电动力学

2026 年 2 月 5 日

目录

符号表

符号	含义	符号	含义
δ_{ij}	克罗内克尔符号		
$\cdot$	点乘		
$\times$	叉乘		
$\boldsymbol{a}$	矢量函数	a_i	矢量 $\boldsymbol{a}$ 的第 i 个分量
f	标量函数	$\boldsymbol{F}$	力
$\boldsymbol{B}$	磁场		
$\delta(x)$	狄拉克函数		
ε_{ijk}	Levi–Civita 符号 (三维)		
ε_0	真空介电常数	ε	相对介电常数
q	电荷量		
χ_e	电极化率		
e	元电荷	e	自然对数的底数
$\boldsymbol{e}_i$	第 i 个单位矢量	$\boldsymbol{E}$	电场强度
∇	梯度算子		
r	$\sqrt{x_i x_i}$		
ϕ	方位角		
θ	极角		
$\boldsymbol{r}$	$x_i\boldsymbol{e}_i$	$\boldsymbol{R}$	旋转矩阵
$\boldsymbol{v}$	速度		
Φ_E	电场强度通量		
λ	电荷线密度		
σ	电荷面密度		
ρ	电荷体密度		
$\boldsymbol{J}$	体电流密度		
U	电势		
$\mathrm{d}\tau$	体积微元		
$\mathbb{N}$	自然数集		
$\mathbb{Z}$	整数集		
$\boldsymbol{d}$	电偶极距	$\boldsymbol{D}$	电位移矢量
$\boldsymbol{p}$	动量	$\boldsymbol{P}$	极化强度矢量
$P_l(x)$	l 阶勒让德多项式		
W	能量		

1 矢量分析

1.1 定义

1.1.1 爱因斯坦求和约定

$$\sum_{i=1}^{3} a_i b_i \equiv a_i b_i \tag{1.1}$$

1.1.2 克罗内克尔符号

$$\delta_{ij} \equiv \begin{cases} 1 \ (i = j) \\ 0 \ (i \neq j) \end{cases} \tag{1.2}$$

性质:

$$a_i b_j \delta_{ij} = a_i b_i \tag{1.3}$$

$$\delta_{ij} \delta_{jk} = \delta_{ik} \tag{1.4}$$

1.1.3 点乘

$$\boldsymbol{a} \cdot \boldsymbol{b} \equiv a_i b_i \tag{1.5}$$

1.1.4 Levi-Civita 符号

$$\varepsilon_{ijk} \equiv \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix} \tag{1.6}$$

性质:

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik} \tag{1.7}$$

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix} \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix} \\
&= \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix}^T \begin{pmatrix} \delta_{1l} & \delta_{2l} & \delta_{3l} \\ \delta_{1m} & \delta_{2m} & \delta_{3m} \\ \delta_{1n} & \delta_{2n} & \delta_{3n} \end{pmatrix} \\
&= \begin{pmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{pmatrix} \begin{pmatrix} \delta_{1l} & \delta_{2l} & \delta_{3l} \\ \delta_{1m} & \delta_{2m} & \delta_{3m} \\ \delta_{1n} & \delta_{2n} & \delta_{3n} \end{pmatrix} \\
&= \begin{pmatrix} \delta_{1l} & \delta_{2l} & \delta_{3l} \\ \delta_{1m} & \delta_{2m} & \delta_{3m} \\ \delta_{1n} & \delta_{2n} & \delta_{3n} \end{pmatrix} \begin{pmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{pmatrix} \\
&= \begin{pmatrix} \delta_{li} & \delta_{lj} & \delta_{lk} \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \end{pmatrix}
\end{aligned} \tag{1.8}
$$

$$
\varepsilon_{ijk}\varepsilon_{lmk} = \sum_{k=1}^{3} \begin{pmatrix} \delta_{li} & \delta_{lj} & \delta_{lk} \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ki} & \delta_{kj} & 1 \end{pmatrix} = \begin{pmatrix} \delta_{li} & \delta_{lj} & 0 \\ \delta_{mi} & \delta_{mj} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi} \tag{1.9}
$$

$$
\varepsilon_{ijk}\varepsilon_{ljk} = \delta_{li}\delta_{jj} - \delta_{lj}\delta_{ji} = 3\delta_{li} - \delta_{li} = 2\delta_{li} \tag{1.10}
$$

1.1.5 叉乘

$$
\boldsymbol{a} \times \boldsymbol{b} \equiv \varepsilon_{ijk} a_i b_j \boldsymbol{e}_k = \begin{pmatrix} \boldsymbol{e}_i & \boldsymbol{e}_j & \boldsymbol{e}_k \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{pmatrix} \tag{1.11}
$$

性质:

$$
\begin{aligned}
\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{c}\cdot\varepsilon_{ijk}a_ib_j\boldsymbol{e}_k \\
&= c_k\varepsilon_{ijk}a_ib_j \\
&= \varepsilon_{ijk}a_ib_jc_k \\
&= \begin{pmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{pmatrix} a_ib_jc_k \\
&= \begin{pmatrix} \delta_{1i}a_i & \delta_{2i}a_i & \delta_{3i}a_i \\ \delta_{1j}b_j & \delta_{2j}b_j & \delta_{3j}b_j \\ \delta_{1k}c_k & \delta_{2k}c_k & \delta_{3k}c_k \end{pmatrix} \\
&= \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}
\end{aligned}
\tag{1.12}
$$

$$
\begin{aligned}
\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{c}\times\varepsilon_{ijk}a_ib_j\boldsymbol{e}_k \\
&= \varepsilon_{lkn}c_l\varepsilon_{ijk}a_ib_j\boldsymbol{e}_n \\
&= -\varepsilon_{lnk}\varepsilon_{ijk}a_ib_jc_l\boldsymbol{e}_n \\
&= -(\delta_{il}\delta_{jn}-\delta_{in}\delta_{jl})a_ib_jc_l\boldsymbol{e}_n \\
&= -\delta_{il}\delta_{jn}a_ib_jc_l\boldsymbol{e}_n+\delta_{in}\delta_{jl}a_ib_jc_l\boldsymbol{e}_n \\
&= -a_ib_jc_i\boldsymbol{e}_j+a_ib_jc_j\boldsymbol{e}_i \\
&= -(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}
\end{aligned}
\tag{1.13}
$$

$$
\boldsymbol{e}\times(\boldsymbol{e}\times\boldsymbol{b}) = -(\boldsymbol{e}\cdot\boldsymbol{e})\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{e})\boldsymbol{e} = -\boldsymbol{b}+(\boldsymbol{b}\cdot\boldsymbol{e})\boldsymbol{e} \tag{1.14}
$$

1.1.6 梯度算子

$$
\boldsymbol{\nabla} \equiv \frac{\partial}{\partial x_i}\boldsymbol{e}_i \tag{1.15}
$$

1.1.7 梯度

设位置矢量 $\boldsymbol{r}=\boldsymbol{r}(u_1,u_2,u_3)$, 拉梅系数为

$$
h_i \equiv \left\|\frac{\partial\boldsymbol{r}}{\partial u_i}\right\|, \qquad \boldsymbol{e} \equiv \frac{1}{h_i}\frac{\partial\boldsymbol{r}}{\partial u_i}
$$

且 $\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij}$。

对标量场 $f(u_1,u_2,u_3)$, 全微分为

$$
\mathrm{d}f = \frac{\partial f}{\partial u_i}\,\mathrm{d}u_i.
$$

无穷小位移 (线元) 为

$$\mathrm{d}\boldsymbol{l} = h_i \ \mathrm{d}u_i \, \boldsymbol{e}_i.$$

设 $\boldsymbol{\nabla} f = G_i \boldsymbol{e}_i$, 则

$$\mathrm{d}f = \boldsymbol{\nabla} f \cdot \boldsymbol{dl} = G_i h_i \ \mathrm{d}u_i.$$

比较系数, 得 $\boldsymbol{G} = \dfrac{1}{h_i}\dfrac{\partial f}{\partial u_i}\boldsymbol{e}_i$, 从而

$$\boldsymbol{\nabla} f = \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i \tag{1.16}$$

对于直角坐标

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x_i}\boldsymbol{e}_i \tag{1.17}$$

性质:

$$\boldsymbol{\nabla} r = \frac{\partial \sqrt{x_i x_i}}{\partial x_i}\boldsymbol{e}_i = \frac{x_i}{r}\boldsymbol{e}_i = \frac{\boldsymbol{r}}{r} \tag{1.18}$$

$$\boldsymbol{\nabla} f(r) = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_i}\boldsymbol{e}_i = \frac{\partial f}{\partial r}\frac{\boldsymbol{r}}{r} \tag{1.19}$$

1.1.8 散度

对微小增量 $\mathrm{d}u_1, \mathrm{d}u_2, \mathrm{d}u_3$, 线元在三方向的实际长度分别为

$$\mathrm{d}l_1 = h_1 \, \mathrm{d}u_1, \quad \mathrm{d}l_2 = h_2 \, \mathrm{d}u_2, \quad \mathrm{d}l_3 = h_3 \, \mathrm{d}u_3$$

因此微小长方体的体元为

$$\mathrm{d}\tau = \mathrm{d}l_1 \, \mathrm{d}l_2 \, \mathrm{d}l_3 = h_1 h_2 h_3 \, \mathrm{d}u_1 \, \mathrm{d}u_2 \, \mathrm{d}u_3. \tag{1.20}$$

设矢量场在该点为

$$\boldsymbol{a} = a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2 + a_3 \boldsymbol{e}_3,$$

其中 a_1, a_2, a_3 为在局部单位基矢下的分量 (均为 u_1, u_2, u_3 的函数)。

下面计算穿过微小长方体六个面的净通量。以 u_1 方向为例, 考察在 u_1 与 $u_1 + \mathrm{d}u_1$ 两个面:

面 u_1(在坐标 u 处, 向外法向为 $-\boldsymbol{e}_1$)：面积元为

$$\mathrm{d}\boldsymbol{a}_1^{(-)} = -\boldsymbol{e}_1 (h_2 \, \mathrm{d}u_2)(h_3 \, \mathrm{d}u_3) = -\boldsymbol{e}_1 h_2 h_3 \, \mathrm{d}u_2 \, \mathrm{d}u_3,$$

对应的通量 (近似取该面中心处分量)：

$$\Phi_1^{(-)} = \boldsymbol{a}(u_1, u_2, u_3) \cdot \mathrm{d}\boldsymbol{a}_1^{(-)} = -a_1(u_1, u_2, u_3) h_2 h_3 \, \mathrm{d}u_2 \, \mathrm{d}u_3.$$

面 $u + \mathrm{d}u$(向外法向为 $+\boldsymbol{e}_1$)：面积元

$$\mathrm{d}\boldsymbol{a}_1^{(+)} = +\boldsymbol{e}_1 h_2 h_3 \, \mathrm{d}u_2 \, \mathrm{d}u_3,$$

通量 (在 $u + du$ 处)：

$$\Phi_1^{(+)} = a_1(u_1 + du_1, u_2, u_3) h_2(u_1 + du_1, u_2, u_3) h_3(u_1 + du_1, u_2, u_3) \, \mathrm{d}u_2 \, \mathrm{d}u_3.$$

因此穿过这对面的净通量为

$$\begin{aligned}
\Delta\Phi_1 &= \Phi_1^{(+)} - \Phi_1^{(-)} \\
&= [a_1(u_1 + \mathrm{d}u_1, u_2, u_3) h_2(u_1 + \mathrm{d}u_1, u_2, u_3) h_3(u_1 + \mathrm{d}u_1, u_2, u_3) - a_1(u_1, u_2, u_3) h_2 h_3] \, \mathrm{d}u_2 \, \mathrm{d}u_3.
\end{aligned}$$

用泰勒展开到一阶并忽略高阶项, 得

$$\Delta\Phi_u = \frac{\partial}{\partial u_1}(h_2 h_3 a_1)\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 + o(\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3).$$

对 u_2 和 u_3 方向做同样的计算, 六个面的净通量近似为三项之和：

$$\Delta\Phi_{\text{total}} = \frac{\partial}{\partial u_i}\left(h_1 h_2 h_3 \frac{a_i}{h_i}\right)\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$$

由于体元按照式(1.20)为 $\mathrm{d}\tau = h_1 h_2 h_3\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3$, 故单位体积的通量密度 (即散度) 为极限：

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{a} &= \lim_{\Delta\tau\to 0}\frac{\Delta\Phi_{\text{total}}}{d\tau} \\ &= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_i}\left(h_1 h_2 h_3 \frac{a_i}{h_i}\right).\end{aligned}$$

于是得到正交曲线坐标系下的散度公式：

$$\boldsymbol{\nabla}\cdot\boldsymbol{a} = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_i}\left(h_1 h_2 h_3 \frac{a_i}{h_i}\right). \tag{1.21}$$

对于直角坐标

$$\boldsymbol{\nabla}\cdot\boldsymbol{a} = \frac{\partial a_i}{\partial x_i} \tag{1.22}$$

性质:

$$\boldsymbol{\nabla}\cdot\boldsymbol{r} = 3 \tag{1.23}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{f}(r) = \frac{\partial f_i(r)}{\partial x_i} = f_i'\frac{\partial r}{\partial x_i} = \boldsymbol{f}'\cdot\frac{\boldsymbol{r}}{r} \tag{1.24}$$

1.1.9 旋度

选取微小矩形位于 u_1-u_2 平面, 固定 u_3。矩形四边的线元素：

$$\begin{aligned}
&\text{下边: } \mathrm{d}\boldsymbol{l}_1 = h_1\,\mathrm{d}u_1\boldsymbol{e}_1, \quad \text{从}(u_1,u_2)\to(u_1+\mathrm{d}u_1,u_2)\\
&\text{右边: } \mathrm{d}\boldsymbol{l}_2 = h_2\,\mathrm{d}u_2\boldsymbol{e}_2, \quad \text{从}(u_1+\mathrm{d}u_1,u_2)\to(u_1+\mathrm{d}u_1,u_2+\mathrm{d}u_2)\\
&\text{上边: } \mathrm{d}\boldsymbol{l}_3 = -h_1\,\mathrm{d}u_1\boldsymbol{e}_1, \quad \text{从}(u_1+\mathrm{d}u_1,u_2+\mathrm{d}u_2)\to(u_1,u_2+\mathrm{d}u_2)\\
&\text{左边: } \mathrm{d}\boldsymbol{l}_4 = -h_2 du_2\boldsymbol{e}_2, \quad \text{从}(u_1,u_2+\mathrm{d}u_2)\to(u_1,u_2)
\end{aligned}$$

沿每一边计算 $\boldsymbol{a}\cdot d\boldsymbol{l}$ 并做泰勒展开 (只保留一阶项):

$$\oint \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} \approx \left[\frac{\partial}{\partial u_1}(h_2 a_2) - \frac{\partial}{\partial u_2}(h_1 a_1)\right]\mathrm{d}u_1\,\mathrm{d}u_2$$

面元面积：

$$dS_3 = h_1 h_2\,\mathrm{d}u_1\,\mathrm{d}u_2$$

旋度在 $\boldsymbol{e}_3$ 方向的分量：

$$(\boldsymbol{\nabla}\times\boldsymbol{a})_3 = \lim_{\mathrm{d}u_1,\mathrm{d}u_2\to 0}\frac{\oint\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}}{\mathrm{d}S_3} = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 a_2) - \frac{\partial}{\partial u_2}(h_1 a_1)\right]$$

同理, 对于其他两个方向：

$$(\boldsymbol{\nabla}\times\boldsymbol{a})_1 = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 a_3) - \frac{\partial}{\partial u_3}(h_2 a_2)\right], \quad (\boldsymbol{\nabla}\times\boldsymbol{a})_2 = \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(h_1 a_1) - \frac{\partial}{\partial u_1}(h_3 a_3)\right]$$

最终正交曲线坐标系下旋度公式为:

$$\nabla \times \boldsymbol{a} = \frac{h_i}{h_1 h_2 h_3} \varepsilon_{ijk} \frac{\partial (h_k a_k)}{\partial u_j} \boldsymbol{e}_i \tag{1.25}$$

对于直角坐标

$$\nabla \times \boldsymbol{a} = \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \boldsymbol{e}_k \tag{1.26}$$

性质:

$$\nabla \times \boldsymbol{r} = 0 \tag{1.27}$$

$$\nabla \times (f\boldsymbol{a}) = \varepsilon_{ijk} \frac{\partial (f a_j)}{\partial x_i} \boldsymbol{e}_k = f \varepsilon_{ijk} \frac{\partial a_j}{\partial x_i} \boldsymbol{e}_k + \varepsilon_{ijk} \frac{\partial f}{\partial x_i} a_j \boldsymbol{e}_k = f(\nabla \times \boldsymbol{a}) + (\nabla f) \times \boldsymbol{a} \tag{1.28}$$

1.1.10 并积

$$\nabla \boldsymbol{a} \equiv \begin{pmatrix} \frac{\partial a_1}{x_1} & \frac{\partial a_2}{x_1} & \frac{\partial a_3}{x_1} \\ \frac{\partial a_1}{x_2} & \frac{\partial a_2}{x_2} & \frac{\partial a_3}{x_2} \\ \frac{\partial a_1}{x_3} & \frac{\partial a_2}{x_3} & \frac{\partial a_3}{x_3} \end{pmatrix} = \frac{\partial a_i}{\partial x_j} \boldsymbol{e}_i \boldsymbol{e}_j \tag{1.29}$$

1.1.11 拉普拉斯算子

定义为梯度的散度, 由1.16和1.21得

$$\begin{aligned} \nabla^2 f &= \nabla \cdot (\nabla f) \\ &= \nabla \cdot \left(\frac{1}{h_i} \frac{\partial f}{\partial u_i} \right) \\ &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(h_1 h_2 h_3 \frac{\frac{1}{h_i} \frac{\partial f}{\partial u_i}}{h_i} \right) \\ &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(h_1 h_2 h_3 \frac{\partial f}{h_i^2 \partial u_i} \right) \end{aligned} \tag{1.30}$$

$$\nabla^2 \boldsymbol{A} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} \left(h_1 h_2 h_3 \frac{\partial A_i \boldsymbol{e}_i}{h_i^2 \partial u_i} \right) \tag{1.31}$$

对于直角坐标

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i^2} \tag{1.32}$$

$$\nabla^2 \boldsymbol{A} = \frac{\partial^2 A_i}{\partial x_i^2} \boldsymbol{e}_i \tag{1.33}$$

性质:

$$
\begin{aligned}
\boldsymbol{\nabla}^2(fg) &= \frac{\partial}{\partial x_i}\frac{\partial fg}{\partial x_i} \\
&= \frac{\partial}{\partial x_i}\left(g\frac{\partial f}{\partial x_i} + f\frac{\partial g}{\partial x_i}\right) \\
&= \frac{\partial}{\partial x_i}\left(g\frac{\partial f}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(f\frac{\partial g}{\partial x_i}\right) \\
&= \frac{\partial}{\partial x_i}\left(g\frac{\partial f}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(f\frac{\partial g}{\partial x_i}\right) \\
&= \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_i} + g\frac{\partial^2 f}{\partial x_i^2} + \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} + a\frac{\partial^2 g}{\partial x_i^2} \\
&= g\frac{\partial^2 f}{\partial x_i^2} + 2\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} + f\frac{\partial^2 g}{\partial x_i^2} \\
&= g\boldsymbol{\nabla}^2 f + 2(\boldsymbol{\nabla} f)\cdot(\boldsymbol{\nabla} g) + f\boldsymbol{\nabla}^2 g
\end{aligned} \tag{1.34}
$$

1.2 微分运算

1.2.1 和规则

$$\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g \tag{1.35}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{\nabla}\cdot\boldsymbol{a} + \boldsymbol{\nabla}\cdot\boldsymbol{b} \tag{1.36}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{\nabla}\times\boldsymbol{a} + \boldsymbol{\nabla}\times\boldsymbol{b} \tag{1.37}$$

1.2.2 积规则

$$\boldsymbol{\nabla}(fg) = g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g \tag{1.38}$$

$$
\begin{aligned}
&\boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) + (\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b} + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a} \\
=&\boldsymbol{a}\times\left(\varepsilon_{ijk}\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_k\right) + \boldsymbol{b}\times\left(\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_k\right) + \left(a_i\frac{\partial}{\partial x_i}\right)\boldsymbol{b} + \left(b_i\frac{\partial}{\partial x_i}\right)\boldsymbol{a} \\
=&\varepsilon_{lkn}a_l\varepsilon_{ijk}\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n + \varepsilon_{lkn}b_l\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n + a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j + b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&-\varepsilon_{lnk}\varepsilon_{ijk}a_l\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n - \varepsilon_{lnk}\varepsilon_{ijk}b_l\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n + a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j + b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&-(\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl})a_l\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n - (\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl})b_l\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n + a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j + b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&\left(-\delta_{il}\delta_{jn}a_l\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n + \delta_{in}\delta_{jl}a_l\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_n\right) + \left(-\delta_{il}\delta_{jn}b_l\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n + \delta_{in}\delta_{jl}b_l\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_n\right) + a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j + b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&\left(-a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j + a_j\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_i\right) + \left(-b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j + b_j\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_i\right) + a_i\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_j + b_i\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_j \\
=&a_j\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_i + b_j\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_i \\
=&\frac{\partial(a_jb_j)}{\partial x_i}\boldsymbol{e}_i \\
=&\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{b})
\end{aligned} \tag{1.39}
$$

$$\nabla\cdot(f\boldsymbol{a})=\frac{\partial fa_i}{\partial x_i}=f\frac{\partial a_i}{\partial x_i}+a_i\frac{\partial f}{\partial x_i}=f\nabla\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\nabla f \tag{1.40}$$

$$\begin{aligned}\boldsymbol{b}\cdot(\nabla\times\boldsymbol{a})-\boldsymbol{a}\cdot(\nabla\times\boldsymbol{b})&=\boldsymbol{b}\cdot(\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_k)-\boldsymbol{a}\cdot(\varepsilon_{ijk}\frac{\partial b_j}{\partial x_i}\boldsymbol{e}_k)\\&=b_k\varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}-a_k\varepsilon_{ijk}\frac{\partial b_j}{\partial x_i}\\&=b_j\varepsilon_{ijk}\frac{\partial a_i}{\partial x_k}+a_i\varepsilon_{ijk}\frac{\partial b_j}{\partial x_k}\\&=\frac{\partial\varepsilon_{ijk}a_ib_j}{\partial x_k}\\&=\nabla\cdot(\varepsilon_{ijk}a_ib_j\boldsymbol{e}_k)\\&=\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b})\end{aligned} \tag{1.41}$$

1.2.3 二阶积规则

$$\nabla\times(\nabla f)=\nabla\times(\frac{\partial f}{\partial x_i}\boldsymbol{e}_i)=\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j}\boldsymbol{e}_k=0 \tag{1.42}$$

$$\nabla\cdot(\nabla\times\boldsymbol{a})=\nabla\cdot(\varepsilon_{ijk}\frac{\partial a_i}{\partial x_j}\boldsymbol{e}_k)=\varepsilon_{ijk}\frac{\partial^2 a_i}{\partial x_j\partial x_k}=0 \tag{1.43}$$

$$\Delta f=\frac{\partial^2 f}{\partial x_i^2} \tag{1.44}$$

$$\Delta\boldsymbol{a}=\frac{\partial^2(a_j\boldsymbol{e}_j)}{\partial x_i^2} \tag{1.45}$$

$$\begin{aligned}\nabla\times(\nabla\times\boldsymbol{a})&=\nabla\times(\varepsilon_{ijk}\frac{\partial a_i}{\partial x_j}\boldsymbol{e}_k)\\&=\varepsilon_{lmn}\varepsilon_{ijl}\frac{\partial^2 a_i}{\partial x_j\partial x_m}\boldsymbol{e}_n\\&=\varepsilon_{mnl}\varepsilon_{ijl}\frac{\partial^2 a_i}{\partial x_j\partial x_m}\boldsymbol{e}_n\\&=(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\frac{\partial^2 a_i}{\partial x_j\partial x_m}\boldsymbol{e}_n\\&=\delta_{im}\delta_{jn}\frac{\partial^2 a_i}{\partial x_j\partial x_m}\boldsymbol{e}_n-\delta_{in}\delta_{jm}\frac{\partial^2 a_i}{\partial x_j\partial x_m}\boldsymbol{e}_n\\&=\frac{\partial^2 a_i}{\partial x_j\partial x_i}\boldsymbol{e}_j-\frac{\partial^2 a_i}{\partial x_j\partial x_j}\boldsymbol{e}_i\\&=\nabla(\nabla\cdot\boldsymbol{a})-\Delta\boldsymbol{a}\end{aligned} \tag{1.46}$$

1.3 积分运算

1.3.1 高斯定理

我们已经得到正交坐标系中的散度局部公式1.21。因此, 对体积元 $\Delta\tau=h_1h_2h_3\,\Delta u_1\Delta u_2\Delta u_3$ 的六个面计算净通量时：

$$\Phi^{(\text{out})}-\Phi^{(\text{in})}=\left[\frac{\partial}{\partial u_i}\left(h_1h_2h_3\frac{a_i}{h_i}\right)\right]\Delta u_1\Delta u_2\Delta u_3=(\nabla\cdot\boldsymbol{a})\,\Delta\tau.$$

将整个区域 V 分割成许多此类体积元, 内部公共面的通量相互抵消, 仅留下外边界 ∂V 的通量。令分割尺寸趋于零, 通量和体积和分别趋于各自积分, 故得到

$$\iiint_V (\boldsymbol{\nabla}\cdot\boldsymbol{a})\,\mathrm{d}\tau = \iint_{\partial V} \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S} \tag{1.47}$$

1.3.2 斯托克斯定理

我们已经得到任意正交坐标系中的旋度局部公式1.25 , 考虑由 (u_1, u_2) 增量生成的一个无穷小曲面元

$$\mathrm{d}\boldsymbol{S} = (h_1 h_2\,\mathrm{d}u_1\,\mathrm{d}u_2)\boldsymbol{e}_3.$$

沿此小曲面元的边界进行线积分, 可得

$$\oiint_{\partial S} \boldsymbol{a}\cdot d\boldsymbol{l} = (\nabla\times\boldsymbol{a})\cdot d\boldsymbol{S}.$$

将整个曲面分割成许多这样的小元, 内部公共边界的线积分相互抵消, 仅留下外边界 ∂S 的线积分。令分割尺寸趋于零, 总和趋于积分, 故得

$$\iint_S (\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S} = \oiint_{\partial S} \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} \tag{1.48}$$

1.4 曲面坐标系

1.4.1 柱坐标系

将柱坐标系坐标转化为直角系坐标的方式为

$$x_1 = s\cos\phi,\ x_2 = s\sin\phi,\ x_3 = z \tag{1.49}$$

其中 s 是坐标点到 z 轴的距离。柱坐标系和直角系坐标无限小元转换方式为

$$\begin{pmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\\ \mathrm{d}x_3\end{pmatrix} = \begin{pmatrix}\frac{x_1}{s} & -x_2 & 0\\ \frac{x_2}{s} & x_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathrm{d}s\\ \mathrm{d}\phi\\ \mathrm{d}z\end{pmatrix} \tag{1.50}$$

其拉梅系数为

$$h_1 = 1,\ h_2 = s,\ h_3 = 1 \tag{1.51}$$

其无限小位移为

$$\mathrm{d}\boldsymbol{l} = h_i\,\mathrm{d}u_i\boldsymbol{e}_i = \mathrm{d}s\boldsymbol{s} + s\,\mathrm{d}\phi\boldsymbol{\phi} + \mathrm{d}z\boldsymbol{z} \tag{1.52}$$

其体积元为

$$\mathrm{d}\tau = s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z \tag{1.53}$$

梯度

$$\boldsymbol{\nabla}f = \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i = \frac{\partial}{\partial s}\boldsymbol{s} + \frac{1}{s}\frac{\partial}{\partial\phi}\boldsymbol{\phi} + \frac{\partial}{\partial z}\boldsymbol{z} \tag{1.54}$$

散度

$$\boldsymbol{\nabla}\cdot\boldsymbol{a} = \frac{1}{s}(\frac{\partial sa_s}{\partial s} + \frac{\partial a_\phi}{\partial\phi} + \frac{\partial sa_z}{\partial z}) = \frac{1}{s}\frac{\partial sa_s}{\partial s} + \frac{1}{s}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} = \frac{a_s}{s} + \frac{\partial a_s}{\partial s} + \frac{1}{s}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \tag{1.55}$$

旋度

$$
\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{a} &= \frac{h_i}{h_1h_2h_3}\varepsilon_{ijk}\frac{\partial(h_ka_k)}{\partial u_j}\boldsymbol{e}_i \\
&= \frac{h_i}{s}\varepsilon_{ijk}\frac{\partial(h_ka_k)}{\partial u_j}\boldsymbol{e}_i \\
&= \frac{1}{s}\left(\varepsilon_{123}\frac{\partial a_3}{\partial u_2}\boldsymbol{e}_1+\varepsilon_{132}\frac{\partial sa_2}{\partial u_3}\boldsymbol{e}_1+\varepsilon_{321}\frac{\partial a_1}{\partial u_2}\boldsymbol{e}_3+s\varepsilon_{213}\frac{\partial a_3}{\partial u_1}\boldsymbol{e}_2+\varepsilon_{312}\frac{\partial sa_2}{\partial u_1}\boldsymbol{e}_3+s\varepsilon_{231}\frac{\partial a_1}{\partial u_3}\boldsymbol{e}_2\right) \\
&= \frac{1}{s}\left(\frac{\partial a_3}{\partial u_2}\boldsymbol{e}_1-\frac{\partial sa_2}{\partial u_3}\boldsymbol{e}_1-\frac{\partial a_1}{\partial u_2}\boldsymbol{e}_3-s\frac{\partial a_3}{\partial u_1}\boldsymbol{e}_2+\frac{\partial sa_2}{\partial u_1}\boldsymbol{e}_3+s\frac{\partial a_1}{\partial u_3}\boldsymbol{e}_2\right) \\
&= \frac{1}{s}\left(\frac{\partial a_z}{\partial \phi}\boldsymbol{s}-\frac{\partial sa_\phi}{\partial z}\boldsymbol{s}-\frac{\partial a_s}{\partial \phi}\boldsymbol{z}-s\frac{\partial a_z}{\partial s}\boldsymbol{\phi}+\frac{\partial sa_\phi}{\partial s}\boldsymbol{z}+s\frac{\partial a_s}{\partial z}\boldsymbol{\phi}\right) \\
&= \frac{1}{s}\left(\frac{\partial a_z}{\partial \phi}-s\frac{\partial a_\phi}{\partial z}\right)\boldsymbol{s}+\left(\frac{\partial a_s}{\partial z}-\frac{\partial a_z}{\partial s}\right)\boldsymbol{\phi}+\frac{1}{s}\left(\frac{\partial sa_\phi}{\partial s}-\frac{\partial a_s}{\partial \phi}\right)\boldsymbol{z} \qquad (1.56)
\end{aligned}
$$

