

STAT8310 - Bayesian Data Analysis

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Preface

Description

This course will cover the topics in the theory and practice of *Bayesian statistical inference*, ranging from a review of fundamentals to questions of current research interest. Motivation for the Bayesian approach. Bayesian computation, Monte Carlo methods, asymptotics. Model checking and comparison. A selection of examples and issues in modelling and data analysis. Discussion of advantages and difficulties of the Bayesian approach. This course will be computationally intensive through analysis of data sets using the R statistical computing language.

Prerequisites

MATH 4752/6752 – Mathematical Statistics II or equivalent, and the ability to program in a high-level language.

Instructor

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Office Hour

10:00–13:00 on Monday, or by appointment.

Grade Distribution

- Homework – 50%
- Exam – 30%
- Final – 20%

Assignment

- A1, due on Jan 29, 2026
 A2 TBA

Midterm

- March 3, 2026

Topics and Corresponding Lectures

Those chapters are based on the lecture notes. This part will be updated frequently.

Status	Topic	Lecture
	Welcome and Overview	1
	Intro to R Programming	2
	Probability and Exchangeability	3–5
	One Parameter Models	6–

Recommended Textbooks

- Gelman, A., Carlin, J., Stern, H., Rubin, D., Dunson, D., and Vehtari, A. (2021). [Bayesian Data Analysis](#), CRC Press, 3rd Ed.
- Hoff, P.D. (2009). [A First Course in Bayesian Statistical Methods](#), Springer.
- McElreath, R. (2018). [Statistical Rethinking: A Bayesian Course with Examples in R and Stan](#), CRC Press.

Side Readings

- TBA

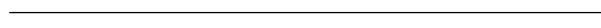
1 Quick Overview

The posterior distribution is obtained from the prior distribution and sampling model via *Bayes' rule*:

$$p(\theta | y) = \frac{p(y | \theta)p(\theta)}{\int_{\Theta} p(y | \theta')p(\theta')d\theta'}.$$

1.1 Why Bayesian?

- **Intuitive probability interpretation:** Directly quantifies uncertainty about parameters as probability distributions
- **Incorporates prior knowledge:** Systematically combines domain expertise with data through the prior distribution
- **Principled inference:** Bayes' rule provides a coherent framework for updating beliefs based on evidence
- **Natural handling of uncertainty:** Posterior distributions capture full uncertainty, not just point estimates
- **Sequential analysis:** Easily updates beliefs as new data arrives (posterior becomes new prior)
- **Small sample inference:** Performs well with limited data by leveraging prior information
- **Prediction with uncertainty:** Generates predictive distributions that quantify uncertainty in future observations
- **Decision-making:** Naturally incorporates loss functions for optimal decision rules
- **Model comparison:** Bayes factors provide a principled approach to comparing competing models



1.2 Some Bayesian Topics and their Computational Focus

Table 1.1: Some of the Bayesian Topics and its computational related focuses.

Topics	Key Concepts / Readings	Computing Focus
Introduction to Bayesian Thinking	Bayesian vs. Frequentist paradigms; Prior, likelihood, posterior	Review of R basics and reproducible workflows
Bayesian Inference for Simple Models	Conjugate priors, Beta-Binomial, Normal-Normal, Poisson-Gamma	Simulating posteriors, visualization
Prior Elicitation and Sensitivity	Informative vs. noninformative priors, Jeffreys prior	Prior sensitivity plots
Monte Carlo Integration	Law of large numbers, sampling-based inference	Random sampling and Monte Carlo approximation
Markov Chain Monte Carlo (MCMC)	Metropolis-Hastings, Gibbs sampler	Implementing MCMC in R
Convergence Diagnostics	Trace plots, autocorrelation, Gelman–Rubin statistic	<code>coda</code> , <code>rstan</code> , and <code>bayesplot</code> packages
Hierarchical Bayesian Models	Partial pooling, shrinkage, multilevel structures	<code>rstanarm</code> / <code>brms</code>
Midterm Project: Bayesian Linear Regression	Posterior inference for regression, model selection	<code>brms</code> , <code>rstanarm</code> , custom Gibbs samplers
Bayesian Model Comparison	Bayes factors, BIC, DIC, WAIC, LOO	Practical comparison via cross-validation
Model Checking and Diagnostics	Posterior predictive checks, residual analysis	<code>pp_check</code> in <code>brms</code>
Advanced Computation	Hamiltonian Monte Carlo (HMC), Variational Inference	Using <code>Stan</code> and <code>CmdStanR</code>
Bayesian Decision Theory	Utility functions, decision rules, loss minimization	Simple decision problems in R
Modern Bayesian Methods	Approximate Bayesian computation (ABC), Bayesian neural networks	Examples via <code>rstan</code> or <code>tensorflow-probability</code>
Student Project Presentations	Applications and case studies	Full workflow demonstration in R

1.3 Interesting Article:

- Goligher, E.C., Harhay, M.O. (2023). [What Is the Point of Bayesian Analysis?](#), American Journal of Respiratory and Critical Care Medicine, 209, 485–487.

2 Belief function and Probability Review

Leading objectives:

be familiar with the following concepts

- Belief Functions
- Probability
- Bayes' Rule
- Random Variables
- Exchangeability

2.1 Belief functions

Probability is a way to express rational beliefs.

A **belief function** $\text{Be}(\cdot)$ is a function that assigns number to statements such that the larger the number, the higher the degree of belief.

Let F, G , and H be three possibly overlapping statements about the world.

For example:

- $F = \{ \text{a person owns a smartphone} \}$
- $G = \{ \text{a person uses social media daily} \}$
- $H = \{ \text{a person works remotely at least part of the time} \}$

or

- $F = \{ \text{a person has a graduate degree} \}$
- $G = \{ \text{a person works in a STEM field} \}$
- $H = \{ \text{a person is employed in the private sector} \}$

The preference over bets involving these statements can be used to define a belief function

- $\text{Be}(F) > \text{Be}(G)$ means you prefer a bet F is true over that G is true.

Also, we want $\text{Be}(\cdot)$ to describe our beliefs under certain conditions

- $\text{Be}(F | H) > \text{Be}(G | H)$ means that if we knew that H were true, then we would prefer to bet that F is also true over G is also true.
- $\text{Be}(F | G) > \text{Be}(F | H)$ means that if we bet on F , we would prefer to do it under the condition that G is true rather than H is true.

Some more notations:

- Let \neg denote negation. That is, $\neg F$ is the statement that F is not true.
- Let $F \vee G$ denote the disjunction (or) of statements F and G , meaning that at least one of F or G is true.
- Let $F \wedge G$ denote the conjunction (and) of statements F and G , meaning that both F and G are true.

It has been argued by many that any function that is to numerically represent our beliefs should have the following properties:

- **B1:** $\text{Be}(\neg H | H) \leq \text{Be}(F | H) \leq \text{Be}(H | H)$
- **B2:** $\text{Be}(F \vee G | H) \geq \max\{\text{Be}(F | H), \text{Be}(G | H)\}$
- **B3:** $\text{Be}(F \wedge G | H)$ can be derived from $\text{Be}(G | H)$ and $\text{Be}(F | G \wedge H)$.

How should we interpret these properties, and do they make sense?

- B1 means that the number we assign to $\text{Be}(F | H)$, our conditional belief in F given H , is bounded below and above by the numbers we assign to complete disbelief $\text{Be}(\neg H | H)$ and complete belief $\text{Be}(H | H)$.
- B2 says that our belief that the truth lies in a given set of possibilities should not decrease as we add to the set of possibilities.
- B3 is a bit trickier. To see why it makes sense, imagine you have to decide whether or not F and G are true, knowing that H is true. You could do this by first deciding whether or not G is true given H , and if so, then deciding whether or not F is true given G and H .

Recall the notation from (elementary) probability that, $F \cup G$ means F or G , and $F \cap G$ means F and G , and \emptyset is the empty set.

- **P1:**

$$0 = \Pr(\neg H | H) \leq \Pr(F | H) \leq \Pr(H | H) = 1$$

- **P2:**

$$\Pr(F \cup G | H) = \Pr(F | H) + \Pr(G | H), \quad \text{if } F \cap G = \emptyset$$

- **P3:**

$$\Pr(F \cap G | H) = \Pr(G | H)\Pr(F | G \cap H)$$

2.1.1 Conclusion

You can see that, a probability function satisfy P1–P3 also satisfies B1–B3. Therefore, probability functions are a special case of belief functions, and we can use it to describe our belief.

2.2 Events, Partitions and Bayes' Rule

A collectiion of sets $\{H_1, \dots, H_K\}$ is a partition of another set \mathcal{H} if

1. $H_i \cap H_j = \emptyset$ for all $i \neq j$ (mutually exclusive);
2. $\bigcup_{i=1}^K H_i = \mathcal{H}$ (collectively exhaustive).

In the context of indetifying which of several statements is true, if \mathcal{H} is the set of all possible truths and $\{H_1, \dots, H_K\}$ is a partition of \mathcal{H} , then exactly one set H_j contains the truth.

Let \mathcal{H} be the status of a statistical model.

Valid partitions include:

- {correctly specified, misspecified}
- {underfitting, well-specified, overfitting}

2.2.1 Partition and Probability

Suppose $\{H_1, \dots, H_K\}$ is a partition of \mathcal{H} , $\Pr(\mathcal{H}) = 1$ and E is some specific event. Then, by the axioms of probability, we have

- Law of total probability

$$\sum_{k=1}^K \Pr(H_k) = \Pr\left(\bigcup_{k=1}^K H_k\right) = \Pr(\mathcal{H}) = 1$$

- Law of marginal probability

$$\Pr(E) = \sum_{k=1}^K \Pr(E \cap H_k) = \sum_{k=1}^K \Pr(E | H_k) \Pr(H_k)$$

- Bayes' rule

$$\Pr(H_j | E) = \frac{\Pr(E | H_j) \Pr(H_j)}{\Pr(E)} = \frac{\Pr(E | H_j) \Pr(H_j)}{\sum_{k=1}^K \Pr(E | H_k) \Pr(H_k)}$$

A subset of the 1996 General Social Survey includes data on the education level and income for a sample of males over 30 years of age. Let $\{H_1, H_2, H_3, H_4\}$ be the events that a randomly selected person in this sample is in, respectively, the lower 25th percentile, the second 25th percentile, the third 25th percentile and the upper 25th percentile in terms of income. By definition,

$$\{\Pr(H_1), \Pr(H_2), \Pr(H_3), \Pr(H_4)\} = \{.25, .25, .25, .25\}.$$

Note that $\{H_1, H_2, H_3, H_4\}$ is a partition and so these probabilities sum to 1. Let E be the event that a randomly sampled person from the survey has a college education. From the survey data, we have

$$\{\Pr(E | H_1), \Pr(E | H_2), \Pr(E | H_3), \Pr(E | H_4)\} = \{.11, .19, .31, .53\}.$$

These probabilities do not sum to 1 - they represent the proportions of people with college degrees in the four different income subpopulations H_1, H_2, H_3 and H_4 . Now let's consider the income distribution of the college-educated population. Using Bayes' rule we can obtain

$\{\Pr(H_1 | E), \Pr(H_2 | E), \Pr(H_3 | E), \Pr(H_4 | E)\} = \{0.09, 0.17, 0.27, 0.47\}$, and we see that the income distribution for people in the college-educated population differs markedly from $\{0.25, 0.25, 0.25, 0.25\}$, the distribution for the general population. Note that these probabilities do sum to 1 - they are the conditional probabilities of the events in the partition, given E .

In Bayesian inference, H_1, \dots, H_K often refer to disjoint hypotheses or states of nature and E refers to the outcome of a survey, study or experiment. To compare hypotheses *post-experimentally*, we often calculate the following ratio:

$$\begin{aligned} \frac{\Pr(H_i | E)}{\Pr(H_j | E)} &= \frac{\Pr(E | H_i) \Pr(H_i) / \Pr(E)}{\Pr(E | H_j) \Pr(H_j) / \Pr(E)} \\ &= \frac{\Pr(E | H_i) \Pr(H_i)}{\Pr(E | H_j) \Pr(H_j)} \\ &= \frac{\Pr(E | H_i)}{\Pr(E | H_j)} \times \frac{\Pr(H_i)}{\Pr(H_j)} \\ &= \text{"Bayes factor"} \times \text{"prior beliefs"}. \end{aligned}$$

This calculation reminds us that Bayes' rule does not determine what our *beliefs should be* after seeing the data, it only tells us how they *should change after seeing the data*.

2.3 Independence

Two events F and G are conditionally independent, if given H , we have $\Pr(F \cap G | H) = \Pr(F | H)\Pr(G | H)$.

How do we interpret conditional independence? By Axiom P3, the following is always true:

$$\begin{aligned} \Pr(G | H) \Pr(F | H \cap G) &\stackrel{\text{always}}{=} \Pr(F \cap G | H) \stackrel{\text{independence}}{=} \Pr(F | H) \Pr(G | H) \\ \Pr(G | H) \Pr(F | H \cap G) &= \\ \Pr(F | H \cap G) &= \frac{\Pr(F | H) \Pr(G | H)}{\Pr(F | H)}. \end{aligned}$$

Thus, conditional independence implies that $\Pr(F | H \cap G) = \Pr(F | H)$. In other words, if we know H is true, and F and G are conditionally independent given H , then knowing G does not change our belief about F .

