Adversarial Search
(or Game Search)

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Lesson 1

GAME SEARCH TERMINOLOGY

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Two-player zero-sum discrete finite deterministic games of perfect information

- **Two player:** well, there are two players…
- **Zero Sum:** In any outcome of any game Player A's gains equals Player B's losses.
- **Discrete:** All game states and decisions are discrete values.
- **Finite:** There are only a finite number of states and decisions.
- **Deterministic:** no chance… no dice rolls… etc
- **Games:** defined shortly….
- **Perfect information:** Both players can see the state, and each decision is made sequentially.

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A Game Defined

- A two-player zero-sum discrete finite deterministic game of perfect information is a quintuplet, (S, I, Succs, T, V) where:
  - S: Finite set of states (must include sufficient information to deduce whose turn it is to move next)
  - I: Initial state
  - Succs: Function that takes a state as input and returns a set of states (legal positions after a move).
    - Must be non-empty if its argument is not a terminal state
  - T: Set of terminal states (i.e., states when game ends and payoff occurs)
  - V: Maps terminal states to real values (player A's gain / player B's loss)

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Example: Nim

- You begin with some number of piles of matches.
- During a turn, the player may remove any number of matches from one pile
- The last person to remove a match loses
- In II-Nim, you begin with two piles each with two matches
- **States of Nim**
  - A(jj,jj); A(j,jj); A(,jj); A(jj,j); A(jj,); A(j,j); A(,j); A(j,); A(,)
  - B(jj,jj); B(j,jj); B(,jj); B(jj,j); B(jj,); B(j,j); B(,j); B(j,); B(,)

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Common Trick: Exploit Symmetry

- **States of Nim**
  - A(jj,jj); A(j,jj); A(,jj); A(jj,j); A(jj,); A(j,j); A(,j); A(j,); A(,)
  - B(jj,jj); B(j,jj); B(,jj); B(jj,j); B(jj,); B(j,j); B(,j); B(j,); B(,)
- **Common Trick: Symmetry**
  - Some states are trivially equivalent (e.g., A(,jj); A(jj,))
  - Use some canonical description to make them one state
    - e.g., right pile always has at least as many matches as right
- **States of Nim Using Symmetry**
  - A(jj,jj); A(j,jj); A(,jj); A(j,j); A(,j); A(,)
  - B(jj,jj); B(j,jj); B(,jj); B(j,j); B(,j); B(,)

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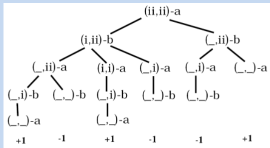
Nim Formalized

- S = {A(jj,jj); A(j,jj); A(,jj); A(j,j); A(,j); A(,); B(jj,jj); B(j,jj); B(,jj); B(j,j); B(,j); B(,)}
- I = A(jj,jj)
- T = {A(,); B(,)}
- V(A(,)) = +1; V(B(,)) = -1
- Succs(A(jj,jj)) = { B(j,jj); B(,jj) }
- Succs(B(j,jj)) = { A(,jj); A(j,j); A(,j) }
- Succs(B(,jj)) = { A(,j); A(,) }
- etc.



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Lesson 2

GAME THEORETIC VALUES AND SOLVING GAMES

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Game Theoretic Value

- We usually assume that all players are perfectly rational
  - Perfect rationality = always chooses action that maximizes expected value of outcome given what we know
  - Useful assumption even though not met in practice
- **Game theoretic value:** The game theoretic (or minimax) value of a game state is the value of the terminal state that will be reached if both players play optimally.
- Game theoretic value of a terminal state is simply the value of the state.
- How do we find the game theoretic value of non-terminal states?

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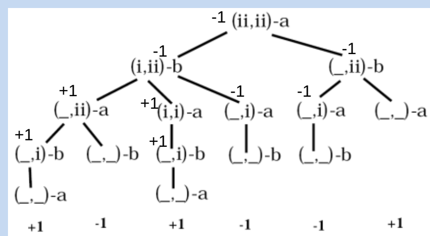
Game Theoretic Value

- How do we find the game theoretic value of non-terminal states?
- Idea: fill in the tree bottom-up.



