Solution Manual

prepared by

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for

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by

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2. The Possion Process

Ex. 2.1

$$\mathbb{P}\{N(h) = 1\} = e^{-\lambda h}\lambda h = \lambda h + \lambda h(e^{-\lambda h} - 1)$$

Since,

$$\lim_{h \to 0} \frac{\lambda h(e^{-\lambda h} - 1)}{h} = 0$$

we have,

$$\mathbb{P}\{N(h) = 1\} = \lambda h + o(h)$$

Similarly,

$$\mathbb{P}\{N(h) \geq 2\} = 1 - e^{-\lambda h}\lambda h - e^{-\lambda h} = o(h)$$

Ex. 2.2 (**a**)

$P_0(t+s) = 1 - \lambda(t+s) - o(t+s) = (1 - \lambda t - o(t))(1 - \lambda s - o(s)) = P_0(t)P_0(s)$

(**b**)

$$\begin{aligned}
P_0(t) &= P_0\left(\lim_{n\to\infty} \sum_{i=1}^{n} \frac{t}{n}\right) = \lim_{n\to\infty}\left(P_0\left(\frac{t}{n}\right)\right)^n = \lim_{n\to\infty} \exp\left(n \log\left(P_0\left(\frac{t}{n}\right)\right)\right) \\
&= \lim_{n\to\infty} \exp(n\log(1 - \lambda t/n + o(t/n))) \\
&= \lim_{n\to\infty} \exp\left(-n\left(\sum_{i=1}^{\infty}(\lambda t/n + o(t/n))^i\right)\right) \\
&= \lim_{n\to\infty} \exp\left(-\lambda t - \frac{to(t/n)}{t/n} - \left(\sum_{i=2}^{\infty}(\lambda t/n + o(t/n))^i\right)\right) \\
&= \exp(-\lambda t)
\end{aligned}$$

$$\mathbb{P}\{X_1 > t\} = P_0(t) = \exp(-\lambda t)$$

$$\begin{aligned}
\mathbb{P}\{X_2 > t | X_1 = s\} &= \mathbb{P}\{0 \text{ event in } (s, s+t] | X_1 = s\} \\
&= \mathbb{P}\{0 \text{ event in } (s, s+t]\} \quad (\because \text{ independent increments}) \\
&= P_0(t) \quad (\because \text{ stationarity}) \\
&= \exp(-\lambda t)
\end{aligned}$$

(**c**)

$$
\begin{aligned}
\mathbb{P}\{N(t) \geq n\} = \mathbb{P}\{S_n \leq t\} &= \int_0^t \frac{\lambda^n x^{n-1} \exp(-\lambda x)}{(n-1)!} dx \\
&= -\frac{\exp(-\lambda t)(\lambda t)^{n-1}}{(n-1)!} - \int_0^t \frac{\lambda^{n-1} x^{n-2} \exp(-\lambda x)}{(n-2)!} dx \\
&. \\
&. \\
&= -\sum_{i=1}^{n-1} \frac{\exp(-\lambda t)(\lambda t)^i}{i!} - \int_0^t \lambda \exp(-\lambda x) dx \\
&= 1 - \sum_{i=0}^{n-1} \frac{\exp(-\lambda t)(\lambda t)^i}{i!}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\{N(t) = n\} &= \mathbb{P}\{N(t) \geq n\} - \mathbb{P}\{N(t) \geq n+1\} \\
&= \frac{\exp(-\lambda t)(\lambda t)^n}{n!}
\end{aligned}
$$

Ex. 2.3

$$
\begin{aligned}
\mathbb{P}\{N(s) = k | N(t) = n\} &= \frac{\mathbb{P}\{N(s) = k, N(t) = n\}}{\mathbb{P}\{N(t) = n\}} = \frac{\mathbb{P}\{N(s) = k, N(t-s) = n-k\}}{\mathbb{P}\{N(t) = n\}} \\
&= \frac{\exp(-\lambda s)(\lambda s)^k}{k!} \frac{\exp(-\lambda(t-s))(\lambda(t-s))^{t-s}}{(n-k)!} \frac{n!}{\exp(-\lambda t)(\lambda t)^n} \\
&= \binom{n}{k} (s/t)^k (1 - s/t)^{n-k}
\end{aligned}
$$

Alternatively, given that $N(t) = n$, those n events have arrival times which are uniformly distributed over $(0, t)$ when considered as unordered random variables. Therefore, given $N(t) = n$ and $s < t$, $N(s)$ follows a binomial distribution with parameters n and $p = \frac{s}{t}$, which is the probability of a randomly chosen event (out of n events) to have an arrival time of less than or equal to s.

Ex. 2.4

$$
\begin{aligned}
\mathbb{E}[N(t)N(t+s)] &= \mathbb{E}[N(t)(N(t+s) - N(t)) + N(t)^2] \\
&= \mathbb{E}[\mathbb{E}[N(t)(N(t+s) - N(t)) | N(t))] + \mathbb{E}[N(t)^2] \\
&= \mathbb{E}[\lambda s N(t)] + \lambda t + (\lambda t)^2 \qquad (\because N(t+s) - N(t) \perp N(t)) \\
&= \lambda^2 t(t+s) + \lambda t
\end{aligned}
$$

Ex. 2.5

$$\begin{aligned}
\mathbb{P}\{N_1(t)+N_2(t)=n\} &= \sum_{k=0}^{\infty} \mathbb{P}\{N_1(t)+N_2(t)=n, N_1(t)=k\} \\
&= \sum_{k=0}^{n} \mathbb{P}\{N_1(t)+N_2(t)=n, N_1(t)=k\} \\
&= \sum_{k=0}^{n} \mathbb{P}\{N_2(t)=n-k, N_1(t)=k\} \\
&= \sum_{k=0}^{n} \mathbb{P}\{N_2(t)=n-k\}\mathbb{P}\{N_1(t)=k\} \qquad (\because N_1 \perp N_2) \\
&= \sum_{k=0}^{n} \frac{\exp(-\lambda_1 t)(\lambda_1 t)^k}{k!} \frac{\exp(-\lambda_2 t)(\lambda_2 t)^{n-k}}{(n-k)!} \\
&= \frac{\exp(-(\lambda_1+\lambda_2)t)t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k} \\
&= \frac{\exp(-(\lambda_1+\lambda_2)t)((\lambda_1+\lambda_2)t)^n}{n!}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\{X_1^{(1)} < X_1^{(2)}\} &= \int_0^\infty \mathbb{P}\{X_1^{(1)} < X_1^{(2)}, X_1^{(2)} = t\} dt \\
&= \int_0^\infty \mathbb{P}\{X_1^{(1)} < t\}\mathbb{P}\{X_1^{(2)} = t\} dt \\
&= \int_0^\infty (1-\exp(-\lambda_1 t))\lambda_2 \exp(-\lambda_2 t) dt \\
&= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}$$

Ex. 2.6 The combined process $N(t)$ will have a rate $\mu_1+\mu_2$ (using **2.5**). Let S_N be the time when the machine fails where N represents the number of components failed by time S_N. Then, we require $\mathbb{E}S_N$ where,

$$\mathbb{E}S_N = \mathbb{E}[\mathbb{E}[S_N|N)] = \mathbb{E}\left[\frac{N}{\mu_1+\mu_2}\right] = \frac{\mathbb{E}N}{\mu_1+\mu_2}$$

Now, $\mathbb{E}N$ is given by,

$$\begin{aligned}
\mathbb{E}N &= \mathbb{E}[N|\text{last event is type-1 fail}]P(\text{last event is type-1 fail}) + \mathbb{E}[N|\text{last event is type-2 fail}]P(\text{last event i} \\
&= \sum_{k=n}^{n+m-1} k\binom{k-1}{n-1}\left(\frac{\mu_1}{\mu_1+\mu_2}\right)^n\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^{k-n} + \sum_{k=m}^{m+n-1} k\binom{k-1}{m-1}\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^m\left(\frac{\mu_1}{\mu_1+\mu_2}\right)^{k-m}
\end{aligned}$$

Ex. 2.7

$$\begin{aligned}
f_{S_1,S_2,S_3}(s_1,s_2,s_3) &= f_{X_1,X_2,X_3}(s_1, s_2-s_1, s_3-s_2) \\
&= f_{X_1}(s_1) f_{X_2}(s_2-s_1) f_{X_3}(s_3-s_2) \quad (X_i \perp X_j) \\
&= \lambda \exp(-\lambda s_1) \lambda \exp(-\lambda(s_2-s_1)) \lambda \exp(-\lambda(s_3-s_2)) \\
&= \lambda^3 \exp(-\lambda s_3)
\end{aligned}$$

Ex. 2.8 (**i**)

$$\begin{aligned}
U_i &= \exp(-\lambda X_i) \\
\left|\frac{dU_i}{dX_i}\right| &= \lambda \exp(-\lambda X_i) \\
f_{X_i}(x) &= \lambda \exp(-\lambda x) \mathbb{I}(\exp(-\lambda x) \in (0,1)) \\
&= \lambda \exp(-\lambda x) \mathbb{I}(x \in (0,\infty))
\end{aligned}$$

(**ii**) Taking negative *log* of the inequality and dividing by λ gives,

$$\begin{aligned}
\sum_{i=1}^{n} X_i \leq 1 < \sum_{i=1}^{n+1} X_i \\
S_n \leq 1 < S_{n+1}
\end{aligned}$$

Thus n represents number of events till time 1 of a poisson process with rate λ. Therefore, $n = N(1)$ where $N(1)$ follows poisson distribution with mean $\lambda \cdot 1 = \lambda$.

