Geometry and Complex Arithmetic - Notes

Dhruv Kohli

August 7, 2018

- Geometric rules for complex numbers: addition $\equiv$ vector addition, multiplication $=$ $R\angle\theta r\angle\phi = Rr\angle(\theta+\phi)$.

- Cardano argument: $ax^2+bx+c=0$ has roots $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If $b^2 < 4ac$ then it leads to impossible numbers so discard such solutions. In the particular case of quadratic equations, this is perfectly fine to do. $ax^2 = -bx-c$, LHS is a parabola and RHS is a line. When $b^2 < 4ac$ then line does not intersect the parabola so there will be no solution (with any physical meaning).

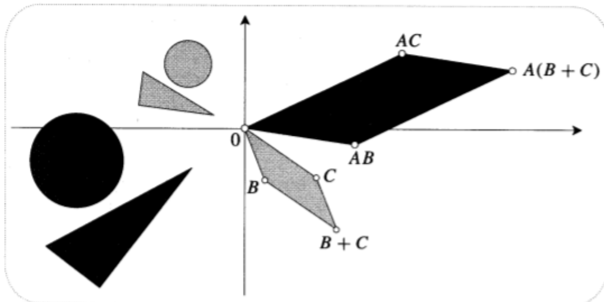
- Bombelli argument: Consider $x^3 = 3px + 2q$. LHS is a cubic and RHS is a line. There will always be atleast one point of intersection. General solution is of form $x = (q+\sqrt{q^2-p^3})^{1/3} + (q-\sqrt{q^2-p^3})^{1/3}$. $q^2 < p^3$ leads to impossible numbers (according to Cardano), but we can't discard solution (because there will be atleast one intersection). Bombelli wild thought: If $q^2 < p^3$ then take $\sqrt{q^2-p^3} = i\sqrt{p^3-q^2}$ (ofcourse he worked out a specific example).

- Representation of complex numbers: Cartesian: $z = x+iy$, unique representation. Polar: $z = r\angle\theta$, non-unique representation, $r = |z|, \theta = \mathbf{arg} z$.

- Symbolic rules: addition: $(a+ib)+(\tilde{a}+i\tilde{b}) = (a+\tilde{a}) + i(b+\tilde{b})$. multiplication: $(a+ib)(\tilde{a}+i\tilde{b}) = (a\tilde{a}-b\tilde{b}) + i(a\tilde{b}+\tilde{a}b)$.

- Geometric rules of addition and multiplication are same as symbolic rules. Equivalence of addition is easy to show. Geometric rule of multiplication means rotation of plane by θ and expansion by R. Symbolic rule of multiplication means $i^2=-1$ and brackets can be multiplied out $A(B+C) = AB+AC$. $G \implies S$: $i\cdot i =$ Rotate i by $\pi/2$ which makes it -1 and parallelograms are preserved by rotation and expansion (see figure). $S \implies G$: If $z = x+iy$ then according to S, $iz = -y+ix$ which is nothing but rotation of z by $\pi/2$. In general, $(a+ib)z = az + b(iz) = \exp_a^n(z) + \exp_a^n(\mathrm{rot}_\theta^n(z))$ [Proved later].

- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$. Moving particle argument: $z(t) = e^{it}$, $z(0) = 1+0i$. $v(t) = \frac{dz(t)}{dt} = ie^{it} = iz(t)$. After $t=\theta$, since, $|z(t)| = 1$ (particle traversing on circle of radius 1) and $|v(t)| = 1$ (speed of particle is 1). Therefore, distance travelled is θ. So, $z(\theta) = \cos\theta + i\sin\theta$; Power series argument: Use power series of $\cos\theta$, $\sin\theta$ and $e^{i\theta}$.