拉普拉斯算子

$$
\begin{aligned}
\boldsymbol{\nabla}^2 f &= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_i}\left(h_1h_2h_3\frac{\partial f}{h_i^2\partial u_i}\right) \\
&= \frac{1}{s}\frac{\partial}{\partial u_i}\left(s\frac{\partial f}{h_i^2\partial u_i}\right) \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{h_s^2\partial s}\right)+\frac{1}{s}\frac{\partial}{\partial \phi}\left(s\frac{\partial f}{h_\phi^2\partial \phi}\right)+\frac{1}{s}\frac{\partial}{\partial z}\left(s\frac{\partial f}{h_3^2\partial z}\right) \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right)+\frac{1}{s}\frac{\partial}{\partial \phi}\left(s\frac{\partial f}{s^2\partial \phi}\right)+\frac{1}{s}\frac{\partial}{\partial z}\left(s\frac{\partial f}{\partial z}\right) \\
&= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 f}{\partial \phi^2}+\frac{\partial^2 f}{\partial z^2} \qquad (1.57)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}^2\boldsymbol{A} &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_i\boldsymbol{e}_i}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 A_i\boldsymbol{e}_i}{\partial\phi^2}+\frac{\partial^2 A_i\boldsymbol{e}_i}{\partial z^2}\\
\boldsymbol{\nabla}^2\boldsymbol{A}_s &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_s\boldsymbol{e}_s}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 A_s\boldsymbol{e}_s}{\partial\phi^2}+\frac{\partial^2 A_s\boldsymbol{e}_s}{\partial z^2}\\
\boldsymbol{\nabla}^2\boldsymbol{A}_s &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_s}{\partial s}\right)\boldsymbol{e}_s+\frac{1}{s^2}\frac{\partial}{\partial\phi}\left(A_s\frac{\partial \boldsymbol{e}_s}{\partial\phi}+\boldsymbol{e}_s\frac{\partial A_s}{\partial\phi}\right)+\frac{\partial^2 A_s}{\partial z^2}\boldsymbol{e}_s\\
\boldsymbol{\nabla}^2\boldsymbol{A}_s &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_s}{\partial s}\right)\boldsymbol{e}_s+\frac{1}{s^2}\frac{\partial}{\partial\phi}\left(A_s\boldsymbol{e}_\phi+\boldsymbol{e}_s\frac{\partial A_s}{\partial\phi}\right)+\frac{\partial^2 A_s}{\partial z^2}\boldsymbol{e}_s\\
\boldsymbol{\nabla}^2\boldsymbol{A}_s &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_s}{\partial s}\right)\boldsymbol{e}_s+\frac{1}{s^2}\left(A_s\frac{\partial \boldsymbol{e}_\phi}{\partial\phi}+\boldsymbol{e}_\phi\frac{\partial A_s}{\partial\phi}+\frac{\partial \boldsymbol{e}_s}{\partial\phi}\frac{\partial A_s}{\partial\phi}+\boldsymbol{e}_s\frac{\partial^2 A_s}{\partial\phi^2}\right)+\frac{\partial^2 A_s}{\partial z^2}\boldsymbol{e}_s\\
\boldsymbol{\nabla}^2\boldsymbol{A}_s &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_s}{\partial s}\right)\boldsymbol{e}_s+\frac{1}{s^2}\left(-A_s\boldsymbol{e}_s+\boldsymbol{e}_\phi\frac{\partial A_s}{\partial\phi}+\boldsymbol{e}_\phi\frac{\partial A_s}{\partial\phi}+\boldsymbol{e}_s\frac{\partial^2 A_s}{\partial\phi^2}\right)+\frac{\partial^2 A_s}{\partial z^2}\boldsymbol{e}_s\\
\boldsymbol{\nabla}^2\boldsymbol{A}_s &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_s}{\partial s}\right)\boldsymbol{e}_s+\frac{1}{s^2}\left(-A_s\boldsymbol{e}_s+2\boldsymbol{e}_\phi\frac{\partial A_s}{\partial\phi}+\boldsymbol{e}_s\frac{\partial^2 A_s}{\partial\phi^2}\right)+\frac{\partial^2 A_s}{\partial z^2}\boldsymbol{e}_s
\end{aligned}\tag{1.58}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}^2\boldsymbol{A}_\phi &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_\phi}{\partial s}\right)\boldsymbol{e}_\phi+\frac{1}{s^2}\frac{\partial}{\partial\phi}\left(A_\phi\frac{\partial \boldsymbol{e}_\phi}{\partial\phi}+\boldsymbol{e}_\phi\frac{\partial A_\phi}{\partial\phi}\right)+\frac{\partial^2 A_\phi}{\partial z^2}\boldsymbol{e}_\phi\\
\boldsymbol{\nabla}^2\boldsymbol{A}_\phi &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_\phi}{\partial s}\right)\boldsymbol{e}_\phi+\frac{1}{s^2}\frac{\partial}{\partial\phi}\left(-A_\phi\boldsymbol{e}_s+\boldsymbol{e}_\phi\frac{\partial A_\phi}{\partial\phi}\right)+\frac{\partial^2 A_\phi}{\partial z^2}\boldsymbol{e}_\phi\\
\boldsymbol{\nabla}^2\boldsymbol{A}_\phi &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_\phi}{\partial s}\right)\boldsymbol{e}_\phi+\frac{1}{s^2}\left(-\boldsymbol{e}_s\frac{\partial A_\phi}{\partial\phi}-A_\phi\frac{\partial \boldsymbol{e}_s}{\partial\phi}+\boldsymbol{e}_\phi\frac{\partial^2 A_\phi}{\partial\phi^2}+\frac{\partial \boldsymbol{e}_\phi}{\partial\phi}\frac{\partial A_\phi}{\partial\phi}\right)+\frac{\partial^2 A_\phi}{\partial z^2}\boldsymbol{e}_\phi\\
\boldsymbol{\nabla}^2\boldsymbol{A}_\phi &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_\phi}{\partial s}\right)\boldsymbol{e}_\phi+\frac{1}{s^2}\left(-\boldsymbol{e}_s\frac{\partial A_\phi}{\partial\phi}-A_\phi\boldsymbol{e}_\phi+\boldsymbol{e}_\phi\frac{\partial^2 A_\phi}{\partial\phi^2}-\boldsymbol{e}_s\frac{\partial A_\phi}{\partial\phi}\right)+\frac{\partial^2 A_\phi}{\partial z^2}\boldsymbol{e}_\phi\\
\boldsymbol{\nabla}^2\boldsymbol{A}_\phi &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_\phi}{\partial s}\right)\boldsymbol{e}_\phi+\frac{1}{s^2}\left(-2\boldsymbol{e}_s\frac{\partial A_\phi}{\partial\phi}-A_\phi\boldsymbol{e}_\phi+\boldsymbol{e}_\phi\frac{\partial^2 A_\phi}{\partial\phi^2}\right)+\frac{\partial^2 A_\phi}{\partial z^2}\boldsymbol{e}_\phi
\end{aligned}\tag{1.59}
$$

$$
(\boldsymbol{\nabla}^2\boldsymbol{A})_s = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_s}{\partial s}\right)\boldsymbol{e}_s+\frac{1}{s^2}\left(-A_s-2\frac{\partial A_\phi}{\partial\phi}+\frac{\partial^2 A_s}{\partial\phi^2}\right)\boldsymbol{e}_s+\frac{\partial^2 A_s}{\partial z^2}\boldsymbol{e}_s \tag{1.60}
$$

$$
(\boldsymbol{\nabla}^2\boldsymbol{A})_\phi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_\phi}{\partial s}\right)\boldsymbol{e}_\phi+\frac{1}{s^2}\left(2\frac{\partial A_\phi}{\partial\phi}-A_\phi+\frac{\partial^2 A_\phi}{\partial\phi^2}\right)\boldsymbol{e}_\phi+\frac{\partial^2 A_\phi}{\partial z^2}\boldsymbol{e}_\phi \tag{1.61}
$$

$$
(\boldsymbol{\nabla}^2\boldsymbol{A})_z = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_z}{\partial s}\right)\boldsymbol{e}_z+\frac{1}{s^2}\frac{\partial^2 A_z}{\partial\phi^2}\boldsymbol{e}_z+\frac{\partial^2 A_z}{\partial z^2}\boldsymbol{e}_z \tag{1.62}
$$

1.4.2 球坐标系

$$
\begin{cases}
\boldsymbol{e}_r = \sin\theta\cos\phi & \sin\theta\sin\phi\cos\theta\\
\boldsymbol{e}_\theta = \cos\theta\cos\phi & \cos\theta\sin\phi-\sin\theta\\
\boldsymbol{e}_\phi = -\sin\phi & \cos\phi 0
\end{cases}\tag{1.63}
$$

将球坐标系坐标转化为直角系坐标的方式为

$$
x_1 = r\cos\phi\sin\theta,\ x_2 = r\sin\phi\sin\theta,\ x_3 = r\cos\theta \tag{1.64}
$$

球坐标系和直角系坐标无限小元转换方式为

$$
\begin{pmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\\ \mathrm{d}x_3\end{pmatrix} = \begin{pmatrix}\dfrac{x_1}{r} & -x_2 & \dfrac{x_1x_3}{r\sin\theta}\\ \dfrac{x_2}{r} & x_1 & \dfrac{x_2x_3}{r\sin\theta}\\ \dfrac{x_3}{r} & 0 & -r\sin\theta\end{pmatrix}\begin{pmatrix}\mathrm{d}r\\ \mathrm{d}\phi\\ \mathrm{d}\theta\end{pmatrix} \tag{1.65}
$$

其拉梅系数为

$$h_1 = 1,\ h_2 = r\sin\theta,\ h_3 = r \tag{1.66}$$

其无限小位移为

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r\boldsymbol{r} + r\sin\theta\,\mathrm{d}\phi\boldsymbol{\phi} + r\,\mathrm{d}\theta\boldsymbol{\theta} \tag{1.67}$$

其体积元为

$$\mathrm{d}\tau = r\sin\theta r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{1.68}$$

梯度

$$\nabla f = \frac{1}{h_i}\frac{\partial f}{\partial u_i}\boldsymbol{e}_i = \frac{\partial f}{\partial r}\boldsymbol{r} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\boldsymbol{\phi} + \frac{1}{r}\frac{\partial f}{\partial\theta}\boldsymbol{\theta} \tag{1.69}$$

散度

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{a} &= \frac{1}{r^2\sin\theta}(\frac{\partial r^2\sin\theta a_r}{\partial r} + \frac{\partial r a_\phi}{\partial\phi} + \frac{\partial r\sin\theta a_\theta}{\partial\theta}) \\
&= \frac{1}{r^2\sin\theta}(\sin\theta\frac{\partial r^2 a_r}{\partial r} + r\frac{\partial a_\phi}{\partial\phi} + r\frac{\partial\sin\theta a_\theta}{\partial\theta}) \\
&= \frac{1}{r^2}\frac{\partial r^2 a_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial\sin\theta a_\theta}{\partial\theta}
\end{aligned} \tag{1.70}$$

旋度

$$\begin{aligned}
&\boldsymbol{\nabla}\times\boldsymbol{a} \\
=&\frac{h_i}{h_1h_2h_3}\varepsilon_{ijk}\frac{\partial(h_k a_k)}{\partial u_j}\boldsymbol{e}_i \\
=&\frac{h_i}{r^2\sin\theta}\varepsilon_{ijk}\frac{\partial(h_k a_k)}{\partial u_j}\boldsymbol{e}_i \\
=&\frac{1}{r^2\sin\theta}\left(\varepsilon_{123}\frac{\partial r a_3}{\partial u_2}\boldsymbol{e}_1 + \varepsilon_{132}\frac{\partial s a_2}{\partial u_3}\boldsymbol{e}_1 + r\varepsilon_{321}\frac{\partial a_1}{\partial u_2}\boldsymbol{e}_3 + s\varepsilon_{213}\frac{\partial r a_3}{\partial u_1}\boldsymbol{e}_2 + r\varepsilon_{312}\frac{\partial s a_2}{\partial u_1}\boldsymbol{e}_3 + s\varepsilon_{231}\frac{\partial a_1}{\partial u_3}\boldsymbol{e}_2\right) \\
=&\frac{1}{r^2\sin\theta}\left(\frac{\partial r a_3}{\partial u_2}\boldsymbol{e}_1 - \frac{\partial r\sin\theta a_2}{\partial u_3}\boldsymbol{e}_1 - r\frac{\partial a_1}{\partial u_2}\boldsymbol{e}_3 - r\sin\theta\frac{\partial r a_3}{\partial u_1}\boldsymbol{e}_2 + r\frac{\partial r\sin\theta a_2}{\partial u_1}\boldsymbol{e}_3 + r\sin\theta\frac{\partial a_1}{\partial u_3}\boldsymbol{e}_2\right) \\
=&\frac{1}{r^2\sin\theta}\left(\frac{\partial r a_\theta}{\partial\phi}\boldsymbol{r} - \frac{\partial r\sin\theta a_\phi}{\partial\theta}\boldsymbol{r} - r\frac{\partial a_r}{\partial\phi}\boldsymbol{\theta} - r\sin\theta\frac{\partial r a_\theta}{\partial r}\boldsymbol{\phi} + r\frac{\partial r\sin\theta a_\phi}{\partial r}\boldsymbol{\theta} + r\sin\theta\frac{\partial a_r}{\partial\theta}\boldsymbol{\phi}\right) \\
=&\frac{1}{r^2\sin\theta}\left(r\frac{\partial a_\theta}{\partial\phi} - r\frac{\partial\sin\theta a_\phi}{\partial\theta}\right)\boldsymbol{r} + \frac{1}{r}\left(\frac{\partial a_r}{\partial\theta} - \frac{\partial r a_\theta}{\partial r}\right)\boldsymbol{\phi} + \frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial r a_\phi}{\partial r} - \frac{\partial a_r}{\partial\phi}\right)\boldsymbol{\theta} \\
=&\frac{1}{r\sin\theta}\left(\frac{\partial a_\theta}{\partial\phi} - \frac{\partial\sin\theta a_\phi}{\partial\theta}\right)\boldsymbol{r} + \frac{1}{r}\left(\frac{\partial a_r}{\partial\theta} - \frac{\partial r a_\theta}{\partial r}\right)\boldsymbol{\phi} + \frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial r a_\phi}{\partial r} - \frac{\partial a_r}{\partial\phi}\right)\boldsymbol{\theta}
\end{aligned} \tag{1.71}$$

拉普拉斯算子

$$\begin{aligned}
\boldsymbol{\nabla}^2 f &= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_i}\left(h_1h_2h_3\frac{\partial f}{h_i^2\partial u_i}\right) \\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial u_i}\left(r^2\sin\theta\frac{\partial f}{h_i^2\partial u_i}\right) \\
&= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\sin\theta\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)
\end{aligned} \tag{1.72}$$

$$
\begin{aligned}
\boldsymbol{\nabla}^2\boldsymbol{A} =& \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_i\boldsymbol{e}_i}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_i\boldsymbol{e}_i}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_i\boldsymbol{e}_i}{\partial\theta}\right)\\
\boldsymbol{\nabla}^2\boldsymbol{A}_r =& \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_r\boldsymbol{e}_r}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r\boldsymbol{e}_r}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r\boldsymbol{e}_r}{\partial\theta}\right)\\
&\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\boldsymbol{e}_r\frac{\partial A_r}{\partial\phi}+A_r\frac{\partial\boldsymbol{e}_r}{\partial\phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\boldsymbol{e}_r\frac{\partial A_r}{\partial\theta}+\sin\theta A_r\frac{\partial\boldsymbol{e}_r}{\partial\theta}\right)\\
=&\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\boldsymbol{e}_r\frac{\partial A_r}{\partial\phi}+\sin\theta A_r\boldsymbol{e}_\phi\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\boldsymbol{e}_r\frac{\partial A_r}{\partial\theta}+\sin\theta A_r\boldsymbol{e}_\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{r^2\sin^2\theta}\left(\frac{\partial\boldsymbol{e}_r}{\partial\phi}\frac{\partial A_r}{\partial\phi}+\boldsymbol{e}_r\frac{\partial^2 A_r}{\partial\phi^2}+\sin\theta\boldsymbol{e}_\phi\frac{\partial A_r}{\partial\phi}+\sin\theta A_r\frac{\partial\boldsymbol{e}_\phi}{\partial\phi}\right)\\
=&\frac{1}{r^2\sin^2\theta}\left(2\sin\theta\frac{\partial A_r}{\partial\phi}\boldsymbol{e}_\phi+\frac{\partial^2 A_r}{\partial\phi^2}\boldsymbol{e}_r-\sin^2\theta A_r\boldsymbol{e}_r-\sin\theta\cos\theta A_r\boldsymbol{e}_\theta\right)\\
=&\frac{1}{r^2\sin\theta}\left(2\frac{\partial A_r}{\partial\phi}\boldsymbol{e}_\phi+\frac{1}{\sin\theta}\frac{\partial^2 A_r}{\partial\phi^2}\boldsymbol{e}_r-\sin\theta A_r\boldsymbol{e}_r-\cos\theta A_r\boldsymbol{e}_\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\boldsymbol{e}_r\frac{\partial A_r}{\partial\theta}+\sin\theta A_r\boldsymbol{e}_\theta\right)\\
=&\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial A_r}{\partial\theta}\boldsymbol{e}_r+\sin\theta\frac{\partial\boldsymbol{e}_r}{\partial\theta}\frac{\partial A_r}{\partial\theta}+\sin\theta\boldsymbol{e}_r\frac{\partial^2 A_r}{\partial\theta^2}+\sin\theta\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}A_r+\cos\theta A_r\boldsymbol{e}_\theta+\frac{\partial A_r}{\partial\theta}\sin\theta\boldsymbol{e}_\theta\right)\\
=&\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial A_r}{\partial\theta}\boldsymbol{e}_r+\sin\theta\boldsymbol{e}_\theta\frac{\partial A_r}{\partial\theta}+\sin\theta\boldsymbol{e}_r\frac{\partial^2 A_r}{\partial\theta^2}-\sin\theta\boldsymbol{e}_r A_r+\cos\theta A_r\boldsymbol{e}_\theta+\frac{\partial A_r}{\partial\theta}\sin\theta\boldsymbol{e}_\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\boldsymbol{e}_r\frac{\partial A_r}{\partial\phi}+\sin\theta A_r\boldsymbol{e}_\phi\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\boldsymbol{e}_r\frac{\partial A_r}{\partial\theta}+\sin\theta A_r\boldsymbol{e}_\theta\right)\\
=&\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial A_r}{\partial\theta}\boldsymbol{e}_r+2\sin\theta\boldsymbol{e}_\theta\frac{\partial A_r}{\partial\theta}+\sin\theta\boldsymbol{e}_r\frac{\partial^2 A_r}{\partial\theta^2}-2\sin\theta\boldsymbol{e}_r A_r+2\frac{\partial A_r}{\partial\phi}\boldsymbol{e}_\phi+\frac{1}{\sin\theta}\frac{\partial^2 A_r}{\partial\phi^2}\boldsymbol{e}_r\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla}^2\boldsymbol{A}_\phi =&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi\boldsymbol{e}_\phi}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi\boldsymbol{e}_\phi}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi\boldsymbol{e}_\phi}{\partial\theta}\right)\\
\boldsymbol{\nabla}^2\boldsymbol{A}_\phi =&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi\boldsymbol{e}_\phi}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_\phi}{\partial\phi}\boldsymbol{e}_\phi+A_\phi\frac{\partial\boldsymbol{e}_\phi}{\partial\phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\boldsymbol{e}_\phi\right)\\
&\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_\phi}{\partial\phi}\boldsymbol{e}_\phi+A_\phi\frac{\partial\boldsymbol{e}_\phi}{\partial\phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\boldsymbol{e}_\phi\right)\\
&=\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_\phi}{\partial\phi}\boldsymbol{e}_\phi-\sin\theta A_\phi\boldsymbol{e}_r-\cos\theta A_\phi\boldsymbol{e}_\theta\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\boldsymbol{e}_\phi\right)\\
&\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_\phi}{\partial\phi}\boldsymbol{e}_\phi-\sin\theta A_\phi\boldsymbol{e}_r-\cos\theta A_\phi\boldsymbol{e}_\theta\right)\\
&=\frac{1}{r^2\sin^2\theta}\left(\frac{\partial\boldsymbol{e}_\phi}{\partial\phi}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial^2 A_\phi}{\partial\phi^2}\boldsymbol{e}_\phi-\frac{\partial A_\phi}{\partial\phi}\sin\theta\boldsymbol{e}_r-\boldsymbol{e}_\phi\sin^2\theta A_\phi-\frac{\partial A_\phi}{\partial\phi}\cos\theta\boldsymbol{e}_\theta-\boldsymbol{e}_\phi\cos^2\theta A_\phi\right)\\
&=\frac{1}{r^2\sin^2\theta}\left(-\sin\theta\boldsymbol{e}_r\frac{\partial A_\phi}{\partial\phi}-\cos\theta\boldsymbol{e}_\theta\frac{\partial A_\phi}{\partial\phi}+\frac{\partial^2 A_\phi}{\partial\phi^2}\boldsymbol{e}_\phi-\frac{\partial A_\phi}{\partial\phi}\sin\theta\boldsymbol{e}_r-\frac{\partial A_\phi}{\partial\phi}\cos\theta\boldsymbol{e}_\theta-\boldsymbol{e}_\phi A_\phi\right)\\
&=\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 A_\phi}{\partial\phi^2}\boldsymbol{e}_\phi-2\frac{\partial A_\phi}{\partial\phi}\sin\theta\boldsymbol{e}_r-2\frac{\partial A_\phi}{\partial\phi}\cos\theta\boldsymbol{e}_\theta-A_\phi\boldsymbol{e}_\phi\right)\\
&\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\phi}{\partial\theta}\boldsymbol{e}_\phi\right)\\
&=\frac{1}{r^2\sin\theta}\left(\frac{\partial\sin\theta}{\partial\theta}\frac{\partial A_\phi}{\partial\theta}\boldsymbol{e}_\phi+\sin\theta\frac{\partial^2 A_\phi}{\partial\theta^2}\boldsymbol{e}_\phi+\sin\theta\frac{\partial A_\phi}{\partial\theta}\frac{\partial\boldsymbol{e}_\phi}{\partial\theta}\right)\\
&=\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial A_\phi}{\partial\theta}\boldsymbol{e}_\phi+\sin\theta\frac{\partial^2 A_\phi}{\partial\theta^2}\boldsymbol{e}_\phi\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta\boldsymbol{e}_\theta}{\partial\phi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\theta\boldsymbol{e}_\theta}{\partial\theta}\right)\\
=&\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_\theta}{\partial\phi}\boldsymbol{e}_\theta+A_\theta\frac{\partial\boldsymbol{e}_\theta}{\partial\phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\theta}{\partial\theta}\boldsymbol{e}_\theta+A_\theta\sin\theta\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}\right)\\
=&\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_\theta}{\partial\phi}\boldsymbol{e}_\theta+\cos\theta A_\theta\boldsymbol{e}_\phi\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\theta}{\partial\theta}\boldsymbol{e}_\theta-A_\theta\sin\theta\boldsymbol{e}_r\right)\\
&\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial A_\theta}{\partial\phi}\boldsymbol{e}_\theta+\cos\theta A_\theta\boldsymbol{e}_\phi\right)\\
=&\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 A_\theta}{\partial\phi^2}\boldsymbol{e}_\theta+\frac{\partial\boldsymbol{e}_\theta}{\partial\phi}\frac{\partial A_\theta}{\partial\phi}+\cos\theta A_\theta\frac{\partial\boldsymbol{e}_\phi}{\partial\phi}+\frac{\partial A_\theta}{\partial\phi}\cos\theta\boldsymbol{e}_\phi\right)\\
=&\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 A_\theta}{\partial\phi^2}\boldsymbol{e}_\theta+2\cos\theta\boldsymbol{e}_\phi\frac{\partial A_\theta}{\partial\phi}-\sin\theta\cos\theta A_\theta\boldsymbol{e}_r-\cos^2\theta A_\theta\boldsymbol{e}_\theta\right)\\
&\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_\theta}{\partial\theta}\boldsymbol{e}_\theta-A_\theta\sin\theta\boldsymbol{e}_r\right)\\
=&\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial A_\theta}{\partial\theta}\boldsymbol{e}_\theta+\sin\theta\frac{\partial^2 A_\theta}{\partial\theta^2}\boldsymbol{e}_\theta+\sin\theta\frac{\partial A_\theta}{\partial\theta}\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}-\frac{\partial A_\theta}{\partial\theta}\sin\theta\boldsymbol{e}_r-\cos\theta A_\theta\boldsymbol{e}_r-\frac{\partial\boldsymbol{e}_r}{\partial\theta}A_\theta\sin\theta\right)\\
=&\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial A_\theta}{\partial\theta}\boldsymbol{e}_\theta+\sin\theta\frac{\partial^2 A_\theta}{\partial\theta^2}\boldsymbol{e}_\theta-\sin\theta\frac{\partial A_\theta}{\partial\theta}\boldsymbol{e}_r-\frac{\partial A_\theta}{\partial\theta}\sin\theta\boldsymbol{e}_r-\cos\theta A_\theta\boldsymbol{e}_r-\boldsymbol{e}_\theta A_\theta\sin\theta\right)\\
=&\frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial A_\theta}{\partial\theta}\boldsymbol{e}_\theta+\sin\theta\frac{\partial^2 A_\theta}{\partial\theta^2}\boldsymbol{e}_\theta-2\sin\theta\frac{\partial A_\theta}{\partial\theta}\boldsymbol{e}_r-\cos\theta A_\theta\boldsymbol{e}_r-A_\theta\sin\theta\boldsymbol{e}_\theta\right)
\end{aligned}
$$

$$(\boldsymbol{\nabla}^2\boldsymbol{A})_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_r}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2} \tag{1.73}$$

$$+ \frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial A_r}{\partial\theta} + \sin\theta\frac{\partial^2 A_r}{\partial\theta^2} - 2\sin\theta A_r - 2\sin\theta\frac{\partial A_\theta}{\partial\theta} - 2\cos\theta A_\theta - 2\frac{\partial A_\phi}{\partial\phi}\right)$$

$$(\boldsymbol{\nabla}^2\boldsymbol{A})_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\phi}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 A_\phi}{\partial\phi^2} - A_\phi + 2\cos\theta\frac{\partial A_\theta}{\partial\phi}\right) \tag{1.74}$$

$$+ \frac{1}{r^2\sin\theta}\left(2\frac{\partial A_r}{\partial\phi} + \cos\theta\frac{\partial A_\phi}{\partial\theta} + \sin\theta\frac{\partial^2 A_\phi}{\partial\theta^2}\right)$$

$$(\boldsymbol{\nabla}^2\boldsymbol{A})_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\theta}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 A_\theta}{\partial\phi^2} - 2\frac{\partial A_\phi}{\partial\phi}\cos\theta - \cos^2\theta A_\theta\right)$$

$$+ \frac{1}{r^2\sin\theta}\left(2\sin\theta\frac{\partial A_r}{\partial\theta} + \cos\theta\frac{\partial A_\theta}{\partial\theta} + \sin\theta\frac{\partial^2 A_\theta}{\partial\theta^2} - A_\theta\sin\theta\right) \tag{1.75}$$

1.5 狄拉克函数与阶跃函数

1.5.1 狄拉克函数

严格来说, 狄拉克 δ 函数并不是一个通常意义下的函数, 而是通过其在积分中的作用来定义的。设 $f(x)$ 是在 $x = 0$ 附近连续, 并且在无穷远处足够快衰减的函数。定义狄拉克 δ 满足

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\ \mathrm{d}x = f(0), \tag{1.76}$$

由定义立即得到

$$\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\ \mathrm{d}x = f(a). \tag{1.77}$$

设 $g(x)$ 在 x_i 处有孤立零点, 且 $g'(x_i) \neq 0$, 则

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}. \tag{1.78}$$

1.5.2 阶跃函数

定义阶跃函数

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \tag{1.79}$$

1.6 阶跃函数的导数

在通常意义下, $H(x)$ 在 $x = 0$ 不可导。但在积分意义下, 我们定义

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}H(x)}{\mathrm{d}x}\, f(x)\ \mathrm{d}x \equiv -\int_{-\infty}^{\infty} H(x)\,\frac{\mathrm{d}f}{\mathrm{d}x}\ \mathrm{d}x. \tag{1.80}$$

对右侧分部积分：

$$\begin{aligned} -\int_{-\infty}^{\infty} H(x)\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x &= -\int_{0}^{\infty} \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x \\ &= f(0). \end{aligned} \tag{1.81}$$

于是得到

$$\frac{\mathrm{d}H(x)}{\mathrm{d}x} = \delta(x), \tag{1.82}$$

1.6.1 多维狄拉克函数

三维 δ 函数定义为

$$\delta^{(3)}(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z), \tag{1.83}$$

并满足

$$\iiint \delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}_0) f(\boldsymbol{r})\,\mathrm{d}\tau = f(\boldsymbol{r}_0). \tag{1.84}$$

在正交曲线坐标 (u_1, u_2, u_3) 中,

$$\mathrm{d}\tau = h_1 h_2 h_3\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3,$$

因此定义

$$\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}_0) = \frac{\delta(u_1-u_{1,0})\delta(u_2-u_{2,0})\delta(u_3-u_{3,0})}{h_1 h_2 h_3}. \tag{1.85}$$

1.7 矢量场理论

1.7.1 亥姆霍兹定理

设矢量场 $\boldsymbol{a}(\boldsymbol{r})$ 定义在整个空间中, 其散度与旋度分别给定为

$$\boldsymbol{\nabla}\cdot\boldsymbol{a}(\boldsymbol{r}) = f(\boldsymbol{r}), \qquad \boldsymbol{\nabla}\times\boldsymbol{a}(\boldsymbol{r}) = \boldsymbol{b}(\boldsymbol{r}).$$

为了使体积分有意义, 仅要求 $f(\boldsymbol{r})$ 与 $\boldsymbol{b}(\boldsymbol{r})$ 在无穷远处趋于零是不够的, 它们的衰减速度必须足够快。

考虑定义在整个空间上的积分

$$\iiint_V X(\boldsymbol{r})\,\mathrm{d}\tau,$$

其中 $X(\boldsymbol{r})$ 表示 $f(\boldsymbol{r})$ 或 $\boldsymbol{b}(\boldsymbol{r})$ 的某一分量。

将积分写为球坐标形式,

$$\iiint_V X(\boldsymbol{r})\,\mathrm{d}\tau = \int_0^\infty \left(\iint_S X(r,\Omega) r^2\,\mathrm{d}\Omega\right)\mathrm{d}r.$$

若存在常数 $A>0$ 及 $R>0$, 使得当 $r>R$ 时对所有方向 Ω 成立

$$|X(r,\Omega)| \geq \frac{A}{r},$$

则有下界估计

$$\iiint_{|\boldsymbol{r}|>R} |X(\boldsymbol{r})|\,\mathrm{d}\tau \geq A\int_R^\infty r\,\mathrm{d}r \iint_S \mathrm{d}\Omega,$$

该积分显然发散。

类似地, 若

$$|X(r,\Omega)| \geq \frac{A}{r^2},$$

则径向积分包含

$$\int_R^\infty \frac{\mathrm{d}r}{r},$$

从而发散。

因此, 为保证体积分收敛, 必须要求

$$X(\boldsymbol{r}) = o\left(\frac{1}{r^2}\right), \qquad r\to\infty, \tag{1.86}$$

该条件需在角向上一致成立。

因此, 为保证积分收敛, 必须要求

$$f(\boldsymbol{r}),\ \boldsymbol{b}(\boldsymbol{r}) = o\left(\frac{1}{r^2}\right), \qquad r \to \infty. \tag{1.87}$$

这一条件同时也足以保证在无穷远处的曲面积分为零。

设矢量场 $\boldsymbol{a}(\boldsymbol{r})$ 具有散度 $f(\boldsymbol{r})$ 与旋度 $\boldsymbol{b}(\boldsymbol{r})$。若取另一矢量场

$$\boldsymbol{a}'(\boldsymbol{r}) = \boldsymbol{a}(\boldsymbol{r}) + \boldsymbol{c}(\boldsymbol{r}),$$

且 $\boldsymbol{c}(\boldsymbol{r})$ 满足

$$\boldsymbol{\nabla} \cdot \boldsymbol{c} = 0, \qquad \boldsymbol{\nabla} \times \boldsymbol{c} = \boldsymbol{0},$$