Ex. 2.9 (**a**) Probability of winning equals the probability of exactly one event in $(s, T]$ which by stationarity of poisson process equals
$h(s) = \exp(-\lambda(T-s))\lambda(T-s)$.
(**b**)

$$\begin{aligned}
\frac{dh(s)}{ds} = 0 \implies s = T - 1/\lambda \\
\left.\frac{d^2h(s)}{ds^2}\right|_{s=T-1/\lambda} = -\lambda^2 e^{-1} < 0
\end{aligned}$$

(**c**)

$$h(T-1/\lambda) = e^{-1}$$

Ex. 2.10 (**a**)

$$T = \begin{cases} X_1 + R, & X_1 \le s \\ s + W, & X_1 > s \end{cases}$$

$$\begin{aligned}
\mathbb{E}T &= \mathbb{E}[T|X_1 \le s]\mathbb{P}\{X_1 \le s\} + \mathbb{E}[T|X_1 > s]\mathbb{P}\{X_1 > s\} \\
&= \mathbb{E}[X_1 + R|X_1 \le s]\mathbb{P}\{X_1 \le s\} + \mathbb{E}[s + W|X_1 > s]\mathbb{P}\{X_1 > s\} \\
&= \mathbb{E}[X_1\mathbb{I}(X_1 \le s)] + R(1 - \exp(-\lambda s)) + (s + W)\exp(-\lambda s) \\
&= \frac{1 - \lambda s \exp(-\lambda s) - \exp(-\lambda s)}{\lambda} + R(1 - \exp(-\lambda s)) + (s + W)\exp(-\lambda s) \\
&= (R + 1/\lambda)(1 - \exp(-\lambda s)) + W\exp(-\lambda s)
\end{aligned}$$

(**b**) When $W < R + 1/\lambda$, minimum is achieved with $\exp(-\lambda s) = 1 \implies s = 0$. When $W > R + 1/\lambda$, minimum is achieved with $1 - \exp(-\lambda s) = 1 \implies s = \infty$. And, when $W = R + 1/\lambda$, then all values of s gives $\mathbb{E}T = W = R + 1/\lambda$.
(**c**) The expected time of arrival of bus is $\mathbb{E}[X_1] = 1/\lambda$. So, intuitively, if $W < R + 1/\lambda$, in order to minimize $\mathbb{E}T$, I will not wait at bus stop at all ($s = 0$), and reach home by walking. On the other hand, if $W > R + 1/\lambda$, I will wait for the bus to arrive indefinitely ($s = \infty$) (since the increase in time increases the likeliness of arrival of bus as $\lim_{t\to\infty} \mathbb{P}\{X_1 > t\} = 0$ and the expected arrival time is $1/\lambda$).

Ex. 2.11

$$W = \begin{cases} 0 & X_1 > T \\ W' + X_1 & X_1 \le T \end{cases}$$

Convince yourself that W' and W have the same distribution and hence the expectation. Also, note that W' is independent of X_1. Therefore,

$$\begin{aligned}
\mathbb{E}W &= \mathbb{E}[W|X_1 > T]\mathbb{P}\{X_1 > T\} + \mathbb{E}[W|X_1 \le T]\mathbb{P}\{X_1 \le T\} \\
&= 0 + \mathbb{E}[W' + X_1|X_1 \le T]\mathbb{P}\{X_1 \le T\} \\
&= \mathbb{E}[W']\mathbb{P}\{X_1 \le T\} + \mathbb{E}[X_1\mathbb{I}(X_1 \le T)]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}W &= \mathbb{E}[W](1 - \exp(-\lambda T)) + \frac{1 - \lambda T\exp(-\lambda T) - \exp(-\lambda T)}{\lambda} \quad (\because \mathbb{E}W = \mathbb{E}W') \\
\mathbb{E}W &= \frac{\exp(\lambda T) - \lambda T - 1}{\lambda}
\end{aligned}$$

Ex. 2.12 Let type-1 events be those which are registered and type-2 events be those which are not registered. An event at arbitrary time s is type-1 event with probability $\mathbb{P}\{0 \text{ event in } [s - b, s)\} = \exp(-\lambda b)$.

(**a**) Since the probability of an event happening at an arbitrary time is classified as a type-1 event with a probability of $p = \exp(-\lambda b)$ which is independent of the time of happening of the event. Therefore, the first k events will be classified as type 1 event with probability $p^k = \exp(-\lambda kb)$. This can also be formally computed as follows:

$$\begin{aligned}
\mathbb{P}\{S_k^{(1)} < X_1^{(2)}\} &= \int_0^\infty \mathbb{P}\{X_1^{(2)} > t | S_k^{(1)} = t\} f_{S_k^{(1)}}(t) dt \\
&= \int_0^\infty \mathbb{P}\{X_1^{(2)} > t\} f_{S_k^{(1)}}(t) dt \quad (\because X_1^{(2)} \perp S_k^{(1)}) \\
&= \int_0^\infty \exp(-\lambda(1-p)t) \frac{(\lambda p)^k t^{k-1} \exp(-\lambda pt)}{(k-1)!} dt \\
&= p^k \int_0^\infty \frac{\lambda^k t^{k-1} \exp(-\lambda t)}{(k-1)!} dt \\
&= p^k \cdot 1 \\
&= \exp(-\lambda kb)
\end{aligned}$$

(**b**)

$$\mathbb{P}\{R(t) \geq n\} = \mathbb{P}\{N_1(t) \geq n\} = \sum_{k=n}^{\infty} \frac{\exp(-\lambda pt)(\lambda pt)^k}{k!}$$

Ex. 2.13 [verify] Let there be two types of events. Type-1 events cause failure with probability p and type-2 events do not cause failure.

$$\begin{aligned}
\mathbb{P}\{N = n | T = t\} &= \mathbb{P}\{N = n | \text{ first type-1 event occurs at } t\} \\
&= \mathbb{P}\{n-1 \text{ type-2 events occur before } t | \text{ first type-1 event occurs at } t\} \\
&= \mathbb{P}\{n-1 \text{ type-2 events occur before } t\} \quad (\because N_1(t) \perp N_2(t)) \\
&= \frac{\exp(-\lambda(1-p)t)(\lambda(1-p)t)^{n-1}}{(n-1)!}
\end{aligned}$$

Ex. 2.14 (**a**)

$$\mathbb{E} O_j = \mathbb{E}[\mathbb{E}[O_j | N_1, N_2, \ldots, N_{j-1}]] = \mathbb{E}\left[\sum_{i=1}^{j-1} P_{ij} N_i\right] = \sum_{i=1}^{j-1} P_{ij} \lambda_i$$

(**b**)

$$O_j \sim \text{Poisson}\left(\sum_{i=1}^{j-1} P_{ij} \lambda_i\right)$$

(**c**) $O_j \perp O_k$.

Ex. 2.15 (**a**) N_i follows negative binomial distribution with parameters n_i and P_i.
(**b**)

$$\begin{aligned}
\mathbb{P}\{N_i = n, N_j = n\} &= \mathbb{P}\{n \text{ flips with ith and jth sides } n_i \text{ and } n_j \text{ times.}\} \\
&= \mathbb{P}\{N_i = n, N_j = n | \text{ end with i}\}\mathbb{P}\{\text{end with i}\} + \\
&\quad \mathbb{P}\{N_i = n, N_j = n | \text{ end with j}\}\mathbb{P}\{\text{end with j}\} \\
&= \binom{n-1}{n_i-1} P_i^{n_i} \binom{n-n_i}{n_j} P_j^{n_j} (1 - P_i - P_j)^{n-n_i-n_j} + \\
&\quad \binom{n-1}{n_j-1} P_j^{n_j} \binom{n-n_j}{n_i} P_i^{n_i} (1 - P_i - P_j)^{n-n_i-n_j} \\
&= P_i^{n_i} P_j^{n_j} (1 - P - i - P_j)^{n-N-i-n_j} \frac{(n-1)!(n_i+n_j)}{n_i! n_j! (n-n_i-n_j)!} \\
&\neq \binom{n-1}{n_i-1} P_i^{n_i} (1-P_i)^{n-n_i} \binom{n-1}{n_j-1} P_j^{n_j} (1-P_j)^{n-n_j} \\
&= \mathbb{P}\{N_i = n\}\mathbb{P}\{N_j = n\}
\end{aligned}$$

So, N_i and N_j are dependent.
(**c**) Now, we have r independent poisson processes $N_i(t), i \in \{1, \dots, r\}$, where $N_i(t)$ has a poisson distribution with mean $\lambda P_i t = P_i t$ (since $\lambda = 1$).

$$\mathbb{P}\{T > t\} = \prod_{i=1}^{r} \mathbb{P}\{S_{n_i}^{(i)} > t\}$$

where $S_{n_i}^{(i)} \sim \text{Gamma}(n_i, P_i)$.
(**d**) $T_i = S_{n_i}^{(i)}$ which are independent since the poisson processes are independent.
(**e**) $\mathbb{E}T = \int_0^\infty \mathbb{P}\{T > t\}$
(**f**)

$$T = \sum_{i=1}^{N} X_i \implies \mathbb{E}T = \mathbb{E}[\mathbb{E}[T|N]] = \frac{1}{\lambda}\mathbb{E}N = \mathbb{E}N$$

Ex. 2.16 Let N be the number of trials to be performed which follows Poisson(λ). Let O_i be the number of trials when ith outcome came up where the probability that a trial results in ith outcome is P_i. Then, O_i will follow Poisson(λP_i).

$$X_j = \sum_{i=1}^{n} \mathbb{I}(O_i = j)$$

$$\mathbb{E}X_j = \sum_{i=1}^{n} \mathbb{P}(O_i = j) = \sum_{i=1}^{n} \frac{\exp(-\lambda P_i)(\lambda P_i)^j}{j!}$$

$$\begin{aligned}\operatorname{Var} X_j &= \mathbb{E}X_j^2 - (\mathbb{E}X_j)^2 \\ &= \sum_{i=1}^{n} \frac{\exp(\lambda P_i)(\lambda P_i)^j}{j!}\left(1 - \frac{\exp(-\lambda P_i)(\lambda P_i)^j}{j!}\right)\end{aligned}$$

Ex. 2.17 (**a**)

$$\begin{aligned}f_{X(i)}(x) &= \mathbb{P}\{i-1 \text{ of the } X\text{'s} \leq x,\ \text{one } X \text{ equals } x,\ \text{remaining } X\text{'s} > x\} \\ &= \binom{n}{i-1}\mathbb{P}\{X \leq x\}^{i-1}\binom{n-(i-1)}{1}\mathbb{P}\{X = x\}\binom{n-i}{n-i}\mathbb{P}\{X > x\}^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!}F(x)^{i-1}f(x)\bar{F}(x)^{n-i}\end{aligned}$$

(**b**) Atleast i X's.
(**c**)

$$\begin{aligned}\mathbb{P}\{X_{(i)} \leq x\} &= \sum_{k=i}^{n} \mathbb{P}\{k \text{ of the } X\text{'s are } \leq x \text{ and remaining are } > x\} \\ &= \sum_{k=i}^{n} \binom{n}{k} F(x)^k \bar{F}(x)^{n-k}\end{aligned}$$

