电动力学

2025 年 12 月 30 日

目录

符号表

符号	含义
δ_{ij}	Kronecker　符号
$\cdot$	点乘
$\times$	叉乘
$\boldsymbol{a}$	矢量函数
f	标量函数
a_i	矢量 $\boldsymbol{a}$ 的第 i 个分量
δ_{ij}	克罗内克尔符号
$\delta(x)$	狄拉克函数
ε_{ijk}	Levi–Civita 符号 (三维)
ε_0	真空介电常数
q	电荷量
x_i	第 i 个坐标分量
e	电子电荷
$\boldsymbol{e}_i$	第 i 个单位矢量
∇	哈密顿算子
r	$\sqrt{x_i x_i}$
φ	方位角
θ	极角
$\boldsymbol{r}$	$x_i \boldsymbol{e}_i$
$\boldsymbol{R}$	旋转矩阵
$\boldsymbol{\Phi}_E$	电场强度通量
λ	电荷线密度
σ	电荷面密度
ρ	电荷体密度
$\boldsymbol{E}$	电场强度
U	电势
$\mathrm{d}\tau$	体积微元
Z	自然数集

1 矢量分析

1.1 定义

1.1.1 爱因斯坦求和约定

$$\sum_{i=1}^{3} a_i b_i \equiv a_i b_i \tag{1.1}$$

1.1.2 克罗内克尔符号

$$\delta_{ij} \equiv \begin{cases} 1 \ (i = j) \\ 0 \ (i \neq j) \end{cases} \tag{1.2}$$

性质:

$$a_i b_j \delta_{ij} = a_i b_i \tag{1.3}$$

$$\delta_{ij} \delta_{jk} = \delta_{ik} \tag{1.4}$$

1.1.3 点乘

$$\boldsymbol{a} \cdot \boldsymbol{b} \equiv a_i b_i \tag{1.5}$$

1.1.4 Levi-Civita 符号

$$\varepsilon_{ijk} \equiv \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix} \tag{1.6}$$

性质:

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik} \tag{1.7}$$

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix} \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix} \\
&= \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix}^T \begin{pmatrix} \delta_{1l} & \delta_{2l} & \delta_{3l} \\ \delta_{1m} & \delta_{2m} & \delta_{3m} \\ \delta_{1n} & \delta_{2n} & \delta_{3n} \end{pmatrix} \\
&= \begin{pmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{pmatrix} \begin{pmatrix} \delta_{1l} & \delta_{2l} & \delta_{3l} \\ \delta_{1m} & \delta_{2m} & \delta_{3m} \\ \delta_{1n} & \delta_{2n} & \delta_{3n} \end{pmatrix} \\
&= \begin{pmatrix} \delta_{1l} & \delta_{2l} & \delta_{3l} \\ \delta_{1m} & \delta_{2m} & \delta_{3m} \\ \delta_{1n} & \delta_{2n} & \delta_{3n} \end{pmatrix} \begin{pmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{pmatrix} \\
&= \begin{pmatrix} \delta_{li} & \delta_{lj} & \delta_{lk} \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \end{pmatrix}
\end{aligned} \tag{1.8}
$$

$$
\varepsilon_{ijk}\varepsilon_{lmk} = \sum_{k=1}^{3} \begin{pmatrix} \delta_{li} & \delta_{lj} & \delta_{lk} \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ki} & \delta_{kj} & 1 \end{pmatrix} = \begin{pmatrix} \delta_{li} & \delta_{lj} & 0 \\ \delta_{mi} & \delta_{mj} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi} \tag{1.9}
$$

$$
\varepsilon_{ijk}\varepsilon_{ljk} = \delta_{li}\delta_{jj} - \delta_{lj}\delta_{ji} = 3\delta_{li} - \delta_{li} = 2\delta_{li} \tag{1.10}
$$

1.1.5 叉乘

$$
\boldsymbol{a} \times \boldsymbol{b} \equiv \varepsilon_{ijk} a_i b_j \boldsymbol{e}_k = \begin{pmatrix} \boldsymbol{e}_i & \boldsymbol{e}_j & \boldsymbol{e}_k \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{pmatrix} \tag{1.11}
$$

性质:

$$\begin{aligned}
\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{c}\cdot\varepsilon_{ijk}a_ib_j\boldsymbol{e}_k \\
&= c_k\varepsilon_{ijk}a_ib_j \\
&= \varepsilon_{ijk}a_ib_jc_k \\
&= \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix} a_ib_jc_k \\
&= \begin{pmatrix} \delta_{1i}a_i & \delta_{2i}a_i & \delta_{3i}a_i \\ \delta_{1j}b_j & \delta_{2j}b_j & \delta_{3j}b_j \\ \delta_{1k}c_k & \delta_{2k}c_k & \delta_{3k}c_k \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \qquad (1.12)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{c}\times\varepsilon_{ijk}a_ib_j\boldsymbol{e}_k \\
&= \varepsilon_{lkn}c_l\varepsilon_{ijk}a_ib_j\boldsymbol{e}_n \\
&= -\varepsilon_{lnk}\varepsilon_{ijk}a_ib_jc_l\boldsymbol{e}_n \\
&= -(\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl})a_ib_jc_l\boldsymbol{e}_n \\
&= -\delta_{il}\delta_{jn}a_ib_jc_l\boldsymbol{e}_n+\delta_{in}\delta_{jl}a_ib_jc_l\boldsymbol{e}_n \\
&= -a_ib_jc_i\boldsymbol{e}_j+a_ib_jc_j\boldsymbol{e}_i \\
&= -(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a} \qquad (1.13)
\end{aligned}$$

$$\boldsymbol{e}\times(\boldsymbol{e}\times\boldsymbol{b}) = -(\boldsymbol{e}\cdot\boldsymbol{e})\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{e})\boldsymbol{e} = -\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{e})\boldsymbol{e} \qquad (1.14)$$

1.1.6 哈密顿算子

$$\boldsymbol{\nabla} \equiv \frac{\partial}{\partial x_i}\boldsymbol{e}_i \qquad (1.15)$$

1.1.7 梯度

设位置矢量 $\boldsymbol{r}=\boldsymbol{r}(u_1,u_2,u_3)$, 拉梅系数为

$$h_i \equiv \left\|\frac{\partial \boldsymbol{r}}{\partial u_i}\right\|, \qquad \boldsymbol{e} \equiv \frac{1}{h_i}\frac{\partial \boldsymbol{r}}{\partial u_i}$$

且 $\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij}$。

对标量场 $f(u_1,u_2,u_3)$, 全微分为

$$\mathrm{d}f = \frac{\partial f}{\partial u_i}\,\mathrm{d}u_i.$$

无穷小位移 (线元) 为

$$\mathrm{d}\boldsymbol{l} = h_i \ \mathrm{d}u_i \, \boldsymbol{e}_i.$$

设 $\boldsymbol{\nabla} f = G_i \boldsymbol{e}_i$, 则

$$\mathrm{d}f = \boldsymbol{\nabla} f \cdot \boldsymbol{dl} = G_i h_i \ \mathrm{d}u_i.$$

比较系数, 得 $\boldsymbol{G} = \dfrac{1}{h_i}\dfrac{\partial f}{\partial u_i}\boldsymbol{e}_i$, 从而

$$\boldsymbol{\nabla} f = \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i \tag{1.16}$$

对于直角坐标

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x_i}\boldsymbol{e}_i \tag{1.17}$$

性质:

$$\boldsymbol{\nabla} r = \frac{\partial \sqrt{x_i x_i}}{\partial x_i}\boldsymbol{e}_i = \frac{x_i}{r}\boldsymbol{e}_i = \frac{\boldsymbol{r}}{r} \tag{1.18}$$

$$\boldsymbol{\nabla} f(r) = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_i}\boldsymbol{e}_i = \frac{\partial f}{\partial r}\frac{\boldsymbol{r}}{r} \tag{1.19}$$

1.1.8 散度

对微小增量 $\mathrm{d}u_1, \mathrm{d}u_2, \mathrm{d}u_3$, 线元在三方向的实际长度分别为

$$\mathrm{d}l_1 = h_1 \, \mathrm{d}u_1, \quad \mathrm{d}l_2 = h_2 \, \mathrm{d}u_2, \quad \mathrm{d}l_3 = h_3 \, \mathrm{d}u_3$$

因此微小长方体的体元为

$$\mathrm{d}\tau = \mathrm{d}l_1 \, \mathrm{d}l_2 \, \mathrm{d}l_3 = h_1 h_2 h_3 \, \mathrm{d}u_1 \, \mathrm{d}u_2 \, \mathrm{d}u_3. \tag{1.20}$$

设矢量场在该点为

$$\boldsymbol{a} = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2 + a_3 \boldsymbol{e}_3,$$

其中 a_1, a_2, a_3 为在局部单位基矢下的分量 (均为 u_1, u_2, u_3 的函数)。

下面计算穿过微小长方体六个面的净通量。以 u_1 方向为例, 考察在 u_1 与 $u_1 + \mathrm{d}u_1$ 两个面:

面 u_1(在坐标 u 处, 向外法向为 $-\boldsymbol{e}_1$):面积元为

$$\mathrm{d}\boldsymbol{a}_1^{(-)} = -\boldsymbol{e}_1 (h_2 \, \mathrm{d}u_2)(h_3 \, \mathrm{d}u_3) = -\boldsymbol{e}_1 h_2 h_3 \, \mathrm{d}u_2 \, \mathrm{d}u_3,$$

对应的通量 (近似取该面中心处分量):

$$\Phi_1^{(-)} = \boldsymbol{a}(u_1, u_2, u_3) \cdot \mathrm{d}\boldsymbol{a}_1^{(-)} = -a_1(u_1, u_2, u_3) h_2 h_3 \, \mathrm{d}u_2 \, \mathrm{d}u_3.$$

面 $u + \mathrm{d}u$(向外法向为 $+\boldsymbol{e}_1$):面积元

$$\mathrm{d}\boldsymbol{a}_1^{(+)} = +\boldsymbol{e}_1 h_2 h_3 \, \mathrm{d}u_2 \, \mathrm{d}u_3,$$

通量 (在 $u + du$ 处):

$$\Phi_1^{(+)} = a_1(u_1 + du_1, u_2, u_3) h_2(u_1 + du_1, u_2, u_3) h_3(u_1 + du_1, u_2, u_3) \, \mathrm{d}u_2 \, \mathrm{d}u_3.$$

因此穿过这对面的净通量为

$$\begin{aligned}
\Delta\Phi_1 &= \Phi_1^{(+)} - \Phi_1^{(-)} \\
&= [a_1(u_1 + \mathrm{d}u_1, u_2, u_3) h_2(u_1 + \mathrm{d}u_1, u_2, u_3) h_3(u_1 + \mathrm{d}u_1, u_2, u_3) - a_1(u_1, u_2, u_3) h_2 h_3] \, \mathrm{d}u_2 \, \mathrm{d}u_3.
\end{aligned}$$

用泰勒展开到一阶并忽略高阶项, 得

$$\Delta\Phi_u = \frac{\partial}{\partial u_1}(h_2h_3a_1)\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 + o(\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3).$$

对 u_2 和 u_3 方向做同样的计算, 六个面的净通量近似为三项之和：

$$\Delta\Phi_{\text{total}} = \frac{\partial}{\partial u_i}\left(h_1h_2h_3\frac{a_i}{h_i}\right)\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$$

由于体元按照式(1.20)为 $\mathrm{d}\tau = h_1h_2h_3\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$, 故单位体积的通量密度 (即散度) 为极限：

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{a} &= \lim_{\Delta\tau\to 0}\frac{\Delta\Phi_{\text{total}}}{d\tau}\\ &= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_i}\left(h_1h_2h_3\frac{a_i}{h_i}\right).\end{aligned}$$

于是得到正交曲线坐标系下的散度公式：

$$\boldsymbol{\nabla}\cdot\boldsymbol{a} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_i}\left(h_1h_2h_3\frac{a_i}{h_i}\right). \tag{1.21}$$

对于直角坐标

$$\boldsymbol{\nabla}\cdot\boldsymbol{a} = \frac{\partial a_i}{\partial x_i} \tag{1.22}$$

性质:

$$\boldsymbol{\nabla}\cdot\boldsymbol{r} = 3 \tag{1.23}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{f}(r) = \frac{\partial f_i(r)}{\partial x_i} = f_i'\frac{\partial r}{\partial x_i} = \boldsymbol{f}'\cdot\frac{\boldsymbol{r}}{r} \tag{1.24}$$

1.1.9 旋度

选取微小矩形位于 u_1-u_2 平面, 固定 u_3。矩形四边的线元素：

下边: $\mathrm{d}\boldsymbol{l}_1 = h_1\,\mathrm{d}u_1\boldsymbol{e}_1,\quad$ 从$(u_1,u_2)\to(u_1+\mathrm{d}u_1,u_2)$

右边: $\mathrm{d}\boldsymbol{l}_2 = h_2\,\mathrm{d}u_2\boldsymbol{e}_2,\quad$ 从$(u_1+\mathrm{d}u_1,u_2)\to(u_1+\mathrm{d}u_1,u_2+\mathrm{d}u_2)$

上边: $\mathrm{d}\boldsymbol{l}_3 = -h_1\,\mathrm{d}u_1\boldsymbol{e}_1,\quad$ 从$(u_1+\mathrm{d}u_1,u_2+\mathrm{d}u_2)\to(u_1,u_2+\mathrm{d}u_2)$

左边: $\mathrm{d}\boldsymbol{l}_4 = -h_2du_2\boldsymbol{e}_2,\quad$ 从$(u_1,u_2+\mathrm{d}u_2)\to(u_1,u_2)$

沿每一边计算 $\boldsymbol{a}\cdot d\boldsymbol{l}$ 并做泰勒展开 (只保留一阶项):

$$\oint\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}\approx\left[\frac{\partial}{\partial u_1}(h_2a_2)-\frac{\partial}{\partial u_2}(h_1a_1)\right]\mathrm{d}u_1\,\mathrm{d}u_2$$

面元面积：

$$dS_3 = h_1h_2\,\mathrm{d}u_1\,\mathrm{d}u_2$$

旋度在 $\boldsymbol{e}_3$ 方向的分量：

$$(\boldsymbol{\nabla}\times\boldsymbol{a})_3 = \lim_{\mathrm{d}u_1,\mathrm{d}u_2\to 0}\frac{\oint\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}}{\mathrm{d}S_3} = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2a_2)-\frac{\partial}{\partial u_2}(h_1a_1)\right]$$

同理, 对于其他两个方向：

$$(\boldsymbol{\nabla}\times\boldsymbol{a})_1 = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3a_3)-\frac{\partial}{\partial u_3}(h_2a_2)\right],\quad(\boldsymbol{\nabla}\times\boldsymbol{a})_2 = \frac{1}{h_3h_1}\left[\frac{\partial}{\partial u_3}(h_1a_1)-\frac{\partial}{\partial u_1}(h_3a_3)\right]$$

最终正交曲线坐标系下旋度公式为：

$$\nabla \times \boldsymbol{a} = \frac{h_i}{h_1 h_2 h_3} \varepsilon_{ijk} \frac{\partial (h_k a_k)}{\partial u_j} \boldsymbol{e}_i \tag{1.25}$$

对于直角坐标

$$\nabla \times \boldsymbol{a} = \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \boldsymbol{e}_k \tag{1.26}$$

性质:

$$\nabla \times \boldsymbol{r} = 0 \tag{1.27}$$

$$\nabla \times (f\boldsymbol{a}) = \varepsilon_{ijk} \frac{\partial (f a_j)}{\partial x_i} \boldsymbol{e}_k = f \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \boldsymbol{e}_k + \varepsilon_{ijk} \frac{\partial f}{\partial x_i} a_j \boldsymbol{e}_k = f(\nabla \times \boldsymbol{a}) + (\nabla f) \times \boldsymbol{a} \tag{1.28}$$

1.1.10 并积

$$\nabla \boldsymbol{a} \equiv \begin{pmatrix} \frac{\partial a_1}{x_1} & \frac{\partial a_2}{x_1} & \frac{\partial a_3}{x_1} \\ \frac{\partial a_1}{x_2} & \frac{\partial a_2}{x_2} & \frac{\partial a_3}{x_2} \\ \frac{\partial a_1}{x_3} & \frac{\partial a_2}{x_3} & \frac{\partial a_3}{x_3} \end{pmatrix} = \frac{\partial a_i}{\partial x_j} \boldsymbol{e}_i \boldsymbol{e}_j \tag{1.29}$$

1.1.11 拉普拉斯算子

定义为梯度的散度, 由1.16和1.21得

$$\begin{aligned} \nabla^2 f &= \nabla \cdot (\nabla f) \\ &= \nabla \cdot \left(\frac{1}{h_i} \frac{\partial f}{\partial u_i} \right) \\ &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(h_1 h_2 h_3 \frac{\frac{1}{h_i} \frac{\partial f}{\partial u_i}}{h_i} \right) \\ &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(h_1 h_2 h_3 \frac{\partial f}{h_i^2 \partial u_i} \right) \end{aligned} \tag{1.30}$$

对于直角坐标

$$\nabla^2 = \frac{\partial^2}{\partial x_i^2} \tag{1.31}$$

性质:

$$\begin{aligned}
\boldsymbol{\nabla}^2(fg) &= \frac{\partial}{\partial x_i}\frac{\partial fg}{\partial x_i} \\
&= \frac{\partial}{\partial x_i}\left(g\frac{\partial f}{\partial x_i}+f\frac{\partial g}{\partial x_i}\right) \\
&= \frac{\partial}{\partial x_i}\left(g\frac{\partial f}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(f\frac{\partial g}{\partial x_i}\right) \\
&= \frac{\partial}{\partial x_i}\left(g\frac{\partial f}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(f\frac{\partial g}{\partial x_i}\right) \\
&= \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_i}+g\frac{\partial^2 f}{\partial x_i^2}+\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}+a\frac{\partial^2 g}{\partial x_i^2} \\
&= g\frac{\partial^2 f}{\partial x_i^2}+2\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}+f\frac{\partial^2 g}{\partial x_i^2} \\
&= g\boldsymbol{\nabla}^2 f+2(\boldsymbol{\nabla}f)\cdot(\boldsymbol{\nabla}g)+f\boldsymbol{\nabla}^2 g \qquad (1.32)
\end{aligned}$$