则 $\boldsymbol{a}'$ 与 $\boldsymbol{a}$ 具有相同的散度与旋度。

因此, 仅给定散度与旋度并不能唯一确定矢量场。然而可以证明：不存在一个非零的矢量场 (见第三章) $\boldsymbol{c}(\boldsymbol{r})$, 它在整个空间中同时满足

$$\boldsymbol{\nabla} \cdot \boldsymbol{c} = 0, \qquad \boldsymbol{\nabla} \times \boldsymbol{c} = \boldsymbol{0},$$

并且在无穷远处趋于零。

因此, 若进一步要求

$$\boldsymbol{a}(\boldsymbol{r}) \to \boldsymbol{0} \qquad \text{当} r \to \infty,$$

则满足给定散度与旋度的矢量场解是唯一的。

现在可以严谨地表述亥姆霍兹定理如下：

> 若矢量场 $\boldsymbol{a}(\boldsymbol{r})$ 的散度 $f(\boldsymbol{r})$ 与旋度 $\boldsymbol{b}(\boldsymbol{r})$ 已知, 且二者在 $r \to \infty$ 时均比 $1/r^2$ 衰减得更快, 同时 $\boldsymbol{a}(\boldsymbol{r})$ 在无穷远处趋于零, 则 $\boldsymbol{a}(\boldsymbol{r})$ 被其散度与旋度唯一确定。

1.7.2 势函数

1. 标量势与无旋场 设向量场 $\boldsymbol{a}(\boldsymbol{r})$ 定义在某一区域。若存在标量函数 $U(\boldsymbol{r})$, 使得

$$\boldsymbol{a} = -\boldsymbol{\nabla} U, \tag{1.88}$$

则称 U 为 $\boldsymbol{a}$ 的**标量势函数**, $\boldsymbol{a}$ 称为**保守场**。由旋度的定义可得

$$\boldsymbol{\nabla} \times \boldsymbol{a} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} U = \boldsymbol{0}. \tag{1.89}$$

因此, 任何具有标量势的向量场必为无旋场。

2. 无旋场的势函数存在性 若

$$\boldsymbol{\nabla} \times \boldsymbol{a} = \boldsymbol{0}, \tag{1.90}$$

并且区域 V 是单连通的 (任意闭合曲线可连续收缩为一点), 则由斯托克斯定理可得

$$\oint_C \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{1.91}$$

线积分与路径无关, 可定义

$$\phi(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}, \tag{1.92}$$

从而得到 $\boldsymbol{a} = -\boldsymbol{\nabla}\phi$。

3. 向量势与无散场 若存在向量函数 $\boldsymbol{b}(\boldsymbol{r})$, 使

$$\boldsymbol{a} = \boldsymbol{\nabla} \times \boldsymbol{b}, \tag{1.93}$$

则称 $\boldsymbol{b}$ 为 $\boldsymbol{F}$ 的**向量势**。由恒等式

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}) = 0 \tag{1.94}$$

可知, 任何具有向量势的向量场必为无散场。

4. 势函数的不唯一性

- 若 $\boldsymbol{a} = -\boldsymbol{\nabla} U$, 则对任意常数 C, $U' = U + C$ 给出同一向量场。
- 若 $\boldsymbol{a} = \boldsymbol{\nabla} \times \boldsymbol{b}$, 则对任意标量函数 $\chi(\boldsymbol{r})$, $\boldsymbol{b}' = \boldsymbol{b} + \boldsymbol{\nabla}\chi$ 产生相同的 $\boldsymbol{a}$(**规范变换**)。

5. 亥姆霍兹分解 在满足适当衰减条件的情况下, 任意向量场 $\boldsymbol{a}$ 可分解为

$$\boldsymbol{a} = -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \boldsymbol{b}, \tag{1.95}$$

其中

$$\boldsymbol{\nabla}^2 U = -\boldsymbol{\nabla} \cdot \boldsymbol{a}, \tag{1.96}$$

$$\boldsymbol{\nabla}^2 \boldsymbol{b} = -\boldsymbol{\nabla} \times \boldsymbol{a}. \tag{1.97}$$

这就是亥姆霍兹定理在势函数语言下的表达。

1.8 习题

1.8.1 证明: $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) = 0$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) = -(\boldsymbol{b}\cdot\boldsymbol{a})\boldsymbol{c} + (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} - (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} + (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a} - (\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a} + (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} = 0$$

1.8.2 证明: $\boldsymbol{Ra} \cdot \boldsymbol{Rb} = \boldsymbol{a} \cdot \boldsymbol{b}$, 其中 $\boldsymbol{R}$ 为旋转矩阵

$$\boldsymbol{Ra} \cdot \boldsymbol{Rb} = (\boldsymbol{Ra})^T(\boldsymbol{Rb}) = \boldsymbol{a}^T\boldsymbol{R}^T\boldsymbol{Rb} = \boldsymbol{a}^T\boldsymbol{R}^{-1}\boldsymbol{Rb} = \boldsymbol{a}^T\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$$

1.8.3 在坐标逆变换 $(x' = -x,\ y' = -y,\ z' = -z)$ 下, 两个矢量的叉乘是如何变换的?

$$\boldsymbol{a}' \times \boldsymbol{b}' = \begin{pmatrix} \boldsymbol{e}_i & \boldsymbol{e}_j & \boldsymbol{e}_k \\ a_i' & a_j' & a_k' \\ b_i' & b_j' & b_k' \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_i & \boldsymbol{e}_j & \boldsymbol{e}_k \\ -a_i & -a_j & -a_k \\ -b_i & -b_j & -b_k \end{pmatrix} = \boldsymbol{a} \times \boldsymbol{b}$$

1.8.4 在坐标逆变换下, 标量三重积 $(\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}))$ 是如何变换的?

$$\boldsymbol{c}' \cdot (\boldsymbol{a}' \times \boldsymbol{b}') = \boldsymbol{c}' \cdot (\boldsymbol{a} \times \boldsymbol{b}) = -\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$$

1.8.5 $\boldsymbol{n}$ **是固定点** (x', y', z') **到点** (x, y, z) **的间隔矢量,** n **是它的长度, 证明:**

(a) $\boldsymbol{\nabla} n^2 = 2\boldsymbol{n}$**.**

(b) $\boldsymbol{\nabla}\frac{1}{n} = -\frac{\boldsymbol{n}}{n^2}$**.**

(c) $\boldsymbol{\nabla} n^a = a\boldsymbol{n}n^{a-2}(a \neq 0)$**.**

(a) $\boldsymbol{\nabla} n^2 = \frac{\partial n^2}{\partial e_i}\boldsymbol{e}_i = 2n\frac{\partial n}{\partial e_i}\boldsymbol{e}_i = 2\boldsymbol{n}$

(c) $\boldsymbol{\nabla} n^a = \frac{\partial n^a}{\partial e_i}\boldsymbol{e}_i = an^{-1}\frac{\partial n}{\partial e_i}\boldsymbol{e}_i = a\boldsymbol{n}n^{a-2}$

1.8.6 **函数** f **只依赖平面坐标** y, z**。令旋转后的坐标记为** $\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}$ **, 求证** $\boldsymbol{\nabla} f' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} y' \\ z' \end{pmatrix}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y'} = \frac{\partial f}{\partial y}\cos\theta + \frac{\partial f}{\partial z}\sin\theta$$

$$\frac{\partial f}{\partial z'} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial z'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial z'} = -\frac{\partial f}{\partial y}\sin\theta + \frac{\partial f}{\partial z}\cos\theta$$

$$\begin{pmatrix} \dfrac{\partial f}{\partial y'} \\ \dfrac{\partial f}{\partial z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{pmatrix}$$

1.8.7 **计算** $(\boldsymbol{\nabla T}) \times (\boldsymbol{\nabla S})$

$$\begin{aligned}
(\boldsymbol{\nabla T}) \times (\boldsymbol{\nabla S}) &= \left(\varepsilon_{ijk}\frac{\partial T_j}{\partial x_i}\boldsymbol{e}_k\right) \times \left(\varepsilon_{lmn}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_n\right) \\
&= \varepsilon_{opq}\varepsilon_{ijo}\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q \\
&= \varepsilon_{pqo}\varepsilon_{ijo}\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q \\
&= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q \\
&= \delta_{ip}\delta_{jq}\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q - \delta_{iq}\delta_{jp}\frac{\partial T_j}{\partial x_i}\varepsilon_{lmp}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_q \\
&= \frac{\partial T_j}{\partial x_i}\varepsilon_{lmi}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_j - \frac{\partial T_j}{\partial x_i}\varepsilon_{lmj}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_i \\
&= \frac{\partial T_i}{\partial x_j}\varepsilon_{lmj}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_i - \frac{\partial T_j}{\partial x_i}\varepsilon_{lmj}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_i \\
&= \left(\frac{\partial T_i}{\partial x_j} - \frac{\partial T_j}{\partial x_i}\right)\varepsilon_{lmj}\frac{\partial S_m}{\partial x_l}\boldsymbol{e}_i
\end{aligned}$$

1.8.8 证明 $\iint_S f(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S}=\iint_S[\boldsymbol{a}\times(\boldsymbol{\nabla}f)]\cdot\mathrm{d}\boldsymbol{S}+\oint_{\partial S}f\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}$

$$\begin{aligned}\oint_{\partial S}f\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}&=\iint_S[\boldsymbol{\nabla}\times(\boldsymbol{f}\boldsymbol{a})]\cdot\mathrm{d}\boldsymbol{S}\\&=\iint_S[f(\boldsymbol{\nabla}\times\boldsymbol{a})+(\boldsymbol{\nabla}f)\times\boldsymbol{a}]\cdot\mathrm{d}\boldsymbol{S}\\\iint_S f(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S}&=\iint_S[\boldsymbol{a}\times(\boldsymbol{\nabla}f)]\cdot\mathrm{d}\boldsymbol{S}+\oint_{\partial S}f\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}\end{aligned}$$

1.8.9 证明 $\iiint_V\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})\,\mathrm{d}\tau=\iiint_V\boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\,\mathrm{d}\tau+\iint(\boldsymbol{a}\times\boldsymbol{b})\cdot\mathrm{d}\boldsymbol{S}_{\partial V}$

$$\begin{aligned}\iint_{\partial V}(\boldsymbol{a}\times\boldsymbol{b})\cdot\mathrm{d}\boldsymbol{S}&=\iiint_V\boldsymbol{\nabla}\cdot(\boldsymbol{a}\times\boldsymbol{b})\,\mathrm{d}\tau\\&=\iiint_V[\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})-\boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})]\,\mathrm{d}\tau\\\iiint_V\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})\,\mathrm{d}\tau&=\iiint_V\boldsymbol{a}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\,\mathrm{d}\tau+\iint_{\partial V}(\boldsymbol{a}\times\boldsymbol{b})\cdot\mathrm{d}\boldsymbol{S}\end{aligned}$$

1.8.10 设 $f=f(r)$**, 求** $\boldsymbol{\nabla}^2\left[f(r)\right]$

$$\begin{aligned}\boldsymbol{\nabla}^2\left[f(r)\right]&=\frac{1}{sr}\frac{\partial}{\partial r}\left(sr\frac{\partial f}{\partial r}\right)+\frac{1}{s^2}\frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial\phi}\right)+\frac{1}{sr^2}\frac{\partial}{\partial\theta}\left(s\frac{\partial f}{\partial\theta}\right)\\&=\frac{1}{sr}\frac{\partial}{\partial r}\left(sr\frac{\partial f}{\partial r}\right)\\&=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\\&=\frac{2}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial r^2}\end{aligned}\tag{1.98}$$

1.8.11 证明:$\iiint_V\boldsymbol{\nabla}\times\boldsymbol{a}\,\mathrm{d}\tau=\oiint\mathrm{d}\boldsymbol{S}\times\boldsymbol{a}$

取任意常向量 $\boldsymbol{c}$。利用式1.41得 $\boldsymbol{\nabla}\times\boldsymbol{c}=\boldsymbol{0}$, 因此

$$\boldsymbol{\nabla}\cdot(\boldsymbol{a}\times\boldsymbol{c})=\boldsymbol{c}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a}).$$

应用高斯定理,

$$\iiint_V\boldsymbol{\nabla}\cdot(\boldsymbol{a}\times\boldsymbol{c})\,\mathrm{d}\tau=\oiint_{\partial V}(\boldsymbol{a}\times\boldsymbol{c})\cdot\mathrm{d}\boldsymbol{S}=\iiint_V\boldsymbol{c}\cdot(\boldsymbol{\nabla}\times\boldsymbol{a})\,\mathrm{d}\tau.$$

利用式1.12得

$$\boldsymbol{c}\cdot\iiint_V(\boldsymbol{\nabla}\times\boldsymbol{a})\,\mathrm{d}\tau=\boldsymbol{c}\cdot\oiint_{\partial V}\mathrm{d}\boldsymbol{S}\times\boldsymbol{a}.$$

由于上述等式对任意常向量 $\boldsymbol{c}$ 成立, 必有

$$\iiint_V(\boldsymbol{\nabla}\times\boldsymbol{a})\,\mathrm{d}\tau=\oiint_{\partial V}\mathrm{d}\boldsymbol{S}\times\boldsymbol{a}.$$

1.8.12 证明: $\iint_S \mathrm{d}\boldsymbol{S} \times \boldsymbol{\nabla} f = \oint_{\partial S} f\,\mathrm{d}\boldsymbol{l}$

取任意常向量 $\boldsymbol{c}$。利用式1.28得

$$\iint_S (\boldsymbol{\nabla} f \times \boldsymbol{c}) \cdot \mathrm{d}\boldsymbol{S} = \iint_S (\boldsymbol{\nabla} \times (f\boldsymbol{c})) \cdot \mathrm{d}\boldsymbol{S}.$$

由斯托克斯定理

$$\iint_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \oint_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l},$$

取 $\boldsymbol{A} = f\boldsymbol{c}$, 得

$$\iint_S (\boldsymbol{\nabla} \times (f\boldsymbol{c})) \cdot \mathrm{d}\boldsymbol{S} = \oint_{\partial S} (f\boldsymbol{c}) \cdot \mathrm{d}\boldsymbol{l} = \oint_{\partial S} f\,\boldsymbol{c} \cdot \mathrm{d}\boldsymbol{l}.$$

又由混合积恒等式

$$\boldsymbol{c} \cdot (\mathrm{d}\boldsymbol{S} \times \boldsymbol{\nabla} f) = (\boldsymbol{\nabla} f \times \boldsymbol{c}) \cdot \mathrm{d}\boldsymbol{S},$$

所以

$$\boldsymbol{c} \cdot \iint_S \mathrm{d}\boldsymbol{S} \times \boldsymbol{\nabla} f = \boldsymbol{c} \cdot \oint_{\partial S} f\ \mathrm{d}\boldsymbol{l}.$$

由于上述等式对任意常向量 $\boldsymbol{c}$ 都成立, 必有

$$\iint_S \mathrm{d}\boldsymbol{S} \times \boldsymbol{\nabla} f = \oint_{\partial S} f\ \mathrm{d}\boldsymbol{l}.$$

1.8.13 证明: $\oiint_{\partial S} (\boldsymbol{a} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l} = 2\iint_S \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S}$

$$\begin{aligned}
\oiint_{\partial S} (\boldsymbol{a} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l} &= \iint_S [\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{r})] \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S [\boldsymbol{\nabla} \times (\varepsilon_{ijk} a_i x_j \boldsymbol{e}_k)] \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S (\varepsilon_{lmn}\varepsilon_{ijm} \frac{\partial a_i x_j}{\partial x_l} \boldsymbol{e}_n) \cdot \mathrm{d}\boldsymbol{S} \\
&= -\iint_S (\varepsilon_{lnm}\varepsilon_{ijm} \frac{\partial a_i x_j}{\partial x_l} \boldsymbol{e}_n) \cdot \mathrm{d}\boldsymbol{S} \\
&= -\iint_S (\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl})(\frac{\partial a_i x_j}{\partial x_l} \boldsymbol{e}_n) \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S (\delta_{in}\delta_{jl} \frac{\partial a_i x_j}{\partial x_l} \boldsymbol{e}_n - \delta_{il}\delta_{jn} \frac{\partial a_i x_j}{\partial x_l} \boldsymbol{e}_n) \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S (\frac{\partial a_i x_j}{\partial x_j} \boldsymbol{e}_i - \frac{\partial a_i x_j}{\partial x_i} \boldsymbol{e}_j) \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S (\frac{\partial a_j x_i}{\partial x_i} \boldsymbol{e}_j - \frac{\partial a_i x_j}{\partial x_i} \boldsymbol{e}_j) \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S (a_j \frac{\partial x_i}{\partial x_i} + x_i \frac{\partial a_j}{\partial x_i} - a_i \frac{\partial x_j}{\partial x_i} - x_j \frac{\partial a_i}{\partial x_i}) \boldsymbol{e}_j \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S (3a_j + x_i \frac{\partial a_j}{\partial x_i} - a_i \delta_{ij} - x_j \frac{\partial a_i}{\partial x_i}) \boldsymbol{e}_j \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S 2a_j \boldsymbol{e}_j \cdot \mathrm{d}\boldsymbol{S} \\
&= \iint_S \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S}
\end{aligned}$$

1.8.14 将下列电荷分布表示成电荷密度函数 ρ:**1.** 电荷量 Q 均匀分布在半径为 R 的球面上; **2.** 电荷均匀分布在半径为 R 的柱面上, 单位长度的电荷量为 λ; **3.** 电荷量均匀分布在半径为 R 的平面圆盘上

$$
\begin{aligned}
&1.\rho = \delta(R\boldsymbol{e}_r - \boldsymbol{r})\frac{Q}{4\pi R^2} \\
&2.\rho = \delta(R\boldsymbol{e}_s - \boldsymbol{s})\frac{\lambda}{2\pi R} \\
&3.\rho = \delta(\boldsymbol{x}_3)H(x+R)H(-x-R)\frac{\lambda}{\pi R^2}
\end{aligned}
$$

1.8.15 对函数 $\boldsymbol{a} = r^2\cos\theta\boldsymbol{r} + r^2\cos\phi\boldsymbol{\theta} - r^2\cos\theta\sin\phi\boldsymbol{\phi}$ 验证散度定理, 体积为半径为 R 在第一卦限的 $1/8$ 球体

$$\boldsymbol{\nabla}\cdot\boldsymbol{a} = 2r\cos\theta - r^2\cos\theta\cos\phi$$

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R\boldsymbol{\nabla}\cdot\boldsymbol{a}\,\mathrm{d}r &= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R\left(2r\cos\theta - r^2\cos\theta\cos\phi\right)\mathrm{d}r \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\left(r^2\cos\theta - \frac{r^3}{3}\cos\theta\cos\phi\right)\Bigg|_0^R \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\left(R^2\cos\theta - \frac{R^3}{3}\cos\theta\cos\phi\right)\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}\phi\left(R^2\sin\theta - \frac{R^3}{3}\sin\theta\cos\phi\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \int_0^{\frac{\pi}{2}}\left(R^2 - \frac{R^3}{3}\cos\phi\right)\mathrm{d}\phi \\
&= \left(R^2\phi - \frac{R^3}{3}\sin\phi\right)\Bigg|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}R^2 - \frac{R^3}{3}
\end{aligned}
$$

计算 XY 平面:

$$
\begin{aligned}
\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R a_\phi\,\mathrm{d}r &= -\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R r^2\cos\theta\sin\phi\,\mathrm{d}r \\
&= -\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R r^2\cos\theta\,\mathrm{d}r \\
&= -\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\,\frac{r^3}{3}\cos\theta\Bigg|_0^R \\
&= -\int_0^{\frac{\pi}{2}}\frac{R^3}{3}\cos\theta\,\mathrm{d}\theta \\
&= -\frac{R^3}{3}
\end{aligned}
$$

计算 XZ 平面:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \mathrm{d}\phi \int_0^R a_\theta \,\mathrm{d}r &= \int_0^{\frac{\pi}{2}} \mathrm{d}\phi \int_0^R r^2 \cos\phi \,\mathrm{d}r \\ &= \int_0^{\frac{\pi}{2}} \mathrm{d}\phi \, \frac{r^3}{3}\cos\phi \Big|_0^R \\ &= \int_0^{\frac{\pi}{2}} \frac{R^3}{3}\cos\phi \,\mathrm{d}\phi \\ &= \frac{R^3}{3}\end{aligned}$$

计算 YZ 平面:

$$\int_0^{\frac{\pi}{2}} \mathrm{d}\phi \int_0^R a_\theta \,\mathrm{d}r = \int_0^{\frac{\pi}{2}} \mathrm{d}\phi \int_0^R r^2 \cos\phi \,\mathrm{d}r = \frac{R^3}{3}$$

1.8.16　对函数 $\boldsymbol{a} = bx_2\boldsymbol{e}_1 + cx_1\boldsymbol{e}_2$ 验证斯托克斯定理, 面的边界线选为处在 XY 平面, 半径为 R, 圆心在原点的圆周线。

$$\boldsymbol{\nabla} \times \boldsymbol{a} = \boldsymbol{\nabla} \times (by\boldsymbol{e}_1 + cx\boldsymbol{e}_2) = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ bx_2 & cx_1 & 0 \end{pmatrix} = (c-b)\boldsymbol{e}_3$$

$$\iint \boldsymbol{\nabla} \times \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \iint (c-b)\boldsymbol{e}_3 \cdot \mathrm{d}\boldsymbol{S} = 2\pi R(c-b)$$

$$\begin{aligned}\oint (bR\sin\phi\boldsymbol{e}_1 + cR\cos\phi\boldsymbol{e}_2) \cdot \mathrm{d}\boldsymbol{l} =& \int_0^{2\pi} (bR\sin\phi\boldsymbol{e}_1 + cR\cos\phi\boldsymbol{e}_2) \cdot (-R\sin\phi\boldsymbol{e}_1 + R\cos\phi\boldsymbol{e}_2)\,\mathrm{d}\phi \\ =& \int_0^{2\pi} (-bR^2\sin^2\phi + cR^2\cos^2\phi)\,\mathrm{d}\phi \\ =& 2\pi R(c-b)\end{aligned}$$

1.8.17　对函数 $\boldsymbol{a} = r\cos^2\theta\boldsymbol{r} - r\sin\theta\cos\theta\boldsymbol{\theta} + 3r\boldsymbol{\phi}$ 验证斯托克斯定理。

$$\begin{aligned}&\boldsymbol{\nabla} \times \boldsymbol{a} \\ =&\frac{1}{sr}\left(r\frac{\partial a_\theta}{\partial\phi} - s\frac{\partial a_\phi}{\partial\theta}\right)\boldsymbol{r} + \frac{1}{r}\left(\frac{\partial a_r}{\partial\theta} - \frac{\partial r a_\theta}{\partial r}\right)\boldsymbol{\phi} + \frac{1}{s}\left(\frac{\partial s a_\phi}{\partial r} - \frac{\partial a_r}{\partial\phi}\right)\boldsymbol{\theta} \\ =&\frac{1}{sr}\left(r\frac{\partial r\sin\theta\cos\theta}{\partial\phi} - s\frac{\partial 3r}{\partial\theta}\right)\boldsymbol{r} + \frac{1}{r}\left(\frac{\partial r\cos^2\theta}{\partial\theta} - \frac{\partial rr\sin\theta\cos\theta}{\partial r}\right)\boldsymbol{\phi} + \frac{1}{s}\left(\frac{\partial s3r}{\partial r} - \frac{\partial r\cos^2\theta}{\partial\phi}\right)\boldsymbol{\theta} \\ =&\frac{1}{s}(6r\sin\theta)\,\boldsymbol{\theta} \\ =&6\boldsymbol{\theta}\end{aligned}$$

计算 XY 平面:

$$\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^R a_\phi \,\mathrm{d}r = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^R 6 \,\mathrm{d}r$$
$$= \frac{3}{2}R$$

1.8.18 对函数 $\boldsymbol{a} = r^2 \sin^2\theta \boldsymbol{r} + 4r^2\cos\theta\boldsymbol{\theta} + r^2\tan\theta\boldsymbol{\phi}$ 验证散度定理。

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{a} &= \frac{1}{sr}(\frac{\partial s r a_r}{\partial r} + r\frac{\partial a_\phi}{\partial\phi} + s\frac{\partial\theta}{\partial\theta}) \\
&= \frac{1}{sr}(\frac{\partial s r r^2\sin^2\theta}{\partial r} + r\frac{\partial 4r^2\cos\theta}{\partial\phi} + s\frac{\partial r^2\tan\theta}{\partial\theta}) \\
&= \frac{1}{sr}(4sr^2\sin^2\theta + sr^2\frac{1}{\cos^2\theta}) \\
&= 4r\sin^2\theta + r\frac{1}{\cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
&\int_0^R r^2\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi\int_0^{\frac{\pi}{6}}(4r\sin^2\theta + r\frac{1}{\cos^2\theta})\sin\theta\,\mathrm{d}\theta \\
=&\int_0^R 2\pi r^3\,\mathrm{d}r\int_0^{\frac{\pi}{6}}(4\sin^2\theta + \frac{1}{\cos^2\theta})\,\mathrm{d}\theta \\
=&\int_0^R 2\pi r^3\ (2\theta - \sin2\theta + \tan\theta)|_0^{\frac{\pi}{6}}\,\mathrm{d}r \\
=&\int_0^R 2\pi r^3(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2})\,\mathrm{d}r \\
=&\int_0^R 2\pi r^3\frac{\pi}{3}\,\mathrm{d}r \\
=&2\pi R^4\frac{\pi}{12}
\end{aligned}$$

1.8.19 证明 $\iiint_V \left[f\boldsymbol{\nabla}^2 g + (\boldsymbol{\nabla}f)\cdot(\boldsymbol{\nabla}g)\right]\mathrm{d}\tau = \oiint_S (f\boldsymbol{\nabla}g)\cdot\mathrm{d}S$

$$\begin{aligned}
\iiint_V(\boldsymbol{\nabla}\cdot\boldsymbol{a})\,\mathrm{d}\tau &= \iint_{\partial V}\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S} \\
\iiint_V[\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla}g)]\,\mathrm{d}\tau &= \iint_{\partial V}(f\boldsymbol{\nabla}g)\cdot\mathrm{d}\boldsymbol{S} \\
\iiint_V\left[f\boldsymbol{\nabla}^2 g + (\boldsymbol{\nabla}f)\cdot(\boldsymbol{\nabla}g)\right]\mathrm{d}\tau &= \iint_{\partial V}(f\boldsymbol{\nabla}g)\cdot\mathrm{d}\boldsymbol{S}
\end{aligned}$$

1.8.20 求 $\boldsymbol{a} = r^n \boldsymbol{e}_r$ 的散度

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{a} &= \frac{\partial a_i}{\partial x_i} \\
&= \frac{\partial x_i r^{n-1}}{\partial x_i} \\
&= 3r^{n-1} + x_i(n-1)r^{n-2}\frac{\partial r}{\partial x_i} \\
&= 3r^{n-1} + x_i(n-1)r^{n-2}\frac{x_i}{r} \\
&= 3r^{n-1} + (n-1)r^{n-1} \\
&= (n+2)r^{n-1}
\end{aligned}
$$

$$
\iint_{\partial V} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \iint_{\partial V} r^n \boldsymbol{e}_r \cdot \mathrm{d}\boldsymbol{S} = 4\pi r^2 r^n =
$$

1.8.21 求 $\boldsymbol{a} = r^n \boldsymbol{e}_r$ 的旋度

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{a} &= \varepsilon_{ijk}\frac{\partial a_j}{\partial x_i}\boldsymbol{e}_k \\
&= \varepsilon_{ijk}\frac{\partial r^{n-1}x_j}{\partial x_i}\boldsymbol{e}_k \\
&= \varepsilon_{ijk}x_j\frac{\partial r^{n-1}}{\partial x_i}\boldsymbol{e}_k \\
&= (n-1)\varepsilon_{ijk}x_j r^{n-2}\frac{\partial r}{\partial x_i}\boldsymbol{e}_k \\
&= (n-1)\varepsilon_{ijk}x_j r^{n-2}\frac{x_i}{r}\boldsymbol{e}_k \\
&= (n-1)r^{n-3}\varepsilon_{ijk}x_i x_j\boldsymbol{e}_k \\
&= 0
\end{aligned}
$$

1.8.22 设 $\boldsymbol{m}$ 是一常矢量, 证明:$\nabla\times\dfrac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}=-\nabla\dfrac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3}$

$$
\begin{aligned}
\nabla\times\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}&=\nabla\times\frac{\varepsilon_{ijk}m_ir_j\boldsymbol{e}_k}{r^3}\\
&=\varepsilon_{lkn}\frac{\partial\dfrac{\varepsilon_{ijk}m_ir_j}{r^3}}{\partial x_l}\boldsymbol{e}_n\\
&=-\varepsilon_{lnk}\varepsilon_{ijk}\frac{\partial\dfrac{m_ir_j}{r^3}}{\partial x_l}\boldsymbol{e}_n\\
&=-(\delta_{li}\delta_{nj}-\delta_{lj}\delta_{ni})\frac{\partial\dfrac{m_ir_j}{r^3}}{\partial x_l}\boldsymbol{e}_n\\
&=-\delta_{li}\delta_{nj}\frac{\partial\dfrac{m_ir_j}{r^3}}{\partial x_l}\boldsymbol{e}_n+\delta_{lj}\delta_{ni}\frac{\partial\dfrac{m_ir_j}{r^3}}{\partial x_l}\boldsymbol{e}_n\\
&=-\frac{\partial\dfrac{m_lr_n}{r^3}}{\partial x_l}\boldsymbol{e}_n+\frac{\partial\dfrac{m_nr_l}{r^3}}{\partial x_l}\boldsymbol{e}_n\\
&=-\frac{m_n}{r^3}\boldsymbol{e}_n+3\frac{m_lr_n}{r^4}\frac{r_l}{r}\boldsymbol{e}_n+3\frac{m_n}{r^3}\boldsymbol{e}_n-3\frac{m_nr_l}{r^4}\frac{r_l}{r}\boldsymbol{e}_n\\
&=3\frac{m_lr_n}{r^4}\frac{r_l}{r}\boldsymbol{e}_n+2\frac{m_n}{r^3}\boldsymbol{e}_n-3\frac{m_n}{r^3}\boldsymbol{e}_n\\
&=3\frac{m_lr_l}{r^4}\frac{r_n}{r}\boldsymbol{e}_n-\frac{m_n}{r^3}\boldsymbol{e}_n
\end{aligned}
$$

$$
\begin{aligned}
\nabla\frac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3}&=\nabla\frac{m_ir_i}{r^3}\\
&=\frac{\partial}{\partial x_j}\frac{m_ir_i}{r^3}\boldsymbol{e}_j\\
&=r_i\frac{\partial}{\partial x_j}\frac{m_i}{r^3}\boldsymbol{e}_j+\frac{1}{r^3}\frac{\partial m_ir_i}{\partial x_j}\boldsymbol{e}_j\\
&=-3r_i\frac{m_ir_j}{r^5}\boldsymbol{e}_j+\frac{1}{r^3}m_i\delta_{ij}\boldsymbol{e}_j\\
&=-3r_i\frac{m_ir_j}{r^5}\boldsymbol{e}_j+\frac{1}{r^3}m_i\boldsymbol{e}_i
\end{aligned}
$$

1.8.23 以匀角速绕轴转动的抛物线形金属丝, 其方程为 $x^2=4ay$。一质量为 m 的小环套在此金属丝上, 可沿着金属丝无摩擦滑动。求小环在 x 方向的运动微分方程。

$$
\begin{aligned}
\frac{\partial\dfrac{mx^2\omega^2}{2}+mgy+\dfrac{m\dot{y}^2}{2}+\dfrac{m\dot{x}^2}{2}}{\partial t}&=0\\
\frac{\partial\dfrac{mx^2\omega^2}{2}+mg\dfrac{x^2}{4a}+\dfrac{mx^2\dot{x}^2}{8a^2}+\dfrac{m\dot{x}^2}{2}}{\partial t}&=0\\
x\dot{x}\omega^2+gx\frac{\dot{x}}{2a}+\frac{x\dot{x}^3}{4a^2}+\frac{x^2\dot{x}\ddot{x}}{4a^2}+mx\dot{x}&=0\\
x\omega^2+\frac{gx}{2a}+\frac{x\dot{x}^2}{4a^2}+\frac{x^2\ddot{x}}{4a^2}+mx&=0
\end{aligned}
$$