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The Minimax Algorithm

- Generate the full Game tree, storing it in memory
- Run through all of the terminal states assigning them values
- Run through all predecessors assigning them values, etc, etc, etc…
- Question: Do we really need to store the whole game tree in memory?
  - NO… Use DFS

```
Minimax-Value(S)
    if (S is a terminal)
        return V(S)
    else
        Let S1,S2,…Sk = Succs(S)
        Let vi=Minimax-Value(Si)
        for each Si.
        If PlayerToMove(S) = A
            return Max(vi)
        else
            return Min(vi)
```

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Dynamic Programming (DP)

- Dynamic Programming---Russell & Norvig's definition:
  - "solutions to subproblems are constructed incrementally from those of smaller subproblems and are cached to avoid recomputation"
- You may have encountered this in other classes (e.g., if you've taken Data Structures & Algorithms II).
  - E.g., Floyd-Warshall for All Pairs Shortest Paths is a dynamic programming algorithm

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Solving Games

- Solving a Game means determining the game theoretic value for the initial state (i.e., determining who will win if both players play optimally)
- Running the minimax algorithm on the initial state will solve a game:
  - Minimax requires that $O(B^L)$ states are expanded (worst case and best case), where B is the average branching factor, and L is the usual length of the game.
  - DFS on simple search problems can get lucky and find a solution quickly but Minimax is a DFS of the entire game tree, so always $O(B^L)$
- What if the total number of game states, N, is much less than B^L ?
  - E.g., for chess, $N = 10^{40}$, while $B^L = 10^{120}$
- In such cases, DP is a better method, assuming you can afford the memory.
  - Runtime of DP for game solving: $O(NL)$. Memory: $O(N)$.

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Example: DP for Chess Endgames

- Playing the endgame of a game of chess is easy.
  - Chess playing computers use an endgame database that stores the "solutions" for all game states with fewer than a certain number of pieces left on the board.
  - Playing the endgame simply involves database lookups.
- Dynamic programming for generating an endgame database
  - Consider that there are only 4 chess pieces left on the board.
  - With sufficient computational resources, you can compute, for all possible positions, whether it is a win for black, white, or a draw.
  - Details next slide….

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DP for Chess Endgames

- Assume there are N positions with no more than 4 pieces left:
  1. Define a 1-to-1 mapping from the N board positions to the integers 0..N-1
  2. Create an of length N (2 bits per entry). Each element can take on one of three values:
     - W: White will eventually win.
     - B: Black will eventually win
     - ?: We don't know who wins from this state
  3. Mark all terminal states with their values, W or B.
  4. Look through all states still marked by "?"
     - If W is about to move, then
       - if all successors are marked with B, mark the state B
       - if any successor state is marked W, then mark the state W
       - else leave the state unchanged (marked "?")
     - if B is about to move, then
       - if all successors are marked with W, mark the state W
       - if any successor state is marked B, then mark the state B
       - else leave the state unchanged (marked "?")
  5. If 4 changed the label of at least one state, then repeat 4.
  6. Any state still marked with "?" is a state from which no one can force a win---thus a draw.

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Lesson 3

ALPHA BETA PRUNING

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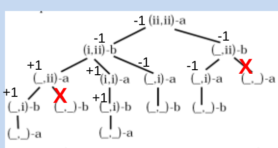
Cutting Off Unnecessary Game States

- We're now returning to minimax with DFS.
- Minimax searches the entire game tree with DFS.
- So we really need to search the entire game tree?
- If we knew that the only possible game values were +1 and -1, can we save computation?
  - Yes (a lot actually, although not much in this example)
  - If any successor is a forced win for the current player, then don't bother expanding any further successors.



