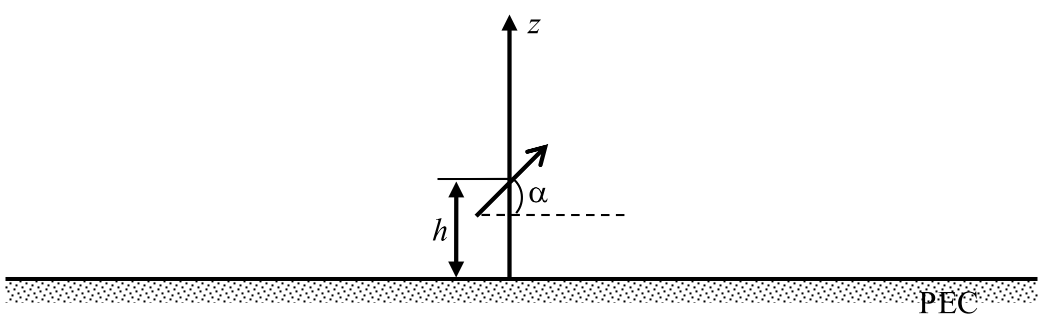
Applied Computational EM
Homework Assignment #2

Problem 1

Consider the short current filament (dipole) $Id\boldsymbol{\ell}$ radiating in free space in the presence of an infinite PEC surface as shown in Figure 1 below. The dipole is oriented at an angle α with respect to the x-axis.



a) Derive the magnetic vector potential radiated by the short dipole.
b) From the vector potential, derive the ***E*** and ***H*** fields radiated in the far-zone.

<u>Solution</u>

a) The current element $Id\boldsymbol{\ell}$ is resolved into two components:

$$Id\boldsymbol{\ell} = Id\ell\cos\alpha\hat{\mathbf{x}} + Id\ell\sin\alpha\hat{\mathbf{z}} \tag{1}$$

According to image theory discussed in class, the image of the x-directed component of the current element $Id\boldsymbol{\ell}$ is directed opposite to the x-directed component of the current element. The z-directed image, on the other hand, is directed in the same direction as the z-directed component of the current element. The overall magnetic vector potential due to the x and z current elements and their images are given by:

$$A_x = \frac{Id\ell\cos\alpha}{4\pi}\mu_0\left(\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}_1'|}}{|\mathbf{r}-\mathbf{r}_1'|} - \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}_2'|}}{|\mathbf{r}-\mathbf{r}_2'|}\right) \tag{2}$$

$$A_z = \frac{Id\ell\sin\alpha}{4\pi}\mu_0\left(\frac{e^{-jk_0|\mathbf{r}-\mathbf{r}_1'|}}{|\mathbf{r}-\mathbf{r}_1'|} + \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}_2'|}}{|\mathbf{r}-\mathbf{r}_2'|}\right) \tag{3}$$

In (2) and (3), $\mathbf{r}_1' = h\hat{\mathbf{z}}$, $\mathbf{r}_2' = -h\hat{\mathbf{z}}$. In the far-field region we have

$$
\begin{aligned}
&\left|\mathbf{r}-\mathbf{r}_1'\right| = \left(r^2 - 2\mathbf{r}\cdot\mathbf{r}_1' + r_1'^2\right)^{1/2} = r\left(1 - \frac{2\mathbf{r}\cdot\mathbf{r}_1'}{r^2} + \frac{r_1'^2}{r^2}\right)^{1/2} \cong r - \hat{\mathbf{r}}\cdot\mathbf{r}_1' = r - h\cos\theta \\
&\left|\mathbf{r}-\mathbf{r}_2'\right| \cong r + h\cos\theta
\end{aligned}
\tag{4}
$$