(**d**) Replace $y = F(x)$ and integrate (**a**).
(**e**) Given $N(t) = n$, for $i \leq n$, S_i follows the distribution of ith order statistic of n random variables uniformly distributed in $(0, t)$. Therefore,
$\mathbb{E}[S_i|N(t) = n] = \frac{i}{n+1}$ when $i \leq n$. Given $N(t) = n$, for $i > n$,
$\mathbb{E}[S_i|N(t) = n] = \mathbb{E}[S_i|S_i > t] = \mathbb{E}[S_i\mathbb{I}(S_i > t)]/\mathbb{P}\{S_i > t\}$ which equals,

$$\frac{\int_t^\infty x\frac{\lambda^i x^{i-1}\exp(-\lambda x)}{(i-1)!}}{\int_t^\infty \frac{\lambda^i x^{i-1}\exp(-\lambda x)}{(i-1)!}} = \frac{i}{\lambda}\frac{\bar{G}(t)}{\bar{F}(t)},\ \text{where } G \sim \text{Gamma}(i+1, \lambda), F \sim \text{Gamma}(i, \lambda)$$

Ex. 2.18

$$
\begin{aligned}
\mathbb{P}\{U_{(i)} = x | U_{(n)} = y\} &= \frac{\mathbb{P}\{U_{(i)} = x, U_{(n)} = y\}}{\mathbb{P}\{U_n = y\}} \mathbb{I}(x \le y) \\
&= \frac{\frac{n!}{(i-1)!(n-i-1)!} f(x) f(y) F(x)^{i-1} (F(y) - F(x))^{n-i-1}}{\frac{n!}{(n-1)!} f(y) F(y)^{n-1}} \mathbb{I}(x \le y) \\
&= \frac{n!}{(i-1)!(n-1-i)!} \frac{x^{i-1}(y-x)^{n-i-1}}{y^{n-1}} \mathbb{I}(x \le y) \\
&= \frac{n!}{(i-1)!(n-1-i)!} \left(\frac{x}{y}\right)^{i-1} \left(1 - \frac{x}{y}\right)^{n-i-1} \mathbb{I}(x \le y)
\end{aligned}
$$

Ex. 2.19 Type-j bus load arrival, where the number of customers in the bus equals j, follows a poisson process $N_j(t)$ having rate $\lambda\alpha_j$. Let the total number of customers arrived by time t is given by $N(t)$. Then,

$$
N(t) = \sum_{j=1}^{\infty} j N_j(t)
$$

Since, $N_j(t) \sim \text{Poisson}(\lambda\alpha_j t)$, $N(t)$ is a sum of poisson random variables and therefore $N(t) \sim \text{Poisson}(\gamma)$ where $\gamma = \lambda \sum_{j=1}^{\infty} j\alpha_j$.
Now, a randomly chosen customer who arrived at time s will be served by time t with probability $G(t-s)$. Let $\beta = \frac{1}{t}\int_0^t G(t-s)ds$, then the poisson process $N'(t)$ having rate $\gamma\beta$ corresponds to the number of customers served by time t. Clearly, $X(t) = N'(t)$.
(**a**)

$$
\mathbb{E}X(t) = \gamma\beta t = \lambda\beta t \sum_{j=1}^{\infty} j\alpha_j
$$

(**b**) $X(t) \sim \text{Poisson}(\lambda\beta t \sum_{j=1}^{\infty} j\alpha_j)$

Ex. 2.20 Let $p_i = \frac{1}{t}\int_0^t P_i(s)ds$. Then,

$$
\begin{aligned}
\mathbb{P}\{N_i(t)=n_i, i\in\{1,\dots,k\}\} &= \sum_m \mathbb{P}\{N_i(t)=n_i, i\in\{1,\dots,k\}|N(t)=m\}\mathbb{P}\{N(t)=m\}\\
&= \mathbb{P}\left\{N_i(t)=n_i, i\in\{1,\dots,k\}|N(t)=\sum_{j=1}^k n_j\right\}\mathbb{P}\left\{N(t)=\sum_{j=1}^k n_j\right\}\\
&= \frac{\left(\sum_{j=1}^k n_j\right)!}{\prod_{j=1}^k n_j!}\prod_{j=1}^k p_j^{n_j}\cdot\exp\left(-\lambda t\right)\frac{(\lambda t)^{\sum_{j=1}^k n_j}}{\left(\sum_{j=1}^k n_j\right)!}\\
&= \prod_{j=1}^k \exp(-\lambda p_j t)\frac{(\lambda p_j t)^{n_j}}{n_j!}
\end{aligned}
$$

Therefore, $N_i(t)\perp N_j(t), i\neq j$ and $N_i(t)\sim \text{Poisson}(\lambda p_i t)$.

Ex. 2.21 We need to show that,

$$
\int_0^s \alpha(s)ds = \mathbb{E}[\text{amount of time individual is in state } i \text{ during its first } t \text{ units in the system}]
$$

Divide interval $(0,t]$ in n equal parts and let $h=t/n$. Then, the amount of time individual is in state i during its first t units in the system equals $\sum_{i=1}^n \mathbb{I}(\text{individual is in state } i \text{ during } ((i-1)h, ih])h$. Therefore,

$$
\begin{aligned}
&\mathbb{E}[\text{amount of time individual is in state } i \text{ during its first } t \text{ units in the system}] =\\
&\quad \lim_{h\to 0}\mathbb{E}[\sum_{i=1}^n \mathbb{I}(\text{individual is in state } i \text{ during } ((i-1)h, ih])h]\\
&= \lim_{h\to 0}\sum_{i=1}^n \mathbb{P}\{\text{individual is in state } i \text{ during } ((i-1)h, ih]\}h\\
&= \int_0^t \alpha(s)ds
\end{aligned}
$$

Ex. 2.22 A car entering at time s will be located in the interval (a,b) at time t when its velocity satisfies $a < V(t-s) < b \implies \frac{a}{t-s} < V < \frac{b}{t-s}$, the probability of which is $P(s) = F(b/(t-s)) - F(a/(t-s))$. Let $p = \frac{1}{t}\int_t^t P(s)ds$, then, the number of cars located in the interval (a,b) at time t will follow poisson distribution with mean λpt.

Ex. 2.23 (**a**) Using $\text{Var}[D(t)] = \text{Var}[\mathbb{E}[D(t)|N(t)]] + \mathbb{E}[\text{Var}[D(t)|N(t)]]$, we get,

$$
\begin{aligned}
\mathrm{Var}[\mathbb{E}[D(t)|N(t)]] &= \mathrm{Var}\left[\frac{N(t)}{\alpha t}(1-\exp(-\alpha t))\mathbb{E}[D]\right] \\
&= \frac{\lambda(1-\exp(-\alpha t))^2\mathbb{E}[D]^2}{\alpha^2 t} \\
\mathrm{Var}[D(t)|N(t)] &= \mathbb{E}[D]^2\exp(-2\alpha t)\,\mathrm{Var}\left[\sum_{i=1}^{N(t)}\exp(\alpha S_i)|N(t)\right] \\
&= \mathbb{E}[D]^2\exp(-2\alpha t)n\left(\frac{\exp(2\alpha t)-1}{2\alpha t}-\frac{(\exp(\alpha t)-1)^2}{\alpha^2 t^2}\right) \\
\mathbb{E}[\mathrm{Var}[D(t)|N(t)]] &= \mathbb{E}[D]^2\exp(-2\alpha t)\lambda\left(\frac{\exp(2\alpha t)-1}{2\alpha}-\frac{(\exp(\alpha t)-1)^2}{\alpha^2 t}\right) \\
\mathrm{Var}[D(t)] &= \frac{\mathbb{E}[D]^2\lambda(1-\exp(-2\alpha t))}{2\alpha}
\end{aligned}
$$

(**b**) Using property of independent increments of poisson process we have,

$$
\begin{aligned}
D(t+s) &= D(t)\exp(-\alpha s) + \sum_{i=N(t)+1}^{N(t+s)} D_i\exp(-\alpha(t+s-S_i)) \\
&= D(t)\exp(-\alpha s) + D'(s)\exp(-\alpha t)
\end{aligned}
$$

where $D'(s) \perp D(t)$ and $D'(s)$ follows the same distribution as $D(s)$. So,

$$
\begin{aligned}
\mathrm{Cov}(D(t), D(t+s)) &= \mathbb{E}[D(t)D(t+s)] - \mathbb{E}[D(t)]\mathbb{E}[D(t+s)] \\
&= \mathbb{E}[D(t)^2\exp(-\alpha s) + D(t)D'(s)\exp(-\alpha t)] \\
&\quad - \mathbb{E}[D(t)]^2\exp(-\alpha s) - \mathbb{E}[D(t)]\mathbb{E}[D'(s)]\exp(-\alpha t) \\
&= \mathrm{Var}[D(t)]\exp(-\alpha s)
\end{aligned}
$$

Ex. 2.24 Let the time taken T by a car to travel the highway of length L follow distribution G. Then, $\mathbb{P}(T \le t) = G(t) = \mathbb{P}(V \ge L/t) = \bar{F}(L/t)$. Let v be the speed of the car that enters the highway at time t. Then, the time taken by the car to travel the highway is $t_v = L/v$. Let s be the time a random car enters the highway and leaving after time T, then, the probability of an encounter with the car entering at time t is,

$$
P(s) = \begin{cases}
\mathbb{P}\{T \ge t-s+t_v\} = \bar{G}(t-s+t_v), & s < t \\
\mathbb{P}\{T \le t+t_v-s\} = G(t-s+t_v), & t \le s < t+t_v \\
0, & \text{otherwise}
\end{cases}
$$

Then, expected number of encounters is given by,

$$\lambda\left(\int_0^t \bar{G}(t-s+t_v)ds + \int_t^{t+t_v} G(t-s+t_v)ds\right) = \lambda\left(1 - \int_{t_v}^{t+t_v} G(s)ds + \int_0^{t_v} G(s)ds\right)$$

The value of t_v minimizing the above, satisfies,

$$G(t_v) - G(t+t_v) + G(t_v) = 0 \implies G(t_v) = 1/2 \text{ as } t \to \infty$$

Thus, as $t \to \infty$,

$$\bar{F}(v) = 1/2 \implies v = F^{-1}(1/2)$$

Ex. 2.25

$$W = \sum_{i=1}^{N(t)} Y_i, \qquad Y_i \sim F_{S_i}$$

$$\mathbb{P}\{W \leq w | N(t) = n\} = \mathbb{P}\left\{\sum_{i=1}^{n} Y_i \leq w \middle| N(t) = n\right\}$$

Given $N(t) = n$, $S_1, S_2, \ldots, S_n$ are uniform$(0, t)$. Therefore, for all i,

$$\begin{aligned}\mathbb{P}\{Y_i \leq y | N(t) = n\} &= \int_0^t \mathbb{P}\{Y_i \leq y | N(t) = n, S_i = s\}\mathbb{P}\{S_i = s\}ds \\ &= \int_0^t F_s(y).\frac{1}{t}ds = \frac{1}{t}\int_0^t F_s(y)ds\end{aligned}$$

Therefore, W can be thought of as a compound poisson random variable, $\sum_{i=1}^{N} X_i$, where X_i are iid with distribution,

$$F(x) = \frac{1}{t}\int_0^t F_s(y)ds$$

and are also independent with N which follows poisson distribution with mean λt.