2 静电学

2.1 电场

2.1.1 库仑定律

设一个静止点电荷 q_1 距检验电荷 q_2 的距离为 r, 那么它作用在 Q 上的力是

$$\boldsymbol{F}=\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^3}\boldsymbol{r} \tag{2.1}$$

常数 ε_0 称为真空介电常数, $\varepsilon_0=8.85\times10^{-12}\ \dfrac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$

2.1.2 电场

$$\boldsymbol{E}\equiv\frac{\boldsymbol{F}}{Q}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r} \tag{2.2}$$

2.1.3 电场强度通量

$\boldsymbol{\Phi}_E$ 为电场强度通量

$$\boldsymbol{\Phi}_E\equiv\iint_S\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} \tag{2.3}$$

2.1.4 高斯定理

ρ 为电荷密度

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{1}{\varepsilon_0}\rho \tag{2.4}$$

2.1.5 两个带电为 q 的电荷, 相距 d 放置, 求垂直于连线中点且距离为 x_2 处的电场, 如果把一个换成 $-q$ 会怎样? 当 $y>>d$ 时会怎样?

$$\begin{aligned}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r}_1+\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r}_2&=2\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}x_2\boldsymbol{e}_2\\&=\frac{1}{2\pi\varepsilon_0}\frac{qx_2}{r^3}\boldsymbol{e}_2\\&=\frac{1}{2\pi\varepsilon_0}\frac{qx_2}{(x_2^2+\frac{d^2}{4})^{\frac{3}{2}}}\boldsymbol{e}_2\end{aligned}$$

$$\begin{aligned}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r}_1+\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\boldsymbol{r}_2&=2\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\frac{d}{2}\boldsymbol{e}_1\\&=\frac{1}{4\pi\varepsilon_0}\frac{qd}{r^3}\boldsymbol{e}_1\\&\approx\frac{1}{4\pi\varepsilon_0}\frac{qd}{x_2^3}\boldsymbol{e}_1\end{aligned}$$

2.1.6　一个长度为 $2L$ 的细杆均匀带电, 电荷线密度为 λ, 求垂直于杆且与杆中心距离为 x_2 处的电场

$$
\begin{aligned}
\int_{-L}^{L} \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^3}\boldsymbol{r}\,\mathrm{d}x &= \int_{-L}^{L} \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r^3}x_2\boldsymbol{e}_2\,\mathrm{d}x \\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-L}^{L}\frac{x_2}{r^3}\,\mathrm{d}x \\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-L}^{L}\frac{x_2}{(x^2+x_2^2)^{\frac{3}{2}}}\,\mathrm{d}x \\
&\xlongequal{x=x_2\tan\theta} \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-\arctan\frac{L}{x_2}}^{\arctan\frac{L}{x_2}}\frac{x_2}{(x_2^2\tan\theta^2+x_2^2)^{\frac{3}{2}}}\,\mathrm{d}x_2\tan\theta \\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-\arctan\frac{L}{x_2}}^{\arctan\frac{L}{x_2}}\frac{x_2^2\cos^3\theta}{x_2^3\cos^2\theta}\,\mathrm{d}\theta \\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\int_{-\arctan\frac{L}{x_2}}^{\arctan\frac{L}{x_2}}\frac{\cos\theta}{x_2^2}\,\mathrm{d}\theta \\
&= \frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_2\left.\frac{\sin\theta}{x_2}\right|_{-\arctan\frac{L}{x_2}}^{\arctan\frac{L}{x_2}} \\
&= \frac{\lambda}{2\pi\varepsilon_0 x_2}\boldsymbol{e}_2\frac{L}{\sqrt{L^2+x_2^2}}
\end{aligned}
$$

2.1.7　一个长度为 L 的细杆均匀带电, 电荷线密度为 λ, 求杆一端上方距杆 y 处的电场

$$
\begin{aligned}
\int_0^L \frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0 r^2}\,\mathrm{d}x_2 &= \int_0^L \frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0(x_2+y)^2}\,\mathrm{d}x_2 \\
&= -\left.\frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0(x_2+y)}\right|_0^L \\
&= \frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0(L+y)} - \frac{\lambda\boldsymbol{e}_y}{4\pi\varepsilon_0 y} \\
&= \frac{\lambda\boldsymbol{e}_y L}{4\pi\varepsilon_0(L+y)y}
\end{aligned}
$$

2.1.8　一个边长为 $2L$ 的正方形线框均匀带电, 电荷线密度为 λ, 求线框中心上方距 x_3 处的电场

$$
4\frac{\lambda}{2\pi\varepsilon_0\sqrt{L^2+x_3^2}}\boldsymbol{e}_3\frac{L}{2\sqrt{2L^2+x_3^2}}\frac{x_3}{x_3\sqrt{L^2+x_3^2}^2} = \frac{\lambda}{\pi\varepsilon_0\sqrt{2L^2+x_3^2}}\boldsymbol{e}_3\frac{2Lx_3}{L^2+x_3^2}
$$

2.1.9 一个半径为 r 的圆线框均匀带电, 电荷线密度为 λ, 求线框中心上方距 x_3 处的电场

$$2\pi r\frac{\lambda}{4\pi\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2+r^2}^3}=\frac{\lambda}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3 r}{\sqrt{x_3^2+r^2}^3}$$

2.1.10 一个半径为 r 的圆片均匀带电, 电荷面密度为 σ, 求圆片中心上方距 x_3 处的电场

$$\begin{aligned}\int_0^r\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3x_1}{\sqrt{x_3^2+x_1^2}^3}\,\mathrm{d}x_1&=-\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2+x_1^2}}\Bigg|_0^r\\&=\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2}}-\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2+r^2}}\\&=\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3-\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{x_3}{\sqrt{x_3^2+r^2}}\end{aligned}$$

2.1.11 一个半径为 r 的球面均匀带电, 电荷面密度为 σ, 求圆球上方距 d 处的电场 (分 $d>r$ 和 $d<r$ 讨论)

$$\begin{aligned}\int_\pi^0\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{(x_3+d)\sqrt{x_1^2+x_2^2}}{\sqrt{(x_3+d)^2+x_1^2+x_2^2}^3}\,\mathrm{d}r\phi&=\int_\pi^0\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{(r\cos\phi+d)\sqrt{x_1^2+x_2^2}}{\sqrt{(r\cos\phi+d)^2+x_1^2+x_2^2}^3}\,\mathrm{d}r\phi\\&=\int_\pi^0\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{(r\cos\phi+d)r\sin\phi}{\sqrt{r^2+2rd\cos\phi+d^2}^3}\,\mathrm{d}r\phi\\&=\int_0^\pi\frac{\sigma}{2\varepsilon_0}\boldsymbol{e}_3\frac{(r\cos\phi+d)r^2\sin\phi}{\sqrt{r^2+2rd\cos\phi+d^2}^3}\,\mathrm{d}\phi\end{aligned}$$

2.1.12 在某个区域电场可以写为 $\boldsymbol{E}=kr^3\boldsymbol{e}_r$, 求电荷密度和包含在半径为 R, 球心在原点的闭合球面内的总电荷

$$\rho=\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}=\varepsilon_0\frac{1}{sr}\frac{\partial srE_r}{\partial r}=\varepsilon_0\frac{1}{sr}\frac{\partial srr^3}{\partial r}=\varepsilon_0\frac{1}{sr}5sr^4=5\varepsilon_0r^3k$$

$$4\pi\int_0^R5\varepsilon_0r^3\,\mathrm{d}r=5\pi\varepsilon_0R^4k$$

2.2 电势

2.2.1 定义

$$U(\boldsymbol{r})\equiv-\int_O^r\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}\tag{2.5}$$

其中 O 为预先设置的标准参考点, 通常为无限远处

2.2.2　一个半径为 R 的均匀带电球体, 总电荷为 q, 求电势 $U(\boldsymbol{r})$

$r \geq R$:

$$\int_r^\infty \frac{q}{4\pi\varepsilon_0 x^2}\,\mathrm{d}x = -\frac{q}{4\pi\varepsilon_0 x}\bigg|_r^\infty = \frac{q}{4\pi\varepsilon_0 r}$$

$r < R$:

$$\begin{aligned}\frac{q}{4\pi\varepsilon_0 R} + \int_r^R \frac{q}{4\pi\varepsilon_0 x^2}\frac{x^3}{R^3}\,\mathrm{d}x &= \frac{q}{4\pi\varepsilon_0 R} + \frac{qx^2}{4\pi\varepsilon_0 R^3}\bigg|_r^R \\ &= \frac{q}{4\pi\varepsilon_0 R} + \frac{qR^2}{4\pi\varepsilon_0 R^3} - \frac{qr^2}{4\pi\varepsilon_0 R^3} \\ &= \frac{q}{2\pi\varepsilon_0 R} - \frac{qr^2}{4\pi\varepsilon_0 R^3}\end{aligned}$$

2.2.3　一条均匀带电的无限长直线, 电荷线密度为 λ, 求电势 $U(\boldsymbol{r})$

$$U = \int_r^\infty \frac{\lambda}{2\pi r}\,\mathrm{d}x = \frac{\lambda}{2\pi}\ln(x)\bigg|_r^\infty = \frac{1}{2\pi}\ln(r)$$

2.2.4　一个长度为 $2L$ 的细杆均匀带电, 电荷线密度为 λ, 求垂直于杆且与杆中心距离为 h 处的电势

$$\begin{aligned}U &= \int_h^\infty \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{x_2\sqrt{L^2+x_2^2}}\,\mathrm{d}x_2 \\ &= -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{x^2+L^2}+L}{x}\right)\bigg|_h^\infty \\ &= \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{h^2+L^2}+L}{h}\right)\end{aligned}$$

2.2.5　一个半径为 r 的圆片均匀带电, 电荷面密度为 σ, 求圆片中心上方距 h 处的电势

$$\begin{aligned}U &= \int_h^\infty \left(\frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0}\frac{x_3}{\sqrt{x_3^2+r^2}}\right)\mathrm{d}x_3 \\ &= \frac{\sigma}{2\varepsilon_0}x_3 - \frac{\sigma}{2\varepsilon_0}\sqrt{x_3^2+r^2}\bigg|_h^\infty \\ &= -\frac{\sigma}{2\varepsilon_0}\frac{r^2}{x_3+\sqrt{x_3^2+r^2}}\bigg|_h^\infty \\ &= \frac{\sigma}{2\varepsilon_0}\frac{r^2}{h+\sqrt{h^2+r^2}} \\ &= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{h^2+r^2}-h\right)\end{aligned}$$

2.2.6 一个半径为 r 的圆线框均匀带电, 电荷线密度为 λ, 求线框中心上方距 h 处的电势

$$U=\int_h^\infty \frac{\lambda}{2\varepsilon_0}\frac{x_3 r}{\sqrt{x_3^2+r^2}^3}\,\mathrm{d}x_3=-\frac{\lambda r}{2\varepsilon_0}\frac{1}{\sqrt{x_3^2+r^2}}\bigg|_h^\infty=\frac{\lambda r}{2\varepsilon_0}\frac{1}{\sqrt{h^2+r^2}}$$

2.2.7 一个尖角向下的圆锥体均匀带电, 电荷面密度为 σ, 圆锥高度等于半径为 h, 求 $z=h$ 处的电势

$$\begin{aligned}
U&=\int_{-h}^0 \frac{\lambda(h+x_3)}{2\varepsilon_0}\frac{1}{\sqrt{(h-x_3)^2+(h+x_3)^2}}\,\mathrm{d}x_3\\
&=\int_{-h}^0 \frac{\lambda(h+x_3)}{2\varepsilon_0}\frac{1}{\sqrt{2h^2+2x_3^2}}\,\mathrm{d}x_3\\
&=\frac{\lambda}{2\sqrt{2}\varepsilon_0}\int_{-h}^0 \frac{h}{\sqrt{h^2+x_3^2}}+\frac{x_3}{\sqrt{h^2+x_3^2}}\,\mathrm{d}x_3\\
&=h\ln\left(x_3+\sqrt{h^2+x_3^2}\right)+\sqrt{h^2+x_3^2}\bigg|_{-h}^0\frac{\lambda}{2\sqrt{2}\varepsilon_0}\\
&=\left[h\ln\left(\sqrt{h^2}\right)+\sqrt{h^2}-h\ln\left(-h+\sqrt{2h^2}\right)-\sqrt{2h^2}\right]\frac{\lambda}{2\sqrt{2}\varepsilon_0}\\
&=\left[h\ln\left(\frac{1}{\sqrt{2}-1}\right)+h-\sqrt{2}h\right]\frac{\lambda}{2\sqrt{2}\varepsilon_0}\\
&=\left[1-\sqrt{2}-\ln\left(\sqrt{2}-1\right)\right]\frac{\lambda h}{2\sqrt{2}\varepsilon_0}
\end{aligned}$$

2.2.8 一个半径为 r 的圆柱均匀带电, 电荷体密度为 ρ, 求圆柱中心上方距 h 处的电势

$$\begin{aligned}
U&=\int_{-h}^0 \frac{\rho}{2\varepsilon_0}\frac{r^2}{h+x_3+\sqrt{(h+x_3)^2+r^2}}\,\mathrm{d}x_3\\
&\xlongequal{u=h+x_3}\int_0^h \frac{\rho}{2\varepsilon_0}\frac{r^2}{u+\sqrt{u^2+r^2}}\,\mathrm{d}u\\
&=\int_0^h \frac{\rho r^2}{2\varepsilon_0}\frac{\sqrt{u^2+r^2}-u}{r^2}\,\mathrm{d}u\\
&=\frac{\rho}{2\varepsilon_0}\int_0^h \sqrt{u^2+r^2}-u\,\mathrm{d}u\\
&=u\sqrt{u^2+r^2}+r^2\ln\left(u+\sqrt{u^2+r^2}\right)-u^2\bigg|_0^h\frac{\rho}{4\varepsilon_0}\\
&=\left[h\sqrt{h^2+r^2}+r^2\ln\left(h+\sqrt{h^2+r^2}\right)-h^2-r^2\ln r\right]\frac{\rho}{4\varepsilon_0}
\end{aligned}$$

2.3 静电场的能量

2.3.1 离散电荷体系的静电能

考虑由 N 个点电荷 q_i 构成的静电体系。将电荷从无穷远逐个缓慢搬运到其最终位置, 外力所做的总功即为体系的静电能。

设在放置第 i 个电荷时, 其余电荷已就位, 则该电荷所处位置的当前电势为

$$\phi_i=\sum_{j=1}^{i-1}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}.$$

因此, 体系的总静电能为

$$W=\sum_{i=1}^{N}\sum_{j=1}^{i-1}\frac{1}{4\pi\varepsilon_0}\frac{q_iq_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}.$$

注意到相互作用能在指标交换下满足 $U_{ij}=U_{ji}$, 而对所有 $i\neq j$ 的求和中每一对指标被计数两次, 故可将上式对称化为

$$W=\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_iq_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}=\frac{1}{2}q_i\phi_i.$$

2.3.2 连续电荷分布的能量表达式

对离散情形的自然推广给出

$$W=\frac{1}{2}\iiint\rho\phi\,\mathrm{d}\tau.$$

2.3.3 场能量

静电势满足泊松方程

$$\nabla^2\phi=-\frac{\rho}{\varepsilon_0}.$$

将其代入能量表达式, 得

$$W=-\frac{\varepsilon_0}{2}\iiint\phi\,\nabla^2\phi\ \mathrm{d}\tau.$$

对右端积分作分部积分。注意到

$$\nabla\cdot(\phi\nabla\phi)=(\nabla\phi)^2+\phi\nabla^2\phi,$$

于是

$$\phi\nabla^2\phi=\nabla\cdot(\phi\nabla\phi)-(\nabla\phi)^2.$$

代入得

$$W=-\frac{\varepsilon_0}{2}\left[\iiint\nabla\cdot(\phi\nabla\phi)\ \mathrm{d}\tau-\iiint(\nabla\phi)^2\ \mathrm{d}\tau\right].$$

当 $r\to\infty$ 时,

$$\phi(\boldsymbol{r})\sim\frac{1}{r},\qquad\nabla\phi\sim\frac{1}{r^2}.$$

因此

$$\phi\nabla\phi\sim\frac{1}{r^3},$$

对应的无穷远处曲面积分

$$\oiint\phi\nabla\phi\cdot\mathrm{d}\boldsymbol{S}\to0$$

$$W=\frac{\varepsilon_0}{2}\iiint(\nabla\phi)^2\ \mathrm{d}\tau.$$

$$W=\frac{\varepsilon_0}{2}\iiint E^2\ \mathrm{d}\tau$$

2.3.4 电场能量密度

这表明：静电能可以视为分布在空间中的电场所携带的能量。

由此自然引入电场的能量密度

$$\boxed{u \equiv \frac{\varepsilon_0}{2}\boldsymbol{E}^2}$$

2.3.5 考虑两个同心球面, 半径分别为 a 和 b, 内球面带有电荷 q , 外球面带有电荷 $-q$, 求总能量

$$W = \frac{\varepsilon_0}{2}\iiint E^2\,\mathrm{d}\tau = \frac{\varepsilon_0 4\pi}{2}\int_a^b \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 r^2\,\mathrm{d}r = \int_a^b \frac{q^2}{8\pi\varepsilon_0 r^2}\,\mathrm{d}r = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right)$$

2.4 导体

2.4.1 导体内部电场为零

若导体内部存在非零电场, 则自由电荷将在电场作用下持续运动, 与静电平衡的假设矛盾。因此, 导体内部必须满足

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi = \boldsymbol{0}.$$

2.4.2 导体内部体电荷密度为零

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0},$$

结合 $\boldsymbol{E} = \boldsymbol{0}$,

$$\rho = 0 \qquad (\text{导体内部}).$$

2.4.3 净电荷只能分布在导体表面

既然导体内部体电荷密度为零, 而导体整体可能带有净电荷, 则这些电荷只能分布在导体的表面上。

2.4.4 导体是等势体

由 $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ 以及导体内部 $\boldsymbol{E} = \boldsymbol{0}$, 可知导体内部电势处处相同

2.4.5 导体表面外侧电场垂直于表面

考虑导体表面上一点的切向电场分量。若存在非零切向分量, 则自由电荷将在表面沿切向运动, 从而破坏静电平衡。

因此, 导体表面外侧的电场只能沿法向：

$$\boldsymbol{E}_{\parallel} = 0$$

2.4.6 空腔中含点电荷的球形导体

设一不带电的球形导体, 半径为 R, 中心位于原点, 其内部挖去一任意形状的空腔。在空腔内某处放置一点电荷 q。求球外区域的电场分布。

导体处于静电平衡时, 金属内部电场为零, 因而导体整体为等势体。取一紧贴导体内壁、位于金属内部的高斯面, 由高斯定律可得

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{a} = 0,$$

故内壁所感应的总电荷必为 $-q$。又由于导体整体不带电, 外表面所带电荷总量为 $+q$。

关键在于确定外表面电荷的分布形式。注意到：在所有满足

$$\boldsymbol{E} = 0 \quad (\text{导体内部}), \qquad \phi = \text{常数} \quad (\text{导体表面})$$

的允许电荷分布中, 实际的静电平衡态对应于体系总静电能的极小值。

球外区域不含自由电荷, 其电势完全由外表面电荷分布决定。若外表面电荷分布破坏球对称性, 则球外电场中将出现非径向分量, 从而在保持总电荷为 q 的约束下增加电场能

$$U = \frac{\varepsilon_0}{2} \int E^2 \, \mathrm{d}\tau.$$

因此, 能量极小所对应的电荷分布必然保持球对称性。

由此可知, 外表面电荷在球面上均匀分布, 其产生的球外电场与位于球心的点电荷 q 完全相同。因此, 球外任意一点处的电场为

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \, \boldsymbol{e}_r, \qquad r > R.$$

可见, 空腔的形状以及点电荷在腔内的具体位置, 均不影响球外区域的电场分布。

2.4.7 一个半径为 R 的金属球, 带有电荷 q, 这个金属球又被一个厚的同心金属球壳所包围 (球壳内径为 a, 外径为 b)。
(a) 分别求出 R, a, b 球面上的电荷面密度 σ。
(b) 求出球心处的电势, 选无限远处为参考点。
(c) 现在球壳的外表面接地, 电势能为零。(a) 和 (b) 所得结果改变为什么?

(a):

$$\sigma_R = \frac{q}{4\pi R^2}$$
$$\sigma_a = \frac{q}{4\pi a^2}$$
$$\sigma_b = \frac{q}{4\pi b^2}$$

(b):

$$U = \int_b^\infty \frac{q}{4\pi\varepsilon_0 r} \, \mathrm{d}r + \int_R^a \frac{q}{4\pi\varepsilon_0 r} \, \mathrm{d}r = \frac{q}{4\pi\varepsilon_0} \ln\left(\frac{bR}{a}\right)$$

(c):

$$\sigma_R = \frac{q}{4\pi R^2}$$
$$\sigma_a = \frac{q}{4\pi a^2}$$
$$\sigma_b = 0$$
$$U = \frac{q}{4\pi\varepsilon_0}\ln\left(\frac{R}{a}\right)$$

2.4.8 一个半径为 R 的导体球体, 其内部有两个半径分别为 **a** 和 **b** 的圆形空洞, 在 a 空洞的中心放有点电荷 q, 在 b 空洞的中心放有点电荷 **q**。
(a) 求出电荷面密度 σ_a, σ_b 和 σ_R。
(b) 导体外面的电场是什么?
(c) 每个空洞内的电场是什么?
(d)q_a 和 q_b 受到的力是什么?
(e) 如果让第三个电荷 q 靠近导体, 上面所得结果哪一个会发生变化?

(a):

$$\sigma_R = \frac{2q}{4\pi R^2}$$
$$\sigma_a = \frac{q}{4\pi a^2}$$
$$\sigma_b = \frac{q}{4\pi b^2}$$

(b):

$$\boldsymbol{E} = \frac{2q\boldsymbol{r}}{4\pi\varepsilon_0 r^3}$$

(c):

$$\boldsymbol{E_a} = \frac{q\boldsymbol{r}}{4\pi\varepsilon_0 r^3}$$
$$\boldsymbol{E_b} = \frac{q\boldsymbol{r}}{4\pi\varepsilon_0 r^3}$$

(d):0

(e): σ_R, b

2.5 电容

$$C \equiv \frac{Q}{U} \tag{2.6}$$

$$W = \int_0^Q \left(\frac{q}{C}\right)\mathrm{d}q = \frac{Q^2}{2C} \tag{2.7}$$

2.5.1 两个同轴金属管壳, 半径分别为 a 和 b, 求出单位长度的电容。

$$
\begin{aligned}
U &= \int_a^b \frac{Q}{2\varepsilon_0 r}\,\mathrm{d}r \\
&= \frac{Q}{2\varepsilon_0}\ln\left(\frac{b}{a}\right) \\
C &= \frac{Q}{U} \\
&= \frac{2\varepsilon_0}{\ln\left(\frac{b}{a}\right)}
\end{aligned}
$$

2.6 拉普拉斯方程

$$
\frac{\mathrm{d}^2 U}{\mathrm{d}x_i^2} = 0 \tag{2.8}
$$

2.6.1 一维拉普拉斯方程

$$
U = ax + b \tag{2.9}
$$

2.6.2 在球坐标下, 对 U 仅依赖于 r 的情况, 求出拉普拉斯方程的一般解。对柱坐标系, 假定 U 仅依赖于 s, 做同样的计算。

$$
\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) &= 0 \\
\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) &= 0 \\
r^2\frac{\partial U}{\partial r} &= C_1 \\
\frac{\partial U}{\partial r} &= \frac{C_1}{r^2} \\
U &= -\frac{C_1}{r} + C_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{s}\frac{\partial}{\partial u_s}\left(s\frac{\partial U}{\partial u_s}\right) &= 0 \\
s\frac{\partial U}{\partial u_s} &= C_1 \\
\frac{\partial U}{\partial u_s} &= \frac{C_1}{s} \\
U &= C_1 \ln s + C_2
\end{aligned}
$$

2.6.3 $U=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}}-\frac{q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right]$ 求出球面上的诱导电荷面密度。对其积分求出总诱导电荷。计算这个构型的能量。

$$
\begin{aligned}
\sigma &= -\varepsilon_0\frac{\partial U}{\partial r}\\
&= -\varepsilon_0\frac{\partial\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}}-\frac{q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right]}{\partial r}\\
&= -\frac{q}{4\pi}\left[-\frac{2r-2a\cos\theta}{2\sqrt{r^2+a^2-2ra\cos\theta}^3}+\frac{2\frac{ra}{R}-2a\cos\theta}{2\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right]\\
&\xlongequal{r=R} -\frac{q}{4\pi}\left[-\frac{2R-2a\cos\theta}{2\sqrt{R^2+a^2-2Ra\cos\theta}^3}+\frac{2\frac{Ra}{R}-2a\cos\theta}{2\sqrt{R^2+\left(\frac{Ra}{R}\right)^2-2Ra\cos\theta}}\right]\\
&\xlongequal{r=R} -\frac{q}{4\pi}\left[\frac{a-R}{\sqrt{R^2+a^2-2Ra\cos\theta}^3}\right]
\end{aligned}
$$

$$
\begin{aligned}
W &= \int_\infty^a q\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}\\
&= \int_\infty^a \frac{q^2R}{4\pi\varepsilon_0 r\left(r-\frac{R^2}{r}\right)^2}\,\mathrm{d}r\\
&= \int_\infty^a \frac{q^2R}{8\pi\varepsilon_0\left(r^2-R^2\right)^2}\,\mathrm{d}r^2\\
&= -\left.\frac{q^2R}{8\pi\varepsilon_0\left(r^2-R^2\right)}\right|_\infty^a\\
&= -\frac{q^2R}{8\pi\varepsilon_0\left(a^2-R^2\right)}
\end{aligned}
$$

2.7 镜像法

2.7.1 一条无限长均匀带电线, 电荷线密度为 λ, 它距一个接地导体板距离为 d。带电线平行于 x 轴并位于 x 轴上方, 导体板为 xy 平面
(a) 求出导体板上方的电势。
(b) 求出导体板上的诱导电荷的面密度。

(a):

$$U=\frac{\lambda}{2\pi\varepsilon_0|x_3-d|}\ln|x_3-d|-\frac{\lambda}{2\pi\varepsilon_0(x_3+d)}\ln(x_3+d)$$

(b):

$$
\begin{aligned}
E_3 &= \frac{\lambda}{\pi\varepsilon_0 d}\\
\sigma &= \varepsilon_0E_3=\frac{\lambda}{\pi d}
\end{aligned}
$$

2.7.2 两个半无限大接地导体板一端相接形成一个直角。在它们之间的区域有一个点电荷 q, 计算这个区域内的电势。作用在 q 上的力是什么? 把 q 从无限远处移到所示位置需做多少功? 假定两板形成的角度不是 $\frac{\pi}{2}$, 而是另外的一些角度, 你还能用镜像法求解问题吗? 如果不能, 对什么样的特殊角度仍然可以用镜像法求解?