Let's consider the conditional dependence of F and G when H is assumed to be true in the following two situations:

Situation 1:

- $F = \{ \text{a hospital patient is a smoker} \}$
- $G = \{ \text{a hospital patient has lung cancer} \}$
- $H = \{ \text{smoking causes lung cancer} \}$

Situation 2:

- $F = \{ \text{a student studies regularly for an exam} \}$
- $G = \{ \text{a student receives a high exam score} \}$
- $H = \{ \text{studying improves exam performance} \}$

Think: In both of these situations, H being true implies a relationship between F and G . What about when H is not true?

2.4 Random Variables

In Bayesian inference a random variable is defined as an unknown numerical quantity about which we make probability statements. For example, the quantitative outcome of a survey, experiment or study is a random variable before the study is performed. Additionally, a fixed but unknown population parameter is also a random variable

2.4.1 Discrete Random variables

Let Y be a random variable and let \mathcal{Y} be the set of all possible values that Y can take. If \mathcal{Y} is countable, meaning that $\mathcal{Y} = \{y_1, y_2, \dots\}$, then Y is a discrete random variable.

The event that the outcome Y of our survey has the value y is expressed as $\{Y = y\}$. For each $y \in \mathcal{Y}$, the shorthand notation for $\Pr(Y = y)$ is $p(y)$, and $p(\cdot)$ is called the **probability mass function** of Y , and with two properties

1. $0 \leq p(y) \leq 1$ for all $y \in \mathcal{Y}$,
2. $\sum_{y \in \mathcal{Y}} p(y) = 1$.

General probability statements about Y can be derived from the pdf/pmf, for example, for any subset $A \subseteq \mathcal{Y}$, we have $\Pr(Y \in A) = \sum_{y \in A} p(y)$. When we have two disjoint subsets A and B of \mathcal{Y} , we have

$$\Pr(Y \in A \cup B) = \Pr(Y \in A) + \Pr(Y \in B) = \sum_{y \in A} p(y) + \sum_{y \in B} p(y).$$

Let Y be the number of successes in n independent Bernoulli trials, each with probability of success θ . Then, Y follows a Binomial distribution with parameters n and θ , denoted as $Y \sim \text{Binomial}(n, p)$. The probability mass function of Y is given by

$$p(y) = \Pr(Y = y) = \binom{n}{y} \theta^y (1 - \theta)^{n-y}, \quad y = 0, 1, 2, \dots, n.$$

If $\theta = 0.3$ and $n = 3$, then the probability of observing exactly 2 successes is

$$p(2) = \Pr(Y = 2 \mid \theta = 0.3) = \binom{3}{2} (0.3)^2 (0.7)^1 = 3 \cdot 0.09 \cdot 0.7 = 0.189.$$

2.4.2 Continuous random variables

If \mathcal{Y} is uncountable, for example, $\mathcal{Y} = \mathbb{R}$ or $\mathcal{Y} = (0, 1)$, then Y is a continuous random variable. In this case, we cannot list all possible values of Y and assign probabilities to each value. Instead, we use a probability distribution to describe the distribution of Y . That is, the cumulative distribution function (cdf) defined as follows.

The **cumulative distribution function** (cdf) of a continuous random variable Y is defined as

$$F(y) = \Pr(Y \leq y), \quad y \in \mathcal{Y}.$$

Note that, for the cdf $F(y)$, we have the following properties:

- $0 \leq F(y) \leq 1$ for all $y \in \mathcal{Y}$,
- $F(y)$ is non-decreasing, meaning that if $y_1 < y_2$, then $F(y_1) \leq F(y_2)$,
- $\lim_{y \rightarrow -\infty} F(y) = 0$
- $\lim_{y \rightarrow \infty} F(y) = 1$.

Probability of various events can be derived from the cdf. For example, for any interval $A = (a, b] \subseteq \mathcal{Y}$, we have

$$\Pr(Y \in A) = \Pr(a < Y \leq b) = F(b) - F(a).$$

Also, $\Pr(Y \leq a) = F(a)$ and $\Pr(Y > a) = 1 - F(a)$.

2.4.3 Description of distributions through quantiles and moments

In this subsection, we discuss a few ways to describe probability distributions: quantiles and moments. They are used to describe the behaviour of the distribution compressing them into summary statistics.

The **expectation** or **mean** of a random variable Y can be thought as the centre of mass or the location of the distribution, which is defined as

- For discrete random variable:

$$E(Y) = \sum_{y \in \mathcal{Y}} yp(y).$$

- For continuous random variable:

$$E(Y) = \int_{\mathcal{Y}} yf(y)dy.$$

Difference between mean, mode and median

- Mean: the centre of mass of the distribution
- Mode: The most probable value of Y
- Median: The value of Y in the middle of the distribution.

In skewed distribution, the three will not equal to each other.

```
library(ggplot2)

# -----
# Theoretical reference lines
# -----
lines_normal <- data.frame(
  value = c(0, 0, 0),
  Statistic = c("Mean", "Median", "Mode")
)

lines_lognormal <- data.frame(
  value = c(exp(1/8), 1, exp(-1/4)),
  Statistic = c("Mean", "Median", "Mode")
)

cols <- c("Mean" = "red", "Median" = "darkgreen", "Mode" = "purple")
```

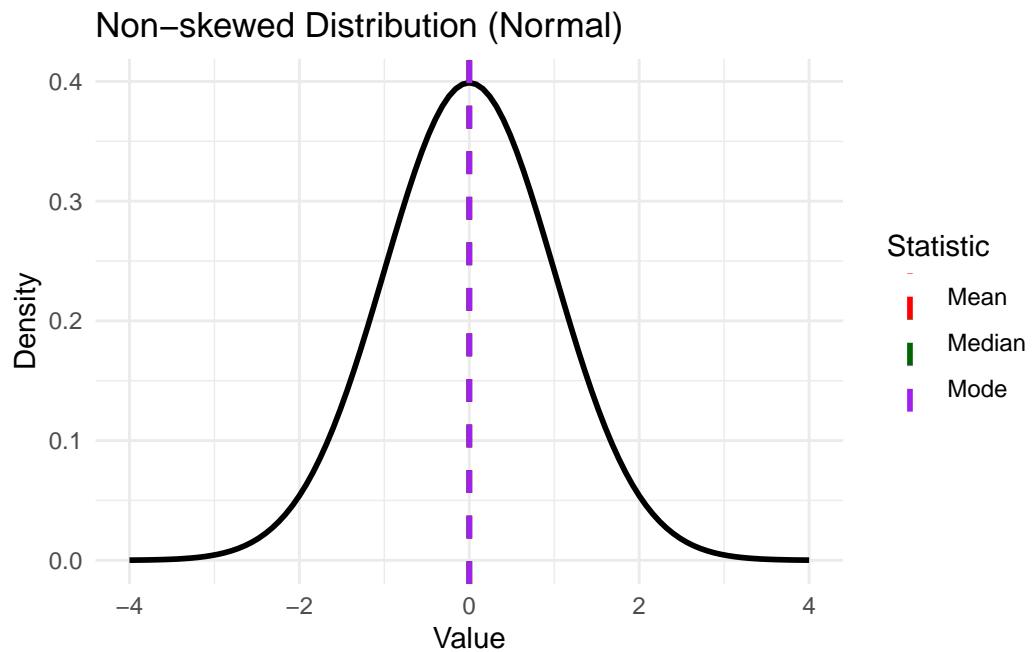
```

# -----
# Normal distribution
# -----
p1 <- ggplot() +
  stat_function(fun = dnorm, size = 1, color = "black") +
  geom_vline(
    data = lines_normal,
    aes(xintercept = value, color = Statistic),
    linetype = "dashed",
    size = 1
  ) +
  scale_color_manual(values = cols) +
  scale_x_continuous(limits = c(-4, 4)) +
  labs(
    title = "Non-skewed Distribution (Normal)",
    x = "Value", y = "Density", color = "Statistic"
  ) +
  theme_minimal()

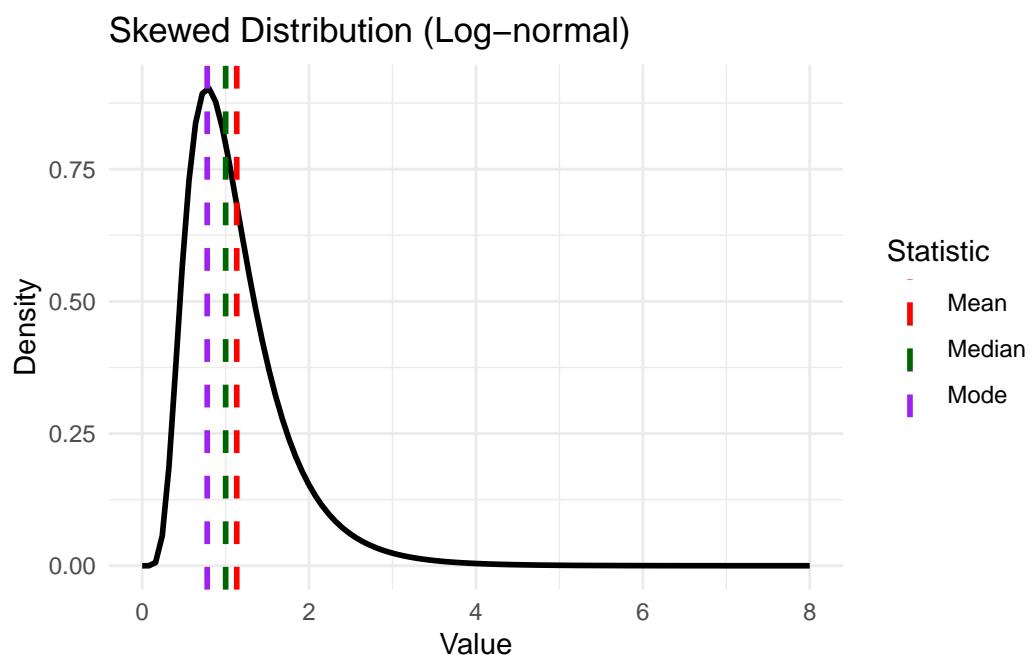
# -----
# Log-normal distribution: LN(0, 0.5)
# -----
p2 <- ggplot() +
  stat_function(
    fun = function(x) dlnorm(x, meanlog = 0, sdlog = 0.5),
    size = 1,
    color = "black"
  ) +
  geom_vline(
    data = lines_lognormal,
    aes(xintercept = value, color = Statistic),
    linetype = "dashed",
    size = 1
  ) +
  scale_color_manual(values = cols) +
  scale_x_continuous(limits = c(0, 8)) +
  labs(
    title = "Skewed Distribution (Log-normal)",
    x = "Value", y = "Density", color = "Statistic"
  ) +
  theme_minimal()

```

p1



p2



Why use mean?

The mean is widely used in statistics and data analysis for several reasons:

1. Mathematical properties: The mean has desirable mathematical properties, such as linearity, which makes it easier to work with in various statistical analyses and models.
2. Sensitivity to all values: The mean takes into account all values in the dataset, providing a comprehensive measure of central tendency. It is also a scaled version of the total, which is often an interest
3. Foundation for other statistical measures: The mean serves as the basis for many other statistical measures, such as variance and standard deviation, which are essential for understanding the spread and variability of data.
4. Mean minimizes the sum of squared deviations: The mean is the value that minimizes the sum of squared deviations (i.e., the expected penalty by choosing one value) from itself, making it a natural choice for summarizing data.
5. May contains full information: In some distributions (e.g., bernoulli distribution), the mean contains all the information about the distribution, making it a sufficient statistic for inference.

The **variance** of a random variable Y measures the spread or dispersion of the distribution, and is defined as

$$\text{Var}(Y) = E[(Y - E(Y))^2] = E[Y^2] - E^2[Y].$$

The standard deviation is the square root of the variance, denoted as $\text{SD}(Y) = \sqrt{\text{Var}(Y)}$.

The **quantile** of order α of a random variable Y is defined as the value y_α such that

$$\Pr(Y \leq y_\alpha) = F(y_\alpha) = \alpha$$

for $0 < \alpha < 1$.

For example, the median is the quantile of order 0.5, denoted as $y_{0.5}$, which satisfies $\Pr(Y \leq y_{0.5}) = 0.5$. Also, $(y_{0.025}, y_{0.975})$ and $(y_{0.25}, y_{0.75})$ contains 95% and 50% of the mass of the distribution, respectively.

2.5 Joint Disitribution

2.5.1 Discrete random variables

Let Y_1 and Y_2 be two random variables with possible values in \mathcal{Y}_1 and \mathcal{Y}_2 , respectively. The **joint distribution** of Y_1 and Y_2 describes the probability of various combinations of values

that (Y_1, Y_2) can take.

