Simple numerical methods introduction (ODE)

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Ordinary differential equations (ODE): initial value problem

- Examples of ODE:
  - $\frac{d^2 a}{dt^2} = f[a, t]$, **with** $a(0), a'(0)$ known.
  - $\frac{d\vec{a}}{dt} = \vec{f}[\vec{a}, t]$, **with** $\vec{a}(0), \vec{a}'(0)$ known.
- Higher than first-order ODEs can always be recast into a set of first-order ODEs:

$$
\begin{aligned}
& \frac{d^2 x}{dt^2} + c_1[x, t]\frac{dx}{dt} + c_0[x, t] = 0 \\
\longrightarrow \quad & \frac{d a_1}{dt} = a_2 \\
& \frac{d a_2}{dt} = -c_1[a_1, t]\, a_2 - c_0[a_1, t], \quad a_1 = x, a_2 = \frac{dx}{dt}
\end{aligned}
$$

Euler method

Given $\frac{da}{dt} = f(a)$

- Tyler expansion:

$$a(t_0 + \delta t) \approx a(t_0) + a'(t_0)\delta t + \frac{1}{2}a''(t_0)\delta t^2 + \mathcal{O}(\delta t^3)$$

- After discretized, $t_n = t_0 + n\Delta t$ and $a_n = a(t_n)$:

$$a_{n+1} = a_n + f(a(t_n))\Delta t + \mathcal{O}(\Delta t^2)$$

- Euler method (forward)(explicit method)

$$a_{n+1} = a_n + f_n \Delta t\,, \qquad f_n = f(a(t_n))$$

Other methods

- Euler method (backward)(implicit method)

$$a_{n+1} = a_n + f_{n+1}\Delta t$$

- Crank-Nichelson method (implicit method) (2nd order)

$$a_{n+1} = a_n + \frac{1}{2}(f_n + f_{n+1})\Delta t$$

- Mid-point method (explicit method) (2nd order)

$$a_{n+1} = a_n + f(a(t_n + \frac{1}{2}\Delta t))\Delta t$$

Runge-Kutta method

Given $\frac{dy}{dx} = f(x, y)$

- Second order:

$$k_1 = h\,f(x_n, y_n)$$
$$k_2 = h\,f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$
$$y_{n+1} = y_n + k_2 + \mathcal{O}(h^3)$$

Runge-Kutta method

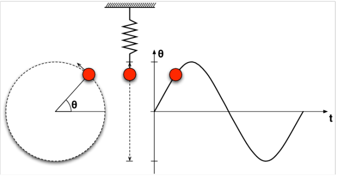
- fourth order:

$$
\begin{aligned}
k_1 &= h\,f(x_n, y_n)\,,\\
k_2 &= h\,f(x_n + \frac{h}{2}\,, y_n + \frac{k_1}{2})\\
k_3 &= h\,f(x_n + \frac{h}{2}\,, y_n + \frac{k_2}{2})\,,\\
k_4 &= h\,f(x_n + h\,, y_n + k_3)\,,\\
y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + \mathcal{O}(h^5)
\end{aligned}
$$

Examples

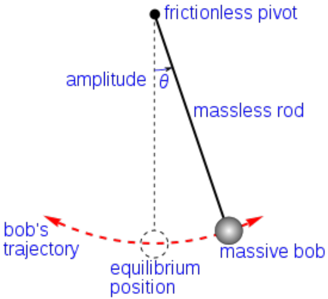
- Simple harmonic motion (SHM): $\theta''(t) = -\theta(t)$

$$\frac{d}{dt}\begin{pmatrix}\theta(t)\\ \frac{d\theta}{dt}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\begin{pmatrix}\theta(t)\\ \frac{d\theta}{dt}\end{pmatrix}$$



- Pendulum: $\theta''(t) = -\sin\theta(t)$

$$\frac{d\theta(t)}{dt} = \omega(t)$$
$$\frac{d\omega(t)}{dt} = -\sin\theta(t)$$



Trying out: more non-linear examples

- Driven oscillator with non-linear term:

$$x''(t) = -\omega_0^2 x + \beta x^2 + f_0 \cos \omega_d t$$

- Driven oscillator with damping and non-linear term:

$$x''(t) = -\gamma x'(t) - \omega_0^2 x + \beta x^2 + f_0 \cos \omega_d t$$