Applied Category Theory

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Contents

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Why?

Why should you care about studying the coming content and applying it to your field?

Physicists:

- The particles in the standard model *are* irreducible representations. So rep theory is crucial to you.
- Monoidal categories give a good framework for understanding QM.

Computer scientists:

- It gives a framework for the Curry-Howard correspondence (proofs are programs).

Me:

- Began as a study of "analogies" and turned into a study of nifty algebraic gadgets.
- It's written in a *wild* language, and learning languages is fun.

As a great motivation, see `https://arxiv.org/abs/0903.0340`

Plan

The **first goal** is to define monoidal categories with some context. The **second goal** is to describe a "skeletal" category defined by diagrammatics. To accomplish the first goal, we will study things including:

- Algebraic objects (groups, vector spaces, ...) and maps between them
- Subobjects, images, combining objects ($\times$, $\otimes$, $\oplus$, ...)
- Categories (Grp, Set, $PoSet$, $\mathbb{N}$, Vec, ...)

I'd like to have as little fat on this as necessary. That is, not get sidetracked studying, for instance, too much of the internal structure of these objects. I want to give many examples and try to build intuition. For the second goal we'll study things including:

- Representations and maps ($T(gv) = gT(v)$)
- Rep(D_3) in detail

Up to this point I have a strong vision of where we're going. After this we can go where the interest steers us.

This plan is incomplete and non-exhaustive.

Groups

This is the best onramp to categories I know of, so bear with me through some basics.

Definition 1. *A* **group** *is a triple*

$$(G, \mu, e)$$

where G is a set, $\mu : G \times G \to G$ is a function, and $e \in G$, such that

$$\forall a, b, c \in G, \quad \mu(\mu(a, b), c) = \mu(a, \mu(b, c)) \tag{Associativity}$$

$$\forall a \in G, \quad \mu(a, e) = \mu(e, a) = a \tag{Identity}$$

$$\forall a \in G, \exists b \in G, \quad \textit{such that} \quad \mu(a, b) = \mu(b, a) = e \tag{Inverse}$$

We often call the element b from 1 by a^{-1}. We also often use the following shorthands:

- $\mu(a, b) = a \cdot b = a \star b = ab$

- $\underbrace{a \cdot a \cdots a}_{n \text{ copies}} = a^n$

Exercise 1. *Translate the three axioms above into the ab notation.*

Exercise 2. *Prove the identity element in a group is unique. That is, if e and e' both satisfy Axiom 1, show that $e' = e$.*

Now some examples. As an excercise, prove that each of the following is a group. The notation $:=$ reads as "is defined to be."

Example 1 (General linear group)**.** (G, μ, e), *where*

- $G = \mathrm{GL}_2(\mathbb{C}) := \{$*invertible* 2×2 *matrices with entries in* $\mathbb{C}\}$
- $\mu(A, B) := AB$
- $e = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2 (General linear group)**.** (G, μ, e), *where*

- $G = \mathrm{GL}_n(\mathbb{C}) := \{$*invertible* $n \times n$ *matrices with entries in* $\mathbb{C}\}$
- $\mu(A, B) := AB$
- $e = I_n$ *(the $n \times n$ identity matrix)*

Example 3 (Integers)**.** (G, μ, e), *where*

- $G = \mathbb{Z}$
- $\mu(a, b) := a + b$
- $e = 0$

Example 4 (Not a group! Why?)**.** (G, μ, e), *where*

- $G = \mathbb{Z}$
- $\mu(a, b) := a \times b$
- $e = 1$

Example 5 (Braid group)**.** $B_n := (G, \mu, e)$, *where*

- $G =$ *n-strand braid diagrams (up to isotopy/wiggling)*
- $\mu =$ *vertical concatenation*
- $e = n$ *unbraided strands*

Steve pointed out that when $n = 2$, B_n is isomorphic to $\mathbb{Z}$. We'll get to that in the next lecture I hope.