Hence,

$$
\begin{aligned}
A_x &= \frac{Id\ell\cos\alpha}{4\pi}\mu_0\frac{e^{-jk_0r}}{r}\left(e^{jk_0h\cos\theta} - e^{-jk_0h\cos\theta}\right) \\
&= j\frac{Id\ell\cos\alpha}{2\pi}\mu_0\frac{e^{-jk_0r}}{r}\sin\left(k_0h\cos\theta\right)
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
A_z &= \frac{Id\ell\sin\alpha}{4\pi}\mu_0\frac{e^{-jk_0r}}{r}\left(e^{jk_0h\cos\theta} + e^{-jk_0h\cos\theta}\right) \\
&= \frac{Id\ell\sin\alpha}{2\pi}\mu_0\frac{e^{-jk_0r}}{r}\cos\left(k_0h\cos\theta\right)
\end{aligned}
\tag{6}
$$

In the far-field, we need the vector potential components A_θ and A_ϕ to compute the far-zone radiated field. These are given by

$$
\begin{aligned}
A_\theta &= A_x\cos\phi\cos\theta - A_z\sin\theta \\
&= -Id\ell\mu_0\frac{e^{-jk_0r}}{2\pi r}\left[\sin\alpha\sin\theta\cos\left(k_0h\cos\theta\right) - j\cos\alpha\cos\phi\cos\theta\sin\left(k_0h\cos\theta\right)\right]
\end{aligned}
\tag{7}
$$

$$
A_\phi = -A_x\sin\phi = -jId\ell\cos\alpha\mu_0\frac{e^{-jk_0r}}{2\pi r}\sin\phi\sin\left(k_0h\cos\theta\right)
\tag{8}
$$

b) In the far-field, the electric and magnetic fields are transverse to each other and to the **r** direction. Hence,

$$
\begin{aligned}
E_\theta &= -j\omega A_\theta \\
E_\phi &= -j\omega A_\phi
\end{aligned}
\tag{9}
$$

and

$$
\mathbf{H} = \frac{1}{\eta_0}\hat{\mathbf{r}}\times\mathbf{E} = \frac{1}{\eta_0}\left(-E_\phi\hat{\boldsymbol{\theta}} + E_\theta\hat{\boldsymbol{\varphi}}\right)
\tag{10}
$$

Problem 2

For the problem of TM^z scattering from a perfectly conducting infinite cylinder treated in class. Take $\theta_0 = 90°$, $\phi_0 = 0$, $E_0 = 1$ V/m, frequency = 1, 5, 10, 20 GHz, radius a = 0.25 m.

a) Calculate the electric surface current density induced on the surface of the cylinder. Note that the surface current density is given by

$$\mathbf{J}_s = \left(\hat{\mathbf{n}} \times \mathbf{H}\right)_{\rho=a} = \hat{\mathbf{n}} \times \left(\mathbf{H}_i + \mathbf{H}_s\right)_{\rho=a}$$

where the unit surface normal is given by $\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}}$.

b) Calculate the physical optics approximation to the surface current density given by

$$\mathbf{J}_{PO} = 2\left(\hat{\mathbf{n}} \times \mathbf{H}_i\right)_{\rho=a}$$

c) Compare the two results and comment on the differences.

d) Repeat steps (a) to (c) for the TE^z case.

