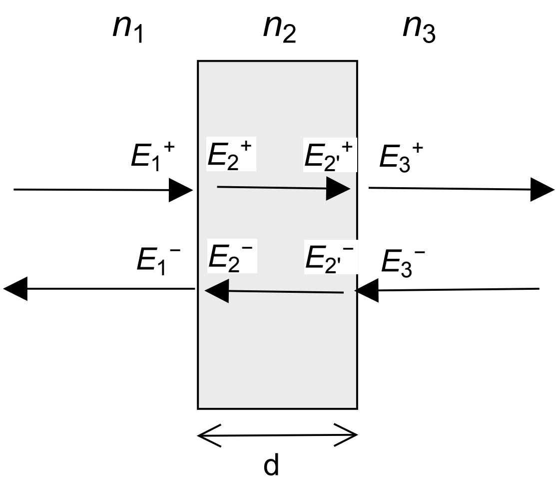
Physical Optics

Lecture 5 Planar multilayer media and thin film optics

1. Multilayered media

Let us consider the following planar multilayer medium.



For normal incidence, one has the following two boundary conditions:

$E_1^+ + E_1^- = E_2^+ + E_2^-$
$n_1 E_1^+ - n_1 E_1^- = n_2 E_2^+ - n_2 E_2^-$

They can be solved to derive the (linear) relation fro the fields on the two sides:

$$\begin{pmatrix} 1 & 1 \\ n_1 & -n_1 \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ n_2 & -n_2 \end{pmatrix} \begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix}$$

$$\begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ n_2 & -n_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ n_1 & -n_1 \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

$$\begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix} = \begin{pmatrix} \frac{n_2+n_1}{2\,n_2} & \frac{n_2-n_1}{2\,n_2} \\ \frac{n_2-n_1}{2\,n_2} & \frac{n_2+n_1}{2\,n_2} \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

$$\begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = \begin{pmatrix} \frac{n_1+n_2}{2\,n_1} & \frac{n_1-n_2}{2\,n_1} \\ \frac{n_1-n_2}{2\,n_1} & \frac{n_1+n_2}{2\,n_1} \end{pmatrix} \begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix}$$

This is called the transfer matrix for the interface .

For free propagation of length d in the medium, the transfer matrix is

$$\begin{pmatrix} E_2'^{+} \\ E_2'^{-} \end{pmatrix} = \begin{pmatrix} \mathrm{Exp}[I\, n_2\, k\, d] & 0 \\ 0 & \mathrm{Exp}[-I\, n_2\, k\, d] \end{pmatrix} \begin{pmatrix} E_2^{+} \\ E_2^{-} \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} E_3^{+} \\ E_3^{-} \end{pmatrix} = \begin{pmatrix} \frac{n_3+n_2}{2 n_3} & \frac{n_3-n_2}{2 n_3} \\ \frac{n_3-n_3}{2 n_3} & \frac{n_3+n_3}{2 n_3} \end{pmatrix} \begin{pmatrix} E_2'^{+} \\ E_2'^{-} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n_3+n_2}{2 n_3} & \frac{n_3-n_2}{2 n_3} \\ \frac{n_3-n_2}{2 n_3} & \frac{n_3+n_2}{2 n_3} \end{pmatrix} \begin{pmatrix} E^{I\, n_2\, k\, d} & 0 \\ 0 & E^{-I\, n_2\, k\, d} \end{pmatrix} \begin{pmatrix} \frac{n_2+n_1}{2 n_2} & \frac{n_2-n_1}{2 n_2} \\ \frac{n_2-n_1}{2 n_2} & \frac{n_2+n_1}{2 n_2} \end{pmatrix} \begin{pmatrix} E_1^{+} \\ E_1^{-} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_1^{+} \\ E_1^{-} \end{pmatrix}$$

$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is the total transfer matrix.

If there is no incidence from the right, then $E_3^{-} = 0$ and thus

$M_{21} E_1^{+} + M_{22} E_1^{-} = 0$

Therefore,

reflection coef : $r = E_1^{-} / E_1^{+} = -M_{21} / M_{22}$

transmission coef : $t = E_3^{+} / E_1^{+} = M_{11} - M_{12} M_{21} / M_{22}$

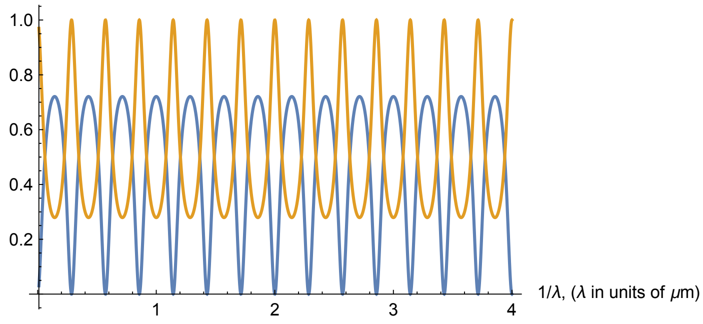
The above theory is the transfer matrix method for analyzing planar multilayered media. One can easily write a program to calculate the transmission and reflection coefficients based on the above formula.

In[ ]:=
```
TI[n2_, n1_] := {{(n2+n1)/(2 n2), (n2-n1)/(2 n2)}, {(n2-n1)/(2 n2), (n2+n1)/(2 n2)}};
TD[n_, d_, k_] := {{E^(I n k d), 0}, {0, E^(-I n k d)}};
TE[nl_, dl_, k_] := Block[{M},
  M = TI[nl[[2]], nl[[1]]];
  Do[
   M = TI[nl[[i + 2]], nl[[i + 1]]].TD[nl[[i + 1]], dl[[i]], k].M;,
   {i, Length[dl]}];
  Return[{-M[[2, 1]] / M[[2, 2]], M[[1, 1]] - M[[1, 2]] × M[[2, 1]] / M[[2, 2]] }];]
```

Example : Fabry-Perot etalon

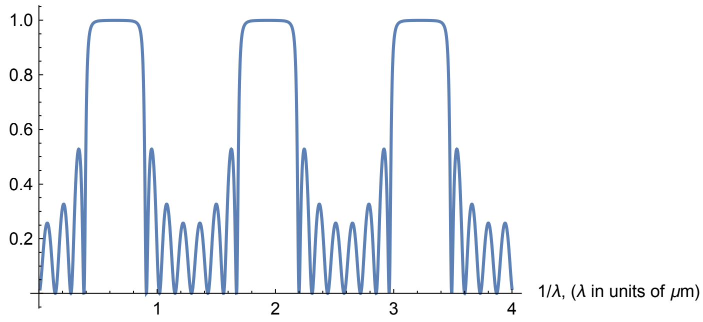
In[ ]:=
```
pa = {n1 → 1., n2 → 3.5, n3 → 1., d → 0.5};
Plot[{Abs[TE[{n1, n2, n3}, {d}, 2 Pi λinv][[1]]]^2 /. pa,
  Abs[TE[{n1, n2, n3}, {d}, 2 Pi λinv][[2]]]^2 /. pa},
 {λinv, 0.01, 4.}, AxesLabel → {"1/λ, (λ in units of μm)"}]
```

Out[ ]=



1 D Photonic crystal :

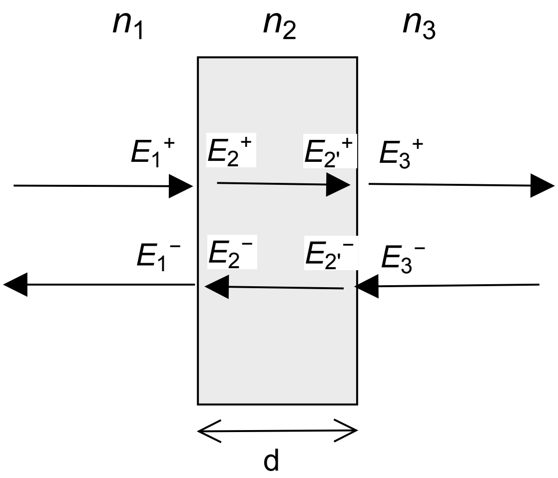
```
λ0 = 1.55;
Plot[
 Abs[TE[{1., 3., 1., 3., 1., 3., 1., 3., 1.}, {λ0 / 4. / 3., λ0 / 4. / 1., λ0 / 4. / 3.,
      λ0 / 4. / 1., λ0 / 4. / 3., λ0 / 4. / 1., λ0 / 4. / 3.}, 2 Pi f][[1]]]^2,
 {f, 0.01, 4.}, AxesLabel → {"1/λ, (λ in units of μm)"}]
```



2. Thin film optics

(1) Anti-reflection coating

For the following thin film structure,



the transfer matrix of a single interface can be rewritten as

$$\begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix} = \begin{pmatrix} \frac{n_2+n_1}{2n_2} & \frac{n_2-n_1}{2n_2} \\ \frac{n_2-n_1}{2n_2} & \frac{n_2+n_1}{2n_2} \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = \frac{n_2+n_1}{2n_2} \begin{pmatrix} 1 & \frac{n_2-n_1}{n_2+n_1} \\ \frac{n_2-n_1}{n_2+n_1} & 1 \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = \frac{1}{1+\Gamma_{21}} \begin{pmatrix} 1 & \Gamma_{21} \\ \Gamma_{21} & 1 \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

