Fall Interschool 2020

FAMAT

Fall 2020

Welcome to the FAMAT Student Delegate 2020 Fall Interschool!

What is this? This test contains 8 sections, each containing a set of test items: questions, puzzles, or problems.

#	Section	Points
1	Algebra	30
2	Calculus	18
3	Counting and Probability	36
4	Trivia	16
5	Science	9
6	Codes and Ciphers	6
7	Logic and Puzzles	21
8	LState Convention	0
	Total	136

Who can participate? Any student or sponsor associated with your chapter can help to solve the problems. However, you may not consult alumni, parents, or any other humans outside of your chapter. That being said, you may use the internet/calculators/ books/etc., provided that you do not get help from another human, which means something like asking a question on an internet forum like Yahoo Answers is not be allowed.

How do we do this? You must submit the answers using the answer document available under the Downloads section of famat.org. You have until 11:59pm on Wednesday, November 4 to submit your answer sheet to robsnow23@yahoo.com.

How do we dispute? Hello, FAMAT Student, we know that you love to dispute our tests. If you have any questions or concerns about the test or if you have any issues using the answer document, you may email the above e-mail address during the testing period with the subject line: "Fall Interschool: section title, question number". We will resolve your issues, if any clarifications must be made during the testing period, an errata will be posted under Downloads.

A note on answer form: All answers must be exact unless otherwise stated. Stick to the generally accepted answer forms (Rationalized denominators, simplified fractions and radicals, etc.). We will accept equivalent answer forms within reason.

1 Algebra

Problem 1 [12 Points]

Consider the functional equation

$$f(xy) = yf(x) + xf(y)$$

for positive integers x and y. Let $f(1) = 0$ and $f(p) = 1$ for all primes p. As a quick note, it is true that all positive integers n have a unique value $f(n)$, but you should try to verify this for yourself!

Problem 1.1 [2 Points]

Compute the value of $f(2021)$.

Problem 1.2 [4 Points]

Compute the value of $f(2^{2021})$. You may want to think about the general case of $f(p^k)$ for a prime p and integer k. Express your answer as a product involving an integer times a (maximal) power of 2.

Problem 1.3 [6 Points]

Let

$$n = 2^4 3^9 5^{25} 7^{49}$$

Compute the value of $f(n)$ in terms of n.

Problem 2 [2 Points]

Let $f(x) = x^2$ and let $g(x)$ be a quadratic function that is tangent to $f(x)$ at exactly one point and satisfies $g(x) \geq f(x)$ for all real values of x. How many unique real solutions for x does the equation

$$f(x)g(x) = f(x)$$

have?

Problem 3 [2 Points]

Let x be a real number. If $\arccos x \arcsin x = \frac{1}{2}$, then compute the maximum possible value of $\arccos^3 x + \arcsin^3 x$.

Problem 4 [2 Points]

A real number x is selected uniformly and at random from the interval $(0, 2)$. The probability that

$$2 \leq \tan(x) \leq 3$$

is p. What is the value of $\tan(p)$?

2 Calculus

Problem 1 [2 Points] When $x = \frac{\pi}{7}$, compute the value of

$$\lim_{n \to \infty} \left(x^n + (1-x)^n \right)^{\frac{1}{n}}$$

Problem 2 [2 Points] Starting at $t = 0$ seconds, Ryan Chlong and Eric Shode start traveling from the pole along the paths $r = \theta$ and $r = 2\theta$ in the polar plane, respectively. They both travel such that their distance from the pole is increasing at a rate of 4 units per second. At time $t = \frac{\pi}{2}$ seconds, what is the rate of change of the distance between Ryan and Eric (in units per second)?

Problem 3 [3 Points]

Compute the value of

$$\lim_{n \to \infty} \frac{1}{n} \int_0^\infty \frac{3x^2 + 4nx + 1}{xe^x} dx$$

Problem 4 [3 Points]

Let $f(x) = \sum_{n=0}^{2020} x^{\frac{n}{2}}$. Compute the value of the expression

$$\lfloor \log_2(f(4) + 4f'(4)) \rfloor$$

Problem 5 [4 Points]

Buff is taking a multiple choice test with n questions, each of which has n answer choices. Buff has decided that he will randomly guess one of the answer choices for all n questions on the test. If there is exactly one correct answer per question, then as n grows large, what is the probability that Buff will get exactly 5 questions right on this exam?