<u>Solution</u>

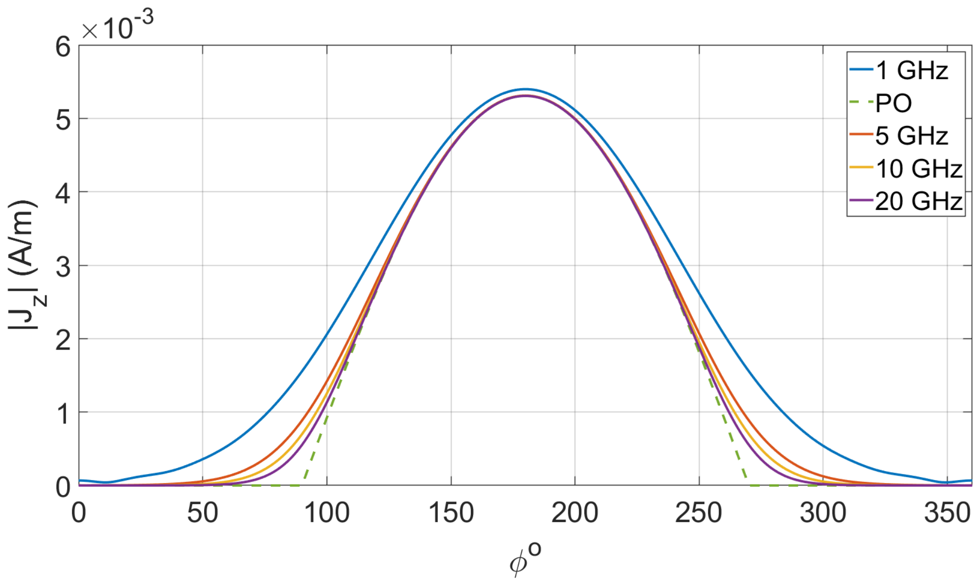
a) The surface current was derived in class. The result is:

$$J_z = \frac{2E_0}{\pi k a \eta} \sum_{n=0}^{\infty} j^{-n} \varepsilon_n \left[\frac{\cos(n\phi)}{H_n^{(2)}(ka)} \right]$$

b) The physical optics current is given by

$$\mathbf{E}_i = E_0 e^{-j\mathbf{k}_i \cdot \mathbf{r}} \hat{\mathbf{z}}$$

$$J_z^{PO} \hat{\mathbf{z}} = 2\left(\hat{\boldsymbol{\rho}} \times \mathbf{H}_i\right)_{\rho=a} = -2\left[\frac{E_0}{\eta} e^{-j\mathbf{k}_i \cdot \mathbf{r}} \left(\hat{\boldsymbol{\rho}} \times \hat{\mathbf{y}}\right)\right]_{\rho=a} = -2\frac{E_0}{\eta} \cos\phi e^{-jka\cos\phi} \hat{\mathbf{z}}$$



c) The difference between the exact current and the physical optics current is due to diffraction effects not accounted for by the physical optics approximation.

d) For the TE^z case, the electric current induced on the surface of the cylinder is given by

$$\mathbf{J} = \hat{\mathbf{n}} \times \left(\mathbf{H}^i + \mathbf{H}^s\right)\Big|_{\rho=a} = \hat{\boldsymbol{\rho}} \times \left(\mathbf{H}^i + \mathbf{H}^s\right)\Big|_{\rho=a} \tag{11}$$

At normal incidence, $\theta_i = \pi/2$, the incident magnetic field component expanded in term of cylindrical wave functions is given by (12). Note that when $\theta_i = \pi/2$, $H_\rho^i = H_\phi^i = 0$, and the only nonzero component of the incident magnetic field is given by

$$H_z^i = H_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n\left(k\rho\right) e^{jn\phi} \tag{12}$$

As shown below, the only non-zero component of the magnetic field is given by

$$H_z^s = -H_0 \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n'\left(k_0 a\right)}{H_n^{(2)\prime}\left(k_0 a\right)} H_n^{(2)}\left(k_0 \rho\right) e^{jn\phi} \tag{13}$$

Substituting (12) and (13) into (11) and evaluating the resulting expression at $\rho = a$ yields

$$
\begin{aligned}
J_\phi &= H_0 \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n'(ka) H_n^{(2)}(ka) - J_n(ka) H_n^{(2)\prime}(ka)}{H_n^{(2)\prime}(ka)} e^{jn\phi} \\
&= \frac{j2}{\pi ka} H_0 \sum_{n=-\infty}^{\infty} j^{-n} \frac{e^{jn\phi}}{H_n^{(2)\prime}(ka)}
\end{aligned}
\tag{14}
$$