1.2 微分运算

1.2.1 和规则

$$\boldsymbol{\nabla}(f+g)=\boldsymbol{\nabla}f+\boldsymbol{\nabla}g \qquad (1.33)$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{a}+\boldsymbol{b})=\boldsymbol{\nabla}\cdot\boldsymbol{a}+\boldsymbol{\nabla}\cdot\boldsymbol{b} \qquad (1.34)$$

$$\boldsymbol{\nabla}\times(\boldsymbol{a}+\boldsymbol{b})=\boldsymbol{\nabla}\times\boldsymbol{a}+\boldsymbol{\nabla}\times\boldsymbol{b} \qquad (1.35)$$

1.2.2 积规则

$$\boldsymbol{\nabla}(fg)=g\boldsymbol{\nabla}f+f\boldsymbol{\nabla}g \qquad (1.36)$$

$$\begin{aligned}
&\boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b})+\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a})+(\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a} \\
=&\boldsymbol{a}\times\left(\varepsilon_{ijk}\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_k\right)+\boldsymbol{b}\times\left(\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_k\right)+\left(a_i\frac{\partial}{\partial x_i}\right)\boldsymbol{b}+\left(b_i\frac{\partial}{\partial x_i}\right)\boldsymbol{a} \\
=&\varepsilon_{lkn}a_l\varepsilon_{ijk}\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n+\varepsilon_{lkn}b_l\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n+a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j+b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&-\varepsilon_{lnk}\varepsilon_{ijk}a_l\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n-\varepsilon_{lnk}\varepsilon_{ijk}b_l\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n+a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j+b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&-(\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl})a_l\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n-(\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl})b_l\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n+a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j+b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&\left(-\delta_{il}\delta_{jn}a_l\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n+\delta_{in}\delta_{jl}a_l\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n\right)+\left(-\delta_{il}\delta_{jn}b_l\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n+\delta_{in}\delta_{jl}b_l\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n\right)+a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j+b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&\left(-a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j+a_j\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_i\right)+\left(-b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j+b_j\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_i\right)+a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j+b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&a_j\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_i+b_j\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_i \\
=&\frac{\partial(a_jb_j)}{\partial x_i}\boldsymbol{e}_i \\
=&\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{b}) \qquad (1.37)
\end{aligned}$$

$$\nabla \cdot (f\boldsymbol{a}) = \frac{\partial f a_i}{\partial x_i} = f\frac{\partial a_i}{\partial x_i} + a_i\frac{\partial f}{\partial x_i} = f\nabla \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \nabla f \tag{1.38}$$

$$\begin{aligned}
\boldsymbol{b} \cdot (\nabla \times \boldsymbol{a}) - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b}) &= \boldsymbol{b} \cdot (\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_k) - \boldsymbol{a} \cdot (\varepsilon_{ijk}\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_k) \\
&= b_k\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i} - a_k\varepsilon_{ijk}\frac{\partial b_j}{\partial x_i} \\
&= b_j\varepsilon_{ijk}\frac{\partial a_i}{\partial x_k} + a_i\varepsilon_{ijk}\frac{\partial b_j}{\partial x_k} \\
&= \frac{\partial \varepsilon_{ijk}a_i b_j}{\partial x_k} \\
&= \nabla \cdot (\varepsilon_{ijk}a_i b_j \boldsymbol{e}_k) \\
&= \nabla \cdot (\boldsymbol{a} \times \boldsymbol{b})
\end{aligned} \tag{1.39}$$

1.2.3 二阶积规则

$$\nabla \times (\nabla f) = \nabla \times (\frac{\partial f}{\partial x_i}\boldsymbol{e}_i) = \varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j}\boldsymbol{e}_k = 0 \tag{1.40}$$

$$\nabla \cdot (\nabla \times \boldsymbol{a}) = \nabla \cdot (\varepsilon_{ijk}\frac{\partial a_i}{\partial x_j}\boldsymbol{e}_k) = \varepsilon_{ijk}\frac{\partial^2 a_i}{\partial x_j \partial x_k} = 0 \tag{1.41}$$

$$\Delta f = \frac{\partial^2 f}{\partial x_i^2} \tag{1.42}$$

$$\Delta \boldsymbol{a} = \frac{\partial^2 (a_j \boldsymbol{e}_j)}{\partial x_i^2} \tag{1.43}$$

$$\begin{aligned}
\nabla \times (\nabla \times \boldsymbol{a}) &= \nabla \times (\varepsilon_{ijk}\frac{\partial a_i}{\partial x_j}\boldsymbol{e}_k) \\
&= \varepsilon_{lmn}\varepsilon_{ijl}\frac{\partial^2 a_i}{\partial x_j \partial x_m}\boldsymbol{e}_n \\
&= \varepsilon_{mnl}\varepsilon_{ijl}\frac{\partial^2 a_i}{\partial x_j \partial x_m}\boldsymbol{e}_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\frac{\partial^2 a_i}{\partial x_j \partial x_m}\boldsymbol{e}_n \\
&= \delta_{im}\delta_{jn}\frac{\partial^2 a_i}{\partial x_j \partial x_m}\boldsymbol{e}_n - \delta_{in}\delta_{jm}\frac{\partial^2 a_i}{\partial x_j \partial x_m}\boldsymbol{e}_n \\
&= \frac{\partial^2 a_i}{\partial x_j \partial x_i}\boldsymbol{e}_j - \frac{\partial^2 a_i}{\partial x_j \partial x_j}\boldsymbol{e}_i \\
&= \nabla(\nabla \cdot \boldsymbol{a}) - \Delta \boldsymbol{a}
\end{aligned} \tag{1.44}$$

1.3 积分运算

1.3.1 高斯定理

我们已经得到正交坐标系中的散度局部公式1.21。因此，对体积元 $\Delta\tau = h_1h_2h_3\,\Delta u_1\Delta u_2\Delta u_3$ 的六个面计算净通量时：

$$\Phi^{(\mathrm{out})} - \Phi^{(\mathrm{in})} = \left[\frac{\partial}{\partial u_i}\left(h_1h_2h_3\frac{a_i}{h_i}\right)\right]\Delta u_1\Delta u_2\Delta u_3 = (\nabla \cdot \boldsymbol{a})\,\Delta\tau.$$

将整个区域 V 分割成许多此类体积元, 内部公共面的通量相互抵消, 仅留下外边界 ∂V 的通量。令分割尺寸趋于零, 通量和体积和分别趋于各自积分, 故得到

$$\iiint_V (\boldsymbol{\nabla}\cdot\boldsymbol{a})\,\mathrm{d}\tau = \iint_{\partial V} \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S} \tag{1.45}$$

1.3.2 斯托克斯定理

我们已经得到任意正交坐标系中的旋度局部公式1.25 , 考虑由 (u_1, u_2) 增量生成的一个无穷小曲面元

$$\mathrm{d}\boldsymbol{S} = (h_1 h_2\,\mathrm{d}u_1\,\mathrm{d}u_2)\boldsymbol{e}_3.$$

沿此小曲面元的边界进行线积分, 可得

$$\oiint_{\partial S} \boldsymbol{a}\cdot d\boldsymbol{l} = (\nabla\times\boldsymbol{a})\cdot d\boldsymbol{S}.$$

将整个曲面分割成许多这样的小元, 内部公共边界的线积分相互抵消, 仅留下外边界 ∂S 的线积分。令分割尺寸趋于零, 总和趋于积分, 故得

$$\iint_S (\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S} = \oiint_{\partial S} \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} \tag{1.46}$$

1.4 曲面坐标系

1.4.1 柱坐标系

将柱坐标系坐标转化为直角系坐标的方式为

$$x_1 = s\cos\phi,\ x_2 = s\sin\phi,\ x_3 = x_3 \tag{1.47}$$

其中 s 是坐标点到 z 轴的距离。柱坐标系和直角系坐标无限小元转换方式为

$$\begin{pmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\\ \mathrm{d}x_3\end{pmatrix} = \begin{pmatrix}\frac{x_1}{s} & -x_2 & 0\\ \frac{x_2}{s} & x_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathrm{d}s\\ \mathrm{d}\phi\\ \mathrm{d}x_3\end{pmatrix} \tag{1.48}$$

其拉梅系数为

$$h_1 = 1,\ h_2 = s,\ h_3 = 1 \tag{1.49}$$

其无限小位移为

$$\mathrm{d}\boldsymbol{l} = h_i\,\mathrm{d}u_i\boldsymbol{e}_i = \mathrm{d}s\boldsymbol{s} + s\,\mathrm{d}\phi\boldsymbol{\phi} + \mathrm{d}x_3\boldsymbol{x}_3 \tag{1.50}$$

其体积元为

$$\mathrm{d}\tau = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}x_3 \tag{1.51}$$

梯度

$$\nabla f = \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i = \frac{\partial}{\partial s}\boldsymbol{s} + \frac{1}{s}\frac{\partial}{\partial\phi}\boldsymbol{\phi} + \frac{\partial}{\partial x_3}\boldsymbol{x}_3 \tag{1.52}$$

散度

$$\boldsymbol{\nabla}\cdot\boldsymbol{a} = \frac{1}{s}(\frac{\partial s a_s}{\partial s} + \frac{\partial a_\phi}{\partial\phi} + \frac{\partial s a_3}{\partial x_3}) = \frac{1}{s}\frac{\partial s a_s}{\partial s} + \frac{1}{s}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_3}{\partial x_3} = \frac{a_s}{s} + \frac{\partial a_s}{\partial s} + \frac{1}{s}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_3}{\partial x_3} \tag{1.53}$$

旋度

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{a} &= \frac{h_i}{h_1 h_2 h_3} \varepsilon_{ijk} \frac{\partial (h_k a_k)}{\partial u_j} \boldsymbol{e}_i \\
&= \frac{h_i}{s} \varepsilon_{ijk} \frac{\partial (h_k a_k)}{\partial u_j} \boldsymbol{e}_i \\
&= \frac{1}{s} \left(\varepsilon_{123} \frac{\partial a_3}{\partial u_2} \boldsymbol{e}_1 + \varepsilon_{132} \frac{\partial s a_2}{\partial u_3} \boldsymbol{e}_1 + \varepsilon_{321} \frac{\partial a_1}{\partial u_2} \boldsymbol{e}_3 + s \varepsilon_{213} \frac{\partial a_3}{\partial u_1} \boldsymbol{e}_2 + \varepsilon_{312} \frac{\partial s a_2}{\partial u_1} \boldsymbol{e}_3 + s \varepsilon_{231} \frac{\partial a_1}{\partial u_3} \boldsymbol{e}_2 \right) \\
&= \frac{1}{s} \left(\frac{\partial a_3}{\partial u_2} \boldsymbol{e}_1 - \frac{\partial s a_2}{\partial u_3} \boldsymbol{e}_1 - \frac{\partial a_1}{\partial u_2} \boldsymbol{e}_3 - s \frac{\partial a_3}{\partial u_1} \boldsymbol{e}_2 + \frac{\partial s a_2}{\partial u_1} \boldsymbol{e}_3 + s \frac{\partial a_1}{\partial u_3} \boldsymbol{e}_2 \right) \\
&= \frac{1}{s} \left(\frac{\partial a_3}{\partial \phi} \boldsymbol{s} - \frac{\partial s a_\phi}{\partial x_3} \boldsymbol{s} - \frac{\partial a_s}{\partial \phi} \boldsymbol{e}_3 - s \frac{\partial a_3}{\partial s} \boldsymbol{\phi} + \frac{\partial s a_\phi}{\partial s} \boldsymbol{e}_3 + s \frac{\partial a_s}{\partial x_3} \boldsymbol{\phi} \right) \\
&= \frac{1}{s} \left(\frac{\partial a_3}{\partial \phi} - s \frac{\partial a_\phi}{\partial x_3} \right) \boldsymbol{s} + \left(\frac{\partial a_s}{\partial x_3} - \frac{\partial a_3}{\partial s} \right) \boldsymbol{\phi} + \frac{1}{s} \left(\frac{\partial s a_\phi}{\partial s} - \frac{\partial a_s}{\partial \phi} \right) \boldsymbol{e}_3
\end{aligned} \tag{1.54}
$$

拉普拉斯算子

$$
\begin{aligned}
\boldsymbol{\nabla}^2 f &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(h_1 h_2 h_3 \frac{\partial f}{h_i^2 \partial u_i} \right) \\
&= \frac{1}{s} \frac{\partial}{\partial u_i} \left(s \frac{\partial f}{h_i^2 \partial u_i} \right) \\
&= \frac{1}{s} \frac{\partial}{\partial u_s} \left(s \frac{\partial f}{h_s^2 \partial u_s} \right) + \frac{1}{s} \frac{\partial}{\partial \phi} \left(s \frac{\partial f}{h_\phi^2 \partial \phi} \right) + \frac{1}{s} \frac{\partial}{\partial u_3} \left(s \frac{\partial f}{h_3^2 \partial u_3} \right) \\
&= \frac{1}{s} \frac{\partial}{\partial u_s} \left(s \frac{\partial f}{\partial u_s} \right) + \frac{1}{s} \frac{\partial}{\partial \phi} \left(s \frac{\partial f}{s^2 \partial \phi} \right) + \frac{1}{s} \frac{\partial}{\partial u_3} \left(s \frac{\partial f}{\partial u_3} \right) \\
&= \frac{1}{s} \frac{\partial}{\partial u_s} \left(s \frac{\partial f}{\partial u_s} \right) + \frac{1}{s^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial u_3^2}
\end{aligned} \tag{1.55}
$$

1.4.2 球坐标系

将球坐标系坐标转化为直角系坐标的方式为

$$
x_1 = r \cos\phi \sin\theta,\ x_2 = r \sin\phi \sin\theta,\ x_3 = r \cos\theta \tag{1.56}
$$

球坐标系和直角系坐标无限小元转换方式为

$$
\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{pmatrix} = \begin{pmatrix} \dfrac{x_1}{r} & -x_2 & \dfrac{x_1 x_3}{r \sin\theta} \\ \dfrac{x_2}{r} & x_1 & \dfrac{x_2 x_3}{r \sin\theta} \\ \dfrac{x_3}{r} & 0 & -r \sin\theta \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\phi \\ \mathrm{d}\theta \end{pmatrix} \tag{1.57}
$$

其拉梅系数为

$$
h_1 = 1,\ h_2 = r \sin\theta,\ h_3 = r \tag{1.58}
$$

其无限小位移为

$$
\mathrm{d}\boldsymbol{l} = \mathrm{d}r \boldsymbol{r} + r \sin\theta \, \mathrm{d}\phi \boldsymbol{\phi} + r \, \mathrm{d}\theta \boldsymbol{\theta} \tag{1.59}
$$

其体积元为

$$
\mathrm{d}\tau = r \sin\theta r \, \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{1.60}
$$

梯度

$$
\nabla f = \frac{1}{h_i} \frac{\partial f}{\partial u_i} \boldsymbol{e}_i = \frac{\partial}{\partial r} \boldsymbol{r} + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \boldsymbol{\phi} + \frac{1}{r} \frac{\partial}{\partial \theta} \boldsymbol{\theta} \tag{1.61}
$$

散度

$$
\begin{aligned}
\nabla\cdot\boldsymbol{a} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial r^2\sin\theta a_r}{\partial r}+\frac{\partial r a_\phi}{\partial\phi}+\frac{\partial r\sin\theta a_\theta}{\partial\theta}\right)\\
&= \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial r^2 a_r}{\partial r}+r\frac{\partial a_\phi}{\partial\phi}+r\frac{\partial\sin\theta a_\theta}{\partial\theta}\right)\\
&= \frac{1}{r^2}\frac{\partial r^2 a_r}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial\sin\theta a_\theta}{\partial\theta} \qquad (1.62)
\end{aligned}
$$

旋度

$$
\begin{aligned}
&\nabla\times\boldsymbol{a} \qquad (1.63)\\
=&\frac{h_i}{h_1h_2h_3}\varepsilon_{ijk}\frac{\partial(h_k a_k)}{\partial u_j}\boldsymbol{e}_i\\
=&\frac{h_i}{r^2\sin\theta}\varepsilon_{ijk}\frac{\partial(h_k a_k)}{\partial u_j}\boldsymbol{e}_i\\
=&\frac{1}{r^2\sin\theta}\left(\varepsilon_{123}\frac{\partial r a_3}{\partial u_2}\boldsymbol{e}_1+\varepsilon_{132}\frac{\partial s a_2}{\partial u_3}\boldsymbol{e}_1+r\varepsilon_{321}\frac{\partial a_1}{\partial u_2}\boldsymbol{e}_3+s\varepsilon_{213}\frac{\partial r a_3}{\partial u_1}\boldsymbol{e}_2+r\varepsilon_{312}\frac{\partial s a_2}{\partial u_1}\boldsymbol{e}_3+s\varepsilon_{231}\frac{\partial a_1}{\partial u_3}\boldsymbol{e}_2\right)\\
=&\frac{1}{r^2\sin\theta}\left(\frac{\partial r a_3}{\partial u_2}\boldsymbol{e}_1-\frac{\partial r\sin\theta a_2}{\partial u_3}\boldsymbol{e}_1-r\frac{\partial a_1}{\partial u_2}\boldsymbol{e}_3-r\sin\theta\frac{\partial r a_3}{\partial u_1}\boldsymbol{e}_2+r\frac{\partial r\sin\theta a_2}{\partial u_1}\boldsymbol{e}_3+r\sin\theta\frac{\partial a_1}{\partial u_3}\boldsymbol{e}_2\right)\\
=&\frac{1}{r^2\sin\theta}\left(\frac{\partial r a_\theta}{\partial\phi}\boldsymbol{r}-\frac{\partial r\sin\theta a_\phi}{\partial\theta}\boldsymbol{r}-r\frac{\partial a_r}{\partial\phi}\boldsymbol{\theta}-r\sin\theta\frac{\partial r a_\theta}{\partial r}\boldsymbol{\phi}+r\frac{\partial r\sin\theta a_\phi}{\partial r}\boldsymbol{\theta}+r\sin\theta\frac{\partial a_r}{\partial\theta}\boldsymbol{\phi}\right)\\
=&\frac{1}{r^2\sin\theta}\left(r\frac{\partial a_\theta}{\partial\phi}-r\frac{\partial\sin\theta a_\phi}{\partial\theta}\right)\boldsymbol{r}+\frac{1}{r}\left(\frac{\partial a_r}{\partial\theta}-\frac{\partial r a_\theta}{\partial r}\right)\boldsymbol{\phi}+\frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial r a_\phi}{\partial r}-\frac{\partial a_r}{\partial\phi}\right)\boldsymbol{\theta}\\
=&\frac{1}{r\sin\theta}\left(\frac{\partial a_\theta}{\partial\phi}-\frac{\partial\sin\theta a_\phi}{\partial\theta}\right)\boldsymbol{r}+\frac{1}{r}\left(\frac{\partial a_r}{\partial\theta}-\frac{\partial r a_\theta}{\partial r}\right)\boldsymbol{\phi}+\frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial r a_\phi}{\partial r}-\frac{\partial a_r}{\partial\phi}\right)\boldsymbol{\theta} \qquad (1.64)
\end{aligned}
$$