Things that came up

- Generators and relations presentations
- Free group/group of words
- Symmetric group/permutation groups
- The natural numbers game:
  `https://www.ma.imperial.ac.uk/~buzzard/xena/natural_number_game/index2.html`
- Peano arithmetic:
  `https://en.wikipedia.org/wiki/Peano_axioms`

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More groups

Recall that, loosely, a group is a set endowed with a binary operation (multiplication), with some associativity, identity, and inversion constraints. Henceforth we will refer to a group (G, μ, e) almost exclusively as G. Usually, μ and e will be understood from context. For some of the following examples we might also write μ_G to refer specifically to multiplication in G.

Example 6 (Nonzero field elements)**.** *The set $\mathbb{C}^\times = \mathbb{C} - \{0\}$ is a group under multiplication. Associativity is known. Its identity element is $1 \in \mathbb{C}$. This is true for any field, i.e. $\mathbb{R}^\times$, $\mathbb{Q}^\times$, and $\mathcal{F}_{p^n}^\times$ are all groups.*

Example 7 (Symmetric groups)**.** *Let X be a set. Then (G, μ, e) is a group, where*

- $G = \{\sigma : X \to X \mid \sigma \text{ is bijective } \}$
- $\mu(\sigma_1, \sigma_2) \coloneqq \sigma_1 \circ \sigma_2$
- $e = id_X$, *defined by* $\forall x \in X$, $id_X(x) = x$

Example 8. *If $X = \{1, \dots, n\}$, then we denote the group S_X by S_n. If $\sigma \in S_n$ then we often denote σ as*

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

Example 9 (Dihedral group)**.** *Let G consist of the following six matrices:*

These matrices permute the points

$$\left\{ (1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \right\}$$

So we can see that it's "the same as" a certain permutation group. There are really 3 things at play in this last example:

- Dihedral groups: 3 wasn't special. We could divide by any positive n and get a group of $2n$ matrices.
- Generators and relations: We could equally express this group as

  $$\langle r, s \mid r^3 = e, s^2 = e, rs = sr^2 \rangle.$$

  In fact, that's usually how dihedral groups are presented.
- Isomorphism: That group of matrices "is" a permutation group.

Homomorphisms

We'll start by definine how two groups can be similar, or the same.

Definition 2. *Let G and H be two groups. Let $f : G \to H$ be a function.*

- *We call f a* **homomorphism** *if*

  $$\forall a, b \in G, f(ab) = f(a)f(b)$$

- *We call f an* **isomorphism** *if it is a bijective homomorphism*
- *If f is a homomorphism, the* **kernel** *of f is the set*

  $$\ker(f) \coloneqq \{x \in G \mid f(x) = e_H\}$$

- *If f is a homomorphism, the* **image** *of f is the set*

  $$f(G) \coloneqq \{f(X) \mid x \in G\} \subseteq H$$

Here are many examples. It would be useful to prove those you don't see immediately. Well, it would probably be good to prove all of them...

Example 10 (Modular arithmetic)**.** *$f : \mathbb{Z} \to \mathbb{Z}_n$, given by $f(x) \coloneqq x \pmod n$*

Example 11 (Multiplication)**.** *$f : \mathbb{Z} \to \mathbb{Z}$, given by $f(x) \coloneqq 4x$. What is special about 4 here? Anything?*

Example 12 (Trivial)**.** *Let G and H be any two groups. Define $f : G \to H$ by $f(x) = e$.*

Example 13 (Identity)**.** *Let G be any group. Define $f : G \to G$ by $f(x) \coloneqq x$.*

Example 14 (Symmetric group inclusion)**.** *Define $f : S_n \to S_{n+1}$ by declaring*

$$f(\sigma) \coloneqq \begin{pmatrix} 1 & 2 & \cdots & n & n+1 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) & n+1 \end{pmatrix}$$

In particular, since this homomorphism is injective, this means we can think of S_n as "sitting inside of" S_{n+1}. This actually holds more generally. If a homomorphism $f : G \to H$ is injective, then there is an isomorphic copy of G inside of H, in the form of the image $f(G)$. We'll define this more precisely in the next subsection.

