Applied Category Theory

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Contents

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Why?

Why should you care about studying the coming content and applying it to your field?
Physicists:

- The particles in the standard model *are* irreducible representations. So rep theory is crucial to you.
- Monoidal categories give a good framework for understanding QM.

Computer scientists:

- It gives a framework for the Curry-Howard correspondence (proofs are programs).

Me:

- Began as a study of "analogies" and turned into a study of nifty algebraic gadgets.
- It's written in a *wild* language, and learning languages is fun.

As a great motivation, see `https://arxiv.org/abs/0903.0340`

Plan

The **first goal** is to define monoidal categories with some context. The **second goal** is to describe a "skeletal" category defined by diagrammatics. To accomplish the first goal, we will study things including:

- Algebraic objects (groups, vector spaces, ...) and maps between them
- Subobjects, images, combining objects ($\times$, $\otimes$, $\oplus$, ...)
- Categories (Grp, Set, $PoSet$, $\mathbb{N}$, Vec, ...)

I'd like to have as little fat on this as necessary. That is, not get sidetracked studying, for instance, too much of the internal structure of these objects. I want to give many examples and try to build intuition. For the second goal we'll study things including:

- Representations and maps ($T(gv) = gT(v)$)
- $\text{Rep}(D_3)$ in detail

Up to this point I have a strong vision of where we're going. After this we can go where the interest steers us.
This plan is incomplete and non-exhaustive.

Groups

This is the best onramp to categories I know of, so bear with me through some basics.

Definition 1. *A* **group** *is a triple*

$$(G, \mu, e)$$

where G is a set, $\mu : G \times G \to G$ is a function, and $e \in G$, such that

$$\forall a, b, c \in G, \quad \mu(\mu(a, b), c) = \mu(a, \mu(b, c)) \tag{Associativity}$$

$$\forall a \in G, \quad \mu(a, e) = \mu(e, a) = a \tag{Identity}$$

$$\forall a \in G, \exists b \in G, \quad \text{such that} \quad \mu(a, b) = \mu(b, a) = e \tag{Inverse}$$

We often call the element b from 1 by a^{-1}. We also often use the following shorthands:

- $\mu(a, b) = a \cdot b = a \star b = ab$

- $\underbrace{a \cdot a \cdots a}_{n \text{ copies}} = a^n$

Exercise 1. *Translate the three axioms above into the ab notation.*

Exercise 2. *Prove the identity element in a group is unique. That is, if e and e' both satisfy Axiom 1, show that $e' = e$.*

Now some examples. As an excercise, prove that each of the following is a group. The notation $:=$ reads as "is defined to be."

Example 1 (General linear group)**.** (G, μ, e), *where*

- $G = \mathrm{GL}_2(\mathbb{C}) := \{$*invertible* 2×2 *matrices with entries in* $\mathbb{C}\}$
- $\mu(A, B) := AB$
- $e = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2 (General linear group)**.** (G, μ, e), *where*

- $G = \mathrm{GL}_n(\mathbb{C}) := \{$*invertible* $n \times n$ *matrices with entries in* $\mathbb{C}\}$
- $\mu(A, B) := AB$
- $e = I_n$ *(the $n \times n$ identity matrix)*

Example 3 (Integers)**.** (G, μ, e), *where*

- $G = \mathbb{Z}$
- $\mu(a, b) := a + b$
- $e = 0$

Example 4 (Not a group! Why?)**.** (G, μ, e), *where*

- $G = \mathbb{Z}$
- $\mu(a, b) := a \times b$
- $e = 1$

Example 5 (Braid group)**.** $B_n := (G, \mu, e)$, *where*

- $G =$ *n-strand braid diagrams (up to isotopy/wiggling)*
- $\mu =$ *vertical concatenation*
- $e = n$ *unbraided strands*

Steve pointed out that when $n = 2$, B_n is isomorphic to $\mathbb{Z}$. We'll get to that in the next lecture I hope.

Things that came up

- Generators and relations presentations
- Free group/group of words
- Symmetric group/permutation groups
- The natural numbers game:
  `https://www.ma.imperial.ac.uk/~buzzard/xena/natural_number_game/index2.html`
- Peano arithmetic:
  `https://en.wikipedia.org/wiki/Peano_axioms`

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More groups

Example 6 (Symmetric groups)**.** *Let X be a set. Then (G, μ, e) is a group, where*

- $G = \{\sigma : X \to X \mid \sigma \text{ is bijective } \}$
- $\mu(\sigma_1, \sigma_2) \coloneqq \sigma_1 \circ \sigma_2$
- $e = id_X$*, defined by* $\forall x \in X$*,* $id_X(x) = x$

Subgroups

There is rich literature about the structure of subgroups of a group. We won't get into it more than necessary.

Definition 2. *Let G and H be two groups...*

Homomorphisms

Example 7 (det)**.**

Example 8 $(n \mapsto kn)$**.**