TEST 2 REVIEW SOLUTIONS

1) a) B_1B_1, B_1B_2, B_1W, B_2B_1, B_2B_2, B_2W, WB_1, WB_2, WW

b) B_1B_1, B_1B_2, B_2B_1, B_2B_2 $\quad P(B,B) = \frac{4}{9}$

c) B_1W, B_2W, WB_1, WB_2 $\quad P(\text{different}) = \frac{4}{9}$

2) $\underset{\text{LETTERS}}{\underline{26}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }}\quad \underset{\text{DIGITS}}{\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }}$ $\quad$ a) $26^4 \cdot 10^3$ $\quad$ b) $\underset{A}{\underline{1}}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{10}\ \underset{O}{\underline{1}}$ $\quad \boxed{26^3 \cdot 10}$

c) $\underline{I}\ \underline{G}\ \underline{I}\ \underline{F}\ \underline{10}\ \underline{10}\ \underline{10}$ $\quad \boxed{10^3}$

3) To prove an existence statement we must find the numbers that satisfy the given conditions

if $\boxed{m = n = 2}$ Then m & n are integers and

$\frac{1}{2} + \frac{1}{2} = 1$ is an integer.

4) Base case $P(1) : 1^2 = \frac{1 \cdot 2 \cdot 3}{6}$ ✓

Induction Step: Assume P_k is true $\forall k$, show $P_{(k+1)}$ is true

Suppose $1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Add $(k+1)^2$ $\quad 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

we now work on the RHS.
we would like to make RHS
$= $ to $\frac{(k+1)(k+2)(2(k+1)+1)}{6}$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6}$$

Factor $(k+1)$ and distribute the rest

$$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$$

(4) continued $1^2+2^2+\dots+(k+1)^2 = \frac{(k+1)(2k^2+7k+6)}{6}$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+2)(2k+2+1)}{6}$$

$$1^2+2^2+\dots+(k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

So $P(k+1)$ is true

By induction $1^2+2^2+\dots n^2 = \frac{n(n+1)(2n+1)}{6}$ $\forall n \in \mathbb{Z}^+$

(5) To determine this we try a few examples.

$\frac{3+5}{2} = \frac{8}{2} = 4$ example of average of 2 odds that is even.

$\frac{9+25}{2} = \frac{34}{2} = 17$ example of average of 2 odd that is odd.

This statement is true for some and false for other integers.

(6) For all integers a and b if $a|b$ then $a^2|b^2$. We may believe this to be true because $3|9$ and $9|81$. We will try to prove this using a direct proof.

(next page)

Suppose $a \mid b$

There $\exists$ $k \in \mathbb{Z}$ such that $b = k \cdot a$

Square both sides

Then $b^2 = k^2 a^2$

Since k^2 is an integer

$a^2 \mid b^2$

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7) a) $C(16,7) = \dfrac{16!}{7!(16-7)!} = \dfrac{16 \cdot \cancel{15}^{5} \cdot \cancel{14}^{2} \cdot 13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10} \cdot \cancel{9!}}{\cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{9!}} = 11{,}440$

b) $C(16,13) = \dfrac{16!}{13!(3!)} = \dfrac{\cancel{16}^{8} \cdot \cancel{15}^{5} \cdot 14 \cdot \cancel{13!}}{\cancel{13!} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 560$

$C(16,14) = \dfrac{16!}{14!\,2!} = \dfrac{\cancel{16}^{8} \cdot 15 \cdot \cancel{14!}}{\cancel{2} \cdot \cancel{14!}} = 120$

$C(16,15) = \dfrac{16!}{15! \cdot 1!} = \dfrac{16 \cdot \cancel{15!}}{\cancel{15!}} = 16$

$C(16,16) = \dfrac{16!}{16! \cdot 0!} = 1$

At least 13 means 13 or 14 or 15 or 16

So $560 + 120 + 16 + 1 = 697$

c) At least one 1 = all possible bit string − # of bit strings with no 1's

There is one bit string with no 1's

and 2^{16} possible bitstrings

So $2^{16} - 1 = 65{,}535$ bit strings contain at least a 1

d) At most one 1 means one 1 or no 1's
There is one bit string that contains no 1's
and $C(16,1) = \frac{16!}{15! \cdot 1!} = 16$ that contain one)

So $16+1 = 17$ bit strings contain
at most one 1.

(8) We will try a direct proof.
Suppose x, y & z are consecutive integers
Then $y = x+1$ and $z = x+2$

$x + x+1 + x+2 = 3x+3 = 3(x+1)$ with $x+1$ an integer.

So $3 \mid (x+x+1+x+3)$

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(9) Proof by cases:
Suppose both m & n are odd
Then $\exists$ a and b integers such that
$m = 2a+1$ and $n = 2b+1$
Then $m+n = 2a+2b+2 = 2(a+b+1)$ - even.
$m-n = 2a+1-2b-1 = 2(a-b)$ - even.

Suppose both m & n are even
Then $\exists$ k & j integers such that
$m = 2k$ and $n = 2j$
Then $m+n = 2(k+j)$ - even
and $m-n = 2(k-j)$ - even.