拉普拉斯算子

$$
\begin{aligned}
\nabla^2 f &= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_i}\left(h_1h_2h_3\frac{\partial f}{h_i^2\partial u_i}\right)\\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial u_i}\left(r^2\sin\theta\frac{\partial f}{h_i^2\partial u_i}\right)\\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\sin\theta\partial\phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \qquad (1.65)
\end{aligned}
$$

1.5 狄拉克函数与阶跃函数

1.5.1 狄拉克函数

严格来说, 狄拉克 δ 函数并不是一个通常意义下的函数, 而是通过其在积分中的作用来定义的。设 $f(x)$ 是在 $x=0$ 附近连续, 并且在无穷远处足够快衰减的函数。定义狄拉克 δ 满足

$$\int_{-\infty}^{\infty}\delta(x)\,f(x)\,\mathrm{d}x=f(0), \qquad (1.66)$$

由定义立即得到

$$\int_{-\infty}^{\infty}\delta(x-a)\,f(x)\,\mathrm{d}x=f(a). \qquad (1.67)$$

设 $g(x)$ 在 x_i 处有孤立零点, 且 $g'(x_i)\neq 0$, 则

$$\delta(g(x))=\sum_i\frac{\delta(x-x_i)}{|g'(x_i)|}. \qquad (1.68)$$

1.5.2 阶跃函数

定义阶跃函数

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \tag{1.69}$$

1.6 阶跃函数的导数

在通常意义下, $H(x)$ 在 $x = 0$ 不可导。但在积分意义下, 我们定义

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}H(x)}{\mathrm{d}x} f(x)\ \mathrm{d}x \equiv -\int_{-\infty}^{\infty} H(x) \frac{\mathrm{d}f}{\mathrm{d}x}\ \mathrm{d}x. \tag{1.70}$$

对右侧分部积分:

$$\begin{aligned} -\int_{-\infty}^{\infty} H(x)\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x &= -\int_{0}^{\infty} \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x \\ &= f(0). \end{aligned} \tag{1.71}$$

于是得到

$$\frac{\mathrm{d}H(x)}{\mathrm{d}x} = \delta(x), \tag{1.72}$$

1.6.1 多维狄拉克函数

三维 δ 函数定义为

$$\delta^{(3)}(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z), \tag{1.73}$$

并满足

$$\iiint \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_0) f(\boldsymbol{r})\ \mathrm{d}\tau = f(\boldsymbol{r}_0). \tag{1.74}$$

在正交曲线坐标 (u_1, u_2, u_3) 中,

$$\mathrm{d}\tau = h_1 h_2 h_3\ \mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3,$$

因此定义

$$\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_0) = \frac{\delta(u_1 - u_{1,0})\delta(u_2 - u_{2,0})\delta(u_3 - u_{3,0})}{h_1 h_2 h_3}. \tag{1.75}$$

1.7 矢量场理论

1.7.1 亥姆霍兹定理

设矢量场 $\boldsymbol{a}(\boldsymbol{r})$ 定义在整个空间中, 其散度与旋度分别给定为

$$\boldsymbol{\nabla} \cdot \boldsymbol{a}(\boldsymbol{r}) = f(\boldsymbol{r}), \qquad \boldsymbol{\nabla} \times \boldsymbol{a}(\boldsymbol{r}) = \boldsymbol{b}(\boldsymbol{r}).$$

为了使体积分有意义, 仅要求 $f(\boldsymbol{r})$ 与 $\boldsymbol{b}(\boldsymbol{r})$ 在无穷远处趋于零是不够的, 它们的衰减速度必须足够快。

考虑定义在整个空间上的积分

$$\iiint_V X(\boldsymbol{r})\,\mathrm{d}\tau,$$

其中 $X(\boldsymbol{r})$ 表示 $f(\boldsymbol{r})$ 或 $\boldsymbol{b}(\boldsymbol{r})$ 的某一分量。

将积分写为球坐标形式,

$$\iiint_V X(\boldsymbol{r})\,\mathrm{d}\tau = \int_0^\infty \left(\iint_S X(r,\Omega) r^2\,\mathrm{d}\Omega\right)\mathrm{d}r.$$

若存在常数 $A>0$ 及 $R>0$, 使得当 $r>R$ 时对所有方向 Ω 成立

$$|X(r,\Omega)| \geq \frac{A}{r},$$

则有下界估计

$$\iiint_{|\boldsymbol{r}|>R} |X(\boldsymbol{r})|\,\mathrm{d}\tau \geq A\int_R^\infty r\,\mathrm{d}r \iint_S \mathrm{d}\Omega,$$

该积分显然发散。

类似地, 若

$$|X(r,\Omega)| \geq \frac{A}{r^2},$$

则径向积分包含

$$\int_R^\infty \frac{\mathrm{d}r}{r},$$

从而发散。

因此, 为保证体积分收敛, 必须要求

$$X(\boldsymbol{r}) = o\left(\frac{1}{r^2}\right), \qquad r\to\infty, \tag{1.76}$$

该条件需在角向上一致成立。

因此, 为保证积分收敛, 必须要求

$$f(\boldsymbol{r}),\ \boldsymbol{b}(\boldsymbol{r}) = o\left(\frac{1}{r^2}\right), \qquad r\to\infty. \tag{1.77}$$

这一条件同时也足以保证在无穷远处的曲面积分为零。

设矢量场 $\boldsymbol{a}(\boldsymbol{r})$ 具有散度 $f(\boldsymbol{r})$ 与旋度 $\boldsymbol{b}(\boldsymbol{r})$。若取另一矢量场

$$\boldsymbol{a}'(\boldsymbol{r}) = \boldsymbol{a}(\boldsymbol{r}) + \boldsymbol{c}(\boldsymbol{r}),$$

且 $\boldsymbol{c}(\boldsymbol{r})$ 满足

$$\boldsymbol{\nabla}\cdot\boldsymbol{c} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{c} = \boldsymbol{0},$$

则 $\boldsymbol{a}'$ 与 $\boldsymbol{a}$ 具有相同的散度与旋度。

因此, 仅给定散度与旋度并不能唯一确定矢量场。然而可以证明：不存在一个非零的矢量场 (见第三章) $\boldsymbol{c}(\boldsymbol{r})$, 它在整个空间中同时满足

$$\boldsymbol{\nabla}\cdot\boldsymbol{c} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{c} = \boldsymbol{0},$$

并且在无穷远处趋于零。

因此, 若进一步要求

$$\boldsymbol{a}(\boldsymbol{r}) \to \boldsymbol{0} \qquad \text{当} r\to\infty,$$

则满足给定散度与旋度的矢量场解是唯一的。

现在可以严谨地表述亥姆霍兹定理如下：

> 若矢量场 $\boldsymbol{a}(\boldsymbol{r})$ 的散度 $f(\boldsymbol{r})$ 与旋度 $\boldsymbol{b}(\boldsymbol{r})$ 已知, 且二者在 $r\to\infty$ 时均比 $1/r^2$ 衰减得更快, 同时 $\boldsymbol{a}(\boldsymbol{r})$ 在无穷远处趋于零, 则 $\boldsymbol{a}(\boldsymbol{r})$ 被其散度与旋度唯一确定。

1.7.2 势函数

1. 标量势与无旋场 设向量场 $\boldsymbol{a}(\boldsymbol{r})$ 定义在某一区域。若存在标量函数 $U(\boldsymbol{r})$, 使得

$$\boldsymbol{a} = -\boldsymbol{\nabla} U, \tag{1.78}$$

则称 U 为 $\boldsymbol{a}$ 的**标量势函数**, $\boldsymbol{a}$ 称为**保守场**。由旋度的定义可得

$$\boldsymbol{\nabla} \times \boldsymbol{a} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} U = \boldsymbol{0}. \tag{1.79}$$

因此, 任何具有标量势的向量场必为无旋场。

2. 无旋场的势函数存在性 若

$$\boldsymbol{\nabla} \times \boldsymbol{a} = \boldsymbol{0}, \tag{1.80}$$

并且区域 V 是单连通的 (任意闭合曲线可连续收缩为一点), 则由斯托克斯定理可得

$$\oint_C \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{1.81}$$

线积分与路径无关, 可定义

$$\phi(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}, \tag{1.82}$$

从而得到 $\boldsymbol{a} = -\boldsymbol{\nabla}\phi$。

3. 向量势与无散场 若存在向量函数 $\boldsymbol{b}(\boldsymbol{r})$, 使

$$\boldsymbol{a} = \boldsymbol{\nabla} \times \boldsymbol{b}, \tag{1.83}$$

则称 $\boldsymbol{b}$ 为 $\boldsymbol{F}$ 的**向量势**。由恒等式

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}) = 0 \tag{1.84}$$

可知, 任何具有向量势的向量场必为无散场。

4. 势函数的不唯一性

- 若 $\boldsymbol{a} = -\boldsymbol{\nabla} U$, 则对任意常数 C, $U' = U + C$ 给出同一向量场。
- 若 $\boldsymbol{a} = \boldsymbol{\nabla} \times \boldsymbol{b}$, 则对任意标量函数 $\chi(\boldsymbol{r})$, $\boldsymbol{b}' = \boldsymbol{b} + \boldsymbol{\nabla}\chi$ 产生相同的 $\boldsymbol{a}$(**规范变换**)。

5. 亥姆霍兹分解 在满足适当衰减条件的情况下, 任意向量场 $\boldsymbol{a}$ 可分解为

$$\boldsymbol{a} = -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \boldsymbol{b}, \tag{1.85}$$

其中

$$\boldsymbol{\nabla}^2 U = -\boldsymbol{\nabla} \cdot \boldsymbol{a}, \tag{1.86}$$

$$\boldsymbol{\nabla}^2 \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{a}. \tag{1.87}$$

这就是亥姆霍兹定理在势函数语言下的表达。

1.8 习题

1.8.1 证明: $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})=0$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})=-(\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c}+(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}-(\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}+(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=0$$

1.8.2 证明: $\boldsymbol{Ra}\cdot\boldsymbol{Rb}=\boldsymbol{a}\cdot\boldsymbol{b}$, 其中 $\boldsymbol{R}$ 为旋转矩阵

$$\boldsymbol{Ra}\cdot\boldsymbol{Rb}=(\boldsymbol{Ra})^T(\boldsymbol{Rb})=\boldsymbol{a}^T\boldsymbol{R}^T\boldsymbol{Rb}=\boldsymbol{a}^T\boldsymbol{R}^{-1}\boldsymbol{Rb}=\boldsymbol{a}^T\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{b}$$

1.8.3 在坐标逆变换 $(x'=-x,\ y'=-y,\ z'=-z)$ 下, 两个矢量的叉乘是如何变换的?

$$\boldsymbol{a}'\times\boldsymbol{b}'=\begin{pmatrix}\boldsymbol{e}_i & \boldsymbol{e}_j & \boldsymbol{e}_k\\ a_i' & a_j' & a_k'\\ b_i' & b_j' & b_k'\end{pmatrix}=\begin{pmatrix}\boldsymbol{e}_i & \boldsymbol{e}_j & \boldsymbol{e}_k\\ -a_i & -a_j & -a_k\\ -b_i & -b_j & -b_k\end{pmatrix}=\boldsymbol{a}\times\boldsymbol{b}$$

1.8.4 在坐标逆变换下, 标量三重积 $(\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}))$ 是如何变换的?

$$\boldsymbol{c}'\cdot(\boldsymbol{a}'\times\boldsymbol{b}')=\boldsymbol{c}'\cdot(\boldsymbol{a}\times\boldsymbol{b})=-\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

1.8.5 $\boldsymbol{n}$ 是固定点 (x',y',z') 到点 (x,y,z) 的间隔矢量, n 是它的长度, 证明:

(a) $\boldsymbol{\nabla}n^2=2\boldsymbol{n}$.

(b) $\boldsymbol{\nabla}\frac{1}{n}=-\frac{\boldsymbol{n}}{n^2}$.

(c) $\boldsymbol{\nabla}n^a=a\boldsymbol{n}n^{a-2}(a\neq 0)$.

(a) $\boldsymbol{\nabla}n^2=\frac{\partial n^2}{\partial e_i}\boldsymbol{e}_i=2n\frac{\partial n}{\partial e_i}\boldsymbol{e}_i=2\boldsymbol{n}$

(c) $\boldsymbol{\nabla}n^a=\frac{\partial n^a}{\partial e_i}\boldsymbol{e}_i=an^{-1}\frac{\partial n}{\partial e_i}\boldsymbol{e}_i=a\boldsymbol{n}n^{a-2}$

1.8.6 函数 f 只依赖平面坐标 y,z。令旋转后的坐标记为 $\begin{pmatrix}y'\\z'\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}y\\z\end{pmatrix}$，求证 $\boldsymbol{\nabla}f'=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{pmatrix}$

$$\begin{pmatrix}y\\z\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}y'\\z'\end{pmatrix}$$

$$\frac{\partial f}{\partial y'}=\frac{\partial f}{\partial y}\frac{\partial y}{\partial y'}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y'}=\frac{\partial f}{\partial y}\cos\theta+\frac{\partial f}{\partial z}\sin\theta$$

$$\frac{\partial f}{\partial z'}=\frac{\partial f}{\partial y}\frac{\partial y}{\partial z'}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial z'}=-\frac{\partial f}{\partial y}\sin\theta+\frac{\partial f}{\partial z}\cos\theta$$

$$\begin{pmatrix}\frac{\partial f}{\partial y'}\\ \frac{\partial f}{\partial z'}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{pmatrix}$$

1.8.7 计算 $(\boldsymbol{\nabla T})\times(\boldsymbol{\nabla S})$

$$
\begin{aligned}
(\boldsymbol{\nabla T})\times(\boldsymbol{\nabla S}) &= \left(\varepsilon_{ijk}\frac{\partial T_j}{\partial x_i}\boldsymbol{e}_k\right)\times\left(\varepsilon_{lmn}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_n\right)\\
&= \varepsilon_{opq}\varepsilon_{ijo}\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q\\
&= \varepsilon_{pqo}\varepsilon_{ijo}\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q\\
&= (\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q\\
&= \delta_{ip}\delta_{jq}\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q-\delta_{iq}\delta_{jp}\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q\\
&= \frac{\partial T_j}{\partial x_i}\varepsilon_{lmi}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_j-\frac{\partial T_j}{\partial x_i}\varepsilon_{lmj}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_i\\
&= \frac{\partial T_i}{\partial x_j}\varepsilon_{lmj}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_i-\frac{\partial T_j}{\partial x_i}\varepsilon_{lmj}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_i\\
&= \left(\frac{\partial T_i}{\partial x_j}-\frac{\partial T_j}{\partial x_i}\right)\varepsilon_{lmj}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_i
\end{aligned}
$$

1.8.8 证明 $\iint_S f(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S}=\iint_S[\boldsymbol{a}\times(\boldsymbol{\nabla}f)]\cdot\mathrm{d}\boldsymbol{S}+\oint_{\partial S}f\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}$

$$
\begin{aligned}
\oint_{\partial S}f\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} &= \iint_S[\boldsymbol{\nabla}\times(\boldsymbol{fa})]\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S[f(\boldsymbol{\nabla}\times\boldsymbol{a})+(\boldsymbol{\nabla}f)\times\boldsymbol{a}]\cdot\mathrm{d}\boldsymbol{S}\\
\iint_S f(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S} &= \iint_S[\boldsymbol{a}\times(\boldsymbol{\nabla}f)]\cdot\mathrm{d}\boldsymbol{S}+\oint_{\partial S}f\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}
\end{aligned}
$$

1.8.9 证明 $\iiint_V\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})\,\mathrm{d}\tau=\iiint_V\boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\,\mathrm{d}\tau+\iint(\boldsymbol{a}\times\boldsymbol{b})\cdot\mathrm{d}\boldsymbol{S}_{\partial V}$

$$
\begin{aligned}
\iint_{\partial V}(\boldsymbol{a}\times\boldsymbol{b})\cdot\mathrm{d}\boldsymbol{S} &= \iiint_V\boldsymbol{\nabla}\cdot(\boldsymbol{a}\times\boldsymbol{b})\,\mathrm{d}\tau\\
&= \iiint_V[\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})-\boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})]\,\mathrm{d}\tau\\
\iiint_V\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})\,\mathrm{d}\tau &= \iiint_V\boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\,\mathrm{d}\tau+\iint_{\partial V}(\boldsymbol{a}\times\boldsymbol{b})\cdot\mathrm{d}\boldsymbol{S}
\end{aligned}
$$

1.8.10 设 $f=f(r)$, 求 $\nabla^2[f(r)]$

$$\begin{aligned}\nabla^2[f(r)] &= \frac{1}{sr}\frac{\partial}{\partial r}\left(sr\frac{\partial f}{\partial r}\right)+\frac{1}{s^2}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial\phi}\right)+\frac{1}{sr^2}\frac{\partial}{\partial\theta}\left(s\frac{\partial f}{\partial\theta}\right)\\ &= \frac{1}{sr}\frac{\partial}{\partial r}\left(sr\frac{\partial f}{\partial r}\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\\ &= \frac{2}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial r^2}\end{aligned} \tag{1.88}$$

1.8.11 证明:$\iiint_V \nabla\times\boldsymbol{a}\,\mathrm{d}\tau=\oiint \mathrm{d}\boldsymbol{S}\times\boldsymbol{a}$

取任意常向量 $\boldsymbol{c}$。利用式1.39得 $\nabla\times\boldsymbol{c}=\boldsymbol{0}$, 因此

$$\nabla\cdot(\boldsymbol{a}\times\boldsymbol{c})=\boldsymbol{c}\cdot(\nabla\times\boldsymbol{a}).$$

应用高斯定理,

$$\iiint_V \nabla\cdot(\boldsymbol{a}\times\boldsymbol{c})\,\mathrm{d}\tau=\oiint_{\partial V}(\boldsymbol{a}\times\boldsymbol{c})\cdot\mathrm{d}\boldsymbol{S}=\iiint_V \boldsymbol{c}\cdot(\nabla\times\boldsymbol{a})\,\mathrm{d}\tau.$$

利用式1.12得

$$\boldsymbol{c}\cdot\iiint_V(\nabla\times\boldsymbol{a})\,\mathrm{d}\tau=\boldsymbol{c}\cdot\oiint_{\partial V}\mathrm{d}\boldsymbol{S}\times\boldsymbol{a}.$$