Or

$$\begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix} = \begin{pmatrix} \frac{n_1+n_2}{2n_1} & \frac{n_1-n_2}{2n_1} \\ \frac{n_1-n_2}{2n_1} & \frac{n_1+n_2}{2n_1} \end{pmatrix} \begin{pmatrix} E_2^+ \\ E_2^- \end{pmatrix} = \frac{1}{1+\Gamma_{12}} \begin{pmatrix} 1 & \Gamma_{12} \\ \Gamma_{12} & 1 \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

Here

$$\Gamma_{21} = \frac{n_2 - n_1}{n_2 + n_1}$$

(field reflection coefficient from 2 to 1)

$$\Gamma_{12} = \frac{n_1 - n_2}{n_2 + n_1}$$

(field reflection coefficient from 1 to 2)

The total transfer matrix can then be calculated. Note that

$$\mathbf{Simplify}\left[\begin{pmatrix} 1 & \Gamma_{32} \\ \Gamma_{32} & 1 \end{pmatrix} . \begin{pmatrix} E^{I n_2 k d} & 0 \\ 0 & E^{-I n_2 k d} \end{pmatrix} . \begin{pmatrix} 1 & \Gamma_{21} \\ \Gamma_{21} & 1 \end{pmatrix}\right] \text{ // MatrixForm}$$

$$\begin{pmatrix} e^{i d k n_2} + e^{-i d k n_2}\Gamma_{21}\Gamma_{32} & e^{-i d k n_2}\left(e^{2 i d k n_2}\Gamma_{21} + \Gamma_{32}\right) \\ e^{-i d k n_2}\left(\Gamma_{21} + e^{2 i d k n_2}\Gamma_{32}\right) & e^{-i d k n_2} + e^{i d k n_2}\Gamma_{21}\Gamma_{32} \end{pmatrix}$$

This is a more simple form suitable for analytic derivation. The net transfer matrix now becomes

$$\begin{pmatrix} E_3^+ \\ E_3^- \end{pmatrix} = \frac{1}{1+\Gamma_{32}} \frac{1}{1+\Gamma_{21}} \begin{pmatrix} 1 & \Gamma_{32} \\ \Gamma_{32} & 1 \end{pmatrix} . \begin{pmatrix} E^{I n_2 k d} & 0 \\ 0 & E^{-I n_2 k d} \end{pmatrix} . \begin{pmatrix} 1 & \Gamma_{21} \\ \Gamma_{21} & 1 \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

$$= \frac{1}{1+\Gamma_{32}} \frac{1}{1+\Gamma_{21}} \begin{pmatrix} e^{i d k n_2} + e^{-i d k n_2}\Gamma_{21}\Gamma_{32} & e^{-i d k n_2}\left(e^{2 i d k n_2}\Gamma_{21} + \Gamma_{32}\right) \\ e^{-i d k n_2}\left(\Gamma_{21} + e^{2 i d k n_2}\Gamma_{32}\right) & e^{-i d k n_2} + e^{i d k n_2}\Gamma_{21}\Gamma_{32} \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

The reflection coef : $r = E_1^- / E_1^+ = -M_{21}/M_{22} = -\frac{\Gamma_{21} + e^{2 i d k n_2} \Gamma_{32}}{1 + e^{2 i d k n_2} \Gamma_{21} \Gamma_{32}}$

For the reflection to be zero, one can choose $d\ k\ n_2 = \pi / 2$ (quarter wavelength thickness) so that the condition becomes

$\Gamma_{21} - \Gamma_{32} = 0$

$\frac{n_2 - n_1}{n_2 + n_1} = \frac{n_3 - n_2}{n_3 + n_2}$

One finally has

$n_2 = \sqrt{n_1 n_3}$

This is the condition for anti-reflection design.

(2) Fabry-Perot etalon

For a symmetric structure ($n_1 = n_3$), one has

$$\text{Simplify}\left[\begin{pmatrix} 1 & -\Gamma_{21} \\ -\Gamma_{21} & 1 \end{pmatrix} . \begin{pmatrix} E^{I n_2 k d} & 0 \\ 0 & E^{-I n_2 k d} \end{pmatrix} . \begin{pmatrix} 1 & \Gamma_{21} \\ \Gamma_{21} & 1 \end{pmatrix}\right] \text{ // MatrixForm}$$

$$\begin{pmatrix} e^{i d k n_2} - e^{-i d k n_2} \Gamma_{21}^2 & e^{-i d k n_2} \left(-1 + e^{2 i d k n_2}\right) \Gamma_{21} \\ -e^{-i d k n_2} \left(-1 + e^{2 i d k n_2}\right) \Gamma_{21} & e^{-i d k n_2} - e^{i d k n_2} \Gamma_{21}^2 \end{pmatrix}$$

$$M = \frac{1}{1 - \Gamma_{21}} \frac{1}{1 + \Gamma_{21}} \begin{pmatrix} e^{i d k n_2} - e^{-i d k n_2} \Gamma_{21}^2 & e^{-i d k n_2} \left(-1 + e^{2 i d k n_2}\right) \Gamma_{21} \\ -e^{-i d k n_2} \left(-1 + e^{2 i d k n_2}\right) \Gamma_{21} & e^{-i d k n_2} - e^{i d k n_2} \Gamma_{21}^2 \end{pmatrix};$$

```
Simplify[-M[[2, 1]] / M[[2, 2]]]
```

$$-\frac{\left(-1 + e^{2 i d k n_2}\right) \Gamma_{21}}{-1 + e^{2 i d k n_2} \Gamma_{21}^2}$$

```
Simplify[M[[1, 1]] - M[[1, 2]] × M[[2, 1]] / M[[2, 2]]]
```

$$\frac{e^{i d k n_2} \left(-1 + \Gamma_{21}\right) \left(1 + \Gamma_{21}\right)}{-1 + e^{2 i d k n_2} \Gamma_{21}^2}$$

In the above case, since $n_3 = n_1$, one has $\Gamma_{32} = \Gamma_{12} = -\Gamma_{21}$. The reflection and transmission coefficients are

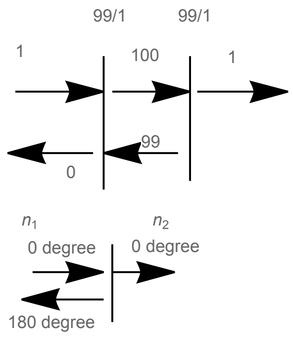
$$r = -\frac{\left(-1 + e^{2 i d k n_2}\right) \Gamma_{21}}{-1 + e^{2 i d k n_2} \Gamma_{21}^2}$$

$$t = \frac{e^{i d k n_2} \left(-1 + \Gamma_{21}{}^2\right)}{-1 + e^{2 i d k n_2} \Gamma_{21}^2}$$

Note that when $d\ k\ n_2 = m\ \pi$, $e^{2 i d k n_2} = 1$ (at resonance) and

$r = 0$

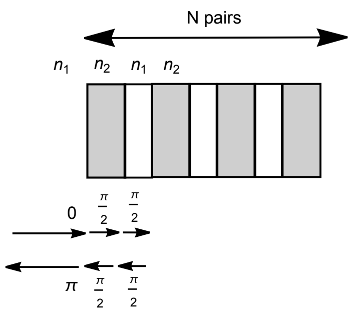
$t = 1$



That is, for a symmetric Fabry-Perot resonator, the incident light can totally transmit when at resonance if there is no internal loss.

(3) High-reflection coating

Consider a stack of quarter wavelength pairs as shown in the figure.



For a single pair, the combined transmission matrix is

```
Simplify[1/(1 - Γ21) 1/(1 + Γ21) {{I, 0}, {0, -I}}.{{1, -Γ21}, {-Γ21, 1}}.{{I, 0}, {0, -I}}.{{1, Γ21}, {Γ21, 1}}] // MatrixForm
```

$$\begin{pmatrix} \frac{1+\Gamma_{21}^2}{-1+\Gamma_{21}^2} & \frac{2\,\Gamma_{21}}{-1+\Gamma_{21}^2} \\ \frac{2\,\Gamma_{21}}{-1+\Gamma_{21}^2} & \frac{1+\Gamma_{21}^2}{-1+\Gamma_{21}^2} \end{pmatrix}$$

The matrix can be decomposed (diagonalized) as follows

$$\texttt{Simplify}\left[\texttt{Eigensystem}\left[\begin{pmatrix} \frac{1+\Gamma_{21}^2}{-1+\Gamma_{21}^2} & \frac{2\,\Gamma_{21}}{-1+\Gamma_{21}^2} \\ \frac{2\,\Gamma_{21}}{-1+\Gamma_{21}^2} & \frac{1+\Gamma_{21}^2}{-1+\Gamma_{21}^2} \end{pmatrix}\right]\right]$$

$$\left\{\left\{\frac{-1+\Gamma_{21}}{1+\Gamma_{21}}, \frac{1+\Gamma_{21}}{-1+\Gamma_{21}}\right\}, \{\{-1, 1\}, \{1, 1\}\}\right\}$$

This can be checked easily:

$$\texttt{Simplify}\left[\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \begin{pmatrix} \frac{-1+\Gamma_{21}}{1+\Gamma_{21}} & 0 \\ 0 & \frac{1+\Gamma_{21}}{-1+\Gamma_{21}} \end{pmatrix} . \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right] \texttt{ // MatrixForm}$$

$$\begin{pmatrix} \frac{1+\Gamma_{21}^2}{-1+\Gamma_{21}^2} & \frac{2\,\Gamma_{21}}{-1+\Gamma_{21}^2} \\ \frac{2\,\Gamma_{21}}{-1+\Gamma_{21}^2} & \frac{1+\Gamma_{21}^2}{-1+\Gamma_{21}^2} \end{pmatrix}$$