Ex. 2.26

$$f_{S_1,S_2,\ldots,S_n|S_n=t}(s_1, s_2, \ldots, s_n) = \begin{cases} f_{S_1,S_2,\ldots,S_n|S_n=t}(s_1, s_2, \ldots, t), & s_1 \leq s_2 \leq \ldots \leq s_n = t \\ 0, & \text{otherwise}\end{cases}$$

$$
\begin{aligned}
f_{S_1,S_2,\ldots,S_n|S_n=t}(s_1,s_2,\ldots,t) &= \frac{f_{S_1,\ldots,S_n}(s_1,\ldots,t)}{f_{S_n}(t)} \\
&= \frac{f_{X_1,\ldots,X_n}(s_1,s_2-s_1,\ldots,t-s_{n-1})}{f_{S_n}(t)} \\
&= \frac{\lambda\exp(-\lambda s_1)\lambda\exp(-\lambda(s_2-s_1))\ldots\lambda\exp(-\lambda(t-s_{n-1}))}{\lambda^n t^{n-1}\exp(-\lambda t)/(n-1)!} \\
&= \frac{(n-1)!}{t^{n-1}}
\end{aligned}
$$

Ex. 2.28 First note that,

$$
\mathbb{E}\left[Y_1+\ldots+Y_k|\sum_{i=1}^n Y_i=y\right]=\mathbb{E}\left[Y_{j_1}+\ldots+Y_{j_k}|\sum_{i=1}^n Y_i=y\right]
$$

Taking every combination of k Y_i's, adding them, taking expectation and then using the linearity of expectation we get,

$$
\begin{aligned}
&\binom{n}{k}\mathbb{E}\left[Y_1+\ldots+Y_k|\sum_{i=1}^n Y_i=y\right]=\binom{n-1}{k-1}\mathbb{E}\left[Y_1+\ldots+Y_n|\sum_{i=1}^n Y_i=y\right]=\binom{n-1}{k-1}y \\
&\implies \mathbb{E}\left[Y_1+\ldots+Y_k|\sum_{i=1}^n Y_i=y\right]=\frac{ky}{n}
\end{aligned}
$$

Ex. 2.29 First we divide the interval $(t,t+s]$ into k equal subintervals and prove that the probability of greater than or equal to 2 events in any of those subintervals approaches 0 as k approaches ∞.

$$
\begin{aligned}
\mathbb{P}\{\geq 2 \text{ events in a subinterval}\} &= \cup_{i=1}^k \mathbb{P}\left\{\geq 2 \text{ events in } \left(t+\frac{(i-1)s}{k},t+\frac{is}{k}\right]\right\} \\
&\leq \sum_{i=1}^k \mathbb{P}\left\{\geq 2 \text{ events in } \left(t+\frac{(i-1)s}{k},t+\frac{is}{k}\right]\right\} \\
&= ko(s/k)=t\frac{o(s/k)}{s/k}\to 0 \text{ as } k\to\infty
\end{aligned}
$$

Let I_j be defined as,

$$
I_j=\begin{cases} 1, & \text{an event in } \left(t+\frac{(i-1)s}{k},t+\frac{is}{k}\right] \\ 0, & 0 \text{ event in } \left(t+\frac{(i-1)s}{k},t+\frac{is}{k}\right] \end{cases}
$$

So, the number of events in $(t, t+s]$, by poisson approaximation of binomial distribution, follows a poisson distribution with mean,

$$\lim_{k\to\infty} \mathbb{E}\left[\sum_{j=1}^{k} I_j\right] = \lim_{k\to\infty} \sum_{j=1}^{k} \lambda\left(t + \frac{js}{k}\right)\frac{s}{k} = \int_{t}^{t+s} \lambda(x)dx = m(t+s) - m(t)$$

Ex. 2.31

$$\mathbb{P}\{N^*(t) = n\} = \mathbb{P}\{N(m^{-1}(t)) = n\} = \frac{\exp(-m(m^{-1}(t)))(m(m^{-1}(t)))^n}{n!} = \frac{\exp(-t)t^n}{n!}$$

Ex. 2.32(a) Let $t_1, t_2, \ldots, t_n$ be such that $0 < t_1 < t_2 < \ldots < t_n < t$ and Δ_i be such that $t_i + \Delta_i < t_{i+1}$, then,

$$\begin{aligned}
&\mathbb{P}\{t_i \le S_i \le t_i + h_i, i \in \{1, 2, \ldots, n\} | N(t) = n\} \\
&= \frac{e^{-m(t_1)}\left(\prod_{i=1}^{n} e^{-(m(t_i+\Delta t_i)-m(t_i))}(m(t_i + \Delta t_i) - m(t_i))\right)e^{-(m(t)-m(t_n+\Delta t_n))}}{e^{-m(t)}m(t)^n/n!} \\
&= \frac{n!\prod_{i=1}^{n}(m(t_i + \Delta t_i) - m(t_i))}{m(t)}
\end{aligned}$$

As $\Delta_i \to 0$,

$$f_{S_1,\ldots,S_{N(t)}|N(t)=n}(t_1, t_2, \ldots, t_n) = \frac{n!\prod_{i=1}^{n}\lambda(t_i)}{m(t)^n}$$

Therefore, the unordered set of arrival times has the same distribution as n iid random variables having distribution function,

$$F(x) = \begin{cases} m(x)/m(t) & x \le t \\ 1 & x > t \end{cases}$$

(**b**) Let $P(s)$ be the probability that a worker injured at time s is out of work at time t. Then,

$$P(s) = \bar{F}(t-s)$$

The two types of Poisson processes: $N_1(t)$ represents the number of workers out of work at time t and $N_2(t)$ represents the number of workers at work at time t. Now, a random worker injured before time t will be out of work at time t with probability p,

$$\begin{aligned}
p &= \int_0^t \mathbb{P}\{\text{out of work at time } t | \text{injured at time } s\}\mathbb{P}\{\text{injured at time } s\} ds \\
&= \int_0^t P(s)\frac{\lambda(s)}{m(t)} ds \\
&= \frac{1}{m(t)}\int_0^t \bar{F}(t-s)\lambda(s) ds
\end{aligned}$$

Finally,
$\mathbb{E}[X(t)] = \mathbb{E}[N_1(t)] = m(t)p = \int_0^t \bar{F}(t-s)\lambda(s)ds = \mathrm{Var}(N_1(t)) = \mathrm{Var}(X(t))$.

Ex. 2.33(a)

$$\mathbb{P}\{X > t\} = \mathbb{P}\{0 \text{ events in } \bar{B}_t(0)\} = \exp(-\lambda\pi t^2)$$

(**b**)

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}\{X > t\}dt = \int_0^\infty \exp(-\lambda\pi t^2)dt = \frac{1}{2\sqrt{\lambda}}$$

(**c**)

$$\mathbb{P}\{\pi R_1^2 > t\} = \mathbb{P}\{R_1 > \sqrt{t}/\sqrt{\pi}\} = \exp(-\lambda\pi t/\pi) = \exp(-\lambda t)$$

$$\begin{aligned}
\mathbb{P}\{\pi R_2^2 - \pi R_1^2 > t | \pi R_1^2 = s\} &= \mathbb{P}\{0 \text{ event in } s/\pi < \|\vec{r}\|^2 \le (s+t)/\pi | \pi R_1^2 = s\} \\
&= \mathbb{P}\{0 \text{ event in } s/\pi < \|\vec{r}\|^2 \le (s+t)/\pi\} \text{ (non-overlapping regions)} \\
&= \exp(-\lambda(\pi(s+t)/\pi - \pi s/\pi)) = \exp(-\lambda t)
\end{aligned}$$

Ex. 2.34

$$W = \sum_{i=1}^{N(t)} Y_i, \qquad Y_i \sim F_{S_i}$$

$$\mathbb{P}\{W \le w | N(t) = n\} = \mathbb{P}\left\{\sum_{i=1}^{n} Y_i \le w \middle| N(t) = n\right\}$$

Using **2.32**(**a**), for all i,

$$\begin{aligned}
\mathbb{P}\{Y_i \le y | N(t) = n\} &= \int_0^t \mathbb{P}\{Y_i \le y | N(t) = n, S_i = s\}\mathbb{P}\{S_i = s\}ds \\
&= \int_0^t F_s(y).\frac{dm(s)}{m(t)} = \frac{1}{m(t)}\int_0^t F_s(y)dm(s)
\end{aligned}$$

Therefore, W can be thought of as a compound poisson random variable, $\sum_{i=1}^{N} X_i$, where X_i are iid with distribution,

$$F(x) = \frac{1}{m(t)} \int_0^t F_s(y) dm(s)$$

and are also independent with N which follows poisson distribution with mean $\lambda(t)$.

Ex. 2.35(a)

$$\begin{aligned} N^*(t+s) - N^*(t) &= N(t+s+\tau) - N(t+\tau) \\ N^*(t) &= N(t+\tau) - N(\tau) \\ N(t+\tau) - N(\tau) \perp N(t+s+\tau) - N(t+\tau) &\implies N^*(t+s) - N^*(t) \perp N^*(t) \end{aligned}$$

(**b**) Last implication is still valid.

3. Renewal Theory

Ex. 3.1(a) True.

(b) True.

(c) If $F(0) = 0$, then true. If $F(0) > 0$, then false.

Ex. 3.2

$N(\infty) + 1 = n \iff N(\infty) = n - 1 \iff X_i < \infty,\ \forall i < n \wedge X_n = \infty$

$$\mathbb{P}\{N(\infty) + 1 = n\} = \mathbb{P}\{X_i < \infty,\ \forall i < n \wedge X_n = \infty\} = \mathbb{P}\{X_n = \infty\} \prod_{i=1}^{n-1} \mathbb{P}\{X_i < \infty\}$$
$$= F(\infty)^{n-1}(1 - F(\infty))$$

Therefore, $N(\infty) + 1$ is geometric with mean $1/(1 - F(\infty))$.