Joint beliefs about Y_1 and Y_2 can be represented with probabilities. For example, for subsets $A \subset \mathcal{Y}_1$ and $B \subset \mathcal{Y}_2$, $\Pr(\{Y_1 \in A\} \cap \{Y_2 \in B\})$ represents our belief that Y_1 takes a value in A and Y_2 takes a value in B . The *joint pdf* or *joint density* of Y_1 and Y_2 is defined as

$$p_{Y_1 Y_2}(y_1, y_2) = \Pr(\{Y_1 = y_1\} \cap \{Y_2 = y_2\}), \text{ for } y_1 \in \mathcal{Y}_1, y_2 \in \mathcal{Y}_2.$$

The *marginal density* of Y_1 can be computed from the joint density:

$$\begin{aligned} p_{Y_1}(y_1) &\equiv \Pr(Y_1 = y_1) \\ &= \sum_{y_2 \in \mathcal{Y}_2} \Pr(\{Y_1 = y_1\} \cap \{Y_2 = y_2\}) \\ &\equiv \sum_{y_2 \in \mathcal{Y}_2} p_{Y_1 Y_2}(y_1, y_2) \end{aligned}$$

The *conditional density* of Y_2 given $\{Y_1 = y_1\}$ can be computed from the joint density and the marginal density:

$$\begin{aligned} p_{Y_2|Y_1}(y_2 | y_1) &= \frac{\Pr(\{Y_1 = y_1\} \cap \{Y_2 = y_2\})}{\Pr(Y_1 = y_1)} \\ &= \frac{p_{Y_1 Y_2}(y_1, y_2)}{p_{Y_1}(y_1)}. \end{aligned}$$

You should be able to see that

- $\{p_{Y_1}, p_{Y_2|Y_1}\}$ can be derived from $p_{Y_1 Y_2}$,
- $\{p_{Y_2}, p_{Y_1|Y_2}\}$ can be derived from $p_{Y_1 Y_2}$
- $p_{Y_1 Y_2}$ can be derived from $\{p_{Y_1}, p_{Y_2|Y_1}\}$
- $p_{Y_1 Y_2}$ can be derived from $\{p_{Y_2}, p_{Y_1|Y_2}\}$

BUT

- $p_{Y_1 Y_2}$ cannot be derived from $\{p_{Y_1}, p_{Y_2}\}$.

The subscripts of density functions are often dropped, in which case the type of density function is determined by the arguments. For example,

- $p(y_1, y_2) = p_{Y_1 Y_2}(y_1, y_2)$ is the joint density of Y_1 and Y_2 ,
- $p(y_1) = p_{Y_1}(y_1)$ is the marginal density of Y_1
- $p(y_2 | y_1) = p_{Y_2|Y_1}(y_2 | y_1)$ is the conditional density of Y_2 given $\{Y_1 = y_1\}$, and so on.

Suppose a sociological study reports the following joint distribution of parents' education level and children's income level in a population.

Joint distribution of education and income Suppose a sociological study reports the following **joint distribution of parents' education level and children's income level** in a population as shown in the Table below

Parent \ Child	Low Income	Middle Income	High Income
High School or Less	0.18	0.22	0.10
College	0.08	0.20	0.12
Graduate School	0.04	0.06	0.10

Suppose we randomly sample a **parent–child pair** from this population.

Let

- Y_1 be the parent's education level
- Y_2 be the child's income level

We are interested in the conditional probability that the child has **high income**, given that the parent has a **college education**.

We may answer this question using the conditional probability formula:

$$\Pr(Y_2 = \text{High Income} | Y_1 = \text{College}) = \frac{\Pr(Y_2 = \text{High Income} \cap Y_1 = \text{College})}{\Pr(Y_1 = \text{College})}$$

From the table,

$$\Pr(Y_2 = \text{High Income} \cap Y_1 = \text{College}) = 0.12$$

$$\Pr(Y_1 = \text{College}) = 0.08 + 0.20 + 0.12 = 0.40$$

Therefore,

$$\Pr(Y_2 = \text{High Income} | Y_1 = \text{College}) = \frac{0.12}{0.40} = 0.30$$

Thus, our conclusion from the table is, among children whose parents have a college education, **30%** attain high income.

2.5.2 Continuous random variables

Let Y_1 and Y_2 be two continuous random variables with possible values in \mathcal{Y}_1 and \mathcal{Y}_2 , respectively. The **joint distribution** of Y_1 and Y_2 describes the probability of various combinations of values that (Y_1, Y_2) can take. We again work with the cumulative distribution function (cdf). The definition is given as follows.

Given a continuous joint cdf $F_{Y_1 Y_2}(y_1, y_2)$, there is a function p_{Y_1, Y_2} such that

$$F_{Y_1, Y_2}(a, b) = \int_{-\infty}^a \int_{-\infty}^b p_{Y_1, Y_2}(y_1, y_2) dy_2 dy_1,$$

and $p_{Y_1, Y_2}(y_1, y_2)$ is called the *joint density function* of Y_1 and Y_2 .

Similar to the discrete case, we can derive marginal and conditional densities from the joint density as

- Marginal density of Y_1 :

$$p_{Y_1}(y_1) = \int_{\mathcal{Y}_2} p_{Y_1, Y_2}(y_1, y_2) dy_2,$$

- Conditional density of Y_2 given $\{Y_1 = y_1\}$:

$$p_{Y_2|Y_1}(y_2 | y_1) = \frac{p_{Y_1, Y_2}(y_1, y_2)}{p_{Y_1}(y_1)}.$$

Think about why $p_{Y_2|Y_1}(y_2 | y_1)$ is an actual pdf.

2.5.3 Mixed continuous and discrete variables

It is possible to have joint distributions involving both discrete and continuous random variables. For example, let Y_1 be a discrete random variable taking values in \mathcal{Y}_1 and Y_2 be a continuous random variable taking values in \mathcal{Y}_2 . The joint distribution of Y_1 and Y_2 can be described by the joint density function $p_{Y_1, Y_2}(y_1, y_2)$, which gives the probability that Y_1 takes the value y_1 and Y_2 takes a value in an infinitesimal interval around y_2 . One such as example is that Y_1 is a binary variable indicating the presence or absence of a disease, and Y_2 is a continuous variable representing the severity of symptoms. Suppose we define

- Marginal density p_{Y_1} from our belief $\Pr(Y_1 = y_1)$
- a conditional density $p_{Y_2|Y_1}$ from $\Pr(Y_2 \leq y_2 | Y_1 = y_1) \doteq F_{Y_2|Y_1}(y_2 | y_1)$.

Then, the joint density can be derived as

$$p_{Y_1, Y_2}(y_1, y_2) = p_{Y_1}(y_1)p_{Y_2|Y_1}(y_2 | y_1),$$

and the probability can be calculated as

$$\Pr(Y_1 \in A, Y_2 \in B) = \int_{y_2 \in B} \left\{ \sum_{y_1 \in A} p_{Y_1, Y_2}(y_1, y_2) \right\} dy_2.$$

2.5.4 Bayes' rule and parameter estimation

Let

- θ : proportion of people in a large population who have a certain characteristic.
- Y : number of people in a small random sample from the population who have the characteristic

Then, in this case, we may threat θ as continuous random variable taking values in $\Theta = (0, 1)$, and Y as a discrete random variable taking values in $\mathcal{Y} = \{0, 1, 2, \dots, n\}$, where n is the sample size. *Bayesian estimation of the parameter θ* derives from the calculate of $p(\theta | y)$ where y is the observed value of Y . In Bayesian, this calculation first requires that we have a joint density $p(y, \theta)$ representing our belief about θ and the survey outcome Y . Often, it is natural to construct this joint density from

- $p(\theta)$: our prior belief about θ before seeing the data, and
- $p(y | \theta)$: belief about Y given θ , often called the likelihood function.

Once we observed $\{Y = y\}$, we need to compute our updated belief about θ , represented by the **posterior density** $p(\theta | y)$ as

$$p(\theta | y) = \frac{p(\theta, y)}{p(y)} = \frac{p(y | \theta)p(\theta)}{p(y)} = \frac{p(y | \theta)p(\theta)}{\int_{\Theta} p(y | \theta)p(\theta)d\theta}.$$

If we have two values θ_1 and θ_2 in Θ that may be true, then the ratio of their posterior densities is given by

$$\frac{p(\theta_1 | y)}{p(\theta_2 | y)} = \frac{p(y | \theta_1)p(\theta_1)/p(y)}{p(y | \theta_2)p(\theta_2)/p(y)} = \frac{p(y | \theta_1)p(\theta_1)}{p(y | \theta_2)p(\theta_2)}.$$

i Note

From this calculation, we notice when we are calculating the relative posterior probability between two parameter values **we do not need** calculate $p(y)$ out.

Another way to think about this is, for a function of θ ,

$$p(\theta | y) \propto p(y | \theta)p(\theta).$$

i Note

We will see that the numerator is the important part, while the denominator is just a normalizing constant to make sure the posterior density integrates to 1.

2.6 Independence Random Variables

Let Y_1, \dots, Y_n be random variables with joint density $p(y_1, \dots, y_n)$, and θ is the parameter describe the conditions under which the random variables are generated. We say that Y_1, \dots, Y_n are conditionally independent given θ if every collection of n sets $\{A_1, \dots, A_n\}$ satisfies

$$\Pr(Y_1 \in A_1, \dots, Y_n \in A_n | \theta) = \prod_{i=1}^n \Pr(Y_i \in A_i | \theta).$$

If we have independence property, then

$$\Pr(Y_i \in A_i | \theta, Y_j \in A_j) = \Pr(Y_i \in A_i | \theta),$$

so the conditional independence can be interpreted as meaning that Y_j gives no additional information about Y_i once we know θ . Also, under independence, the joint density can be factorized as

$$p(y_1, \dots, y_n | \theta) = \prod_{i=1}^n p_{Y_i}(y_i | \theta).$$

If the samples are also identically distributed, meaning that each Y_i has the same marginal density $p_Y(y | \theta)$, then the joint density can be further simplified as

$$p(y_1, \dots, y_n | \theta) = \prod_{i=1}^n p_Y(y_i | \theta).$$

In this case , we say that Y_1, \dots, Y_n are **independent and identically distributed** (i.i.d.) given θ , with notation

$$Y_1, \dots, Y_n | \theta \stackrel{i.i.d.}{\sim} p_Y(y | \theta).$$

2.7 Exchangeability

A sequence of random variables Y_1, Y_2, \dots, Y_n is **exchangeable** if for any permutation π of the indices $\{1, 2, \dots, n\}$, we have

$$p(y_1, y_2, \dots, y_n) = p(y_{\pi(1)}, y_{\pi(2)}, \dots, y_{\pi(n)}).$$

In other words, the joint density of an exchangeable sequence is invariant to the order of the random variables. That is, the labels contains no information about the outcome.

Suppose a factory produces a large batch of items. Each item may be either **defective** or **non-defective**.

Let

$$Y_i = \begin{cases} 1, & \text{if the } i\text{th inspected item is defective,} \\ 0, & \text{otherwise.} \end{cases}$$

We inspect $n = 10$ items chosen at random from the batch and record Y_1, Y_2, \dots, Y_{10} .

Consider the following three observed sequences:

1. $p(1, 0, 1, 0, 1, 0, 0, 1, 0, 1)$
2. $p(0, 1, 0, 1, 0, 1, 1, 0, 0, 1)$
3. $p(1, 1, 0, 0, 1, 0, 1, 0, 0, 1)$

Each sequence contains **5 defective items** and **5 non-defective items**.

Question: Is there a reason to assign these three sequences *different probabilities*?

If the inspection order conveys no additional information about quality, then *only the number of defective items matters*, not their positions in the sequence. This motivates the concept of exchangeability.

2.7.1 Independence versus dependence

Consider the probability assignments

$$\begin{cases} \Pr(Y_{10} = 1) = a, \\ \Pr(Y_{10} = 1 \mid Y_1 = \dots = Y_9 = 1) = b. \end{cases}$$

If $a \neq b$, then Y_{10} is **not independent** of Y_1, \dots, Y_9 .

However, lack of independence does **not** imply lack of exchangeability.

Question: should we have $a = b$, $a > b$ or $a < b$?

2.7.2 A latent-parameter model

Suppose the defect rate θ of the factory is unknown.

Conditional on θ ,

$$Y_1, \dots, Y_{10} \mid \theta \sim \text{i.i.d. Bernoulli}(\theta).$$

Then

$$\Pr(Y_1 = y_1, \dots, Y_{10} = y_{10} \mid \theta) = \theta^{\sum y_i} (1 - \theta)^{10 - \sum y_i}.$$

If our uncertainty about θ is described by a prior distribution $p(\theta)$, the marginal joint distribution is

$$p(y_1, \dots, y_{10}) = \int \theta^{\sum y_i} (1 - \theta)^{10 - \sum y_i} p(\theta) d\theta.$$

This probability depends **only on the number of defective items**, not their order.

Thus, we have exchangeability, even though the Y_i are not independent under this model of belief.

Conditional i.i.d. given a latent parameter implies marginal exchangeability. That is, if $\theta \sim p(\theta)$ and Y_1, \dots, Y_n are conditionally i.i.d. given θ , then Y_1, \dots, Y_n (i.e., unconditional on θ) are exchangeable.

For the Proof, see page 28 in Hopf (2009).

