Solutions of "counting problem"

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1   Solution via matrix multiplication

The problem from YouTube video "Olympiad level counting - How many subsets of {1,…,2000} have a sum divisible by 5" by 3Blue1Brown (url: https://www.youtube.com/watch?v=bOXCLR3Wric) can be described via matrix multiplication.

We want to name the counts of sums congruent to (0, 1, …, d-1) modulo d (d=5 in the video) as $c_0, c_1, \cdots c_{d-1}$.

As per definition in video, an empty set has a sum of 0 and is a multiple of d.

If we know the counts $c_0, c_1, \cdots c_{d-1}$ in a set of given numbers (set C), and we've got another set with counts $a_0, a_1, \cdots a_{d-1}$ (set A), the total counts $t_0, t_1, \cdots t_{d-1}$ for the union (set T) of these two sets will be (e. g. d=5):

$$
\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} c_0 & c_4 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_4 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_4 & c_3 \\ c_3 & c_2 & c_1 & c_0 & c_4 \\ c_4 & c_3 & c_2 & c_1 & c_0 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \quad .
$$

For example, for the count of sums congruent to 2 in the new set T, we can combine:
- sums congruent to 2 from set C and sums congruent to 0 from set A,
- sums congruent to 1 from set C and sums congruent to 1 from set A,
- sums congruent to 0 from set C and sums congruent to 2 from set A,
- sums congruent to 4 from set C and sums congruent to 3 from set A, and
- sums congruent to 3 from set C and sums congruent to 4 from set A.

In total, it will be $c_2 \cdot a_0 + c_1 \cdot a_1 + c_0 \cdot a_2 + c_4 \cdot a_3 + c_3 \cdot a_4$, which is third row ($t_2$) from the result of matrix multiplication.

1.1 Calculation for any number

First of all, we know that for an empty set, there is one sum (= 0), so c_0 is 1 and $c_1 = c_2 = c_3 = c_4 = 0$.

Then we combine this set with a set with only the number "1". We've got two possibilities here to:

- either don't use the new number (congruent 0 mod 5, $a_0=1$), or

- or use the new number (congruent 1 mod 5, $a_1=1$).

So for the new set as union of empty set and number "1", we've got the following new counts in a solution vector.

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad .$$

It works similarly when combining this set with number "2":

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad .$$

For sets with 3, 4, and 5 numbers:

$$n=3 : \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, n=4 : \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, n=5 : \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 6 \\ 6 \\ 6 \end{pmatrix} \quad .$$

For sets with numbers n > 5, we can repeat the matrix generation and matrix multiplication with the solution vector for n=5. If there is remainder for n divided by 5, we will apply another matrix multiplication with solution vector of n=1, …, 4.

For n = 2000, we will need 2000 / 5 = 400 steps.

But we can also calculate solution vectors for $n=5^1$, 5^2, 5^4, etc. by multiplying the generated matrix with the new solution vector coming out from the matrix multiplication.
For n = 2000 = 5 * 400 and 400 = 1 1001 0000$_b$, we need to multiply with solution vectors of 5^{16}, 5^{128}, and 5^{256}.

2 Solution via generating function

In the YouTube video "Olympiad level counting - How many subsets of {1,…,2000} have a sum divisible by 5" by 3Blue1Brown (url: https://www.youtube.com/watch?v=bOXCLR3Wric), it shows a solution by using a generating function:

$$f(x)=(1+x)(1+x^2)(1+x^3)...(1+x^{2000})$$
$$f(x)=\sum_{n=0}^{N} c_n x^n = 1+x+x^2+2x^3+2x^4+... \quad .$$

With $\zeta = e^{\frac{2\pi i}{5}}$, the sum f($\zeta^0$)+f($\zeta^1$)+f($\zeta^2$)+f($\zeta^3$)+f($\zeta^4$) results in 5 times of the sum of coefficients which is multiple of 5:

$$f\left(\xi^0\right) + f\left(\xi^1\right) + f\left(\xi^2\right) + f\left(\xi^3\right) + f\left(\xi^4\right) = 5 \cdot \left(c_0 + c_5 + c_{10} + \ldots\right)$$

For numbers n which is multiple of 5, the video gives the answer that there are $\frac{1}{5}\left(2^n + 4 \cdot 2^{\frac{n}{5}}\right)$ sums which are divisible by 5. E. g. for n=2000, there are $\frac{1}{5}\left(2^{2000} + 4 \cdot 2^{400}\right)$ of them.