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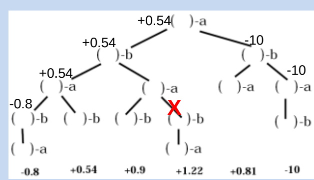
What if possible terminal values are unknown?

- Do DFS, but if something is discovered that implies your parent would not choose you, then don't bother expanding further successors.
- More generally, not just your parent, but any ancestors.



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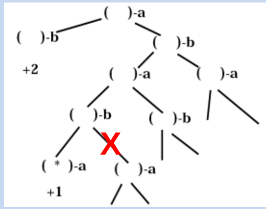
An ancestor causing cutoff….

- Suppose we've done a full DFS, expanding left-most successors first and that we are currently at the search state marked by the *
- What can we cut off in the rest of the search?
- If either player has a better alternative at an ancestor of a given search node, then it will not be visited.



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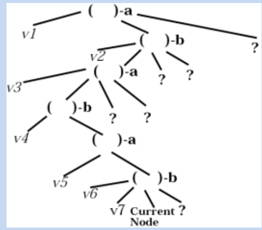
A general cutoff rule



- In this example:
  - let α=max(v1,v3,v5)
  - let β=min(v6,v7)
  - if β <= α, then we can be certain that it is a waste of time searching the "current node" or its sibling to the right
- In general:
  - if at a B-move node,
    - let α=max of all A's choices on current path
    - let β=min of all B's choices including those at current node
    - Cut-off if β<= α
  - Converse rule if at an A-move node

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Alpha Beta Pruning

```
function Max-Value(s,α,β)
inputs:
   s: current state in game, A about to play
   α: best score (highest) for A along path to s
   β: best score (lowest) for B along path to S
output: min(β , best-score (for A) available from s)

   if ( s is a terminal state )
   then return ( terminal value of s )
   else for each s' in Succs(s)
            α := max( α , Min-value(s',α,β) )
            if ( α >= β ) then return β
      return α
```

```
function Min-Value(s,α,β)
output: max(α , best-score (for B) available from s)

   if ( s is a terminal state )
   then return(terminal value of S)
   else for each s' in Succs(s)
            β:= min( β, Max-value(s',α,β) )
            if ( β <= α) then return α
      return β
```

- Alpha-Beta Pruning from Russell & Norvig.
- Assumes players alternate moves.
- Top level call: **Max-Value(I,-∞ ,+ ∞)**

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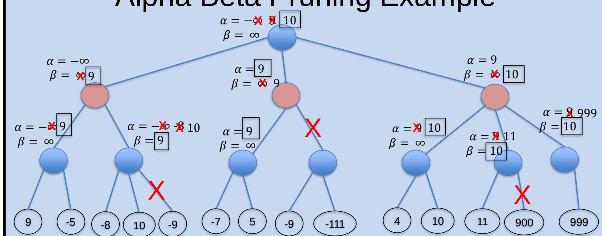
Alpha Beta Pruning Example



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How Useful is Alpha Beta Pruning?

- What is the best case performance of alpha-beta?
- How much of the tree would you examine if you were very lucky in the order you tried successors?
- Best case:
  - The number of nodes you need to search in the tree is $O(B^{L/2})$.
  - The square root of the recursive minimax cost.
  - Large real-sized games with a huge number of states are still problematic (e.g., chess).

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Lesson 4

FROM GAME SOLVING TO GAME PLAYING

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A Few Solved Games

- So far, we've entirely focused on game solving (determining what the outcome of a game will be if both players play optimally).
- **Solving a game** means proving the game-theoretic value of the start state.
- A few solved games:
  - By brute-force dynamic programming
    - Four-in-a-row
    - Chess endgames (certainly NOT chess itself though)
  - By brute-force DP to create endgame database, plus alpha-beta search
    - nine men's morris
  - By brute-force DP plus game specific analysis
    - Connect-Four

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Checkers has been Solved: Draw

- The largest game that has been solved is Checkers.
- There are approximately $5 * 10^{20}$ states in the game of checker.
- Checkers was solved by a system called Chinook.
  - Mostly brute-force Dynamic Programming
  - Dozens of computers during the years 1989-2007 (18 years, with a couple year break in the middle)
  - First computer program in any game that won the right to play in a human world championship (1990).
  - Lost in 1992, but became the world champion in 1994.
  - Retired from playing in human tournaments in 1996 (clear no human could win).
  - Game theoretic value of checkers is 0 (a draw).