2.8 de Finetti's Theorem

As of now, we have seen that conditional i.i.d. given a latent parameter implies marginal exchangeability. For example,

$$\left\{ \begin{array}{l} Y_1, \dots, Y_n \mid \theta \stackrel{\text{i.i.d.}}{\sim} \\ \theta \sim p(\theta) \end{array} \right. \implies Y_1, \dots, Y_n \text{ are exchangeable.}$$

The converse is also true, as stated in de Finetti's theorem.

Let $Y_i \in \mathcal{Y}$ for all $i \in \{1, 2, \dots, n\}$ be an exchangeable sequence of random variables. Then, there exists a parameter space Θ and a prior distribution $p(\theta)$ on Θ such that the joint distribution of Y_1, \dots, Y_n can be represented as

$$p(y_1, \dots, y_n) = \int_{\Theta} \left\{ \prod_{i=1}^n p_Y(y_i \mid \theta) \right\} p(\theta) d\theta,$$

where $p_Y(y | \theta)$ is a probability density function on \mathcal{Y} for each $\theta \in \Theta$. The prior and sampling model depend on the form of the belief model $p(y_1, \dots, y_n)$.

The probability distribution $p(\theta)$ represents our belief about the outcomes $\{Y_1, Y_2, \dots, Y_n\}$, induced by our belief model $p(y_1, \dots, y_n)$. That is,

- $p(\theta)$ represents our belief about $\lim_{n \rightarrow \infty} \sum Y_i/n$ in the binary sense
- $p(\theta)$ represents our belief about $\lim_{n \rightarrow \infty} \sum(Y_i \leq c)/n$ for each c in the general case.

The main idea of this and the previous section is as follows

$$Y_1, \dots, Y_n \mid \theta \stackrel{\text{i.i.d.}}{\sim} p(\cdot | \theta), \quad \Leftrightarrow \quad Y_1, \dots, Y_n \text{ are exchangeable for all } n.$$

$$\theta \sim p(\theta)$$

Question: When is the condition of “exchangeability for all n ” reasonable?

- Have exchangeability and repeatability
 - Exchangeability holds if the labels convey no information
 - repeatability holds includes the follows
 1. Y_1, \dots, Y_n are outcomes of a repeatable experiment
 2. Y_1, \dots, Y_n are sampled from a finite population **with replacement**
 3. Y_1, \dots, Y_n are sampled from an infinite population without replacement.

i In large finite population

Note, if Y_1, \dots, Y_n are exchangeable and sampled from a finite population of size N that is way bigger than n without replacement, then they can be modelled as *approximate* being conditional i.i.d.

This Chapter follows closely with Chapter 2 in Hoff (2009).

3 Bayesian Inference for single parameter models

Leading objectives:

Understand how to perform Bayesian inference on a single parameter model.

- Binomial model with given n
- Poission model
- Exponential family

Recall the important ingredients of Bayesian inference:

1. **Prior distribution:** $\pi(\theta)$
2. **Likelihood function:** $p(y | \theta)$
3. **Posterior distribution:** $p(\theta | y) \propto p(y | \theta)\pi(\theta)$

3.1 Three basic ingredients of Bayesian inference

3.1.1 Prior

The prior distribution encodes our beliefs about the parameter θ *before* conduct any experiments.

i Prior and Data are independent

Note that, the prior distribution is independent of the data. It represents our knowledge or beliefs about the parameter before seeing the data.

How do we choose a prior?

1. **Informative priors:** Based on previous studies or expert knowledge
2. **Weakly informative priors:** Provide some regularization without dominating the data
3. **Non-informative priors:** Attempt to be “objective” (e.g., uniform, Jeffreys prior)

3.1.2 Likelihood

The likelihood function represents the probability of observing the data given the parameter θ . It can be derived from the assumed statistical model for the data or experiment, i.e., $y \sim p(y | \theta)$, or we can estimate this non-parametrically (i.e., without assuming the underlying distribution is the one we know.).

i Likelihood is NOT a probability distribution for θ

Note that, the likelihood function is not a probability distribution for θ itself. It is a function of θ for fixed data y .

3.1.3 Posterior

The posterior distribution combines the prior and likelihood to update our beliefs about θ after observing the data. It is given by Bayes' theorem:

$$p(\theta | y) = \frac{p(y | \theta)\pi(\theta)}{p(y)},$$

where $p(y) = \int p(y | \theta)\pi(\theta)d\theta$ is the marginal likelihood or evidence.

3.1.4 An simple example

Examples:

- Beta prior + Binomial likelihood \rightarrow Beta posterior
- Normal prior + Normal likelihood (known variance) \rightarrow Normal posterior
- Gamma prior + Poisson likelihood \rightarrow Gamma posterior

Advantages: - Analytical posteriors (no numerical integration needed) - Interpretable parameters - Computationally efficient

Limitations:

- May not reflect true prior beliefs
- Modern computing makes non-conjugate priors feasible

Let's look a simple example to illustrate the convenience of conjugate priors. Consider a Binomial model with unknown success probability θ and known number of trials n . We can use a Beta prior for θ .

Suppose we have a Binomial model with known number of trials n and unknown success probability θ . We can use a Beta prior for θ .

- **Prior:** $\theta \sim \text{Beta}(\alpha, \beta)$
- **Likelihood:** $y | \theta \sim \text{Binomial}(n, \theta)$

The derivation of the posterior is as follows:

$$p(y | \theta) = \binom{n}{y} \theta^y (1 - \theta)^{n-y},$$

$$\pi(\theta) = \frac{\theta^{\alpha-1} (1 - \theta)^{\beta-1}}{B(\alpha, \beta)},$$

where $B(\alpha, \beta)$ is the Beta function. Then the posterior is proportional to:

$$p(\theta | y) \propto p(y | \theta) \pi(\theta) \propto \theta^{y+\alpha-1} (1 - \theta)^{n-y+\beta-1}.$$

This is the kernel of a Beta distribution with parameters $(\alpha + y, \beta + n - y)$. Thus, the posterior distribution is:

$$\theta | y \sim \text{Beta}(\alpha + y, \beta + n - y).$$

Thus, the **Posterior** is $\theta | y \sim \text{Beta}(\alpha + y, \beta + n - y)$.

3.2 Happiness Data – the first example of Bayesian inference procedure

We study Bayesian inference for a binomial proportion θ when the sample size n is fixed. In this example, we want to see what is the procedure of doing Bayesian inference

In the 1998 General Social Survey, each female respondent aged 65 or over was asked whether she was generally happy.

Define the response variable

$$Y_i = \begin{cases} 1, & \text{if respondent } i \text{ reports being generally happy,} \\ 0, & \text{otherwise,} \end{cases} \quad i = 1, \dots, n,$$

where $n = 129$.

Because we lack information that distinguishes individuals, it is reasonable to treat the responses as **exchangeable**.

That is, before observing the data, the labels or ordering of respondents carry no information.

Since the sample size n is small relative to the population size N of senior women, results from the previous chapter justify the following modeling approximation.

Modeling Assumptions: Our beliefs about (Y_1, \dots, Y_{129}) are described by:

- **An unknown population proportion**

$$\theta = \frac{1}{N} \sum_{i=1}^N Y_i,$$

where θ represents the proportion of generally happy individuals in the population.

- **A sampling model given θ**

Conditional on θ , the responses Y_1, \dots, Y_{129} are independent and identically distributed Bernoulli random variables with

$$\Pr(Y_i = 1 | \theta) = \theta.$$

Given the population proportion θ , each respondent independently reports being happy with probability θ .

Likelihood: Under this model, the probability of observing data $\{y_1, \dots, y_{129}\}$ given θ is

$$p(y_1, \dots, y_{129} | \theta) = \theta^{\sum_{i=1}^{129} y_i} (1 - \theta)^{129 - \sum_{i=1}^{129} y_i}.$$

This expression depends on the data only through the sufficient statistic

$$S = \sum_{i=1}^{129} Y_i,$$

the total number of respondents who report being generally happy.

For the happiness data,

$$S = 118,$$

so the likelihood simplifies to

$$p(y_1, \dots, y_{129} | \theta) = \theta^{118} (1 - \theta)^{11}.$$

Q: Which prior to be used?

A prior distribution is **conjugate** to a likelihood if the posterior distribution belongs to the same family as the prior. For the binomial likelihood, the **Beta distribution** is conjugate. But we have another choice of prior, to use *non-informative prior*.

A Uniform Prior Distribution: Suppose our prior information about θ is very weak, in the sense that all subintervals of $[0, 1]$ with equal length are equally plausible.

Symbolically, for any $0 \leq a < b < b + c \leq 1$,

$$\Pr(a \leq \theta \leq b) = \Pr(a + c \leq \theta \leq b + c).$$

This implies a **uniform prior**:

$$\pi(\theta) = 1, \quad 0 \leq \theta \leq 1.$$

Posterior Distribution: Bayes' rule gives

$$p(\theta | y_1, \dots, y_{129}) = \frac{p(y_1, \dots, y_{129} | \theta) \pi(\theta)}{p(y_1, \dots, y_{129})}.$$

With a uniform prior, this reduces to

$$p(\theta | y_1, \dots, y_{129}) \propto \theta^{118} (1 - \theta)^{11}.$$

Key idea: with a uniform prior, the posterior has the **same shape** as the likelihood.

To obtain a proper probability distribution, we must normalize.

Normalizing Constant and the Beta Distribution: Using the identity

$$\int_0^1 \theta^{a-1} (1 - \theta)^{b-1} d\theta = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

we find

$$p(y_1, \dots, y_{129}) = \frac{\Gamma(119)\Gamma(12)}{\Gamma(131)}.$$

Therefore, the posterior density is

$$p(\theta | y_1, \dots, y_{129}) = \frac{\Gamma(131)}{\Gamma(119)\Gamma(12)} \theta^{119-1} (1 - \theta)^{12-1}.$$

That is,

$$\theta | y \sim \text{Beta}(119, 12).$$

Recall that, a random variable $\theta \sim \text{Beta}(a, b)$ distribution if

$$\pi(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1 - \theta)^{b-1}.$$

For $\theta \sim \text{Beta}(a, b)$, the expectation (i.e., mean or the first moment) is $\mathbb{E}(\theta) = \frac{a}{a+b}$, and the variance is $\text{Var}(\theta) = \frac{ab}{(a+b)^2(a+b+1)}$.

In our example, the happiness data, the posterior distribution is

$$\theta | y \sim \text{Beta}(119, 12).$$

Thus, the posterior mean is $\mathbb{E}(\theta | y) = 0.915$, and the posterior standard deviation is $\text{sd}(\theta | y) = 0.025$.

These summaries quantify both our **best estimate** of the population proportion and our **remaining uncertainty** after observing the data.

3.2.1 Inference about exchangeable binary data

Posterior Inference under a Uniform Prior

Suppose $Y_1, \dots, Y_n \mid \theta \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(\theta)$, and we place a uniform prior on θ . The posterior distribution of θ given the observed data y_1, \dots, y_n is proportional to

$$\begin{aligned} p(\theta \mid y_1, \dots, y_n) &= \frac{p(y_1, \dots, y_n \mid \theta)\pi(\theta)}{p(y_1, \dots, y_n)} \\ &= \theta^{\sum_i y_i} (1-\theta)^{n-\sum_i y_i} \times \frac{\pi(\theta)}{p(y_1, \dots, y_n)} \\ &\propto \theta^{\sum_i y_i} (1-\theta)^{n-\sum_i y_i}. \end{aligned}$$

Consider two parameter values θ_a and θ_b . The ratio of their posterior densities is

$$\begin{aligned} \frac{p(\theta_a \mid y_1, \dots, y_n)}{p(\theta_b \mid y_1, \dots, y_n)} &= \frac{\theta_a^{\sum_i y_i} (1-\theta_a)^{n-\sum_i y_i} \times p(\theta_a) / p(y_1, \dots, y_n)}{\theta_b^{\sum_i y_i} (1-\theta_b)^{n-\sum_i y_i} \times p(\theta_b) / p(y_1, \dots, y_n)} \\ &= \left(\frac{\theta_a}{\theta_b} \right)^{\sum_i y_i} \left(\frac{1-\theta_a}{1-\theta_b} \right)^{n-\sum_i y_i} \frac{p(\theta_a)}{p(\theta_b)}. \end{aligned}$$

This expression shows that the data affect the posterior distribution **only through the sum of the data** $\sum_{i=1}^n y_i$ based on the relative probability density at θ_a to θ_b .

As a result, for any set A , one can show that

$$\Pr(\theta \in A \mid Y_1 = y_1, \dots, Y_n = y_n) = \Pr\left(\theta \in A \mid \sum_{i=1}^n Y_i = \sum_{i=1}^n y_i\right).$$

This means that $\sum_{i=1}^n Y_i$ contains **all the information** in the data relevant for inference about θ . We therefore say that $Y = \sum_{i=1}^n Y_i$ is a **sufficient statistic** for θ . The term *sufficient* is used because knowing $\sum_{i=1}^n Y_i$ is sufficient to carry out inference about θ ; no additional information from the individual observations Y_1, \dots, Y_n is required.

