Solutions of ”counting problem”

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1 Solution via matrix multiplication

The problem from YouTube video "Olympiad level counting - How many subsets of {1,…,2000} have a sum divisible by 5" by 3Blue1Brown (url: https://www.youtube.com/watch?v=bOXCLR3Wric) can be described via matrix multiplication.

We want to name the counts of sums congruent to (0, 1, …, d-1) modulo d (d=5 in the video) as $c_0, c_1, \cdots c_{d-1}$.

As per definition in video, an empty set has a sum of 0 and is a multiple of d.

If we know the counts $c_0, c_1, \cdots c_{d-1}$. in a set of given numbers (set C), and we’ve got another set with counts $a_0, a_1, \cdots a_{d-1}$. (set A), the total counts $t_0, t_1, \cdots t_{d-1}$. for the union (set T) of these two sets will be (e. g. d=5):

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} c_0 & c_4 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_4 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_4 & c_3 \\ c_3 & c_2 & c_1 & c_0 & c_4 \\ c_4 & c_3 & c_2 & c_1 & c_0 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} .$$

For example, for the count of sums congruent to 2 in the new set T, we can combine:

- sums congruent to 2 from set C and sums congruent to 0 from set A,
- sums congruent to 1 from set C and sums congruent to 1 from set A,
- sums congruent to 0 from set C and sums congruent to 2 from set A,
- sums congruent to 4 from set C and sums congruent to 3 from set A, and
- sums congruent to 3 from set C and sums congruent to 4 from set A.

In total, it will be $c_2 \cdot a_0 + c_1 \cdot a_1 + c_0 \cdot a_2 + c_4 \cdot a_3 + c_3 \cdot a_4$, which is third row (t_2) from the result of matrix multiplication.

1.1 Calculation for any number

First of all, we know that for an empty set, there is one sum (= 0), so c_0 is 1 and $c_1 = c_2 = c_3 = c_4 = 0$.

Then we combine this set with a set with only the number “1”. We’ve got two possibilities here to:

- either don't use the new number (congruent 0 mod 5, $a_0=1$), or
- or use the new number (congruent 1 mod 5, $a_1=1$).

So for the new set as union of empty set and number "1", we've got the following new counts in a solution vector.

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

It works similarly when combining this set with number "2":

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} .$$

For sets with 3, 4, and 5 numbers:

$$n=3: \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, n=4: \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, n=5: \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 6 \\ 6 \\ 6 \end{pmatrix} .$$

For sets with numbers n > 5, we can repeat the matrix generation and matrix multiplication with the solution vector for n=5. If there is remainder for n divided by 5, we will apply another matrix multiplication with solution vector of n=1, …, 4.

For n = 2000, we will need 2000 / 5 = 400 steps.

But we can also calculate solution vectors for $n=5^1$, 5^2, 5^4, etc. by multiplying the generated matrix with the new solution vector coming out from the matrix multiplication.
For n = 2000 = 5 * 400 and 400 = $1\ 1001\ 0000_b$, we need to multiply with solution vectors of 5^{16}, 5^{128}, and 5^{256}.

2 Solution via generating function

In the YouTube video "Olympiad level counting - How many subsets of {1,…,2000} have a sum divisible by 5" by 3Blue1Brown (url: https://www.youtube.com/watch?v=bOXCLR3Wric), it shows a solution by using a generating function:

$$f(x)=(1+x)(1+x^2)(1+x^3)\ldots(1+x^{2000})$$
$$f(x)=\sum_{n=0}^{N} c_n x^n = 1+x+x^2+2x^3+2x^4+\ldots .$$

With $\zeta = e^{\frac{2\pi i}{5}}$, the sum $f(\zeta^0)+f(\zeta^1)+f(\zeta^2)+f(\zeta^3)+f(\zeta^4)$ results in 5 times of the sum of coefficients which is multiple of 5:

$$f(\zeta^0)+f(\zeta^1)+f(\zeta^2)+f(\zeta^3)+f(\zeta^4)=5\cdot(c_0+c_5+c_{10}+\dots)$$

For numbers n which is multiple of 5, the video gives the answer that there are $\frac{1}{5}\left(2^n+4\cdot 2^{\frac{n}{5}}\right)$ sums which are divisible by 5. E. g. for n=2000, there are $\frac{1}{5}\left(2^{2000}+4\cdot 2^{400}\right)$ of them.

The video doesn’t show the case if n is not a multiple of d, but this approach is still valid then. However, the generating function f will have a remaining product p_r (r is remainder of n divided by d), e. g. $p_3(x)=(1+x)(1+x^2)(1+x^3)$ for r is 3, and the formula will look like this:

$$\frac{1}{5}\left(2^{n-r}\cdot p_r(\zeta^0)+2^{\frac{n-r}{5}}\cdot\left[p_r(\zeta^1)+p_r(\zeta^2)+p_r(\zeta^3)+p_r(\zeta^4)\right]\right) .$$

As $p_r(\zeta^0)=2^r$, the formula becomes $\frac{1}{5}\left(2^n+2^{\frac{n-r}{5}}\cdot\left[p_r(\zeta^1)+p_r(\zeta^2)+p_r(\zeta^3)+p_r(\zeta^4)\right]\right)$.