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Game Playing vs Game Solving

- Two very different activities
- Game Solving: finding the true game-theoretic value of a state.
- What about game playing?
- Game solving often very different from playing a game well.
- Example, what do real chess playing programs do?
- Some features that the search algorithms covered so far don't have:
  - Cannot possibly find a guaranteed solution (not enough time).
  - Must make decisions quickly in real-time.
  - It is not possible to pre-compute a solution.

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Heuristic Evaluation Functions

- Popular solution: use heuristic evaluation functions
- An evaluation function maps a state to a real value.
  - The higher the evaluation, the higher the true game-theoretic value is estimated to be.
- Note: this is not the same as the heuristic in A*…
  - no notion of admissibility
  - not an estimate of path cost to reach a goal
- Search the game tree as deeply as time allows
- Leaves of tree you search are not leaves of game tree, but are intermediate nodes
- The values assigned to leaves are from the heuristic evaluation function

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Heuristic Evaluation Intuition

- **Visibility:**
  - Evaluation function will be more accurate nearer the end of the game.
  - So worth using heuristic estimates from there.
- **Filtering:**
  - If we used the evaluation function without searching, we'd be using a handful of inaccurate estimates (near the root).
  - By searching, we're combining thousands of these estimates, hopefully eliminating noise.
- Can give counter-examples… But often works very well in practice for real games…

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Heuristic Evaluation Example

- A simple heuristic for chess:
  - The typical introductory chess book heuristic: a bishop or knight worth the value of 3 pawns; a rook worth 5; a queen worth 9
  - This leads to a simple **weighted linear evaluation** function
- More sophisticated chess heuristics consider other state features:
  - good pawn structure might be worth value of a pawn
  - "king safety" might be worth a pawn
- Or **nonlinear evaluation functions** are possible:
  - two bishops might be worth slightly more than twice the value of a single bishop
  - a bishop near the end of the game may be worth more than earlier in the game (e.g., more powerful in open space)
- Machine learning also applicable here
  - E.g., to learn a heuristic (then use alpha-beta pruning search)

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Monte Carlo Tree Search

- Monte Carlo Tree Search, an alternative to using a heuristic
- Common for the game of Go, as well as games with chance.
- Pure Monte Carlo Tree Search:
  - Let's say player has N alternative moves they can make.
  - Simulate making move 1, and then both players playing randomly to the end of the game, record the outcome (1 for win, -1 for loss, 0 for draw), repeat this process k times, and average the outcomes.
  - Repeat for move 2, move 3, … move N.
  - Pick the move that has the highest average outcome.
- One simple improvement to Pure Monte Carlo Tree Search:
  - Run Alpha-Beta Pruning search to depth D. If the states at that depth are not terminal, then use Monte Carlo Tree Search to evaluate the states at depth D.

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Some Other Issues for Game Playing

- How to determine how far to search if you only have a fixed time to make a decision?
- **Quiescence:** What if you stop the search at a state where subsequent moves drastically change the evaluation?
  - e.g., you search to depth d in chess, but at depth d+1, a queen is taken…
- **Quiescence search:** an extra bit of search to attempt to reach a quiescent state
  - e.g., in chess, continue search only considering "capture" moves to resolve any uncertainties in position