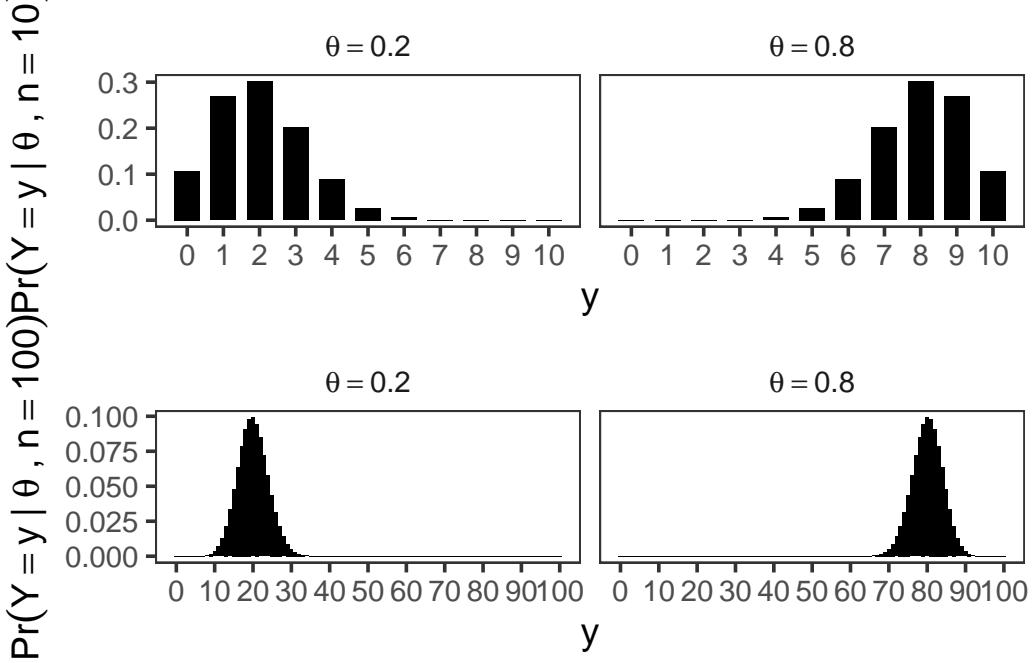
In the case where $Y_1, \dots, Y_n \mid \theta$ are i.i.d. Bernoulli(θ) random variables, the sufficient statistic $Y = \sum_{i=1}^n Y_i$ follows a **binomial distribution** with parameters (n, θ) .

The Binomial Model

Because each Y_i is Bernoulli(θ) and the observations are independent, the sufficient statistic $Y = \sum_{i=1}^n Y_i$ follows a **binomial distribution** with parameters (n, θ) .

That is, $\Pr(Y = y \mid \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$, $y = 0, 1, \dots, n$. For a binomial(n, θ) random variable Y ,

- $\mathbb{E}[Y \mid \theta] = n\theta$,
- $\text{Var}(Y \mid \theta) = n\theta(1 - \theta)$.



Posterior inference under a uniform prior distribution

Having observed $Y = y$ our task is to obtain the posterior distribution of θ . By Bayes' theorem,

$$p(\theta \mid y) = \frac{p(y \mid \theta), \pi(\theta)}{p(y)}.$$

For a binomial model with $Y \sim \text{Binomial}(n, \theta)$, the likelihood is

$$p(y \mid \theta) = \binom{n}{y} \theta^y (1 - \theta)^{n-y}.$$

Therefore,

$$p(\theta \mid y) = \frac{\binom{n}{y} \theta^y (1 - \theta)^{n-y} \pi(\theta)}{p(y)} = c(y) \theta^y (1 - \theta)^{n-y} \pi(\theta),$$

where $c(y)$ is a normalizing constant that depends only on y , not on θ . When using the uniform

distribution, $\pi(\theta)$, we can calculate $c(y)$ easily as

$$\begin{aligned} 1 &= \int_0^1 c(y)\theta^y(1-\theta)^{n-y}d\theta \\ &= c(y) \int_0^1 \theta^y(1-\theta)^{n-y}d\theta \quad . \\ &= c(y) \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)} \end{aligned}$$

Hence, $c(y) = \Gamma(n+2)/\{\Gamma(y+1)\Gamma(n-y+1)\}$, and the posterior distribution is

$$\begin{aligned} p(\theta | y) &= \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\theta^y(1-\theta)^{n-y} \\ &= \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\theta^{(y+1)-1}(1-\theta)^{(n-y+1)-1}, \end{aligned}$$

Which is exactly the beta($y+1, n-y+1$). In the happiness example, we have $n = 129$ and $Y = \sum Y_i = 118$, so the posterior distribution is beta(119, 12), written as

$$n = 129, Y \equiv \sum Y_i = 118 \quad \Rightarrow \quad \theta | \{Y = 118\} \sim \text{beta}(119, 12).$$

This confirms the sufficiency result for this model and prior distribution, by showing that if $\sum y_i = y = 118$, $p(\theta | y_1, \dots, y_n) = p(\theta | y) = \text{beta}(119, 12)$. That is, the information contained in $\{Y_1 = y_1, \dots, Y_n = y_n\}$ is the same as the information contained in $\{Y = y\}$, where $Y = \sum Y_i$ and $y = \sum y_i$. This show the posterior when we use **uniform prior**. One may ask, what if we use a different prior?

Posterior distributions under beta prior distributions

The uniform prior distribution has $\pi(\theta) = 1$ for all $\theta \in [0, 1]$. This distribution can be thought of as a beta prior distribution with parameters $a = 1, b = 1$

$$\pi(\theta) = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\theta^{1-1}(1-\theta)^{1-1} = \frac{1}{1 \times 1}1 \times 1 = 1$$

for all $\theta \in [0, 1]$.

The gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.$$

It satisfies the following properties:

- $\Gamma(n) = (n-1)!$ for any positive integer n .
- $\Gamma(x+1) = x\Gamma(x)$ for any $x > 0$.

- $\Gamma(1/2) = \sqrt{\pi}$.
- $\Gamma(1) = 1$ by convention.

Now, from the previous part, recall that we have,

$$\text{if } \left\{ \begin{array}{l} \theta \sim \text{beta}(1, 1) \text{ (uniform)} \\ Y \sim \text{binomial}(n, \theta) \end{array} \right\}, \text{ then } \{\theta | Y = y\} \sim \text{beta}(1 + y, 1 + n - y).$$

To get the posterior distribution under a general beta prior distribution, we just need to add the number of 1's to the α parameter and the number of 0's to the β parameter. To see this, assume $\theta \sim \text{beta}(\alpha, \beta)$, and $Y | \theta \sim \text{binomial}(n, \theta)$. Then, once we observed $\{Y = y\}$, by Bayes' theorem, the posterior distribution is

$$\begin{aligned} p(\theta | y) &= \frac{\pi(\theta)p(y | \theta)}{p(y)} \\ &= \frac{1}{p(y)} \times \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1} \times \binom{n}{y} \theta^y (1-\theta)^{n-y} \\ &= c(n, y, a, b) \times \theta^{a+y-1} (1-\theta)^{b+n-y-1} \\ &\propto \beta(a+y, b+n-y). \end{aligned}$$

i One-to-one correspondence between the distribution

Note that, there is a one-to-one correspondence between the prior distribution parameters and the posterior distribution parameters. Two distributions are said to be the same if

- Their CDFs are the same.
- Their PDFs are the same.
- All of their moments are the same. This implies that they are equal if and only if the moment generating function or the probability generating functions are the same.

We have seen the beta-binomial example twice, which is an example of **conjugate prior**, let's definite this formally,

A class \mathcal{P} of prior distribution for θ is said **conjugate** for the likelihood function $p(y | \theta)$ if for every prior distribution $\pi(\theta) \in \mathcal{P}$, the corresponding posterior distribution $p(\theta | y)$ is also in \mathcal{P} , that is

$$\pi(\theta) \in \mathcal{P} \Rightarrow p(\theta | y) \in \mathcal{P}.$$

i Note

Conjugate priors simplify posterior calculations, but they may not accurately reflect genuine prior beliefs. Still, mixtures of conjugate priors offer substantially greater flexibility while remaining computationally tractable.

If the likelihood $\theta \mid \{Y = y\} \sim \text{beta}(a + y, b + n - y)$, recall that

- $E[\theta \mid y] = \frac{a+y}{a+b+n}$
- $\text{mode}[\theta \mid y] = \frac{a+y-1}{a+b+n-2}$
- $\text{Var}[\theta \mid y] = \frac{E[\theta|y]E[1-\theta|y]}{a+b+n+1}$

The posterior mean can be expressed as a weighted average of the prior mean and the maximum likelihood estimate (MLE) of θ :

$$\begin{aligned} E[\theta \mid y] &= \frac{a+y}{a+b+n} \\ &= \frac{a+b}{a+b+n} \times \frac{a}{a+b} + \frac{n}{a+b+n} \times \frac{y}{n} \\ &= \frac{a+b}{a+b+n} \times \text{prior expectation} + \frac{n}{a+b+n} \times \text{data mean} \end{aligned}$$

For this model and prior distribution, the posterior expectation (also known as the posterior mean) can be expressed as a weighted average of the prior expectation and the sample mean. The weights are proportional to the prior sample size $a + b$ and the observed sample size n , respectively. This representation leads to a natural interpretation of the Beta prior parameters as prior data:

- $a \approx \text{"prior \# of 1's,"}$
- $b \approx \text{"prior \# of 0's,"}$
- $a + b \approx \text{"prior sample size."}$

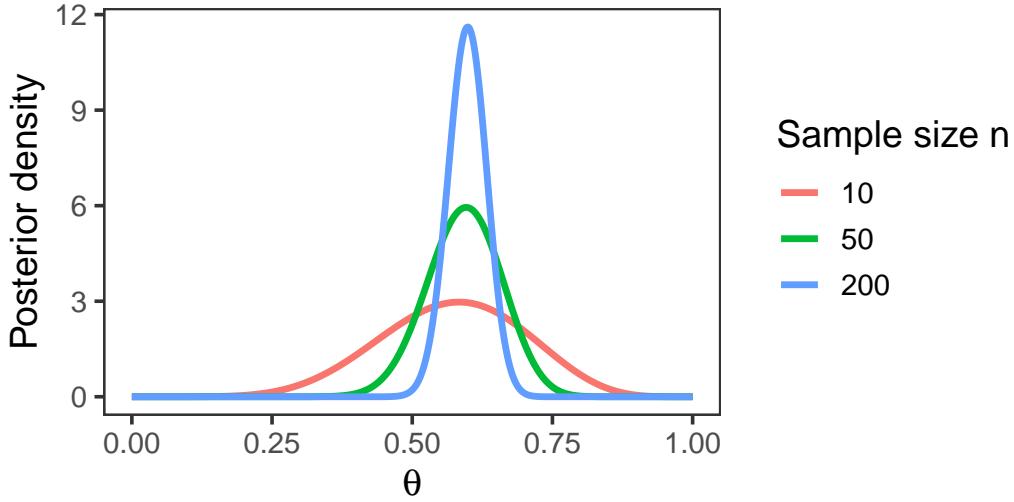
When $n \gg a + b$, it is reasonable to expect that most of the information about θ should come from the data rather than from the prior distribution. This intuition is confirmed mathematically. In particular, when $n \gg a + b$,

- $\frac{a+b}{a+b+n} \approx 0,$
- $E[\theta \mid y] \approx \frac{y}{n},$
- $\text{Var}(\theta \mid y) \approx \frac{1}{n} \frac{y}{n} (1 - \frac{y}{n}).$

Thus, in large samples, the posterior mean approaches the sample proportion and the posterior variance shrinks at rate $1/n$, reflecting increasing information from the data.

Effect of sample size on the posterior distribution

Prior: Beta(2,2), observed proportion $\hat{\theta} = 0.6$



Prediction

An important feature of Bayesian inference is the existence of a **predictive distribution** for new observations.

The posterior predictive distribution for a new observation Y_{new} given the observed data y is obtained by integrating over the posterior distribution of θ .

Returning to our notation for binary data, let y_1, \dots, y_n be the observed outcomes from a sample of n binary rvs, and let $\tilde{Y} \in \{0, 1\}$ denote a future observation from the same population that has not yet been observed. The **predictive distribution** of \tilde{Y} is defined as the conditional distribution of \tilde{Y} given the observed data $\{Y_1 = y_1, \dots, Y_n = y_n\}$. For conditionally i.i.d. binary observations, the predictive distribution can be derived by integrating out the unknown parameter θ :

$$\begin{aligned}
 \Pr(\tilde{Y} = 1 | y_1, \dots, y_n) &= \int \Pr(\tilde{Y} = 1, \theta | y_1, \dots, y_n) d\theta \\
 &= \int \Pr(\tilde{Y} = 1 | \theta, y_1, \dots, y_n) p(\theta | y_1, \dots, y_n) d\theta \\
 &= \int p(\theta | y_1, \dots, y_n) \theta d\theta \\
 &= E[\theta | y_1, \dots, y_n] \\
 &= \frac{a + \sum_{i=1}^n y_i}{a + b + n}.
 \end{aligned}$$

Hence, we also have,

$$\Pr(\tilde{Y} = 0 \mid y_1, \dots, y_n) = 1 - \mathbb{E}[\theta \mid y_1, \dots, y_n] = \frac{b + \sum_{i=1}^n (1 - y_i)}{a + b + n}.$$

i Properties of the predictive distribution

1. **It does not depend on any unknown quantities.** If it did, it could not be used to make predictions.
2. **It depends on the observed data.** In particular, \tilde{Y} is not independent of Y_1, \dots, Y_n , because the observed data provide information about θ , which in turn influences \tilde{Y} . If \tilde{Y} were independent of the observed data, learning from data would be impossible.