- $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ and $\cos\theta = (e^{i\theta} - e^{-i\theta})/2$.
- Application of complex numbers is trigonometry:

$$\begin{aligned}
\cos(\theta+\phi) + i\sin(\theta+\phi) &= e^{i(\theta+\phi)} \\
&= e^{i\theta}e^{i\phi} \\
&= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\
&= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)
\end{aligned}$$

$$\begin{aligned}
\cos 4\theta + i\sin 4\theta &= e^{i4\theta} \\
&= (e^{i\theta})^4 \\
&= (\cos\theta + i\sin\theta)^4
\end{aligned}$$

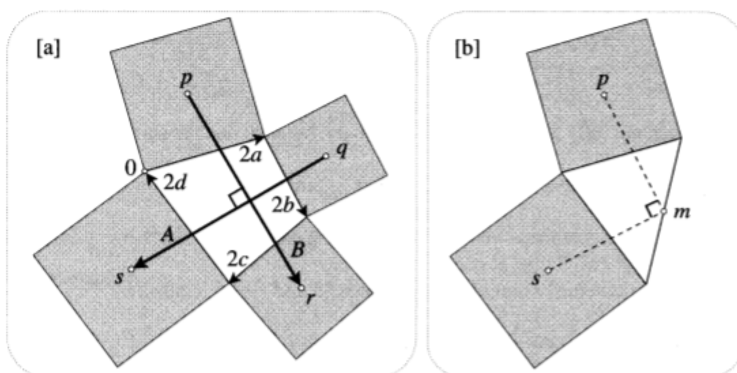
$$\cos^4\theta = ((e^{i\theta} + e^{-i\theta})/2)^4 = \dots$$

$$\begin{aligned}
T &= \tan\theta \\
z &= 1 + iT \\
\tan\theta &= \frac{\mathbf{Im} z}{\mathbf{Re} z} \\
\tan 3\theta &= \frac{\mathbf{Im}(z^3)}{\mathbf{Re}(z^3)}
\end{aligned}$$

- Application of complex numbers in geometry: Consider a quadrilateral, draw squares on the edges of the quadrilateral outwards, the lines joining the centres of the opposite squares are equal in length and perpendicular to each other. Using complex numbers to show this: Let $2a, 2b, 2c, 2d$ be the complex number representing consecutive edges of the quadrilateral. Let the one of the vertex be origin (from where edge $2a$ originates and edge $2d$ ends). Then $2a + 2b + 2c + 2d = 0$. Let centres of the squares be p, q, r, s.

Then,

$$\begin{aligned}
p &= a + ia \\
q &= 2a + b + ib \\
r &= 2a + 2b + c + ic \\
s &= 2a + 2b + 2c + d + id \\
A &= r - p \\
&= a - ia + 2b + c + ic \\
B &= s - q \\
&= b - ib + 2c + d + id \\
A - iB &= a - ia + 2b + c + ic - (ib + b + 2ic + id - d) \\
&= a + b + c + d - i(a + b + c + d) \\
&= 0
\end{aligned}$$



- A naive approach to solve above problem is as follows. Translation transformation is given by,

$$T_v(z) = z + v$$

Rotational transformation about a point a is given by,

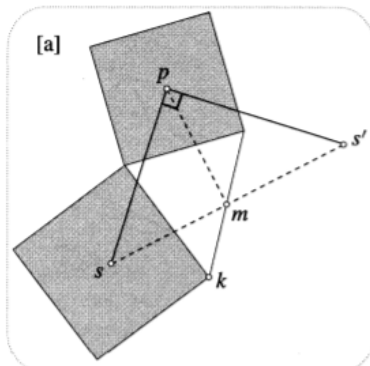
$$\begin{aligned}
R_a^\theta(z) &= T_a \circ R_0^\theta \circ T_{-a}(z) \\
&= e^{i\theta}(z - a) + a
\end{aligned}$$

Note that,

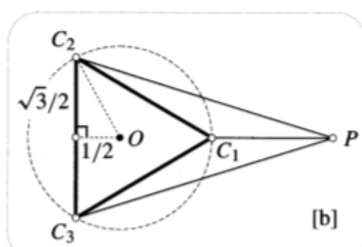
$$\begin{aligned}
R_a^{\theta_1} \circ R_b^{\theta_2}(z) &= R_a^{\theta_1}(e^{i\theta_2}(z - b) + b) \\
&= e^{i(\theta_1+\theta_2)}z + e^{i\theta_1}b(1 - e^{i\theta_2}) + a(1 - e^{i\theta_1}) \\
&= z + c \qquad \text{when } \theta_1 + \theta_2 = 2\pi
\end{aligned}$$

In general, *when* $M = R_{a_1}^{\theta_1} \circ R_{a_2}^{\theta_2} \ldots \circ R_{a_n}^{\theta_n}$ *and* $\sum_{i=1}^{n} \theta_i = 2\pi$, *then* $M = T_v$.

