n!$, then $f^{(n)}(a)$ is the first nonvanishing derivative of f at a and, therefore, a is also a critical point of f of order $n - 1$.
- Little geometric intuition of above point - If $f'(a) \neq 0$ the an infinitesimal disc centered at a is amplitwisted to an infinitesimal disc centered at 0, so that points close to a cannot map to 0.



- If root a of polynomial $P(z)$ has multiplicity n then P may be factorizaed as $(z - a)^n\Omega(z)$, where $\Omega(a) \neq 0$.
- If a is a p-point of f of algebraic multiplicity n then, by equation (6) we have, $f(z) - p = \Omega(z)(z-a)^n$ where $\Omega(z) = \frac{f^{(n)}(a)}{n!} + \frac{f^{(n+1)}(a)}{(n+1)!}(z-a) + \dots$.
- Note that from this point of view, the only difference between analytic mapping and a polynomial is that the latter has a single "once and for all" factorization while the former requires different factorization in the neighbourhood of each p-point.

4.2 Counting Preimages Geometrically

- We can further generalize (4) for mappings that are just continuous. The very notion of algebraic multiplicity is meaningless for such general mappings (because Taylor series expansion may not converge). So, we need a geometric way of counting preimages that will agree with the previous defintion of algebraic multiplicity (5) if we specialize to analytic mappings.

- Condider an analytic mapping f with p-point a. Consider an infinitesimal circle C_a centred at a. If a is simple i.e. $f'(a) \neq 0$ then C_a is amplitwisted to an infinitesimal circle at p. The winding number of this image circle around p is 1 which is same as the algebraic multiplicity of a.

- Now, suppose algebraic multiplicity of a is n. Then, $f(z) = p + \Omega(z)(z-a)^n$ with $\Omega(a) \neq 0$. Thus, as z revolves around a once, $z - a$ revolves 0 once and $(z-a)^n$ revolves around 0 n times. As C_a is made smaller approaching to point a, $\Omega(C_a)$ approaches to $\Omega(a) \neq 0$. Therefore, we can choose C_a small enough so that winding number of $\Omega(C_a)$ about 0 is 0. Finally we have,

$$\nu[f(C_a), p] = \nu[f(C_a) - p, 0] = \nu[(C_a - a)^n \Omega(C_a), 0] = \nu[z^n, 0] + \nu[\Omega(C_a), 0] = n \quad (7)$$

- Now to define multiplicity of a mapping $h(z)$ that is merely continuous - Let Γ_a be any simple loop round a that does not contain other p-points. If we continuously deform Γ_a to $\tilde{\Gamma}_a$ with crossing a or any other p-point then $h(\Gamma_a)$ will continuously deform into $h(\tilde{\Gamma}_a)$ without ever crossing p, and so $\nu[h(\tilde{\Gamma}_a), p] = \nu[h(\Gamma_a), p]$.

- Thus, without specifying Γ_a further, we may define topological multiplicity (local degree of h at a) of a to be $\nu(a) = \nu[h(\Gamma_a), p]$.

- By deforming Γ_a into infinitesimal circle C_a for an analytic mapping one can note that the two types of multiplicites agree with each other.

4.3 What's Topologically Special About Analytic Functions?

- $$\begin{array}{l} \nu(a) \textit{ is always positive for analytic functions,} \\ \textit{while it can be negative for nonanalytic functions.} \end{array} \quad (8)$$

- For example, $h(z) = \bar{z}$ has topologically multiplicity of -1 everywhere.