Problem 6 [4 Points]

Compute the value of

$$\int_0^1 (x \ln x)^{2020} dx$$

Express your answer in the form $\frac{m!}{n^p}$ for positive integers m, n and p.

3 Counting and Probability

Problem 1 [9 Points]

Define a string composed of only '0's and '1's to be *trendy* if the following conditions are satisfied:

- Reading left to right, every '0' is immediately followed by a '1'

- The string is palindromic (reads the same from left to right and right to left)

Problem 1.1 [2 Points] How many trendy strings of length 12 exist?

Problem 1.2 [3 Points] Let S be the set of all trendy strings of length 2020. If each string is interpreted as a **binary** number with any leading zeros ignored (so the actual binary representation might not be a trendy string), then (to the nearest percent) what percent of the strings in S are divisible by 3?

Problem 1.3 [4 Points] Let S be the set of all trendy strings of length 2021. If each string is interpreted as a **binary** number with any leading zeros ignored (so the actual binary representation might not be a trendy string), then (to the nearest percent) what percent of the strings in S are divisible by 4?

Problem 2 [9 Points]

Four numbers are randomly generated from the interval $[0, 1]$ to be used as roots for a (monic) quartic polynomial. The polynomial can be written as $f(x) = x^4 + mx^3 + nx^2 + px + q$ for real numbers m, n, p, and q.

Problem 2.1 [1 Point]

What is the expected value of m?

Problem 2.2 [2 Points]

What is the expected value of $f(1)$?

Problem 2.3 [3 Points]

Let the second largest root of $f(x)$ be r. What is the probability that $r < \frac{1}{2}$?

Problem 2.4 [3 Points]

Mr. Thetoad notices that no matter which two of the four roots that he selects and sums up, he always gets a value greater than 1! What is the probability that all pairwise sums of the roots are greater than 1?

4 Trivia

4.1 Movie Madness [1 point each]

How well do you know your movies? Identify the movies that these scenes were taken from.

a.

b.

c.

d.

4.2 Mr. Worldwide [1 point each]

The world as we know it wouldn't be quite the same without certain well-known brands –
give the names of the multinational corporations being referred to for one point each.

a. This convenience store chain values its hours highly - so highly that it went under a
whole name change to indicate it.
b. This huge food company added an nontraditional summer treat for dogs to its product
line in 2011.
c. This company draws its name from a swift-running breed of antelope, perfect for its
product.
d. 2008 was not a great year for many companies, but this financial institution kept afloat
by merging with the fourth largest bank holding company at the time.

4.3 Odd One Out [2 points each]

All but one in each list have something in common – 1. what do they have in common, and
2. which element does not fit?

a. Parabola, Hippopede, Tractrix, Golden Ratio, Witch of Agnesi
b. Binghamton, Bismarck, Jackson, Montgomery, Raleigh
c. Ingredient, Agent, Sniper, Service, Santa
d. anthropology, canyon, hemisphere, organism, victory

5 Science

5.1 Biology [1 pt each]

a. A sample of DNA is measured and determined to be composed of 22 percent adenine bases. What percentage of the bases in this sample are cytosine bases?

b. Which class of DNA enzymes is primarily responsible with the formation of phosphodiester bonds between two strands of DNA?

c. Suppose that there are many red, blue, and green feathered penguins that live in a certain antarctic area. During a hunting season, the population of penguins in this area is decimated, and all of the red penguins are wiped out. The population of penguins eventually rebounds to normal numbers, but there are now no red feathered penguins in the population. What is the name of the phenomenon described in this situation?