由于上述等式对任意常向量 $\boldsymbol{c}$ 成立, 必有

$$\iiint_V(\nabla\times\boldsymbol{a})\,\mathrm{d}\tau=\oiint_{\partial V}\mathrm{d}\boldsymbol{S}\times\boldsymbol{a}.$$

1.8.12 证明:$\iint_S \mathrm{d}\boldsymbol{S}\times\nabla f=\oint_{\partial S} f\,\mathrm{d}\boldsymbol{l}$

取任意常向量 $\boldsymbol{c}$。利用式1.28得

$$\iint_S(\nabla f\times\boldsymbol{c})\cdot\mathrm{d}\boldsymbol{S}=\iint_S(\nabla\times(f\boldsymbol{c}))\cdot\mathrm{d}\boldsymbol{S}.$$

由斯托克斯定理

$$\iint_S(\nabla\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{S}=\oint_{\partial S}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l},$$

取 $\boldsymbol{A}=f\boldsymbol{c}$, 得

$$\iint_S(\nabla\times(f\boldsymbol{c}))\cdot\mathrm{d}\boldsymbol{S}=\oint_{\partial S}(f\boldsymbol{c})\cdot\mathrm{d}\boldsymbol{l}=\oint_{\partial S}f\,\boldsymbol{c}\cdot\mathrm{d}\boldsymbol{l}.$$

又由混合积恒等式

$$\boldsymbol{c}\cdot(\mathrm{d}\boldsymbol{S}\times\nabla f)=(\nabla f\times\boldsymbol{c})\cdot\mathrm{d}\boldsymbol{S},$$

所以

$$\boldsymbol{c}\cdot\iint_S\mathrm{d}\boldsymbol{S}\times\nabla f=\boldsymbol{c}\cdot\oint_{\partial S}f\ \mathrm{d}\boldsymbol{l}.$$

由于上述等式对任意常向量 $\boldsymbol{c}$ 都成立, 必有

$$\iint_S\mathrm{d}\boldsymbol{S}\times\nabla f=\oint_{\partial S}f\ \mathrm{d}\boldsymbol{l}.$$

1.8.13 证明:$\oiint_{\partial S}(\boldsymbol{a}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{l}=2\iint_S\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S}$

$$\begin{aligned}
\oiint_{\partial S}(\boldsymbol{a}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{l} &= \iint_S[\boldsymbol{\nabla}\times(\boldsymbol{a}\times\boldsymbol{r})]\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S[\boldsymbol{\nabla}\times(\varepsilon_{ijk}a_ix_j\boldsymbol{e}_k)]\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S(\varepsilon_{lmn}\varepsilon_{ijm}\frac{\partial a_ix_j}{\partial x_l}\boldsymbol{e}_n)\cdot\mathrm{d}\boldsymbol{S}\\
&= -\iint_S(\varepsilon_{lnm}\varepsilon_{ijm}\frac{\partial a_ix_j}{\partial x_l}\boldsymbol{e}_n)\cdot\mathrm{d}\boldsymbol{S}\\
&= -\iint_S(\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl})(\frac{\partial a_ix_j}{\partial x_l}\boldsymbol{e}_n)\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S(\delta_{in}\delta_{jl}\frac{\partial a_ix_j}{\partial x_l}\boldsymbol{e}_n-\delta_{il}\delta_{jn}\frac{\partial a_ix_j}{\partial x_l}\boldsymbol{e}_n)\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S(\frac{\partial a_ix_j}{\partial x_j}\boldsymbol{e}_i-\frac{\partial a_ix_j}{\partial x_i}\boldsymbol{e}_j)\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S(\frac{\partial a_jx_i}{\partial x_i}\boldsymbol{e}_j-\frac{\partial a_ix_j}{\partial x_i}\boldsymbol{e}_j)\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S(a_j\frac{\partial x_i}{\partial x_i}+x_i\frac{\partial a_j}{\partial x_i}-a_i\frac{\partial x_j}{\partial x_i}-x_j\frac{\partial a_i}{\partial x_i})\boldsymbol{e}_j\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S(3a_j+x_i\frac{\partial a_j}{\partial x_i}-a_i\delta_{ij}-x_j\frac{\partial a_i}{\partial x_i})\boldsymbol{e}_j\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S 2a_j\boldsymbol{e}_j\cdot\mathrm{d}\boldsymbol{S}\\
&= \iint_S\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S}
\end{aligned}$$

1.8.14 将下列电荷分布表示成电荷密度函数 ρ**:1.** 电荷量 Q 均匀分布在半径为 R 的球面上; **2.** 电荷均匀分布在半径为 R 的柱面上, 单位长度的电荷量为 λ; **3.** 电荷量均匀分布在半径为 R 的平面圆盘上

$$\begin{aligned}
&1.\rho=\delta(R\boldsymbol{e}_r-\boldsymbol{r})\frac{Q}{4\pi R^2}\\
&2.\rho=\delta(R\boldsymbol{e}_s-\boldsymbol{s})\frac{\lambda}{2\pi R}\\
&3.\rho=\delta(\boldsymbol{x}_3)H(x+R)H(-x-R)\frac{\lambda}{\pi R^2}
\end{aligned}$$

1.8.15 对函数 $\boldsymbol{a} = r^2\cos\theta\boldsymbol{r} + r^2\cos\phi\boldsymbol{\theta} - r^2\cos\theta\sin\phi\boldsymbol{\phi}$ 验证散度定理, 体积为半径为 R 在第一卦限的 1/8 球体

$\boldsymbol{\nabla}\cdot\boldsymbol{a} = 2r\cos\theta - r^2\cos\theta\cos\phi$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R\boldsymbol{\nabla}\cdot\boldsymbol{a}\,\mathrm{d}r &= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R\big(2r\cos\theta - r^2\cos\theta\cos\phi\big)\,\mathrm{d}r \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\left(r^2\cos\theta - \frac{r^3}{3}\cos\theta\cos\phi\right)\Bigg|_0^R \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\left(R^2\cos\theta - \frac{R^3}{3}\cos\theta\cos\phi\right)\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\left(R^2\sin\theta - \frac{R^3}{3}\sin\theta\cos\phi\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \int_0^{\frac{\pi}{2}}\left(R^2 - \frac{R^3}{3}\cos\phi\right)\mathrm{d}\phi \\
&= \left(R^2\phi - \frac{R^3}{3}\sin\phi\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}R^2 - \frac{R^3}{3}
\end{aligned}$$

计算 XY 平面:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R a_\phi\,\mathrm{d}r &= -\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R r^2\cos\theta\sin\phi\,\mathrm{d}r \\
&= -\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R r^2\cos\theta\,\mathrm{d}r \\
&= -\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\,\frac{r^3}{3}\cos\theta\Bigg|_0^R \\
&= -\int_0^{\frac{\pi}{2}}\frac{R^3}{3}\cos\theta\,\mathrm{d}\theta \\
&= -\frac{R^3}{3}
\end{aligned}$$

计算 XZ 平面:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^R a_\theta\,\mathrm{d}r &= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^R r^2\cos\phi\,\mathrm{d}r \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\,\frac{r^3}{3}\cos\phi\Bigg|_0^R \\
&= \int_0^{\frac{\pi}{2}}\frac{R^3}{3}\cos\phi\,\mathrm{d}\phi \\
&= \frac{R^3}{3}
\end{aligned}$$

计算 YZ 平面:

$$\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^R a_\theta\,\mathrm{d}r = \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^R r^2\cos\phi\,\mathrm{d}r = \frac{R^3}{3}$$

1.8.16 对函数 $\boldsymbol{a} = bx_2\boldsymbol{e}_1 + cx_1\boldsymbol{e}_2$ 验证斯托克斯定理, 面的边界线选为处在 XY 平面, 半径为 R, 圆心在原点的圆周线。

$$\nabla \times \boldsymbol{a} = \nabla \times (by\boldsymbol{e}_1 + cx\boldsymbol{e}_2) = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ bx_2 & cx_1 & 0 \end{pmatrix} = (c-b)\boldsymbol{e}_3$$

$$\iint \nabla \times \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \iint (c-b)\boldsymbol{e}_3 \cdot \mathrm{d}\boldsymbol{S} = 2\pi R(c-b)$$

$$\begin{aligned}\oint (bR\sin\phi\boldsymbol{e}_1 + cR\cos\phi\boldsymbol{e}_2)\cdot \mathrm{d}\boldsymbol{l} &= \int_0^{2\pi} (bR\sin\phi\boldsymbol{e}_1 + cR\cos\phi\boldsymbol{e}_2)\cdot(-R\sin\phi\boldsymbol{e}_1 \ + R\cos\phi\boldsymbol{e}_2\ )\,\mathrm{d}\phi \\ &= \int_0^{2\pi} (-bR^2\sin^2\phi + cR^2\cos^2\phi)\,\mathrm{d}\phi \\ &= 2\pi R(c-b)\end{aligned}$$

1.8.17 对函数 $\boldsymbol{a} = r\cos^2\theta\boldsymbol{r} - r\sin\theta\cos\theta\boldsymbol{\theta} + 3r\boldsymbol{\phi}$ 验证斯托克斯定理。

$$\begin{aligned}&\nabla \times \boldsymbol{a} \\ =&\frac{1}{sr}\left(r\frac{\partial a_\theta}{\partial\phi} - s\frac{\partial a_\phi}{\partial\theta}\right)\boldsymbol{r} + \frac{1}{r}\left(\frac{\partial a_r}{\partial\theta} - \frac{\partial r a_\theta}{\partial r}\right)\boldsymbol{\phi} + \frac{1}{s}\left(\frac{\partial s a_\phi}{\partial r} - \frac{\partial a_r}{\partial\phi}\right)\boldsymbol{\theta} \\ =&\frac{1}{sr}\left(r\frac{\partial r\sin\theta\cos\theta}{\partial\phi} - s\frac{\partial 3r}{\partial\theta}\right)\boldsymbol{r} + \frac{1}{r}\left(\frac{\partial r\cos^2\theta}{\partial\theta} - \frac{\partial rr\sin\theta\cos\theta}{\partial r}\right)\boldsymbol{\phi} + \frac{1}{s}\left(\frac{\partial s3r}{\partial r} - \frac{\partial r\cos^2\theta}{\partial\phi}\right)\boldsymbol{\theta} \\ =&\frac{1}{s}\left(6r\sin\theta\right)\boldsymbol{\theta} \\ =&6\boldsymbol{\theta}\end{aligned}$$

计算 XY 平面:

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R a_\phi\,\mathrm{d}r &= \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R 6\,\mathrm{d}r \\ &= \frac{3}{2}R\end{aligned}$$

1.8.18 对函数 $\boldsymbol{a} = r^2\sin^2\theta\boldsymbol{r} + 4r^2\cos\theta\boldsymbol{\theta} + r^2\tan\theta\boldsymbol{\phi}$ 验证散度定理。

$$\begin{aligned}\nabla\cdot\boldsymbol{a} &= \frac{1}{sr}(\frac{\partial s r a_r}{\partial r} + r\frac{\partial a_\phi}{\partial\phi} + s\frac{\partial\theta}{\partial\theta}) \\ &= \frac{1}{sr}(\frac{\partial s r r^2\sin^2\theta}{\partial r} + r\frac{\partial 4r^2\cos\theta}{\partial\phi} + s\frac{\partial r^2\tan\theta}{\partial\theta}) \\ &= \frac{1}{sr}(4sr^2\sin^2\theta + sr^2\frac{1}{\cos^2\theta}) \\ &= 4r\sin^2\theta + r\frac{1}{\cos^2\theta}\end{aligned}$$

$$
\begin{aligned}
&\int_0^R r^2\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^{\frac{\pi}{6}} (4r\sin^2\theta + r\frac{1}{\cos^2\theta})\sin\theta\,\mathrm{d}\theta \\
=&\int_0^R 2\pi r^3\,\mathrm{d}r \int_0^{\frac{\pi}{6}} (4\sin^2\theta + \frac{1}{\cos^2\theta})\,\mathrm{d}\theta \\
=&\int_0^R 2\pi r^3\ (2\theta - \sin 2\theta + \tan\theta)|_0^{\frac{\pi}{6}}\,\mathrm{d}r \\
=&\int_0^R 2\pi r^3(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2})\,\mathrm{d}r \\
=&\int_0^R 2\pi r^3\frac{\pi}{3}\,\mathrm{d}r \\
=&2\pi R^4\frac{\pi}{12}
\end{aligned}
$$

1.8.19 证明 $\iiint_V \left[f\boldsymbol{\nabla}^2 g + (\boldsymbol{\nabla} f)\cdot(\boldsymbol{\nabla} g)\right]\mathrm{d}\tau = \oiint_S (f\boldsymbol{\nabla} g)\cdot\mathrm{d}S$

$$
\begin{aligned}
\iiint_V (\boldsymbol{\nabla}\cdot\boldsymbol{a})\,\mathrm{d}\tau &= \iint_{\partial V}\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S} \\
\iiint_V [\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla} g)]\,\mathrm{d}\tau &= \iint_{\partial V}(f\boldsymbol{\nabla} g)\cdot\mathrm{d}\boldsymbol{S} \\
\iiint_V \left[f\boldsymbol{\nabla}^2 g + (\boldsymbol{\nabla} f)\cdot(\boldsymbol{\nabla} g)\right]\mathrm{d}\tau &= \iint_{\partial V}(f\boldsymbol{\nabla} g)\cdot\mathrm{d}\boldsymbol{S}
\end{aligned}
$$

1.8.20 求 $\boldsymbol{a} = r^n\boldsymbol{e}_r$ **的散度**

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{a} &= \frac{\partial a_i}{\partial x_i} \\
&= \frac{\partial x_i r^{n-1}}{\partial x_i} \\
&= 3r^{n-1} + x_i(n-1)r^{n-2}\frac{\partial r}{\partial x_i} \\
&= 3r^{n-1} + x_i(n-1)r^{n-2}\frac{x_i}{r} \\
&= 3r^{n-1} + (n-1)r^{n-1} \\
&= (n+2)r^{n-1}
\end{aligned}
$$

$$
\iint_{\partial V}\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S} = \iint_{\partial V} r^n\boldsymbol{e}_r\cdot\mathrm{d}\boldsymbol{S} = 4\pi r^2 r^n =
$$

1.8.21 求 $\boldsymbol{a} = r^n \boldsymbol{e}_r$ 的旋度

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{a} &= \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \boldsymbol{e}_k \\
&= \varepsilon_{ijk} \frac{\partial r^{n-1} x_j}{\partial x_i} \boldsymbol{e}_k \\
&= \varepsilon_{ijk} x_j \frac{\partial r^{n-1}}{\partial x_i} \boldsymbol{e}_k \\
&= (n-1) \varepsilon_{ijk} x_j r^{n-2} \frac{\partial r}{\partial x_i} \boldsymbol{e}_k \\
&= (n-1) \varepsilon_{ijk} x_j r^{n-2} \frac{x_i}{r} \boldsymbol{e}_k \\
&= (n-1) r^{n-3} \varepsilon_{ijk} x_i x_j \boldsymbol{e}_k \\
&= 0
\end{aligned}
$$

1.8.22 以匀角速绕轴转动的抛物线形金属丝, 其方程为 $x^2 = 4ay$。一质量为 m 的小环套在此金属丝上, 可沿着金属丝无摩擦滑动。求小环在 x 方向的运动微分方程。

$$
\begin{aligned}
\frac{\partial \dfrac{mx^2\omega^2}{2} + mgy + \dfrac{m\dot{y}^2}{2} + \dfrac{m\dot{x}^2}{2}}{\partial t} &= 0 \\
\frac{\partial \dfrac{mx^2\omega^2}{2} + mg\dfrac{x^2}{4a} + \dfrac{mx^2\dot{x}^2}{8a^2} + \dfrac{m\dot{x}^2}{2}}{\partial t} &= 0 \\
x\dot{x}\omega^2 + gx\frac{\dot{x}}{2a} + \frac{x\dot{x}^3}{4a^2} + \frac{x^2\dot{x}\ddot{x}}{4a^2} + mx\dot{x} &= 0 \\
x\omega^2 + \frac{gx}{2a} + \frac{x\dot{x}^2}{4a^2} + \frac{x^2\ddot{x}}{4a^2} + mx &= 0
\end{aligned}
$$

2 静电学

2.1 电场

2.1.1 库仑定律

设一个静止点电荷 q_1 距检验电荷 q_2 的距离为 r, 那么它作用在 Q 上的力是

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^3}\boldsymbol{r} \tag{2.1}$$

常数 ε_0 称为真空介电常数, $\varepsilon_0 = 8.85\times10^{-12}\ \dfrac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$

2.1.2 电场

$$\boldsymbol{E} \equiv \frac{\boldsymbol{F}}{Q} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r} \tag{2.2}$$

2.1.3 电场强度通量

$\boldsymbol{\Phi}_E$ 为电场强度通量

$$\boldsymbol{\Phi}_E \equiv \iint_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} \tag{2.3}$$

2.1.4 高斯定理

ρ 为电荷密度

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{1}{\varepsilon_0}\rho \tag{2.4}$$

2.1.5 两个带电为 q 的电荷, 相距 d 放置, 求垂直于连线中点且距离为 x_2 处的电场, 如果把一个换成 $-q$ 会怎样? 当 $y >> d$ 时会怎样?

$$\begin{aligned}
\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r}_1 + \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r}_2 &= 2\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}x_2\boldsymbol{e}_2 \\
&= \frac{1}{2\pi\varepsilon_0}\frac{qx_2}{r^3}\boldsymbol{e}_2 \\
&= \frac{1}{2\pi\varepsilon_0}\frac{qx_2}{(x_2^2+\frac{d^2}{4})^{\frac{3}{2}}}\boldsymbol{e}_2
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r}_1 + \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r}_2 &= 2\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\frac{d}{2}\boldsymbol{e}_1 \\
&= \frac{1}{4\pi\varepsilon_0}\frac{qd}{r^3}\boldsymbol{e}_1 \\
&\approx \frac{1}{4\pi\varepsilon_0}\frac{qd}{x_2^3}\boldsymbol{e}_1
\end{aligned}$$