This approach works similarly with every divisor d ≥ 2.

2.1 Calculation for remainder of sums other than 0

I’ve have modified this approach by adding factors while adding the generating functions. With $\alpha_{l,k}=\zeta^{k(d-l)}$ and k is the remainder of sums when dividing by d (e. g. 5), the sum $\alpha_{0,k}f(\zeta^0)+\alpha_{1,k}f(\zeta^1)+\alpha_{2,k}f(\zeta^2)+\alpha_{3,k}f(\zeta^3)+\alpha_{4,k}f(\zeta^4)$ results in 5 times of the sum of coefficients with remainders is k = (0, …, 4) when divided by 5:

$$\begin{aligned}
k=0: \alpha_{0,0}f(\zeta^0)+\alpha_{1,0}f(\zeta^1)+\alpha_{2,0}f(\zeta^2)+\alpha_{3,0}f(\zeta^3)+\alpha_{4,0}f(\zeta^4)&=5\cdot(c_0+c_5+c_{10}+\dots)\\
k=1: \alpha_{0,1}f(\zeta^0)+\alpha_{1,1}f(\zeta^1)+\alpha_{2,1}f(\zeta^2)+\alpha_{3,1}f(\zeta^3)+\alpha_{4,1}f(\zeta^4)&=5\cdot(c_1+c_6+c_{11}+\dots)\\
k=2: \alpha_{0,2}f(\zeta^0)+\alpha_{1,2}f(\zeta^1)+\alpha_{2,2}f(\zeta^2)+\alpha_{3,2}f(\zeta^3)+\alpha_{4,2}f(\zeta^4)&=5\cdot(c_2+c_7+c_{12}+\dots)\\
k=3: \alpha_{0,3}f(\zeta^0)+\alpha_{1,3}f(\zeta^1)+\alpha_{2,3}f(\zeta^2)+\alpha_{3,3}f(\zeta^3)+\alpha_{4,3}f(\zeta^4)&=5\cdot(c_3+c_8+c_{13}+\dots)\\
k=4: \alpha_{0,4}f(\zeta^0)+\alpha_{1,4}f(\zeta^1)+\alpha_{2,4}f(\zeta^2)+\alpha_{3,4}f(\zeta^3)+\alpha_{4,4}f(\zeta^4)&=5\cdot(c_4+c_9+c_{14}+\dots)
\end{aligned}$$

For k=0, the formula $\frac{1}{5}\left(2^n+4\cdot 2^{\frac{n}{5}}\right)$ remains the same since all factors $\alpha_{l,0}$ equal to 1. For k=1..4, we’ve got the formula $\frac{1}{5}\left(2^n-2^{\frac{n}{5}}\right)$ for number of sums with remainder k when divided by 5.

These formulas applies for numbers n which is multiple of 5.

If n is not a multiple of d, the generating function f will have a remaining product p_r and the formulas must be modified accordingly (see Chapter 2). For k=1..4, we’ll get formulas which are very unhandy: $\frac{1}{5}\left(2^n+2^{\frac{n-r}{5}}\cdot\left[\alpha_{1,k}\cdot p_r(\zeta^1)+\alpha_{2,k}\cdot p_r(\zeta^2)+\alpha_{3,k}\cdot p_r(\zeta^3)+\alpha_{4,k}\cdot p_r(\zeta^4)\right]\right)$. Functions in my program will apply matrix multiplication with solution vector of n=1, …, d-1 instead (see Section 1.1).

This approach works similarly with every divisor d ≥ 2.

2.2 Calculation of generating function using gcd(k, d)

Given the generating function for divisor d and numbers from 1 to n (with n is a multiple of d):

$f(x)=\prod_{r=1}^{n}(1+x^r)$, the video by 3Blue1Brown shows that when we plug in $\zeta^k=e^{\frac{k\cdot 2\pi i}{d}}$ for x, with k=0, …, d-1, the function can be expressed as $f(\zeta^k)=\left(\prod_{r=1}^{d}(1+\zeta^{rk})\right)^{\frac{n}{d}}$, since $\zeta^{(r+d)k}=\zeta^{rk}$.

Let call the product $g(\zeta^k)=\prod_{r=1}^{d}(1+\zeta^{rk})$. The larger d, the more terms need to be multiplied in $g(\zeta^k)$.