5.2 Chemistry [1 pt each]

a. In a titration experiment, it takes 120 mL of 0.5 M sulfuric acid to reach equivalence. How many mL of 10 M hydrochloric acid would it have taken to reach equivalence in the same experiment?

b.

$$Cu_{(s)} + 2H^+_{(aq)} \rightleftharpoons Cu^{2+}_{(aq)} + H_{2(g)}$$
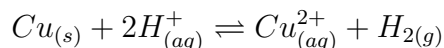
The above reaction occurs under standard conditions (1 atm, 298 K) and has a standard cell potential $E^0_{cell} = -0.34$. The (m rounded to nearest integer) value of the equilibrium constant K for this reaction can be written as $m \cdot 10^n$ for integers m and n. What is the value of $m + n$?

c. In a ground state Cl atom, how many completely filled p orbitals are there?
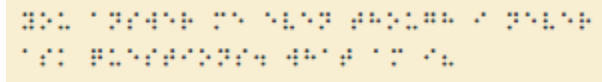
5.3 Physics [1 pt each]

a. A ball is thrown off a 30 meter tall cliff at an angle of 45 degrees above the horizon. If the ball hits the ground at a speed of 25 meters per second, what was the initial speed of the ball when thrown off the cliff, in meters per second? Assume no air resistance and use $g = 10$ meters per second squared.

b. A sample of gas at a temperature of 300 K is composed entirely of hydrogen gas. If each H_2 molecule in the sample is replaced by a molecule of oxygen gas moving at the same speed, then, rounded to the nearest hundred, what will be the new temperature (in Kelvin) of the sample?

c. What is the SI unit for magnetic flux?

6 Codes and Ciphers

Provide the answer to the following questions/descriptions that are in code.

6.1 [2 pts]

6.2 [1 pt]

ejsy fp upi hry ejrm upi vp,nomr djogy smf yjr nsvl wipyr lru

6.3 [1 pt]

84433007777332226666300866600555277778008444444664

6.4 [2 pts]

YHQAGXWQRMTTTILNSRAFFTECNWRQNMURQSBIEZT
(hint: FAMAT)

7 Logic and Puzzles

7.1 Definitely Not Algebra [1 point each]

Amy loves doing math problems in her diary. However, afraid that someone would read it and steal her answers, she encoded her equations by replacing each digit (1-9, no zeros!) with a letter. Find the number on the right side of the equal sign for all of her equations (ex. BAB for part a).

a. $AA * BA/A - 15 = BAB$
b. $CD * C + DD - D = DCD$
c. $EF * FE + EE + FE + 79 = FEFE$

7.2 Crossing the River [4 pts]

Arjun, Bahar, Caroline, and Daniel are trying the cross the Pythagoras river in a canoe that can hold only two people at a time. On all 3 forward trips across the river, 2 people were on the canoe, and on both return trips, one person was on the canoe. Arjun, having stage fright, was unable to row when someone else was also in the canoe with him, and Bahar, having skipped arm day, was unable to row when anyone but Caroline was with her. Each person rows at least once. Who rowed twice?

7.3 Rebus Puzzles [1 point each]

These rebus puzzles use the size and position of different words to suggest a certain phrase. Give the word or phrase of each puzzle as your answer.

a.

b.

c.