假设在 $(a,a),(-a,-a)$ 处有电荷 q, $(a,-a),(-a,a)$ 有电荷 $-q$, 当 $x=0$, $z=0$ 时

$$U=\frac{q}{4\pi}\left(\frac{1}{\sqrt{a^2+(y-a)^2}}+\frac{1}{\sqrt{a^2+(y+a)^2}}-\frac{1}{\sqrt{a^2+(y-a)^2}}-\frac{1}{\sqrt{a^2+(y+a)^2}}\right)=0$$

$$\begin{aligned}
\boldsymbol{F}=q\boldsymbol{E}&=-\frac{q^2\boldsymbol{e}_1}{4\pi\varepsilon_0a^2}-\frac{q^2\boldsymbol{e}_2}{4\pi\varepsilon_0a^2}+\frac{\frac{\sqrt{2}}{2}q^2\boldsymbol{e}_1+\frac{\sqrt{2}}{2}q^2\boldsymbol{e}_2}{4\pi\varepsilon_0 2a^2}\\
|\boldsymbol{F}|&=\left(\sqrt{2}-\frac{1}{2}\right)\frac{q^2}{4\pi\varepsilon_0a^2}\\
W&=\int_\infty^a\left(\sqrt{2}-\frac{1}{2}\right)\frac{\sqrt{2}q^2}{4\pi\varepsilon_0r^2}\,\mathrm{d}r\\
&=-\left(\sqrt{2}-\frac{1}{2}\right)\frac{\sqrt{2}q^2}{4\pi\varepsilon_0r}\bigg|_\infty^a\\
&=-\left(\sqrt{2}-\frac{1}{2}\right)\frac{\sqrt{2}q^2}{4\pi\varepsilon_0a}
\end{aligned}$$

在无源区域内, 电势满足拉普拉斯方程:

$$\nabla^2V(x,y)=0.$$

在二维情况下, 可引入复势

$$W(z)=\Phi(x,y)+i\Psi(x,y),$$

其中

$$\Phi=V.$$

$W(z)$ 为解析函数的充要条件是 Φ 与 Ψ 满足 Cauchy–Riemann 条件, 而这等价于 Φ 满足拉普拉斯方程。因此, 求解二维静电问题等价于构造合适的解析函数 $W(z)$。

二维中, 点电荷对应的 Green 函数为对数型:

$$W_0(z)=-\frac{q}{2\pi\epsilon_0}\ln(z-z_0),$$

其电势为

$$U(z)=-\frac{q}{2\pi\epsilon_0}\ln|z-z_0|.$$

该表达式在 $z=z_0$ 处具有对数奇点, 对应于二维点电荷。

定义映射

$$w=z^{\pi/\alpha}.$$

该映射具有如下性质:

- 若 $0<\arg z<\alpha$, 则 $0<\arg w<\pi$;
- 楔形区域被映射为上半平面;
- $\arg z=0,\alpha$ 被映射为实轴。

因此, 楔形导体边界在 w 平面中对应于接地的实轴。

在上半平面中, 实轴接地, 位于 $w_0(\Im w_0 > 0)$ 的点电荷的复势为

$$W(w) = -\frac{q}{2\pi\epsilon_0}\ln\frac{w-w_0}{w-\overline{w_0}}.$$

该表达式满足:

- 在上半平面内调和;
- 在实轴上 $|w-w_0| = |w-\overline{w_0}|$, 因而 $V=0$;
- 在 $w=w_0$ 处具有正确的对数奇点。

这是由唯一性定理保证的解。

将 $w = z^{\pi/\alpha}$ 代回, 得到楔形区域内的复势:

$$W(z) = -\frac{q}{2\pi\epsilon_0}\ln\frac{z^{\pi/\alpha}-z_0^{\pi/\alpha}}{z^{\pi/\alpha}-\overline{z_0}^{\pi/\alpha}}.$$

电势为其实部:

$$U(z) = -\frac{q}{2\pi\epsilon_0}\ln\left|\frac{z^{\pi/\alpha}-z_0^{\pi/\alpha}}{z^{\pi/\alpha}-\overline{z_0}^{\pi/\alpha}}\right|.$$

这是任意楔角 α 下的严格解。

若

$$\alpha = \frac{\pi}{n}, \quad n\in\mathbb{N},$$

则

$$w = z^n$$

是单值多项式映射。

利用因式分解:

$$z^n - z_0^n = \prod_{k=0}^{n-1}\left(z - z_0 e^{2\pi ik/n}\right),$$

电势可写为有限和:

$$U(z) = -\frac{q}{2\pi\epsilon_0}\sum_{k=0}^{n-1}\ln\left|\frac{z-z_0 e^{2\pi ik/n}}{z-\overline{z_0}e^{2\pi ik/n}}\right|.$$

这正对应于有限多个镜像电荷的叠加。

若 $\alpha\neq\pi/n$, 则 $z^{\pi/\alpha}$ 为多值函数, 解析延拓将产生无限多个像点, 镜像法不再以有限求和形式成立。

2.8 分离变量法

2.8.1 直角坐标系

两个无限大接地金属平板平行于 xz 平面放置, 一个位于 $y=0$, 另一个位于 $y=a$。在 $x=0$ 两板的左端点, 被与两板绝缘的无限长带封闭, 带子上维持特定的电势 $U_0(y)$。求出这个“夹缝”中的电势。

由于几何结构和边界条件在 z 方向具有平移对称性, 且 U_0 与 z 无关, 物理解必然满足

$$\frac{\partial\Phi}{\partial z} = 0.$$

因此问题严格退化为二维拉普拉斯方程：

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

其边界条件为

$$\begin{aligned}
&U \xlongequal{y=0} 0\\
&U \xlongequal{y=a} 0\\
&U \xlongequal{x=0} U_0(y)\\
&U \xrightarrow{x\to\infty} 0
\end{aligned}$$

设解为

$$U(x,y)=X(x)Y(y).$$

代入 Laplace 方程得

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=0.$$

因此两项必须分别等于常数, 记为 $-k^2$：

$$\begin{aligned}
Y''(y)+k^2Y(y)&=0,\\
X''(x)-k^2X(x)&=0.
\end{aligned}$$

非平凡解存在当且仅当

$$k=\frac{n\pi}{a},\quad n=1,2,3,\ldots$$

对应本征函数为

$$Y_n(y)=\sin\frac{n\pi y}{a}.$$

对每个 $k_n=n\pi/a$, 横向方程为

$$X_n''(x)-\left(\frac{n\pi}{a}\right)^2X_n(x)=0.$$

通解为

$$X_n(x)=A_n\mathrm{e}^{-(n\pi/a)x}+B_n\mathrm{e}^{+(n\pi/a)x}.$$

由远处边界条件, 要求 $V\to 0$,

$$B_n=0.$$

利用线性叠加原理, 电势的一般解为

$$V(x,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{a}\,\mathrm{e}^{-(n\pi/a)x}.$$

在 $x=0$ 处, 要求

$$V(0,y)=U_0(y).$$

因此

$$U_0(y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{a}.$$

$$A_n = \frac{2}{a}\int_0^a U_0(y)\sin\frac{n\pi y}{a}\,dy$$

综上, 夹缝区域中的电势为

$$U = \sum_{n=1}^{\infty}\left[\frac{2}{a}\int_0^a U_0(y')\sin\frac{n\pi y'}{a}\,\mathrm{d}y'\right]\sin\frac{n\pi y}{a}\mathrm{e}^{-(n\pi/a)x}$$

若问题在 z 方向不具平移对称性, 可进一步设

$$\Phi = X(x)Y(y)Z(z),$$

并引入

$$Z'' + \lambda^2 Z = 0, \qquad Z(z) = \mathrm{e}^{i\lambda z}, \quad \lambda\in\mathbb{R}.$$

此时解的结构变为

$$\Phi(x,y,z) = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} C_n(\lambda)\sin\frac{n\pi y}{a}\,\mathrm{e}^{-\sqrt{(n\pi/a)^2+\lambda^2}\,x}\,\mathrm{e}^{i\lambda z}\,d\lambda.$$

2.8.2 球坐标系

在球坐标系中, 拉普拉斯算符为

$$\boldsymbol{\nabla}^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}$$

设电势可以写成完全分离的形式, 即 $U = R(r)\Theta(\theta)\Phi(\phi)$

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} &= 0\\
\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} &= 0\\
\frac{\partial}{\partial r}\left(r^2\frac{\partial R\Theta\Phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial R\Theta\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 R\Theta\Phi}{\partial\phi^2} &= 0\\
\frac{1}{Rr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{\Theta r^2\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} &= 0\\
-\left[\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}\right] &= \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)
\end{aligned}$$

由于左边仅依赖 r, 右边仅依赖角变量, 两边必须等于同一个常数。
引入分离常数 $l(l+1)$, $l\in\mathbb{N}$
于是得到:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - l(l+1)R = 0. \tag{2.10}$$

$$\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + l(l+1) = 0. \tag{2.11}$$

继续分离 $\Theta(\theta)\Phi(\phi)$, 令

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -m^2, \qquad m\in\mathbb{Z}.$$

于是得到:

ϕ 方程

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} + m^2\Phi = 0, \tag{2.12}$$

其解为

$$\Phi_m(\phi) = C_1\mathrm{e}^{im\phi}$$

θ 方程

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0. \tag{2.13}$$

这是关联勒让德方程, 其在 $\theta \in [0,\pi]$ 上正则的解为

$$\Theta_{lm}(\theta) = P_{lm}(\cos\theta), \qquad l = 0,1,2,\ldots, \qquad |m| \le l.$$

其中,

$$P_{lm}(x) \equiv \sqrt{(1+x^2)^m}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x) \tag{2.14}$$

$P_l(x)$ 可由罗德里格 (Rodrigue) 公式计算:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l (x^2-1)^l \tag{2.15}$$

罗德里格公式显然仅对非负的整数 l 成立。另外, 它仅提供给我们一个解。但是式2.13应当有两个解。情况是那些另外的解在 $\theta = 0$ 和/或 $\theta = \pi$ 发散

为了方便, 将角向部分合并, 定义球谐函数

$$Y_{lm} \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)\mathrm{e}^{im\phi} \tag{2.16}$$

径向方程为

$$r^2R'' + 2rR' - l(l+1)R = 0,$$

通解为

$$R_l(r) = A_l r^l + B_l r^{-(l+1)} \tag{2.17}$$

将各部分组合, 得到球坐标下拉普拉斯方程的一般解

$$U(r,\theta,\phi) = \left(A_{lm}r^l + B_{lm}r^{-(l+1)}\right)Y_{lm}(\theta,\phi) \tag{2.18}$$

当 $m = 0$ 时, 式2.13退化为

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + l(l+1)\Theta = 0 \tag{2.19}$$

式2.13的解退化为

$$\Theta(\theta) = P_l(\cos\theta) \tag{2.20}$$

球坐标下拉普拉斯方程的一般解退化为

$$U(r,\theta) = \left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta) \tag{2.21}$$

2.8.3 柱坐标系

$$\nabla^2 U=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial U}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 U}{\partial\phi^2}+\frac{\partial^2 U}{\partial z^2}$$

设电势可以写成完全分离的形式, 即 $U=S(s)Z(z)\Phi(\phi)$

$$\begin{aligned}0&=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S(s)Z(z)\Phi(\phi)}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 S(s)Z(z)\Phi(\phi)}{\partial\phi^2}+\frac{\partial^2 S(s)Z(z)\Phi(\phi)}{\partial z^2}\\&=\frac{1}{sS(s)}\frac{\partial}{\partial s}\left(s\frac{\partial S(s)}{\partial s}\right)+\frac{1}{s^2\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}\end{aligned}$$

设 $\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=-m^2$

$$-\left[\frac{1}{sS(s)}\frac{\partial}{\partial s}\left(s\frac{\partial S(s)}{\partial s}\right)-\frac{m^2}{s^2}\right]=\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$

由于左边仅依赖 s, 右边仅依赖 z, 两边必须等于同一个常数, 设为 $-k^2$

$$\frac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2}+k^2 Z(z)=0$$

$$\begin{aligned}\frac{1}{sS(s)}\frac{\partial}{\partial s}\left(s\frac{\partial S(s)}{\partial s}\right)-\frac{m^2}{s^2}&=k^2\\\frac{s}{S(s)}\frac{\partial}{\partial s}\left(s\frac{\partial S(s)}{\partial s}\right)-k^2s^2&=m^2\\s^2\frac{\partial^2 S(s)}{\partial s^2}+s\frac{\partial S(s)}{\partial s}-k^2s^2S(s)-m^2S(s)&=0\end{aligned}$$

设 $S(s)=\sum\limits_{n=0}^{\infty}a_n s^{n+c}$

$$\begin{aligned}0&=\sum_{n=0}^{\infty}a_n(n+c)(n+c-1)s^{n+c}+\sum_{n=0}^{\infty}a_n(n+c)s^{n+c}-k^2\sum_{n=0}^{\infty}a_n s^{n+c+2}-m^2\sum_{n=0}^{\infty}a_n s^{n+c}\\&=\sum_{n=0}^{\infty}a_n\left[(n+c)^2-m^2\right]s^{n+c}-k^2\sum_{n=0}^{\infty}a_n s^{n+c+2}\\&=a_0\left[c^2-m^2\right]s^c+a_1\left[(1+c)^2-m^2\right]s^{1+c}+\sum_{n=2}^{\infty}\left\{a_n\left[(n+c)^2-m^2\right]-k^2a_{n-2}\right\}s^{n+c}\end{aligned}$$

$$a_{n+2}=\frac{k^2}{(n+c)^2-m^2}$$

当处于 $\frac{\partial U}{\partial z}=0$ 的简化条件下时

$$s^2\frac{\partial^2 S(s)}{\partial s^2}+s\frac{\partial S(s)}{\partial s}-m^2S(s)=0\tag{2.22}$$

解得

$$S(s)=C_m s^{-m}+D_m s^m\tag{2.23}$$

$$\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = -m^2$$

$$\Phi = E_m\cos m\phi + F_m\sin m\phi,\ (m\in\mathbb{N}) \tag{2.24}$$

将各部分组合, 得到一般解

$$U = \left(C_m s^{-m} + D_m s^m\right)\left(E_m\cos m\phi + F_m\sin m\phi\right),\ (m\in\mathbb{N}) \tag{2.25}$$

2.8.4 勒让德多项式性质的证明

$l\in\mathbb{N}$ 的证明

$$\begin{aligned}
0 &= \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + l(l+1)\Theta \\
&\xlongequal{u=\cos\theta} \frac{1}{\sqrt{1-u^2}}\frac{\mathrm{d}}{\mathrm{d}\arccos u}\left(\sqrt{1-u^2}\frac{\mathrm{d}\Theta}{\mathrm{d}\arccos u}\right) + l(l+1)\Theta \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left[\left(1-u^2\right)\frac{\mathrm{d}\Theta}{\mathrm{d}u}\right] + l(l+1)\Theta \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}\Theta}{\mathrm{d}u}\right) - \frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\frac{\mathrm{d}\Theta}{\mathrm{d}u}\right) + l(l+1)\Theta \\
&= \frac{\mathrm{d}^2\Theta}{\mathrm{d}u^2} - u^2\frac{\mathrm{d}^2\Theta}{\mathrm{d}u^2} - 2u\frac{\mathrm{d}\Theta}{\mathrm{d}u} + l(l+1)\Theta
\end{aligned}$$

设

$$\begin{aligned}
\Theta &= \sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
0 &= \frac{\mathrm{d}^2\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n}{\mathrm{d}u^2} - u^2\frac{\mathrm{d}^2\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n}{\mathrm{d}u^2} - 2u\frac{\mathrm{d}\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n}{\mathrm{d}u} + l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
&= \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n - u^2\sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n - 2u\sum_{n=0}^{\infty}\frac{a_{n+1}}{n!}u^n + l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
&= \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n - \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^{n+2} - 2\sum_{n=0}^{\infty}\frac{a_{n+1}}{n!}u^{n+1} + l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
&= \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n - \sum_{n=2}^{\infty}\frac{a_n}{(n-2)!}u^n - 2\sum_{n=1}^{\infty}\frac{a_n}{(n-1)!}u^n + l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n \\
&= \sum_{n=0}^{\infty}\frac{a_{n+2}}{n!}u^n - \sum_{n=0}^{\infty}\frac{a_n}{(n-2)!}u^n - 2\sum_{n=0}^{\infty}\frac{a_n}{(n-1)!}u^n + l(l+1)\sum_{n=0}^{\infty}\frac{a_n}{n!}u^n\ (n>1) \\
&= \frac{a_{n+2}}{n!} - \frac{a_n}{(n-2)!} - 2\frac{a_n}{(n-1)!} + l(l+1)\frac{a_n}{n!} \\
&= a_{n+2} - a_n n(n-1) - 2a_n n + l(l+1)a_n \\
a_{n+2} &= [n(n+1) - l(l+1)]a_n
\end{aligned}$$

由高斯判别法可得级数在 $u=\pm1$ 时发散, 因此 $l\in Z$

勒让德多项式的求和形式 由递推关系 $a_{n+2} = -(l-n)(l+n+1)a_n$, 若级数在 $n=l$ 时截断, 则 $\Theta(u)$ 成为一个 l 阶多项式。规定最高次项 u^l 的系数为 $\frac{(2l)!}{2^l(l!)^2}$, 通过逆向递推最终得到 l 阶勒让德多项式的求和形式:

$$P_l(x) = \sum_{k=0}^{M}(-1)^k\frac{(2l-2k)!}{2^l k!(l-k)!(l-2k)!}x^{l-2k} \tag{2.26}$$

其中 $M = l/2$(l 为偶数) 或 $(l-1)/2$(l 为奇数)。

正交性的证明 取两个不同阶数的勒让德函数 P_a、P_b, 满足:

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right] + a(a+1)P_a \\
0 &= \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right] + b(b+1)P_b \\
0 &= \left\{\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right] + a(a+1)P_a\right\}P_b - \left\{\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right] + b(b+1)P_b\right\}P_a \\
&= \int_{-1}^{1}\left\{\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right] + a(a+1)P_a\right\}P_b - \left\{\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right] + b(b+1)P_b\right\}P_a\,\mathrm{d}u \\
&= \int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right]P_b + a(a+1)P_aP_b - \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right]P_a - b(b+1)P_bP_a\,\mathrm{d}u \\
&= \int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right]P_b - \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right]P_a\,\mathrm{d}u + \left[a(a+1)-b(b+1)\right]\int_{-1}^{1}P_aP_b\,\mathrm{d}u
\end{aligned}
$$

由分部积分可得

$$
\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_a'\right]P_b - \frac{\mathrm{d}}{\mathrm{d}u}\left[(1-u^2)P_b'\right]P_a\,\mathrm{d}u = (1-u^2)(P_bP_a' - P_aP_b')\Big|_{-1}^{1} = 0
$$

得证

$$
\left[a(a+1)-b(b+1)\right]\int_{-1}^{1}P_aP_b\,\mathrm{d}u = 0\,\mathrm{d}u
$$

完备性的证明略过

证明 $P_l(1) = 1$

$$
\begin{aligned}
P_l(x) &= \frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l (x^2-1)^l \\
&\xlongequal{u=x-1} \frac{1}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^l \left[u^l(u+2)^l\right]
\end{aligned}
$$

根据莱布尼茨公式, 只有当 u^l 被求导 l 次变成 $l!$ 时, 且剩下部分不含 u(即 $u \to 0$ 时不为 0), 该项才有贡献:

$$
P_l(1) = \frac{1}{2^l l!}\cdot l!\cdot (u+2)^l\Big|_{u=0} = \frac{1}{2^l}\cdot 2^l = 1
$$

同理可证 $P_l(-1) = (-1)^l$。

2.8.5 两个无限长接地金属板, 分别在 $y=0$ 和 $y=a$ 放置, 在 $x=\pm b$ 的侧边连接有电势为 U_0 的两个金属带。求出这个矩形管中的电势。
若侧面上的电势为零, 顶面的电势为非零常数 U_0, 求出管内的电势。
当 $a=b$ 时, 求底面单位长度的电荷

此时边界条件为:

$$
\begin{aligned}
U &\xlongequal{y=0} 0 \\
U &\xlongequal{y=a} 0 \\
U &\xlongequal{x=b} U_0 \\
U &\xlongequal{x=-b} U_0
\end{aligned}
$$

做法同前解得

$$U=(A\mathrm{e}^{\frac{n\pi x}{a}}+B\mathrm{e}^{-\frac{n\pi x}{a}})(C\sin\frac{n\pi}{a}y+D\cos\frac{n\pi}{a}y)$$

因为 $U(-x)=U(x)$, $U\xlongequal{y=0}0$, 所以 $A=B$, $D=0$, 并把系数吸进 C 得

$$U=C\cosh\frac{n\pi}{a}x\sin\frac{n\pi}{a}y$$

余下的事是构造一般的叠加解, 设定系数 C_n, 使其拟合边界条件

$$U(b,y)=\sum_{n=1}^{\infty}C_n\cosh\frac{n\pi}{a}b\sin\frac{n\pi}{a}y=U_0$$

因为 $U(b,y)=U(b,-y)$, 所以 $\sin\frac{n\pi}{a}y$ 为偶函数, n 为奇数, 即

$$U(b,y)=\sum_{n=0}^{\infty}C_n\cosh\frac{2n+1}{a}\pi b\sin\frac{2n+1}{a}\pi y=U_0$$

$$\begin{aligned}
C_n\cosh\frac{2n+1}{a}\pi b&=\frac{2}{a}\int_0^a U_0\sin\frac{2n+1}{a}\pi y\,\mathrm{d}y\\
&=\frac{4U_0}{(2n+1)\pi}\left(\cosh\frac{2n+1}{a}\pi b\right)^{-1}\\
U&=\sum_{n=0}^{\infty}\frac{4U_0}{(2n+1)\pi}\left(\cosh\frac{2n+1}{a}\pi b\right)^{-1}\cosh\frac{n\pi}{a}x\sin\frac{n\pi}{a}y
\end{aligned}$$

此时边界条件为:

$$\begin{aligned}
U&\xlongequal{y=0}0\\
U&\xlongequal{y=a}U_0\\
U&\xlongequal{x=b}0\\
U&\xlongequal{x=-b}0
\end{aligned}$$

做法同前解得

$$U=(A\mathrm{e}^{\frac{n\pi}{b}y}+B\mathrm{e}^{-\frac{n\pi}{b}y})(C\sin\frac{n\pi}{b}x+D\cos\frac{n\pi}{b}x)$$

因为 $U(-x)=U(x)$, $U\xlongequal{y=0}0$, 所以 $A=-B$, $D=0$, 并把系数吸进 D 得

$$U=C\sinh\frac{n\pi}{b}y\sin\frac{n\pi}{b}x$$

余下的事是构造一般的叠加解, 设定系数 C_n, 使其拟合边界条件

$$U(x,a)=\sum_{n=1}^{\infty}C_n\sinh\frac{n\pi}{b}a\sin\frac{n\pi}{b}x=U_0$$

$$\begin{aligned}
\int_{-b}^{b}C_n\sinh\frac{n\pi}{b}a\sin^2\frac{n\pi}{b}x\,\mathrm{d}x&=\int_{-b}^{b}U_0\sin\frac{n\pi}{b}x\,\mathrm{d}x\\
bC_n\sinh\frac{n\pi}{b}a&=-\frac{2b}{n\pi}U_0\cos n\pi\\
C_n&=-\frac{2}{n\pi\sinh\frac{n\pi}{b}a}U_0\cos n\pi
\end{aligned}$$

$$
\begin{aligned}
U &= \frac{4U_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{\sinh\frac{(2n+1)\pi}{2b}y}{\sinh\frac{(2n+1)\pi}{2b}a}\sin\frac{(2n+1)\pi(x+b)}{2b}\\
&= \frac{4U_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{\sinh\frac{(2n+1)\pi}{2b}y}{\sinh\frac{(2n+1)\pi}{2}}\sin\frac{(2n+1)\pi(x+b)}{2b}
\end{aligned}
$$

$$
\begin{aligned}
\sigma &= -\varepsilon_0\frac{\partial U}{\partial y}\\
\sigma &= -\varepsilon_0\frac{\partial\frac{4U_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{\sinh\frac{(2n+1)\pi}{2b}y}{\sinh\frac{(2n+1)\pi}{2}}\sin\frac{(2n+1)\pi(x+b)}{2b}}{\partial y}\\
\sigma &= -\varepsilon_0\frac{4U_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{(2n+1)\pi}{2b}\frac{1}{\sinh\frac{(2n+1)\pi}{2}}\sin\frac{(2n+1)\pi(x+b)}{2b}\\
\sigma &= -4U_0\varepsilon_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2b}\frac{1}{\sinh\frac{(2n+1)\pi}{2}}\sin\frac{(2n+1)\pi(x+b)}{2b}\\
\int_{-b}^{b}\sigma\,\mathrm{d}x &= \int_{-b}^{b}-4U_0\varepsilon_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2b}\frac{1}{\sinh\frac{(2n+1)\pi}{2}}\sin\frac{(2n+1)\pi(x+b)}{2b}\,\mathrm{d}x
\end{aligned}
$$

2.8.6 一个半径为 R 的球面上的电势为 $U_0 = k\cos 3\theta$。求出球面内外的电势以及球面上的电荷面密度 $\sigma(\theta)$。(假定球内和球外没有电荷分布。)

球内:

$$
\begin{aligned}
U(R,\theta) &= k\cos 3\theta\\
\left(A_lR^l + B_lR^{-(l+1)}\right)P_l(\cos\theta) &= k\cos 3\theta\\
A_lR^lP_l(\cos\theta) &= k\cos 3\theta\\
\int_0^\pi A_l^2R^{2l}P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta A_lR^lP_l(\cos\theta)\sin\theta\,\mathrm{d}\theta\\
\int_0^\pi A_lR^lP_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta\\
\int_0^\pi A_0R^0P_0^2(\cos\theta)\sin\theta\,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta P_0(\cos\theta)\sin\theta\,\mathrm{d}\theta\\
\int_0^\pi A_0\sin\theta\,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta\sin\theta\,\mathrm{d}\theta\\
A_0 &= 0\\
\int_0^\pi A_1RP_1^2(\cos\theta)\sin\theta\,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta P_1(\cos\theta)\sin\theta\,\mathrm{d}\theta\\
\int_0^\pi A_1R\cos^2\theta\sin\theta\,\mathrm{d}\theta &= \int_0^\pi k\cos 3\theta\cos\theta\sin\theta\,\mathrm{d}\theta
\end{aligned}
$$

不想算积分了

2.8.7 假定一个球面上的电势为 $U_0(\theta)$，并且球内球外没有电荷分布。证明球面上的电荷面密度为 $\sigma = \frac{\varepsilon_0}{2R}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta'$

$$
\begin{aligned}
U(R,\theta) &= U_0(\theta) \\
\left(A_l R^l + B_l R^{-(l+1)}\right) P_l(\cos\theta) &= U_0(\theta)
\end{aligned}
$$

球外:

$$
\begin{aligned}
B_l R^{-(l+1)} P_l(\cos\theta) &= U_0(\theta) \\
\int_0^\pi B_l^2 R^{-2(l+1)} P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta &= \int_0^\pi B_l R^{-(l+1)} P_l(\cos\theta) U_0(\theta)\sin\theta\,\mathrm{d}\theta \\
\int_0^\pi B_l R^{-(l+1)} P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta &= \int_0^\pi P_l(\cos\theta) U_0(\theta)\sin\theta\,\mathrm{d}\theta \\
B_l &= \frac{\int_0^\pi P_l(\cos\theta) U_0(\theta)\sin\theta\,\mathrm{d}\theta}{\int_0^\pi R^{-(l+1)} P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta}
\end{aligned}
$$

球内:

$$
\begin{aligned}
A_l R^l P_l(\cos\theta) &= U_0(\theta) \\
\int_0^\pi A_l R^l P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta &= \int_0^\pi U_0(\theta) P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta \\
A_l &= \frac{\int_0^\pi U_0(\theta) P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta}{\int_0^\pi R^l P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta}
\end{aligned}
$$

由连续性可得

$$
\begin{aligned}
A_l R^l P_l(\cos\theta) &= B_l R^{-(l+1)} P_l(\cos\theta) \\
A_l R^l &= B_l R^{-(l+1)}
\end{aligned}
$$

$$
\begin{aligned}
\sigma &= -\varepsilon_0 \frac{\partial U}{\partial r} \\
\frac{\varepsilon_0}{2R}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= -\varepsilon_0 \frac{\partial U}{\partial r} \\
\frac{1}{2R}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= -\left.\frac{\partial B_l r^{-(l+1)} P_l(\cos\theta)}{\partial r} + \frac{\partial A_l r^l P_l(\cos\theta)}{\partial r}\right|_{r=R} \\
\frac{1}{2R}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= (l+1)B_l R^{-(l+2)} P_l(\cos\theta) + l A_l R^{l-1} P_l(\cos\theta) \\
\frac{1}{2}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= (l+1)B_l R^{-(l+1)} P_l(\cos\theta) + l A_l R^l P_l(\cos\theta) \\
\frac{1}{2}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= (l+1)B_l R^{-(l+1)} P_l(\cos\theta) + l B_l R^{-(l+1)} P_l(\cos\theta) \\
\frac{1}{2}(2l+1)^2 P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= (2l+1)B_l R^{-(l+1)} P_l(\cos\theta)
\end{aligned}
$$

逐项比较可得 (不使用爱因斯坦求和约定)

$$\begin{aligned}\frac{1}{2}(2l+1)^2P_l(\cos\theta)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= (2l+1)B_lR^{-(l+1)}P_l(\cos\theta)\\ \frac{1}{2}(2l+1)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= B_lR^{-(l+1)}\\ \frac{1}{2}(2l+1)\int_0^\pi U_0(\theta')P_l(\cos\theta')\sin\theta'\,\mathrm{d}\theta' &= \frac{\int_0^\pi P_l(\cos\theta)U_0(\theta)\sin\theta\,\mathrm{d}\theta}{\int_0^\pi R^{-(l+1)}P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta}R^{-(l+1)}\\ \frac{1}{2}(2l+1) &= \frac{1}{\int_0^\pi P_l^2(\cos\theta)\sin\theta\,\mathrm{d}\theta}\end{aligned}$$

2.8.8 一个带电金属球 (电荷为 Q, 半径为 R) 置于均匀外电场 $\boldsymbol{E}_0$ 中, 求球外的电势。

边界条件:

$$\begin{aligned}\lim_{r\to\infty}U &\to E_0r\cos\theta\\ U(R) &= 0\\ \left(A_lR^l+B_lR^{-(l+1)}\right)P_l(\cos\theta) &= 0\\ \left(A_lR^l+B_lR^{-(l+1)}\right)P_l(\cos\theta) &= 0\end{aligned}$$

显然 $A_1=-E_0$。, 其余诸 A_l 为零

$$\begin{aligned}-E_0R^1+B_1R^{-2} &= 0\\ B_1 &= E_0R^3\end{aligned}$$

2.8.9 在习题2.2.5中求出了一个均匀带电盘轴线上的电势。
(a) 对带电盘不在轴线上的电势, 计算在展开式中的前三项, 假设 $r>R$。
(b) 求出 $r<R$ 的电势。

(a):

$$\begin{aligned}\frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2}-r\right) &= \left(A_lr^l+B_lr^{-(l+1)}\right)P_l(1)\\ \frac{\sigma r}{2\varepsilon_0}\left(\sqrt{1+\frac{R^2}{r^2}}-1\right) &= \sum_{l=0}^{\infty}B_lr^{-(l+1)}\\ \frac{\sigma R^2}{2\varepsilon_0 2r} &= B_0r^{-1}\\ 0 &= B_1r^{-2}\\ -\frac{\sigma R^4}{2\varepsilon_0 8r^3} &= B_2r^{-3}\end{aligned}$$

(b):

$$\begin{aligned}\frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2}-r\right) &= \left(A_lr^l+B_lr^{-(l+1)}\right)P_l(1)\\ \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2}-r\right) &= A_lr^l\end{aligned}$$

2.8.10 一个半径为 R 的球壳在“北半球”带有均匀的面电荷, 电荷面密度为 σ, 在“南半球”也带有均匀的面电荷, 电荷面密度为 $-\sigma$。求出球壳内外的电势。

由连续性可得 $A_l r^l = B_l r^{-(l+1)}$

$$\sigma(\theta) = -\varepsilon_0 \frac{\partial U}{\partial r}$$
$$\sigma(\theta) = -\varepsilon_0 \frac{\partial\left(-A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)}{\partial r}$$
$$\sigma(\theta) = -\varepsilon_0 \left[-lA_l r^{l-1} - (l+1) B_l r^{-(l+2)}\right] P_l(\cos\theta)$$
$$\sigma(\theta) = \varepsilon_0 (2l+1) B_l r^{-(l+2)} P_l(\cos\theta)$$

2.8.11 一个横截面为圆形 (半径为 R) 的长金属管, 管面沿圆周分为相等的四份, 其中三份接地, 余下的一份维持常数电势 U_0 求出相对于 U_0 面的面上单位长度的电荷。

$$U(\phi) = \left(C_m s^{-m} + D_m s^m\right)\left(E_m \cos m\phi + F_m \sin m\phi\right)$$
$$U(\phi) = \sum_{m=0}^{\infty} D_m s^m \cos m\phi$$
$$U(\phi) = \sum_{m=0}^{\infty} D_m R^m \cos m\phi$$
$$\int_0^{2\pi} U(\phi)\cos m\phi \,\mathrm{d}\phi = \int_0^{2\pi} D_m R^m \cos^2 m\phi \,\mathrm{d}\phi$$
$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} U_0 \cos m\phi \,\mathrm{d}\phi = \pi D_m R^m$$
$$\frac{2U_0}{m}\sin\frac{m}{4} = \pi D_m R^m$$
$$\frac{2U_0}{m\pi R^m}\sin\frac{m}{4} = D_m$$
$$U(\phi) = \frac{U_0}{4} + \sum_{m=1}^{\infty} \frac{2U_0 s^m}{m\pi R^m}\sin\frac{m}{4}\cos m\phi$$

$$\sigma(\phi) = -\varepsilon_0 \frac{\partial U}{\partial r}$$
$$\sigma(\phi) = -\varepsilon_0 \frac{\partial \sum\limits_{m=0}^{\infty} \frac{2U_0}{m\pi R^m}\sin\frac{m}{4} r^m \cos m\phi}{\partial r}$$
$$\sigma(\phi) = \varepsilon_0 \sum_{m=0}^{\infty} \frac{2U_0}{\pi R^m}\sin\frac{m}{4} r^{m-1}\cos m\phi$$
$$\sigma(\phi) = \varepsilon_0 \sum_{m=0}^{\infty} \frac{2U_0}{\pi R}\sin\frac{m}{4}\cos m\phi$$
$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} R\sigma(\phi)\,\mathrm{d}\phi = \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} \varepsilon_0 \sum_{m=0}^{\infty} R\frac{2U_0}{\pi R}\sin\frac{m}{4}\cos m\phi \,\mathrm{d}\phi$$
$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} R\sigma(\phi)\,\mathrm{d}\phi = \varepsilon_0 \sum_{m=0}^{\infty} \frac{4U_0}{\pi m\pi}\sin\frac{m}{4}\sin\frac{5m}{4}$$

2.9 多极展开

设电荷分布 $\rho(\boldsymbol{r}')$ 局域在有限区域内, 取坐标原点在该区域附近。空间中任意点 $\boldsymbol{r}$ 处的静电势为

$$U(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}r' \tag{2.27}$$

当观测点满足

$$r \gg r'$$

时, 可展开电荷:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma) \tag{2.28}$$

代入电势表达式, 得

$$U(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int r'^l P_l(\cos\gamma)\rho(\boldsymbol{r}')\,\mathrm{d}r' \tag{2.29}$$

该展开称为**多极展开**。其中每一阶 l 对应一个多极矩项, 并按 $r^{-(l+1)}$ 的幂次递减。

2.9.1 单极项

注意 $P_0(\cos\gamma)=1$, 于是

$$U_0(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(\boldsymbol{r}')\ \mathrm{d}^3r' = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}, \tag{2.30}$$

其中

$$Q = \int \rho(\boldsymbol{r}')\ \mathrm{d}^3r'$$

为体系的总电荷。

2.9.2 偶极项

利用

$$P_1(\cos\gamma) = \cos\gamma = \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r\,r'},$$