This approach works similarly with every divisor d ≥ 2 and when n is multiple of d.

2.1 Calculation for remainder of sums other than 0

I've have modified this approach by adding factors while adding the generating functions. With $\alpha_{l,k} = \zeta^{k(d-l)}$ and k is the remainder of sums when dividing by d (e. g. 5), the sum $\alpha_{0,k}$f(ζ^0)+$\alpha_{1,k}$f(ζ^1)+$\alpha_{2,k}$f(ζ^2)+$\alpha_{3,k}$f(ζ^3)+$\alpha_{4,k}$f(ζ^4) results in 5 times of the sum of coefficients with remainders is k = (0, …, 4) when divided by 5:

$$k=0: \alpha_{0,0}f\left(\xi^0\right) + \alpha_{1,0}f\left(\xi^1\right) + \alpha_{2,0}f\left(\xi^2\right) + \alpha_{3,0}f\left(\xi^3\right) + \alpha_{4,0}f\left(\xi^4\right) = 5 \cdot \left(c_0 + c_5 + c_{10} + \ldots\right)$$
$$k=1: \alpha_{0,1}f\left(\xi^0\right) + \alpha_{1,1}f\left(\xi^1\right) + \alpha_{2,1}f\left(\xi^2\right) + \alpha_{3,1}f\left(\xi^3\right) + \alpha_{4,1}f\left(\xi^4\right) = 5 \cdot \left(c_1 + c_6 + c_{11} + \ldots\right)$$
$$k=2: \alpha_{0,2}f\left(\xi^0\right) + \alpha_{1,2}f\left(\xi^1\right) + \alpha_{2,2}f\left(\xi^2\right) + \alpha_{3,2}f\left(\xi^3\right) + \alpha_{4,2}f\left(\xi^4\right) = 5 \cdot \left(c_2 + c_7 + c_{12} + \ldots\right)$$
$$k=3: \alpha_{0,3}f\left(\xi^0\right) + \alpha_{1,3}f\left(\xi^1\right) + \alpha_{2,3}f\left(\xi^2\right) + \alpha_{3,3}f\left(\xi^3\right) + \alpha_{4,3}f\left(\xi^4\right) = 5 \cdot \left(c_3 + c_8 + c_{13} + \ldots\right)$$
$$k=4: \alpha_{0,4}f\left(\xi^0\right) + \alpha_{1,4}f\left(\xi^1\right) + \alpha_{2,4}f\left(\xi^2\right) + \alpha_{3,4}f\left(\xi^3\right) + \alpha_{4,4}f\left(\xi^4\right) = 5 \cdot \left(c_4 + c_9 + c_{14} + \ldots\right)$$

For k=0, the formula $\frac{1}{5}\left(2^n + 4 \cdot 2^{\frac{n}{5}}\right)$ remains the same since all factors $\alpha_{l,0}$ equal to 1. For k=1..4, we've got the formula $\frac{1}{5}\left(2^n - 2^{\frac{n}{5}}\right)$ for number of sums with remainder k when divided by 5.

This approach works similarly with every divisor d ≥ 2 and when n is multiple of d. If n is not a multiple of d, we'll have to apply matrix multiplication with solution vector of n=1, …, d-1 (see 1.1).

2.2 Calculation of generating function using gcd(k, d)

Given the generating function for divisor d and numbers from 1 to n (with n is a multiple of d):

$f(x) = \prod_{r=1}^{n}\left(1 + x^r\right)$, the video by 3Blue1Brown shows that when we plug in $\zeta^k = e^{\frac{k \cdot 2\pi i}{d}}$ for x,

with k=0, …, d-1, the function can be expressed as $f\left(\xi^k\right) = \left(\prod_{r=1}^{d}\left(1 + \zeta^{rk}\right)\right)^{\frac{n}{d}}$, since $\zeta^{(r+d)k} = \zeta^{rk}$.