d.
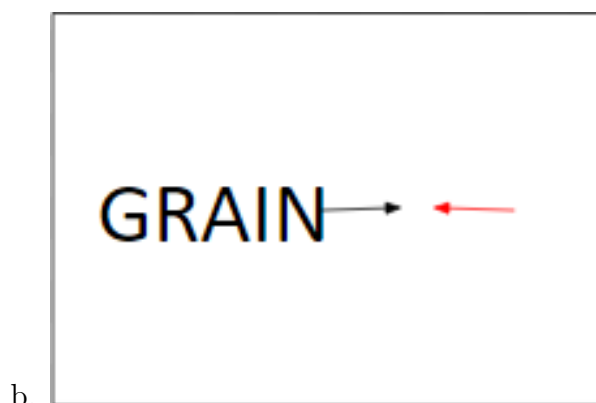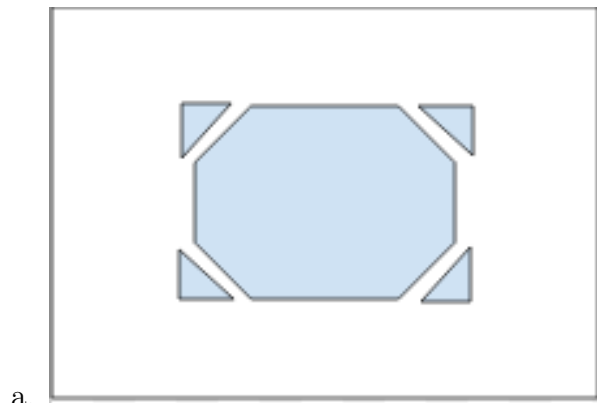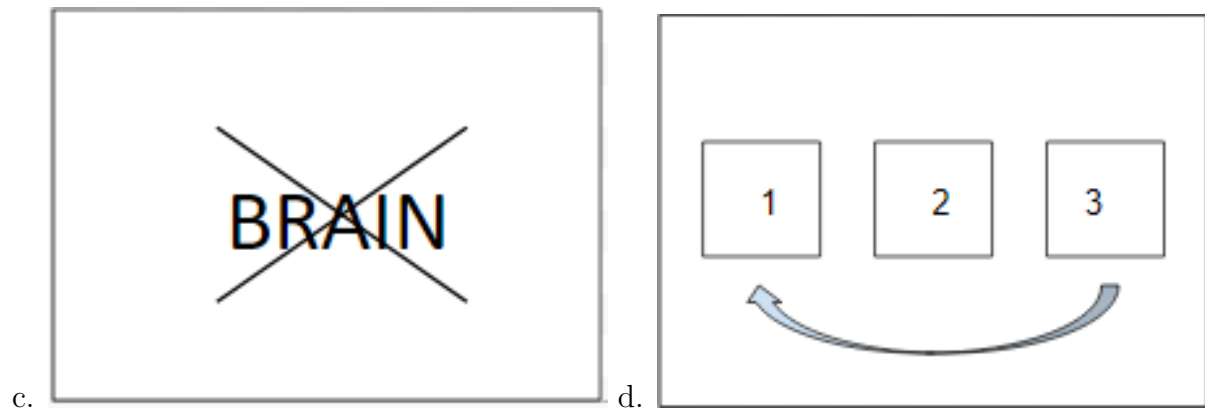
7.4 Math Team Field Trip [10 points]

Four friends on a math team decide to go on a field trip to the beach. They all brought their own lunch, but they seem to all have gotten scrambled! Help them sort out everyone's lunch.

- David likes apples and food with holes.

- The vegetarian is very health conscious so they choose an all healthy meal. Even the cheese has veggies in it!

- One of the boys likes junk food, but not donuts.

- Andy and Annie like a similarly shaped dessert.

- One girl likes a soft sandwich and her meat choice is perfect for their location.

- The person who had beef did not have cheddar.

- The tuna was served with the only soft cheese.

	Cola	Apple juice	Lemonade	Water	Cupcake	Donut	Muffin	None	Swiss	Cheddar	Pepper Jack	Brie	Ham	Tuna	Beef	Veggie
Andy																
Annie																
David																
Jenny																
Ham																
Tuna																
Beef																
Veggie																
Swiss																
Cheddar																
Pepper Jack																
Brie																
Cupcake																
Donut																
Muffin																
None																

8   State Convention

Thank you for participating in the 2020 FAMAT Fall Interschool! We have two more tasks for you. These will not count for any points, but they may carry financial rewards for the winning schools.

- Come up with a 2021 State Convention **poster topic**. Each school may submit more than one topic suggestion. Members of the FAMAT Board will review all suggestions and choose the topic.

- Create a design for the 2021 State Convention **T-shirt/Program Cover**. Black and white, camera ready, pictures may be placed on standard white copy paper. A second copy of the same picture with suggested colors can also be turned in.

The school whose design is selected by the FAMAT Board for the front of the State Convention t-shirt will receive two free student registrations for the State Convention while the school whose design is selected for the back of the t-shirt will receive one free student registration. Schools may submit more than one design.

Poster topic and t-shirt/program logos must be emailed by **Wednesday, November 4** to famatconvention@gmail.com with the subject "2021 State Convention Poster Topic Suggestions" for poster topics and "2021 State Convention T-Shirt Designs" for t-shirt designs.

Please remember to include contact information when sending your email. For each t-shirt design attachment, please save it as:
SchoolName_ArtistName_BW (change BW to Color for the color file)

And, once again, remember to email your answer sheet for this test by [insert date] to robsnow23@yahoo.com.