Now

$$M = \begin{pmatrix} \frac{1+\Gamma_{21}^2}{-1+\Gamma_{21}^2} & \frac{2\,\Gamma_{21}}{-1+\Gamma_{21}^2} \\ \frac{2\,\Gamma_{21}}{-1+\Gamma_{21}^2} & \frac{1+\Gamma_{21}^2}{-1+\Gamma_{21}^2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \begin{pmatrix} \frac{-1+\Gamma_{21}}{1+\Gamma_{21}} & 0 \\ 0 & \frac{1+\Gamma_{21}}{-1+\Gamma_{21}} \end{pmatrix} . \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \begin{pmatrix} -\frac{n_1}{n_2} & 0 \\ 0 & -\frac{n_2}{n_1} \end{pmatrix} . \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

The diagonalization of the matrix
M can let us calculate M^N analytically as follow.

$$M_{tol} = M^N = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \begin{pmatrix} -\frac{n_1}{n_2} & 0 \\ 0 & -\frac{n_2}{n_1} \end{pmatrix}^N . \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \begin{pmatrix} \left(-\frac{n_1}{n_2}\right)^N & 0 \\ 0 & \left(-\frac{n_2}{n_1}\right)^N \end{pmatrix} . \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\left(\left(-\frac{n_1}{n_2}\right)^N + \left(-\frac{n_2}{n_1}\right)^N\right) & \frac{1}{2}\left(-\left(-\frac{n_1}{n_2}\right)^N + \left(-\frac{n_2}{n_1}\right)^N\right) \\ \frac{1}{2}\left(-\left(-\frac{n_1}{n_2}\right)^N + \left(-\frac{n_2}{n_1}\right)^N\right) & \frac{1}{2}\left(\left(-\frac{n_1}{n_2}\right)^N + \left(-\frac{n_2}{n_1}\right)^N\right) \end{pmatrix}$$

The reflection coefficient is simply

$$r = \frac{\left(-\frac{n_1}{n_2}\right)^N - \left(-\frac{n_2}{n_1}\right)^N}{\left(-\frac{n_1}{n_2}\right)^N + \left(-\frac{n_2}{n_1}\right)^N}$$

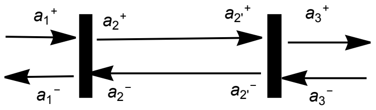
When N is large,

$r \to 1$ or -1 (depending $n_1 > n_2$ or $n_2 > n_1$)

High reflection around the designed wavelength can be achieved when N is large enough.

3. Fabry-Perot resonator by two mirrors

Fabry - Perot resonators can also be formed by two mirrors in free space as shown in the figure below.



To analyze it in terms of the mirror reflectivities, one can derive the transfer matrix from the scattering matrix first. Then the same analysis can be applied. The scattering matrix of a partially transmitting mirror is given by

$$\begin{pmatrix} a_1^- \\ a_2^+ \end{pmatrix} = \begin{pmatrix} -r & I\sqrt{1-r^2} \\ I\sqrt{1-r^2} & -r \end{pmatrix} \begin{pmatrix} a_1^+ \\ a_2^- \end{pmatrix}$$

The transfer matrix can be obtained to be

$$\begin{pmatrix} a_2^+ \\ a_2^- \end{pmatrix} = \begin{pmatrix} \frac{I}{\sqrt{1-r^2}} & \frac{I r}{\sqrt{1-r^2}} \\ -\frac{I r}{\sqrt{1-r^2}} & -\frac{I}{\sqrt{1-r^2}} \end{pmatrix} \begin{pmatrix} a_1^+ \\ a_1^- \end{pmatrix}$$

If the two mirrors are identical, then the total transfer matrix is

$$\text{Simplify}\left[\begin{pmatrix} \frac{I}{\sqrt{1-r^2}} & \frac{I r}{\sqrt{1-r^2}} \\ -\frac{I r}{\sqrt{1-r^2}} & -\frac{I}{\sqrt{1-r^2}} \end{pmatrix} . \begin{pmatrix} E^{I n_2 k d} & 0 \\ 0 & E^{-I n_2 k d} \end{pmatrix} . \begin{pmatrix} \frac{I}{\sqrt{1-r^2}} & \frac{I r}{\sqrt{1-r^2}} \\ -\frac{I r}{\sqrt{1-r^2}} & -\frac{I}{\sqrt{1-r^2}} \end{pmatrix}\right] \text{ // MatrixForm}$$

$$\begin{pmatrix} \frac{e^{-i d k n_2}\left(e^{2 i d k n_2}-r^2\right)}{-1+r^2} & \frac{e^{-i d k n_2}\left(-1+e^{2 i d k n_2}\right) r}{-1+r^2} \\ -\frac{e^{-i d k n_2}\left(-1+e^{2 i d k n_2}\right) r}{-1+r^2} & \frac{e^{-i d k n_2}\left(1-e^{2 i d k n_2} r^2\right)}{-1+r^2} \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{e^{-i d k n_2}\left(e^{2 i d k n_2}-r^2\right)}{-1+r^2} & \frac{e^{-i d k n_2}\left(-1+e^{2 i d k n_2}\right) r}{-1+r^2} \\ -\frac{e^{-i d k n_2}\left(-1+e^{2 i d k n_2}\right) r}{-1+r^2} & \frac{e^{-i d k n_2}\left(1-e^{2 i d k n_2} r^2\right)}{-1+r^2} \end{pmatrix};$$

The reflection coefficient is

```
Simplify[-M[[2, 1]] / M[[2, 2]]]
```

$$\frac{\left(-1+e^{2\,i\,d\,k\,n_2}\right)\,r}{1-e^{2\,i\,d\,k\,n_2}\,r^2}$$

The transmission coefficient is

```
Simplify[M[[1, 1]] - M[[1, 2]] × M[[2, 1]] / M[[2, 2]]]
```

$$-\frac{e^{i\,d\,k\,n_2}\,\left(-1+r^2\right)}{-1+e^{2\,i\,d\,k\,n_2}\,r^2}$$

So

$$r = \frac{\left(-1+e^{2 i d k n_2}\right) r}{1-e^{2 i d k n_2}\, r^2}$$

$$t = -\frac{e^{i d k n_2}\left(-1+r^2\right)}{-1+e^{2 i d k n_2}\, r^2}$$

When at resonance, $2\,d\,k\,n_2 = 2\,m\,\pi$, r=0 and t=1.

This agrees with what we have seen above.

4. Oblique TE incidence

The above formula are derived for normal incidence. To generalize the transfer matrix formula to the oblique TE incidence case, one only needs to adopt the following substitution:

$$n_1 \to k_{1,z}/k_0 = \left(n_1^2\, k_0^2 - k_x^2 - k_y^2\right)^{1/2}/k_0 = \left(n_1^2 - n_t^2\right)^{1/2}$$

$$n_2 \to k_{2,z}/k_0 = \left(n_2^2\, k_0^2 - k_x^2 - k_y^2\right)^{1/2}/k_0 = \left(n_2^2 - n_t^2\right)^{1/2}$$

$$n_3 \to k_{3,z}/k_0 = \left(n_3^2\, k_0^2 - k_x^2 - k_y^2\right)^{1/2}/k_0 = \left(n_3^2 - n_t^2\right)^{1/2}$$

Here

$$n_t^2 = \left(k_x^2 + k_y^2\right)/k_0^2 = n_1^2 \sin^2(\theta_1) = n_2^2 \sin^2(\theta_2) = n_3^2 \sin^2(\theta_3)$$

The transfer matrix formula can also be generalized to calculate the thin film waveguide modes. For this application, we will set $n_t^2 = n_{eff}^2$, where $n_{eff} = \beta/k_0$ is the effective mode index for the TE waveguide modes, which is one of the physical quantity to be determined.

For guiding modes, the fields exponentially decay outside the core. Thus one has

$$k_{1,z} = \sqrt{k_1^2 - \beta^2} = i\,\gamma_{1,z}$$

$$k_{2,z} = \sqrt{k_2^2 - \beta^2}$$

$$k_{3,z} = \sqrt{k_3^2 - \beta^2} = i\,\gamma_{3,z}$$

The substitution formula become

$$n_1 \to k_{1,z}/k_0 = i\left(n_{eff}^2 - n_1^2\right)^{1/2}$$

$$n_2 \to k_{2,z}/k_0 = \left(n_2^2 - n_{eff}^2\right)^{1/2}$$

$$n_3 \to k_{3,z}/k_0 = i\left(n_{eff}^2 - n_3^2\right)^{1/2}$$

From the incidence (boundary) conditions $E_3^- = 0$ and $E_1^+ = 0$, one can derive a characteristic equation for solving n_{eff} (or β).

