Test 1 - In-Class Review Solutions

1) a) -1 b) 4 c) DNE d) -1 e) Undefined f) ∞ g) $-\infty$
h) $-\infty$ i) $-\infty$ j) 0 K) 0

2) a)
$x=-6$ Jump Discontinuity
$x=-3$ Removable Discontinuity
$x=-1$ Removable Discontinuity
$x=3$ Infinite Discontinuity
$x=8$ Infinite Discontinuity

b)
$x=-6$ Discontinuity
$x=-3$ Discontinuity
$x=-1$ Discontinuity
$x=1$ Kink
$x=3$ Discontinuity
$x=8$ Discontinuity

3) Given $\lim\limits_{x\to1} g(x)=-8$ and $\lim\limits_{x\to1} h(x)=-4$, find:

Note: Show ALL your work

a) $$\lim_{x\to1}\frac{2}{h(x)}=\frac{\lim\limits_{x\to1}2}{\lim\limits_{x\to1}h(x)}=\frac{2}{-4}=-\frac{1}{2}$$

b) $$\lim_{x\to1}\sqrt[3]{g(x)}=\sqrt[3]{\lim_{x\to1}g(x)}=\sqrt[3]{-8}=-2$$

c) $$\lim_{x\to1}\left[5g(x)-2h(x)\right]=5\lim_{x\to1}g(x)-2\lim_{x\to1}h(x)=5(-8)-2(-4)=-32$$

d) $$\lim_{x\to1}g(x)h(x)=\left[\lim_{x\to1}g(x)\right]\left[\lim_{x\to1}h(x)\right]=(-8)(-4)=32$$

4) Evaluate the given limits

a) $$\lim_{x\to1}\sin\left(\frac{\pi x}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

b) $$\lim_{x\to4}\frac{\sqrt{x+1}}{x-4}=\text{DNE}$$

Note: $\lim\limits_{x\to4^+}\frac{\sqrt{x+1}}{x-4}=+\infty$ and $\lim\limits_{x\to4^-}\frac{\sqrt{x+1}}{x-4}=-\infty$

c) $\lim_{h\to 4} \dfrac{\sqrt{h+5}-3}{h-4} = \lim_{h\to 4} \dfrac{\sqrt{h+5}-3}{h-4} \cdot \dfrac{\sqrt{h+5}+3}{\sqrt{h+5}+3} = \lim_{h\to 4} \dfrac{h+5-9}{(h-4)\sqrt{h+5}+3}$

$= \lim_{h\to 4} \dfrac{\cancel{(h-4)}}{\cancel{(h-4)}\sqrt{h+5}+3} = \lim_{h\to 4} \dfrac{1}{\sqrt{h+5}+3} = \dfrac{1}{6}$

d) $\lim_{x\to 0} e^{x}\cos(2x) = e^{0}\cos(0) = 1$

e) $\lim_{x\to 0} \dfrac{\frac{1}{x+4}-\frac{1}{4}}{x} = \lim_{x\to 0} \left[\frac{4}{4}\left(\frac{1}{x+4}\right) - \frac{1}{4}\left(\frac{x+4}{x+4}\right)\right] \div x = \lim_{x\to 0} \dfrac{4-(x+4)}{4(x+4)} \div x$

$= \lim_{x\to 0} \dfrac{\cancel{4}-x-\cancel{4}}{4(x+4)} \cdot \dfrac{1}{x} = \lim_{x\to 0} \dfrac{-1}{4(x+4)} = \dfrac{-1}{16}$

f) $\lim_{x\to 3} \dfrac{x^2-x-6}{x^2-5x+6} = \lim_{x\to 3} \dfrac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x-2)} = \lim_{x\to 3} \dfrac{x+2}{x-2} = \dfrac{5}{1} = 5$

g) $\lim_{x\to\infty} \dfrac{6x^2-2x-1}{2x^2+3x+2} = \lim_{x\to\infty} \dfrac{(6x^2-2x-1)\div x^2}{(2x^2+3x+2)\div x^2} = \lim_{x\to\infty} \dfrac{6-\frac{2}{x}-\frac{1}{x^2}}{2+\frac{3}{x}+\frac{2}{x^2}} = \dfrac{6}{2} = 3$