Ex. 3.3

$$\begin{aligned}
\mathbb{P}\{X_{N(t)+1} \geq x\} &= \mathbb{P}\{X_{N(t)+1} \geq x | S_{N(t)} = 0\}\bar{F}(t) + \int_0^t \mathbb{P}\{X_{N(t)+1} \geq x | S_{N(t)} = y\}\bar{F}(t-y)dm(y) \\
&= \mathbb{P}\{X_1 \geq x | X_1 > t\}\bar{F}(t) + \int_0^t \mathbb{P}\{X \geq x | X > t - y\}\bar{F}(t-y)dm(y) \\
&= \mathbb{I}(x \leq t)(\bar{F}(t) + \int_0^{t-x} \bar{F}(t-y)dm(y) + \int_{t-x}^t dm(y)) + \mathbb{I}(x > t)\bar{F}(x)(1 + m(t)) \\
&= \mathbb{I}(x \leq t)(\mathbb{P}\{X \leq t - x\} + m(t) - m(t-x)) + \mathbb{I}(x > t)\bar{F}(x)(1 + m(t))
\end{aligned}$$

Ex. 3.4

$$\begin{aligned}
m(t) &= \sum_{n=1}^{\infty} F_n(t) = F(t) + F(t) * \left(\sum_{n=1}^{\infty} F_n(t)\right) = F(t) + F(t) * m(t) \\
&= F(t) + \int_0^t m(t-x)dF(x)
\end{aligned}$$

Ex. 3.5

$$\begin{aligned}
m &= F + m * F \\
F &= m - m * F \\
F &= m - m_2 + m_2 * F \\
F(t) &= \sum_{n=1}^{\infty} (-1)^{n+1} m_n(t)
\end{aligned}$$

Ex. 3.6

$\forall s \le t,\ \mathbb{E}[N(s)|N(t)] = \frac{s}{t}N(t) \implies m(s) = \mathbb{E}[N(s)] = \mathbb{E}[\mathbb{E}[N(s)|N(t)]] = \frac{s}{t}m(t)$

Therefore, $m(t) = kt$ where k is a positive constant.
Using **3.4**, we get,

$$\begin{aligned} kt &= F(t) + \int_0^t k(t-x)dF(x) \\ k &= \frac{dF(t)}{dt} + kF(t) \\ F(t) &= 1 - \exp(-kt) \end{aligned}$$

Hence, the interarrival times distribution is exponential and therefore $\{N(t), t \ge 0\}$ is a Poisson process.

Ex. 3.7 Using **3.4**, for $t \in [0,1]$, we get,

$$\begin{aligned} m(t) &= t + \int_0^t m(t-x)dt \\ \frac{dm(t)}{dt} &= 1 + m(t) \\ m(t) &= \exp(t) - 1 \end{aligned}$$

$$\mathbb{E}(N(1)+1) = m(1) + 1 = e$$

Ex. 3.10 (**a**)

$$\lim_{m\to\infty} \frac{\sum\limits_{i=1}^{N_1+N_2+\ldots+N_m} X_i}{N_1+N_2+\ldots+N_m} = \mathbb{E}[X_1]$$

(**b**)

$$\lim_{m\to\infty} \frac{\sum\limits_{i=1}^{m} S_i}{m} . \frac{m}{\sum\limits_{i=1}^{m} N_i} = \frac{\mathbb{E}[S_1]}{\mathbb{E}[N_1]}$$

(**c**)

$$\mathbb{E}[X_1] = \frac{\mathbb{E}[S_1]}{\mathbb{E}[N_1]} \implies \mathbb{E}[S_1] = \mathbb{E}[X_1]\mathbb{E}[N_1]$$

Ex. 3.11 (**a**)

$$X_i = \begin{cases} 2 & w.p.\ 1/3 \\ 4 & w.p.\ 1/3 \\ 8 & w.p.\ 1/3 \end{cases}$$
$$N = \min\{n : X_n = 2\}$$

(**b**)

$$\mathbb{E}[T] = \mathbb{E}[X_1]\mathbb{E}[N] = \frac{14}{3}\frac{1}{1/3} = 14$$

(**c**)

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i | N = n\right] = (4+8)\frac{1}{2}(n-1) + 2 = 6n - 4$$
$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \frac{14n}{3}$$

(**d**)

$$\mathbb{E}\left[T\right] = \mathbb{E}[\mathbb{E}[T|N]] = \mathbb{E}[6N-4] = 6.3 - 4 = 14$$

Ex. 3.12

$h(t) = \mathbb{I}(t \le a)$

$$\lim_{t\to\infty} \int_0^t h(t-x)dm(x) = \lim_{t\to\infty} \int_{t-a}^t dm(x) = \lim_{t\to\infty} m(t) - m(t-a) = \lim_{t\to\infty} \frac{1}{\mu} \int_0^t h(x)dx = \frac{a}{\mu}$$

Ex. 3.13

$$\frac{\mathbb{E}[T_i]}{\mathbb{E}[\sum_{i=1}^n T_i]} = \frac{\mu_i}{\sum_{j=1}^n \mu_j}$$

Ex. 3.14 (**a**)

$$(t - x, t]$$

(**b**)

$$(t, t + x]$$

(**c**)

$$\mathbb{P}\{Y(t) > x\} = \mathbb{P}\{A(t+x) > x\}$$

(**d**)

$$\begin{aligned}\mathbb{P}\{Y(t) > y, A(t) > x\} &= \mathbb{P}\{\text{No event in } (t, t+y], \text{No event in } (t-x, t]\} \\ &= \mathbb{P}\{\text{No event in } (t-x, t+y]\} \\ &= \exp(-\lambda(x+y))\end{aligned}$$

Ex. 3.15 (**a**)

$$\mathbb{P}\{Y(t) > x | S_{N(t)} = t-s\} = \mathbb{P}\{X > x+s | X > s\} = \frac{\bar{F}(x+s)}{\bar{F}(x)}$$

(**b**)

$$\mathbb{P}\{Y(t) > x | A(t+x/2) = s\} = \begin{cases} 0 & s < x/2 \\ \mathbb{P}\{X > x/2 + s | X > s - x/2\} = \frac{\bar{F}(s+x/2)}{\bar{F}(s-x/2)} & s \geq x/2 \end{cases}$$

(**c**)

$$\begin{aligned}\mathbb{P}\{Y(t) > x | A(t+x) > s\} &= \mathbb{P}\{\text{No event in } (t, t+x] | \text{No event in } (t+x-s, t+x]\} \\ &= \begin{cases} 1 & s \geq x \\ \frac{\mathbb{P}\{\text{No event in } (t,t+x]\}}{\mathbb{P}\{\text{No event in} (t+x-s,t+x]\}} = \frac{\exp(-\lambda x)}{\exp(-\lambda s)} = \exp(-\lambda(x-s)) & s < x \end{cases}\end{aligned}$$

(**d**)

$$\begin{aligned}\mathbb{P}\{Y(t) > x, A(t) > y\} &= \mathbb{P}\{Y(t > x), S_{N(t)} < t-y\} \\ &= \int_0^{t-y} \mathbb{P}\{Y(t) > x | S_{N(t)} = s\} dF_{S_{N(t)}}(s) \\ &= \int_0^{t-y} \frac{\bar{F}(x+t-s)}{\bar{F}(x)} \cdot \bar{F}(t-s) dm(s)\end{aligned}$$

(**e**)

$$\frac{A(t)}{t} = \frac{t - S_{N(t)}}{t} = 1 - \frac{S_{N(t)}}{N(t)} \cdot \frac{N(t)}{t} \to 1 - \mu \cdot \frac{1}{\mu} = 0$$

Ex. 3.16

$$\mathbb{E}[Y(t)] \to \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} = \frac{n/\lambda^2 + n^2/\lambda^2}{2n/\lambda} = \frac{n+1}{2\lambda}$$

Ex. 3.17

$$g = h + g * F = h + F * (h + F * g) = \ldots = h + h * \sum_{n=1}^{\infty} F_n = h + h * m$$

(**a**)

$$\begin{aligned}\mathbb{P}\{\text{on at } t\} &= \mathbb{P}\{\text{on at } t | S_{N(t)} = 0\} \bar{F}(t) + \int_0^t \mathbb{P}\{\text{on at } t | S_{N(t)} = y\} \bar{F}(t-y) dm(y) \\ &= \bar{H}(t) + \int_0^t \bar{H}(t-y) dm(y) \to \frac{\mu_H}{\mu_F}\end{aligned}$$

(**b**)

$$\begin{aligned}\mathbb{E}[A(t)] &= \mathbb{E}[A(t)|S_{N(t)} = 0]\bar{F}(t) + \int_0^t \mathbb{E}[A(t)|S_{N(t)} = y]\bar{F}(t-y)dm(y) \\ &= t\bar{F}(t) + \int_0^t (t-y)\bar{F}(t-y)dm(y) \\ &\to \frac{\int_0^\infty t\bar{F}(t)dt}{\mu} = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]}\end{aligned}$$

Ex. 3.19

$$\begin{aligned}\mathbb{P}\{S_{N(t)} \le s\} &= \sum_{n=0}^{\infty} \mathbb{P}\{S_n \le s, S_{n+1} > t\} \\ &= \mathbb{P}\{S_1 > t\} + \sum_{n=1}^{\infty} \int_0^\infty \mathbb{P}\{S_n \le s, S_{n+1} > t | S_n = y\} \mathbb{P}\{S_n = y\} dy \\ &= \bar{G}(t) + \sum_{n=1}^{\infty} \int_0^s \mathbb{P}\{S_{n+1} > t | S_n = y\} d(G * F_{n-1})(y) \\ &= \bar{G}(t) + \int_0^s \bar{F}(t-y) dm_D(y)\end{aligned}$$

Ex. 3.20 (**a**)

$$\mathbb{E}[T_{\cdot \to HHTHHTT}] = \mathbb{E}[T_{HHTHHTT \to HHTHHTT}] = 2^7$$

(**b**)

$$\begin{aligned}&\mathbb{E}[T_{\cdot \to HHTT}] = 2^4 \\ &\mathbb{E}[T_{\cdot \to HTHT}] = \mathbb{E}[T_{\cdot \to HT}] + \mathbb{E}[T_{HT \to HTHT}] = 2^2 + 2^4\end{aligned}$$

Ex. 3.21 $T_{\cdot \to WWWWWWW}$ is, by definition, stopping time. Using Wald's equation,

(**a**)

$$\mathbb{E}\left[\sum_{i=1}^{T_{\cdot\to WWWWWWW}} X_i\right] = \mathbb{E}[X_i]\mathbb{E}[T_{\cdot\to WWWWWWW}] = (2p-1)(\sum_{i=1}^{7} p^{-i})$$

(**b**)

$$\mathbb{E}\left[\sum_{i=1}^{T_{\cdot\to WWWWWWW}} Y_i\right] = \mathbb{E}[Y_i]\mathbb{E}[T_{\cdot\to WWWWWWW}] = p(\sum_{i=1}^{7} p^{-i})$$