5. Oblique TM incidence

To generalize the transfer matrix formula to the oblique TM incidence, one only needs to adopt the following substitution:

$$n_1 \to \frac{k_{1,z}}{n_1^2\, k_0} = \frac{\left(n_1^2 - n_t^2\right)^{1/2}}{n_1^2}$$

$$n_2 \to \frac{k_{2,z}}{n_2^2\, k_0} = \frac{\left(n_2^2 - n_t^2\right)^{1/2}}{n_2^2}$$

$$n_3 \to \frac{k_{3,z}}{n_3^2\, k_0} = \frac{\left(n_3^2 - n_t^2\right)^{1/2}}{n_3^2}$$

Here

$$n_t^2 = \left(k_x^2 + k_y^2\right)/k_0^2 = n_1^2 \sin^2(\theta_1) = n_2^2 \sin^2(\theta_2) = n_3^2 \sin^2(\theta_3)$$

The transfer matrix formula can also be generalized to calculate the TM thin film waveguide modes. For this application, we will set $n_t^2 = n_{eff}^2$, where $n_{eff} = \beta/k_0$ is the effective mode index for the TM waveguide modes, which is to be determined. The formula for TM modes are different from those for TE modes.

$$k_{1,z} = \sqrt{k_1^2 - \beta^2} = i\,\gamma_{1,z}$$

$$k_{2,z} = \sqrt{k_2^2 - \beta^2}$$

$$k_{3,z} = \sqrt{k_3^2 - \beta^2} = i\,\gamma_{3,z}$$

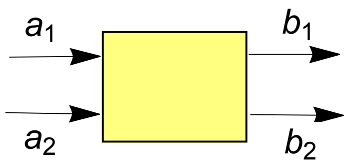
$$n_1 \to k_{1,z}/\left(n_1^2\, k_0\right) = i\left(n_{eff}^2 - n_1^2\right)^{1/2}/n_1^2$$

$$n_2 \to k_{2,z}/\left(n_2^2\, k_0\right) = \left(n_2^2 - n_{eff}^2\right)^{1/2}/n_2^2$$

$$n_3 \to k_{3,z}/\left(n_3^2\, k_0\right) = i\left(n_{eff}^2 - n_3^2\right)^{1/2}/n_2^2$$

From the incidence (boundary) conditions, one can derive a characteristic equation for solving n_{eff} (or β).

6. Scattering matrix



In general, the scattering matrix of a two port linear system is defined as the linear relation between the two inputs and the two outputs :

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Here $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$

$|b_1|^2 = |S_{11}|^2 \, |a_1|^2, \quad |b_2|^2 = |S_{21}|^2 \, |a_1|^2$ if $a_2 = 0$

$|S_{11}|^2 + |S_{21}|^2 = 1, \quad |S_{12}|^2 + |S_{22}|^2 = 1$ if lossless

is the scattering matrix and all the fields are normalized such that their absolute squared values represent their powers.

$a_1 \propto E_{\text{input}\,1}$, $|a_1|^2$ is the power of the input 1.

$a_2 \propto E_{\text{input}\,2}$, $|a_2|^2$ is the power of the input 2.

$b_1 \propto E_{\text{output}\,1}$, $|b_1|^2$ is the power of the output 1.

$b_2 \propto E_{\text{output}\,2}$, $|b_2|^2$ is the power of the output 2.

The scattering matrix has to obey certain properties if the system is lossless and reciprocal. The detailed derivation can be found in the Chapter 3 of Haus' s book (H.A.Haus, "Waves and Fields in Optoelectronics").

For example, if the system is lossless, then the total input power is equal to the total output power.

$$|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2$$

This condition is equivalent to

$S^{T*} S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (S is unitary)

Note that

$\begin{pmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{pmatrix} . \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ **// MatrixForm**

$$\begin{pmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{12}(S_{11})^* + S_{22}(S_{21})^* \\ S_{11}(S_{12})^* + S_{21}(S_{22})^* & |S_{12}|^2 + |S_{22}|^2 \end{pmatrix}$$

So

$$|S_{11}|^2 + |S_{21}|^2 = 1$$
$$|S_{12}|^2 + |S_{22}|^2 = 1$$
$$S_{11}(S_{12})^* + S_{21}(S_{22})^* = 0$$
$$S_{12}(S_{11})^* + S_{22}(S_{21})^* = 0$$

Furthermore, if the system is reciprocal, then

$S = S^T$ (S is symmetric)

The scattering matrix of a partially transmitting mirror is usually written as

reflection : 180 degree
transmission : 90 degree

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -r & i\sqrt{1-r^2} \\ i\sqrt{1-r^2} & -r \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -r & it \\ it & -r \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$t = \sqrt{1-r^2}$

$|r|^2$: power reflection coef., $|t|^2$: power transmision coef.

For example,

$$S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ for a } 50/50 \text{ mirror}$$

One can easily verify that it satisfies the above requirements. This form of the scattering matrix has the property that the coefficients for the same port number have the same phase (=π) and the coefficients for the different port number also have the same phase (but = π/2).

Note that if one changes the phase references of the two input and two output ports,

$$a_1 = \bar{a}_1 e^{i\theta_{a1}}$$
$$a_2 = \bar{a}_2 e^{i\theta_{a2}}$$
$$b_1 = \bar{b}_1 e^{i\theta_{b1}}$$
$$b_2 = \bar{b}_2 e^{i\theta_{b2}}$$

one then has

$$\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} = \begin{pmatrix} -r\,e^{i\theta_{a1} - i\theta_{b1}} & i\sqrt{1-r^2}\,e^{i\theta_{a2} - i\theta_{b1}} \\ i\sqrt{1-r^2}\,e^{i\theta_{a1} - i\theta_{b2}} & -r\,e^{i\theta_{a2} - i\theta_{b2}} \end{pmatrix} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \end{pmatrix}$$

So the phases of the scattering matrix components can be different if the phase references are chosen differently. For example, if

$$\theta_{b1} = 0$$
$$\theta_{b2} = -\frac{\pi}{2}$$
$$\theta_{a1} = -\pi$$

$$\theta_{a2} = -\frac{\pi}{2}$$

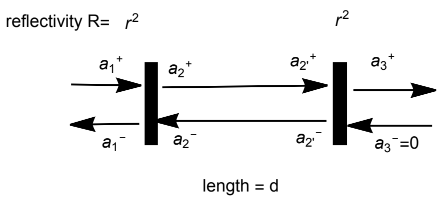
then

$$\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} = \begin{pmatrix} r & \sqrt{1-r^2} \\ \sqrt{1-r^2} & -r \end{pmatrix} \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \end{pmatrix}$$

This is another popular form for the scattering matrix of a partially transmitting mirror. It is sometimes used due to its simplicity.

7. Fabry-Perot resonator analyzed by the scattering matrix formulation

Consider again the Fabry - Perot resonator formed by two mirrors in free space as shown in the figure below.



One can analyze it in terms of the scattering matrix formulation directly without using the transfer matrix concept, especially when there is only one input ($a_3^- = 0$). To do it, one only needs to write down the equations at the interfaces and then solve them directly.

$$a_2^+ = -r\,a_2^- + i\,t\,a_1^+$$
$$a_1^- = -r\,a_1^+ + i\,t\,a_2^-$$
$$a_{2'}^+ = a_2^+\,e^{i\theta}$$
$$a_2^- = a_{2'}^-\,e^{i\theta}$$
$$a_{2'}^- = -r\,a_{2'}^+$$

$$a_2^+ = -r\,a_2^- + i\,t\,a_1^+$$
$$a_1^- = -r\,a_1^+ + i\,t\,a_2^-$$
$$a_2^- = -r\,a_2^+\,e^{i2\theta}$$

$$a_2^+ = -r\left(-r\,a_2^+\,e^{i2\theta}\right) + i\,t\,a_1^+$$
$$a_1^- = -r\,a_1^+ + i\,t\left(-r\,a_2^+\,e^{i2\theta}\right)$$

$$a_2^+ = \frac{i\,t\,a_1^+}{1 - r^2\,e^{i2\theta}}$$

$$a_1^- = -r\,a_1^+ + i\,t\left(-r\,\frac{i\,t\,a_1^+}{1 - r^2\,e^{i2\theta}}\,e^{i2\theta}\right)$$

$$=-r\,a_1{}^{+}+\left(\frac{r\,t^2\,e^{i\,2\,\theta}}{1-r^2\,e^{i\,2\,\theta}}\right)a_1{}^{+}$$

$$=r\left(-1+\frac{\left(1-r^2\right)e^{i\,2\,\theta}}{1-r^2\,e^{i\,2\,\theta}}\right)a_1{}^{+}$$

$$=r\left(\frac{-1+e^{i\,2\,\theta}}{1-r^2\,e^{i\,2\,\theta}}\right)a_1{}^{+}$$

$r_{tot}=a_1{}^{-}/a_1{}^{+}=r\left(\frac{-1+e^{i2\theta}}{1-r^2\,e^{i2\theta}}\right)$

$t_{tot}=a_3{}^{+}/a_1{}^{+}=i\,t\,e^{i\,\theta}a_2{}^{+}/a_1{}^{+}=-\frac{t^2}{1-r^2\,e^{i2\theta}}\,e^{i\,\theta}=-\frac{(1-r^2)}{1-r^2\,e^{i2\theta}}\,e^{i\,\theta}$

$R_{tol}=\ |\,r_{tot}\,|^2=R\ \left|\ \frac{-1+e^{i2\theta}}{1-R\,e^{i2\theta}}\ \right|^2$

$T_{tol}=|\ t_{tot}\ |^2\ =\ \left|\ \frac{(1-R)}{1-Re^{i2\theta}}\ \right|^2$

Here t= $\sqrt{1-r^2}$ is the field transmission coefficient, and $\theta=n_2\,k_0\,d$ is the single-pass phase shift and $2\,\theta=2\,n_2\,k_0\,d$ is the roundtrip phase shift of the resonator.