Example 15 (Linear maps)**.** *Let V and W be vector spaces over $\mathbb{C}$. The definition of vector spaces says that $(V, +)$ and $(W, +)$ are, in particular, abelian groups. Let $T : V \to W$ be a linear map. The condition*

$$T(x + y) = T(x) + T(y)$$

implies that T is a homomorphism of (abelian) groups.

Example 16 (All linear maps)**.** *$f : \mathbb{Z} \to \mathbb{Z}_n$, given by $f(x) \coloneqq x \pmod n$*

Here's a fact that we'll have uses for.

Exercise 3. *Let $f : G \to H$ be a homomorphim of groups. For every $a \in G$, the inverse of $f(a)$ is $f(a^{-1})$. In equation form, that's*

$$[f(a)]^{-1} = f(a^{-1}).$$

Subgroups

As we saw above, S_n "sits inside of" S_{n+1}. Here's how we say that precisely.

Definition 3. *Let G be a group. A nonempty subset $A \subseteq G$ is called a subgroup of G if*

$$\forall a, b \in A, \quad ab \in A \tag{Closure}$$

$$\forall a \in A, \quad a^{-1} \in A \tag{Inversion}$$

The first consequence of this definition is that if A is a subgroup of G, then $e \in A$. Why? Well, take any element $a \in A$. Its inverse, a^{-1} also is in A by definition. Their product also must be in A. But their product is e.

Here's a fact that will be of use.

Proposition 1. *Let $f : G \to H$ be a homomorphism. Then*

1. *The kernel $\ker(f)$ is a subgroup of G.*
2. *The image $f(G)$ is a subgroup of H.*

Let's see some examples of subgroups in action.

Example 17 (Dihedral groups)**.** *Let $n \geq 1$ be a positive integer. The set of $2n$ matrices of the forms*

$$\begin{bmatrix} \cos 2\pi\frac{k}{3} & -\sin 2\pi\frac{k}{3} \\ \sin 2\pi\frac{k}{3} & \cos 2\pi\frac{k}{3} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \cos 2\pi\frac{k}{3} & \sin 2\pi\frac{k}{3} \\ \sin 2\pi\frac{k}{3} & -\cos 2\pi\frac{k}{3} \end{bmatrix}$$

for $k = 0, 1, \ldots, n-1$ is a subgroup of $\mathrm{GL}_2(\mathbb{C})$.

Example 18 (Special linear group)**.** *The set $SL_N(\mathbb{C}) \coloneqq \{M \in \mathrm{GL}_N(\mathbb{C}) \mid \det(M) = 1\}$ is a subgroup of the general linear gorup $\mathrm{GL}_N(\mathbb{C})$.*

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I'd like to define a representaiton today, since we have all the prerequisite knowledge. The crux of the problem is this: group multiplication is hard. It's generally undecidable whether a given group element is even the identity. But if we can translate a group's multiplication into something more concrete (like matrices!), then we can learn a lot more about the group.

We'll begin by recalling that a homomorphism between groups G and H is a function $f : G \to H$ satisfying

$$f(\underbrace{ab}_{\in G}) = \underbrace{f(a)f(b)}_{\in H}.$$

It's important to remember which group the multiplication is happening in.

There's a cool way to visualize this property that I'll draw on the board. I won't include it here becasue it's time-consuming to create...

Representations

Groups are meant to *act* on things, that is, to encode structure-preserving permutations. We've seen examples of this already:

- Permutation groups: all permutations of an abstract set X. The structure being preserved here is cardinality.
- Matrix groups: an invertible (bijective!) matrix is a permutation of $\mathbb{C}^n$.

A representation of a group is a sort of middle ground between these two. That is, it translates some (maybe not all) of the structure of a group into a matrix group, which permutes a vector space.

Definition 4. *Let G be a group. A (linear)* **representation** *of G is a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$.*

Remark 1. *Soemtimes we will write $\rho(g)$ as ρ_g and $[\rho(g)](v)$ as any one of $\rho_g v$, $g.v$, or even just gv.*

Every group has at least one representation.