Ex. 3.22

$$\begin{aligned}
\mathbb{E}[N_A] &= \mathbb{E}[N_{HH}] + p^{-4}q^{-2} = p^{-1} + p^{-2} + p^{-4}q^{-2} \\
\mathbb{E}[N_B] &= p^{-2}q^{-3} \\
\mathbb{E}[N_{A|B}] &= \mathbb{E}[N_A] \\
\mathbb{E}[N_{B|A}] &= \mathbb{E}[N_{B|H}] = \mathbb{E}[N_B] - \mathbb{E}[N_H] = \mathbb{E}[N_B] - p^{-1}
\end{aligned}$$

$M = \min\{N_A, N_B\}$ and $a = \mathbb{P}\{A \text{ before } B\}$

(**a**)(**b**)

$$\begin{aligned}
\mathbb{E}[N_A] &= \mathbb{E}[N_A - M] + \mathbb{E}[M] = \mathbb{E}[N_A - M|M = N_B]a + \mathbb{E}[M] \\
70 &= \mathbb{E}[N_{A|B}]a + \mathbb{E}[M] = (2 + 2^2 + 2^6)a + \mathbb{E}[M] = 70a + \mathbb{E}[M] \\
\mathbb{E}[N_B] &= \mathbb{E}[N_{B|A}](1-a) + \mathbb{E}[M] \\
32 &= 30(1-a) + \mathbb{E}[M]
\end{aligned}$$

So, $a = 0.68$ and $\mathbb{E}[M] = 22.4$

Ex. 3.23 Let A be the set of binary strings of length k where $1 \equiv H$ and $0 \equiv T$. Let σ be the binary string corresponding to first k flips of coin and F be the number of additional flips required to obtain the same pattern.

$$\mathbb{E}[F] = \sum_{a\in A} \mathbb{E}[F|\sigma = a]\mathbb{P}\{\sigma = a\} = \sum_{a\in A} \frac{1}{\mathbb{P}\{\sigma = a\}}\mathbb{P}\{\sigma = a\} = 2^k$$

Ex. 3.24 Let a renewal correspond to last 4 cards being of same suit. Let L denote the suit of the last renewal (i.e. of the 4 consecutive cards of same suit), N denote the suit of the card just after the last renewal. T be the time to get the first renewal i.e. the first time 4 consecutive cards of same suit appear. Let T' be the time between two renewals. Since,

$$\begin{aligned}
\mathbb{E}[T'|L = i] &= \mathbb{E}[T'|L = i, N = i]\mathbb{P}\{N = i|L = i\} + \mathbb{E}[T'|L = i, N \neq i]\mathbb{P}\{N \neq i|L = i\} \\
&= 1 \cdot 1/4 + \mathbb{E}[T] \cdot 3/4
\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[T'] &= \sum_{i=1}^{4} \mathbb{E}[T|L=i]\mathbb{P}\{L=i\} \\ &= 1/4 + \frac{3}{4}\mathbb{E}[T]\end{aligned}$$

Finally, using $\mathbb{E}[T'] = \lim_{n\to\infty} \mathbb{P}\{\text{renewal at } n\}^{-1} = 4^3$, we have, $\mathbb{E}[T] = 85$.

Ex. 3.25 (**a**)

$$m_D = G * \sum_{n=1}^{\infty} F_{n-1} = G + G * \sum_{n=1}^{\infty} F_n = G + G * m$$

(**b**)

$$\begin{aligned}\mathbb{E}[A_D(t)] &= t\bar{G}(t) + \int_0^t (t-y)\bar{F}(t-y)dm_D(y) \\ &\to 0 + \frac{\int_0^\infty x^2 dF(x)}{2\int_0^\infty x dF(x)} \quad \text{(by key-renewal theorem of delayed renewal process)}\end{aligned}$$

(**c**)

$$\begin{aligned}\mathbb{E}[X] - \int_0^t xg(x)dx = \int_t^\infty xg(x)dx \geq t\int_t^\infty g(x)dx = t\bar{G}(t) \\ 0 \leq \lim_{t\to\infty} t\bar{G}(t) \leq \lim_{t\to\infty}\left(\mathbb{E}[X] - \int_0^t xg(x)dx\right) = 0\end{aligned}$$

Ex. 3.26 The proof is similar to the proof of $m(t+a) - m(t) \to \frac{a}{\mathbb{E}[X]}$.

$$\mathbb{E}[R(t+a)] - \mathbb{E}[R(t)] \to a \lim_{t\to\infty} \frac{R(t)}{t} = a\frac{\mathbb{E}[R]}{\mathbb{E}[X]}$$

Ex. 3.27

$$\begin{aligned}
\mathbb{E}[R_{N(t)+1}] &= \mathbb{E}[R_{N(t)+1}|S_N(t)=0]\bar{F}(t) + \int_0^t \mathbb{E}[R_{N(t)+1}|S_N(t)=y]\bar{F}(t-y)dm(y) \\
&= \mathbb{E}[R_1|X_1>t]\bar{F}(t) + \int_0^t \mathbb{E}[R|X>t-y]\bar{F}(t-y)dm(y) \\
&\to \frac{1}{\mu}\int_0^\infty \mathbb{E}[R|X>t]\bar{F}(t)dt \\
&= \frac{1}{\mu}\int_0^\infty \int_{-\infty}^\infty \int_t^\infty r dF_{R,X}(r,x)dt \\
&= \frac{1}{\mu}\int_0^\infty \int_{-\infty}^\infty \int_0^x dt r dF_{R,X}(r,x) \\
&= \frac{\mathbb{E}[RX]}{\mu}
\end{aligned}$$

Ex. 3.28

$$N^* = \sqrt{\frac{2K}{\mu c}}$$
$$\mathbb{E}[\text{cost}(N^*)] = \sqrt{\frac{2Kc}{\mu}} - c/2$$

$$\begin{aligned}
\mathbb{E}[\text{cost}] &= \frac{c\mu\mathbb{E}[N(T)^2 - N(T)]/2 + K}{T} = \frac{c\mu^3T^2/2 + K}{T} \\
T^* &= \sqrt{\frac{2\mu K}{c}} \\
\mathbb{E}[T^*] &= \sqrt{\frac{2Kc}{\mu}}
\end{aligned}$$

Ex. 3.29 (**a**)

$$\begin{aligned}
\mathbb{E}[\text{cycle time}] &= \mathbb{E}[\min(A,X)] \\
\mathbb{E}[\text{reward in a cycle}] &= C_1\mathbb{P}\{X \geq A\} + (C_1+C_2)\mathbb{P}\{X<A\}
\end{aligned}$$

(**b**)

$$\begin{aligned}
\mathbb{E}[\text{cycle time}] &= \left(\frac{1}{\mathbb{P}\{X<A\}} - 1\right)A + \mathbb{E}[X|X<A] = \frac{\mathbb{E}[\min(A,X)]}{\mathbb{P}\{X<A\}} \\
\mathbb{E}[\text{reward in a cycle}] &= \left(\frac{1}{\mathbb{P}\{X<A\}} - 1\right)C_1 + C_1 + C_2 = \frac{C_1\mathbb{P}\{X\geq A\} + (C_1+C_2)\mathbb{P}\{X<A\}}{\mathbb{P}\{X<A\}}
\end{aligned}$$

Ex. 3.30 Let T be the time to get m consecutive tails. Then, long run proportion of the number of heads is,

$$\frac{N_H(t)}{t} \to \frac{\mathbb{E}[\sum_{n=1}^{T} \mathbb{I}(X_n = H)]}{\mathbb{E}[T]} = \frac{\int_0^1 \sum_{k=1}^{m} p(1-p)^{-k} C p^{n-1} (1-p)^{m-1} dp}{\int_0^1 \sum_{k=1}^{m} (1-p)^{-k} C p^{n-1} (1-p)^{m-1} dp}$$

$$= 1 - \frac{\int_0^1 \sum_{k=1}^{m} (1-p)^{-(k-1)} C p^{n-1} (1-p)^{m-1} dp}{\int_0^1 \sum_{k=1}^{m} (1-p)^{-k} C p^{n-1} (1-p)^{m-1} dp} = 1 - \frac{\to \text{const} < \infty}{\to \infty} = 1$$

The denominator reaches infinity when $k = m$.

4. Markov Chains

Ex. 4.1

$$\mathbb{P}\{X_{n+1} = y | X_n = x\} = \begin{cases} \alpha_{S-y} & x < s, y < S \\ \alpha_0 + \sum_{j=S+1}^{\infty} \alpha_j & x < s, y = S \\ \alpha_{x-y} & x \geq s, y < x \\ \alpha_0 + \sum_{j=x+1}^{\infty} \alpha_j & x \geq s, y = x \end{cases}$$

Ex. 4.2 Markovian property: Past is independent of future given present.

Ex. 4.3 Let the minimum number of steps required to reach j from i is k steps i.e. ($P_{ij}^k > 0$) such that $k > n$. Since the number of states are n, there must exist atleast one state which is visited twice on the path from i to j. Let m be such a state. Then, there is a closed path starting and ending on m and removing this path (except the state m) from the path from i to j still connects i and j in u steps where $u < k$ i.e. ($P_{ij}^u > 0$) which contradicts our assumption. Therefore, $\exists\ k \leq n$ such that $P_{ij}^k > 0$.