- Recall that local effect of a nonanalytic mapping that is differentiable in the real sense (i.e. $u(x, y)$ and $v(x, y)$ are differentiable functions of x and y) at a p-point a consist of (after translation to p) a stretch by some factor ξ_a in one direction, another stretch by some factor η_a in perpendicular direction and a rotation through some angle ϕ_a. For such a mapping, an infinitesimal circle C_a centred at a will distort into an infinitesimal ellipse E_p centred at p. If both expansion factors have same sign then the mapping preserves orientation so that E_p circulates in the same sense as C_a and $\nu(a) = +1$ otherwise orientation is reversed and $\nu(a) = -1$. Note that ξ_a and η_a are NOT the eigenvalues of $J(a)$. Also, note that $\mathbf{det}J(a)$ measure the local expansion factor of the area at a (including sign for the orientation). In summary,

$$\begin{aligned}\nu(a) &= \text{the sign of } (\xi_a \eta_a) \\ &= \text{the sign of } \mathbf{det}[J(a)] \\ &= \text{the sign of } (\lambda_1(a)\lambda_2(a))\end{aligned} \tag{9}$$

- If $\mathbf{det}[J(a)] = 0$ then the formula cannot be used. The behaviour of the mapping is locally crushing at a. As for analytic mappings, such a place is called a critical point.
- But, local crushing at a critical point of an analytic mapping is perfectly symmetrical in all directions (both expansion factors/eigenvalues are equal), this may not be the case for the nonanalytic mappings that are differentiable in the real sense. For example, if $f(x + iy) = x - iy^3$ then $\mathbf{det}[J] = -3y^2$ clearly horizontal separation of points are left as it is, but all points on the real axis are critical points as a result of crushing in the vertical direction.

The critical points of an analytic mapping can be distinguished purely on the basis of topological multiplicity; those of a nonanalytic mapping cannot. (10)

- For analytic functions $\nu(a) = +1$ iff a is not critical. In the nonanalytic case $\nu(a) = \pm 1$ if a is not critical, but it is also possible for a critical point to have these multiplicities. As in the example above the points on the real axis are critical but have topological multiplicity of -1.

$\nu(a)$ is never zero for analytic mappings, but it can vanish for non-analytic mappings. (11)

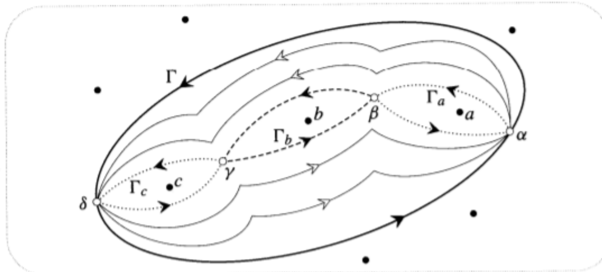
- For example, points on real axis of $f(x, y) = x + i|y|$ have multiplicity of 0.

4.4 A Topological Argument Principle

- Let Γ be a simple loop, and let $h(z)$ be a continuous mapping such that only a finite number of its p-points lie inside Γ.

The total number of p-points inside Γ (counted with their topological multiplicities) is equal to the winding number of $h(\Gamma)$ around p. (12)

- Proof:
  - Consider three p-points a, b and c lying inside Γ while others lie scattered outside.



- Γ can be deformed into doubly pinched loop $\alpha\beta\gamma\delta\gamma\beta\alpha$ which we call $\tilde{\Gamma}$.
- Since no p-points were crossed $h(\Gamma)$ will wind round p same number of times as $h(\tilde{\Gamma})$.
- $\tilde{\Gamma}$ is made up of $\Gamma_a = \alpha\beta\alpha$, $\Gamma_b = \beta\gamma\beta$, $\Gamma_c = \gamma\delta\gamma$.
- winding numbers of their images round p are, by definition, topological multiplicities of a, b and c.
- Let $\mathcal{R}(K)$ be the net rotation of $h(z)$ round p as z traverses K. If K is closed then $\mathcal{R}(K) = 2\pi\nu[h(K), p]$. Then,