The uniform prior distribution on $[0,1]$, also known as the Beta(1,1) prior, can be interpreted as containing the same information as a hypothetical prior dataset consisting of one success (“1”) and one failure (“0”).

Under this prior, the posterior predictive probability of a future success is

$$\Pr(\tilde{Y} = 1 \mid Y = y) = \mathbb{E}[\theta \mid Y = y] = \frac{2}{2+n} \cdot \frac{1}{2} + \frac{n}{2+n} \cdot \frac{y}{n}.$$

This expression highlights that the predictive probability is a weighted average of:

- the prior mean $1/2$, and
- the sample proportion y/n ,

with weights proportional to the prior sample size 2 and the observed sample size n , respectively.

The posterior mode under this prior is

$$\text{mode}(\theta \mid Y = y) = \frac{y}{n},$$

where

$$Y = \sum_{i=1}^n Y_i.$$

At first glance, the discrepancy between these two posterior summaries may seem surprising. However, it reflects the fact that different summaries capture different features of the posterior distribution.

To see this clearly, consider the case $Y = 0$. In this case,

$$\text{mode}(\theta \mid Y = 0) = 0,$$

but the predictive probability remains

$$\Pr(\tilde{Y} = 1 \mid Y = 0) = \frac{1}{2+n}.$$

Thus, even when no successes have been observed, the Bayesian predictive distribution assigns a positive probability to a future success due to the prior information. This illustrates how Bayesian prediction naturally balances prior beliefs with observed data.

3.2.2 Confidence Regions: Bayesian v.s. Frequentist

If it often desirable to identify the regions of the parameter space that are likely to contain the true value of the parameter. To do this, after observing the data $Y = y$, we can construct an interval $[\ell(y), u(y)]$ that is likely to contain the true value of θ , i.e., the probability that $\ell(y) < \theta < u(y)$ is large. There are two different ways to interpret this probability, leading to the concepts of **Bayesian coverage** and **frequentist coverage**.

An interval $[\ell(y), u(y)]$, based on the observed data $Y = y$, has $100(1-\alpha)\%$ Bayesian coverage for θ if

$$\Pr(\ell(y) < \theta < u(y) \mid Y = y) = 1 - \alpha.$$

A random interval $[\ell(Y), u(Y)]$ has $100(1-\alpha)\%$ frequentist coverage for θ if, before the data are gathered,

$$\Pr(\ell(Y) < \theta < u(Y) \mid \theta) = 1 - \alpha.$$

i Note

In a sense, the frequentist and Bayesian notions of coverage describe **pre** experimental and **post** experimental perspectives, respectively.

3.3 Frequentist vs Bayesian Coverage

You may recall an important point often emphasized in introductory statistics courses. Suppose we observe data $Y = y$ and compute a frequentist confidence interval

$$[l(y), u(y)].$$

Once the data are observed, the parameter θ is **treated as fixed, not random**. Therefore,

$$\Pr(l(y) < \theta < u(y) \mid \theta) = \begin{cases} 1, & \text{if } \theta \in [l(y), u(y)], \\ 0, & \text{if } \theta \notin [l(y), u(y)]. \end{cases}$$

This highlights a key limitation of frequentist confidence intervals:

They do not admit a post-experimental probability interpretation.

After observing the data, it is *not meaningful*, from a frequentist perspective, to say that there is a 95% probability that θ lies in the computed interval.

What Frequentist Coverage Means

Although this interpretation may feel unsatisfying, frequentist coverage is still useful in many situations. Imagine repeatedly running many independent experiments and constructing a confidence interval for each one.

If each interval procedure has 95% frequentist coverage, then:

About 95% of the intervals will contain the true parameter value.

This is a **long-run, repeated-sampling interpretation**, not a statement about any single observed interval.

Can Bayesian and Frequentist Coverage Agree?

A natural question is whether a confidence interval can simultaneously have:

- a Bayesian interpretation, i.e., a $100(1-\alpha)\%$ posterior probability that θ lies in the interval, and
- approximately $100(1-\alpha)\%$ frequentist coverage.

Hartigan (1966) showed that, for the types of intervals considered in Hopf (2009), an interval that has 95% Bayesian coverage additionally has the property that

$$\Pr(l(Y) < \theta < u(Y) \mid \theta) = 0.95 + \varepsilon_n,$$

where the error term satisfies $|\varepsilon_n| < a/n$ for some constant a . This result implies that, an interval with 95% Bayesian coverage, will also have approximately 95% frequentist coverage, at least asymptotically, as the sample size n grows.

In other words, under suitable conditions, **Bayesian credible intervals and frequentist confidence intervals can agree in large samples**, even though their interpretations are fundamentally different. Keep in mind that most non-Bayesian methods of constructing $100(1-\alpha)\%$ confidence intervals also only achieve their nominal coverage probability asymptotically.

Reminder

This reconciliation is important, but it should not obscure the conceptual distinction:

- frequentist coverage is a *pre-experimental* property of a procedure,
- Bayesian coverage is a *post-experimental* probability statement about θ given the data.

Quantile-Based Credible Intervals

In fact, this is also true for most classical 95% confidence intervals.

For further discussion of the similarities between Bayesian and frequentist intervals, see Severini (1991) and Sweeting (2001).

3.4 Posterior Quantile Intervals

One of the simplest ways to construct a Bayesian credible interval is to use **posterior quantiles**. To form a $100(1 - \alpha)\%$ credible interval for θ , find numbers $\theta_{\alpha/2} < \theta_{1-\alpha/2}$ such that

1. $\Pr(\theta < \theta_{\alpha/2} | Y = y) = \alpha/2,$
2. $\Pr(\theta > \theta_{1-\alpha/2} | Y = y) = \alpha/2,$

where $\theta_{\alpha/2}$ and $\theta_{1-\alpha/2}$ are the $\alpha/2$ and $1 - \alpha/2$ posterior quantiles of θ . By construction,

$$\begin{aligned}\Pr(\theta \in [\theta_{\alpha/2}, \theta_{1-\alpha/2}] | Y = y) &= 1 - \Pr(\theta \notin [\theta_{\alpha/2}, \theta_{1-\alpha/2}] | Y = y) \\ &= 1 - [\Pr(\theta < \theta_{\alpha/2} | Y = y) + \Pr(\theta > \theta_{1-\alpha/2} | Y = y)] \\ &= 1 - \alpha.\end{aligned}$$

Suppose we observe $n = 10$ conditionally independent Bernoulli trials and obtain $Y = 2$ successes. Using a uniform prior for θ , $\theta \sim \text{Beta}(1, 1)$, the posterior distribution is

$$\theta | \{Y = 2\} \sim \text{Beta}(1 + 2, 1 + 8) = \text{Beta}(3, 9).$$

A 95% posterior confidence interval can be obtained from by 2.5% and 97.5% quantiles of this Beta distribution $[\theta_{0.025}, \theta_{0.975}]$. In this case,

$$\theta_{0.025} \approx 0.06, \quad \theta_{0.975} \approx 0.52,$$

so

$$\Pr(0.06 \leq \theta \leq 0.52 | Y = 2) = 0.95.$$

This interval has a direct probabilistic interpretation: **given the observed data**, there is a 95% posterior probability that θ lies in this range.

```
b <- a <- 1 # prior parameter
n <- 10 ; y <- 2 # data
qbeta(c(0.025, 0.975), a + y, b + n - y)
```

[1] 0.06021773 0.51775585

```

a_post <- a + y
b_post <- b + (n - y)

# 95% quantile-based credible interval
ci <- qbeta(c(0.025, 0.975), a_post, b_post)
ci_low <- ci[1]
ci_high <- ci[2]

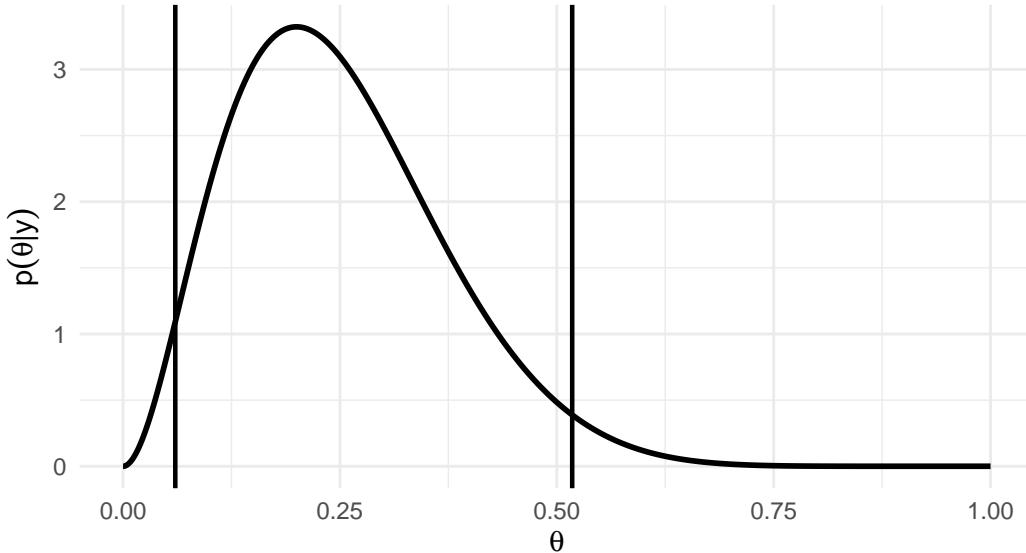
# Grid for plotting posterior density
theta <- seq(0, 1, length.out = 2000)
df <- data.frame(theta = theta, density = dbeta(theta, a_post, b_post))

# Plot: posterior density curve + two vertical CI bars
ggplot(df, aes(x = theta, y = density)) +
  geom_line(linewidth = 1) +
  geom_vline(xintercept = ci_low, linewidth = 0.8) +
  geom_vline(xintercept = ci_high, linewidth = 0.8) +
  labs(
    title = "Beta Posterior with 95% Quantile-Based Credible Interval",
    subtitle = sprintf(
      "Data: n=%d, y=%d | Prior: Beta(%d,%d) | Posterior: Beta(%d,%d) | 95% CI: [%f, %f]",
      n, y, a, b, a_post, b_post, ci_low, ci_high
    ),
    x = expression(theta),
    y = expression(p(theta * "|" * y))
  ) +
  theme_minimal()

```

Beta Posterior with 95% Quantile-Based Credible Interval

Data: n=10, y=2 | Prior: Beta(1,1) | Posterior: Beta(3,9) | 95% CI: [0.060, 0.



Highest posterior density (HPD) region

The Figure above illustrates the posterior distribution of θ for the binomial example with a uniform prior, together with a 95% quantile-based credible interval. Notice an important feature of the plot:

There exist values of θ *outside* the quantile-based interval that have *higher posterior density* than some values *inside* the interval.

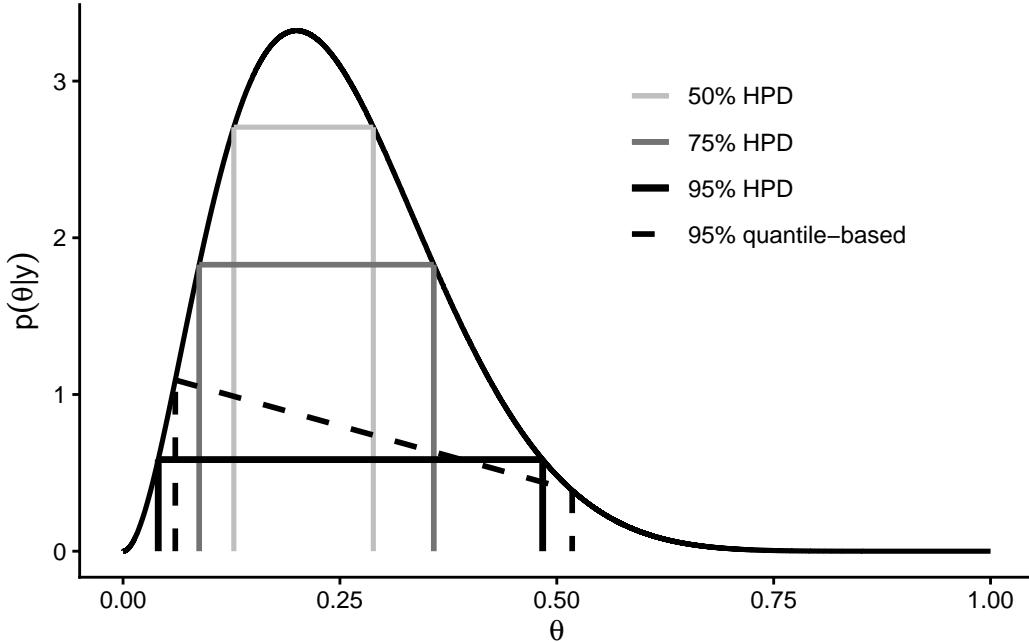
This observation suggests that the quantile-based interval may not be the most efficient way to summarize posterior uncertainty. In particular, it motivates a more restrictive type of credible region that concentrates on the most plausible parameter values.