Coming back to the problem, we will show that $pm = sm$ and $pm \perp sm$. Consider $M = R_m^{\pi} \circ R_p^{\pi/2} \circ R_s^{\pi/2}$. Using the above result, $M = T_v$. To find v we just need to find image of one point. Note that $M(k) = k$ so, $v = 0$. So, $R_p^{\pi/2} \circ R_s^{\pi/2} = R_m^{-\pi}$. Consider $s' = R_m^{-\pi}(s)$. Then $s' = R_p^{\pi/2} \circ R_s^{\pi/2}(s) = R_p^{\pi/2}(s)$. Therefore, $ps = ps'$. $\Delta sps'$ is then an isosceles triangle with $\angle sps' = \pi/2$. This proves the result.



- Cotes results: An approach to write $x^n - 1$ as product of linear and quadratic terms with real coeddicients. Note that, $x^n - 1 = \prod_{i=1}^{n}(x - a_i)$ where a_i are roots of $x^n = 1$. Since complex roots of such equation must occur in conjugates, if $a_i \in \mathbb{C}$, then $\exists j \neq i$ s.t. $a_j = \bar{a_i}$ so that $(x - a_i)(x - a_j)$ is a quadratic with real coefficients. Cotes followed this problem in a remarkable way. Consider a regular n-gon $C_1, C_2, \ldots, C_n$ s.t. these points lie on a circle of unit radius. Let P be a point on the line passing from 0 through C_1 at a distance $x > 1$ from 0. Then, $U_n(x) = x^n - 1 = \prod_{i=1}^{n} PC_i$. Note that, in the figure, $C_2 = \bar{C_3}$ and $PC_2 = PC_3$ so that PC_2PC_3 is a quadratic in x.



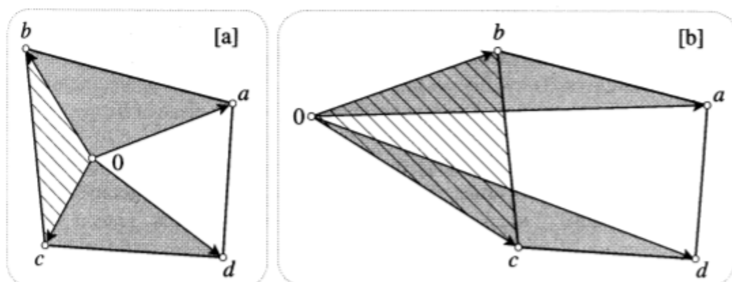
- Vectorial operations: $a.b = |a||b|\cos\theta$ and $a \times b = |a||b|\sin\theta$ (defined differently in general). Cross product represents the signed area of a parallelogram. Now, consider, $a = r_1e^{i\theta_1}$ and $b = r_2e^{i\theta_2}$, where $\theta_1 - \theta_2 = \theta$ then,

$$
\begin{aligned}
a\bar{b} &= r_1 r_2 e^{i(\theta_1-\theta_2)} \\
&= r_1 r_2 e^{i\theta} \\
&= r_1 r_2 (\cos\theta + i\sin\theta) \\
&= a.b + ia \times b
\end{aligned}
$$

- Area of polygon whose vertices are represented by a, b, c, d is given by,

$$
\begin{aligned}
&\frac{1}{2}(a \times b + b \times c + c \times d + d \times a) \\
&= \frac{1}{2}\mathbf{Im}(a\bar{b} + b\bar{c} + c\bar{d} + d\bar{a})
\end{aligned}
$$

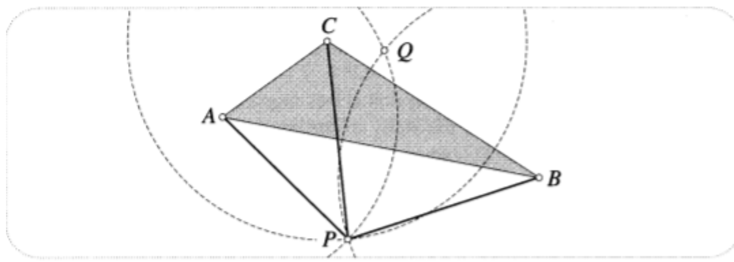
  Since the each of the term represents a signed area, the case when 0 is outside the polygon, is implicitly handled.