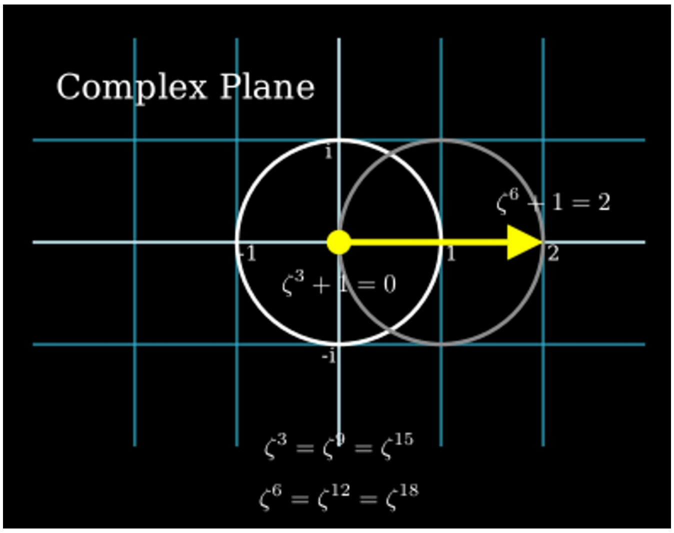
However, $g(\zeta^k)$ can be calculated without actual multiplying the terms. Instead we can apply the following three rules.

Rule 1: $k=0\Rightarrow g(\zeta^k)=\prod_{r=1}^{d}(1+1)=2^d$

Rule 2: If d has 2 as factor more often than k (in other words: $\frac{d}{gcd(k,d)}$ is even, where *gcd* means the greatest common divisor), then at least one of the $(1+\zeta^{rk})$ is 0 and $g(\zeta^k)=0$.

Example 1: d = 6, k = 3

Since gcd(3, 6) = 3 and 6 / 3 = 2 is even, so are $(1+\zeta^{rk})=0 \text{ for } r\in\{1,3,5\}$ and $g(\zeta^k)=0$.

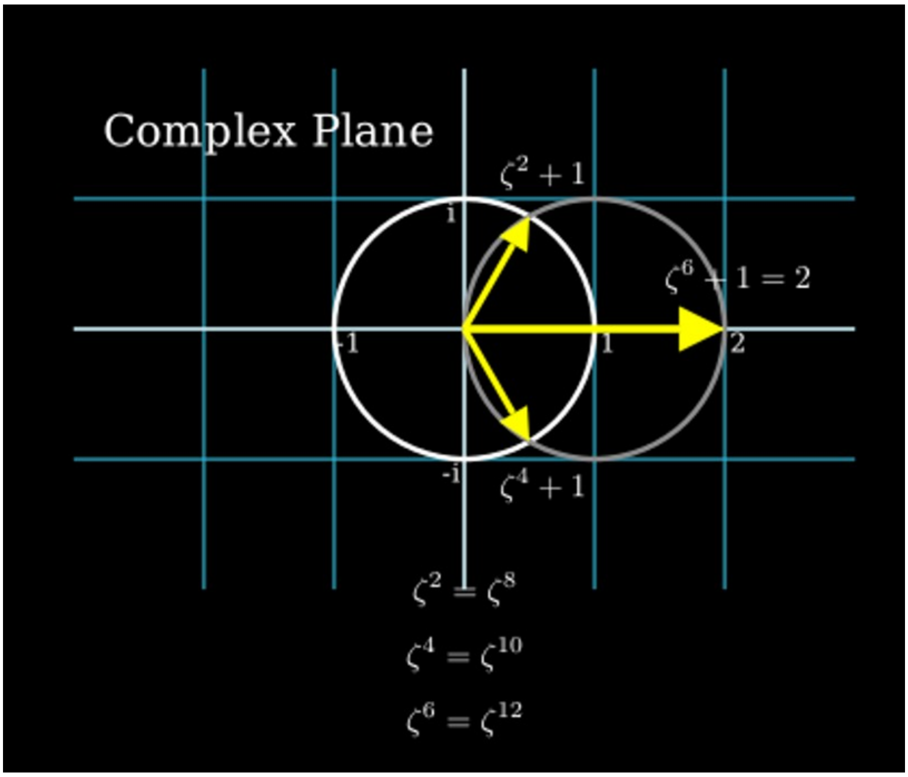


Example 2: Rules 2 can be applied if $d=2^s, s\in Z^+$, since d/gcd(k, d) is even for k = 1, …, d-1, and so is $g(\zeta^k)=0$.

Rule 3: In the remaining cases, the following applies: $g(\zeta^k)=2^{gcd(k,d)}$.

Example 3: d = 6, k = 2

Since gcd(2, 6) = 2 and 6 / 2 = 3 is not even, so is $g(\zeta^k)=2^{gcd(2,6)}=2^2=4$.



Example 4: Rules 3 can be applied if d is an odd prime number, since gcd(k, d) is 1 for k = 1, …, d-1. So is $g(\zeta^k)=2^{gcd(k,d)}=2^1=2$.