Example 19. *Let G be any group. Define $\rho : G \to \mathrm{GL}_1(\mathbb{C})$ by $\rho(g) := [1]$ for every $g \in G$. Then ρ defines a representation, since it's a homomorphism:*

$$\begin{aligned}\rho(gh) &= [1]\\ &= [1][1]\\ &\rho(g)\rho(h)\end{aligned}$$

A similar construction gives a representation of G on $\mathrm{GL}_n(\mathbb{C})$ by sending $g \mapsto I_n$.

Here's an example that translates modular arithmetic into matrix multiplication.

Example 20. *Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ with addition modulo 4 as the operation. Define $\rho : G \to \mathrm{GL}_1(\mathbb{C})$ by $\rho(k) := [e^{2\pi i \frac{k}{4}}]$. Then ρ defines a representation, since it's a homomorphism:*

$$\begin{aligned}\rho(j+k) &= [e^{2\pi i \frac{j+k}{4}}]\\ &= [e^{2\pi i \frac{j}{4}} e^{2\pi i \frac{k}{4}}]\\ &= [e^{2\pi i \frac{j}{4}}][e^{2\pi i \frac{k}{4}}]\\ &\rho(j)\rho(k)\end{aligned}$$

Inner automorphisms

I planned to end this lecture talking about the geometric representation of D_3. Instead, due to a question from Marek, we ended up talking about *inner automorphisms.* The goal here is to morally prove that groups are made to act on things. We'll do that by constructing, for *any* group G, a homomorphism $G \to S_G$ into the set of permutations of the group's elements. This homomorphism won't be surjective; not every permutation of the set G can be viewed as a group element itself. We'll actually be mapping into a subgroup of S_G known as the automorphisms. More specifically, inner automorphisms.

Henceforth, G will be an arbitrary, but fixed, group.

Definition 5 (Automorphism group)**.** *Let G be any group, and define the set* $\mathrm{Aut}(G)$ *by*

$$\mathrm{Aut}(G) \coloneqq \{\varphi : G \to G \mid \varphi \textit{ is an isomorphism}\}.$$

A self-isomorphism $G \to G$ is called an **automorphism***. Note that, in particular, an automorphism is a permutation.*

Proposition 2. *Fix a group G. The set* $\mathrm{Aut}(G)$ *of automorphisms of G forms a group, with the operation being function composition and the identity element being the identity homomorphism id_G.*

Proof. We have three things to check: closure under the group operation, existsence of an identity element, and invertibility. I'll outline the logic here.

(Closure) This reduces to checking that the composition of homomorphisms is a homomorphism, and that the composition of bijective functions is bijective.

(Identity) Check that the identity map id_G given by $id_G(x) = x$ is a bijective homomorphism.

(Inversion) Check that, for any automorphism f of G, the inverse function f^{-1} (which only exists because f is assumed bijective!!) is a bijective homomorphism. □

Now, for any $x \in G$, let $\sigma_x : G \to G$ be the function defined by

$$\sigma_x(g) \coloneqq xgx^{-1}.$$

A function of this form is called an **inner automorphism**. Its name assumes the following fact.

Proposition 3. *For any fixed $x \in G$, the function $\sigma_x : G \to G$ is an automorphism.*

Proof. This entails proving that $\sigma_x(gh) = \sigma_x(g)\sigma_x(h)$, and that σ_x is a bijective function. To prove it's bijective, it suffices to find an inverse function. Naturally enough, one can show that the function $\sigma_{x^{-1}}$ is the inverse function to σ_x. The homomorphism condition is a neat exercise. □

Now the result mentioned earlier.

Theorem 1. *Define the function $\Sigma : G \to \mathrm{Aut}(G)$ by $\Sigma(x) \coloneqq \sigma_x$. Then Σ is an injective homomorphism. In particular, Σ exactly defines an isomprphism*

$$G \xrightarrow{\cong} \Sigma(G).$$

So G is isomorphic to a subgroup of S_G.