2.1.6 一个长度为 $2L$ 的细杆均匀带电, 电荷线密度为 λ, 求垂直于杆且与杆中心距离为 x_2 处的电场

$$\begin{aligned}
\int_{-L}^{L}\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^3}\boldsymbol{r}\,\mathrm{d}x &= \int_{-L}^{L}\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^3}x_2\boldsymbol{e}_2\,\mathrm{d}x\\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-L}^{L}\frac{x_2}{r^3}\,\mathrm{d}x\\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-L}^{L}\frac{x_2}{(x^2+x_2^2)^{\frac{3}{2}}}\,\mathrm{d}x\\
&\xlongequal{x=x_2\tan\theta} \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-\arctan\frac{L}{x_2}}^{\arctan\frac{L}{x_2}}\frac{x_2}{(x_2^2\tan\theta^2+x_2^2)^{\frac{3}{2}}}\,\mathrm{d}x_2\tan\theta\\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-\arctan\frac{L}{x_2}}^{\arctan\frac{L}{x_2}}\frac{x_2^2\cos^3\theta}{x_2^3\cos^2\theta}\,\mathrm{d}\theta\\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-\arctan\frac{L}{x_2}}^{\arctan\frac{L}{x_2}}\frac{\cos\theta}{x_2^2}\,\mathrm{d}\theta\\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\left.\frac{\sin\theta}{x_2}\right|_{-\arctan\frac{L}{x_2}}^{\arctan\frac{L}{x_2}}\\
&= \frac{\lambda}{2\pi\varepsilon_0 x_2}\boldsymbol{e}_2\frac{L}{\sqrt{L^2+x_2^2}}
\end{aligned}$$

2.1.7 一个长度为 L 的细杆均匀带电, 电荷线密度为 λ, 求杆一端上方距杆 y 处的电场

$$\begin{aligned}
\int_0^L\frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0 r^2}\,\mathrm{d}x_2 &= \int_0^L\frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0(x_2+y)^2}\,\mathrm{d}x_2\\
&= -\left.\frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0(x_2+y)}\right|_0^L\\
&= \frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0(L+y)}-\frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0 y}\\
&= \frac{\lambda\boldsymbol{e}_y L}{4\pi\varepsilon_0(L+y)y}
\end{aligned}$$

2.1.8 一个边长为 $2L$ 的正方形线框均匀带电, 电荷线密度为 λ, 求线框中心上方距 x_3 处的电场

$$4\frac{\lambda}{2\pi\varepsilon_0\sqrt{L^2+x_3^2}}\boldsymbol{e}_3\frac{L}{2\sqrt{2L^2+x_3^2}}\frac{x_3}{x_3\sqrt{L^2+x_3^2}^2}=\frac{\lambda}{\pi\varepsilon_0\sqrt{2L^2+x_3^2}}\boldsymbol{e}_3\frac{2Lx_3}{L^2+x_3^2}$$

2.1.9 一个半径为 r 的圆线框均匀带电, 电荷线密度为 λ, 求线框中心上方距 x_3 处的电场

$$2\pi r\frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2+r^2}}=\frac{\lambda}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3 r}{\sqrt{x_3^2+r^2}^3}$$

2.1.10 一个半径为 r 的圆片均匀带电, 电荷面密度为 σ, 求圆片中心上方距 x_3 处的电场

$$\begin{aligned}\int_0^r\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3x_1}{\sqrt{x_3^2+x_1^2}^3}\,\mathrm{d}x_1&=-\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2+x_1^2}}\Bigg|_0^r\\&=\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2}}-\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2+r^2}}\\&=\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3-\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2+r^2}}\end{aligned}$$

2.1.11 一个半径为 r 的球面均匀带电, 电荷面密度为 σ, 求圆球上方距 d 处的电场 (分 $d>r$ 和 $d<r$ 讨论)

$$\begin{aligned}\int_\pi^0\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{(x_3+d)\sqrt{x_1^2+x_2^2}}{\sqrt{(x_3+d)^2+x_1^2+x_2^2}^3}\,\mathrm{d}r\phi&=\int_\pi^0\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{(r\cos\phi+d)\sqrt{x_1^2+x_2^2}}{\sqrt{(r\cos\phi+d)^2+x_1^2+x_2^2}^3}\,\mathrm{d}r\phi\\&=\int_\pi^0\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{(r\cos\phi+d)r\sin\phi}{\sqrt{r^2+2rd\cos\phi+d^2}^3}\,\mathrm{d}r\phi\\&=\int_0^\pi\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{(r\cos\phi+d)r^2\sin\phi}{\sqrt{r^2+2rd\cos\phi+d^2}^3}\,\mathrm{d}\phi\end{aligned}$$

2.1.12 在某个区域电场可以写为 $\boldsymbol{E}=kr^3\boldsymbol{e_r}$, 求电荷密度和包含在半径为 R, 球心在原点的闭合球面内的总电荷

$$\rho=\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}=\varepsilon_0\frac{1}{sr}\frac{\partial srE_r}{\partial r}=\varepsilon_0\frac{1}{sr}\frac{\partial srr^3}{\partial r}=\varepsilon_0\frac{1}{sr}5sr^4=5\varepsilon_0r^3k$$

$$4\pi\int_0^R5\varepsilon_0r^3\,\mathrm{d}r=5\pi\varepsilon_0R^4k$$

2.2 电势

2.2.1 定义

$$U(\boldsymbol{r})\equiv-\int_O^r\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}\tag{2.5}$$

其中 O 为预先设置的标准参考点, 通常为无限远处

2.2.2 一个半径为 R 的均匀带电球体, 总电荷为 q, 求电势 $U(\boldsymbol{r})$

$r \geq R$:

$$\int_r^\infty \frac{q}{4\pi\varepsilon_0 x^2}\,\mathrm{d}x = -\frac{q}{4\pi\varepsilon_0 x}\bigg|_r^\infty = \frac{q}{4\pi\varepsilon_0 r}$$

$r < R$:

$$\begin{aligned}\frac{q}{4\pi\varepsilon_0 R} + \int_r^R \frac{q}{4\pi\varepsilon_0 x^2}\frac{x^3}{R^3}\,\mathrm{d}x &= \frac{q}{4\pi\varepsilon_0 R} + \frac{qx^2}{4\pi\varepsilon_0 R^3}\bigg|_r^R \\ &= \frac{q}{4\pi\varepsilon_0 R} + \frac{qR^2}{4\pi\varepsilon_0 R^3} - \frac{qr^2}{4\pi\varepsilon_0 R^3} \\ &= \frac{q}{2\pi\varepsilon_0 R} - \frac{qr^2}{4\pi\varepsilon_0 R^3}\end{aligned}$$

2.2.3 一条均匀带电的无限长直线, 电荷线密度为 λ, 求电势 $U(\boldsymbol{r})$

$$U = \int_r^\infty \frac{\lambda}{2\pi r}\,\mathrm{d}x = \frac{\lambda}{2\pi}\ln(x)\bigg|_r^\infty = \frac{1}{2\pi}\ln(r)$$

2.2.4 一个长度为 $2L$ 的细杆均匀带电, 电荷线密度为 λ, 求垂直于杆且与杆中心距离为 h 处的电势

$$\begin{aligned}U &= \int_h^\infty \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{x_2\sqrt{L^2+x_2^2}}\,\mathrm{d}x_2 \\ &= -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{x^2+L^2}+L}{x}\right)\bigg|_h^\infty \\ &= \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{h^2+L^2}+L}{h}\right)\end{aligned}$$

2.2.5 一个半径为 r 的圆片均匀带电, 电荷面密度为 σ, 求圆片中心上方距 h 处的电势

$$\begin{aligned}U &= \int_h^\infty \left(\frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0}\frac{x_3}{\sqrt{x_3^2+r^2}}\right)\mathrm{d}x_3 \\ &= \frac{\sigma}{2\varepsilon_0}x_3 - \frac{\sigma}{2\varepsilon_0}\sqrt{x_3^2+r^2}\bigg|_h^\infty \\ &= -\frac{\sigma}{2\varepsilon_0}\frac{r^2}{x_3+\sqrt{x_3^2+r^2}}\bigg|_h^\infty \\ &= \frac{\sigma}{2\varepsilon_0}\frac{r^2}{h+\sqrt{h^2+r^2}}\end{aligned}$$

2.2.6 一个半径为 r 的圆线框均匀带电，电荷线密度为 λ，求线框中心上方距 h 处的电势

$$U=\int_h^{\infty}\frac{\lambda}{2\varepsilon_0}\frac{x_3 r}{\sqrt{x_3^2+r^2}^3}\,\mathrm{d}x_3=-\frac{\lambda r}{2\varepsilon_0}\frac{1}{\sqrt{x_3^2+r^2}}\bigg|_h^{\infty}=\frac{\lambda r}{2\varepsilon_0}\frac{1}{\sqrt{h^2+r^2}}$$

2.2.7 一个尖角向下的圆锥体均匀带电，电荷面密度为 σ，圆锥高度等于半径为 h，求 $z=h$ 处的电势

$$\begin{aligned}
U&=\int_{-h}^{0}\frac{\lambda(h+x_3)}{2\varepsilon_0}\frac{1}{\sqrt{(h-x_3)^2+(h+x_3)^2}}\,\mathrm{d}x_3\\
&=\int_{-h}^{0}\frac{\lambda(h+x_3)}{2\varepsilon_0}\frac{1}{\sqrt{2h^2+2x_3^2}}\,\mathrm{d}x_3\\
&=\frac{\lambda}{2\sqrt{2}\varepsilon_0}\int_{-h}^{0}\frac{h}{\sqrt{h^2+x_3^2}}+\frac{x_3}{\sqrt{h^2+x_3^2}}\,\mathrm{d}x_3\\
&=h\ln\left(x_3+\sqrt{h^2+x_3^2}\right)+\sqrt{h^2+x_3^2}\bigg|_{-h}^{0}\frac{\lambda}{2\sqrt{2}\varepsilon_0}\\
&=\left[h\ln\left(\sqrt{h^2}\right)+\sqrt{h^2}-h\ln\left(-h+\sqrt{2h^2}\right)-\sqrt{2h^2}\right]\frac{\lambda}{2\sqrt{2}\varepsilon_0}\\
&=\left[h\ln\left(\frac{1}{\sqrt{2}-1}\right)+h-\sqrt{2}h\right]\frac{\lambda}{2\sqrt{2}\varepsilon_0}\\
&=\left[1-\sqrt{2}-\ln\left(\sqrt{2}-1\right)\right]\frac{\lambda h}{2\sqrt{2}\varepsilon_0}
\end{aligned}$$

2.2.8 一个半径为 r 的圆柱均匀带电，电荷体密度为 ρ，求圆柱中心上方距 h 处的电势

$$\begin{aligned}
U&=\int_{-h}^{0}\frac{\rho}{2\varepsilon_0}\frac{r^2}{h+x_3+\sqrt{(h+x_3)^2+r^2}}\,\mathrm{d}x_3\\
&\xlongequal{u=h+x_3}\int_0^h\frac{\rho}{2\varepsilon_0}\frac{r^2}{u+\sqrt{u^2+r^2}}\,\mathrm{d}u\\
&=\int_0^h\frac{\rho r^2}{2\varepsilon_0}\frac{\sqrt{u^2+r^2}-u}{r^2}\,\mathrm{d}u\\
&=\frac{\rho}{2\varepsilon_0}\int_0^h\sqrt{u^2+r^2}-u\,\mathrm{d}u\\
&=u\sqrt{u^2+r^2}+r^2\ln\left(u+\sqrt{u^2+r^2}\right)-u^2\bigg|_0^h\frac{\rho}{4\varepsilon_0}\\
&=\left[h\sqrt{h^2+r^2}+r^2\ln\left(h+\sqrt{h^2+r^2}\right)-h^2-r^2\ln r\right]\frac{\rho}{4\varepsilon_0}
\end{aligned}$$

2.3 静电场的能量

2.3.1 离散电荷体系的静电能

考虑由 N 个点电荷 q_i 构成的静电体系。将电荷从无穷远逐个缓慢搬运到其最终位置，外力所做的总功即为体系的静电能。

设在放置第 i 个电荷时, 其余电荷已就位, 则该电荷所处位置的当前电势为

$$\varphi_i = \sum_{j=1}^{i-1} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

因此, 体系的总静电能为

$$U = \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

注意到相互作用能在指标交换下满足 $U_{ij} = U_{ji}$, 而对所有 $i \neq j$ 的求和中每一对指标被计数两次, 故可将上式对称化为

$$U = \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} = \frac{1}{2} q_i \varphi_i.$$

2.3.2 连续电荷分布的能量表达式

对离散情形的自然推广给出

$$U = \frac{1}{2} \iiint \rho\varphi \,\mathrm{d}\tau.$$

2.3.3 场能量

静电势满足泊松方程

$$\nabla^2 \varphi = -\frac{\rho}{\varepsilon_0}.$$

将其代入能量表达式, 得

$$U = -\frac{\varepsilon_0}{2} \iiint \varphi \, \nabla^2 \varphi \ \mathrm{d}\tau.$$

对右端积分作分部积分。注意到

$$\nabla \cdot (\varphi \nabla \varphi) = (\nabla \varphi)^2 + \varphi \nabla^2 \varphi,$$

于是

$$\varphi \nabla^2 \varphi = \nabla \cdot (\varphi \nabla \varphi) - (\nabla \varphi)^2.$$

代入得

$$U = -\frac{\varepsilon_0}{2} \left[\iiint \nabla \cdot (\varphi \nabla \varphi) \ \mathrm{d}\tau - \iiint (\nabla \varphi)^2 \ \mathrm{d}\tau \right].$$

当 $r \to \infty$ 时,

$$\varphi(\boldsymbol{r}) \sim \frac{1}{r}, \qquad \nabla\varphi \sim \frac{1}{r^2}.$$

因此

$$\varphi \nabla \varphi \sim \frac{1}{r^3},$$

对应的无穷远处曲面积分

$$\oiint \varphi \nabla \varphi \cdot \mathrm{d}\boldsymbol{S} \to 0$$

$$U = \frac{\varepsilon_0}{2} \iiint (\nabla \varphi)^2 \ \mathrm{d}\tau.$$

$$U = \frac{\varepsilon_0}{2} \iiint E^2 \ \mathrm{d}\tau$$

2.3.4 电场能量密度

这表明：静电能可以视为分布在空间中的电场所携带的能量。

由此自然引入电场的能量密度

$$\boxed{u \equiv \frac{\varepsilon_0}{2}\boldsymbol{E}^2}$$

2.3.5 考虑两个同心球面, 半径分别为 a 和 b, 内球面带有电荷 q , 外球面带有电荷 $-q$, 求总能量

$$U = \frac{\varepsilon_0}{2}\iiint E^2\,\mathrm{d}\tau = \frac{\varepsilon_0 4\pi}{2}\int_a^b \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 r^2\,\mathrm{d}r = \int_a^b \frac{q^2}{8\pi\varepsilon_0 r^2}\,\mathrm{d}r = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right)$$

2.4 导体

2.4.1 导体内部电场为零

若导体内部存在非零电场, 则自由电荷将在电场作用下持续运动, 与静电平衡的假设矛盾。因此, 导体内部必须满足

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi = \boldsymbol{0}.$$

2.4.2 导体内部体电荷密度为零

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0},$$

结合 $\boldsymbol{E} = \boldsymbol{0}$,

$$\rho = 0 \qquad (\text{导体内部}).$$

2.4.3 净电荷只能分布在导体表面

既然导体内部体电荷密度为零, 而导体整体可能带有净电荷, 则这些电荷只能分布在导体的表面上。

2.4.4 导体是等势体

由 $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$ 以及导体内部 $\boldsymbol{E} = \boldsymbol{0}$, 可知导体内部电势处处相同

2.4.5 导体表面外侧电场垂直于表面

考虑导体表面上一点的切向电场分量。若存在非零切向分量, 则自由电荷将在表面沿切向运动, 从而破坏静电平衡。

因此, 导体表面外侧的电场只能沿法向：

$$\boldsymbol{E}_\parallel = 0$$

2.4.6 空腔中含点电荷的球形导体

设一不带电的球形导体, 半径为 R, 中心位于原点, 其内部挖去一任意形状的空腔。在空腔内某处放置一点电荷 q。求球外区域的电场分布。

导体处于静电平衡时, 金属内部电场为零, 因而导体整体为等势体。取一紧贴导体内壁、位于金属内部的高斯面, 由高斯定律可得

$$\oint \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{a}=0,$$

故内壁所感应的总电荷必为 $-q$。又由于导体整体不带电, 外表面所带电荷总量为 $+q$。

关键在于确定外表面电荷的分布形式。注意到：在所有满足

$$\boldsymbol{E}=0\quad(\text{导体内部}),\qquad \varphi=\text{常数}\quad(\text{导体表面})$$

的允许电荷分布中, 实际的静电平衡态对应于体系总静电能的极小值。

球外区域不含自由电荷, 其电势完全由外表面电荷分布决定。若外表面电荷分布破坏球对称性, 则球外电场中将出现非径向分量, 从而在保持总电荷为 q 的约束下增加电场能

$$U=\frac{\varepsilon_0}{2}\int E^2\ \mathrm{d}\tau.$$

因此, 能量极小所对应的电荷分布必然保持球对称性。

由此可知, 外表面电荷在球面上均匀分布, 其产生的球外电场与位于球心的点电荷 q 完全相同。因此, 球外任意一点处的电场为

$$\boldsymbol{E}(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\boldsymbol{e}_r,\qquad r>R.$$

可见, 空腔的形状以及点电荷在腔内的具体位置, 均不影响球外区域的电场分布。

2.4.7 一个半径为 R 的金属球, 带有电荷 q, 这个金属球又被一个厚的同心金属球壳所包围 (球壳内径为 a, 外径为 b)。
(a) 分别求出 R, a, b 球面上的电荷面密度 σ。
(b) 求出球心处的电势, 选无限远处为参考点。
(c) 现在球壳的外表面接地, 电势能为零。(a) 和 (b) 所得结果改变为什么?

(a):

$$\sigma_R=\frac{q}{4\pi R^2}$$
$$\sigma_a=\frac{q}{4\pi a^2}$$
$$\sigma_b=\frac{q}{4\pi b^2}$$

(b):

$$U=\int_b^{\infty}\frac{q}{4\pi\varepsilon_0 r}\,\mathrm{d}r+\int_R^{a}\frac{q}{4\pi\varepsilon_0 r}\,\mathrm{d}r=\frac{q}{4\pi\varepsilon_0}\ln\left(\frac{bR}{a}\right)$$

(c):

$$
\begin{aligned}
\sigma_R &= \frac{q}{4\pi R^2} \\
\sigma_a &= \frac{q}{4\pi a^2} \\
\sigma_b &= 0 \\
U &= \frac{q}{4\pi\varepsilon_0} \ln\left(\frac{R}{a}\right)
\end{aligned}
$$

2.4.8 一个半径为 R 的导体球体, 其内部有两个半径分别为 **a** 和 **b** 的圆形空洞, 在 a 空洞的中心放有点电荷 q, 在 b 空洞的中心放有点电荷 **q**。
(a) 求出电荷面密度 σ_a, σ_b 和 σ_R。
(b) 导体外面的电场是什么?
(c) 每个空洞内的电场是什么?
(d)q_a 和 q_b 受到的力是什么?
(e) 如果让第三个电荷 q 靠近导体, 上面所得结果哪一个会发生变化?