$$\begin{aligned}
2\pi\nu[h(\Gamma), p] &= 2\pi\nu[h(\tilde{\Gamma}), p] \\
&= \mathcal{R}(\alpha\beta\gamma\delta\gamma\beta\alpha) \\
&= \mathcal{R}(\alpha\beta) + \mathcal{R}(\beta\gamma) + \mathcal{R}(\gamma\delta) + \mathcal{R}(\delta\gamma) + \mathcal{R}(\gamma\beta) + \mathcal{R}(\beta\alpha) \\
&= \mathcal{R}(\alpha\beta\alpha) + \mathcal{R}(\beta\gamma\beta) + \mathcal{R}(\gamma\delta\gamma) \\
&= \mathcal{R}(\Gamma_a) + \mathcal{R}(\Gamma_b) + \mathcal{R}(\Gamma_c) \\
&= 2\pi[\nu(a) + \nu(b) + \nu(c)]
\end{aligned}$$

- This extends to any number of p-points a_1, a_2 etc. lying inside Γ:

$$\nu[h(\Gamma), p] = \sum_{\text{inside } \Gamma} \nu(a_j)$$

- A consequence of above principle is *Darboux's Theorem*:

> *Let Γ be a simple loop. If an analytic function h maps Γ onto $h(\Gamma)$*
> *in one-to-one fashion, then it also maps the interior of Γ*
> *onto the interior of $h(\Gamma)$ in one-to-one fashion.*

Firstly, since Γ is a simple loop and $\Gamma \to h(\Gamma)$ is a one to one map i.e. $h(x_1) = h(x_2) \implies x_1 = x_2$ for all $x_1, x_2 \in \Gamma$, we have $h(\Gamma)$ is also a simple loop. Now, if p inside $h(\Gamma)$ has more than one preimage then it must have a winding number of more than 1 which is a contradiction (because $h(\Gamma)$ is a simple loop p has winding number of 1). So, p must have exactly one preimage inside Γ. Done.

5 Rouche's Theorem

5.1 The Result

- Let Γ be a simple loop. If $|g(z)| < |f(z)|$ on Γ, then $\nu[(f+g)(\Gamma), 0] = \nu[f(\Gamma), 0]$. This can be seen by the analogy of tree-man-dog-walk.

- By the Argument Principle, we have, *Rouche's Theorem.*

  If $|g(z)| < |f(z)|$ on Γ, then $(f+g)$ must have the same number of zeros inside Γ as f.

  $|g(z)| < |f(z)|$ is a sufficient, not a necessary condition for $(f+g)$ to have same number of roots as f. Example, $g(z) = 2f(z)$ has same number of roots but $|g(z)| = 2|f(z)| \geq |f(z)|$.

5.2 The Fundamental Theorem of Algebra

- Using Rouche's theorem one can prove the fundamental theorem of algebra, which states that a polynomial

$$P(z) = z^n + Az^{n-1} + Bz^{n-2} + \ldots + E$$

  of degree n always has n roots.

- Proof: let $f(z) = z^n$ and $g(z) = Az^{n-1} + Bz^{n-2} + \ldots + E$. Let C be the circle $|z| = 1 + |A| + |B| + \ldots + |E|$. Since $|z| > 1$ on C,

$$\begin{aligned}
|g(z)| &= |Az^{n-1} + Bz^{n-2} + \ldots + E| \\
&\leq |A||z^{n-1}| + |B||z^{n-2}| + \ldots |E| \\
&< |A||z|^{n-1} + |B||z|^{n-1} + \ldots + |E||z|^{n-1} \\
&= |f(z)|
\end{aligned}$$

  Using Rouche's theorem we have the number of preimages of $f(z)$ in C (which is n) is equal to the number of preimages of $f(z) + g(z) = P(z)$ in C.

5.3 Brouwer's Fixed Point Theorem

- *Any continuous mapping of the disc to itself will have a fixed point.* Physical analogy: Talcum powder on coffee.

- Showing that there must be a fixed point if the disc is mapped into its interior and there are at most a finite number of fixed points.