h) $\lim_{x\to\infty} 9e^{-x}+3 = \lim_{x\to\infty} \dfrac{9}{e^{x}} + 3 = 3$

i) $\lim_{x\to 2^+} \dfrac{x-3}{x-2} = -\infty$

j) $\lim_{x\to 0^-} 1+\dfrac{1}{x} = -\infty$

k) $\lim_{x\to 0} \dfrac{\sin(7x)}{x} = \lim_{x\to 0} \dfrac{\sin(7x)}{x} \cdot \dfrac{7}{7} = 7 \lim_{x\to 0} \dfrac{\sin(7x)}{7x} = 7(1) = 7$

l) $\lim_{x\to 0} \dfrac{1-\cos(4x)}{x} = \lim_{x\to 0} \dfrac{1-\cos(4x)}{x} \cdot \dfrac{4}{4} = 4 \lim_{x\to 0} \dfrac{1-\cos(4x)}{4x} = 4\cdot 0 = 0$

m) $\lim_{x\to 0} \dfrac{x+2}{\cos(x)} = \dfrac{0+2}{\cos(0)} = \dfrac{2}{1} = 2$

n) $\lim_{x\to \frac{\pi}{2}} \sin^2 x + \cos^2 x = \lim_{x\to \frac{\pi}{2}} 1 = 1$

o) $h(x) = \dfrac{|x+8|}{2x+16}$

$$h(x) = \begin{cases} \dfrac{x+8}{2(x+8)} & x > -8 \\ \dfrac{-(x+8)}{2(x+8)} & x < -8 \end{cases} \rightarrow h(x) = \begin{cases} 1/2 & x > -8 \\ -1/2 & x < -8 \end{cases}$$

i) $\lim_{x \to -8^+} h(x) = 1/2$ ii) $\lim_{x \to -8^-} h(x) = -1/2$ iii) $\lim_{x \to -8} h(x) = \text{DNE}$

iv) $\lim_{x \to 1} h(x) = 1/2$ v) $\lim_{x \to -11} h(x) = -1/2$

p) $\lim_{x \to 3} f(x)$, $f(x) = \begin{cases} \dfrac{x+2}{2} & x \leq 3 \\ \dfrac{12-2x}{3} & x > 3 \end{cases}$

Since $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \dfrac{x+2}{2} = \dfrac{5}{2}$ and $\lim_{x \to 3^+} \dfrac{12-2x}{3} = \dfrac{6}{3} = 2$

therefore, $\lim_{x \to 3} f(x) = \text{DNE}$

5) Squeeze Theorem

Find $\lim_{x \to 0} f(x)$ given that $4 - x^2 \leq f(x) \leq 4 + x^2$

Since $\lim_{x \to 0} 4 - x^2 = 4$ and $\lim_{x \to 0} 4 + x^2 = 4$ then $\lim_{x \to 0} f(x) = 4$

6) Continuity

A function f is continuous at a if the following three conditions are met.

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

7) a) $g(x) = 7x^2 - 3x^3 + 4x + 5 \quad a = 1$

① $g(1) = 13$

② $\lim\limits_{x \to 1} g(x) = 13$

③ $g(1) = \lim\limits_{x \to 1} g(x)$

$\therefore$ g(x) is continuous at $a=1$

b) $h(x) = \dfrac{4x^2 + 3x - 1}{4x - 1} \quad a = 1/4$

① $h(1/4) =$ undefined

② $\lim\limits_{x \to 1/4} \dfrac{4x^2+3x-1}{4x-1} = \lim\limits_{x \to 1/4} \dfrac{\cancel{(4x-1)}(x+1)}{\cancel{(4x-1)}} = \lim\limits_{x \to 1/4} x+1 = \frac{1}{4}+1 = 5/4$