(a):

$$
\begin{aligned}
\sigma_R &= \frac{2q}{4\pi R^2} \\
\sigma_a &= \frac{q}{4\pi a^2} \\
\sigma_b &= \frac{q}{4\pi b^2}
\end{aligned}
$$

(b):

$$
\boldsymbol{E} = \frac{2q\boldsymbol{r}}{4\pi\varepsilon_0 r^3}
$$

(c):

$$
\begin{aligned}
\boldsymbol{E_a} &= \frac{q\boldsymbol{r}}{4\pi\varepsilon_0 r^3} \\
\boldsymbol{E_b} &= \frac{q\boldsymbol{r}}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

(d):0

(e): σ_R, b

2.5 电容

$$
C \equiv \frac{Q}{U} \tag{2.6}
$$

$$
W = \int_0^Q \left(\frac{q}{C}\right) \mathrm{d}q = \frac{Q^2}{2C} \tag{2.7}
$$

2.5.1 两个同轴金属管壳, 半径分别为 a 和 b, 求出单位长度的电容。

$$
\begin{aligned}
U &= \int_a^b \frac{Q}{2\varepsilon_0 r}\,\mathrm{d}r \\
&= \frac{Q}{2\varepsilon_0}\ln\left(\frac{b}{a}\right) \\
C &= \frac{Q}{U} \\
&= \frac{2\varepsilon_0}{\ln\left(\frac{b}{a}\right)}
\end{aligned}
$$

2.6 拉普拉斯方程

$$
\frac{\mathrm{d}^2U}{\mathrm{d}x_i^2} = 0 \tag{2.8}
$$

2.6.1 一维拉普拉斯方程

$$
U = ax + b \tag{2.9}
$$

2.6.2 在球坐标下，对 U 仅依赖于 r 的情况，求出拉普拉斯方程的一般解。对柱坐标系，假定 U 仅依赖于 s，做同样的计算。

$$
\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) &= 0 \\
\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) &= 0 \\
r^2\frac{\partial U}{\partial r} &= C_1 \\
\frac{\partial U}{\partial r} &= \frac{C_1}{r^2} \\
U &= -\frac{C_1}{r} + C_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{s}\frac{\partial}{\partial u_s}\left(s\frac{\partial U}{\partial u_s}\right) &= 0 \\
s\frac{\partial U}{\partial u_s} &= C_1 \\
\frac{\partial U}{\partial u_s} &= \frac{C_1}{s} \\
U &= C_1\ln s + C_2
\end{aligned}
$$

2.6.3 $U=\dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{q}{\sqrt{r^2+a^2-2ra\cos\theta}}-\dfrac{q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right]$ 求出球面上的诱导电荷面密度。对其积分求出总诱导电荷。计算这个构型的能量。

$$
\begin{aligned}
\sigma &= -\varepsilon_0\frac{\partial U}{\partial r}\\
&= -\varepsilon_0\frac{\partial \dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{q}{\sqrt{r^2+a^2-2ra\cos\theta}}-\dfrac{q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right]}{\partial r}\\
&= -\frac{q}{4\pi}\left[-\frac{2r-2a\cos\theta}{2\sqrt{r^2+a^2-2ra\cos\theta}^3}+\frac{2\frac{ra}{R}-2a\cos\theta}{2\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right]\\
&\xlongequal{r=R} -\frac{q}{4\pi}\left[-\frac{2R-2a\cos\theta}{2\sqrt{R^2+a^2-2Ra\cos\theta}^3}+\frac{2\frac{Ra}{R}-2a\cos\theta}{2\sqrt{R^2+\left(\frac{Ra}{R}\right)^2-2Ra\cos\theta}}\right]\\
&\xlongequal{r=R} -\frac{q}{4\pi}\left[\frac{a-R}{\sqrt{R^2+a^2-2Ra\cos\theta}^3}\right]
\end{aligned}
$$

$$
\begin{aligned}
W &= \int_\infty^a q\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{l}\\
&= \int_\infty^a \frac{q^2R}{4\pi\varepsilon_0 r\left(r-\frac{R^2}{r}\right)^2}\,\mathrm{d}r\\
&= \int_\infty^a \frac{q^2R}{8\pi\varepsilon_0\left(r^2-R^2\right)^2}\,\mathrm{d}r^2\\
&= -\left.\frac{q^2R}{8\pi\varepsilon_0\left(r^2-R^2\right)}\right|_\infty^a\\
&= -\frac{q^2R}{8\pi\varepsilon_0\left(a^2-R^2\right)}
\end{aligned}
$$

2.6.4 一条无限长均匀带电线，电荷线密度为 λ，它距一个接地导体板距离为 d。带电线平行于 x 轴并位于 x 轴上方，导体板为 xy 平面

(a) 求出导体板上方的电势。

(b) 求出导体板上的诱导电荷的面密度。

(a):

$$U=\frac{\lambda}{2\pi\varepsilon_0|x_3-d|}\ln|x_3-d|-\frac{\lambda}{2\pi\varepsilon_0(x_3+d)}\ln(x_3+d)$$

(b):

$$
\begin{aligned}
E_3 &= \frac{\lambda}{\pi\varepsilon_0 d}\\
\sigma = \varepsilon_0 E_3 &= \frac{\lambda}{\pi d}
\end{aligned}
$$

2.6.5 **两个半无限大接地导体板一端相接形成一个直角。在它们之间的区域有一个点电荷 q，计算这个区域内的电势。作用在 q 上的力是什么？把 q 从无限远处移到所示位置需做多少功？假定两板形成的角度不是 $\frac{\pi}{2}$，而是另外的一些角度，你还能用镜像法求解问题吗？如果不能，对什么样的特殊角度仍然可以用镜像法求解？**

假设在 $(a,a),(-a,-a)$ 处有电荷 q, $(a,-a),(-a,a)$ 有电荷 $-q$, 当 $x=0$, $z=0$ 时

$$U = \frac{q}{4\pi}\left(\frac{1}{\sqrt{a^2+(y-a)^2}}+\frac{1}{\sqrt{a^2+(y+a)^2}}-\frac{1}{\sqrt{a^2+(y-a)^2}}-\frac{1}{\sqrt{a^2+(y+a)^2}}\right)=0$$

$$\begin{aligned}
\boldsymbol{F} &= q\boldsymbol{E} = -\frac{q^2\boldsymbol{e}_1}{4\pi\varepsilon_0 a^2}-\frac{q^2\boldsymbol{e}_2}{4\pi\varepsilon_0 a^2}+\frac{\frac{\sqrt{2}}{2}q^2\boldsymbol{e}_1+\frac{\sqrt{2}}{2}q^2\boldsymbol{e}_2}{4\pi\varepsilon_0 2a^2}\\
|\boldsymbol{F}| &= \left(\sqrt{2}-\frac{1}{2}\right)\frac{q^2}{4\pi\varepsilon_0 a^2}\\
W &= \int_\infty^a \left(\sqrt{2}-\frac{1}{2}\right)\frac{\sqrt{2}q^2}{4\pi\varepsilon_0 r^2}\,\mathrm{d}r\\
&= -\left(\sqrt{2}-\frac{1}{2}\right)\frac{\sqrt{2}q^2}{4\pi\varepsilon_0 r}\bigg|_\infty^a\\
&= -\left(\sqrt{2}-\frac{1}{2}\right)\frac{\sqrt{2}q^2}{4\pi\varepsilon_0 a}
\end{aligned}$$

在无源区域内，电势满足拉普拉斯方程：

$$\nabla^2 V(x,y)=0.$$

在二维情况下，可引入复势

$$W(z)=\Phi(x,y)+i\Psi(x,y),$$

其中

$$\Phi = V.$$

$W(z)$ 为解析函数的充要条件是 Φ 与 Ψ 满足 Cauchy–Riemann 条件，而这等价于 Φ 满足拉普拉斯方程。因此，求解二维静电问题等价于构造合适的解析函数 $W(z)$。

二维中，点电荷对应的 Green 函数为对数型：

$$W_0(z) = -\frac{q}{2\pi\epsilon_0}\ln(z-z_0),$$

其电势为

$$U(z) = -\frac{q}{2\pi\epsilon_0}\ln|z-z_0|.$$

该表达式在 $z=z_0$ 处具有对数奇点，对应于二维点电荷。

定义映射

$$w = z^{\pi/\alpha}.$$

该映射具有如下性质：

- 若 $0<\arg z<\alpha$，则 $0<\arg w<\pi$；
- 楔形区域被映射为上半平面；

- $\arg z = 0, \alpha$ 被映射为实轴。

因此，楔形导体边界在 w 平面中对应于接地的实轴。
在上半平面中，实轴接地，位于 w_0（$\Im w_0 > 0$）的点电荷的复势为

$$W(w) = -\frac{q}{2\pi\epsilon_0} \ln \frac{w - w_0}{w - \overline{w_0}}.$$

该表达式满足：

- 在上半平面内调和；
- 在实轴上 $|w - w_0| = |w - \overline{w_0}|$，因而 $V = 0$；
- 在 $w = w_0$ 处具有正确的对数奇点。

这是由唯一性定理保证的解。
将 $w = z^{\pi/\alpha}$ 代回，得到楔形区域内的复势：

$$W(z) = -\frac{q}{2\pi\epsilon_0} \ln \frac{z^{\pi/\alpha} - z_0^{\pi/\alpha}}{z^{\pi/\alpha} - \overline{z_0}^{\pi/\alpha}}.$$

电势为其实部：

$$U(z) = -\frac{q}{2\pi\epsilon_0} \ln \left| \frac{z^{\pi/\alpha} - z_0^{\pi/\alpha}}{z^{\pi/\alpha} - \overline{z_0}^{\pi/\alpha}} \right|.$$

这是任意楔角 α 下的严格解。
若

$$\alpha = \frac{\pi}{n}, \quad n \in \mathbb{N},$$

则

$$w = z^n$$

是单值多项式映射。
利用因式分解：

$$z^n - z_0^n = \prod_{k=0}^{n-1} \left(z - z_0 e^{2\pi i k/n} \right),$$

电势可写为有限和：

$$U(z) = -\frac{q}{2\pi\epsilon_0} \sum_{k=0}^{n-1} \ln \left| \frac{z - z_0 e^{2\pi i k/n}}{z - \overline{z_0} e^{2\pi i k/n}} \right|.$$

这正对应于有限多个镜像电荷的叠加。
若 $\alpha \neq \pi/n$，则 $z^{\pi/\alpha}$ 为多值函数，解析延拓将产生无限多个像点，镜像法不再以有限求和形式成立。

2.7 分离变量法

2.7.1 直角坐标系

两个无限大接地金属平板平行于 xz 平面放置，一个位于 $y = 0$，另一个位于 $y = a$。在 $x = 0$ 两板的左端点，被与两板绝缘的无限长带封闭，带子上维持特定的电势 $U_0(y)$。求出这个“夹缝”中的电势。

由于几何结构和边界条件在 z 方向具有平移对称性，且 U_0 与 z 无关，物理解必然满足

$$\frac{\partial \Phi}{\partial z} = 0.$$

因此问题严格退化为二维拉普拉斯方程：

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

其边界条件为

$$U \xlongequal{y=0} 0$$
$$U \xlongequal{y=a} 0$$
$$U \xlongequal{x=0} U_0(y)$$
$$U \xrightarrow{x\to\infty} 0$$

设解为

$$U(x, y) = X(x)Y(y).$$

代入 Laplace 方程得

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0.$$

因此两项必须分别等于常数，记为 $-k^2$：

$$Y''(y) + k^2 Y(y) = 0,$$
$$X''(x) - k^2 X(x) = 0.$$

非平凡解存在当且仅当

$$k = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots$$

对应本征函数为

$$Y_n(y) = \sin\frac{n\pi y}{a}.$$

对每个 $k_n = n\pi/a$，横向方程为

$$X_n''(x) - \left(\frac{n\pi}{a}\right)^2 X_n(x) = 0.$$

通解为

$$X_n(x) = A_n e^{-(n\pi/a)x} + B_n e^{+(n\pi/a)x}.$$

由远处边界条件，要求 $V \to 0$，

$$B_n = 0.$$

利用线性叠加原理，电势的一般解为

$$V(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{a}\, e^{-(n\pi/a)x}.$$

在 $x = 0$ 处，要求

$$V(0, y) = U_0(y).$$

因此

$$U_0(y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{a}.$$

$$A_n=\frac{2}{a}\int_0^a U_0(y)\sin\frac{n\pi y}{a}\,dy$$

综上，夹缝区域中的电势为

$$U=\sum_{n=1}^{\infty}\left[\frac{2}{a}\int_0^a U_0(y')\sin\frac{n\pi y'}{a}\,\mathrm{d}y'\right]\sin\frac{n\pi y}{a}e^{-(n\pi/a)x}$$

若问题在 z 方向不具平移对称性，可进一步设

$$\Phi=X(x)Y(y)Z(z),$$

并引入

$$Z''+\lambda^2Z=0,\qquad Z(z)=e^{i\lambda z},\quad \lambda\in\mathbb{R}.$$

此时解的结构变为

$$\Phi(x,y,z)=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}C_n(\lambda)\sin\frac{n\pi y}{a}\,e^{-\sqrt{(n\pi/a)^2+\lambda^2}\,x}\,e^{i\lambda z}\,d\lambda.$$

2.7.2 球坐标系

在球坐标系中，拉普拉斯算符为

$$\boldsymbol{\nabla}^2U=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2U}{\partial\varphi^2}$$

设电势可以写成完全分离的形式, 即 $U=R(r)\Theta(\theta)\Phi(\varphi)$

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{U}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2U}{\partial\varphi^2}&=0\\
\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2U}{\partial\varphi^2}&=0\\
\frac{\partial}{\partial r}\left(r^2\frac{\partial R\Theta\Phi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial R\Theta\Phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2R\Theta\Phi}{\partial\varphi^2}&=0\\
\frac{1}{Rr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{\Theta r^2\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\frac{1}{\Phi r^2\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}&=0\\
-\left[\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\frac{1}{\Phi\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}\right]&=\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)
\end{aligned}$$

由于左边仅依赖 r，右边仅依赖角变量，两边必须等于同一个常数。
引入分离常数 $l(l+1)$, $l\in Z$
于是得到：

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)-l(l+1)R=0.\tag{2.10}$$

$$\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+l(l+1)=0.\tag{2.11}$$

继续分离 $\Theta(\theta)\Phi(\varphi)$，令

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=-m^2,\qquad m\in Z.$$

于是得到：

φ 方程

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}+m^2\Phi=0, \tag{2.12}$$

其解为

$$\Phi_m(\varphi)=C_1e^{im\varphi}$$

θ 方程

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0. \tag{2.13}$$

这是关联勒让德方程，其在 $\theta\in[0,\pi]$ 上正则的解为

$$\Theta_{lm}(\theta)=P_{lm}(\cos\theta),\qquad l=0,1,2,\ldots,\qquad |m|\le l.$$

其中,

$$P_{lm}(x)\equiv\sqrt{(1+x^2)^m}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x) \tag{2.14}$$

$P_l(x)$ 由罗德里格 (Rodrigue) 公式定义:

$$P_l(x)\equiv\frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l(x^2-1)^l \tag{2.15}$$

罗德里格公式显然仅对非负的整数 l 成立。另外，它仅提供给我们一个解。但是式2.13应当有两个解。情况是那些另外的解在 $\theta=0$ 和/或 $\theta=\pi$ 发散

为了方便，将角向部分合并，定义球谐函数

$$Y_{lm}=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\varphi} \tag{2.16}$$

径向方程为

$$r^2R''+2rR'-l(l+1)R=0,$$

通解为

$$R_l(r)=A_lr^l+B_lr^{-(l+1)} \tag{2.17}$$

将各部分组合，得到球坐标下拉普拉斯方程的一般解

$$U(r,\theta,\varphi)=\left(A_{lm}r^l+B_{lm}r^{-(l+1)}\right)Y_{lm}(\theta,\varphi) \tag{2.18}$$

当 $m=0$ 时, 式2.13退化为

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+l(l+1)\Theta=0 \tag{2.19}$$

式2.13的解退化为

$$\Theta(\theta)=P_l(\cos\theta) \tag{2.20}$$

球坐标下拉普拉斯方程的一般解退化为

$$U(r,\theta)=\left(A_lr^l+B_lr^{-(l+1)}\right)P_l(\cos\theta) \tag{2.21}$$

2.7.3 勒让德多项式性质的证明

$l \in Z$ 的证明

$$
\begin{aligned}
0 &= \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+l(l+1)\Theta \\
&\xlongequal{u=\cos\theta} \frac{1}{\sqrt{1-u^2}}\frac{\mathrm{d}}{\mathrm{d}\arccos u}\left(\sqrt{1-u^2}\frac{\mathrm{d}\Theta}{\mathrm{d}\arccos u}\right)+l(l+1)\Theta \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left[\left(1-u^2\right)\frac{\mathrm{d}\Theta}{\mathrm{d}u}\right]+l(l+1)\Theta \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}\Theta}{\mathrm{d}u}\right)-\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\frac{\mathrm{d}\Theta}{\mathrm{d}u}\right)+l(l+1)\Theta \\
&= \frac{\mathrm{d}^2\Theta}{\mathrm{d}u^2}-u^2\frac{\mathrm{d}^2\Theta}{\mathrm{d}u^2}-2u\frac{\mathrm{d}\Theta}{\mathrm{d}u}+l(l+1)\Theta
\end{aligned}
$$

设

$$
\begin{aligned}
\Theta &= \sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
0 &= \frac{\mathrm{d}^2\sum\limits_{n=0}^{\infty}\frac{a_n}{n!}u^n}{\mathrm{d}u^2}-u^2\frac{\mathrm{d}^2\sum\limits_{n=0}^{\infty}\frac{a_n}{n!}u^n}{\mathrm{d}u^2}-2u\frac{\mathrm{d}\sum\limits_{n=0}^{\infty}\frac{a_n}{n!}u^n}{\mathrm{d}u}+l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
&= \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n-u^2\sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n-2u\sum_{n=0}^{\infty}\frac{a_{n+1}}{n!}u^n+l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
&= \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n-\sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^{n+2}-2\sum_{n=0}^{\infty}\frac{a_{n+1}}{n!}u^{n+1}+l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
&= \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n-\sum_{n=2}^{\infty}\frac{a_n}{(n-2)!}u^n-2\sum_{n=1}^{\infty}\frac{a_n}{(n-1)!}u^n+l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
&= \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n-\sum_{n=0}^{\infty}\frac{a_n}{(n-2)!}u^n-2\sum_{n=0}^{\infty}\frac{a_n}{(n-1)!}u^n+l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n\ (n>1) \\
&= \frac{a_{n+2}}{n!}-\frac{a_n}{(n-2)!}-2\frac{a_n}{(n-1)!}+l(l+1)\frac{a_n}{n!} \\
&= a_{n+2}-a_n n(n-1)-2a_n n+l(l+1)a_n \\
a_{n+2} &= [n(n+1)-l(l+1)]a_n
\end{aligned}
$$