Ex. 4.4 Condition on the number of steps for visiting j from i for the first time.

$$\begin{aligned} P_{ij}^n = \mathbb{P}\{X_n = j | X_0 = i\} &= \sum_{k=0}^{n} \mathbb{P}\{X_k = j, X_u \neq j, u < k | X_0 = i\} \mathbb{P}\{X_n = j | X_k = j\} \\ &= \sum_{k=0}^{n} f_{ij}^k P_{jj}^{n-k} \end{aligned}$$

Ex. 4.5 (**a**) Probability of reaching j from i in n steps without visiting k,
(**b**) Condition on the number of steps for the last visit to i from i.

$$\begin{aligned} P_{ij}^n = \mathbb{P}\{X_n = j | X_0 = i\} &= \sum_{k=0}^{n} \mathbb{P}\{X_k = i | X_0 = i\} \mathbb{P}\{X_n = j, X_u \neq i, u > k | X_k = i\} \\ &= \sum_{k=0}^{n} P_{ii}^k P_{ij/i}^{n-k} \end{aligned}$$

Ex. 4.6

$$
\begin{aligned}
P^{2n}_{(0,0)(0,0)} &= \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!}\left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n}\left(\frac{1}{4}\right)^{2n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} \\
&= \binom{2n}{n}^2\left(\frac{1}{4}\right)^{2n} \\
&\approx \frac{(2n)^{4n+1}e^{-4n}2\pi}{n^{4n+2}e^{-4n}(2\pi)^2 4^{2n}} = \frac{1}{2\pi n}
\end{aligned}
$$

$$
\therefore \sum_{n=0}^{\infty} P^{2n}_{(0,0),(0,0)} = \infty
$$

$$
\begin{aligned}
P^{2n}_{(0,0,0)(0,0,0)} &= \sum_{k_1+k_2=0,k_i\geq 0}^{n} \frac{(2n)!}{k_1!k_1!k_2!k_2!(n-k_1-k_2)!(n-k_1-k_2)!}\left(\frac{1}{6}\right)^{2n} \\
&= \binom{2n}{n}\left(\frac{1}{6}\right)^{2n}\sum_{k_1+k_2=0,k_i\geq 0}^{n}\binom{k_1+k_2}{k_1}^2\binom{n}{k_1+k_2}^2 \\
&= \binom{2n}{n}\left(\frac{1}{6}\right)^{2n}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{k}{m}^2\binom{n}{k}^2 \\
&= \binom{2n}{n}\left(\frac{1}{6}\right)^{2n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2 \\
&\propto \frac{1}{n^{3/2}}
\end{aligned}
$$

$$
\therefore \sum_{n=0}^{\infty} P^{2n}_{(0,0,0),(0,0,0)} < \infty
$$

Ex. 4.7 (**a**)

$$
\lim_{n\to\infty} P^{2n}_{00} = \frac{1}{\mu_{00}} \implies \mu_{00} = \lim_{n\to\infty}\frac{2^{2n}}{\binom{2n}{n}} = \lim_{n\to\infty}\frac{2^{2n}n^{2n+1}e^{-2n}2\pi}{(2n)^{2n+1/2}e^{-2n}\sqrt{2\pi}} = \lim_{n\to\infty}\sqrt{n\pi} = \infty
$$

(**b**) Using complex integration.

$$
\mathbb{E}[N_{2n}] = \sum_{k=0}^{n} u_k\left(\frac{1}{2}\right)^{2k}
$$

(**c**)

$$
\mathbb{E}[N_n] \to \frac{2n+1}{\sqrt{n\pi}} - 1 \propto \sqrt{n}
$$

Ex. 4.9 Multiple applications of Markovian Property.

$$\begin{aligned}
\mathbb{P}\{X_k = i_k | X_j = i_j, \forall j \neq k\} &= \mathbb{P}\{X_k = i_k | X_j = i_j, \forall j \geq k-1\} \\
&= \frac{\mathbb{P}\{X_{k-1} = i_{k-1} | X_k = i_k, X_j = i_j, \forall j \geq k+1\}\mathbb{P}\{X_k = i_k | X_j = i_j, \forall j \geq k+1}{\mathbb{P}\{X_{k-1} = i_{k-1} | X_j = i_j, \forall j \geq k+1\}} \\
&= \frac{\mathbb{P}\{X_{k-1} = i_{k-1} | X_k = i_k, X_{k+1} = i_{k+1}\}\mathbb{P}\{X_k = i_k | X_{k+1} = i_{k+1}\}}{\mathbb{P}\{X_{k-1} = i_{k-1} | X_{k+1} = i_{k+1}\}} \\
&= \mathbb{P}\{X_k = i_k | X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\}
\end{aligned}$$

Ex. 4.11 (**a**)

$$\begin{aligned}
\sum_{n=1}^{\infty} P_{ij}^n = \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{I}(X_n = j) | X_0 = i\right] &= \sum_{k=1}^{\infty} \mathbb{E}\left[\sum_{n=k}^{\infty} \mathbb{I}(X_n = j) | X_k = j\right] f_{ij}^k \\
&= \sum_{k=1}^{\infty} \frac{f_{ij}^k}{1 - f_{jj}} = f_{ij}/(1 - f_{jj}) < \infty
\end{aligned}$$

(**b**)

$$\frac{1}{1 - f_{jj}} = \mathbb{E}\left[\sum_{n=0}^{\infty} \mathbb{I}(X_n = j) | X_0 = j\right] = 1 + \sum_{n=1}^{\infty} P_{jj}^n$$

Ex. 4.12

$$\vec{1}P = \vec{1}$$

Therefore, $\pi_i = \frac{1}{n}$.

Ex. 4.13 Let m and n be such that $P_{ij}^m > 0$, $P_{ji}^n > 0$. Since $d(i) = d(j)$, assume that $d(i) = k$.

$$\begin{aligned}
&\pi_i > 0 \implies \lim_{s\to\infty} P_{jj}^{m+n+sk} \geq \lim_{s\to\infty} P_{ji}^n P_{ii}^{sk} P_{ij}^m > 0 \implies \pi_j > 0 \\
&\pi_i = 0 \implies \lim_{s\to\infty} P_{ii}^{m+n+sk} \geq \lim_{s\to\infty} P_{ij}^m P_{jj}^{sk} P_{ji}^n \implies 0 \geq P_{ij}^m P_{ji}^n \pi_j \implies \pi_j = 0
\end{aligned}$$

Ex. 4.14 If i is a null recurrent state, then the corresponding class of states C will be null recurrent implying $P_{ij}^n \to 0, \forall j \in C$. This implies $\sum_{j\in C} P_{ij}^n \to 0$ contradicting $\sum_{j\in C} P_{ij}^n = 1$. All transient states will imply that some state is visited infinitely many times contradicting that a transient state can only be visited a finite number of times.

Ex. 4.15

$$\sum_{i=0}^{\infty} i\pi_i = \lambda\mathbb{E}[S] + \frac{\lambda^2\mathbb{E}[S^2]}{2(1-\lambda\mathbb{E}[S])}$$

Ex. 4.19 (**a**) enters state j from state i.
(**b**) enters a state in A^c from A.
(**c**) If a transition from A to A^c is denoted by $+1$ and from A^c to A is denoted by -1 with all other transitions denoted by 0, then the sum can only be $-1, 0$ or 1 depdending on the initial and final state of the chain.
(**d**)

$$\begin{aligned}
&\lim_{n\to\infty} |N_n(A, A^c)/n - N_n(A^c, A)/n| \le \lim_{n\to\infty} 1/n = 0 \\
&\implies \lim_{n\to\infty} N_n(A, A^c)/n = \lim_{n\to\infty} N_n(A^c, A)/n \\
&\implies \sum_{j\in A^c}\sum_{i\in A} \pi_i P_{ij} = \sum_{j\in A^c}\sum_{i\in A} \pi_j P_{ji}
\end{aligned}$$

Ex. 4.21

$$\pi_0 = (1-p_1)\pi_1,\ \pi_j = \pi_{j-1}p_{j-1} + \pi_{j+1}(1-p_{j+1}) \implies \pi_j = \pi_0 \prod_{i=0}^{j-1} \frac{p_i}{1-p_{i+1}}, j > 0$$

$$\sum_n \pi_n = 1 \implies \pi_0 = \frac{1}{1+\sum_{n=1}^{\infty}\prod_{i=0}^{n-1}\frac{p_i}{1-p_{i+1}}} \implies \sum_{n=1}^{\infty}\prod_{i=0}^{n-1}\frac{p_i}{1-p_{i+1}} < \infty$$

Ex. 4.22

$$\mathbb{E}[B] = \frac{1}{2p-1}\left\{\frac{n[1-(q/p)^i]}{1-(q/p)^n} - 1\right\}$$

Ex. 4.23

$$\begin{aligned}
\mathbb{P}\{W|i,N\} &= \frac{\mathbb{P}\{N|i,W\}\mathbb{P}\{W|i\}}{\mathbb{P}\{N|i\}} = \frac{\mathbb{P}\{N|i+1\}\mathbb{P}\{W|i\}}{\mathbb{P}\{N|i\}} \\
&= \begin{cases} \frac{(1-(q/p)^{i+1})/(1-(q/p)^N)\cdot p}{(1-(q/p)^i)/(1-(q/p)^N)} & p \ne 1/2 \\ \frac{(i+1)/N\cdot 1/2}{i/N} & \end{cases}
\end{aligned}$$

Ex. 4.24 (**a**)

$$M_0 = I \implies M_n = I + M_{n-1}Q = \ldots = I + Q + Q + \ldots + Q^n$$

(**b**)

$$M_n - I = Q + Q + \ldots + Q^n + Q^{n+1} - Q^{n+1} \implies M_n - I + Q^{n+1} = Q(I + Q + \ldots + Q^n)$$

(**c**)

$$\begin{aligned} M_n &= I - Q^{n+1} + Q(I-Q)^{-1}(I - Q^{n+1}) = (I + Q(I-Q)^{-1})(I - Q^{n+1}) \\ &= ((I-Q)(I-Q)^{-1} + Q(I-Q)^{-1})(I - Q^{n+1}) = (I-Q)^{-1}(I - Q^{n+1}) \end{aligned}$$

Ex. 4.25

$$\begin{aligned} Q &= \begin{bmatrix} 0 & 0.7 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 & 0 \\ 0 & 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 0 & 0.3 & 0 \end{bmatrix} \\ M &= (I - Q)^{-1} \\ M_n &= I + Q + Q^2 + \ldots + Q^n \end{aligned}$$

(**a**)

$$M_{3,5}$$

(**b**)

$$M_{3,1}$$

(**c**)

$$M_{7_{3,5}}$$

(**d**)

$$f_{3,1} = \frac{M_{3,1}}{M_{1,1}}$$

Ex. 4.26 (**a**)

$$\mu_{i,n} = P_{in} + \sum_{j \neq n} (1 + \mu_{j,n}) P_{ij} = 1 + \sum_{j \neq n} P_{ij} \mu_{j,n} = 1 + p\mu_{i+1,n} + (1-p)\mu_{i-1,n}, \quad \forall i \notin \{0, n\}$$

$$\mu_{0,n} = 1 + \mu_{1,n}$$

$$\mu_{n,n} = 1 + \mu_{n-1,n}$$

(**b**)

$$
\begin{aligned}
m_i &= P_{i,i+1} + (1 + m_{i-1} + m_i)P_{i,i-1} = 1 + (m_{i-1} + m_i)(1-p) \\
pm_i &= 1 + (1-p)m_{i-1} \\
m_0 &= 1 \\
m_1 &= \frac{1}{p} + \frac{1-p}{p} \\
m_2 &= \frac{1}{p} + \frac{1-p}{p}\left(\frac{1}{p} + \frac{1-p}{p}\right) = \frac{1}{p} + \frac{1-p}{p^2} + \left(\frac{1-p}{p}\right)^2 \\
m_3 &= \frac{1}{p} + \frac{1-p}{p^2} + \frac{(1-p)^2}{p^3} + \left(\frac{1-p}{p}\right)^3 \\
m_i &= \frac{1}{p}\sum_{j=0}^{i-1}\left(\frac{1-p}{p}\right)^j + \left(\frac{1-p}{p}\right)^i = \frac{1}{p}\left(\frac{1-(q/p)^i}{1-q/p}\right) + (q/p)^i
\end{aligned}
$$