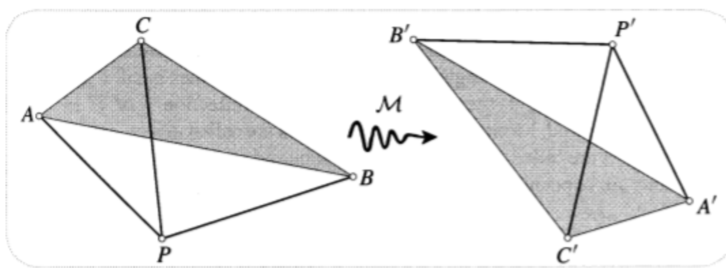
- M=motion = mapping/transformation of a plane to itself that preserves distance i.e. $d(A, B) = d(f(A), f(B))$.
- $F \cong F'$ if there exists M s.t. $F' = M(F)$.
- Geometric properties of a figure are those which are unaltered by set of all possible motions.
- Geometric equalty can in general be denoted by an equivalence relation between figures. Keeping this general definition of geometric equality in mind, the motion can itself be defined in general as a family G of transformations which forms a group.
- Note that distance preserving transformations do form a group (identity preserved distance, composition of distance preserving transformation is another distance preserving transformation, inverse of a distance preserving transformation preserves distance $c = f(a), d = f(b), d(f^{-1}(c), f^{-1}(d)) = d(a, b) = d(f(a), f(b)) = d(c, d)$) and thus constitute a special family of transformations.
- Klein's idea: Take a group of transformations G and define the corresponsing geometry as the study of invariants of G.

- *A motion is uniquely determined by its effect on any trianle (i.e. on any three non-collinear points).*

- Note that a point P is uniquely determined by its distances from vertices of a triangle. Given distances from two vertices, there are only two choices of location for P. Distance from third vertex finalizes which among the two choices should be the exact location of P.



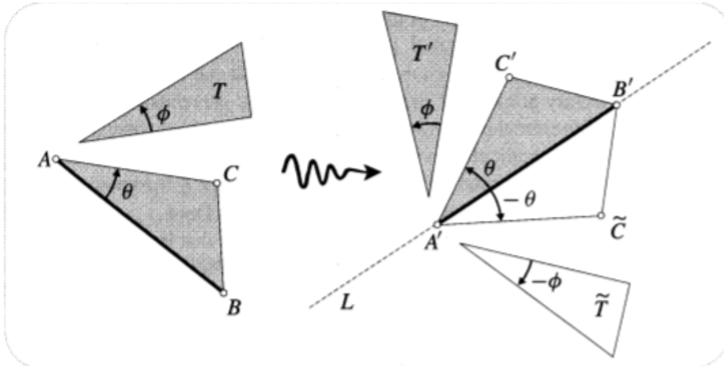
- Now, given that we know the image of a triangle. Let P be a general point. Since motion is distance preserving, the distance of image of P will be at same distances from images of A, B, C as the distance of P from A, B, C. Thus we know we know where each point on the plane will map.