由高斯判别法可得级数在 $u=\pm1$ 时发散, 因此 $l \in Z$

正交性的证明 取两个不同阶数的勒让德函数 P_a、P_b，满足：

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right] + a(a+1)P_a \\
0 &= \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right] + b(b+1)P_b \\
0 &= \left\{\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right] + a(a+1)P_a\right\}P_b - \left\{\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right] + b(b+1)P_b\right\}P_a \\
&= \int_{-1}^{1}\left\{\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right] + a(a+1)P_a\right\}P_b - \left\{\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right] + b(b+1)P_b\right\}P_a\,\mathrm{d}u \\
&= \int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right]P_b + a(a+1)P_aP_b - \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right]P_a - b(b+1)P_bP_a\,\mathrm{d}u \\
&= \int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right]P_b - \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right]P_a\,\mathrm{d}u + \left[a(a+1)-b(b+1)\right]\int_{-1}^{1}P_aP_b\,\mathrm{d}u
\end{aligned}
$$

由分部积分可得

$$
\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right]P_b - \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right]P_a\,\mathrm{d}u = \left.(1-u^2)(P_bP_a' - P_aP_b')\right|_{-1}^{1} = 0
$$

得证

$$
\left[a(a+1)-b(b+1)\right]\int_{-1}^{1}P_aP_b\,\mathrm{d}u = 0\,\mathrm{d}u
$$

完备性的证明略过

2.7.4 两个无限长接地金属板，分别在 $y=0$ 和 $y=a$ 放置，在 $x=\pm b$ 的侧边连接有电势为 U_0 的两个金属带。求出这个矩形管中的电势。

此时边界条件为:

$$
\begin{aligned}
U &\xlongequal{y=0} 0 \\
U &\xlongequal{y=a} 0 \\
U &\xlongequal{x=b} U_0 \\
U &\xlongequal{x=-b} U_0
\end{aligned}
$$

做法同前解得

$$
U = (Ae^{kx} + Be^{-kx})(C\sin\frac{n\pi}{a}x + D\cos\frac{n\pi}{a}x)
$$

因为 $U(-x) = U(x)$, $U \xlongequal{y=0} 0$, 所以 $A = B$, $D = 0$, 并把系数吸进 C 得

$$
U = C\cosh\frac{n\pi}{a}x\sin\frac{n\pi}{a}y
$$

余下的事是构造一般的叠加解, 设定系数 C_n，使其拟合边界条件

$$
U(b,y) = \sum_{n=1}^{\infty} C_n\cosh\frac{n\pi}{a}b\sin\frac{n\pi}{a}y = U_0
$$

因为 $U(b,y) = U(b,-y)$, 所以 $\sin\frac{n\pi}{a}y$ 为偶函数, n 为奇数, 即

$$
U(b,y) = \sum_{n=0}^{\infty} C_n\cosh\frac{2n+1}{a}\pi b\sin\frac{2n+1}{a}\pi y = U_0
$$

$$
\begin{aligned}
C_n \cosh \frac{2n+1}{a}\pi b &= \frac{2}{a}\int_0^a U_0 \sin \frac{2n+1}{a}\pi y \,\mathrm{d}y \\
&= \frac{4U_0}{(2n+1)\pi}\left(\cosh \frac{2n+1}{a}\pi b\right)^{-1} \\
U &= \sum_{n=0}^{\infty} \frac{4U_0}{(2n+1)\pi}\left(\cosh \frac{2n+1}{a}\pi b\right)^{-1} \cosh \frac{n\pi}{a}x \sin \frac{n\pi}{a}y
\end{aligned}
$$

2.7.5 一个半径为 R 的球面上的电势为 $U_0 = k\cos 3\theta$。求出球面内外的电势以及球面上的电荷面密度 $\sigma(\theta)$。（假定球内和球外没有电荷分布。）

球内:

$$
\begin{aligned}
U(R,\theta) &= k\cos 3\theta \\
\left(A_l R^l + B_l R^{-(l+1)}\right) P_l(\cos\theta) &= k\cos 3\theta \\
A_l R^l P_l(\cos\theta) &= k\cos 3\theta \\
\int_0^\pi A_l^2 R^{2l} P_l^2(\cos\theta)\sin\theta \,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta A_l R^l P_l(\cos\theta)\sin\theta \,\mathrm{d}\theta \\
\int_0^\pi A_l R^l P_l^2(\cos\theta)\sin\theta \,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta P_l(\cos\theta)\sin\theta \,\mathrm{d}\theta \\
\int_0^\pi A_0 R^0 P_0^2(\cos\theta)\sin\theta \,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta P_0(\cos\theta)\sin\theta \,\mathrm{d}\theta \\
\int_0^\pi A_0 \sin\theta \,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta \sin\theta \,\mathrm{d}\theta \\
A_0 &= 0 \\
\int_0^\pi A_1 R P_1^2(\cos\theta)\sin\theta \,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta P_1(\cos\theta)\sin\theta \,\mathrm{d}\theta \\
\int_0^\pi A_1 R\cos^2\theta \sin\theta \,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta \cos\theta \sin\theta \,\mathrm{d}\theta
\end{aligned}
$$

不想算积分了

2.7.6 假定一个球面上的电势为 $U_0(\theta)$，并且球内球外没有电荷分布。证明球面上的电荷面密度为 $\sigma = \frac{\varepsilon_0}{2R}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta' \,\mathrm{d}\theta'$

$$
\begin{aligned}
U(R,\theta) &= U_0(\theta) \\
\left(A_l R^l + B_l R^{-(l+1)}\right) P_l(\cos\theta) &= U_0(\theta)
\end{aligned}
$$

球外:

$$
\begin{aligned}
B_l R^{-(l+1)} P_l(\cos\theta) &= U_0(\theta) \\
\int_0^\pi B_l^2 R^{-2(l+1)} P_l^2(\cos\theta)\sin\theta \,\mathrm{d}\theta &= \int_0^\pi B_l R^{-(l+1)} P_l(\cos\theta) U_0(\theta)\sin\theta \,\mathrm{d}\theta \\
\int_0^\pi B_l R^{-(l+1)} P_l^2(\cos\theta)\sin\theta \,\mathrm{d}\theta &= \int_0^\pi P_l(\cos\theta) U_0(\theta)\sin\theta \,\mathrm{d}\theta \\
B_l &= \frac{\int_0^\pi P_l(\cos\theta) U_0(\theta)\sin\theta \,\mathrm{d}\theta}{\int_0^\pi R^{-(l+1)} P_l^2(\cos\theta)\sin\theta \,\mathrm{d}\theta}
\end{aligned}
$$

球内:

$$A_l R^l P_l(\cos\theta) = U_0(\theta)$$
$$\int_0^\pi A_l R^l P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta = \int_0^\pi U_0(\theta)P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta$$
$$A_l = \frac{\int_0^\pi U_0(\theta)P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta}{\int_0^\pi R^l P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta}$$

由连续性可得

$$A_l R^l P_l(\cos\theta) = B_l R^{-(l+1)} P_l(\cos\theta)$$
$$A_l R^l = B_l R^{-(l+1)}$$

$$\sigma = -\varepsilon_0 \frac{\partial U}{\partial r}$$
$$\frac{\varepsilon_0}{2R}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = -\varepsilon_0\frac{\partial U}{\partial r}$$
$$\frac{1}{2R}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = -\left.\frac{\partial B_l r^{-(l+1)}P_l(\cos\theta)}{\partial r} + \frac{\partial A_l r^l P_l(\cos\theta)}{\partial r}\right|_{r=R}$$
$$\frac{1}{2R}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = (l+1)B_l R^{-(l+2)}P_l(\cos\theta) + lA_l R^{l-1}P_l(\cos\theta)$$
$$\frac{1}{2}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = (l+1)B_l R^{-(l+1)}P_l(\cos\theta) + lA_l R^{l}P_l(\cos\theta)$$
$$\frac{1}{2}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = (l+1)B_l R^{-(l+1)}P_l(\cos\theta) + lB_l R^{-(l+1)}P_l(\cos\theta)$$
$$\frac{1}{2}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = (2l+1)B_l R^{-(l+1)}P_l(\cos\theta)$$

逐项比较可得 (不使用爱因斯坦求和约定)

$$\frac{1}{2}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = (2l+1)B_l R^{-(l+1)}P_l(\cos\theta)$$
$$\frac{1}{2}(2l+1)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = B_l R^{-(l+1)}$$
$$\frac{1}{2}(2l+1)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' = \frac{\int_0^\pi P_l(\cos\theta)U_0(\theta)\sin\theta\,\mathrm{d}\theta}{\int_0^\pi R^{-(l+1)}P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta}R^{-(l+1)}$$
$$\frac{1}{2}(2l+1) = \frac{1}{\int_0^\pi P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta}$$

2.7.7 一个带电金属球 (电荷为 Q，半径为 R) 置于均匀外电场 $\boldsymbol{E}_0$ 中, 求出球外的电势。

边界条件:

$$\lim_{r\to\infty} U \to E_0 r\cos\theta$$
$$U(R) = 0$$
$$\left(A_l R^l + B_l R^{-(l+1)}\right) P_l(\cos\theta) = 0$$
$$\left(A_l R^l + B_l R^{-(l+1)}\right) P_l(\cos\theta) = 0$$

显然 $A_1=-E_0$。, 其余诸 A_l 为零

$$
\begin{aligned}
-E_0R^1+B_1R^{-2}=0\\
B_1=E_0R^3
\end{aligned}
$$

2.8 习题

2.8.1 一个均匀带电、边长为 $2a$ 的正方形面, 电荷面密度为 σ。求出距中心高度为 z 处的电场。

$$
\begin{aligned}
&\int_0^a \frac{\sigma}{\pi\varepsilon_0\sqrt{2L^2+z^2}}\boldsymbol{e}_3\frac{2Lz}{L^2+z^2}\,\mathrm{d}L\\
\xlongequal{2L^2=z^2\tan^2\theta}&\int_0^{\arctan\frac{a}{\sqrt{2}z}} \frac{\sigma}{\pi\varepsilon_0\sqrt{z^2\tan^2\theta+z^2}}\boldsymbol{e}_3\frac{2z\tan\theta z}{z^2\tan^2\theta+2z^2}\,\mathrm{d}z\tan\theta\\
=&\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{\sigma\cos\theta}{\pi\varepsilon_0 z\cos^2\theta}\boldsymbol{e}_3\frac{2z\tan\theta z}{z^2\tan^2\theta+2z^2}\,\mathrm{d}z\theta\\
=&\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{\sigma}{\pi\varepsilon_0\cos\theta}\boldsymbol{e}_3\frac{2\tan\theta}{\tan^2\theta+2}\,\mathrm{d}\theta\\
=&\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\frac{\tan\theta}{\tan^2\theta\cos\theta+2\cos\theta}\,\mathrm{d}\theta\\
=&\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\frac{\sin\theta}{\sin^2\theta+2\cos^2\theta}\,\mathrm{d}\theta\\
=&-\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\frac{1}{1+\cos^2\theta}\,\mathrm{d}\cos\theta\\
=&-\frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\arctan\cos\theta\Big|_0^{\arctan\frac{\sqrt{2}a}{z}}\\
=&\frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\left(\frac{\pi}{4}-\arctan\frac{z}{\sqrt{2a^2+z^2}}\right)
\end{aligned}
$$

2.8.2 已知电场 $\boldsymbol{E}=\dfrac{A\boldsymbol{e}_r+B\sin\theta\cos\phi\boldsymbol{e}_\phi}{r}$, 求电荷密度

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E}==&\frac{1}{sr}\frac{\partial srE_r}{\partial r}+\frac{1}{s}\frac{\partial E_\phi}{\partial\phi}\\
=&\frac{1}{sr}\frac{\partial sA}{\partial r}+\frac{1}{sr}\frac{\partial B\sin\theta\cos\phi}{\partial\phi}\\
=&\frac{A}{r^2}-\frac{1}{r^2}B\sin\phi
\end{aligned}
$$

2.8.3 一个均匀带电球体, 求出南半球与北半球之间的净相互作用力

$$
\begin{aligned}
E_z &= E_r \cos\theta = \frac{\rho\cos\theta}{4\pi\varepsilon_0 r^2}\frac{4\pi r^3}{3} = \frac{\rho\cos\theta r}{3\varepsilon_0} \\
F &= \int_0^R \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^{\frac{\pi}{2}} r^2 \sin\theta \rho E_z \,\mathrm{d}\theta \\
&= \int_0^R \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^{\frac{\pi}{2}} r^3 \sin\theta \rho \frac{\rho\cos\theta}{3\varepsilon_0} \,\mathrm{d}\theta \\
&= \pi \int_0^R r^3 \,\mathrm{d}r \int_0^{\frac{\pi}{2}} \sin\theta \rho^2 \frac{2\cos\theta}{3\varepsilon_0} \,\mathrm{d}\theta \\
&= \pi R^4 \int_0^{\frac{\pi}{2}} \sin\theta \rho^2 \frac{\cos\theta}{6\varepsilon_0} \,\mathrm{d}\theta \\
&= \pi R^4 \int_0^{\frac{\pi}{2}} \rho^2 \frac{\sin 2\theta}{24\varepsilon_0} \,\mathrm{d}2\theta \\
&= -\rho^2 R^4 \pi \frac{\cos 2\theta}{24\varepsilon_0}\bigg|_0^{\frac{\pi}{2}} \\
&= \frac{\rho^2 R^4 \pi}{12\varepsilon_0}
\end{aligned}
$$

2.8.4 一个半径为 R 的倒置半球面均匀带电, 电荷面密度为 σ。求出北极与球心处的电势差

$$
\begin{aligned}
U_{\text{半球北极}} &= \mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\frac{\sigma R\sin\theta}{2\varepsilon_0}\frac{1}{\sqrt{R^2\left(1-\cos\theta\right)^2+R^2\sin^2\theta}}R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\frac{\sigma\sin^2\theta}{2\varepsilon_0}\frac{1}{\sqrt{\left(1-\cos\theta\right)^2+\sin^2\theta}}R^2\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\frac{\sigma\sin\theta}{2\varepsilon_0}\frac{1}{\sqrt{2-2\cos\theta}}R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\frac{\sigma 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sqrt{2}\varepsilon_0}\frac{1}{\sqrt{1-\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}}R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\frac{\sigma\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sqrt{2}\varepsilon_0}\frac{1}{\sqrt{2\sin^2\frac{\theta}{2}}}R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\frac{\sigma\cos\frac{\theta}{2}}{2\varepsilon_0}R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\frac{\sigma}{\varepsilon_0}R\,\mathrm{d}\sin\frac{\theta}{2} \\
&= \left.\frac{\sigma\sin\frac{\theta}{2}}{\varepsilon_0}R\right|_0^{\frac{\pi}{2}} \\
&= \frac{\sqrt{2}\sigma}{2\varepsilon_0}R \\
U_{\text{半球球心}} &= \frac{2\pi R^2\sigma}{4\pi\varepsilon_0 R} = \frac{\sigma R}{2\varepsilon_0} \\
\Delta U &= \frac{\sigma}{2\varepsilon_0}\frac{R^2}{\sqrt{R^2}}+\frac{\sigma R}{2\varepsilon_0}-\frac{\sigma}{2\varepsilon_0}\frac{R^2}{R+\sqrt{R^2+R^2}}-\frac{\sqrt{2}\sigma}{2\varepsilon_0}R \\
&= \frac{\sigma R}{2\varepsilon_0}-\frac{\sqrt{2}\sigma}{2\varepsilon_0}R \\
&= \frac{\sigma R}{2\varepsilon_0}-\frac{\sqrt{2}\sigma}{2\varepsilon_0}R
\end{aligned}
$$

2.8.5 一个半径为 R 的球体, 电荷密度 $\rho = kr$, 求能量

$$
\begin{aligned}
|\boldsymbol{E}| &= \begin{cases} \dfrac{1}{4\pi\varepsilon_0 x_3^2}\displaystyle\int_0^R 4\pi k r^3 \,\mathrm{d}r \,(x_3 \geq R) \\ \dfrac{1}{4\pi\varepsilon_0 x_3^2}\displaystyle\int_0^{x_3} 4\pi k r^3 \,\mathrm{d}r \,(x_3 < R) \end{cases} \\
&= \begin{cases} \dfrac{1}{4\varepsilon_0 x_3^2} k R^4 \,(x_3 \geq R) \\ \dfrac{1}{4\varepsilon_0 x_3^2} k x_3^4 \,(x_3 < R) \end{cases} \\
&= \begin{cases} \dfrac{kR^4}{4\varepsilon_0 x_3^2} \,(x_3 \geq R) \\ \dfrac{k x_3^2}{4\varepsilon_0} \,(x_3 < R) \end{cases} \\
U &= \frac{\varepsilon_0}{2} 4\pi \int_0^R \left(\frac{k x_3^2}{4\varepsilon_0}\right)^2 x_3^2 \,\mathrm{d}x_3 + \frac{\varepsilon_0}{2} 4\pi \int_R^\infty \left(\frac{kR^4}{4\varepsilon_0 x_3^2}\right)^2 x_3^2 \,\mathrm{d}x_3 \\
&= 2\pi \int_0^R \frac{k^2 x_3^6}{16\varepsilon_0} \,\mathrm{d}x_3 + 2\pi \int_R^\infty \frac{k^2 R^8}{16\varepsilon_0 x_3^2} \,\mathrm{d}x_3 \\
&= \pi \frac{k^2 R^7}{56\varepsilon_0} + \pi \frac{k^2 R^8}{8\varepsilon_0 R} \\
&= \pi \frac{k^2 R^7}{7\varepsilon_0}
\end{aligned}
$$