If we define ϕ=2θ=2 $n_2\,k_0\,d$ as the roundtrip phase, then

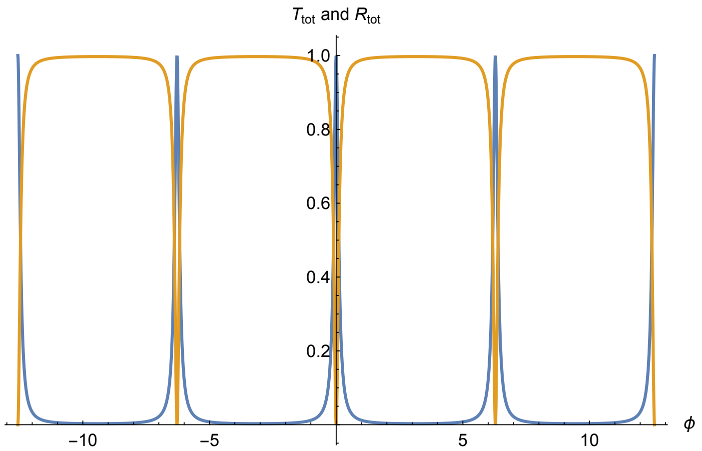
$R_{tol}(\phi)=\ |\,r_{tot}\,|^2=R\ \left|\ \frac{-1+e^{i\phi}}{1-R\,e^{i\phi}}\ \right|^2$

$T_{tol}(\phi)=|\ t_{tot}\ |^2\ =\ \left|\ \frac{(1-R)}{1-Re^{i\phi}}\ \right|^2$

R_{tol}+T_{tol}=1 (since there is no internal loss)

In[ ]:=
```
Plot[{Abs[(1 - R) / (1 - R Exp[I ϕ])] ^2 /. {R → 0.9},
  R * Abs[(-1 + Exp[I ϕ]) / (1 - R Exp[I ϕ])] ^2 /. {R → 0.9}},
 {ϕ, -4 * Pi, 4. * Pi}, AxesLabel → {"ϕ", "Ttot and Rtot"}]
```

Out[ ]=



Note that when $\phi=2\,n_2\,k_0\,d=2\,n_2\,\frac{\omega}{c}\,d=2\,n_2\,\frac{2\,\pi\,f}{c}\,d=2\,n_2\,\frac{2\,\pi}{\lambda}\,d$=2 m π,

$R_{tol}=0$

$T_{tol}=1$

If ϕ=2 m $\pi+\Delta\phi$

$T_{tol}(\Delta\phi)=\left|\ \frac{(1-R)}{1-Re^{i2\theta}}\ \right|^2=\left|\ \frac{(1-R)}{1-Re^{i\Delta\phi}}\ \right|^2$

$$\frac{(1-R)}{1-Re^{i\Delta\phi}} \approx \frac{(1-R)}{1-R(1+i\Delta\phi)} = \frac{(1-R)}{(1-R)-iR\Delta\phi}$$

$$\left|\frac{(1-R)}{1-Re^{i\Delta\phi}}\right|^2 = \frac{(1-R)^2}{(1-R)^2+R^2\Delta\phi^2} = \frac{\frac{(1-R)^2}{R^2}}{\frac{(1-R)^2}{R^2}+\Delta\phi^2} = \frac{1}{1+\left(\frac{\Delta\phi}{W_\phi}\right)^2}$$

$$\frac{1}{1+\left(\frac{\Delta\phi}{W_\phi}\right)^2} = \frac{1}{2} \text{ if } \Delta\phi = W_\phi$$

Lorentzian spectral shape

$$W_{\Delta\phi} = \frac{(1-R)}{R}$$

$$FWHM_{\Delta\phi} = 2W_{\Delta\phi} = 2\frac{(1-R)}{R}$$

The derived transmission spectrum is a Lorentzian function. This approximation is quite valid when the $FWHM_{\Delta\phi}$ is small compared to 2π, which is the free spectral range in term of ϕ. The period of the periodic spectral transmission peaks is called the free spectral range (FSR). Since

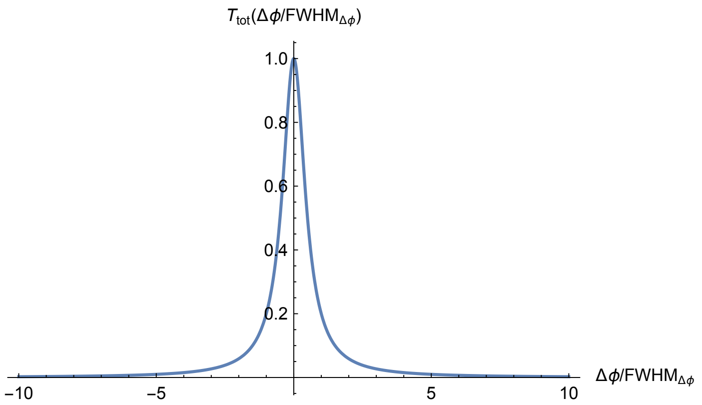
$\phi = 2 n_2 k_0 d = 2 n_2 \omega d / c$

$FSR_\omega = FSR_\phi / (2 n_2 d / c) = \pi c / (n_2 d)$

$FSR_f = \frac{1}{2\pi} FSR_\omega = c/(2 n_2 d) = 1/T_R$

Here $T_R = (2 n_2 d)/c$ is the roundtrip time.

```
Plot[1/(1 + (2 * x)^2), {x, -10, 10}, PlotRange → All,
 AxesLabel → {"Δϕ/FWHMΔϕ", "Ttot(Δϕ/FWHMΔϕ)"}]
```



8. Fabry-Perot resonator as a sensor

Note that the roundtrip phase inside the resonator is given by $\Delta\phi = 2 n_2 k_0 d - 2 m \pi$.
So when n_2, $k_0 = \frac{2\pi}{\lambda} = \frac{\omega}{c}$, or d is changed, $\Delta\phi$ will be changed. This property can be utilized for optical sensors.

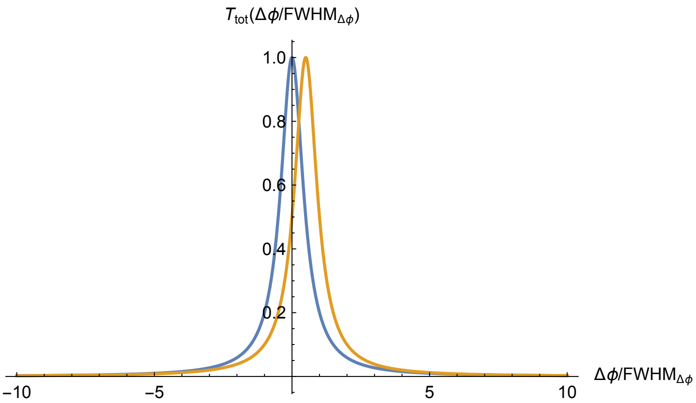
For example, if we want to detect the change of d, then the resonance frequency of the Fabry-Perot resonator will be changed when d is changed. If the change is small, one can analyze the frequency shift by linearization as follows:

$\phi = 2\, n_2\, k_0\, d - 2\, m\, \pi$

$\Delta\phi$=$2\, n_2\, \frac{\omega}{c}\, \Delta\mathrm{d} + 2\, n_2\, \frac{\Delta\omega}{c}\, d$=0 --> $\Delta\omega = -\, \omega\, \frac{\Delta \mathrm{d}}{d}$

As a rough estimate, this center frequency shift needs to be larger than the half FWHM bandwidth ($\mathrm{FWHM}_{\Delta\omega}/2$) of the resonator in order to be resolved.

```
Plot[{1/(1 + (2 * x)^2), 1/(1 + (2 * (x - 0.5))^2)}, {x, -10, 10},
 PlotRange → All, AxesLabel → {"Δϕ/FWHM_Δϕ", "T_tot (Δϕ/FWHM_Δϕ)"}]
```



$$\mathrm{FWHM}_{\Delta\phi} = 2\, \mathrm{W}_{\Delta\phi} = 2\, \frac{(1-\mathrm{R})}{\mathrm{R}}$$

$$\mathrm{FWHM}_{\Delta\omega} = 2\, \frac{(1-\mathrm{R})}{\mathrm{R}}\, \frac{1}{2\, \mathrm{n}_2\, \mathrm{d}/\mathrm{c}} = \frac{(1-\mathrm{R})}{\mathrm{R}}\, \frac{\mathrm{c}}{\mathrm{n}_2\, \mathrm{d}}$$

$$\omega\, \frac{\Delta \mathrm{d}}{d} \geq \frac{1}{2}\, \frac{(1-\mathrm{R})}{\mathrm{R}}\, \frac{\mathrm{c}}{\mathrm{n}_2\, \mathrm{d}}$$

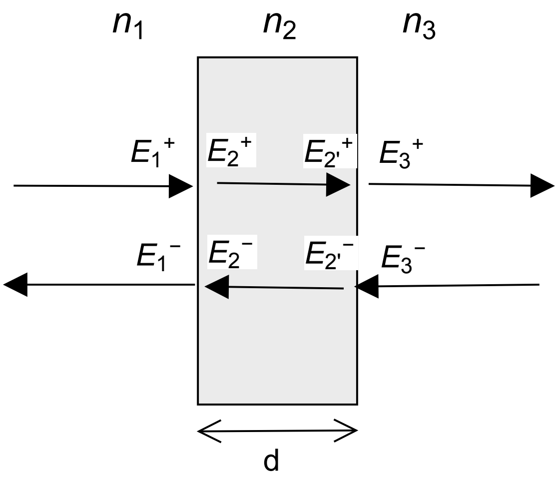
$$\frac{\Delta \mathrm{d}}{d} \geq \frac{1}{2}\, \frac{(1-\mathrm{R})}{\mathrm{R}}\, \frac{\mathrm{c}}{\mathrm{n}_2\, \mathrm{d}\, \omega} = \frac{1}{2}\, \frac{(1-\mathrm{R})}{\mathrm{R}}\, \frac{\lambda}{2\, \pi\, \mathrm{n}_2\, \mathrm{d}}$$

$$\Delta \mathrm{d} \geq \frac{1}{2}\, \frac{(1-\mathrm{R})}{\mathrm{R}}\, \frac{\mathrm{c}}{\mathrm{n}_2\, \mathrm{d}\, \omega} = \frac{1}{2}\, \frac{(1-\mathrm{R})}{\mathrm{R}}\, \frac{\lambda}{2\, \pi\, \mathrm{n}_2}$$

This is the theoretical sensitivity of such measurement.