Let call the product $g(\xi^k) = \prod_{r=1}^{d}\left(1 + \zeta^{rk}\right)$. The larger d, the more terms need to be multiplied in $g\left(\xi^k\right)$.
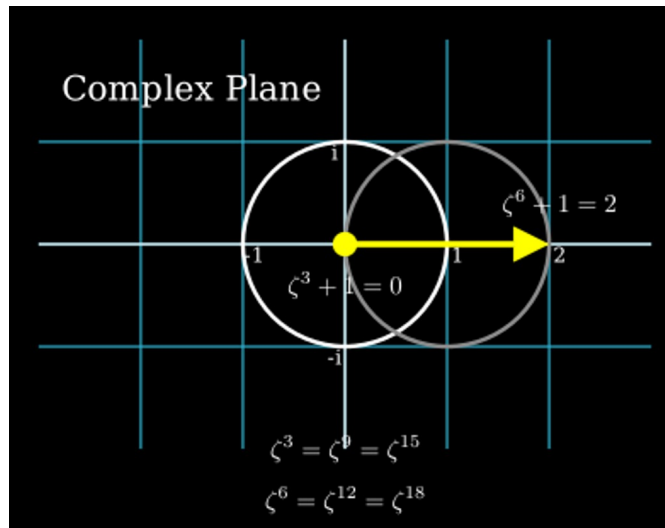
However, $g(\xi^k)$ can be calculated without actual multiplying the terms. Instead we can apply the following three rules.

Rule 1: $k=0 \Rightarrow g(\xi^k) = \prod_{r=1}^{d}(1+1) = 2^d$

Rule 2: If d has 2 as factor more often than k (in other words: $\dfrac{d}{gcd(k,d)}$ is even, where *gcd* means the greatest common divisor), then at least one of the $(1+\xi^{rk})$ is 0 and $g(\xi^k)=0$.

Example 1: d = 6, k = 3

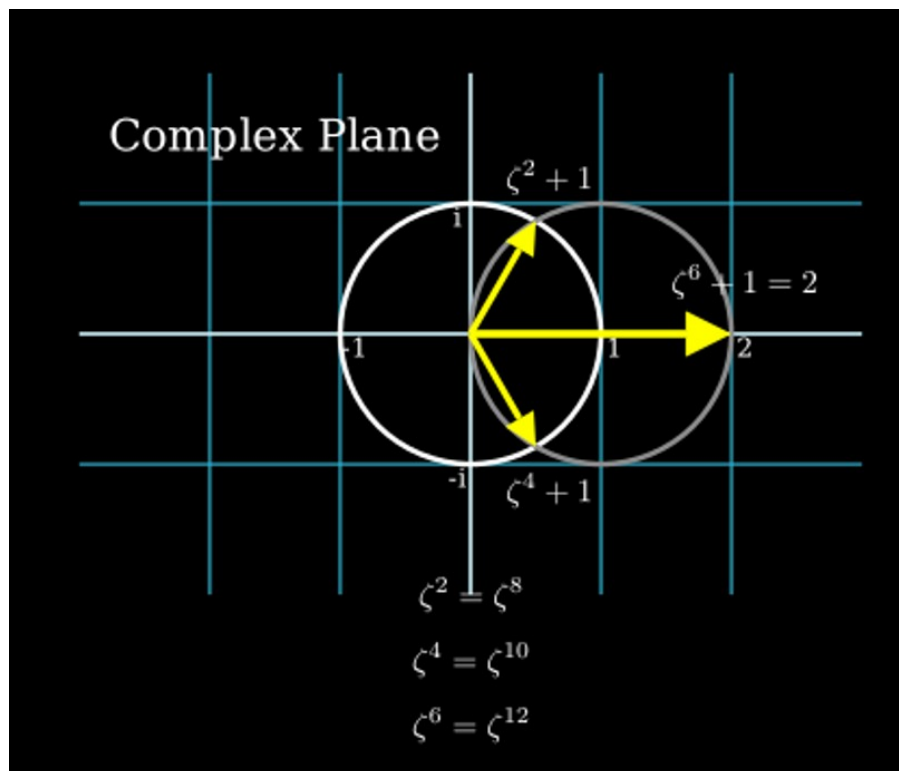Since gcd(3, 6) = 3 and 6 / 3 = 2 is even, so are $(1+\xi^{rk})=0$ for $r \in \{1,3,5\}$ and $g(\xi^k)=0$.

Example 2: Rules 2 can be applied if $d=2^s, s \in Z^+$, since d/gcd(k, d) is even for k = 1, ..., d-1, and so is $g(\xi^k)=0$.

Rule 3: In the remaining cases, the following applies: $g(\xi^k)=2^{gcd(k,d)}$.

Example 3: d = 6, k = 2

Since gcd(2, 6) = 2 and 6 / 2 = 3 is not even, so is $g(\xi^k)=2^{gcd(2,6)}=2^2=4$.

Example 4: Rules 3 can be applied if d is an odd prime number, since gcd(k, d) is 1 for k = 1, …, d-1. So is $g(\zeta^k)=2^{gcd(k,d)}=2^1=2$.