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More Game Playing Issues

- **The horizon problem:**
  - Consider a state in which it is inevitable that your opponent will be able to do something bad to you.
    - e.g., an inevitable queening of a pawn
  - Now consider that you have some delaying tactics.
  - The search algorithm won't recognize the inevitable if the number of delaying steps exceeds the search depth limit…
  - Thus not recognizing the badness of the search state.
- **Endgames:** Are easy to play well. How?
  - An end game database
    - essentially a lookup table (e.g., generated by DP)
- **Openings:** Are easy to play well. How?
  - An opening book
  - e.g., for chess, based on hundreds of years of human chess playing knowledge

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Putting it all together….

```
MaxValue(s, α, β, d)
  if s is a terminal state
     return V(s)
  else if s is in endgame DB
     return Endgame(s)
  else if d = 0
     return Heuristic(s)
  for each s' in Succs(s)
     α=max(α, MinValue(s',α,β,d-1)
     if α >= β then return β
  return α
```

```
MinValue(s, α, β, d)
  if s is a terminal state
     return V(s)
  else if s is in endgame DB
     return Endgame(s)
  else if d = 0
     return Heuristic(s)
  for each s' in Succs(s)
     β=min(β, MaxValue(s',α,β,d-1)
     if α >= β then return α
  return β
```

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How to choose a move

```
ChooseMovePlayerA(s,d)
  bestMove = null
  bestValue = -∞
  if s is in opening book
     return opening(s)
  for s' in Succs(s)
     v = MinValue(s',-∞,∞,d-1)
     if v > bestValue
        bestValue = v
        bestMove = s'
  return bestMove
```

```
ChooseMovePlayerB(s,d)
  bestMove = null
  bestValue = ∞
  if s is in opening book
     return opening(s)
  for s' in Succs(s)
     v = MaxValue(s',-∞,∞,d-1)
     if v < bestValue
        bestValue = v
        bestMove = s'
  return bestMove
```

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Lesson 5

GAMES INVOLVING CHANCE

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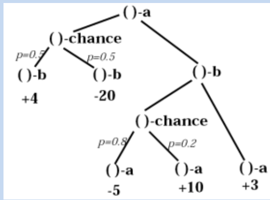
2-player zero-sum finite nondeterministic games of perfect information

- The game tree now includes states in which neither player makes a choice.
- Instead, a random decision is made according to a known set of outcome probabilities.
- Game-theoretic value is the **expected** final outcome if both players are optimal.



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Expectiminimax

- Obvious generalization of minimax:
  - Expectiminimax(s) =
    - Value(s) if s is a terminal state
    - max{s' in succs(s)} Expectiminimax(s') if s is a player A node
    - min{s' in succs(s)} Expectiminimax(s') if s is a player B node
    - Sum{s' in succs(s)} P(s') Expectiminimax(s') if s is a chance node
- Can we use alpha-beta pruning? yes
  - for Min and Max nodes it works unchanged
  - for chance nodes, if we have a bound on terminal values
    - compute an upper bound on the value of a chance node without looking at all of its children

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Bad News for Expectiminimax

- Assume a game with dice rolls.
- Expectiminimax considers all possible dice roll sequences, then it is:
  - $O(b^m n^m)$ where n is the number of distinct dice rolls
- Example: Backgammon
  - n=21
  - b usually 20, but as high as 4000 for dice rolls that are doubles
  - can probably only manage about m=6 (3 moves each player)
- The equivalent of Alpha-Beta pruning helps the situation a bit but not much
- State-of-the-art Backgammon programs rely heavily on sophisticated evaluation heuristics utilizing machine learning techniques

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TD-Gammon (Gerald Tesauro, 1992)

- Machine learning to learn a heuristic board evaluation function
  - A neural network trained by TD-lambda
    - TD-lambda: A form of "temporal difference learning", a type of reinforcement learning.
- TD-Gammon stopped improving performance after 1.5 million games against itself.
- Only 2-ply look ahead used.
- Reached level equivalent to the top human players at the time.
- Made human expert players rethink how to play
  - Discovered strategies that human players never considered or mistakenly thought bad
  - Tournament players changed how they play
  - Began using TD-gammon to study and learn new strategies
- Strong positional player, sometimes plays poorly in end game

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