- Let D be $|z| \leq 1$. Let g map D to its interior i.e. $|g(z)| < 1$ for all $z \in D$. Let $m(z)$ be the movement of z under g i.e. $m(z) = g(z) - z$. A fixed point corresponds to no movement. Let $f(z) = -z$. Then, on the boundary of D, $|g(z)| < 1 = |f(z)|$. So, $m(z) = g(z) + f(z)$ has same number of roots inside D as f i.e. one.
- If g is merely continuous then there can be several fixed points, some of which will necessarily have negative multiplicities, while if g is analytic then there can literally be one fixed point.

6 Maxima and Minima

6.1 Maximum-Modulus Theorem

- *If f is analytic inside and on a simple loop Γ then no point outside $f(\Gamma)$ can have a preimage inside* Γ. Proof: $\sum_{\text{preimages } a_j \text{ inside } \gamma} \nu(a_j) = 0$. Since, $\nu(a_j) > 0$ for analytic mapping, therefore, no preimage inside γ.

- If p lies inside $f(\Gamma)$ then $\nu[f(\Gamma), p] \neq 0$ and so there must be atleast one preimage inside Γ. This is true for nonanalytic mappings too.

- *The maximum of $|f(z)|$ on a region where f is analytic is always achieved by points on the boundary, never ones inside.* This is called *Maximum-Modulus Theorem.* Only exception is the trivial analytic mapping $z \to$ const. So, if an analytic f achieves maximum at an interior point then f must be constant.

- The modular surface of analytic f does not even have a local maximum on an interior point, because if it does then for a small loop about that point, the highest point will fail to lie on the boundary of the loop. Another way to see this is by drawing a small loop γ about p and the image loop $f(\gamma)$. Since there are points away from $f(p)$ in all directions, a ray emanating from origin pointing directly away from origin, will meet the $f(\gamma)$ at some point which will correspond to the point with maximum modulus.

- *For analytic function, unless there is a 0-point inside Γ, at which $|f(z)| = 0$, the points Q closest to origin (minimum $|f(z)|$) must also be the image of a point q lying on the boundary* Γ. This is called *Minimum-Modulus Theorem.* Thus, there can be no pits in the modulur surface of analytic function, unless the surface actually hits the complex plane at an interior 0-point of f.

- Only exception is the constant function. So, if an analytic f achieves minimum (positive) modulus at an interior point then f must be constant.