可得

$$U_1(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{r^3}\cdot\int \boldsymbol{r}'\rho(\boldsymbol{r}')\ \mathrm{d}\tau \tag{2.31}$$

定义电偶极矩

$$\boldsymbol{d} \equiv \int \boldsymbol{r}'\rho(\boldsymbol{r}')\ \mathrm{d}\tau$$

于是

$$U_1(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{d}\cdot\boldsymbol{r}}{r^3} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}. \tag{2.32}$$

相应电场为

$$E_r = -\frac{\partial U}{\partial r} = \frac{2d\cos\theta}{4\pi\varepsilon_0 r^3}, \tag{2.33}$$

$$E_\theta = -\frac{1}{r}\frac{\partial U}{\partial\theta} = \frac{d\sin\theta}{4\pi\varepsilon_0 r^3}, \tag{2.34}$$

$$E_\phi = 0, \tag{2.35}$$

即

$$\boldsymbol{E} = \frac{d}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\boldsymbol{e}_r + \sin\theta\,\boldsymbol{e}_\theta\right) \tag{2.36}$$

2.9.3 四极项

由

$$P_2(\cos\gamma) = \frac{1}{2}\left(3\cos^2\gamma - 1\right) = \frac{1}{2}\left(\frac{3(\boldsymbol{r}\cdot\boldsymbol{r}')^2}{r^2r'^2} - 1\right),$$

可得四极项电势

$$U_2(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2r^5}\int \left[3(\boldsymbol{r}\cdot\boldsymbol{r}')^2 - r^2r'^2\right]\rho(\boldsymbol{r}')\,\mathrm{d}\tau \tag{2.37}$$

将其改写为张量形式, 引入四极矩张量

$$Q_{ij} \equiv \int \left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\boldsymbol{r}')\,\mathrm{d}\tau \tag{2.38}$$

其中 $Q_{ij} = Q_{ji}$ 且满足无迹条件 $Q_{ii} = 0$。

于是四极项电势可写为

$$U_2(\boldsymbol{r}) = \frac{1}{8\pi\varepsilon_0}\frac{1}{r^5}Q_{ij}x_ix_j. \tag{2.39}$$

2.9.4 四极矩矩阵的推导

在远场条件 $r \gg r'$ 下, 多极展开中 $l = 2$ 项为

$$U_2(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int r'^2P_2(\cos\gamma)\,\rho(\boldsymbol{r}')\,\mathrm{d}\tau \tag{2.40}$$

其中

$$\cos\gamma = \frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r\,r'}.$$

利用

$$P_2(\cos\gamma) = \frac{1}{2}\left(3\cos^2\gamma - 1\right),$$

得

$$r'^2P_2(\cos\gamma) = \frac{1}{2}\left(\frac{3(\boldsymbol{r}\cdot\boldsymbol{r}')^2}{r^2} - r'^2\right).$$

因此

$$U_2(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2r^5}\int \left[3(\boldsymbol{r}\cdot\boldsymbol{r}')^2 - r^2r'^2\right]\rho(\boldsymbol{r}')\,\mathrm{d}\tau \tag{2.41}$$

将标量平方写成指标形式,

$$(\boldsymbol{r}\cdot\boldsymbol{r}')^2 = (x_ix_i')(x_jx_j') = x_ix_j\,x_i'x_j',$$

于是

$$3(\boldsymbol{r}\cdot\boldsymbol{r}')^2 - r^2r'^2 = x_ix_j\left(3x_i'x_j' - r'^2\delta_{ij}\right).$$

代回电势表达式, 可得

$$U_2(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2r^5}x_ix_j\int \left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\boldsymbol{r}')\,\mathrm{d}\tau \tag{2.42}$$

由此自然定义四极矩张量

$$Q_{ij} \equiv \int \left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\boldsymbol{r}')\,\mathrm{d}\tau \tag{2.43}$$

由定义可见

$$Q_{ij} = Q_{ji}, \qquad Q_{ii} = 0,$$

即 Q_{ij} 为对称无迹二阶张量。

最终, 四极项电势可写为矩阵形式

$$U_2(\boldsymbol{r}) = \frac{1}{8\pi\varepsilon_0}\frac{1}{r^5}Q_{ij}x_ix_j. \tag{2.44}$$

若记

$$\mathbf{Q} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

则

$$U_2(\boldsymbol{r}) = \frac{1}{8\pi\varepsilon_0}\frac{1}{r^5}\boldsymbol{r}^\top\mathbf{Q}\boldsymbol{r}, \qquad \mathrm{Tr}\,\mathbf{Q} = 0. \tag{2.45}$$

2.9.5 四个粒子 (一个带电为 q, 一个为 $3q$, 另两个为 $-2q$), 每一个距原点距离为 a。求出远离原点处电势的简单近似表达式。

$$U = \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{3a^l}{r^{l+1}}P_l(\cos\theta) + \frac{(-a)^l}{r^{l+1}}P_l(\cos\theta) - \frac{2a^l}{r^{l+1}}P_l(\sin\theta) - \frac{2(-a)^l}{r^{l+1}}P_l(\sin\theta)$$

$l = 0$:

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{3}{r^1} + \frac{1}{r^1} - \frac{2}{r^1} - \frac{2}{r^1}\right] = 0$$

$l = 1$:

$$\begin{aligned} &= \frac{q}{4\pi\varepsilon_0}\left[\frac{3a}{r^2}P_1(\cos\theta) + \frac{(-a)}{r^2}P_1(\cos\theta) - \frac{2a}{r^2}P_1(\sin\theta) - \frac{2(-a)}{r^2}P_1(\sin\theta)\right] \\ &= \frac{q}{4\pi\varepsilon_0}\frac{2a}{r^2}\cos\theta \end{aligned}$$

2.9.6 确定电偶极子电势中的四极矩项

$$U = \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta) - \frac{(-a)^l}{r^{l+1}}P_l(\cos\theta)$$

$l = 3$:

$$\begin{aligned} &= \frac{q}{4\pi\varepsilon_0}\left[\frac{a^3}{r^4}P_3(\cos\theta) - \frac{(-a)^3}{r^4}P_3(\cos\theta)\right] \\ &= \frac{q}{4\pi\varepsilon_0}\left[\frac{a^3}{r^4}\frac{3}{2}\sqrt{1-\cos^2\theta}(5\cos^2\theta - 1) - \frac{(-a)^3}{r^4}\frac{3}{2}\sqrt{1-\cos^2\theta}(5\cos^2\theta - 1)\right] \\ &= \frac{3q}{4\pi\varepsilon_0}\frac{a^3}{r^4}\sqrt{1-\cos^2\theta}(5\cos^2\theta - 1) \end{aligned}$$

2.9.7 一个“纯粹”的偶极子 d 位于原点, 指向 z 方向。
(a) 作用在位于 $(a,0,0)$(直角坐标系) 的点电荷 q 上的力是什么?
(b) 作用在位于 $(0,0,a)$(直角坐标系) 的点电荷 q 上的力是什么?
(c) 把点电荷 q 从 $(a,0,0)$ 移动到 $(0,0,a)$ 所需的功为多少?

(a)

$$\boldsymbol{F}=q\boldsymbol{E}=\frac{d}{4\pi\varepsilon_0 r^2}(2\cos\theta\boldsymbol{e}_r+\sin\theta\boldsymbol{e}_\theta)=\frac{d}{4\pi\varepsilon_0 a^2}\boldsymbol{e}_\theta$$

(b)

$$\boldsymbol{F}=q\boldsymbol{E}=\frac{d}{4\pi\varepsilon_0 r^2}(2\cos\theta\boldsymbol{e}_r+\sin\theta\boldsymbol{e}_\theta)=\frac{d}{2\pi\varepsilon_0 a^2}\boldsymbol{e}_r$$

(c)

$$W=q\Delta U=q\frac{d}{4\pi\varepsilon_0 a^2}$$

2.9.8 两个电量为 $-q$ 的点电荷位于 $(0,a,0),(0,-a,0)$, 一个电量为 q 的点电荷位于 $(0,0,a)$ 求出远离原点处的近似电场, 计算到多极展开中的最低两项。

$$\begin{aligned}Q&=q\\ d&=-aq\boldsymbol{e}_2+aq\boldsymbol{e}_2+aq\boldsymbol{e}_3=aq\boldsymbol{e}_3\end{aligned}$$

2.9.9 证明一个式**2.36**可以写为不依赖于坐标系的形式 $\boldsymbol{E}=\dfrac{3(\boldsymbol{d}\cdot\boldsymbol{e}_r)\boldsymbol{e}_r-\boldsymbol{d}}{4\pi\varepsilon_0 r^3}$

$$\begin{aligned}\frac{3(\boldsymbol{d}\cdot\boldsymbol{e}_r)\boldsymbol{e}_r-\boldsymbol{d}}{4\pi\varepsilon_0 r^3}&=\frac{3(d\boldsymbol{e}_3\cdot\boldsymbol{e}_r)\boldsymbol{e}_r-d\boldsymbol{e}_3}{4\pi\varepsilon_0 r^3}\\&=\frac{3(dr\cos\theta)\boldsymbol{e}_r-d\cos\theta\boldsymbol{e}_r+d\sin\theta\boldsymbol{e}_\theta}{4\pi\varepsilon_0 r^3}\\&=\frac{2d\cos\theta\boldsymbol{e}_r+d\sin\theta\boldsymbol{e}_\theta}{4\pi\varepsilon_0 r^3}\\&=\frac{2d\cos\theta\boldsymbol{e}_r+d\sin\theta\boldsymbol{e}_\theta}{4\pi\varepsilon_0 r^3}\end{aligned}$$

2.10 Green 互易定理

设在同一空间区域 V 内, 有两组电荷分布

$$\rho_1(\boldsymbol{r}),\qquad \rho_2(\boldsymbol{r}),$$

它们分别产生电势

$$U_1(\boldsymbol{r}),\qquad U_2(\boldsymbol{r}),$$

且二者满足相同的边界条件。

则有

$$\boxed{\int_V \rho_1(\boldsymbol{r})\, U_2(\boldsymbol{r})\ \mathrm{d}\tau = \int_V \rho_2(\boldsymbol{r})\, U_1(\boldsymbol{r})\ \mathrm{d}\tau}$$

这就是 Green 互易定理。

$$\nabla^2 U_1 = -\frac{\rho_1}{\varepsilon_0}, \qquad \nabla^2 U_2 = -\frac{\rho_2}{\varepsilon_0}.$$

将第一式乘以 U_2, 第二式乘以 U_1, 得到

$$U_2\nabla^2 U_1 = -\frac{U_2\rho_1}{\varepsilon_0}, \qquad U_1\nabla^2 U_2 = -\frac{U_1\rho_2}{\varepsilon_0}.$$

两式相减:

$$U_2\nabla^2 U_1 - U_1\nabla^2 U_2 = -\frac{1}{\varepsilon_0}\big(U_2\rho_1 - U_1\rho_2\big).$$

注意到如下恒等式 (可直接验证):

$$\nabla\cdot\big(U_2\nabla U_1 - U_1\nabla U_2\big) = U_2\nabla^2 U_1 - U_1\nabla^2 U_2.$$

因此有

$$\nabla\cdot\big(U_2\nabla U_1 - U_1\nabla U_2\big) = -\frac{1}{\varepsilon_0}\big(U_2\rho_1 - U_1\rho_2\big).$$

对体积 V 积分:

$$\int_V \nabla\cdot\big(U_2\nabla U_1 - U_1\nabla U_2\big)\,\tau = -\frac{1}{\varepsilon_0}\int_V \big(U_2\rho_1 - U_1\rho_2\big)\,\tau.$$

左侧应用高斯定理:

$$\oint_{\partial V} \big(U_2\nabla U_1 - U_1\nabla U_2\big)\cdot \mathrm{d}\boldsymbol{S} = -\frac{1}{\varepsilon_0}\int_V \big(U_2\rho_1 - U_1\rho_2\big)\,\tau.$$

若满足以下任一条件:

- Dirichlet 边界条件: $U_1 = U_2 = 0$ 于 ∂V
- 或 Neumann 边界条件: $\dfrac{\partial U_1}{\partial n} = \dfrac{\partial U_2}{\partial n} = 0$

则表面积分为零。

于是得到

$$\int_V U_2\rho_1\,\tau = \int_V U_1\rho_2\,\tau,$$

即 Green 互易定理。

若 ρ_1 或 ρ_2 包含点电荷, 例如

$$\rho_1(\boldsymbol{r}) = q\,\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}_0),$$

则体积分应理解为分布作用:

$$\int \rho_1 U_2\,\tau = q\,U_2(\boldsymbol{r}_0).$$

在此意义下, 互易定理仍然成立。

考虑两块无限大、相互平行的理想导体平板, 分别位于

$$z = 0, \qquad z = d,$$

且二者均接地：

$$U = 0 \quad 于 z = 0, d.$$

在两板之间放置一个点电荷 q, 其位置为

$$\boldsymbol{r}_0 = (0, 0, x), \qquad 0 < x < d.$$

求：两块导体平板上分别感应出的总电荷。

总诱导电荷：

$$Q_1 = \int_{z=0} \sigma_1 \ \mathrm{d}S, \qquad Q_2 = \int_{z=d} \sigma_2 \ \mathrm{d}S.$$

我们构造两种静电系统：

系统 A 两平板接地, 在 $\boldsymbol{r}_0$ 处放置点电荷 q

系统 B 去掉点电荷, 仅将 其中一块平板置于单位电势, 另一块保持接地

随后对系统 A 与系统 B 使用 Green 互易定理。

情形 1：下板 $z = 0$ 置于单位电势 由于几何与边界条件均为平移对称, 系统 B 的电势仅与 z 有关, 满足拉普拉斯方程：

$$\frac{\mathrm{d}^2 U}{\mathrm{d}z^2} = 0, \quad U(0) = 1, \quad U(d) = 0.$$

解：

$$U_B^{(1)}(z) = 1 - \frac{z}{d}$$

情形 2：上板 $z = d$ 置于单位电势 同理：

$$U_B^{(2)}(z) = \frac{z}{d}$$

Green 互易定理给出：

$$\int \rho_A U_B \ \mathrm{d}\tau = \int \rho_B U_A \ \mathrm{d}\tau$$

计算左边 系统 A 的电荷分布仅包含点电荷：

$$\rho_A(\boldsymbol{r}) = q\, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_0)$$

因此：

$$\int \rho_A U_B \ \mathrm{d}\tau = q\, U_B(\boldsymbol{r}_0)$$

计算右边 系统 B 中没有体电荷, 只有导体表面的诱导电荷。由于平板接地, 导体表面电势恒为零, 因此：

$$\int \rho_B U_A \ \mathrm{d}\tau = \sum_i \int_{\text{plate } i} \sigma_i(\boldsymbol{r}_\parallel)\, U_A(\boldsymbol{r}_\parallel) \ \mathrm{d}S$$

但在系统 A 中：

$$U_A = 0 \quad \text{于导体表面}$$

唯一剩下的贡献来自于系统 B 中被置于单位电势的那块板, 其总电荷记为 Q。
于是：

$$\int \rho_B U_A \,\mathrm{d}\tau = Q$$

下板的诱导电荷:

$$q\,U_B^{(1)}(x) = Q_1$$

代入：

$$Q_1 = q\left(1 - \frac{x}{d}\right)$$

上板的诱导电荷：

$$q\,U_B^{(2)}(x) = Q_2$$

代入：

$$Q_2 = q\,\frac{x}{d}$$

2.11 习题

2.11.1 一个均匀带电、边长为 $2a$ 的正方形面, 电荷面密度为 σ。求出距中心高度为 z 处的电场。

$$\begin{aligned}
&\int_0^a \frac{\sigma}{\pi\varepsilon_0\sqrt{2L^2+z^2}}\boldsymbol{e}_3\frac{2Lz}{L^2+z^2}\,\mathrm{d}L \\
\xlongequal{2L^2=z^2\tan^2\theta}&\int_0^{\arctan\frac{a}{\sqrt{2}z}} \frac{\sigma}{\pi\varepsilon_0\sqrt{z^2\tan^2\theta+z^2}}\boldsymbol{e}_3\frac{2z\tan\theta z}{z^2\tan^2\theta+2z^2}\,\mathrm{d}z\tan\theta \\
=&\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{\sigma\cos\theta}{\pi\varepsilon_0 z\cos^2\theta}\boldsymbol{e}_3\frac{2z\tan\theta z}{z^2\tan^2\theta+2z^2}\,\mathrm{d}z\theta \\
=&\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{\sigma}{\pi\varepsilon_0\cos\theta}\boldsymbol{e}_3\frac{2\tan\theta}{\tan^2\theta+2}\,\mathrm{d}\theta \\
=&\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\frac{\tan\theta}{\tan^2\theta\cos\theta+2\cos\theta}\,\mathrm{d}\theta \\
=&\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\frac{\sin\theta}{\sin^2\theta+2\cos^2\theta}\,\mathrm{d}\theta \\
=&-\int_0^{\arctan\frac{\sqrt{2}a}{z}} \frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\frac{1}{1+\cos^2\theta}\,\mathrm{d}\cos\theta \\
=&-\frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\arctan\cos\theta\Big|_0^{\arctan\frac{\sqrt{2}a}{z}} \\
=&\frac{2\sigma}{\pi\varepsilon_0}\boldsymbol{e}_3\left(\frac{\pi}{4}-\arctan\frac{z}{\sqrt{2a^2+z^2}}\right)
\end{aligned}$$

2.11.2 已知电场 $\boldsymbol{E} = \dfrac{A\boldsymbol{e}_r + B\sin\theta\cos\phi\boldsymbol{e}_\phi}{r}$, 求电荷密度

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E} = &= \frac{1}{sr}\frac{\partial srE_r}{\partial r} + \frac{1}{s}\frac{\partial E_\phi}{\partial\phi} \\
&= \frac{1}{sr}\frac{\partial sA}{\partial r} + \frac{1}{sr}\frac{\partial B\sin\theta\cos\phi}{\partial\phi} \\
&= \frac{A}{r^2} - \frac{1}{r^2}B\sin\phi
\end{aligned}$$

2.11.3 一个均匀带电球体, 求出南半球与北半球之间的净相互作用力

$$\begin{aligned}
E_z &= E_r\cos\theta = \frac{\rho\cos\theta}{4\pi\varepsilon_0 r^2}\frac{4\pi r^3}{3} = \frac{\rho\cos\theta r}{3\varepsilon_0} \\
F &= \int_0^R \mathrm{d}r \int_0^{2\pi}\mathrm{d}\phi\int_0^{\frac{\pi}{2}} r^2\sin\theta\rho E_z\,\mathrm{d}\theta \\
&= \int_0^R \mathrm{d}r \int_0^{2\pi}\mathrm{d}\phi\int_0^{\frac{\pi}{2}} r^3\sin\theta\rho\frac{\rho\cos\theta}{3\varepsilon_0}\,\mathrm{d}\theta \\
&= \pi\int_0^R r^3\,\mathrm{d}r\int_0^{\frac{\pi}{2}}\sin\theta\rho^2\frac{2\cos\theta}{3\varepsilon_0}\,\mathrm{d}\theta \\
&= \pi R^4\int_0^{\frac{\pi}{2}}\sin\theta\rho^2\frac{\cos\theta}{6\varepsilon_0}\,\mathrm{d}\theta \\
&= \pi R^4\int_0^{\frac{\pi}{2}}\rho^2\frac{\sin 2\theta}{24\varepsilon_0}\,\mathrm{d}2\theta \\
&= -\rho^2R^4\pi\frac{\cos 2\theta}{24\varepsilon_0}\bigg|_0^{\frac{\pi}{2}} \\
&= \frac{\rho^2R^4\pi}{12\varepsilon_0}
\end{aligned}$$

2.11.4 一个半径为 R 的倒置半球面均匀带电, 电荷面密度为 σ。求出北极与球心处的电势差

$$
\begin{aligned}
U_{\text{半球北极}} &= \mathrm{d}\phi \int_0^{\frac{\pi}{2}} \frac{\sigma R \sin\theta}{2\varepsilon_0} \frac{1}{\sqrt{R^2\left(1-\cos\theta\right)^2 + R^2\sin^2\theta}} R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\sigma \sin^2\theta}{2\varepsilon_0} \frac{1}{\sqrt{\left(1-\cos\theta\right)^2 + \sin^2\theta}} R^2\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\sigma \sin\theta}{2\varepsilon_0} \frac{1}{\sqrt{2-2\cos\theta}} R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\sigma 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sqrt{2}\varepsilon_0} \frac{1}{\sqrt{1-\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}} R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\sigma \sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sqrt{2}\varepsilon_0} \frac{1}{\sqrt{2\sin^2\frac{\theta}{2}}} R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\sigma \cos\frac{\theta}{2}}{2\varepsilon_0} R\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{\sigma}{\varepsilon_0} R\,\mathrm{d}\sin\frac{\theta}{2} \\
&= \left.\frac{\sigma\sin\frac{\theta}{2}}{\varepsilon_0} R\right|_0^{\frac{\pi}{2}} \\
&= \frac{\sqrt{2}\sigma}{2\varepsilon_0} R \\
U_{\text{半球球心}} &= \frac{2\pi R^2\sigma}{4\pi\varepsilon_0 R} = \frac{\sigma R}{2\varepsilon_0} \\
\Delta U &= \frac{\sigma}{2\varepsilon_0}\frac{R^2}{\sqrt{R^2}} + \frac{\sigma R}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0}\frac{R^2}{R+\sqrt{R^2+R^2}} - \frac{\sqrt{2}\sigma}{2\varepsilon_0} R \\
&= \frac{\sigma R}{2\varepsilon_0} - \frac{\sqrt{2}\sigma}{2\varepsilon_0} R \\
&= \frac{\sigma R}{2\varepsilon_0} - \frac{\sqrt{2}\sigma}{2\varepsilon_0} R
\end{aligned}
$$

2.11.5 一个半径为 R 的球体, 电荷密度 $\rho = kr$, 求能量

$$
\begin{aligned}
|\boldsymbol{E}| &= \begin{cases} \dfrac{1}{4\pi\varepsilon_0 x_3^2}\displaystyle\int_0^R 4\pi k r^3\,\mathrm{d}r\,(x_3 \geq R) \\ \dfrac{1}{4\pi\varepsilon_0 x_3^2}\displaystyle\int_0^{x_3} 4\pi k r^3\,\mathrm{d}r\,(x_3 < R) \end{cases} \\
&= \begin{cases} \dfrac{1}{4\varepsilon_0 x_3^2} k R^4\,(x_3 \geq R) \\ \dfrac{1}{4\varepsilon_0 x_3^2} k x_3^4\,(x_3 < R) \end{cases} \\
&= \begin{cases} \dfrac{kR^4}{4\varepsilon_0 x_3^2}\,(x_3 \geq R) \\ \dfrac{k x_3^2}{4\varepsilon_0}\,(x_3 < R) \end{cases} \\
U &= \frac{\varepsilon_0}{2}4\pi\int_0^R \left(\frac{k x_3^2}{4\varepsilon_0}\right)^2 x_3^2\,\mathrm{d}x_3 + \frac{\varepsilon_0}{2}4\pi\int_R^\infty \left(\frac{kR^4}{4\varepsilon_0 x_3^2}\right)^2 x_3^2\,\mathrm{d}x_3 \\
&= 2\pi\int_0^R \frac{k^2 x_3^6}{16\varepsilon_0}\,\mathrm{d}x_3 + 2\pi\int_R^\infty \frac{k^2 R^8}{16\varepsilon_0 x_3^2}\,\mathrm{d}x_3 \\
&= \pi\frac{k^2 R^7}{56\varepsilon_0} + \pi\frac{k^2 R^8}{8\varepsilon_0 R} \\
&= \pi\frac{k^2 R^7}{7\varepsilon_0}
\end{aligned}
$$

2.11.6 电势为 $U = \frac{A\mathrm{e}^{-\lambda r}}{r}$, 求电场, 电荷密度, 总电荷

$$
\begin{aligned}
\boldsymbol{E} &= -\boldsymbol{\nabla} U \\
&= \frac{\partial \dfrac{A\mathrm{e}^{-\lambda r}}{r}}{\partial r}\boldsymbol{e}_r \\
&= \frac{\lambda r A\mathrm{e}^{-\lambda r} + A\mathrm{e}^{-\lambda r}}{r^2}\boldsymbol{e}_r \\
\rho &= \varepsilon_0 \boldsymbol{\nabla}^2 U \\
&= \varepsilon_0 \frac{1}{r^2}\frac{\partial r^2 a_r}{\partial r} \\
&= \varepsilon_0 \frac{1}{r^2}\frac{\partial \lambda r A\mathrm{e}^{-\lambda r} + A\mathrm{e}^{-\lambda r}}{\partial r} \\
&= \varepsilon_0 \frac{\lambda A\mathrm{e}^{-\lambda r} - \lambda^2 r A\mathrm{e}^{-\lambda r} - \lambda A\mathrm{e}^{-\lambda r}}{r^2} \\
&= -\frac{\varepsilon_0 \lambda^2 A\mathrm{e}^{-\lambda r}}{r}\,(r \neq 0) \\
\rho &= \frac{\varepsilon_0 A 4\pi \delta^{(3)}(r)}{r} - \frac{\varepsilon_0 \lambda^2 A\mathrm{e}^{-\lambda r}}{r} \\
Q &= 4\pi\int_0^\infty \rho r^2\,\mathrm{d}r \\
&= 0
\end{aligned}
$$

2.11.7 两条平行于 z 轴的无限长均匀带电线, 电荷线密度分别为 $+\lambda$ 和 $-\lambda$, 距离为 $2d$。
(a) 求出任意一点的电势。
(b) 证明等势面为圆柱面, 对给定的电势 U, 给出圆柱面的半径和轴的位置。

(a):

$$\begin{aligned}
U_{+\lambda} &= -\frac{\lambda}{2\pi\varepsilon_0}\ln|\boldsymbol{r}-d\boldsymbol{e}_1| \\
U_{-\lambda} &= \frac{\lambda}{2\pi\varepsilon_0}\ln|\boldsymbol{r}+d\boldsymbol{e}_1| \\
U &= -\frac{\lambda}{2\pi\varepsilon_0}\ln|\boldsymbol{r}-d\boldsymbol{e}_1| + \frac{\lambda}{2\pi\varepsilon_0}\ln|\boldsymbol{r}+d\boldsymbol{e}_1| \\
&= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(x_1+d)^2+x_2^2}{(x_1-d)^2+x_2^2}\right]
\end{aligned}$$

(b):

$$\begin{aligned}
(x_1+d)^2+x_2^2 &= k\left[(x_1-d)^2+x_2^2\right] \\
x_1^2+2x_1d+d^2+x_2^2 &= kx_1^2-2kx_1d+kd^2+kx_2^2 \\
0 &= (k-1)x_1^2-2(k+1)x_1d+(k-1)d^2+(k-1)x_2^2 \\
0 &\xlongequal{u=\frac{k+1}{k-1}} x_1^2-2ux_1d+d^2+x_2^2 \\
u^2-d^2 &= (x_1-u)^2+x_2^2 \\
R &= \sqrt{u^2-d^2} \\
&= \sqrt{\left(\frac{k+1}{k-1}\right)^2-d^2} \\
&= \sqrt{\left(\frac{\mathrm{e}^{\frac{4\pi\varepsilon_0 U}{\lambda}}+1}{\mathrm{e}^{\frac{4\pi\varepsilon_0 U}{\lambda}}-1}\right)^2-d^2}
\end{aligned}$$

2.11.8 在一个真空二极管中, 电子从阴极面“热蒸发”后向阳极面加速运动, 阴极电势为零, 对面阳极的电势为 U_0。在两极间隙中所形成的电子云 (称为空间电荷) 很快会达到一种分布状态, 使得阴极面上的电场为零。然后在两极板之间形成稳定的电流 **I**。假定两个极板面积 A 远大于它们之间的距离 $d(A >> d)$, 所以边界效应可以忽略。则 U, ρ, v(电子速度) 都仅是 x 的函数。

(a) 写出在两极板之间空间的泊松方程。

(b) 假定电子从阴极是从静止开始运动的, 那么在点 x, 这里电势为 $U(x)$, 电子速度为多少?

(c) 在稳定状态下, 电流 I 不依赖于 x。那么 ρ 和 v 之间的关系是什么?