The above result can be also written as follows

$$
J_\phi = \frac{j2}{\pi ka} H_0 \left[\sum_{n=1}^{\infty} j^{n} \frac{e^{-jn\phi}}{H_{-n}^{(2)\prime}(ka)} + \sum_{n=0}^{\infty} j^{-n} \frac{e^{jn\phi}}{H_n^{(2)\prime}(ka)} \right] \tag{15}
$$

From the recurrence relations of the Hankel function we have

$$
\begin{aligned}
H_{-n}^{(2)\prime}(ka) &= -H_{-(n-1)}^{(2)}(ka) - \frac{n}{ka} H_{-n}^{(2)}(ka) \\
&= e^{-jn\pi} \left[H_{(n-1)}^{(2)}(ka) - \frac{n}{ka} H_n^{(2)}(ka) \right] \\
&= (-1)^n H_n^{(2)\prime}(ka)
\end{aligned}
\tag{16}
$$

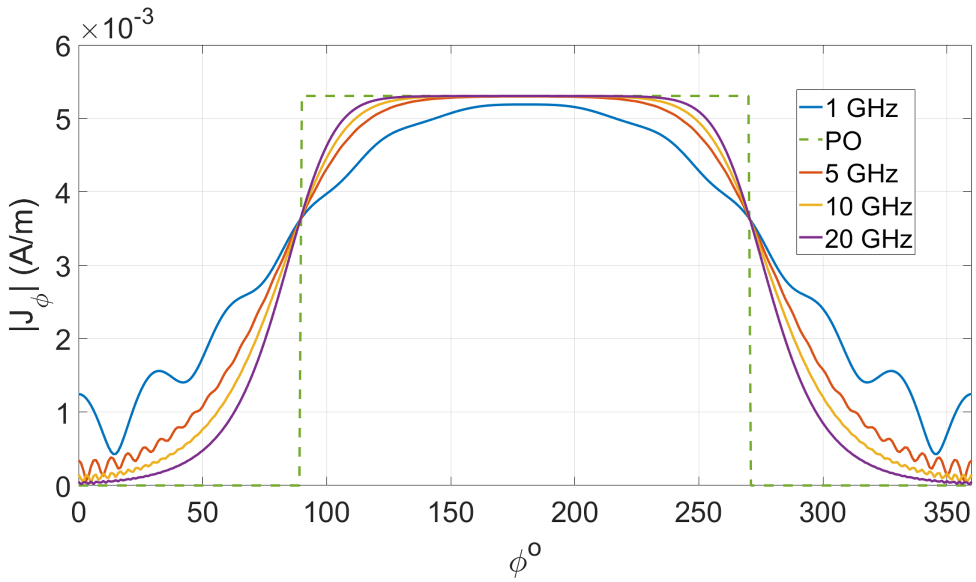
Substituting (16) into (15) we obtain

$$
J_\phi = \frac{j2}{\pi ka} H_0 \sum_{n=0}^{\infty} \varepsilon_n j^{-n} \frac{\cos(n\phi)}{H_n^{(2)\prime}(ka)} \tag{17}
$$

The physical optics current in this case is given by

$$
J_\phi^{PO} = \begin{cases} -2H_0 e^{-jka\cos\phi}, & \text{Lit region} \\ 0, & \text{Shadow region} \end{cases} \tag{18}
$$

These currents are plotted below



Derivation of the TE^z Case Using Hertz Vectors

The procedure is analogous to the TM^z case, but we work with the incident magnetic field instead. In this case, the incident magnetic field is given by

$$\begin{aligned} \mathbf{H}_i &= -H_0 e^{-j\mathbf{k}_i \cdot \mathbf{r}} \hat{\boldsymbol{\theta}}_0 \\ H_{zi} &= \mathbf{H}_i \cdot \hat{\mathbf{z}} = H_0 \sin\theta_0 e^{-j\mathbf{k}_i \cdot \mathbf{r}} \end{aligned} \tag{19}$$