The same principle can be applied to detect the change of n_2. As an exercise, you can try to derive the theoretical sensitivity of detecting the change of n_2 by a Fabry-Perot resonator.

9. Numerical calculation of normal incidence



$$\begin{pmatrix} E_3^+ \\ E_3^- \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is the total transfer matrix.

Since $E_3^- = 0$, so

$M_{21} E_1^+ + M_{22} E_1^- = 0$

Therefore,

reflection coef : $r = E_1^- / E_1^+ = -M_{21} / M_{22}$

transmission coef : $t = E_3^+ / E_1^+ = M_{11} - M_{12} M_{21} / M_{22}$

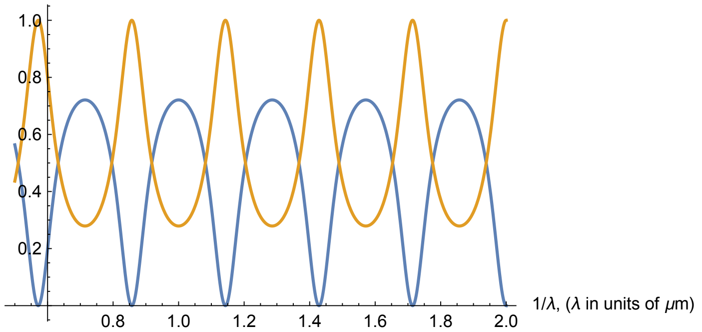
$$\text{TNI[n2\_, n1\_] :=} \begin{pmatrix} \frac{n2+n1}{2\,n2} & \frac{n2-n1}{2\,n2} \\ \frac{n2-n1}{2\,n2} & \frac{n2+n1}{2\,n2} \end{pmatrix};$$

$$\text{TND[n\_, d\_, k\_] :=} \begin{pmatrix} E^{I\,n\,k\,d} & 0 \\ 0 & E^{-I\,n\,k\,d} \end{pmatrix};$$

```
TN[nl_, dl_, k_] := Block[{M},
  M = TNI[nl⟦2⟧, nl⟦1⟧];
  Do[
   M = TNI[nl⟦i + 2⟧, nl⟦i + 1⟧].TND[nl⟦i + 1⟧, dl⟦i⟧, k].M;,
   {i, Length[dl]}];
  Return[{-M⟦2, 1⟧ / M⟦2, 2⟧, M⟦1, 1⟧ - M⟦1, 2⟧ × M⟦2, 1⟧ / M⟦2, 2⟧ }];]
```

Example : Fabry-Perot etalon

```
pa = {n1 → 1., n2 → 3.5, n3 → 1., d → 0.5};
Plot[{Abs[TN[{n1, n2, n3}, {d}, 2 Pi λinv][[1]]]^2 /. pa,
   Abs[TN[{n1, n2, n3}, {d}, 2 Pi λinv][[2]]]^2 /. pa},
 {λinv, 0.5, 2.}, AxesLabel → {"1/λ, (λ in units of μm)"}]
```



$\mathrm{FSR}_f = c/(2\, n_2\, d)$
Since $f\,\lambda = c$,
$\mathrm{FSR}_{1/\lambda} = 1/(2\, n_2\, d) = 1/(2 \times 3.5 \times 0.5\ \mu\mathrm{m}) = 0.286\ \mu\mathrm{m}^{-1}$

```
1 / (2 × 3.5 × 0.5)
0.285714
```

10. Numerical calculation of oblique TE incidence

To generalize the transfer matrix formula to the oblique TE incidence, one only needs to adopt the following substitution:

$n_1 \to k_{1,z}/k_0 = \left(n_1^2\, k_0^2 - k_x^2 - k_y^2\right)^{1/2}/k_0 = \left(n_1^2 - n_t^2\right)^{1/2}$

$n_2 \to k_{2,z}/k_0 = \left(n_2^2\, k_0^2 - k_x^2 - k_y^2\right)^{1/2}/k_0 = \left(n_2^2 - n_t^2\right)^{1/2}$

$n_3 \to k_{3,z}/k_0 = \left(n_3^2\, k_0^2 - k_x^2 - k_y^2\right)^{1/2}/k_0 = \left(n_3^2 - n_t^2\right)^{1/2}$

Here

$n_t^2 = \left(k_x^2 + k_y^2\right)/k_0^2 = n_1^2 \sin^2(\theta_1) = n_2^2 \sin^2(\theta_2) = n_3^2 \sin^2(\theta_3)$

In[∘]:=

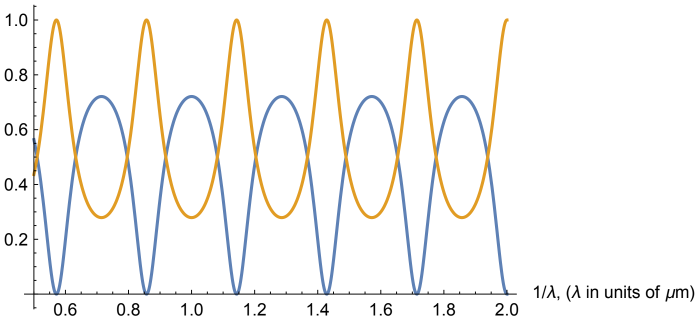
```
TEI[n2_, n1_, nt_] :=
  ( (n2+n1)/(2 n2)   (n2-n1)/(2 n2)
    (n2-n1)/(2 n2)   (n2+n1)/(2 n2) ) /. {n1 → Sqrt[n1^2 - nt^2], n2 → Sqrt[n2^2 - nt^2]};

TED[n_, d_, k_, nt_] := ( E^(I Sqrt[n^2-nt^2] k d)   0
                          0                           E^(-I Sqrt[n^2-nt^2] k d) );

TE[nl_, dl_, k_, nt_] := Block[{M},
  M = TEI[nl[[2]], nl[[1]], nt];
  Do[
   M = TEI[nl[[i + 2]], nl[[i + 1]], nt].TED[nl[[i + 1]], dl[[i]], k, nt].M;,
   {i, Length[dl]}];
  Return[{-M[[2, 1]] / M[[2, 2]], M[[1, 1]] - M[[1, 2]] × M[[2, 1]] / M[[2, 2]] }];]
```

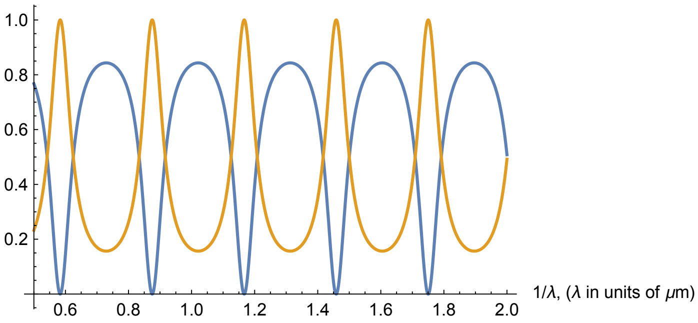
In[ ]:=
```
pa = {n1 → 1., n2 → 3.5, n3 → 1., d → 0.5, nt → 1.0 * Sin[Pi / 180. * 0.]};
Plot[{Abs[TE[{n1, n2, n3}, {d}, 2 Pi λinv, nt][[1]]]^2 /. pa,
  Abs[TE[{n1, n2, n3}, {d}, 2 Pi λinv, nt][[2]]]^2 /. pa},
 {λinv, 0.5, 2.}, AxesLabel → {"1/λ, (λ in units of μm)"}]
```

