# Eigenvalues and Eigenvectors

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# Outline

# Introduction



- $Ax = \lambda x$ is a nonlinear equation. Given $\lambda$, it becomes linear in $x$.
- Solving $Ax = \lambda x \iff (A - \lambda I)x = 0$ is to find $x$ in $N(A - \lambda I)$ where $\lambda$ is chosen so that $A - \lambda I$ has a nullspace. $x = 0$ is always a solution, but is not interesting.
- $N(A - \lambda I)$ must contain non-zero vector. It must be singular i.e. $\lambda$ is an eval. of $A \iff det(A - \lambda I) = 0$ which is the characterisitc equation. Each $\lambda$ is associated with an evec. $x$.
- Examples - Diagonal and triangular matrices have evals. on their diagonal and evals. of Projection matrices are 1 and 0 (Why?).
- Geometrically, we find $\lambda$ and $x$ s.t. $Ax \parallel x$.

# Introduction contd.

## Result 1

A matrix $A_{n\times n}$ has $n$ evals. $\{\lambda_i\}_{i=1}^{n}$ where,

$$\sum_{i=1}^{n} \lambda_i = tr(A), \quad \prod_{i=1}^{n} \lambda_i = det(A)$$

*Proof - Hints*:

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & . & a_{1n} \\ a_{21} & a_{22} - \lambda & . & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & . & a_{nn} - \lambda \end{bmatrix}$$

$\Sigma$ evals. $= (-1)^{n-1} \times$ coefficient of $\lambda^{n-1}$ in $det(A - \lambda I) = 0$ which equals $tr(A)$. $\prod$ evals. $=$ constant term in $det(A - \lambda I)$. Or put $\lambda = 0$ in $det(A - \lambda I) = \prod_{i=1}^{n}(\lambda_1 - \lambda)$ (Why ?).

# Diagonalization of a Matrix

## Result 2

Suppose $A_{n\times n}$ has $n$ linearly independent vectors which are placed in the columns of a matrix $S$, then $S^{-1}AS$ is a diagonal matrix $\Lambda$ whose diagonal entries are evals. of $A$.

*Proof - Hints*:

Note that $S$ is invertible.

$AS = [Ax_1, \ldots, Ax_n] = [\lambda_1 x_1, \ldots, \lambda_n x_n] = S\ diag(\lambda_1, \ldots, \lambda_n)$

Therefore, $AS = S\Lambda$. Also, note that $A = S\Lambda S^{-1}$.

## Result 3

If $A_{n\times n}$ has $n$ distinct evals. then $n$ evecs. are linearly independent.

*Proof - Hints*: For $n = 2$,

$$0 = c_1x_1 + c_2x_2 \Rightarrow A0 = 0 = c_1\lambda_1x_1 + c_2\lambda_2x_2$$

# Diagonalization of a Matrix contd.



## Result 4

The evals. of $A^k$ are $\{\lambda_i^k\}_{i=1}^n$ and every evec. of $A$ is an evec. of $A^k$. If $S$ diagonalizes $A$, then, it also diagonalizes $A^k$.
*Proof - Hints*: Let $\lambda_i$ be an eval. of $A$ and $x_i$ be the associated evec. Then, $A^k x_i = A^{k-1}\lambda_i x_i \ldots = \lambda_i^k x_i$. If $S$ diagonalizes $A$, then, $S^{-1}A^kS = S^{-1}ASS^{-1}A^{k-1}S \ldots = \Lambda^k$.

- If $A$ is invertible, this rule also applies to its inverse.
- Analogy of this rule to product of two different matrices does not hold (construct an example) unless their evecs. are same.

# Diagonalization of a Matrix contd.



## Result 5

Diagonalizable matrices share the same evec. matrix $S$ if and only if $AB = BA$ i.e. they commute.

*Proof - Hints*:

$(\implies)$ $AB = S^{-1}\Lambda_1 S S^{-1}\Lambda_2 S = S^{-1}\Lambda_1\Lambda_2 S = BA$

$(\impliedby)$ Let $x$ be evec of $A$, then $ABx = BAx = \lambda Bx$, therefore, $Bx$ is an evec. of $A$. If we assume that all evals. of $A$ are distinct, then all eigenspaces are one-dimensional, and since $x$ and $Bx$ are evec. of $A$ with same eval. $\lambda$, $Bx$ must be a multiple of $x$. So, $x$ is evec. of $B$ (try to prove when evals. are repeated).

# Complex Matrices



- $\bar{x}^T y$ is different from $\bar{y}^T x$.
- $A$ Hermitian is $A^H = \bar{A}^T$ and $A$ is said to be Hermitian if $A = A^H$ and it contains real diagonal entries and the off-diagonal entries are mirror images across main diagonal.
- Inner product of $x$ and $y$ is $x^H y$. Orthogonal vectors have $x^H y = 0$.
- The squared length of $x$ is $x^H x = \sum_{i=1}^{n} |x_i|^2$.
- $(AB)^H = B^H A^H$.