(9) Suppose that one is even and the other odd. Without loss of generality let m be even and n be odd.

Then $\exists$ c & d integers such that

$m = 2c$ and $n = 2d+1$

Then $m+n = 2c+2d+1 = 2(c+d)+1$ - odd.

and $m-n = 2c-2d-1$

Note: $2(c-d)-1$ is odd but if we want to put the number in $2k+1$ form, we do this

$= 2c-2d-1+2-2$

$= 2c-2d-2+1 = 2(c-d-1)+1$ - odd

since $c-d-1$ is an integer.

So $m+n$ and $m-n$ are either both even or both odd.

(10) Proof by cases, again.

Suppose n is an even integer.

Then $\exists$ $k \in \mathbb{Z}$ s.t. $n = 2k$.

Then $(2k)^2 - 2k + 3 = 4k^2 - 2k + 2 + 1 = 2(2k^2-k+1)+1$

odd.

Suppose n is odd,

Then $\exists$ $k \in \mathbb{Z}$ such that $n = 2k+1$

$(2k+1)^2 - (2k+1) + 3 = 4k^2 + 4k + 1 - 2k - 1 + 3 = 2(2k^2+k+1)+1$

odd.

So n^2-n+3 is odd for all integers.

11) $a \equiv 4 \bmod 7$

That is $7 \mid a-4$

So $a-4=7k$ for some integer k.

Multiply by 5

$$5a-20=35k=7\cdot 5k$$

$7 \mid (5a-20)$

So $5a \equiv 20 \bmod 7$

$$\begin{array}{r} 2 \\ 7\,\overline{)\,20} \\ \underline{14} \\ 6 \end{array}$$

$20 \equiv 6 \bmod 7$

So $5a \equiv 6 \bmod 7$

1st

12) a) The theorem refered to here is Theorem 2 on page 202.

If $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^{+}$ then $\exists\, q$ & $r \in \mathbb{Z}$, unique with $0 \le R < d$ s.t.

$a = dq + R$

To show that a # is div. by 3 using this theorem we use $d=3$ and show that $R=0$

Let $n, n+1, n+2$ be 3 consecutive integers

Then $n+n+1+n+2 = 3n+3 = 3(n+1)+0$

where $n+1$ is the quotient and $R=0$.

b) $n+n+1+n+2 \equiv 0 \bmod 3$

just kidding. This is #12 as a "sum"

1st #12

a) & b) let n, $n+1$, $n+2$ be the 3 consecutive integers

The remainder of n mod 3 can be 0, 1 or 2

Case 1 Suppose $n \equiv 0 \mod 3$ ← mod notation (part b)

Using Theorem 2 (p. 202) $n = 3q + 0$ for some integer q

Then $n(n+1)(n+2) = 3q\,(n+1)(n+2) + 0$

So $3 \mid n(n+1)(n+2)$

Case 2 Suppose $n \equiv 1 \mod 3$ ← mod notation (part b)

OR $n = 3q + 1$ for some integer q

Add 2 to both sides $n + 2 = 3q + 3 = 3(q+1)$

So $n(n+1)(n+2) = n(n+1) \cdot 3(q+1)$

$= 3\left(n(n+1)(q+1)\right) + 0$

So $3 \mid n(n+1)(n+2)$

Case 3 Suppose $n \equiv 2 \mod 3$ ← mod notation (part b)

OR $n = 3q + 2$ for some integer q

Then $n+1 = 3q + 2 + 1 = 3(q+1)$

So $n(n+1)(n+2) = n(n+2) \cdot 3(q+1)$

$= 3\left(n(n+2)(q+1)\right) + 0$

So $3 \mid n(n+1)(n+2)$

So $n(n+1)(n+2)$ is divisible by 3 for any integer n.

2nd #12 is a mistake. It is a repeat of # 4.

(13) Base case: $P(1)$ $7^1 - 2^1 = 5$ $5|5$ ✓

Inductive step: Suppose $7^k - 2^k$ is div. by 5

Then $\exists$ $m \in \mathbb{Z}$ such that $7^k - 2^k = 5m$

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k$$
$$= (2+5)7^k - 2 \cdot 2^k.$$
$$= 2 \cdot 7^k + 5 \cdot 7^k - 2 \cdot 2^k.$$
$$= 2 \cdot 7^k - 2 \cdot 2^k + 5 \cdot 7^k.$$
$$= 2(7^k - 2^k) + 5 \cdot 7^k.$$
$$= 2 \cdot (5m) + 5 \cdot 7^k.$$
$$= 5(2m + 7^k) \text{ where } 2m + 7^k \in \mathbb{Z}.$$

So $5 \mid (7^{k+1} - 2^{k+1})$

By induction $5 \mid 7^n - 2^n$ $\forall n \in \mathbb{Z}, n \geq 1$

(14) $|C.S.| = 38$, $|M| = 23$, $|C.S. \cap M| = 7$

$$|C.S. \cup M| = |C.S.| + |M| - |C.S. \cap M|$$
$$= 38 + 23 - 7$$
$$= 54 \text{ students in the class}$$