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I want to take a step back and examine some parts of the definition of a representation. We defined a (complex, linear) representation as a homomorphism $G \to \mathrm{GL}_n(\mathbb{C})$. I did this to be a bit more concrete and avoid discussing vector spaces in their own right. But I now realize that was a tactical mistake, and we should discuss vector spaces as such. The reason is that in order to build a category theory that doesn't seem conjured, we should have a few more examples of objects, subobjects, and maps. Discussing vector spaces in their own right does this twofold: once for vector spaces, and then we'll build on that for representations.

Definition 6. *A (complex)* **vector space** *is a tuple* $(V, +, s)$ *where* V *is a set, and* $s : \mathbb{C} \times V \to V$ *is a function, satisfying the following properties:*

$$\forall u, v \in V, \quad u + v = v + u \in V \tag{Commutative}$$

$$\forall u, v, w \in V, \quad (u + v) + w = u + (v + w) \in V \tag{Associative}$$

$$\exists \vec{0} \in V; \forall v \in V \quad \vec{0} + v = v + \vec{0} = v \tag{Zero}$$

$$\forall u \in V, \exists v \in V; \quad u + v = v + u = \vec{0} \tag{Negative}$$

$$\forall \alpha, \beta \in \mathbb{C}, v \in V \quad \alpha\beta(v) = \alpha(\beta v) \tag{Scalar assoc.}$$

$$\forall \alpha, \beta \in \mathbb{C}, v \in V \quad (\alpha + \beta)v = \alpha v + \beta v \tag{Left distribute}$$

$$\forall \alpha \in \mathbb{C}, v, w \in V \quad \alpha(v + w) = \alpha v + \alpha w \tag{Right distribute}$$

where we have shortened $s(\lambda, v)$ *to* λv.

In particular, $(V, +)$ is an abelian group (forgetting the scaling structure). Some expected properties follow immediately from this definition.

Exercise 4. *Prove the following.*

- $0v = \vec{0}$
- $v + (-1)v = \vec{0}$

Note: It is not anywhere in the definition that multiplying v *by* -1 *gives the additive inverse of* v*. That's what the second part of this exercise is showing.*

There are many familiar examples.

Example 21 (Zero)**.** $\{\vec{0}\}$

Example 22 ($\mathbb{C}$)**.**

Example 23 (Tuples)**.** *Define* $\mathbb{C}^n := \{(z_1, z_2, \ldots, z_n) \mid z_i \in \mathbb{C}\}$*, with addition and scalar multiplication given pointwise:*

$$(w_1, \ldots, w_n) + (z_1, \ldots, z_n) := (w_1 + z + 1, \ldots, w_n + z_n) \quad \text{and} \quad \alpha(z_1, \ldots, z_n) := (\alpha z_1, \ldots \alpha z_n).$$

Example 24 (Matrices)**.** $M_{m \times n}(\mathbb{C})$

Example 25 (Polynomials)**.** *Define the vector space* $\mathbb{C}[x] := \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{C}\}$ *with addition and scalar multiplication defined pointwise on scalars.*

Example 26 (Functions)**.** *Define the vector space* $\mathbb{C}^X := \{f : X \to \mathbb{C}\}$*, with addition and scalar multiplication defined pointwise:*

$$(f + g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha \cdot f)(x) := \alpha \cdot (f(x)).$$

Sometimes a vector space sits inside another vector space.

Definition 7 (Subspace)**.** *Let V be a vector space, and let U be a subset of V. Call U a* **vector subspace** *of V if*

$$\forall u, v \in V, \quad u + v \in V \tag{Addition}$$

$$\forall u \in U, \lambda \in \mathbb{C}, \quad \lambda u \in U \tag{Scalar multiplication}$$

Exercise 5. *Let V be a vector space, and let U be a subspace of V, and suppose $u \in U$. Prove that:*

- $\vec{0} \in U$
- $-u \in U$

What kinds of functions do we care about for vector spaces? Well, we have two sorts of structure now, so we want a function to respect both.