2.8.6 电势为 $U = \frac{Ae^{-\lambda r}}{r}$, 求电场, 电荷密度, 总电荷

$$
\begin{aligned}
\boldsymbol{E} &= -\boldsymbol{\nabla} U \\
&= \frac{\partial \dfrac{Ae^{-\lambda r}}{r}}{\partial r} \boldsymbol{e}_r \\
&= \frac{\lambda r A e^{-\lambda r} + A e^{-\lambda r}}{r^2} \boldsymbol{e}_r \\
\rho &= \varepsilon_0 \boldsymbol{\nabla}^2 U \\
&= \varepsilon_0 \frac{1}{r^2} \frac{\partial r^2 a_r}{\partial r} \\
&= \varepsilon_0 \frac{1}{r^2} \frac{\partial \lambda r A e^{-\lambda r} + A e^{-\lambda r}}{\partial r} \\
&= \varepsilon_0 \frac{\lambda A e^{-\lambda r} - \lambda^2 r A e^{-\lambda r} - \lambda A e^{-\lambda r}}{r^2} \\
&= -\frac{\varepsilon_0 \lambda^2 A e^{-\lambda r}}{r} \,(r \neq 0) \\
\rho &= \frac{\varepsilon_0 A 4\pi \delta^{(3)}(r)}{r} - \frac{\varepsilon_0 \lambda^2 A e^{-\lambda r}}{r} \\
Q &= 4\pi \int_0^\infty \rho r^2 \,\mathrm{d}r \\
&= 0
\end{aligned}
$$

2.8.7 两条平行于 **z** 轴的无限长均匀带电线, 电荷线密度分别为 $+\lambda$ 和 $-\lambda$, 距离为 $2d$。
(a) 求出任意一点的电势。
(b) 证明等势面为圆柱面, 对给定的电势 U, 给出圆柱面的半径和轴的位置。

(a):

$$\begin{aligned}
U_{+\lambda} &= -\frac{\lambda}{2\pi\varepsilon_0}\ln|\boldsymbol{r}-d\boldsymbol{e}_1| \\
U_{-\lambda} &= \frac{\lambda}{2\pi\varepsilon_0}\ln|\boldsymbol{r}+d\boldsymbol{e}_1| \\
U &= -\frac{\lambda}{2\pi\varepsilon_0}\ln|\boldsymbol{r}-d\boldsymbol{e}_1| + \frac{\lambda}{2\pi\varepsilon_0}\ln|\boldsymbol{r}+d\boldsymbol{e}_1| \\
&= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(x_1+d)^2+x_2^2}{(x_1-d)^2+x_2^2}\right]
\end{aligned}$$

(b):

$$\begin{aligned}
(x_1+d)^2+x_2^2 &= k\left[(x_1-d)^2+x_2^2\right] \\
x_1^2+2x_1d+d^2+x_2^2 &= kx_1^2-2kx_1d+kd^2+kx_2^2 \\
0 &= (k-1)x_1^2-2(k+1)x_1d+(k-1)d^2+(k-1)x_2^2 \\
0 &\xlongequal{u=\frac{k+1}{k-1}} x_1^2-2ux_1d+d^2+x_2^2 \\
u^2-d^2 &= (x_1-u)^2+x_2^2 \\
R &= \sqrt{u^2-d^2} \\
&= \sqrt{\left(\frac{k+1}{k-1}\right)^2-d^2} \\
&= \sqrt{\left(\frac{e^{\frac{4\pi\varepsilon_0 U}{\lambda}}+1}{e^{\frac{4\pi\varepsilon_0 U}{\lambda}}-1}\right)^2-d^2}
\end{aligned}$$

2.8.8 在一个真空二极管中, 电子从阴极面“热蒸发”后向阳极面加速运动, 阴极电势为零, 对面阳极的电势为 U_0。在两极间隙中所形成的电子云 (称为空间电荷) 很快会达到一种分布状态, 使得阴极面上的电场为零。然后在两极板之间形成稳定的电流 **I**。假定两个极板面积 A 远大于它们之间的距离 $d(A >> d)$, 所以边界效应可以忽略。则 U, ρ, v(电子速度) 都仅是 x 的函数。

(a) 写出在两极板之间空间的泊松方程。

(b) 假定电子从阴极是从静止开始运动的, 那么在点 x, 这里电势为 $U(x)$, 电子速度为多少?

(c) 在稳定状态下, 电流 I 不依赖于 x。那么 ρ 和 v 之间的关系是什么?

(d) 利用上面的结果, 消去 ρ 和 v, 得出 U 满足的微分方程。

(e) 作为 x, U_0, d 的函数, 求出 U 的解。并与没有空间电荷的情况比较。另外作为 x 的函数, 求出 ρ 和 v。

(f) 证明 $1 = KU_0^{\frac{3}{2}}$ 求出常数 K。

(a):

$$\frac{\partial^2 U}{\partial x^2} = \frac{\rho}{\varepsilon_0}$$

(b):

$$\begin{aligned} eU &= \frac{1}{2}mv^2 \\ v &= \sqrt{\frac{2eU}{m}} \end{aligned}$$

(c):

$$\begin{aligned} I &= \frac{\Delta q}{\Delta t} \\ &= \rho A v \end{aligned}$$

(d):

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} &= \frac{\rho}{\varepsilon_0} \\ \frac{\partial^2 U}{\partial x^2} &= \frac{I}{Av\varepsilon_0} \\ \frac{\partial^2 U}{\partial x^2} &= \frac{I}{A\sqrt{\frac{2eU}{m}}\varepsilon_0} \end{aligned}$$

(e):

$$
\begin{aligned}
2\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2} &= 2\frac{\partial U}{\partial x}\frac{I}{A\sqrt{\frac{2eU}{m}}\varepsilon_0} \\
\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)^2 &= 4\frac{\partial U^{\frac{1}{2}}}{\partial x}\frac{I}{A\sqrt{\frac{2e}{m}}\varepsilon_0} \\
\left(\frac{\partial U}{\partial x}\right)^2 &= 4\frac{IU^{\frac{1}{2}}}{A\sqrt{\frac{2e}{m}}\varepsilon_0} \\
\frac{\partial U}{\partial x} &= 2\frac{I^{\frac{1}{2}}U^{\frac{1}{4}}}{A^{\frac{1}{2}}\left(\frac{2e}{m}\right)^{\frac{1}{4}}\varepsilon_0^{\frac{1}{2}}} \\
U^{-\frac{1}{4}}\partial U &= 2\frac{I^{\frac{1}{2}}m^{\frac{1}{4}}\partial x}{A^{\frac{1}{2}}(2e)^{\frac{1}{4}}\varepsilon_0^{\frac{1}{2}}} \\
\frac{4}{3}U^{\frac{3}{4}} &= 2\frac{I^{\frac{1}{2}}m^{\frac{1}{4}}x}{A^{\frac{1}{2}}(2e)^{\frac{1}{4}}\varepsilon_0^{\frac{1}{2}}} \\
U &= \frac{3^{\frac{4}{3}}I^{\frac{2}{3}}x^{\frac{4}{3}}m^{\frac{1}{4}}}{2A^{\frac{1}{2}}e^{\frac{1}{3}}\varepsilon_0^{\frac{2}{3}}} \\
v &= \sqrt{\frac{2eU}{m}} \\
&= \sqrt{\frac{2e\frac{3^{\frac{4}{3}}I^{\frac{2}{3}}x^{\frac{4}{3}}m^{\frac{1}{4}}}{2A^{\frac{1}{2}}e^{\frac{1}{3}}\varepsilon_0^{\frac{2}{3}}}}{m}} \\
&= \sqrt{\frac{3^{\frac{4}{3}}I^{\frac{2}{3}}x^{\frac{4}{3}}e^{\frac{2}{3}}}{A^{\frac{1}{2}}\varepsilon_0^{\frac{2}{3}}m^{\frac{3}{4}}}} \\
&= \frac{3^{\frac{2}{3}}I^{\frac{1}{3}}x^{\frac{2}{3}}e^{\frac{1}{3}}}{A^{\frac{1}{4}}\varepsilon_0^{\frac{1}{3}}m^{\frac{3}{8}}} \\
\rho &= \varepsilon_0\frac{\partial^2 U}{\partial x^2} \\
&= \frac{2I^{\frac{2}{3}}m^{\frac{1}{4}}\varepsilon_0^{\frac{1}{3}}}{3^{\frac{2}{3}}A^{\frac{1}{2}}e^{\frac{1}{3}}x^{\frac{2}{3}}}
\end{aligned}
$$

(f):

$$
\begin{aligned}
U &= \frac{3^{\frac{4}{3}}I^{\frac{2}{3}}x^{\frac{4}{3}}m^{\frac{1}{4}}}{2A^{\frac{1}{2}}e^{\frac{1}{3}}\varepsilon_0^{\frac{2}{3}}} \\
I^{\frac{2}{3}} &= \frac{U2A^{\frac{1}{2}}e^{\frac{1}{3}}\varepsilon_0^{\frac{2}{3}}}{3^{\frac{4}{3}}x^{\frac{4}{3}}m^{\frac{1}{4}}} \\
I &= \frac{U^{\frac{3}{2}}2^{\frac{3}{2}}A^{\frac{3}{4}}e^{\frac{1}{2}}\varepsilon_0^{\frac{2}{3}}}{3^2x^2m^{\frac{3}{2}}} \\
I &= \frac{U_0^{\frac{3}{2}}2^{\frac{3}{2}}A^{\frac{3}{4}}e^{\frac{1}{2}}\varepsilon_0^{\frac{2}{3}}}{3^2d^2m^{\frac{3}{2}}}
\end{aligned}
$$

2.8.9 假设现在极精确的测量已经揭示出库仑定律的误差。两个点电荷之间的作用力为 $\boldsymbol{F}=\frac{q_1q_2}{4\pi\varepsilon_0r^2}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\boldsymbol{e}_r$ 式中，入是一个新的自然常数。你的任务是按照这个新发现重新表述静电学。假定叠加原理仍然成立。

(a) 电场是什么?

(b) 求出一个点电荷的电势。

(c) 对一个位于原点的点电荷，证明 $\oiint_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S}+\frac{1}{\lambda^2}\iiint_V U\,\mathrm{d}\tau=\frac{q}{\varepsilon_0}$

(a):

$$\boldsymbol{E}=\frac{q}{4\pi\varepsilon_0r^2}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\boldsymbol{e}_r$$

(b):

$$\begin{aligned}
U&=\int_R^\infty\frac{q}{4\pi\varepsilon_0r^2}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\,\mathrm{d}r\\
&=\frac{q}{4\pi\varepsilon_0\lambda}\int_R^\infty\left(\frac{\lambda^2}{r^2}+\frac{\lambda}{r}\right)e^{-\frac{r}{\lambda}}\,\mathrm{d}\frac{r}{\lambda}\\
&=-\frac{q}{4\pi\varepsilon_0\lambda}\int_R^\infty\left(\frac{\lambda^2}{r^2}--\frac{\lambda}{r}\right)e^{-\frac{r}{\lambda}}\,\mathrm{d}-\frac{r}{\lambda}\\
&\xlongequal{u=-\frac{r}{\lambda}}-\frac{q}{4\pi\varepsilon_0\lambda}\int_{-\frac{R}{\lambda}}^{-\infty}\left(\frac{1}{u^2}-\frac{1}{u}\right)e^u\,\mathrm{d}u\\
&=-\left.\frac{1}{u}e^u\right|_{-\frac{R}{\lambda}}^{-\infty}\frac{q}{4\pi\varepsilon_0\lambda}\\
&=e^{-\frac{R}{\lambda}}\frac{q}{4\pi\varepsilon_0R}
\end{aligned}$$

(c):

$$\begin{aligned}
\oiint_S\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S}&=4\pi R^2\frac{q}{4\pi\varepsilon_0R^2}\left(1+\frac{R}{\lambda}\right)e^{-\frac{r}{\lambda}}=\frac{q}{\varepsilon_0}\left(1+\frac{R}{\lambda}\right)e^{-\frac{r}{\lambda}}\\
\frac{1}{\lambda^2}\iiint_V U\,\mathrm{d}\tau&=4\pi\frac{1}{\lambda^2}\int_0^R e^{-\frac{r}{\lambda}}\frac{q}{4\pi\varepsilon_0r}r^2\,\mathrm{d}r\\
&=\int_0^R-\frac{r}{\lambda}e^{-\frac{r}{\lambda}}\frac{q}{\varepsilon_0}\,\mathrm{d}-\frac{r}{\lambda}\\
&\xlongequal{u=-\frac{r}{\lambda}}\int_0^{-\frac{R}{\lambda}}ue^u\frac{q}{\varepsilon_0}\,\mathrm{d}u\\
&=\left.(u-1)\,e^u\frac{q}{\varepsilon_0}\right|_0^{-\frac{R}{\lambda}}\\
&=\left(-\frac{R}{\lambda}-1\right)e^{-\frac{R}{\lambda}}\frac{q}{\varepsilon_0}+\frac{q}{\varepsilon_0}\\
\frac{q}{\varepsilon_0}&=\oiint_S\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S}+\frac{1}{\lambda^2}\iiint_V U\,\mathrm{d}\tau
\end{aligned}$$

2.8.10 假设一个电场 $E_1=ax,\,E_2=E_3=0$，电荷密度为什么?

$$\varepsilon_0\frac{\partial ax}{\partial x}=a$$

2.8.11 所有的静电学特性都是从库仑定律的 $\frac{1}{r^2}$ 以及叠加原理导出的。因此也可以对牛顿万有引力构建类似的理论。什么是一个半径为 R, 质量为 M 的球体的引力能？假设质量密度是均匀的。利用所得结果估计太阳的引力能

根据库仑定律与万有引力定律的数学相似性, 我们可以通过替换常数直接得到均匀球体的引力自能。

在静电学中, 均匀带电球体的静电能为：

$$W_e = \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R} \tag{2.22}$$

利用类比关系得到质量为 M 的均匀球体的引力能：

$$W_g = -\frac{3}{5}\frac{GM^2}{R} \tag{2.23}$$

假设球体密度为 $\rho = \frac{M}{\frac{4}{3}\pi R^3}$。考虑已形成半径为 r 的球核, 其质量为 $m(r) = \frac{4}{3}\pi\rho r^3$。现从无穷远处移近一层厚度为 dr 的薄壳, 其质量为 $dm = 4\pi\rho r^2 dr$。此过程引力所做的功为：

$$\begin{aligned} dW &= -\frac{Gm(r)dm}{r} &(2.24)\\ &= -\frac{G}{r}\left(\frac{4}{3}\pi\rho r^3\right)\left(4\pi\rho r^2 dr\right) &(2.25)\\ &= -\frac{16}{3}\pi^2 G\rho^2 r^4 dr &(2.26)\end{aligned}$$

对整个球体从 0 到 R 积分：

$$\begin{aligned} W_g &= \int_0^R -\frac{16}{3}\pi^2 G\rho^2 r^4 dr &(2.27)\\ &= -\frac{16}{3}\pi^2 G\rho^2\left[\frac{1}{5}r^5\right]_0^R &(2.28)\\ &= -\frac{16}{15}\pi^2 G\rho^2 R^5 &(2.29)\end{aligned}$$

将 $\rho^2 = \frac{M^2}{\frac{16}{9}\pi^2 R^6}$ 代入上式, 化简得：

$$W_g = -\frac{3}{5}\frac{GM^2}{R} \tag{2.30}$$

取太阳参数：$M \approx 2.0e30kg$, $R \approx 7.0e8m$, 引力常数 $G \approx 6.67e-11m^3.kg^{-1}.s^{-2}$。代入公式得：

$$\begin{aligned} W &= -\frac{3}{5}\frac{6.67\times10^{-11}\times(2.0\times10^{30})^2}{7.0\times10^8}\\ &\approx -2.28e41J\end{aligned}$$

2.8.12 我们知道导体上的电荷是分布于其表面的，但是在表面上是如何分布的不是很容易确定的。电荷面密度可以直接计算的著名例子是椭圆面：$\frac{x_1^2}{r_1^2}+\frac{x_2^2}{r_2^2}+\frac{x_3^2}{r_3^2}=1$ 对这种情况 $\sigma=\frac{Q}{4\pi r_1r_2r_3}\left(\frac{x_1^2}{r_1^4}+\frac{x_2^2}{r_2^4}+\frac{x_3^2}{r_3^4}\right)^{-\frac{1}{2}}$

(a) 一个半径为 R 的圆盘的净电荷面密度 σ;

(b) 位于 x,y 平面一条无限长的导体“丝带”的电荷面密度 σ, 丝带沿 y 轴放置, 宽度 $2a$;

(c) 求出一个从 $x_3=-a$ 到 $x_3=a$ 的导体“针”每单位长度的电荷。

(a): 令 $r_1=r_2$

$$\begin{aligned}
\sigma &= \frac{Q}{4\pi r_1^2r_3}\left(\frac{x_1^2+x_2^2}{r_1^4}+\frac{x_3^2}{r_3^4}\right)^{-\frac{1}{2}}\\
&= \frac{Q}{4\pi r_1^2r_3}\left(\frac{x_1^2+x_2^2}{r_1^4}+\frac{r_1^2-x_1^2-x_2^2}{r_3^2r_1^2}\right)^{-\frac{1}{2}}\\
&= \frac{Q}{4\pi r_1^2}\left(\frac{x_1^2+x_2^2}{r_1^4}r_3^2+\frac{r_1^2-x_1^2-x_2^2}{r_1^2}\right)^{-\frac{1}{2}}\\
&\xlongequal{r_3\to 0} \frac{Q}{4\pi r_1^2}\left(\frac{r_1^2-x_1^2-x_2^2}{r_1^2}\right)^{-\frac{1}{2}}
\end{aligned}$$

(b): 令 $r_2=\infty$

$$\begin{aligned}
\sigma &= \frac{\lambda}{4\pi r_1r_3}\left(\frac{x_1^2}{r_1^4}+\frac{x_3^2}{r_3^4}\right)^{-\frac{1}{2}}\\
&= \frac{\lambda}{2\pi\sqrt{r_1^2-x_1^2}}
\end{aligned}$$

(c):

$$\begin{aligned}
\sigma &= \frac{Q}{4\pi r_1^2}\left(\frac{x_1^2+x_2^2}{r_1^4}r_3^2+\frac{r_1^2-x_1^2-x_2^2}{r_1^2}\right)^{-\frac{1}{2}}\\
&= \frac{Q}{4\pi r_1^2}\left(\frac{r_1^2r_3^2+r_1^2x_3^2}{r_1^4}+\frac{x_3^2}{r_1^2r_3^2}\right)^{-\frac{1}{2}}\\
&\xlongequal{r_1\to 0} \frac{Q}{2\pi\sqrt{a^2-x_3^2}}
\end{aligned}$$