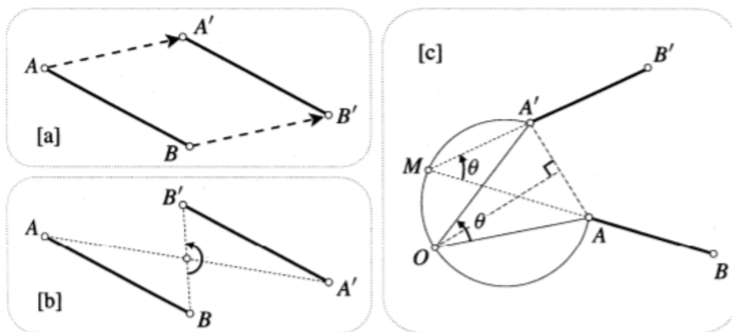


- Classifying motions in two types: Suppose a motion sends A to A' and B to B'. Still, the motion is not determined. Now, motion will send a third point C to one of the two choices, C' and its reflection $\tilde{C}$ in the line through A' and B'. Thus there are two motions that map A, B to A', B': $\mathcal{M}$ sends C to C' and $\tilde{\mathcal{M}}$ sends C to $\tilde{C}$. The two motions differ in the sense that $\mathcal{M}$ preserves sense of the angle θ which $\tilde{\mathcal{M}}$ reverses it. $\mathcal{M}$ preserves all angles while $\tilde{\mathcal{M}}$ reverse all angles. Motion that preserve angles are called *direct* and those that reverse angles are called *opposite*. Thus rotations and translations are direct, while reflections are opposite.

- *There is exactly one direct motion $\mathcal{M}$ (and exactly one opposite motion $\tilde{\mathcal{M}}$) that maps a given line-segment AB to another line segment $A'B'$ of equal length. Furthermore, $\tilde{\mathcal{M}} = \mathcal{M}$ followed by reflection in the line $A'B'$.*

- *Every direct motion is a rotation, or else (exceptionally) a translation.* See figure for proof. Based on the rotational transformation that we did before, we have, *Every direct motion can be expresed as a complex function of the form $\mathcal{M}(z) = e^{i\theta}z + v$.*

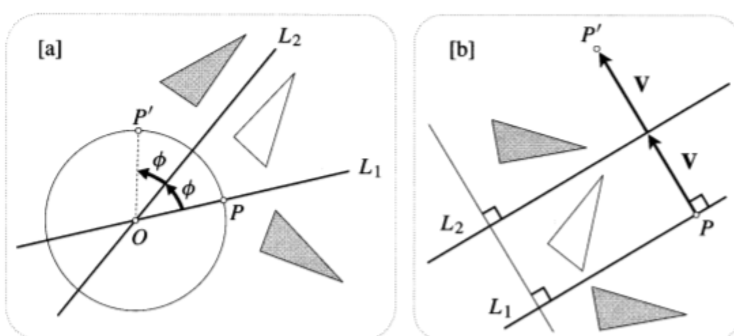




- Set of direction motions form a subgroup of the group of motions while set of opposite motions do not.

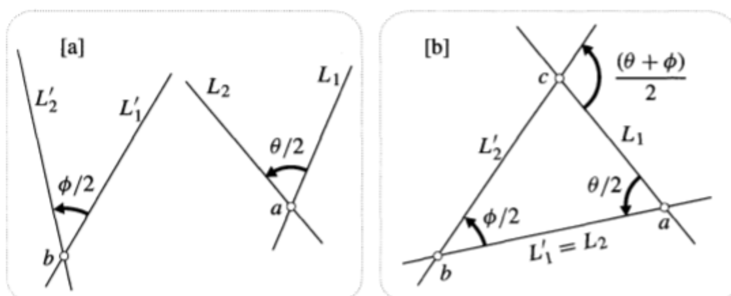
- *Every direct motion is the composition of two reflections.* Therefore, every opposite motion is a composition of three reflections. This is three reflections theorem.

- *If L_1 and L_2 intersect at O, and the angle from L_1 to L_2 is ϕ, then, $\mathcal{R}_{L_2} \circ \mathcal{R}_{L_1}$ is a rotation of 2ϕ about O. If L_1 and L_2 are parallel, and V is the perpendicular connectng vector from L_1 to L_2, then $\mathcal{R}_{L_2} \circ \mathcal{R}_{L_1}$ is a translation of $2V$.*



- Rotation of θ can be represented as $\mathcal{R}_{L_2} \circ \mathcal{R}_{L_1}$, where L_1, L_2 is any pair of lines that pass through the centre of the rotation and that contain an angle $\theta/2$.

- A translation of T corresponds to any pair of parallel lines separated by $T/2$.



[a]
L'_2
L'_1
$\phi/2$
b
L_2
L_1
$\theta/2$
a
[b]
c
$\frac{(\theta+\phi)}{2}$
L'_2
L_1
$\theta/2$
$\phi/2$
a
b
$L'_1 = L_2$