Definition 8 (Linear transformation)**.** *Let V and W be vector spaces. A function $T : V \to W$ is called a* **linear transformation** *(or* **linear map***) if*

$$T(u + v) = T(u) + T(w), \quad \textit{and} \quad T(\lambda v) = \lambda T(v).$$

If $T : V \to W$ is a linear map, its **kernel** *is the set*

$$\ker(T) \coloneqq \{v \in V \mid T(v) = \vec{0}\}.$$

Its **image** *is the set*

$$T(V) \coloneqq \{T(v) \mid v \in V\}.$$

Often we will denote the target $T(v)$ of a vector $v \in V$ simply by Tv.

Next, a couple of results that should remind you of group theory.

Proposition 4. *Let V and W be vector spaces, and suppose $T : V \to W$ is a linear map. Then*

- $\ker(T)$ *is a subspace of V*
- $T(V)$ *is a subspace of W*
- $T(\vec{0}) = \vec{0}$

Here are a few quick examples of linear maps between some of the example spaces above.

Example 27 (Trivial)**.** *For a vector space V, there is always exactly one linear map $\{\vec{0}\} \to V$ and exactly one linear map $V \to \{\vec{0}\}$.*

Example 28 (Embedding)**.** *Define the map $\mathbb{C} \to \mathbb{C}^n$ by defining*

$$\alpha \mapsto \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Example 29 (Matrix)**.** *Let M be an $m \times n$ complex matrix. View tuples in $\mathbb{C}^k$ as column vectors. Define the linear transformation $T_M : \mathbb{C}^n \to \mathbb{C}^m$ by*

$$T_M(v) \coloneqq Mv.$$

In elementary linear algebra courses, linear maps are almost always just given by matrices. This is possible when we are dealing with **finite-dimensional** vector spaces. We will mostly have cause only to deal with such vector spaces, but there are many interesting examples of infinite-dimensional vector spaces (polynomials, for example). For more on the correspondence between linear maps and matrices, see this link.

The reason we're going through all of this is to put ourselves on more sure footing when defining representations, and the maps between them. Let's start by defining some notations that will carry through to our study of categories more generally.

Definition 9. *Let V and W be two vector spaces. The set of linear maps between V and W is*

$$\mathrm{Hom}_{\mathrm{Vec}}(V \to W) := \{T : V \to W \mid T \text{ linear}\}.$$

We also define **endomorphisms** *of vector spaces to be (not necesarrily invertible!) self-maps:*

$$\mathrm{End}_{\mathrm{Vec}}(V) := \mathrm{Hom}_{\mathrm{Vec}}(V \to V).$$

And finally, the invertible self-maps, **automorphisms***:*

$$\mathrm{GL}(V) := \{T \in \mathrm{End}_{\mathrm{Vec}}(V) \mid T \text{ invertible}\}.$$

Note that, since invertible linear maps are bijective, they are permutations. Thus $\mathrm{GL}(V)$ is a very special subset of S_V, the symmetric group on V. In fact, it's a sub**group**.

Proposition 5. *Let V be any vector space. The collection $\mathrm{GL}(V)$ is a group.*

The proof is very similar to proving that $\mathrm{Aut}(G)$ is a group for any group G.

Now we're in a position to ask whether, given a group G and a vector space V, there might be a homomorphism $G \to \mathrm{GL}(V)$. This is exactly what a representation is.

Definition 10 (Representation)**.** *Let G be a group and V a vector space. A* **representation** *of G on V is a homomorphism*

$$\rho : G \to \mathrm{GL}(V).$$

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NOT UP TO DATE

Then we'll define categories in general, and discuss how our big three examples fit in the framework.

Definition 11 ((Small, lcoally-small) Categories)**.**

Example 30 (Groups)**.**

- $ob(\mathcal{C}) =$
- $\mathrm{Hom}_{\mathcal{C}}(G \to H) = \{\}$

Example 31 (Vector Spaces)**.**

Example 32 (1-Cobordisms)**.**

Example 33 (Sets)**.**

I personally happen to be interested in categories more similar to $1Cob$ than Set. And I (Baez, really...) claim physicists ought to be, too. Why? Map reversal.