(**c**)

$$\mu_{i,n} = m_i + \mu_{i+1,n} \implies \mu_{i,n} = \sum_{j=i}^{n-1} m_j$$

(**d**)

$$\mathbb{E}[X_j] = 1 + \frac{1}{2p-1}\left[\frac{n(1-q/p)}{1-(q/p)^n} - 1\right]$$

(**e**)

$$\mathbb{E}[N] = (1-(q/p)^n)/(1-q/p)$$

(**f**)

$$\mu_{0,n} = \mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}[X_i]\mathbb{E}[N] = \frac{1}{2p-1}\left[n - 2q\left(\frac{1-(q/p)^n}{1-q/p}\right)\right]$$

(**g**) Use (**a**)

Ex. 4.27 Assuming $p, q > 0$ so that $f_{i,j} = 1$.

$$
\begin{aligned}
\mathbb{P}\{\text{last node is } i\} &= \mathbb{P}\{\text{last node is } i | i-1 \text{ is visited before } i+1\}\mathbb{P}\{i-1 \text{ is visited before } i+1\} \\
&\quad + \mathbb{P}\{\text{last node is } i | i+1 \text{ is visited before } i-1\}\mathbb{P}\{i+1 \text{ is visited before } i-1\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\{i-1 \text{ is visited before } i+1\} &= \frac{1-\left(\frac{q}{p}\right)^{m-i}}{1-\left(\frac{q}{p}\right)^{m-1}} \\
\mathbb{P}\{i+1 \text{ is visited before } i-1\} &= 1-\mathbb{P}\{i-1 \text{ is visited before } i+1\} \\
\mathbb{P}\{\text{last node is } i|i-1 \text{ is visited before } i+1\} &= \mathbb{P}\{i+1 \text{ is visited before } i|\text{Start from } i-1\}f_{i+1,i} \\
&= \frac{1-\left(\frac{p}{q}\right)}{1-\left(\frac{p}{q}\right)^{m}} \\
\mathbb{P}\{\text{last node is } i|i+1 \text{ is visited before } i-1\} &= \frac{1-\left(\frac{q}{p}\right)}{1-\left(\frac{q}{p}\right)^{m}}
\end{aligned}
$$

Ex. 4.28

$$
\mathbb{E}[T_{00}] = \sum_{i=1}^{m} \mathbb{E}[T_{00}|\text{last node is } i]\mathbb{P}\{\text{last node is } i\}
$$

$$
\mathbb{E}[T_{00}|\text{last node is } i] = \frac{1}{2p-1}\left\{\frac{(m+1)[1-(q/p)^i]}{1-(q/p)^{m+1}} - 1\right\}
$$

Ex. 4.30 Let $Z_i = X_i - Y_i$, so $\mathbb{P}\{Z_i = 1\} = P_1(1-P_2)$,
$\mathbb{P}\{Z_i = -1\} = (1-P_1)P_2$ and $\mathbb{P}\{Z_i = 0\} = P_1P_2 + (1-P_1)(1-P_2)$.

$$
\begin{aligned}
\mathbb{P}\{\text{error}\} &= \mathbb{P}\{\text{reach } -M \text{ before } M|\text{start from } 0\} \\
f_{i,-M} &= f_{i-1,-M}(1-P_1)P_2 + f_{i+1,M}(1-P_2)P_1 + f_{i,M}(P_1P_2 + (1-P_1)(1-P_2)) \\
f_{i,-M} - f_{i-1,-M} &= \frac{(1-P_2)P_1}{(1-P_1)P_2}(f_{i+1,-M} - f_{i,-M}) = \lambda(f_{i+1,-M} - f_{i,-M}) \\
\mathbb{P}\{\text{error}\} &= \frac{1-\lambda^M}{1-\lambda^{2M}} = \frac{1}{1+\lambda^M} \\
\mathbb{E}[N] &= \mathbb{E}\left[\sum_{i=1}^{N} Z_i\right]/\mathbb{E}[Z_1] = \frac{M(\lambda^M-1)}{\lambda^M+1}/(P_1-P_2)
\end{aligned}
$$

Ex. 4.31

$$
P = \begin{bmatrix} 1 & 0 & 0 \\ 0.54 & 0.28 & 0.18 \\ 0.54 & 0.18 & 0.28 \end{bmatrix}
$$

Since $det(P) \neq 0$, P is non-singular and therefore we can find invertible matrix S and diagonal matrix Λ such that $P = S\Lambda S^{-1}$. Therefore, $P^n = S\Lambda^n S^{-1}$.
(**a**)

$$P^n_{2,2}$$

(**b**)

$$\begin{aligned}
\mu_{1,0} &= P_{1,0} + (1+\mu_{1,0})P_{1,1} + (1+\mu_{2,0})P_{1,2} = 1 + 0.28\mu_{1,0} + 0.18\mu_{2,0} \\
\mu_{2,0} &= P_{2,0} + (1+\mu_{1,0})P_{2,1} + (1+\mu_{2,0})P_{2,2} = 1 + 0.18\mu_{1,0} + 0.28\mu_{2,0} \\
\mu_{1,0} &= \frac{1}{0.54}
\end{aligned}$$

Ex. 4.32 $f_{0,0} = 1$ and $f_{N,0} = f_{N-1,0}$

$$\begin{aligned}
f_{n,0} = \mathbb{P}\{N_0(\infty) > 0 | X_0 = n\} &= \sum_{j=0}^{n} \mathbb{P}\{N_0(\infty) > 0 | X_1 = j\}\mathbb{P}\{X_1 = j | X_0 = n\} \\
&= f_{n-1,0}p + f_{n,0}q + f_{n+1,0}p \\
f_{n+1,0} - f_{n,0} &= f_{n,0} - f_{n-1,0} \\
f_{N,0} &= f_{N-1,0} = f_{N-2,0} = \ldots = f_{1,0} = f_{0,0} = 1
\end{aligned}$$

Let $a_n = \mathbb{E}[T|X_0 = n]$. $a_0 = 0$ and $a_N = 1 + a_{N-1}$.

$$\begin{aligned}
\mathbb{E}[T|X_0 = n] &= (1 + \mathbb{E}[T|X_1 = n])q + (1 + \mathbb{E}[T|X_1 = n-1])p + (1 + \mathbb{E}[T|X_1 = n+1])p \\
a_n &= 1 + a_n q + p a_{n-1} + p a_{n+1} \\
a_{n+1} - a_n &= -\frac{1}{p} + a_n - a_{n-1} \\
a_1 &= 1 + \frac{N-1}{p} \\
a_n &= n + \frac{n}{p}(2N - n - 1)
\end{aligned}$$

Ex. 4.33 (**a**)

$$\pi_0 = 1 \iff \mu \leq 1 \text{ and } \mu > 1 \implies \mathbb{E}[X_n] \to \infty$$

(**b**)

$$\begin{aligned}
a_n = \mathrm{Var}(X_n|X_0 = 1) &= \mathbb{E}[\mathrm{Var}(X_n|X_1 = m)|X_0 = 1] + \mathrm{Var}(\mathbb{E}[X_n|X_1 = m]|X_0 = 1) \\
&= \mathbb{E}[m a_{n-1}|X_0 = 1] + \mathrm{Var}(m\mu^{n-1}|X_0 = 1) \\
&= a_{n-1}\mu + \mu^{2n-2}\sigma^2 \\
&= \begin{cases} n\sigma^2 & \text{if } \mu = 1 \\ \sigma^2\mu^{n-1}\frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1 \end{cases}
\end{aligned}$$

Ex. 4.34 (**a**)

$$\begin{aligned}
&\pi_0 = \pi_0^0(1-p)^2 + \pi_0^1 2p(1-p) + \pi_0^2 p^2 \\
&p^2\pi_0^2 + \pi_0(2p(1-p)-1) + (1-p)^2 = 0 \\
&\pi_* = \frac{(1-p)^2}{p^2}
\end{aligned}$$

(**b**) Use iterative conditioning.

$$\pi_0 = \sum_{n=0}^{\infty} \pi_*^n \exp(-\lambda)\frac{\lambda^n}{n!} = \exp(\lambda(\pi_* - 1)) = \exp(\lambda(1-2p)/p^2)$$

Ex. 4.35 (**a**)

$$\pi_0 = \sum_{n=0}^{\infty} \pi_0^n \exp(-\lambda)\frac{\lambda^n}{n!} = \exp(\lambda(\pi_0 - 1)) \implies \lambda\pi_0 \exp(-\lambda\pi_0) = \lambda\exp(-\lambda)$$

(**b**)

$$\mathbb{P}\{X_1 = n | X_0 = 1, \pi_0 = 1\} = \exp(-\lambda)\frac{\lambda^n}{n!} = \exp(-a)\frac{a^n}{n!}$$

Ex. 4.36

$$\begin{aligned}
\mathbb{E}[T_N] \approx \log N &= \log\left(\frac{n^{n+1/2}e^{-n}\sqrt{2\pi}}{m^{m+1/2}e^{-m}\sqrt{2\pi}(n-m)^{n-m+1/2}e^{-(n-m)}\sqrt{2\pi}}\right) \\
&\approx (n+1/2)\log n - (m+1/2)\log m - (n-m+1/2)\log(n-m) \\
&\approx n\log\frac{n}{n-m} + m\log\frac{n-m}{m} \\
&= m\left[c\log\left(\frac{c}{c-1}\right) + \log(c-1)\right]
\end{aligned}$$

5. Continuous-Time Markov Chains

Ex. 5.1

Ex. 5.2

Ex. 5.3

Ex. 5.4

Ex. 5.5

Ex. 5.6

Ex. 5.7

Ex. 5.8

Ex. 5.9

Ex. 5.10

Ex. 5.11

Ex. 5.12

Ex. 5.13

Ex. 5.14

Ex. 5.15

Ex. 5.16

Ex. 5.17

Ex. 5.18

Ex. 5.19

Ex. 5.20

Ex. 5.21

Ex. 5.22

Ex. 5.23

Ex. 5.24

Ex. 5.25

Ex. 5.26

Ex. 5.27

Ex. 5.28

Ex. 5.29

Ex. 5.30

Ex. 5.31

Ex. 5.32

Ex. 5.33

Ex. 5.34

Ex. 5.35

Ex. 5.36

Ex. 5.37