# Complex Matrices contd.

## Result 6

If $A = A^H$, then for all complex real vectors $x$, the number $x^H Ax$ is real and therefore, evals. of $A$ are real.
*Proof - Hints*: $(x^H Ax)^H = x^H A^H x = x^H Ax$, therefore, $x^H Ax$ is real. Let $\lambda$ be an eval. of $A$ which is possibly complex. Then, $Ax = \lambda x \Rightarrow x^H Ax = \lambda x^H x \Rightarrow \lambda = \frac{x^H Ax}{x^H x}$. The denominator is real by definition and the numerator is real because $A$ is Hermitian, therefore, $\lambda$ is real.

## Result 7

Two evecs. of a real symmetric matrix or a Hermitian matrix, if they come from different evals., are orthogonal to one another.
*Proof - Hints*: Let $x$ and $y$ be evecs. associated with different evals. Then, $x^H \lambda_2 y = x^H Ay = x^H A^H y = (Ax)^H y = \lambda_1 x^H y$, therefore, $x^H y(\lambda_1 - \lambda_2) = 0$.

# Complex Matrices contd.



- When $A$ is Hermitian, the diagonalizing matrix can be chosen so that the columns are orthonormal.
- If $A$ is real-symmetric then evecs. are also real.
- Spectral Theorem: A real symmetric matrix can be factored into $Q\Lambda Q^T$ where columns of $Q$ are orthonormal evecs and evals in $\Lambda$ $(Q^{-1} = Q^T)$. Also, $A = Q\Lambda Q^T$ which can be written as the combinations one dimensional projections onto line through evec $x_i$, i.e. $\sum_{i=1}^{n} \lambda_i x_i x_i^T$.
- Surely, if the eigenvalues of a symmetric matrix are distinct then $A = Q\Lambda Q^T$, but, even if the symmetric matrix has repeated evals., it still has a complete set of orthonormal evecs. [We will see soon.]
- Complex matrix with orthonormal columns is called Unitary matrix. $U^H U = I, UU^H = I$ and $U^H = U^{-1}$.

# Complex Matrices contd.



### Result 8

Unitary matrix preserve distances, have evals. with absolute value of 1 and evecs. corresponding to different evals. are orthogonal.

*Proof - Hints*:

$$(Ux)^H(Uy) = x^H U^H Uy = x^H y, (Ux)^H(Ux) = x^H x$$
$$Ux = \lambda x \Rightarrow (Ux)^H(Ux) = (\lambda x)^H(\lambda x) \Rightarrow |\lambda| = 1$$
$$x^H y = (Ux)^H Uy = \lambda_1^H \lambda_2 x^H y \Rightarrow x^H y(\lambda_1^H \lambda_2 - 1) = 0$$

Since $\lambda_1 \neq \lambda_2$ and $|\lambda_1| = |\lambda_2| = 1$ $\lambda_1^H \lambda_2 \neq 1$, therefore, $x^H y = 0$.

# Complex Matrices contd.

## Result 9

Skew-Hermitian matrix have $K^H = -K$. If $A$ is Hermitian then $K = iA$ is skew-Hermitian and eigenvalues of a skew-Hermitian matrix are imaginary.
*Proof - Hints*: $K^H = A^H(-i) = -iA^H = -iA = -K$. Note that $x^H Kx$ is imaginary $\because (x^H Kx)^H = x^H K^H x = -x^H Kx$. Therefore, $\lambda = \frac{x^H Kx}{x^H x}$ has imaginary numerator and real denominator.

- Diagonal entries of $K$ are imaginary (allowing zero).
- Evecs. of skew-Hermitian corresponding to different evals. are still orthogonal (easy proof) and $K$ can be decomposed into $K = U\Lambda U^H$ with unitary $U$ even if evals. are repeated (We will see this soon).

## Complex Matrices contd.



- A Hermitian or symmetric matrix can be compared to a real number (evals. are real).
- A Unitary matrix can be compared to a number on unit circle i.e. a complex number of absolute value 1 (evals. have absolute value of 1).
- A skew-Hermitian matric can be compared with pure imaginary numbers (evals. are imaginary).
- Normal matrices can be compared with all complex numbers (evals. are of form $a + ib$).
- A nonnormal matrix without orthogonal evecs. belong to none of these classes and is outside the whole analogy.

# Similarity Transformation

- The matrices $A$ and $M^{-1}AM$ are similar. Going from one to the other is similarity transformation.
- These combination $M^{-1}AM$ arise upon change of variables in differential or difference equation.

$$\frac{du}{dt} = Au \xrightarrow{u=Mv} M\frac{dv}{dt} = AMv \text{ or } \frac{dv}{dt} = M^{-1}AMv$$

**Result 10**

Suppose $B = M^{-1}AM$ then both $A$ and $B$ have same evals. Every evec. $x$ of $A$ corresponds to evec. $M^{-1}x$ of $B$.
*Proof - Hints*: $Ax = \lambda x \Rightarrow MBM^{-1}x = \lambda x \Rightarrow BM^{-1}\lambda M^{-1}x$.
Alternatively, $det(A - \lambda I) = det(MBM^{-1} - \lambda MM^{-1}) = det(M)det(B - \lambda I)det(M^{-1}) = det(B - \lambda I)$

- Every $M^{-1}AM$ has same number of independent evecs. as $A$ (each evec. is multiplied with $M^{-1}$).