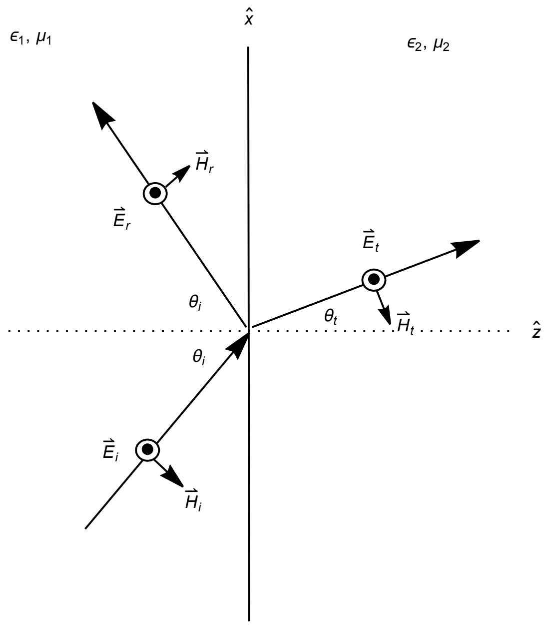
Out[ ]=



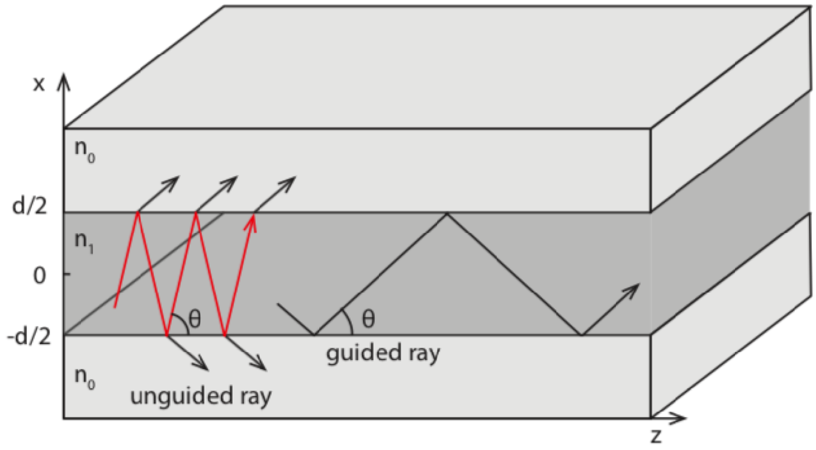
In[ ]:=
```
pa = {n1 → 1., n2 → 3.5, n3 → 1., d → 0.5, nt → 1.0 * Sin[Pi / 180. * 45.]};
Plot[{Abs[TE[{n1, n2, n3}, {d}, 2 Pi λinv, nt][[1]]]^2 /. pa,
  Abs[TE[{n1, n2, n3}, {d}, 2 Pi λinv, nt][[2]]]^2 /. pa},
 {λinv, 0.5, 2.}, AxesLabel → {"1/λ, (λ in units of μm)"}]
```

Out[ ]=

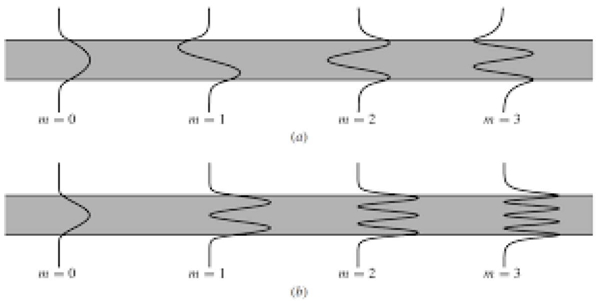




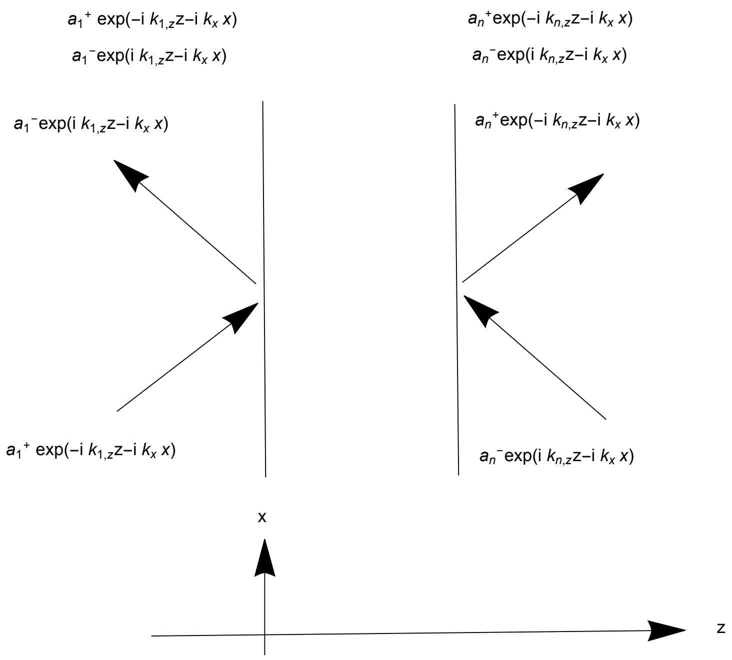
3. TE Dielectric Slab Waveguide



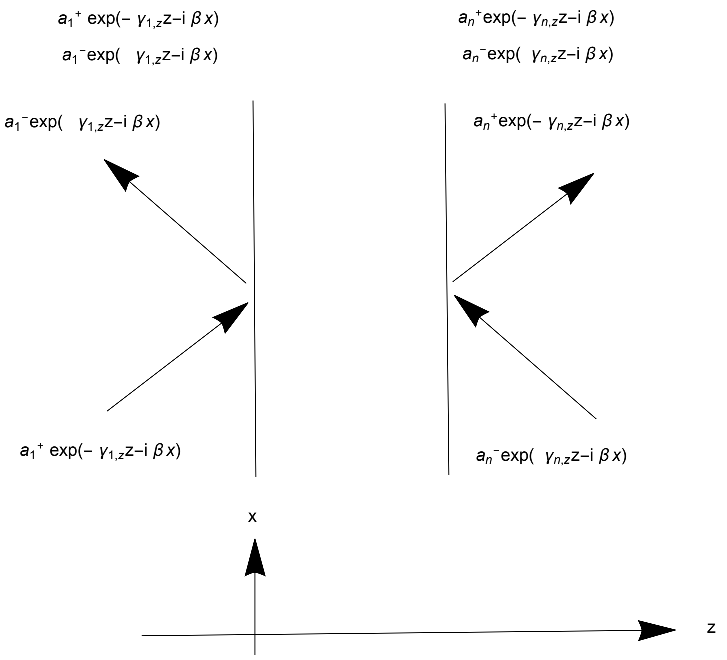
An infite dielectric slab wave guide is defined by some material n_1 surrounded by refractive index material n_0, as shown above.



Thin film optics



Thin film waveguides



For thin film waveguides, if $n_t^2 = n_{eff}^2$ where $n_{eff} = \beta/k_0$ is the effective mode index for the TE waveguide modes, then the above transfer matrix formula can also be used for analyzing the TE thin film waveuide modes. In this case,

$k_{1,z} = \sqrt{k_1^2 - \beta^2} = i\,\gamma_{1,z}$

$k_{2,z} = \sqrt{k_2^2 - \beta^2}$

$k_{3,z} = \sqrt{k_3^2 - \beta^2} = i\,\gamma_{3,z}$

$n_1 \to k_{1,z}/k_0 = i\left(n_{eff}^2 - n_1^2\right)^{1/2}$

$n_2 \to k_{2,z}/k_0 = \left(n_2^2 - n_{eff}^2\right)^{1/2}$

$n_3 \to k_{3,z}/k_0 = i\left(n_{eff}^2 - n_3^2\right)^{1/2}$

$$\begin{pmatrix} E_3^+ \\ E_3^- \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} E_1^+ \\ E_1^- \end{pmatrix}$$

Since $E_3^+(z) \propto e^{i(i\gamma_{3,z})z} = e^{-\gamma_{3,z}z}$, $E_3^-(z) \propto e^{-i(i\gamma_{3,z})z} = e^{\gamma_{3,z}z}$,
for guided modes, $E_3^- = 0$ (also $E_1^+ = 0$), therefore the condition for guided TE modes to exist is
$M_{22} = 0$

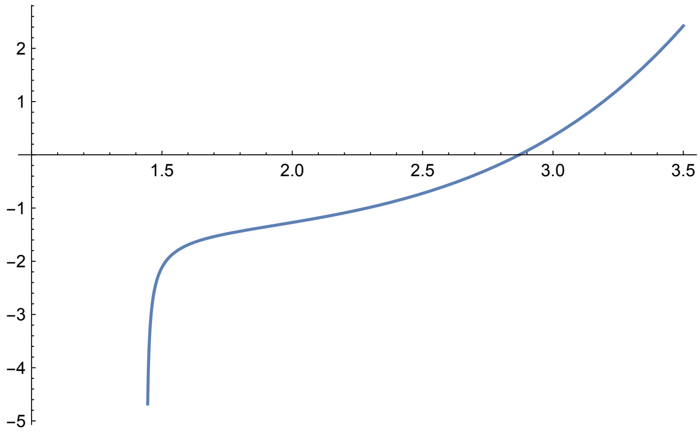
$$\texttt{TI[n2\_, n1\_] :=} \begin{pmatrix} \frac{n2+n1}{2\,n2} & \frac{n2-n1}{2\,n2} \\ \frac{n2-n1}{2\,n2} & \frac{n2+n1}{2\,n2} \end{pmatrix};$$

$$\texttt{TD[n\_, d\_, k\_] :=} \begin{pmatrix} E^{I\,n\,k\,d} & 0 \\ 0 & E^{-I\,n\,k\,d} \end{pmatrix};$$

```
TE[nl_, dl_, k_, neff_] := Block[{M},
   M = TI[Sqrt[nl〚2〛^2 - neff^2 ], Sqrt[nl〚1〛^2 - neff^2]];
   Do[
    M = TI[Sqrt[nl〚i + 2〛^2 - neff^2], Sqrt[nl〚i + 1〛^2 - neff^2 ]].
       TD[Sqrt[nl〚i + 1〛^2 - neff^2 ], dl〚i〛, k].M;,
    {i, 1, Length[dl]}];
   Return[M〚2, 2〛];];
```

Single waveguide :

```
Plot[TE[{1.44, 3.5, 1.44}, {0.22}, 2 Pi / 1.55, neff], {neff, 1., 3.5}]
```



```
FindRoot[TE[{1.44, 3.5, 1.44}, {0.22}, 2 Pi / 1.55, neff] == 0, {neff, 3.4}]
```

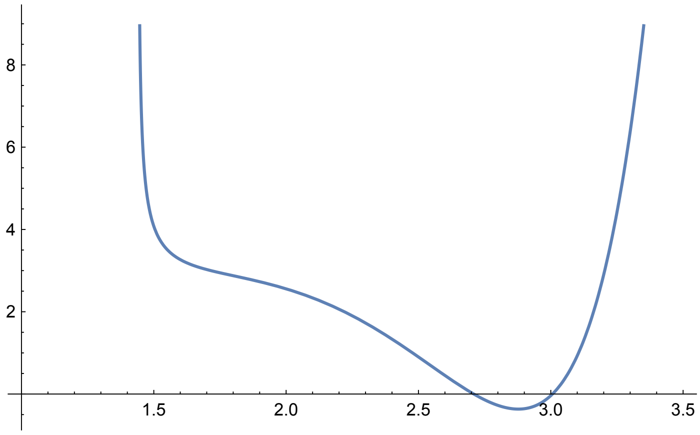
$\{neff \to 2.87117 + 1.12656 \times 10^{-17}\, i\}$

Physical insights :

- When the refractive index of the core is larger than the cladding (the surrounding layers), the guided modes exist and the analysis can also be carried out based on the transfer matrix method.