(d) 利用上面的结果, 消去 ρ 和 v, 得出 U 满足的微分方程。

(e) 作为 x, U_0, d 的函数, 求出 U 的解。并与没有空间电荷的情况比较。另外作为 x 的函数, 求出 ρ 和 v。

(f) 证明 $1 = KU_0^{\frac{3}{2}}$ 求出常数 K。

(a):

$$\frac{\partial^2 U}{\partial x^2} = \frac{\rho}{\varepsilon_0}$$

(b):

$$\begin{aligned} eU &= \frac{1}{2}mv^2 \\ v &= \sqrt{\frac{2eU}{m}} \end{aligned}$$

(c):

$$\begin{aligned} I &= \frac{\Delta q}{\Delta t} \\ &= \rho A v \end{aligned}$$

(d):

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} &= \frac{\rho}{\varepsilon_0} \\ \frac{\partial^2 U}{\partial x^2} &= \frac{I}{Av\varepsilon_0} \\ \frac{\partial^2 U}{\partial x^2} &= \frac{I}{A\sqrt{\frac{2eU}{m}}\varepsilon_0} \end{aligned}$$

(e):

$$
\begin{aligned}
2\frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2} &= 2\frac{\partial U}{\partial x}\frac{I}{A\sqrt{\frac{2eU}{m}}\varepsilon_0} \\
\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)^2 &= 4\frac{\partial U^{\frac{1}{2}}}{\partial x}\frac{I}{A\sqrt{\frac{2e}{m}}\varepsilon_0} \\
\left(\frac{\partial U}{\partial x}\right)^2 &= 4\frac{IU^{\frac{1}{2}}}{A\sqrt{\frac{2e}{m}}\varepsilon_0} \\
\frac{\partial U}{\partial x} &= 2\frac{I^{\frac{1}{2}}U^{\frac{1}{4}}}{A^{\frac{1}{2}}\left(\frac{2e}{m}\right)^{\frac{1}{4}}\varepsilon_0^{\frac{1}{2}}} \\
U^{-\frac{1}{4}}\partial U &= 2\frac{I^{\frac{1}{2}}m^{\frac{1}{4}}\partial x}{A^{\frac{1}{2}}(2e)^{\frac{1}{4}}\varepsilon_0^{\frac{1}{2}}} \\
\frac{4}{3}U^{\frac{3}{4}} &= 2\frac{I^{\frac{1}{2}}m^{\frac{1}{4}}x}{A^{\frac{1}{2}}(2e)^{\frac{1}{4}}\varepsilon_0^{\frac{1}{2}}} \\
U &= \frac{3^{\frac{4}{3}}I^{\frac{2}{3}}x^{\frac{4}{3}}m^{\frac{1}{4}}}{2A^{\frac{1}{2}}\mathrm{e}^{\frac{1}{3}}\varepsilon_0^{\frac{2}{3}}} \\
v &= \sqrt{\frac{2eU}{m}} \\
&= \sqrt{\frac{2e\frac{3^{\frac{4}{3}}I^{\frac{2}{3}}x^{\frac{4}{3}}m^{\frac{1}{4}}}{2A^{\frac{1}{2}}\mathrm{e}^{\frac{1}{3}}\varepsilon_0^{\frac{2}{3}}}}{m}} \\
&= \sqrt{\frac{3^{\frac{4}{3}}I^{\frac{2}{3}}x^{\frac{4}{3}}\mathrm{e}^{\frac{2}{3}}}{A^{\frac{1}{2}}\varepsilon_0^{\frac{2}{3}}m^{\frac{3}{4}}}} \\
&= \frac{3^{\frac{2}{3}}I^{\frac{1}{3}}x^{\frac{2}{3}}\mathrm{e}^{\frac{1}{3}}}{A^{\frac{1}{4}}\varepsilon_0^{\frac{1}{3}}m^{\frac{3}{8}}} \\
\rho &= \varepsilon_0\frac{\partial^2 U}{\partial x^2} \\
&= \frac{2I^{\frac{2}{3}}m^{\frac{1}{4}}\varepsilon_0^{\frac{1}{3}}}{3^{\frac{2}{3}}A^{\frac{1}{2}}\mathrm{e}^{\frac{1}{3}}x^{\frac{2}{3}}}
\end{aligned}
$$

(f):

$$
\begin{aligned}
U &= \frac{3^{\frac{4}{3}}I^{\frac{2}{3}}x^{\frac{4}{3}}m^{\frac{1}{4}}}{2A^{\frac{1}{2}}\mathrm{e}^{\frac{1}{3}}\varepsilon_0^{\frac{2}{3}}} \\
I^{\frac{2}{3}} &= \frac{U2A^{\frac{1}{2}}\mathrm{e}^{\frac{1}{3}}\varepsilon_0^{\frac{2}{3}}}{3^{\frac{4}{3}}x^{\frac{4}{3}}m^{\frac{1}{4}}} \\
I &= \frac{U^{\frac{3}{2}}2^{\frac{3}{2}}A^{\frac{3}{4}}\mathrm{e}^{\frac{1}{2}}\varepsilon_0^{\frac{2}{3}}}{3^2x^2m^{\frac{3}{2}}} \\
I &= \frac{U_0^{\frac{3}{2}}2^{\frac{3}{2}}A^{\frac{3}{4}}\mathrm{e}^{\frac{1}{2}}\varepsilon_0^{\frac{2}{3}}}{3^2d^2m^{\frac{3}{2}}}
\end{aligned}
$$

2.11.9 假设现在极精确的测量已经揭示出库仑定律的误差。两个点电荷之间的作用力为 $\boldsymbol{F}=\frac{q_1q_2}{4\pi\varepsilon_0r^2}\left(1+\frac{r}{\lambda}\right)\mathrm{e}^{-\frac{r}{\lambda}}\boldsymbol{e}_r$ 式中，λ是一个新的自然常数。你的任务是按照这个新发现重新表述静电学。假定叠加原理仍然成立。

(a) 电场是什么？

(b) 求出一个点电荷的电势。

(c) 对一个位于原点的点电荷，证明 $\oint\!\!\!\oint_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S}+\frac{1}{\lambda^2}\iiint_V U\,\mathrm{d}\tau=\frac{q}{\varepsilon_0}$

(a):

$$\boldsymbol{E}=\frac{q}{4\pi\varepsilon_0r^2}\left(1+\frac{r}{\lambda}\right)\mathrm{e}^{-\frac{r}{\lambda}}\boldsymbol{e}_r$$

(b):

$$\begin{aligned}
U&=\int_R^{\infty}\frac{q}{4\pi\varepsilon_0r^2}\left(1+\frac{r}{\lambda}\right)\mathrm{e}^{-\frac{r}{\lambda}}\,\mathrm{d}r\\
&=\frac{q}{4\pi\varepsilon_0\lambda}\int_R^{\infty}\left(\frac{\lambda^2}{r^2}+\frac{\lambda}{r}\right)\mathrm{e}^{-\frac{r}{\lambda}}\,\mathrm{d}\frac{r}{\lambda}\\
&=-\frac{q}{4\pi\varepsilon_0\lambda}\int_R^{\infty}\left(\frac{\lambda^2}{r^2}--\frac{\lambda}{r}\right)\mathrm{e}^{-\frac{r}{\lambda}}\,\mathrm{d}-\frac{r}{\lambda}\\
&\xlongequal{u=-\frac{r}{\lambda}}-\frac{q}{4\pi\varepsilon_0\lambda}\int_{-\frac{R}{\lambda}}^{-\infty}\left(\frac{1}{u^2}-\frac{1}{u}\right)\mathrm{e}^u\,\mathrm{d}u\\
&=-\left.\frac{1}{u}\mathrm{e}^u\right|_{-\frac{R}{\lambda}}^{-\infty}\frac{q}{4\pi\varepsilon_0\lambda}\\
&=\mathrm{e}^{-\frac{R}{\lambda}}\frac{q}{4\pi\varepsilon_0R}
\end{aligned}$$

(c):

$$\begin{aligned}
\oint\!\!\!\oint_S\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S}&=4\pi R^2\frac{q}{4\pi\varepsilon_0R^2}\left(1+\frac{R}{\lambda}\right)\mathrm{e}^{-\frac{r}{\lambda}}=\frac{q}{\varepsilon_0}\left(1+\frac{R}{\lambda}\right)\mathrm{e}^{-\frac{r}{\lambda}}\\
\frac{1}{\lambda^2}\iiint_V U\,\mathrm{d}\tau&=4\pi\frac{1}{\lambda^2}\int_0^R\mathrm{e}^{-\frac{r}{\lambda}}\frac{q}{4\pi\varepsilon_0r}r^2\,\mathrm{d}r\\
&=\int_0^R-\frac{r}{\lambda}\mathrm{e}^{-\frac{r}{\lambda}}\frac{q}{\varepsilon_0}\,\mathrm{d}-\frac{r}{\lambda}\\
&\xlongequal{u=-\frac{r}{\lambda}}\int_0^{-\frac{R}{\lambda}}u\mathrm{e}^u\frac{q}{\varepsilon_0}\,\mathrm{d}u\\
&=\left.(u-1)\,\mathrm{e}^u\frac{q}{\varepsilon_0}\right|_0^{-\frac{R}{\lambda}}\\
&=\left(-\frac{R}{\lambda}-1\right)\mathrm{e}^{-\frac{R}{\lambda}}\frac{q}{\varepsilon_0}+\frac{q}{\varepsilon_0}\\
\frac{q}{\varepsilon_0}&=\oint\!\!\!\oint_S\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S}+\frac{1}{\lambda^2}\iiint_V U\,\mathrm{d}\tau
\end{aligned}$$

2.11.10 假设一个电场 $E_1=ax,\ E_2=E_3=0$ ，电荷密度为什么？

$$\varepsilon_0\frac{\partial ax}{\partial x}=a$$

2.11.11 **所有的静电学特性都是从库仑定律的 $\frac{1}{r^2}$ 以及叠加原理导出的。因此也可以对牛顿万有引力构建类似的理论。什么是一个半径为 R, 质量为 M 的球体的引力能? 假设质量密度是均匀的。利用所得结果估计太阳的引力能**

根据库仑定律与万有引力定律的数学相似性, 我们可以通过替换常数直接得到均匀球体的引力自能。

在静电学中, 均匀带电球体的静电能为:

$$W_e = \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R} \tag{2.46}$$

利用类比关系得到质量为 M 的均匀球体的引力能:

$$W_g = -\frac{3}{5}\frac{GM^2}{R} \tag{2.47}$$

假设球体密度为 $\rho = \frac{M}{\frac{4}{3}\pi R^3}$。考虑已形成半径为 r 的球核, 其质量为 $m(r) = \frac{4}{3}\pi\rho r^3$。现从无穷远处移近一层厚度为 dr 的薄壳, 其质量为 $dm = 4\pi\rho r^2 dr$。此过程引力所做的功为:

$$\begin{aligned}
dW &= -\frac{Gm(r)dm}{r} && (2.48)\\
&= -\frac{G}{r}\left(\frac{4}{3}\pi\rho r^3\right)\left(4\pi\rho r^2 dr\right) && (2.49)\\
&= -\frac{16}{3}\pi^2 G\rho^2 r^4 dr && (2.50)
\end{aligned}$$

对整个球体从 0 到 R 积分:

$$\begin{aligned}
W_g &= \int_0^R -\frac{16}{3}\pi^2 G\rho^2 r^4 dr && (2.51)\\
&= -\frac{16}{3}\pi^2 G\rho^2\left[\frac{1}{5}r^5\right]_0^R && (2.52)\\
&= -\frac{16}{15}\pi^2 G\rho^2 R^5 && (2.53)
\end{aligned}$$

将 $\rho^2 = \frac{M^2}{\frac{16}{9}\pi^2 R^6}$ 代入上式, 化简得:

$$W_g = -\frac{3}{5}\frac{GM^2}{R} \tag{2.54}$$

取太阳参数: $M \approx 2.0e30kg$, $R \approx 7.0e8m$, 引力常数 $G \approx 6.67e-11m^3.kg^{-1}.s^{-2}$。代入公式得:

$$\begin{aligned}
W &= -\frac{3}{5}\frac{6.67\times10^{-11}\times(2.0\times10^{30})^2}{7.0\times10^8}\\
&\approx -2.28e41J
\end{aligned}$$

2.11.12 我们知道导体上的电荷是分布于其表面的，但是在表面上是如何分布的不是很容易确定的。电荷面密度可以直接计算的著名例子是椭圆面：$\frac{x_1^2}{r_1^2}+\frac{x_2^2}{r_2^2}+\frac{x_3^2}{r_3^2}=1$ 对这种情况 $\sigma = \frac{Q}{4\pi r_1 r_2 r_3}\left(\frac{x_1^2}{r_1^4}+\frac{x_2^2}{r_2^4}+\frac{x_3^2}{r_3^4}\right)^{-\frac{1}{2}}$

(a) 一个半径为 R 的圆盘的净电荷面密度 σ;

(b) 位于 x, y 平面一条无限长的导体“丝带”的电荷面密度 σ，丝带沿 y 轴放置，宽度 $2a$;

(c) 求出一个从 $x_3=-a$ 到 $x_3=a$ 的导体“针”每单位长度的电荷。

(a): 令 $r_1=r_2$

$$
\begin{aligned}
\sigma &= \frac{Q}{4\pi r_1^2 r_3}\left(\frac{x_1^2+x_2^2}{r_1^4}+\frac{x_3^2}{r_3^4}\right)^{-\frac{1}{2}} \\
&= \frac{Q}{4\pi r_1^2 r_3}\left(\frac{x_1^2+x_2^2}{r_1^4}+\frac{r_1^2-x_1^2-x_2^2}{r_3^2 r_1^2}\right)^{-\frac{1}{2}} \\
&= \frac{Q}{4\pi r_1^2}\left(\frac{x_1^2+x_2^2}{r_1^4}r_3^2+\frac{r_1^2-x_1^2-x_2^2}{r_1^2}\right)^{-\frac{1}{2}} \\
&\xlongequal{r_3\to 0} \frac{Q}{4\pi r_1^2}\left(\frac{r_1^2-x_1^2-x_2^2}{r_1^2}\right)^{-\frac{1}{2}}
\end{aligned}
$$

(b): 令 $r_2=\infty$

$$
\begin{aligned}
\sigma &= \frac{\lambda}{4\pi r_1 r_3}\left(\frac{x_1^2}{r_1^4}+\frac{x_3^2}{r_3^4}\right)^{-\frac{1}{2}} \\
&= \frac{\lambda}{2\pi\sqrt{r_1^2-x_1^2}}
\end{aligned}
$$

(c):

$$
\begin{aligned}
\sigma &= \frac{Q}{4\pi r_1^2}\left(\frac{x_1^2+x_2^2}{r_1^4}r_3^2+\frac{r_1^2-x_1^2-x_2^2}{r_1^2}\right)^{-\frac{1}{2}} \\
&= \frac{Q}{4\pi r_1^2}\left(\frac{r_1^2 r_3^2+r_1^2 x_3^2}{r_1^4}+\frac{x_3^2}{r_1^2 r_3^2}\right)^{-\frac{1}{2}} \\
&\xlongequal{r_1\to 0} \frac{Q}{2\pi\sqrt{a^2-x_3^2}}
\end{aligned}
$$

2.11.13 一个质量为 m, 带电为 q 的粒子, 从与一个无限大接地导体平板距离为 d 处由静止开始释放. 求这个粒子碰击到板的时间为多少?

$$
\begin{aligned}
\frac{\partial^2 x}{\partial t^2} &= \frac{q^2}{4\pi\varepsilon_0(2x)^2 m} \\
v\frac{dv}{dx} &= \frac{q^2}{16\pi\varepsilon_0 x^2 m} \\
\frac{v^2}{2} &= -\frac{q^2}{16\pi\varepsilon_0 x m} + C_1 \\
v &= \sqrt{-\frac{q^2}{8\pi\varepsilon_0 x m} + C_1} \\
v &= \sqrt{\frac{q^2}{8\pi\varepsilon_0 d m} - \frac{q^2}{8\pi\varepsilon_0 x m}} \\
t &= \int_0^d \sqrt{\frac{q^2}{8\pi\varepsilon_0 d m} - \frac{q^2}{8\pi\varepsilon_0 x m}}\, \mathrm{d}x \\
&= \frac{\pi\sqrt{2\pi\varepsilon_0 m d^3}}{q}
\end{aligned}
$$

2.11.14 两个无限大接地导体平板平行放置。一个点电荷 q 位于原点, 其中一个位于 $z=-a$, 另一个位于 $z=b$。求出作用在 q 上的力。

由电像法可得下方电像位于 $-2n(a+b), -2a-2n(a+b)$ 上方电像位于 $-2n(a+b), -2b-2n(a+b), n\in\mathbb{N}$

$$
\begin{aligned}
F &= \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{[2n(a+b)]^2} + \frac{1}{[2a+2n(a+b)]^2} - \frac{1}{[2n(a+b)]^2} - \frac{1}{[2b+2n(a+b)]^2} \\
&= \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{[2a+2n(a+b)]^2} - \frac{1}{[2b+2n(a+b)]^2}
\end{aligned}
$$

2.11.15 两条长直线, 载有均匀分布异号电荷, 电荷线密度为 $\pm\lambda$。两直线分别放置在一个长导体柱的两边。导体柱半径为 R, 带电线距导体柱轴线的距离为 a。求出 r 点处的电势

$$
U(r,\phi) = \sum_{n=1}^{\infty} A_n \left(\frac{R}{r}\right)^n \sin n\phi
$$

导体柱为等势体, 取其表面电势为零:

$$
U(R,\phi) = 0
$$

同时, 在 $r=a$ 处存在两条线电荷, 其电势在无导体情况下可展开为傅里叶级数:

$$
U_0(r,\phi) = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r^2+a^2-2ar\cos\phi}{r^2+a^2+2ar\cos\phi}
$$

在 $r > R$ 区域展开为

$$U_0(r,\phi) = -\frac{2\lambda}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a}{r}\right)^n \sin n\phi$$

为满足导体边界条件, 修正解应为

$$U(r,\phi) = -\frac{2\lambda}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{a}{r}\right)^n - \left(\frac{R^2}{ar}\right)^n\right]\sin n\phi$$

柱外区域 $r > R$ 的电势为

$$U(r,\phi) = -\frac{2\lambda}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{a}{r}\right)^n - \left(\frac{R^2}{ar}\right)^n\right]\sin n\phi$$

2.11.16 一个半径为 a 的导体球, 电势为 U_0。一个半径为 $b(b > a)$ 的薄球壳包围着导体球, 在球壳上分布有电荷面密度 $\sigma(\theta) = k\cos\theta$。
(a) 求出 $r > b$, $a < r < b$ 区域内的电势。
(b) 求出导体球表面上的诱导电荷面密度 σ。
(c) 这个体系的总电荷为多少?

(a):

$a < r < b$:

$$U(r,\theta) = \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

$r > b$:

$$U(r,\theta) = \left(A_l' r^l + B_l' r^{-(l+1)}\right) P_l(\cos\theta)$$
$$U(r,\theta) = B_l' r^{-(l+1)} P_l(\cos\theta)$$

由边界条件可得:

$$U_0 = \left(A_l a^l + B_l a^{-(l+1)}\right) P_l(\cos\theta)$$
$$\left(A_l b^l + B_l b^{-(l+1)}\right) P_l(\cos\theta) = B_l' b^{-(l+1)} P_l(\cos\theta)$$
$$A_l b^l + B_l b^{-(l+1)} = B_l' b^{-(l+1)}$$
$$A_l b^{2l+1} + B_l = B_l'$$

$$\sigma = -\varepsilon_0\frac{\partial U}{\partial r}$$
$$\sigma = -\varepsilon_0\frac{\partial\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta)}{\partial r} + \varepsilon_0\frac{\partial B_l' r^{-(l+1)}P_l(\cos\theta)}{\partial r}$$
$$\sigma = -\varepsilon_0\left[A_l l b^{l-1} - (l+1)B_l b^{-(l+2)}\right]P_l(\cos\theta) - (l+1)\varepsilon_0 B_l' b^{-(l+2)} P_l(\cos\theta)$$
$$\sigma = -\varepsilon_0\left[A_l l b^{l-1} - (l+1)B_l b^{-(l+2)}\right]P_l(\cos\theta) - (l+1)\varepsilon_0(A_l b^l + B_l b^{-(l+1)})b^{-1}P_l(\cos\theta)$$
$$\sigma = -\varepsilon_0\left[A_l l b^{l-1} - (l+1)A_l b^{l-1}\right]P_l(\cos\theta)$$
$$\sigma = \varepsilon_0 A_l b^{l-1} P_l(\cos\theta)$$
$$\frac{k}{\varepsilon_0} = A_1$$

$$
\begin{aligned}
U_0 &= \left(A_l a^l + B_l a^{-(l+1)}\right) P_l(\cos\theta) \\
U_0 &= B_0 a^{-(0+1)} \\
aU_0 &= B_0 \\
B_1 &= -\frac{ka^3}{\varepsilon_0} \\
B_0' &= A_0 b^1 + B_0 \\
B_0' &= aU_0 \\
B_1' &= A_1 b^3 + B_1 \\
B_1' &= \frac{k}{\varepsilon_0} b^3 - \frac{k}{\varepsilon_0} a^3
\end{aligned}
$$

$a < r < b$:

$$
\begin{aligned}
U &= B_0 r^{-1} P_0(\cos\theta) + \left(A_1 r^1 + B_1 r^{-2}\right) P_1(\cos\theta) \\
U &= \frac{aU_0}{r} + \left(\frac{kr}{\varepsilon_0} - \frac{ka^3}{\varepsilon_0 r^2}\right)\cos\theta
\end{aligned}
$$

$r > b$:

$$
U = aU_0 r^{-1} + \left(\frac{k}{\varepsilon_0} b^3 - \frac{k}{\varepsilon_0} a^3\right) r^{-2}\cos\theta
$$

2.11.17 一个半径为 R 的无限长柱壳上半壳载有均匀分布的电荷, 电荷面密度为 σ_0, 下半壳载有均匀分布的电荷面密度为 $-\sigma_0$ 的电荷求出柱壳内外的电势。

柱壳内:

$$
\begin{aligned}
U(r,\phi) &= \left(C_m s^{-m} + D_m s^m\right)\left(E_m \cos m\phi + F_m \sin m\phi\right) \\
U(r,\phi) &= \sum_{m=0}^{\infty} D_m s^m \cos m\phi
\end{aligned}
$$

柱壳外:

$$
U(r,\phi) = \sum_{m=0}^{\infty} D_m' s^{-m} \cos m\phi
$$

由连续性可得:

$$
\begin{aligned}
D_m R^m \cos m\phi &= D_m' R^{-m} \cos m\phi \\
D_m R^{2n} &= D_m'
\end{aligned}
$$

$$\sigma(\phi)=\sum_{m=0}^{\infty}-\varepsilon_0\frac{\partial U}{\partial s}$$
$$\sigma(\phi)=\sum_{m=0}^{\infty}-\varepsilon_0\frac{\partial D_m's^{-m}\cos m\phi-D_ns^m\cos m\phi}{\partial s}$$
$$\sigma(\phi)=\sum_{m=0}^{\infty}-\varepsilon_0(-mD_m's^{-m-1}\cos m\phi-mD_ms^{m-1}\cos m\phi)$$
$$\sigma(\phi)=\sum_{m=0}^{\infty}-\varepsilon_0(-mD_mR^{2m}R^{-m-1}\cos m\phi-mD_mR^{m-1}\cos m\phi)$$
$$\sigma(\phi)=\sum_{m=0}^{\infty}2\varepsilon_0mD_mR^{m-1}\cos m\phi$$
$$\int_0^{\pi}\sigma(\phi)2\varepsilon_0mD_mR^{m-1}\cos m\phi\,\mathrm{d}\phi=\int_0^{\pi}2^2\varepsilon_0^2m^2D_m^2R^{2m-2}\cos^2 m\phi\,\mathrm{d}\phi$$
$$2\int_0^{\frac{\pi}{2}}\sigma(\phi)\cos m\phi\,\mathrm{d}\phi=\int_0^{\pi}2\varepsilon_0mD_mR^{m-1}\cos^2 m\phi\,\mathrm{d}\phi$$
$$2\int_0^{\frac{\pi}{2}}\frac{\sigma_0}{m}\,\mathrm{d}\sin m\phi=\int_0^{\pi}2\varepsilon_0mD_mR^{m-1}\cos^2 m\phi\,\mathrm{d}\phi$$

m 为奇数时:

$$2\int_0^{\frac{\pi}{2}}\frac{\sigma_0}{m}\,\mathrm{d}\sin m\phi=\int_0^{\pi}2\varepsilon_0mD_mR^{m-1}\cos^2 m\phi\,\mathrm{d}\phi$$
$$2\frac{\sigma_0}{m}=\pi\varepsilon_0mD_mR^{m-1}$$
$$D_m=\frac{2\sigma_0}{\pi\varepsilon_0R^{m-1}m^2}$$

2.11.18　一个细绝缘杆, 端点位于 $z=a$ 和 $z=-a$, 细杆载有均匀电荷, 电荷线密度为 λ。对下列情况求出多极展开电势中的首项：(a)$\lambda=k\cos(\frac{\pi z}{2a})$, (b)$\lambda=k\sin(\frac{\pi z}{2a})$, (c)$\lambda=k\cos(\frac{\pi z}{a})$

$$U=\int_{-a}^{a}\frac{\lambda}{4\pi\varepsilon_0\sqrt{r^2+a^2-2ra\cos\theta}}\,\mathrm{d}z'$$

不想写

2.11.19　一个理想的电偶极子位于原点, 方向沿 z 轴方向。一个电荷从 xy 平面上的一点由静止开始释放。证明它在一个半圆弧上往复运动, 就像挂在原点的一个单摆一样。

由于体系绕 z 轴轴对称, 且初始条件取在某一固定平面内, 可令

$$\dot{\phi}=0,$$

从而运动限制在 (r,θ) 平面中。

系统的拉格朗日量为

$$L=\frac{m}{2}\left(\dot{r}^2+r^2\dot{\theta}^2\right)-\frac{A\cos\theta}{r^2}. \tag{2.55}$$

考虑如下无穷小变换:

$$\delta r=\varepsilon\,r\cos\theta,\qquad\delta\theta=-\varepsilon\,\sin\theta. \tag{2.56}$$

在该变换下, 势能的变分为

$$\delta V = \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial \theta}\delta\theta = -\frac{2A\cos\theta}{r^3}(r\cos\theta) + \frac{A\sin\theta}{r^2}(\sin\theta) = 0.$$

动能的变分可以直接计算, 结果为一个全时间导数：

$$\delta T = \varepsilon m\frac{\mathrm{d}}{\mathrm{d}t}\big(r\dot r\cos\theta - r^2\dot\theta\sin\theta\big).$$

因此

$$\delta L = \frac{\mathrm{d}F}{\mathrm{d}t},$$

该变换是拉格朗日量的一个连续对称性。

由 Noether 定理, 对应的守恒量为

$$Q = \frac{\partial L}{\partial \dot r}\delta r + \frac{\partial L}{\partial \dot\theta}\delta\theta = m\dot r(r\cos\theta) - mr^2\dot\theta\sin\theta.$$

注意到

$$\frac{\mathrm{d}}{\mathrm{d}t}(r\sin^2\theta) = \sin^2\theta\,\dot r + 2r\sin\theta\cos\theta\,\dot\theta,$$

可以验证

$$Q = -\frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(r\sin^2\theta\big).$$

由于 Q 为常数, 得到几何不变量

$$r\sin^2\theta = \text{const}. \tag{2.57}$$

在包含 z 轴的运动平面内, 引入

$$\rho = r\sin\theta, \qquad z = r\cos\theta.$$

守恒关系变为

$$\frac{\rho^2}{r} = C, \qquad C = \text{const}.$$

结合

$$r = \sqrt{\rho^2+z^2},$$

可化为

$$(\rho^2 - \tfrac{C}{2})^2 + z^2 = (\tfrac{C}{2})^2,$$

这是一个圆的方程, 其圆心位于 $(\rho, z) = (C/2, 0)$, 半径为 $C/2$, 且圆经过原点。

因此, 试探电荷的轨迹是一条以原点为端点的半圆弧。

3 物质中的电场

3.1 极化

3.1.1 诱导偶极子

诱导偶极矩:

$$\boldsymbol{d} = \alpha \boldsymbol{E}$$

α 称为原子极化率

考虑一个简单的原子模型：原子核位于原点, 带电量 $+q$；电子云为半径为 a 的均匀带电球体, 总电荷为 $-q$。

在无外场时, 体系整体电中性, 电偶极矩为零。

现施加一弱、均匀的外电场 E_0 假设外场足够弱, 使电子云仅发生微小整体位移而保持形状不变。

$$\boldsymbol{E}_0 = \frac{\boldsymbol{d}}{4\pi\varepsilon_0 a^3}$$

$$\boldsymbol{d} = q\boldsymbol{s} = 4\pi\varepsilon_0 a^3 \boldsymbol{E}_0$$

$$\alpha = 4\pi\varepsilon_0 a^3 \tag{3.1}$$

3.1.2 偶极子在电场

偶极子在电场中的力矩:

$$\boldsymbol{N} = \boldsymbol{d} \times \boldsymbol{E} \tag{3.2}$$

偶极子在电场中的能量:

$$\begin{aligned} U &= \int \rho U_{ext} \,\mathrm{d}\tau \\ &= -\int \boldsymbol{d} \cdot \boldsymbol{\nabla} \delta(\boldsymbol{r}) U_{ext} \,\mathrm{d}\tau \\ &= \int \boldsymbol{d} \cdot \delta(\boldsymbol{r}) \boldsymbol{\nabla} U_{ext} \,\mathrm{d}\tau \\ &= -\boldsymbol{d} \cdot \boldsymbol{E} \end{aligned}$$

偶极子在电场中的力:

$$\begin{aligned} \boldsymbol{F} &= -\boldsymbol{\nabla} U \\ &= \boldsymbol{\nabla}(\boldsymbol{d} \cdot \boldsymbol{E}) \\ &= (\boldsymbol{d} \cdot \boldsymbol{\nabla})\boldsymbol{E} + (\boldsymbol{E} \cdot \boldsymbol{\nabla})\boldsymbol{d} \\ &= (\boldsymbol{d} \cdot \boldsymbol{\nabla})\boldsymbol{E} \end{aligned}$$

3.1.3 一个氢原子 (其玻尔半径为 a) 位于两个相距 l 的金属板中间, 估计在这个装置中电离原子所需的电压为多少。

$$
\begin{aligned}
qs &= 4\pi\varepsilon_0 a^3 E_0 \\
U &= \frac{4\pi\varepsilon_0 a^3 l}{qs}
\end{aligned}
$$

3.1.4 基态氢原子电子云其电荷密度为 $\rho = \dfrac{e}{\pi a^3}\mathrm{e}^{-\frac{2r}{a}}$, 求极化率

$$
\begin{aligned}
E &= \frac{1}{4\pi\varepsilon_0 s^2}\int_0^s 4\pi r^2 \frac{e}{\pi a^3}\mathrm{e}^{-\frac{2r}{a}}\,\mathrm{d}r \\
&= \frac{1}{4\pi\varepsilon_0 s^2}\int_0^s -\frac{4r^2}{a^2}\frac{e}{2}\mathrm{e}^{-\frac{2r}{a}}\,\mathrm{d} - \frac{2r}{a} \\
&= -\frac{1}{4\pi\varepsilon_0 s^2}\int_0^{-\frac{2s}{a}} \frac{e}{2}u^2\mathrm{e}^u\,\mathrm{d}u \\
&= -\frac{1}{4\pi\varepsilon_0 s^2}\ \frac{e}{2}(u^2-2u+2)\mathrm{e}^u\Big|_0^{-\frac{2s}{a}} \\
&= -\frac{1}{4\pi\varepsilon_0 s^2}\frac{e}{2}(\frac{4s^2}{a^2}+\frac{4s}{a}+2)\mathrm{e}^{-\frac{2s}{a}} + \frac{1}{4\pi\varepsilon_0 s^2}e \\
&\xlongequal{s<<a} \frac{1}{4\pi\varepsilon_0 s^2}\frac{e}{2}(\frac{4s^2}{a^2}+\frac{4s}{a}+2)\frac{2s}{a} + \frac{e}{4\pi\varepsilon_0 s^2} \\
&= \frac{ea}{2\pi\varepsilon_0 s} \\
\alpha &= \frac{es2\pi\varepsilon_0 s}{ea} = \frac{2\pi\varepsilon_0 s^2}{a}
\end{aligned}
$$

3.1.5 证明相距为 $\boldsymbol{r}$ 的两个偶极子的相互作用能为 $U = \dfrac{\boldsymbol{d}_1\cdot\boldsymbol{d}_2 - 3(\boldsymbol{d}_1\cdot\boldsymbol{e}_r)(\boldsymbol{d}_2\cdot\boldsymbol{e}_r)}{4\pi\varepsilon_0 r^3}$

$$
\begin{aligned}
U &= -\boldsymbol{d}_1\cdot\boldsymbol{E}_2 \\
&= -\boldsymbol{d}_1\cdot\frac{3(\boldsymbol{d}_2\cdot\boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{d}_2}{4\pi\varepsilon_0 r^3} \\
&= \frac{\boldsymbol{d}_1\cdot\boldsymbol{d}_2 - 3(\boldsymbol{d}_1\cdot\boldsymbol{e}_r)(\boldsymbol{d}_2\cdot\boldsymbol{e}_r)}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

3.1.6 一个偶极子 $\boldsymbol{d}$ 与一个点电荷 q 相距为, $\boldsymbol{d}$ 与 $\boldsymbol{r}$ 的夹角为 θ。
(a) 作用在 $\boldsymbol{d}$ 上的力为多少?
(b) 作用在 q 上的力为多少?

$$
\begin{aligned}
\boldsymbol{F} &= (\boldsymbol{d}\cdot\boldsymbol{\nabla})\boldsymbol{E} \\
&= (d_i\frac{\partial}{\partial x_i})E_j\boldsymbol{e}_j \\
&= (d_i\frac{\partial}{\partial x_i})\frac{qx_j}{4\pi\varepsilon_0 r^3}\boldsymbol{e}_j \\
&= d_i\frac{q}{4\pi\varepsilon_0 r^3}\boldsymbol{e}_j\frac{\partial x_j}{\partial x_i} + (d_i x_j\frac{\partial}{\partial x_i})\frac{q}{4\pi\varepsilon_0 r^3}\boldsymbol{e}_j \\
&= d_i\frac{q}{4\pi\varepsilon_0 r^3}\boldsymbol{e}_j\delta_{ij} - d_i x_j\frac{qx_i}{\pi\varepsilon_0 r^4}\boldsymbol{e}_j \\
&= d_i\frac{q}{4\pi\varepsilon_0 r^3}\boldsymbol{e}_i - \boldsymbol{d}\cdot\boldsymbol{x}x_j\frac{q}{\pi\varepsilon_0 r^4}\boldsymbol{e}_j \\
&= \boldsymbol{d}\frac{q}{4\pi\varepsilon_0 r^3} - \boldsymbol{d}\cdot\boldsymbol{x}\boldsymbol{x}\frac{q}{\pi\varepsilon_0 r^4}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F} &= q\boldsymbol{E} \\
&= q\frac{3(\boldsymbol{d}\cdot\boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{d}}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