A $100(1 - \alpha)\%$ HPD region is a subset of the sample space, $s(y) \subset \Theta$ such that:

1. $\Pr(\theta \in s(y) | Y = y) = 1 - \alpha$, and
2. If $\theta_a \in s(y)$ and $\theta_b \notin s(y)$, then $p(\theta_a | Y = y) \geq p(\theta_b | Y = y)$.

In words, an HPD region contains the parameter values with the *largest posterior density*, subject to containing probability mass $1 - \alpha$.

Observed that, all points inside an HPD region are at least as plausible as any point outside the region, according to the posterior distribution. This property distinguishes HPD regions from quantile-based intervals, which are defined purely by cumulative probability and may include low-density values while excluding higher-density ones.



An HPD region can be constructed conceptually as follows:

i Algorithm to construct an HPD region

1. Begin with a horizontal line above the posterior density curve.
2. Gradually lower the line.
3. At each height, include all values of θ whose posterior density exceeds the line.
4. Stop lowering the line once the total posterior probability of the included region reaches $1 - \alpha$.

This procedure guarantees that the retained region consists of the most probable values of θ .

HPD Regions and Multimodality

If the posterior density is **unimodal**, the HPD region will typically be a single interval. However, if the posterior density is **multimodal** (having multiple peaks), the HPD region need not be an interval; it may consist of several disjoint subsets of the parameter space.

In the binomial example with $n = 10$, $Y = 2$, and a uniform prior, the posterior distribution is Beta(3, 9).

For this posterior:

- The 95% quantile-based credible interval is approximately [0.06, 0.52].
- The 95% HPD region is approximately [0.04, 0.48].

The HPD region is *narrower*, and therefore more precise, than the quantile-based interval, while still containing 95% of the posterior probability.

Both intervals are valid Bayesian credible intervals, but they summarize posterior uncertainty in different ways.

3.5 The Poisson Model

Another commonly used distribution is the *Poisson*, in this case, the measurement are the integer numbers. Some examples include number of coin tosses, the number of friends they have, or the number of birthday celebrations have a person have. In these situations, the sample space is $\mathcal{Y} = \{0, 1, 2, \dots\}$. There are other possible models for those situation, but perhaps the simplest probability model on \mathcal{Y} is the Poisson model.

Poisson distribution

Recall from the previous chapter, that a random variable Y has a Poisson distribution with mean θ if

$$\Pr(Y = y | \theta) = \text{dpois}(y, \theta) = \frac{\theta^y e^{-\theta}}{y!}, \quad y \in \{0, 1, 2, \dots\}.$$

For such a random variable,

$$\mathbb{E}(Y) = \theta \quad \text{and} \quad \text{Var}(Y) = \theta.$$

People sometimes use the Poisson distribution to model count data because of its simplicity and its ability to model events that occur independently over a fixed interval of time or space. The Poisson distribution is particularly useful when the events being counted are rare or infrequent, and when the average rate of occurrence is known. Note that, in this model, the mean and the variance are the same, which is a property that can be useful in certain applications; One may call this property as “mean-variance relationship”.

```
library(ggplot2)
library(dplyr)
library(patchwork)
```

```
Attaching package: 'patchwork'
```

```
The following object is masked from 'package:cowplot':
```

```
align_plots
```

```

set.seed(8670)

# -----
# Parameters
# -----
theta <- 1.83
y1 <- 0:8
y2 <- 0:50

# -----
# Left panel: Poisson vs empirical
# -----

# Poisson model
pois_df <- data.frame(
  y = y1,
  prob = dpois(y1, theta),
  type = "Poisson model"
)

# Fake empirical distribution (for illustration)
empirical_counts <- c(18, 22, 30, 17, 8, 3, 1, 1, 0)
emp_df <- data.frame(
  y = y1,
  prob = empirical_counts / sum(empirical_counts),
  type = "Empirical distribution"
)

left_df <- bind_rows(pois_df, emp_df)

p_left <- ggplot(left_df, aes(x = y, y = prob, color = type)) +
  geom_segment(aes(xend = y, yend = 0),
               linewidth = 2,
               position = position_dodge(width = 0.35)) +
  scale_color_manual(
    values = c("black", "grey70"),
    breaks = c("Poisson model", "Empirical distribution")
) +
  labs(
    x = "number of children",
    y = expression(Pr(Y[i] == y[i])),
    color = NULL

```

```

) +
theme_minimal(base_size = 13) +
theme(
  legend.position = c(0.65, 0.85),
  panel.grid.minor = element_blank()
)

# -----
# Right panel: sum of 10 Poissons
# -----


sum_df <- data.frame(
  y = y2,
  prob = dpois(y2, 10 * theta)
)

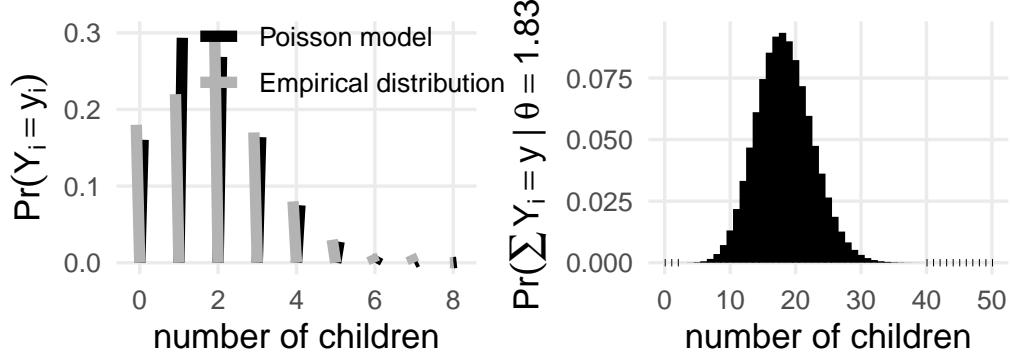
p_right <- ggplot(sum_df, aes(x = y, y = prob)) +
  geom_segment(aes(xend = y, yend = 0), linewidth = 1.3) +
  labs(
    x = "number of children",
    y = expression(Pr(sum(Y[i]) == y ~ " | " ~ theta == 1.83))
  ) +
  theme_minimal(base_size = 13) +
  theme(
    panel.grid.minor = element_blank()
)

# -----
# Combine panels
# -----


p_left + p_right +
  plot_annotation(
    title = "Poisson distributions and the mean-variance relationship",
    caption =
      "Left: Poisson pmf with mean = 1.83 (black) overlaid with an empirical distribution (grey). Right: Distribution of the sum of 10 i.i.d. Poisson(1.83) variables; by additivity this is Poisson(18.3). The increased spread illustrates the Poisson mean-variance relationship: larger means imply larger variance." ) &
  theme(
    plot.caption = element_text(size = 10, hjust = 0),
    plot.margin = margin(5.5, 5.5, 20, 5.5) # extra bottom margin so caption isn't cut
  )

```

Poisson distributions and the mean–variance relationship



Left: Poisson pmf with mean $\lambda = 1.83$ (black) overlaid with an empirical distribution (grey)

Right: Distribution of the sum of 10 i.i.d. Poisson(1.83) variables; by additivity this is Pois

The increased spread illustrates the Poisson mean–variance relationship: larger means

3.5.1 Inference on the Posterior

3.6 Posterior inference for the Poisson model

Suppose we observe data

$$Y_1, \dots, Y_n,$$

and model them as conditionally independent Poisson random variables with common mean θ :

$$Y_i | \theta \sim \text{Poisson}(\theta), \quad i = 1, \dots, n.$$

3.6.1 Likelihood

The joint probability mass function of the data, given θ , is

$$\Pr(Y_1 = y_1, \dots, Y_n = y_n | \theta) = \prod_{i=1}^n p(y_i | \theta).$$

Using the Poisson pmf,

$$p(y_i | \theta) = \frac{\theta^{y_i} e^{-\theta}}{y_i!},$$

we obtain

$$\Pr(Y_1 = y_1, \dots, Y_n = y_n \mid \theta) = \prod_{i=1}^n \frac{\theta^{y_i} e^{-\theta}}{y_i!} = c(y_1, \dots, y_n) \theta^{\sum_{i=1}^n y_i} e^{-n\theta},$$

where

$$c(y_1, \dots, y_n) = \prod_{i=1}^n \frac{1}{y_i!}$$

does not depend on θ .

This expression shows that the likelihood depends on the data only through the statistic

$$S = \sum_{i=1}^n Y_i.$$

3.6.2 Sufficiency

As in the binomial model, the statistic $S = \sum_{i=1}^n Y_i$ contains all information in the data about θ .

Indeed,

$$\sum_{i=1}^n Y_i \mid \theta \sim \text{Poisson}(n\theta),$$

and we therefore say that S is a sufficient statistic for θ .

3.6.3 Comparing posterior beliefs

To compare two values θ_a and θ_b *a posteriori*, consider the posterior odds:

$$\frac{p(\theta_a \mid y_1, \dots, y_n)}{p(\theta_b \mid y_1, \dots, y_n)}.$$

By Bayes' rule,

$$p(\theta \mid y_1, \dots, y_n) \propto p(\theta) p(y_1, \dots, y_n \mid \theta) \propto p(\theta) \theta^{\sum_{i=1}^n y_i} e^{-n\theta}.$$

Therefore,

$$\frac{p(\theta_a \mid y)}{p(\theta_b \mid y)} = \frac{\theta_a^{\sum y_i} e^{-n\theta_a} p(\theta_a)}{\theta_b^{\sum y_i} e^{-n\theta_b} p(\theta_b)}.$$

This expression highlights how posterior beliefs balance prior information with evidence from the data.

3.7 Conjugate prior for the Poisson model

We now seek a prior distribution for θ that leads to a posterior distribution of the same functional form.

From the likelihood,

$$p(\theta | y) \propto p(\theta) \theta^{\sum y_i} e^{-n\theta},$$

we see that a conjugate prior must involve terms of the form

$$\theta^{c_1} e^{-c_2 \theta}$$

for some constants c_1 and c_2 .

The simplest family of distributions with this structure is the **Gamma family**.

3.7.1 Gamma distribution

A positive random variable θ has a $\text{Gamma}(a, b)$ distribution if

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, \quad \theta > 0,$$

where $a > 0$ is the shape parameter and $b > 0$ is the rate parameter.

For a $\text{Gamma}(a, b)$ random variable, - Mean:

$$\mathbb{E}(\theta) = \frac{a}{b},$$

- Variance:

$$\text{Var}(\theta) = \frac{a}{b^2}.$$

3.7.2 Posterior distribution

If the prior is

$$\theta \sim \text{Gamma}(a, b),$$

then combining the prior with the Poisson likelihood yields

$$p(\theta | y) \propto \theta^{a-1 + \sum y_i} e^{-(b+n)\theta}.$$

Thus, the posterior distribution is

$$\theta | y_1, \dots, y_n \sim \text{Gamma}\left(a + \sum_{i=1}^n y_i, b + n\right).$$

This shows that the Gamma distribution is **conjugate** to the Poisson likelihood.

3.8 Interpretation

Posterior inference for the Poisson model is therefore straightforward:

- The data enter only through the sufficient statistic $\sum Y_i$;
- The posterior mean is

$$\mathbb{E}(\theta | y) = \frac{a + \sum y_i}{b + n};$$

- Increasing the sample size n reduces posterior uncertainty.

This conjugate structure makes the Poisson–Gamma model a convenient and interpretable starting point for Bayesian analysis of count data.

3.9 Posterior inference for the Poisson model

Suppose we observe count data

$$Y_1, \dots, Y_n \mid \theta \stackrel{\text{i.i.d.}}{\sim} \text{Poisson}(\theta),$$

where $\theta > 0$ is an unknown mean parameter.

The joint probability mass function of the sample, conditional on θ , is

$$\Pr(Y_1 = y_1, \dots, Y_n = y_n \mid \theta) = \prod_{i=1}^n \Pr(Y_i = y_i \mid \theta) = \prod_{i=1}^n \frac{\theta^{y_i} e^{-\theta}}{y_i!}.$$

This expression can be rewritten as

$$\Pr(Y_1 = y_1, \dots, Y_n = y_n \mid \theta) = c(y_1, \dots, y_n) \theta^{\sum_{i=1}^n y_i} e^{-n\theta},$$

where $c(y_1, \dots, y_n)$ does not depend on θ .

3.9.1 Sufficient statistic

The likelihood depends on the data only through the sum

$$S = \sum_{i=1}^n Y_i.$$

Thus, S is a **sufficient statistic** for θ .

Moreover,

$$S \mid \theta \sim \text{Poisson}(n\theta).$$

All information about θ contained in the data is captured by S .

3.10 Conjugate prior for the Poisson model

We now seek a class of prior distributions for θ that is **conjugate** to the Poisson likelihood.