Coupled waveguides :

```
Plot[TE[{1.44, 3.5, 1.44, 3.5, 1.44},
  {0.22, 0.1, 0.22}, 2 Pi / 1.55, neff], {neff, 1., 3.5}]
```



```
FindRoot[TE[{1.44, 3.5, 1.44, 3.5, 1.44}, {0.22, 0.1, 0.22}, 2 Pi / 1.55, neff] == 0,
 {neff, 3.49}]
```

$\{neff \to 3.0063 - 1.76073 \times 10^{-17}\, i\}$

```
FindRoot[TE[{1.44, 3.5, 1.44, 3.5, 1.44}, {0.22, 0.1, 0.22}, 2 Pi / 1.55, neff] == 0,
 {neff, 2.8}]
```

$\{neff \to 2.70894 - 2.99651 \times 10^{-17}\, i\}$

Physical insights :

- The coupling of two originally degenerate modes produce two new modes. one mode (the symmetric mode) is with a larger effective index and another mode (the anti-symmetric mode) is with a smaller effective index
- The stronger the coupling is, the larger the effective index separation is.

11. Numerical calculation of oblique TM incidence

To generalize the transfer matrix formula to the oblique TM incidence, one only needs to adopt the following substitution:

$$n_1 \to \frac{k_{1,z}}{n_1^2\, k_0} = \frac{\left(n_1^2 - n_t^2\right)^{1/2}}{n_1^2}$$

$$n_2 \to \frac{k_{2,z}}{n_2^2\, k_0} = \frac{\left(n_2^2 - n_t^2\right)^{1/2}}{n_2^2}$$

$$n_3 \to \frac{k_{3,z}}{n_3^2\, k_0} = \frac{\left(n_3^2 - n_t^2\right)^{1/2}}{n_3^2}$$

Here

$$n_t^2 = \left(k_x^2 + k_y^2\right)/k_0^2 = n_1^2 \sin^2(\theta_1) = n_2^2 \sin^2(\theta_2) = n_3^2 \sin^2(\theta_3)$$

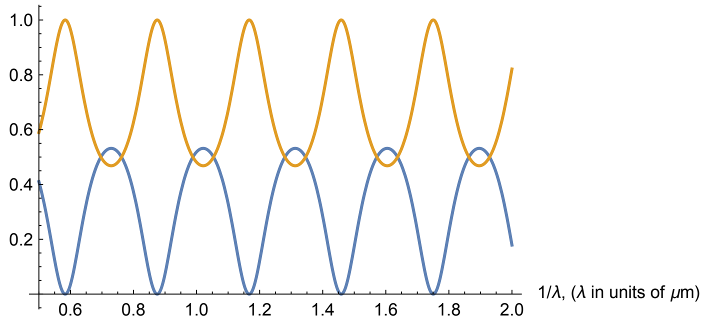
```
In[]:= TMI[n2_, n1_, nt_] :=
  ( (n2+n1)/(2 n2)   (n2-n1)/(2 n2)
    (n2-n1)/(2 n2)   (n2+n1)/(2 n2) ) /. {n1 → Sqrt[n1^2 - nt^2] / n1^2, n2 → Sqrt[n2^2 - nt^2] / n2^2};

TMD[n_, d_, k_, nt_] := ( E^(I Sqrt[n^2-nt^2] k d)   0
                          0   E^(-I Sqrt[n^2-nt^2] k d) );

TM[nl_, dl_, k_, nt_] := Block[{M},
  M = TMI[nl[[2]], nl[[1]], nt];
  Do[
   M = TMI[nl[[i + 2]], nl[[i + 1]], nt].TMD[nl[[i + 1]], dl[[i]], k, nt].M;,
   {i, Length[dl]}];
  Return[{-M[[2, 1]] / M[[2, 2]], M[[1, 1]] - M[[1, 2]] × M[[2, 1]] / M[[2, 2]] }];]
```

In[ ]:=
```
pa = {n1 → 1., n2 → 3.5, n3 → 1., d → 0.5, nt → 1.0 * Sin[Pi / 180. * 45.]};
Plot[{Abs[TM[{n1, n2, n3}, {d}, 2 Pi λinv, nt][[1]]]^2 /. pa,
   Abs[TM[{n1, n2, n3}, {d}, 2 Pi λinv, nt][[2]]]^2 /. pa},
  {λinv, 0.5, 2.}, AxesLabel → {"1/λ, (λ in units of μm)"}]
```

Out[ ]=



Brewster angle :

```
{θ1 / Pi * 180., θ2 / Pi * 180.} /.
 FindRoot[{Sin[θ1] == 3.5 Sin[θ2], Cos[θ1] == Cos[θ2] / 3.5},
  {θ1, Pi / 180. * 40.}, {θ2, Pi / 180. * 70.}]
```

{74.0546, 15.9454}

For thin film waveguides, if $n_t^2 = n_{eff}^2$ where $n_{eff} = \beta / k_0$ is the effective mode index for the TM waveguide modes, then the above transfer matrix formula can also be used for analyzing the TM thin film waveuide modes. In this case,

$k_{1,z} = \sqrt{k_1^2 - \beta^2} = i\,\gamma_{1,z}$

$k_{2,z} = \sqrt{k_2^2 - \beta^2}$

$k_{3,z} = \sqrt{k_3^2 - \beta^2} = i\,\gamma_{3,z}$

$n_1 \to k_{1,z} / (n_1^2 k_0) = i\,(n_{eff}^2 - n_1^2)^{1/2} / n_1^2$

$n_2 \to k_{2,z} / (n_2^2 k_0) = (n_2^2 - n_{eff}^2)^{1/2} / n_2^2$

$n_3 \to k_{3,z} / (n_3^2 k_0) = i\,(n_{eff}^2 - n_3^2)^{1/2} / n_2^2$

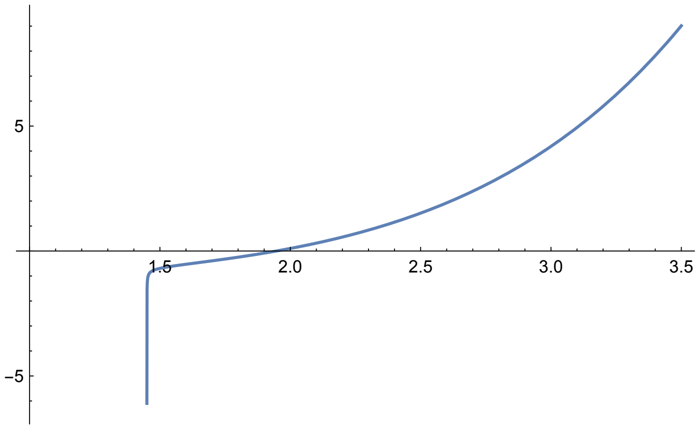
$$TI[n2\_, n1\_] := \begin{pmatrix} \frac{n2+n1}{2\,n2} & \frac{n2-n1}{2\,n2} \\ \frac{n2-n1}{2\,n2} & \frac{n2+n1}{2\,n2} \end{pmatrix};$$

$$TD[n\_, d\_, k\_] := \begin{pmatrix} E^{I\,n\,k\,d} & 0 \\ 0 & E^{-I\,n\,k\,d} \end{pmatrix};$$

```
TM[nl_, dl_, k_, neff_] := Block[{M},
   M = TI[Sqrt[nl[[2]]^2 - neff^2] / nl[[2]]^2, Sqrt[nl[[1]]^2 - neff^2]] / nl[[1]]^2;
   Do[
    M = TI[Sqrt[nl[[i + 2]]^2 - neff^2] / nl[[i + 2]]^2, Sqrt[nl[[i + 1]]^2 - neff^2] /
         nl[[i + 1]]^2].TD[Sqrt[nl[[i + 1]]^2 - neff^2], dl[[i]], k].M;,
    {i, 1, Length[dl]}];
   Return[M[[2, 2]]];];
```

Single waveguide :

```
Plot[TM[{1.45, 3.5, 1.45}, {0.22}, 2 Pi / 1.55, neff], {neff, 1., 3.5}]
```




```
FindRoot[TM[{1.45, 3.5, 1.45}, {0.22}, 2 Pi / 1.55, neff] == 0, {neff, 3.4}]
```

```
{neff → 1.94679 + 0. ⅈ}
```