# Similarity Transformation contd.

- Every linear transformation is represented by a matrix. The matrix depends on the choice of basis. If we change the basis by $M$ we change the matrix $A$ to a similar matrix $B$.
- **Similar Matrices represent the same transformation $T$ with respect to different bases.**

## Change of Basis = Similarity Transformation

The matrices $A$ and $B$ which represent the same linear transformation with respect to different bases are similar:

$$\begin{aligned} [T]_{V \text{ to } v} &= [I]_{v \text{ to } V}[T]_{v \text{ to } v}[I]_{V \text{ to } v} \\ B \quad &= M^{-1} \quad A \quad M \end{aligned}$$

# Similarity Transformation contd.



- The aim is to find an $M$ which can most simplify $A$ i.e. $M^{-1}AM$ becomes diagonal which is equivalent to finding evecs. of $A$ and fill in the columns of $M$ with them. The algebraist says the same thing in the language of Linear Transformation: Choose a basis consisting of evecs.
- $M^{-1}AM$ do not arise in solving $Ax = b$. There we multiply $A$ on LHS by a matrix that subtracts a multiple of one row from another. Such a transformation preserved null space and row space but normally changes eigenvalues.
- Eigenvalues are calculated by a sequence of similarities. The matrix goes gradually towards a triangular form, and the eigenvalues gradually appear on the diagonal.

# Similarity Transformation contd.

## Result 11 - Schur's Lemma

There is a unitary matrix $M = U$ s.t $U^{-1}AU = T$ is triangular. The eigenvalues of $A$ appear along the diagonal of this similar matrix $T$.

*Proof - Hints*: $A$ will have atleast one eval. and therefore has atleast one **unit** evec. Put this evec. in the first column of a matrix $U_1$ and fill the rest of the matrix such that $U_1$ becomes unitary.

$$U_1 A = U_1 \begin{bmatrix} \lambda_1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ 0 & * & \dots & * \end{bmatrix} \implies U_1^{-1} A U_1 = \begin{bmatrix} \lambda_1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ 0 & * & \dots & * \end{bmatrix}$$

# Similarity Transformation contd.

## Result 11 contd.

Now, work recursively with RHS matrix with first column and first row removed. Let $M_2$ be the unitary matrix corresponding to the submatrix then,

$$U_2 = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & & & \\ 0 & & M_2 & \\ 0 & & & \end{bmatrix} \implies U_2^{-1} U_1^{-1} A U_1 U_2 = \begin{bmatrix} \lambda_1 & * & . & * \\ 0 & \lambda_2 & . & * \\ \vdots & \vdots & . & \vdots \\ 0 & 0 & . & * \end{bmatrix}$$

The product $U = U_1 U_2 U_3 \dots$ is still a unitary matrix.

This result applies to all matrices, with no assumption that $A$ is diagonalizable. This can also be used to prove that the powers $A^k$ approach zero when all $|\lambda_i| < 1$, and the exponentials $e^{At}$ approach 0 when all $Re\lambda_i < 0$ - even without full set of evecs.

# Similarity Transformation contd.

## Result 12 - Spectral Theorem

Every real symmetric matrix can be diagonalized by an orthogonal matrix and every hermitian matrix can be diagonalized with unitary matrix.

$$(real) \quad Q^{-1}AQ = \Lambda \text{ or } A = Q\Lambda Q^T$$

$$(complex) \quad U^{-1}AU = \Lambda \text{ or } A = U\Lambda U^H$$

*Proof - Hints*: From Schur's lemma, $U^{-1}AU = T$. Since $A = A^H$, $T = T^H$. Therefore, $T$ is diagonal and equals $\Lambda$.

Every Hermitian matrix with $k$ different evals. has a spectral decomposition into $A = \sum_{i=1} \lambda_i P_i$, where $P_i$ is the projection onto the eigenspace for $\lambda_i$. Since there is a full set of evecs., the projection add up to the identity and since the eigenspaces are orthogonal, $P_j P_i = 0$.

# Similarity Transformation contd.

- For which matrices $T = \Lambda$? Hermitian, skew-Hermitian and unitary matrices are in this class. They correspond to numbers on real axis, imaginary axis and the unit circle.
- The whole class contain matrices (corresponding to all complex numbers) which are called normal.

**Result 13**

Matrix $N$ is normal if it commutes with $N^H$ : $NN^H = N^H N$. For such matrices and no others $T = U^{-1}NU$ is diagonal $\Lambda$. Normal matrices are exactly those that have a complete set of orthonormal evecs.
*Proof - Hints*: If $N$ is normal then $T$ is normal. A triangular normal matrix is a diagonal matrix.

# Bibliography



Gilbert Strang. *Linear algebra and its applications*. Belmont, CA: Thomson, Brooks/Cole, 2006.