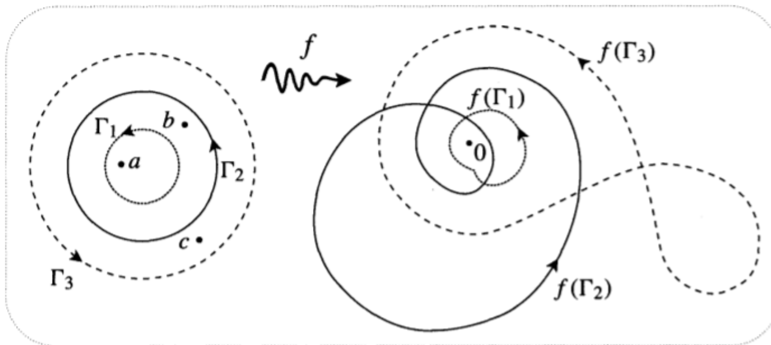
7 Schwarz-Pick Lemma

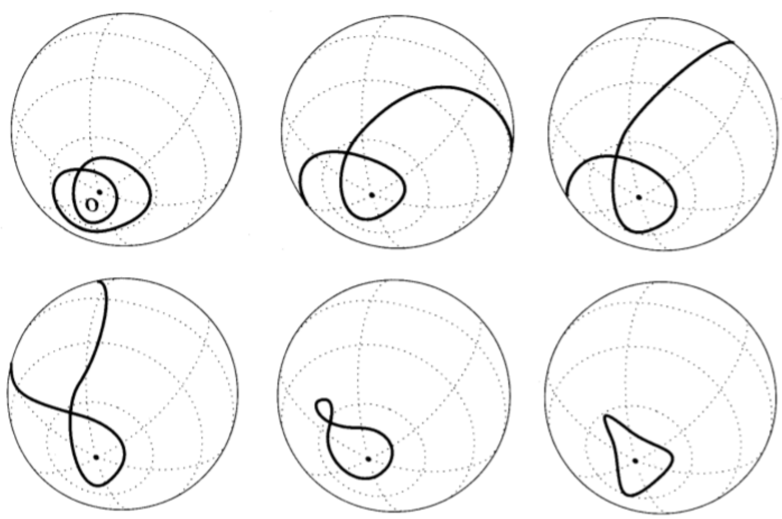
8 The Generalized Argument Principle

8.1 Rational Functions

- *Let f be analytic on a simple loop Γ and analytic inside except for a finite number of poles. If N and M are the number of p-points and poles, both counted with their multiplicities, then $\nu[f(\Gamma), p] = N - M$.*

- Figure shows how it works in case of the mapping $f(z) = ((z-a)(z-b))/(z-c)$. As such $f(z)$ goes to ∞ as $z \to c$ but after that it comes back from ∞ and unwinds.





8.2 Poles and Essential Singularities

- Two kinds of singularities exist for an otherwise analytic function:

  1. Pole: It is the type to which (17) applies to. If $f(z)$ approaches infinity as z approaches a from any directions then a is a pole of f. In the modular surface, there will be an infinitely high spike, a "pole" at a. Since f is analytic $F = 1/f$ is also analytic and has a root at a. If this root has multiplicity m then $F(z) = (z-a)^m \Omega(z)$ where $\Omega(a) \neq 0$. The local behaviour of f near a is thus,

$f(z) = \tilde{\Omega}(z)/(z-a)^m$ where $\tilde{\Omega}(z) = 1/\Omega(z)$ is analytic and nonzero at a. m is defined as the algebraic multiplicity or order of the pole. This is the order of first nonvanishing derivative of $1/f$. The function f is called meromorphic in a region if the only singularities present in the region are poles.

2. Essential Singularities: If analytic f has an essential singularity at f then it must be unbounded in the vicinity of s (otherwise it will not be a singularity) and f must not approach to infinity from all directions as $z \to s$ (otherwise it will be a pole). Example $g(z) = e^{1/z}$. $|g(z)| = e^{\cos\theta}/r$. If $z \to 0$ along imaginary axis $|g(z)| = 1$. But if the approach is along a path on the left of imaginary axis then $|g(z)| \to \infty$. *The rate at which is zooms off to ∞ is beyond the ken of any pole.*

8.3 The Explanation

- If $f(z) = \Omega(z)(z-a)^m$ where $\Omega(a) \neq 0$ then as z traces a small circle around a, f will wind around origin m times. Projecting on Riemann sphere, we get a loop winding around south pole m times. Now if we apply complex inversion to f to get $F(z) = \tilde{\Omega}(z)/(z-a)^m$, on Riemann sphere, since complex inversion results in rotation of π about real axis, the loop winding south pole will now wind north pole m times in counterclockwise sense as seen from inside the sphere. The stereographic projection of this loop will be a a large loop winding around 0, m times in clockwise sense.

- *If a is a pole of order m and Γ_a is any simple loop containing a but no p-points and no other poles, then $\nu[f(\Gamma_a), p] = -m$.*

$$\begin{aligned}\nu[f(\Gamma), p] &= \sum_{p\text{-points}} \nu[f(\Gamma_{a_j}), p] + \sum_{\text{poles}} \nu[f(\Gamma_{s_j}), p] \\ &= [\text{no. of } p\text{-points inside } \Gamma] - [\text{no. of poles inside } \Gamma]\end{aligned}$$