③ $h(1/4) \neq \lim\limits_{x \to 1/4} h(x)$

$\therefore$ h(x) is discontinuous at $a = 1/4$
"Removable discontinuity"

c) $f(x) = \begin{cases} \dfrac{x^2-25}{x-5} & x \neq 5 \\ 0 & x = 5 \end{cases} \quad a = 5$

① $f(5) = 0$

② $\lim\limits_{x \to 5} f(x) = \lim\limits_{x \to 5} \dfrac{x^2-25}{x-5} = \lim\limits_{x \to 5} \dfrac{(x+5)\cancel{(x-5)}}{\cancel{(x-5)}} = \lim\limits_{x \to 5} x+5 = 10$

③ $f(5) \neq \lim\limits_{x \to 5} f(x)$

$\therefore$ f(x) is discontinuous at $a=5$
"Removable discontinuity"

d) $f(x) = \begin{cases} \dfrac{e^x}{e^x+1} & x \leq 0 \\ x^2+1 & x > 0 \end{cases} \quad a = 0$

① $f(0) = \dfrac{e^0}{e^0+1} = \dfrac{1}{2}$

② $\lim\limits_{x \to 0} f(x) = \text{DNE}$

(Note: $\lim\limits_{x \to 0^-} \dfrac{e^x}{e^x+1} = \dfrac{1}{2}$ and $\lim\limits_{x \to 0^+} x^2+1 = 1$)

③ $f(0) \neq \lim\limits_{x \to 0} f(x)$

$\therefore$ f(x) is discontinuous at $a=0$
"Jump Discontinuity"

8) a) $f(x) = \begin{cases} e^{2x+c} & x \geq 0 \\ x+2 & x<0 \end{cases}$

By continuity, $\lim_{x\to 0} f(x)$ must exist. Then,

$$\lim_{x\to 0^+} e^{2x+c} = \lim_{x\to 0^-} x+2$$

$$e^{2(0)+c} = 0+2$$

$$e^c = 2$$

$$\ln e^c = \ln 2$$

$$c \ln e = \ln 2$$

$$c = \ln 2$$

b) $f(x) = \begin{cases} \sin(x+c) & x<1 \\ x^2-1 & x\geq 1 \end{cases}$

By continuity, $\lim_{x\to 1} f(x)$ must exist. Then,

$$\lim_{x\to 1^-} \sin(x+c) = \lim_{x\to 1^+} x^2-1$$

$$\sin(1+c) = 1^2-1$$

$$\sin(1+c) = 0$$

$$\sin^{-1}(\sin(1+c)) = \sin^{-1}(0)$$

$$1+c = \sin^{-1}(0)$$

$$c = \sin^{-1}(0) - 1$$

$$c = \left(\frac{\pi}{2} + n\pi\right) - 1 \quad \text{for } n=0,1,\ldots$$

$$c = \left(\frac{\pi}{2} - 1\right) + n\pi \quad \text{for } n=0,1,\ldots$$

9) a) $f(x) = 4 - 3x^3 + 4x^5$ Continuous on $(-\infty, \infty)$

b) $g(x) = \dfrac{10}{-3x^2-5x+2} = \dfrac{10}{-(3x-1)(x+2)}$ Continuous on $(-\infty,-2) \cup (-2, 1/3) \cup (1/3, \infty)$

c) $h(x) = \sqrt{5-9x}$ Note: $5-9x \geq 0$

$$\frac{-9x}{-9} \geq \frac{-5}{-9}$$

$$x \leq 5/9$$

Continuous on $(-\infty, 5/9]$

d) $y = \cos(x)$ Continuous on $(-\infty, \infty)$

10) Intermediate Value Theorem

$f(x) = 4x^3 - 6x^2 + 3x - 2 \qquad [1, 2]$

$f(x)$ is continuous on $[-1, 2]$. Since $f(1) = -1$ and $f(2) = 12$ it follows that $f(1) < 0$ and $f(2) > 0$. Therefore by the Intermediate Value Theorem there must be some c in $[1, 2]$ such that $f(c) = 0$.