Recall that a class of priors is conjugate if the posterior distribution belongs to the same family as the prior.

By Bayes' rule,

$$p(\theta | y_1, \dots, y_n) \propto p(\theta) p(y_1, \dots, y_n | \theta) \propto p(\theta) \theta^{\sum y_i} e^{-n\theta}.$$

Therefore, any conjugate prior must have the form

$$p(\theta) \propto \theta^{c_1} e^{-c_2 \theta}$$

for constants c_1 and c_2 .

The simplest family of distributions with this form is the **Gamma family**.

3.11 The Gamma distribution

A positive random variable θ has a Gamma distribution with parameters (a, b) if

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, \quad \theta > 0,$$

where $a > 0$ is the shape parameter and $b > 0$ is the rate parameter.

For a $\text{Gamma}(a, b)$ random variable:

- Mean:

$$\mathbb{E}(\theta) = \frac{a}{b},$$

- Variance:

$$\text{Var}(\theta) = \frac{a}{b^2},$$

- Mode:

$$\text{mode}(\theta) = \begin{cases} \frac{a-1}{b}, & a > 1, \\ 0, & a \leq 1. \end{cases}$$

3.12 Posterior distribution

Assume

$$\theta \sim \text{Gamma}(a, b), \quad Y_1, \dots, Y_n \mid \theta \sim \text{Poisson}(\theta).$$

Combining the prior and likelihood,

$$p(\theta \mid y_1, \dots, y_n) \propto \theta^{a-1} e^{-b\theta} \cdot \theta^{\sum y_i} e^{-n\theta} = \theta^{a+\sum y_i - 1} e^{-(b+n)\theta}.$$

Hence, the posterior distribution is

$$\theta \mid y_1, \dots, y_n \sim \text{Gamma}\left(a + \sum_{i=1}^n Y_i, b + n\right).$$

This confirms that the Gamma distribution is conjugate to the Poisson sampling model.

3.13 Interpretation

Posterior inference for the Poisson model closely parallels the binomial case:

- The data influence inference only through the sufficient statistic $\sum Y_i$.
- The posterior mean is

$$\mathbb{E}(\theta \mid y) = \frac{a + \sum Y_i}{b + n},$$

which is a **convex combination** of the prior mean a/b and the sample mean \bar{Y} .

- As the sample size n increases, posterior uncertainty decreases.

This conjugate structure makes the Poisson–Gamma model a convenient and powerful tool for Bayesian analysis of count data.

3.14 Posterior mean and interpretation

For the Poisson–Gamma model,

$$\theta \mid y_1, \dots, y_n \sim \text{Gamma}\left(a + \sum_{i=1}^n y_i, b + n\right).$$

The posterior mean is

$$\mathbb{E}(\theta \mid y_1, \dots, y_n) = \frac{a + \sum_{i=1}^n y_i}{b + n}.$$

This can be rewritten as a weighted average:

$$\mathbb{E}(\theta \mid y_1, \dots, y_n) = \frac{b}{b+n} \cdot \frac{a}{b} + \frac{n}{b+n} \cdot \bar{y},$$

where

$$\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

is the sample mean.

This decomposition gives a useful interpretation of the prior parameters:

- b acts like the **number of prior observations**;
- a acts like the **total count** from those b observations;
- a/b is the prior mean.

As n becomes large relative to b , the data dominate the prior:

$$n \gg b \implies \mathbb{E}(\theta \mid y) \approx \bar{y}, \quad \text{Var}(\theta \mid y) \approx \frac{\bar{y}}{n}.$$

3.15 Posterior predictive distribution

Bayesian prediction for a future observation \tilde{Y} is based on the **posterior predictive distribution**,

$$p(\tilde{y} \mid y_1, \dots, y_n) = \int_0^\infty p(\tilde{y} \mid \theta) p(\theta \mid y_1, \dots, y_n) d\theta.$$

For the Poisson model,

$$p(\tilde{y} \mid \theta) = \text{Poisson}(\theta), \quad p(\theta \mid y) = \text{Gamma}\left(a + \sum y_i, b + n\right).$$

Substituting,

$$p(\tilde{y} | y) = \int_0^\infty \text{dpois}(\tilde{y}, \theta) \text{dgamma}(\theta, a + \sum y_i, b + n) d\theta.$$

Writing this integral explicitly,

$$p(\tilde{y} | y) = \int_0^\infty \frac{\theta^{\tilde{y}} e^{-\theta}}{\tilde{y}!} \cdot \frac{(b+n)^{a+\sum y_i}}{\Gamma(a + \sum y_i)} \theta^{a+\sum y_i - 1} e^{-(b+n)\theta} d\theta.$$

Combining terms,

$$p(\tilde{y} | y) = \frac{(b+n)^{a+\sum y_i}}{\tilde{y}! \Gamma(a + \sum y_i)} \int_0^\infty \theta^{a+\sum y_i + \tilde{y} - 1} e^{-(b+n+1)\theta} d\theta.$$

3.16 Evaluating the integral

Recall the Gamma integral identity:

$$\int_0^\infty \theta^{\alpha-1} e^{-\beta\theta} d\theta = \frac{\Gamma(\alpha)}{\beta^\alpha}, \quad \alpha, \beta > 0.$$

Applying this with

$$\alpha = a + \sum y_i + \tilde{y}, \quad \beta = b + n + 1,$$

we obtain

$$\int_0^\infty \theta^{a+\sum y_i + \tilde{y} - 1} e^{-(b+n+1)\theta} d\theta = \frac{\Gamma(a + \sum y_i + \tilde{y})}{(b+n+1)^{a+\sum y_i + \tilde{y}}}.$$

Substituting back and simplifying,

$$p(\tilde{y} | y_1, \dots, y_n) = \frac{\Gamma(a + \sum y_i + \tilde{y})}{\Gamma(a + \sum y_i) \tilde{y}!} \left(\frac{b+n}{b+n+1} \right)^{a+\sum y_i} \left(\frac{1}{b+n+1} \right)^{\tilde{y}}.$$

3.17 Interpretation

The posterior predictive distribution has a **negative binomial form**. It accounts for uncertainty in θ by averaging over its posterior distribution.

Key takeaways:

- Prediction uncertainty is larger than plug-in prediction using $\hat{\theta} = \bar{y}$.
- As n increases, the posterior predictive distribution approaches a Poisson distribution with mean \bar{y} .
- Bayesian prediction naturally incorporates both **sampling variability** and **parameter uncertainty**.

This completes posterior inference and prediction for the Poisson–Gamma model.

3.18 Posterior predictive moments

Recall that the posterior predictive distribution for a future observation \tilde{Y} under the Poisson–Gamma model is a **negative binomial** distribution with parameters

$$\left(a + \sum_{i=1}^n y_i, b + n \right),$$

defined for $\tilde{y} \in \{0, 1, 2, \dots\}$.

3.18.1 Posterior predictive mean

The predictive mean is

$$\mathbb{E}(\tilde{Y} \mid y_1, \dots, y_n) = \frac{a + \sum_{i=1}^n y_i}{b + n} = \mathbb{E}(\theta \mid y_1, \dots, y_n).$$

Thus, the expected value of a new observation equals the posterior mean of the Poisson rate parameter θ .

3.18.2 Posterior predictive variance

The predictive variance is

$$\text{Var}(\tilde{Y} \mid y_1, \dots, y_n) = \frac{a + \sum_{i=1}^n y_i}{b+n} \cdot \frac{b+n+1}{b+n}.$$

This can be written as

$$\text{Var}(\tilde{Y} \mid y) = \mathbb{E}(\theta \mid y) \cdot \frac{b+n+1}{b+n}.$$

3.18.3 Interpretation of predictive uncertainty

The predictive variance reflects **two sources of uncertainty**:

1. **Sampling variability**

For a Poisson model, the variance of Y given θ is equal to θ .

2. **Parameter uncertainty**

When θ is unknown, uncertainty about θ inflates the variance of future observations.

For large n , the data dominate the prior:

$$\frac{b+n+1}{b+n} \approx 1,$$

so predictive uncertainty is driven primarily by sampling variability.

For small n , posterior uncertainty about θ is substantial, and

$$\frac{b+n+1}{b+n} > 1,$$

leading to larger predictive variance than under a fixed- θ Poisson model.

During the 1990s, the General Social Survey (GSS) collected data on the number of children for women aged 40 at the time of the survey.

These women were in their 20s during the 1970s, a period characterized by historically low fertility rates.

The data are separated into two groups based on educational attainment:

- women with **less than a bachelor's degree**;
- women with **a bachelor's degree or higher**.

Let Y denote the number of children for a randomly selected woman from one of these groups. Because Y takes nonnegative integer values, the Poisson model provides a natural starting point for analysis.

In the following sections, we will:

- model the counts using Poisson likelihoods;
 - specify Gamma priors for the group-specific fertility rates;
 - compare posterior distributions and posterior predictive behavior between the two education groups.
-

This Chapter follows closely with Chapter 3 in Hoff (2009).

4 Summary

In summary, this book has no content whatsoever.

```
1 + 1
```

```
[1] 2
```

Part I

Appendix

5 Appendix: Introduction to R

5.1 R

For conducting analyses with data sets of hundreds to thousands of observations, calculating by hand is not feasible and you will need a statistical software. **R** is one of those. **R** can also be thought of as a high-level programming language. In fact, **R** is [one of the top languages](#) to be used by data analysts and data scientists. There are a lot of analysis packages in **R** that are currently developed and maintained by researchers around the world to deal with different data problems. Most importantly, **R** is free! In this section, we will learn how to use **R** to conduct basic statistical analyses.

5.2 IDE

5.2.1 Rstudio

RStudio is an integrated development environment (IDE) designed specifically for working with the **R** programming language. It provides a user-friendly interface that includes a source editor, console, environment pane, and tools for plotting, debugging, version control, and package management. RStudio supports both **R** and Python and is widely used for data analysis, statistical modeling, and reproducible research. It also integrates seamlessly with tools like **R** Markdown, Shiny, and Quarto, making it popular among data scientists, statisticians, and educators.

5.2.2 Visual Studio Code (VS Code)

VS Code is a versatile code editor that supports multiple programming languages, including **R**. With the **R** extension for VS Code, users can write and execute **R** code, access **R**'s console, and utilize features like syntax highlighting, code completion, and debugging. While not as specialized as RStudio for **R** development, VS Code offers a lightweight alternative with extensive customization options and support for various programming tasks.

5.2.3 Positron

Positron IDE is the next-generation integrated development environment developed by Posit, the company behind RStudio. Designed to be a modern, extensible, and language-agnostic IDE, Positron builds on the strengths of RStudio while supporting a broader range of languages and workflows, including **R**, Python, and Quarto.

5.3 RStudio Layout

RStudio consists of several panes:

- **Source:** Where you write scripts and markdown documents.
- **Console:** Where you type and execute **R** commands.
- **Environment/History:** Shows your variables and command history.
- **Files/Plots/Packages/Help/Viewer:** For file management, viewing plots, managing packages, accessing help, and viewing web content.

5.4 R Scripts

R scripts are plain text files containing **R** code. You can create a new script in RStudio by clicking **File > New File > R Script**.

5.5 R Help

Use `?function_name` or `help(function_name)` to access help for any **R** function. For example:

```
?mean  
help(mean)
```

5.6 R Packages

Packages extend **R**'s functionality. There are thousands of packages available in **R** ecosystem. You may install them from different sources.

5.6.1 With Comprehensive R Archive Network (CRAN)

CRAN is the primary repository for **R** packages. It contains thousands of packages that can be easily installed and updated.

Install a package with:

```
install.packages("package_name")
```

5.6.2 With Bioconductor

Bioconductor is a repository for bioinformatics packages in **R**. It provides tools for the analysis and comprehension of high-throughput genomic data.

Install Bioconductor packages using the `BiocManager` package:

```
BiocManager::install("package_name")
```

5.6.3 From GitHub

Many of the authors of **R** packages host their work on GitHub. You can install these packages using the `devtools` package:

```
devtools::install_github("username/package_name")
```

5.6.4 Load a package

Once a package is installed, you need to load it into your **R** session to use its functions:

```
library(package_name)
```

Alternatively, you may use a function in the package with `package_name::function_name()` without loading the entire package.

5.7 R Markdown

R Markdown allows you to combine text, code, and output in a single document. Create a new **R** Markdown file in RStudio via `File > New File > R Markdown....`

Recently, the posit team has developed a new version of the **R** Markdown called quarto document, with the file extension `.qmd`. It is still under rapid development.

5.8 Vectors

Vectors are the most basic data structure in R.

```
x <- c(1, 2, 3, 4, 5)
x
```

```
[1] 1 2 3 4 5
```

You can perform operations on vectors:

```
x * 2
```

```
[1] 2 4 6 8 10
```

5.9 Data Sets

Data frames are used for storing data tables. Create a data frame:

```
df <- data.frame(Name = c("Alice", "Bob"), Score = c(90, 85))
df
```

	Name	Score
1	Alice	90
2	Bob	85

You can import data from files using `read.csv()` or `read.table()`.

This appendix is adapted from [Why R?](#).

References