The magnetic Hertz potential (Π_{zi}^m) associated with the above incident magnetic field is derived in a manner analogous to the derivation of the (Π_{zi}^e) for the TM^z case. The result is

$$\Pi_{zi}^m = \frac{H_0}{k^2 \sin\theta_0} e^{-jkz\cos\theta_0} \sum_{n=-\infty}^{\infty} j^{-n} J_n\left(k\rho\sin\theta_0\right) e^{jn(\phi-\phi_0)} \tag{20}$$

The scattered Hertz potential is given by

$$\Pi_{zs}^m = \sum_{n=-\infty}^{\infty} j^{-n} A_{sn} e^{-jkz\cos\theta_0} H_n^{(2)}\left(k\rho\sin\theta_0\right) e^{jn(\phi-\phi_0)} \tag{21}$$

The boundary condition is that the tangential electric field E_ϕ must be zero at the surface of the cylinder. This field component is given by

$$E_\phi\Big|_{\rho=a} = j\omega\mu \frac{\partial \Pi_z^m}{\partial\rho}\Big|_{\rho=a} = 0 = \frac{\partial}{\partial\rho}\left(\Pi_{zi}^m + \Pi_{zs}^m\right)\Big|_{\rho=a} = 0 \tag{22}$$

In the above

$$\left.\frac{\partial \Pi_{zs}^{m}}{\partial \rho}\right|_{\rho=a} = k\sin\theta_0 \sum_{n=-\infty}^{\infty} j^{-n} A_{sn} e^{-jkz\cos\theta_0} H_n'^{(2)}\left(ka\sin\theta_0\right) e^{jn(\phi-\phi_0)} \tag{23}$$

$$\left.\frac{\partial \Pi_{zi}^{m}}{\partial \rho}\right|_{\rho=a} = \frac{H_0}{k} e^{-jkz\cos\theta_0} \sum_{n=-\infty}^{\infty} j^{-n} J_n'\left(ka\sin\theta_0\right) e^{jn(\phi-\phi_0)} \tag{24}$$

Substituting (23) and (24) into (22) and solving for A_{sn} we obtain

$$A_{sn} = -\frac{H_0}{k^2\sin\theta_0} \frac{J_n'\left(ka\sin\theta_0\right)}{H_n'^{(2)}\left(ka\sin\theta_0\right)} \tag{25}$$

The scattered magnetic field is given in terms of the scattered Hertz vector by

$$\begin{aligned} H_\rho^s &= \frac{\partial^2 \Pi_{zs}^m}{\partial z \partial \rho} \\ &= jH_0\cos\theta_0 \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n'\left(ka\sin\theta_0\right)}{H_n'^{(2)}\left(ka\sin\theta_0\right)} e^{-jkz\cos\theta_0} H_n'^{(2)}\left(k\rho\sin\theta_0\right) e^{jn(\phi-\phi_0)} \end{aligned} \tag{26}$$

$$\begin{aligned} H_\phi^s &= \frac{1}{\rho}\frac{\partial^2 \Pi_{zs}^m}{\partial z \partial \phi} \\ &= -H_0 \frac{\cot\theta_0}{k\rho} \sum_{n=-\infty}^{\infty} nj^{-n} \frac{J_n'\left(ka\sin\theta_0\right)}{H_n'^{(2)}\left(ka\sin\theta_0\right)} e^{-jkz\cos\theta_0} H_n^{(2)}\left(k\rho\sin\theta_0\right) e^{jn(\phi-\phi_0)} \end{aligned} \tag{27}$$

$$\begin{aligned} H_z^s &= \left(k^2 + \frac{\partial^2}{\partial z^2}\right)\Pi_{zs}^m \\ &= -H_0\sin\theta_0 \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n'\left(ka\sin\theta_0\right)}{H_n'^{(2)}\left(ka\sin\theta_0\right)} e^{-jkz\cos\theta_0} H_n^{(2)}\left(k\rho\sin\theta_0\right) e^{jn(\phi-\phi_0)} \end{aligned} \tag{28}$$