11) a) $y = 3x^2 - x \qquad (1, 2)$

Step 1: Find m_{tan} at $a = 1$

$$m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3(1+h)^2 - (1+h) - [3(1)^2 - 1]}{h}$$

$$\lim_{h \to 0} \frac{3(1+2h+h^2) - 1 - h - 2}{h} = \lim_{h \to 0} \frac{\cancel{3} + 6h + 3h^2 - \cancel{3} - h}{h}$$

$$\lim_{h \to 0} \frac{h(3h+5)}{h} = \lim_{h \to 0} 3h + 5 = 5 \qquad m_{tan} = 5$$

Step 2: Find equation

$y - y_1 = m(x - x_1)$

$y - 2 = 5(x - 1)$

$y = 5x - 5 + 2$

$\to y = 5x - 3$

b) $y = \dfrac{8}{x+4} \qquad (0, 2)$

Step 1: Find m_{tan} at $a = 0$

$$m_{tan} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{8}{h+4} - \frac{8}{4}}{h}$$

$$= \lim_{h \to 0} \frac{8(4) - 8(h+4)}{4(h+4)} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{\cancel{32} - 8h - \cancel{32}}{4h(h+4)}$$

$$= \lim_{h \to 0} \frac{-2}{h+4} = \frac{-2}{4} = \frac{-1}{2}$$

$m_{tan} = \frac{-1}{2}$

Step 2: Find the equation

$y - 2 = -\frac{1}{2}(x - 0)$

$y = -\frac{1}{2}x + 2$

12) $h(t) = -16t^2 + 64t + 80$

a) $h(0) = 80$ ft/s

b) Average Velocity

i) $[0, 1]$ $\quad$ $\text{Avg. Velocity} = \dfrac{h(1) - h(0)}{1 - 0} = \dfrac{128 - 80}{1} = 48 \text{ ft/s}$

ii) $[0, 0.001]$ $\quad$ $\text{Avg. Velocity} = \dfrac{h(0.001) - h(0)}{0.001 - 0} = \dfrac{80.064 - 80}{0.001} = 64 \text{ ft/s}$

c) Instantaneous Velocity at $t = 0$.

$$\text{Inst. Velocity} = \lim_{\Delta t \to 0} \frac{h(0 + \Delta t) - h(0)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-16(\Delta t)^2 + 64(\Delta t) + \cancel{80} - \cancel{80}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\Delta t}(-16\Delta t + 64)}{\cancel{\Delta t}} = \lim_{\Delta t \to 0} -16(\Delta t) + 64 = 64 \text{ ft/s}$$

13) The derivative of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{provided the limit exists.}$$

Note: f' is a function of x for all x for which this limit exists.

14) $f(x)=\sqrt{1-9x}$

a) $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{1-9(x+h)}-\sqrt{1-9x}}{h}$

$=\lim_{h\to 0}\frac{\sqrt{1-9x-9h}-\sqrt{1-9x}}{h}\cdot\frac{\sqrt{1-9x-9h}+\sqrt{1-9x}}{\sqrt{1-9x-9h}+\sqrt{1-9x}}$

$=\lim_{h\to 0}\frac{1-9x-9h-(1-9x)}{h(\sqrt{1-9x-9h}+\sqrt{1-9x})}=\lim_{h\to 0}\frac{-9}{\sqrt{1-9x-9h}+\sqrt{1-9x}}$

$=\frac{-9}{\sqrt{1-9x}+\sqrt{1-9x}}=\frac{-9}{2\sqrt{1-9x}}$

therefore, $f'(x)=\frac{-9}{2\sqrt{1-9x}}$

b) Equation.

$m_{tan}=f'(0)=\frac{-9}{2\sqrt{1-9(0)}}=\frac{-9}{2}$

$y-1=-9/2\,(x-0)$

$y=-9/2\,x+1$

15) Conditions

a) $\lim_{x\to -4}f(x)=\infty$ V.A. $x=-4$

b) $\lim_{x\to \pm\infty}f(x)=0$ H.A $y=0$

d) $\lim_{x\to 2^+}f(x)=-\infty$ V.A $x=2$

c) $f(0)=2$ $(0,2)$

e) $\lim_{x\to 2^-}f(x)=\infty$ V.A. $x=2$