3.2 极化物体的电场

$$
U = \frac{1}{4\pi\varepsilon_0}\left(\iiint_V \frac{\rho_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\tau' + \oiint_S \frac{\sigma_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}S'\right) \tag{3.3}
$$

表面电荷:

$$
\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} \tag{3.4}
$$

内部电荷:

$$
\rho_b = -\boldsymbol{\nabla}\cdot\boldsymbol{P} \tag{3.5}
$$

3.2.1 一个半径为 R 的球的极化强度矢量为 $\boldsymbol{P} = k\boldsymbol{r}$
(a) 计算束缚电荷 σ_b 和 ρ_b。
(b) 求出球内部和外部的电场。

(a):

$$
\begin{aligned}
\sigma_b &= \boldsymbol{P}\cdot\hat{\boldsymbol{n}} \\
&= kR \\
\rho_b &= -\boldsymbol{\nabla}\cdot\boldsymbol{P} \\
&= -3k
\end{aligned}
$$

(b):

$$\begin{aligned}\boldsymbol{E} &= \frac{\int_0^r 4\pi k r'^3 \,\mathrm{d}r'}{4\pi r^2}\boldsymbol{e}_r \\ &= \frac{kr^2}{4}\boldsymbol{e}_r\end{aligned}$$

球内体束缚电荷与表面束缚电荷的总量为

$$Q = \int_V \rho_b \,\mathrm{d}V + \oint_S \sigma_b \,\mathrm{d}S = (-3k)\frac{4\pi R^3}{3} + kR\cdot 4\pi R^2 = 0.$$

3.2.2 一个圆柱体, 半径为 a, 长度为 L, 具有均匀极化强度 $\boldsymbol{P}$, 方向垂直圆柱面轴线

(a) 求电场强度

(b) 证明 $\boldsymbol{E} = \dfrac{a^2}{2\varepsilon_0 s^2}[2(\boldsymbol{P}\cdot\boldsymbol{e}_s)\boldsymbol{e}_s - \boldsymbol{P}]$

(a): 边界条件

$$\sigma_b = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = P\cos\phi$$

球内

$$\begin{aligned}U &= \sum_{m=0}^{\infty} D_m s^m \cos m\phi + D'_m s^{-m}\cos m\phi \\ -\varepsilon_0\frac{\partial U}{\partial s} &= \sum_{m=0}^{\infty} \varepsilon_0 m D_m s^{m-1}\cos m\phi - \varepsilon_0 m D'_m s^{-m-1}\cos m\phi \\ \sigma_b &= \sum_{m=0}^{\infty} \varepsilon_0 m D_m s^{m-1}\cos m\phi - \varepsilon_0 m D'_m s^{-m-1}\cos m\phi \\ P\cos\phi &= \sum_{m=0}^{\infty} \varepsilon_0 m D_m s^{m-1}\cos m\phi - \varepsilon_0 m D'_m s^{-m-1}\cos m\phi \\ P &= \varepsilon_0 D_1 - \varepsilon_0 D'_1 s^{-2}\end{aligned}$$

球外

$$D'_1 = -\frac{a^2 P}{2\varepsilon_0}$$

(b):

$$\begin{aligned}U &= -\frac{a^2 P}{2\varepsilon_0}s^{-1}\cos\phi \\ \boldsymbol{E} &= -\boldsymbol{\nabla} U \\ \boldsymbol{E} &= \frac{\partial U}{\partial s}\boldsymbol{s} + \frac{1}{s}\frac{\partial U}{\partial \phi}\boldsymbol{\phi} \\ \boldsymbol{E} &= -\frac{\partial \frac{a^2 P}{2\varepsilon_0}s^{-1}\cos\phi}{\partial s}\boldsymbol{s} - \frac{1}{s}\frac{\partial \frac{a^2 P}{2\varepsilon_0}s^{-1}\cos\phi}{\partial\phi}\boldsymbol{\phi} \\ \boldsymbol{E} &= \frac{a^2 P}{2\varepsilon_0}s^{-2}\cos\phi\boldsymbol{s} + \frac{1}{s}\frac{a^2 P}{2\varepsilon_0}s^{-1}\sin\phi\boldsymbol{\phi}\end{aligned}$$

$$\begin{aligned}\frac{a^2}{2\varepsilon_0 s^2}[2(\boldsymbol{P}\cdot\boldsymbol{e}_s)\boldsymbol{e}_s - \boldsymbol{P}] &= \frac{a^2}{2\varepsilon_0 s^2}(2P_s\boldsymbol{e}_s - P_s\boldsymbol{e}_s - P_\phi\boldsymbol{e}_\phi) \\ &= \frac{a^2}{2\varepsilon_0 s^2}(P_s\boldsymbol{e}_s - P_\phi\boldsymbol{e}_\phi)\end{aligned}$$

3.3 电位移矢量

$$\boldsymbol{D} \equiv \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \tag{3.6}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho \tag{3.7}$$

$$\boldsymbol{\nabla} \times \boldsymbol{D} = \boldsymbol{P} \tag{3.8}$$

3.3.1 边界条件

$$D^{\perp}_{上} - D^{\perp}_{下} = \sigma_f \tag{3.9}$$

$$D^{\parallel}_{上} - D^{\parallel}_{下} = P^{\parallel}_{上} - P^{\parallel}_{下} \tag{3.10}$$

$$E^{\perp}_{上} - E^{\perp}_{下} = \frac{\sigma}{\varepsilon_0} \tag{3.11}$$

$$E^{\parallel}_{上} - E^{\parallel}_{下} = 0 \tag{3.12}$$

3.4 线性电介质

3.4.1 电极化率, 介电常数, 相对介电常数和能量

$$\chi_e \equiv \frac{P}{\varepsilon_0 E} \tag{3.13}$$

$$\varepsilon_r \equiv 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0} \Rightarrow \boldsymbol{D} = \varepsilon \boldsymbol{E} \tag{3.14}$$

$$W = \frac{\varepsilon_0}{2} \iiint \boldsymbol{D} \cdot \boldsymbol{E} \, \mathrm{d}\tau \tag{3.15}$$

3.4.2 一个线性均匀的球形电介质材料放置于一个均匀的外电场 E_0 中, 求球内部的电场强度

边界条件:

$$\begin{aligned}
&U_{内} = U_{外}, && r = R \\
&\varepsilon \frac{\partial U_{内}}{\partial r} = \varepsilon_0 \frac{\partial U_{外}}{\partial r}, && r = R \\
&U_{外} \to -E_0 r \cos\theta, && r \to \infty
\end{aligned}$$

$$\begin{aligned}
U_{内} &= A_l r^l P_l(\cos\theta) \\
U_{外} &= B_l r^{-(l+1)} P_l(\cos\theta) - E_0 r \cos\theta \\
A_l R^l P_l(\cos\theta) &= B_l R^{-(l+1)} P_l(\cos\theta) - E_0 R \cos\theta \\
\varepsilon \frac{\partial U_{内}}{\partial r} &= \varepsilon_0 \frac{\partial U_{外}}{\partial r} \\
\varepsilon \frac{\partial A_l r^l P_l(\cos\theta)}{\partial r} &= \varepsilon_0 \frac{\partial B_l r^{-(l+1)} P_l(\cos\theta) - E_0 r \cos\theta}{\partial r} \\
\varepsilon l A_l R^{l-1} P_l(\cos\theta) &= -\varepsilon_0 B_l (l+1) R^{-(l+2)} P_l(\cos\theta) - E_0 \cos\theta
\end{aligned}$$

解得

$$
\begin{aligned}
A_1 &= -\frac{3}{\varepsilon_r+2}E_0 \\
B_1 &= \frac{\varepsilon_r-1}{\varepsilon_r+2}E_0
\end{aligned}
$$

3.4.3 在一个半径为 a 的不带电导体球外面覆盖一层绝缘外壳，相对介电常数为 ε_r，外壳半径为 b，放置在均匀外场 E_0 中，求绝缘外壳内部的电场

边界条件:

$$
\begin{aligned}
&U_{a内} = U_{a外}, && r=a \\
&U_{b内} = U_{b外}, && r=b \\
&\varepsilon\frac{\partial U_{a内}}{\partial r} - \varepsilon_0\frac{\partial U_{a外}}{\partial r} = \sigma_f, && r=a \\
&U_{外} \to -E_0 r\cos\theta, && r\to\infty
\end{aligned}
$$

由静电屏蔽并设球心处电势为零可得

$$
\begin{aligned}
U_{a内} &= 0 \\
U_{a外b内} &= A_{l2}r^l P_l(\cos\theta) + B_{l2}r^{-l-1}P_l(\cos\theta) \\
U_{b外} &= B_{l3}r^{-l-1}P_l(\cos\theta) - E_0 r\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
U_{a内} &= U_{a外} \\
0 &= A_{l2}a^l P_l(\cos\theta) + B_{l2}a^{-l-1}P_l(\cos\theta) \\
A_{l2} + B_{l2}a^{-2l-1} &= 0
\end{aligned}
$$

$$
\begin{aligned}
U_{b内} &= U_{b外} \\
A_{l2}b^l P_l(\cos\theta) + B_{l2}b^{-l-1}P_l(\cos\theta) &= B_{l3}b^{-l-1}P_l(\cos\theta) - E_0 b\cos\theta \\
A_{l2}b^l + B_{l2}b^{-l-1} &= B_{l3}b^{-l-1},\ (l\neq 1)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon\frac{\partial U_{b内}}{\partial r} &= \varepsilon_0\frac{\partial U_{b外}}{\partial r} \\
\varepsilon\frac{\partial A_{l2}r^l P_l(\cos\theta) + B_{l2}r^{-l-1}P_l(\cos\theta)}{\partial r} &= \varepsilon_0\frac{\partial B_{l3}r^{-l-1}P_l(\cos\theta) - E_0 r\cos\theta}{\partial r} \\
\varepsilon l A_{l2}b^{l-1}P_l(\cos\theta) - \varepsilon(l+1)B_{l2}b^{-l-2}P_l(\cos\theta) &= -\varepsilon_0(l+1)B_{l3}b^{-l-2}P_l(\cos\theta) - \varepsilon_0 E_0\cos\theta \\
\varepsilon l A_{l2}b^{l-1} - \varepsilon(l+1)B_{l2}b^{-l-2} &= -\varepsilon_0(l+1)B_{l3}b^{-l-2},\ (l\neq 1)
\end{aligned}
$$

$$
\begin{cases}
A_{l2} + B_{l2}a^{-2l-1} = 0 \\
A_{l2}b^l + B_{l2}b^{-l-1} = B_{l3}b^{-l-1} \\
\varepsilon l A_{l2}b^{l-1} - \varepsilon(l+1)B_{l2}b^{-l-2} = -\varepsilon_0(l+1)B_{l3}b^{-l-2}
\end{cases},\ (l\neq 1)
$$

由于线性无关, 所以 $A_{l2}=B_{l2}=B_{l3}=0$

$$\begin{cases} A_{12}+B_{12}a^{-3}=0 \\ A_{12}b+B_{12}b^{-2}=B_{13}b^{-2}-E_0b \\ \varepsilon A_{12}-\varepsilon 2B_{12}b^{-3}=-\varepsilon_0 2B_{13}b^{-3}-\varepsilon_0 E_0 \end{cases}$$

$$\begin{cases} A_{12}+a^{-3}B_{12}=0 \\ bA_{12}+b^{-2}B_{12}-b^{-2}B_{13}=-E_0b \\ \varepsilon A_{12}-\varepsilon 2b^{-3}B_{12}+\varepsilon_0 2b^{-3}B_{13}=-\varepsilon_0 E_0 \end{cases}$$

$$\begin{cases} -a^{-3}B_{12}b+b^{-2}B_{12}-b^{-2}B_{13}=-E_0b \\ -\varepsilon a^{-3}B_{12}-\varepsilon 2b^{-3}B_{12}+\varepsilon_0 2b^{-3}B_{13}=-\varepsilon_0 E_0 \end{cases}$$

$$\begin{cases} -\varepsilon_0 a^{-3}B_{12}+\varepsilon_0 b^{-3}B_{12}-\varepsilon_0 b^{-3}B_{13}=-\varepsilon_0 E_0 \\ -\varepsilon a^{-3}B_{12}-\varepsilon 2b^{-3}B_{12}+\varepsilon_0 2b^{-3}B_{13}=-\varepsilon_0 E_0 \end{cases}$$

$$\begin{cases} -\varepsilon_0 a^{-3}B_{12}+\varepsilon_0 b^{-3}B_{12}-\varepsilon_0 b^{-3}B_{13}=-\varepsilon_0 E_0 \\ -(\varepsilon+2\varepsilon_0)a^{-3}B_{12}-(\varepsilon-2\varepsilon_0)2b^{-3}B_{12}=-3\varepsilon_0 E_0 \end{cases}$$

3.4.4 一个半径为 a, 带电荷为 Q 的导体球被一个半径为 b 的线性电介质包裹, 求总能量

$$W=\int_a^b 4\pi r\varepsilon_0\left(\frac{Q}{4\pi\varepsilon r^2}\right)^2\mathrm{d}R+\int_b^\infty 4\pi r\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2\mathrm{d}R=\left(\frac{1}{a^4}-\frac{1}{b^4}\right)\frac{\varepsilon_0 Q^2}{16\pi\varepsilon^2}+\frac{1}{b^4}\frac{Q^2}{16\pi\varepsilon_0}$$

3.4.5 内径为 a 外径为 b 的同轴圆柱导体管竖直放置在充满密度为 ρ 的油性电介质的桶中, 内部的金属管电势恒为 U_0, 外管接地, 求两管之间油上升的高度 h

$$W=\iiint\frac{D^2\pi(b^2-a^2)}{2\varepsilon}\,\mathrm{d}z\,\mathrm{d}S+\int_0^h\rho g\pi(b^2-a^2)z\,\mathrm{d}z$$
$$W=\iiint\frac{D^2\pi(b^2-a^2)}{2\varepsilon}\,\mathrm{d}z\,\mathrm{d}S+\frac{1}{2}\rho g\pi(b^2-a^2)h^2$$

$$
\begin{aligned}
\frac{\partial W}{\partial h} &= 0 \\
\frac{\partial \iiint \frac{D^2\pi(b^2-a^2)}{2\varepsilon}\,\mathrm{d}z\,\mathrm{d}S + \frac{1}{2}\rho g\pi(b^2-a^2)h^2}{\partial h} &= 0 \\
\frac{\partial -\iint \frac{D^2\pi(b^2-a^2)\chi_e h}{2\varepsilon}\,\mathrm{d}S + \frac{1}{2}\rho g\pi(b^2-a^2)h^2}{\partial h} &= 0 \\
-\iint \frac{D^2\pi(b^2-a^2)\chi_e}{2\varepsilon}\,\mathrm{d}S + \rho g\pi(b^2-a^2)h &= 0 \\
-\int_a^b r\left(\frac{U_0\varepsilon}{r\ln b - r\ln a}\right)^2 \frac{\chi_e}{2\varepsilon}\,\mathrm{d}r + \rho g h &= 0 \\
\rho g h &= \frac{U_0^2\varepsilon}{\ln b - \ln a}\frac{\chi_e}{2} \\
h &= \frac{U_0^2\varepsilon\chi_e}{2\rho g\ln b - 2\rho g\ln a}
\end{aligned}
$$

3.5 习题

3.5.1 电荷 q 位于一个线性均匀电介质球的中心, 求球内电场强度, 极化强度和体束缚电荷

$$
\begin{aligned}
D &= \frac{q}{4\pi r^2} \\
E &= \frac{D}{\varepsilon} = \frac{q}{4\pi r^2\varepsilon} \\
P &= D - \varepsilon_0 E = \frac{q\chi_e}{4\pi r^2\varepsilon_r} \\
\rho_b &= -\boldsymbol{\nabla}\cdot\boldsymbol{P} = 0
\end{aligned}
$$

3.5.2 在两个不同介电常数电介质表面, 电场线会发生弯折, 如果在边界没有自由电荷, 证明: $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\varepsilon_1}{\varepsilon_2}$

$$
\begin{aligned}
D_1^\perp &= D_2^\perp && (3.16) \\
\varepsilon_1 E_1^\perp &= \varepsilon_2 E_2^\perp && (3.17) \\
E_1^\parallel &= E_2^\parallel && (3.18) \\
\frac{\tan\theta_1}{\tan\theta_2} &= \frac{\frac{E_2^\perp}{E_2^\parallel}}{\frac{E_1^\perp}{E_1^\parallel}} && (3.19) \\
&= \frac{E_1^\parallel E_2^\perp}{E_1^\perp E_2^\parallel} && (3.20) \\
&= \frac{E_2^\perp}{E_1^\perp} && (3.21) \\
&= \frac{\varepsilon_1}{\varepsilon_2} && (3.22)
\end{aligned}
$$

3.5.3 一个点电偶极子 d 镶嵌在一个线性均匀介电球 (电极化率为 χ_e, 半径为 R) 的中心。求在球体内部和外部的电势能

边界条件:

$$
\begin{aligned}
&U_{内} = U_{外}, && r = R \\
&\varepsilon \frac{\partial U_{内}}{\partial r} = \varepsilon_0 \frac{\partial U_{外}}{\partial r}, && r = R \\
&U_{外} \to 0, && r \to \infty \\
&U_{内} \to \frac{d\cos\theta}{4\pi\varepsilon_0 r^2}, && r \to 0
\end{aligned}
$$

$$
\begin{aligned}
U_{内} &= A_l r^l P_l(\cos\theta) + \frac{d\cos\theta}{4\pi\varepsilon_0 r^2} \\
U_{b外} &= B_l r^{-l-1} P_l(\cos\theta) \\
A_l R^l P_l(\cos\theta) + \frac{d\cos\theta}{4\pi\varepsilon_0 R^2} &= B_l R^{-l-1} P_l(\cos\theta) \\
A_l R^l &= B_l R^{-l-1}, \ (l \neq 1) \\
\varepsilon \frac{\partial A_l r^l P_l(\cos\theta) + \dfrac{d\cos\theta}{4\pi\varepsilon_0 r^2}}{\partial r} &= \varepsilon_0 \frac{\partial B_l r^{-l-1} P_l(\cos\theta)}{\partial r} \\
\varepsilon l A_l r^{l-1} P_l(\cos\theta) - \frac{2\varepsilon d\cos\theta}{4\pi\varepsilon_0 r^3} &= -(l+1)\varepsilon_0 B_l r^{-l-2} P_l(\cos\theta) \\
\varepsilon l A_l R^{l-1} &= -(l+1)\varepsilon_0 B_l R^{-l-2}, \ (l \neq 1)
\end{aligned}
$$

由于线性无关, 所以 $A_l = B_l = 0$

$$
\begin{aligned}
A_1 R P_1(\cos\theta) + \frac{d\cos\theta}{4\pi\varepsilon_0 R^2} &= B_1' R^{-2} P_l(\cos\theta) \\
A_1 R + \frac{d}{4\pi\varepsilon_0 R^2} &= B_1 R^{-2} \\
\varepsilon A_1 P_1(\cos\theta) - \frac{2\varepsilon d\cos\theta}{4\pi\varepsilon_0 R^3} &= -2\varepsilon_0 B_1 R^{-3} P_1(\cos\theta) \\
\varepsilon A_1 - \frac{\varepsilon d}{2\pi\varepsilon_0 R^3} &= -2\varepsilon_0 B_1 R^{-3} \\
B_1 &= \frac{3d\varepsilon}{4\pi\varepsilon_0(\varepsilon + 2\varepsilon_0)} \\
A_1 &= \frac{2d(\varepsilon - \varepsilon_0)}{4\pi\varepsilon_0(\varepsilon + 2\varepsilon_0) R^3}
\end{aligned}
$$

4 静磁学

4.1 洛伦兹力定律

4.1.1 磁力

$$\boldsymbol{F} = \boldsymbol{v} \times \boldsymbol{B} \tag{4.1}$$

$$\boldsymbol{F} = I \int \mathrm{d}\boldsymbol{l} \times \boldsymbol{B} \tag{4.2}$$

$$\boldsymbol{F} = \int \boldsymbol{J} \times \boldsymbol{B} \, \mathrm{d}\tau \tag{4.3}$$

4.1.2 电流

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} \tag{4.4}$$

4.1.3 电流 I 沿半径为 a 的导线流动,
(a) 如果电流均匀分布在导线表面上, 那么面电流密度 K 为多少?
(b) 如果体电流密度分布反比于到中心轴的距离, 那么 J 为什么?

(a):$\frac{I}{2\pi a}$
(b):

$$I = 2\pi \int_0^a \frac{k}{r} r \, \mathrm{d}r = 2\pi a k \to k = \frac{I}{2\pi a}$$
$$J = \frac{I}{2\pi a r}$$

4.1.4 (a) 一个留声机唱片表面有均匀的电荷面密度 σ。如果它以角速度 ω 旋转, 那么离中心距离为 r 处的面电流密度 K 为多少?
(b) 电荷 Q 均匀分布在半径为 R 的固体球内, 中心在原点, 并以恒定角速度 ω 绕轴旋转。求出球内任意点 (r, θ, ϕ) 处的电流密度 J。

(a):$K = \sigma v = \sigma \omega r$
(b):$J = \rho v = \frac{4Q}{3\pi R^3} r \sin\phi \omega$

4.2 比奥萨伐尔定律

4.2.1 稳恒电流

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0 \tag{4.5}$$

4.2.2 稳恒电流的磁场

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3} \tag{4.6}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{4.7}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{4.8}$$

4.2.3 求出通有稳恒电流 I 的 n 边规则多边形线框中心的磁场, 同样 R 为中心到边的距离。

$$B = 2n\frac{\mu_0 I}{4\pi R}\sin\frac{2\pi}{2n} = \frac{n\mu_0 I}{2\pi R}\sin\frac{\pi}{n}$$

4.2.4 一个大平板的厚度从 $z=-a$ 到 $z=a$, 它的电流密度为 $\boldsymbol{J} = J\hat{\boldsymbol{x}}$, 求出平板内外的磁场。

$$B = \begin{cases} aJ & ,|z| > a \\ zJ & ,|z| <= a \end{cases}$$

4.2.5 一个大平行板电容器, 上极板有均匀电荷 σ, 下极板有均匀电荷 $-\sigma$, 两极板以恒定速度 v 运动

(a) 求出两极板间的磁场

(b) 求出作用在上极板单位面积上的磁力及其方向

(c) 为了使磁力和电场力平衡, 速度应为多大

(a):$B = \mu_0 J = \mu_0 \sigma v$

(b):$F = \sigma v B = \mu_0 \sigma^2 v^2$

(c):$\mu_0 \sigma^2 v^2 = \dfrac{\sigma}{2\varepsilon_0} \to v^2 = \dfrac{1}{2\mu_0 \sigma \varepsilon_0}$

4.3 磁矢势

在静磁学中，磁场满足 $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$. 这意味着磁场是一个*无源场*。

根据向量分析中的 Helmholtz 定理：

> 在单连通区域内，任何散度为零的光滑矢量场，都可以表示为某个矢量场的旋度。

因此存在矢量场 $\boldsymbol{A}(\boldsymbol{r})$，使得

$$\boxed{\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}.} \tag{4.9}$$

该矢量场 $\boldsymbol{A}$ 称为磁矢势。

磁矢势并非唯一。若 $\boldsymbol{A}$ 给出磁场 $\boldsymbol{B}$，则对任意标量场 $\Lambda(\boldsymbol{r})$，定义

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda, \tag{4.10}$$

有

$$\nabla \times \boldsymbol{A}' = \nabla \times \boldsymbol{A} = \boldsymbol{B}. \tag{4.11}$$

磁场 $\boldsymbol{B}$ 是可观测量，而磁矢势 $\boldsymbol{A}$ 本身不是。不同的 $\boldsymbol{A}$ 描述的是同一个物理磁场。

为了固定这种不唯一性，通常施加附加条件。最常用的是库仑规范：

$$\nabla \cdot \boldsymbol{A} = 0. \tag{4.12}$$

静磁场满足安培定律：

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \tag{4.13}$$

代入 $\boldsymbol{B} = \nabla \times \boldsymbol{A}$，得到

$$\nabla \times (\nabla \times \boldsymbol{A}) = \mu_0 \boldsymbol{J}. \tag{4.14}$$

利用恒等式

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A},$$

并在库仑规范下 $\nabla \cdot \boldsymbol{A} = 0$，得

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}. \tag{4.15}$$

这是一组三个形式上独立的泊松方程。

在无穷远处 $\boldsymbol{A} \to 0$ 的条件下，上式的解为

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3 r'. \tag{4.16}$$

该表达式在形式上与静电学中电势的解完全类似。

4.3.1 分离变量法

在没有电流时

$$\nabla^2 \boldsymbol{A} = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \boldsymbol{A}}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \boldsymbol{A}}{\partial \phi^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \boldsymbol{A}}{\partial \theta} \right) = 0$$

$\boldsymbol{r}$ 方向

4.3.2 含表面电流时径向磁矢势的边界条件

对麦克斯韦方程在一条跨越界面的无穷小曲面上积分，并取极限 $\epsilon \to 0$，得到边界条件

$$\boldsymbol{B}_{||\text{out}} - \boldsymbol{B}_{||\text{in}} = \mu_0 \boldsymbol{K}. \tag{4.17}$$

代入 $\boldsymbol{B} = \nabla \times \boldsymbol{A}$，得到边界条件

$$\nabla \times \boldsymbol{A}_{||\text{out}} - \nabla \times \boldsymbol{A}_{||\text{in}} = \mu_0 \boldsymbol{K} \tag{4.18}$$

直角坐标系下的形式

设界面为 $z=0$ 平面，表面电流为 $\boldsymbol{K}=K_x\hat{\boldsymbol{x}}+K_y\hat{\boldsymbol{y}}$。
旋度在直角坐标系中为

$$\nabla\times\boldsymbol{A}=\Big(\partial_y A_z-\partial_z A_y\ \partial_z A_x-\partial_x A_z\ \partial_x A_y-\partial_y A_x\Big). \tag{4.19}$$

代入普适条件并逐分量比较，得到

$$\frac{\partial A_x}{\partial z}\bigg|_{||\text{out}}-\frac{\partial A_x}{\partial z}\bigg|_{||\text{in}}=-\mu_0K_y,\ \frac{\partial A_y}{\partial z}\bigg|_{||\text{out}}-\frac{\partial A_y}{\partial z}\bigg|_{||\text{in}}\qquad=\mu_0K_x. \tag{4.20}$$

三、球坐标系下的形式

设界面为球面 $r=R$。并设电流与 ϕ 无关, 则

$$\nabla\times\boldsymbol{A}=-\frac{1}{r\sin\theta}\left(\frac{\partial a_\theta}{\partial\phi}-\frac{\partial\sin\theta a_\phi}{\partial\theta}\right)\boldsymbol{r}+\frac{1}{r}\left(\frac{\partial a_r}{\partial\theta}-\frac{\partial ra_\theta}{\partial r}\right)\boldsymbol{\phi}+\frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial ra_\phi}{\partial r}-\frac{\partial a_r}{\partial\phi}\right)\boldsymbol{\theta}$$

代入普适边界条件，得到

$$\boxed{\frac{\partial}{\partial r}\big(rA_{\varphi,\text{out}}\big)\Big|*r=R-\frac{\partial}{\partial r}\big(rA*\varphi,\text{in}\big)\Big|*r=R=-\mu_0RK*\varphi} \tag{4.21}$$

这是旋转带电球壳等问题中使用的标准形式。

四、柱坐标系下的形式

设界面为圆柱面 $\rho=a$，法向 $\hat{\boldsymbol{n}}=\hat{\boldsymbol{\rho}}$。若

$$\boldsymbol{K}=K_z(\varphi,z),\hat{\boldsymbol{z}},\qquad\boldsymbol{A}=A_z(\rho),\hat{\boldsymbol{z}}, \tag{4.22}$$

则

$$(\nabla\times\boldsymbol{A})_\varphi=-\frac{\partial A_z}{\partial\rho}. \tag{4.23}$$

边界条件化为

$$\boxed{\frac{\partial A_{z,\text{out}}}{\partial\rho}\bigg|*\rho=a-\frac{\partial A*z,\text{in}}{\partial\rho}\bigg|_{\rho=a}=-\mu_0K_z} \tag{4.24}$$

4.3.3　一半径为 R 的球壳，表面带有均匀电荷，电荷面密度为 σ，当它以恒定角速度 ω 旋转时，求出它在产生的矢势

边界条件:

$$\begin{aligned}
&\boldsymbol{A}_{\text{内}}=\boldsymbol{A}_{\text{外}}, && r=R\\
&\frac{\partial A_{\theta\text{外}}}{\partial r}-\frac{\partial A_{\theta\text{内}}}{\partial r}=-\mu_0\sigma\omega R\sin\theta, && r=R\\
&\boldsymbol{A}_{\text{外}}\to 0, && r\to\infty\\
&\boldsymbol{A}_{\text{内}}\to\frac{d\cos\theta}{4\pi\varepsilon_0 r^2}, && r\to 0
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\nabla}^2 \boldsymbol{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \boldsymbol{A}}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \boldsymbol{A}}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \boldsymbol{A}}{\partial \theta}\right) \\
\boldsymbol{\nabla}^2 \boldsymbol{A}_\theta &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \boldsymbol{A}_\theta}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \boldsymbol{A}_\theta}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \boldsymbol{A}_\theta}{\partial \theta}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\theta \boldsymbol{e}_\theta}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta \boldsymbol{e}_\theta}{\partial \phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial A_\theta \boldsymbol{e}_\theta}{\partial \theta}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_\theta}{\partial r}\right)\boldsymbol{e}_\theta + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial \phi^2}\boldsymbol{e}_\theta + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial A_\theta \boldsymbol{e}_\theta}{\partial \theta}\right)
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
\boldsymbol{A}_{内} &= \boldsymbol{B}_l r^l P_l(\cos\theta) \\
\boldsymbol{A}_{外} &= \boldsymbol{C}_l r^{-(l+1)} P_l(\cos\theta) \\
\boldsymbol{A}_{内} &= \boldsymbol{A}_{外} \\
\boldsymbol{B}_l R^l P_l(\cos\theta) &= \boldsymbol{C}_l R^{-(l+1)} P_l(\cos\theta) \\
\frac{\partial A_{\phi外}}{\partial r} - \frac{\partial A_{\phi内}}{\partial r} &= -\mu_0\sigma\omega R\sin\theta \\
\frac{\partial \boldsymbol{B}_l r^l P_l(\cos\theta)}{\partial r} - \frac{\partial \boldsymbol{C}_l r^{-(l+1)} P_l(\cos\theta)}{\partial r} &= -\mu_0\sigma\omega R\sin\theta \\
l\boldsymbol{B}_l R^{l-1} P_l(\cos\theta) + (l+1)\boldsymbol{C}_l R^{-(l+2)} P_l(\cos\theta) &= -\mu_0\sigma\omega R\sin\theta \\
l\boldsymbol{B}_l R^{l} P_l(\cos\theta) + (l+1)\boldsymbol{C}_l R^{-(l+1)} P_l(\cos\theta) &= -\mu_0\sigma\omega R^2\sin\theta \\
l\boldsymbol{C}_l R^{-(l+1)} P_l(\cos\theta) + (l+1)\boldsymbol{C}_l R^{-(l+1)} P_l(\cos\theta) &= -\mu_0\sigma\omega R^2\sin\theta \\
\boldsymbol{C}_l R^{-2} P_1(\cos\theta) + 2\boldsymbol{C}_l R^{-2} P_1(\cos\theta) &= -\mu_0\sigma\omega R^2\sin\theta \\
\boldsymbol{C}_l R^{-2} P_1(\cos\theta) + 2\boldsymbol{C}_l R^{-2} P_1(\cos\theta) &= -\mu_0\sigma\omega R^2\sin\theta
\end{aligned}
$$