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UNSTABLE

We'll begin by finishing most of what we need from representation theory. Then I'll try and actually sketch the so-called "graphical correspondence."

To understand the definition of the maps in the category of G representations, it first will be helpful to recall some shorthand. We defined the following three things to be the same:

- $\rho(g)(v)$
- $\rho_g(v)$
- $g \cdot v$ or gv when ρ is understood.

Definition 12 (Homomorphism/intertwiner). *Let $\rho : G \to \mathrm{GL}(V)$ and $\pi : G \to \mathrm{GL}(W)$ be two representations of the same gorup. A linear map $T : V \to W$ is called a* **homomorphism** *of representations if, for every $g \in G$, we have $T(\rho_g(v)) = \pi_g(T(v))$. We express this by saying the following diagram "commutes":*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho_g"] & W \arrow[d, "\pi_g"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

Equivalently, we may express this equality by stating

$$\forall g \in G, v \in V, \quad T(g \cdot v) = g \cdot T(v).$$

We may express this in a more element-free way by merely writing

$$\forall g \in G, \quad T \circ \rho_g = \pi_g \circ T. \tag{1}$$

As promised, the representations of a group form a category. It's actually a (rigid, pivotal, etc...) tensor category, but we'll get to some of those adjectives later.

Example 34. *Let G be a group, and let $\mathrm{Rep}(G)$ denote the class of finite-dimensional representations of G.*

- *$ob(\mathrm{Rep}(G)) = \{(V, \rho) \mid \rho : G \to \mathrm{GL}(V)$ is a representation$\}$*
- *$\mathrm{Hom}_{\mathrm{Rep}(G)}(V \to W) = \{T : V \to W \mid T$ is an intertwiner$\}$*

For now we fix a finite group G. We will see how representations of this fixed group interact.

Definition 13 (Irreducibility). *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. A subspace U of V is called a* **subrepresentation** *if*

$$\forall u \in U, g \in G, \quad \rho_g(u) \in U.$$

Equivalently, we have, for every $g \in G$, $\rho_g|_U \in \mathrm{GL}(U)$.

A representation with no nontrivial subrepresentations is called **irreducible**.

Irreducible representations of a group are the atomic objects in a representation category. Every (finite-dimensional) representation of G is a *direct sum* of irreducible representations.

Here's a lemma that we'll use to count multiplicities of subrepresentaitons.

Lemma 1 (Schur's lemma). *Let $\rho : G \to \mathrm{GL}(V)$ and $\pi : G \to \mathrm{GL}(W)$ be irreducible representations. Suppose $T : V \to W$ is a homomorphism of representations. Then either $T = 0$ or T is an isomorphism.*

Proof. It suffices to prove that $\ker(T)$ is a subrepresentation of V, and $T(V)$ is a subrepresentation of W. If T is nonzero, the former must be $\{\vec{0}\}$. Now, it can't be that the image $T(V)$ is anything but all of W. Hence T has trivial kernel, and is surjective; an isomorphism. □

Now here's the form we'll use this lemma in.

Corollary 1. *Let $\rho : G \to V$ be an irreducible representation, and let $\pi : G \to X$ be any representation. If there is a nonzero homomorphism of representations*

$$T : V \to X$$

then X has a subrepresentation isomorphic to V.

In this case, we say "X contains a copy of V," and will sometimes write $V \leq X$.

The effect of this lemma is to allow us to find "decompositions" of representations by findind maps from irreducibles. Let's set the stage.

Recall how we defined the dihedral group:

$$D_3 = \langle r, s \mid r^3 = s^2 = e, sr = r^2 s\rangle.$$

Now we'll define three representations by saying what they do to the generators r and s.

- Define $\pi_t : G \to \mathrm{GL}(\mathbb{C}_t)$ by
$$\pi_t(r) \coloneqq 1 \quad \text{and} \quad \pi_t(s) \coloneqq 1.$$
- Define $\pi_s : G \to \mathrm{GL}(\mathbb{C}_s)$ by
$$\pi_s(r) \coloneqq 1 \quad \text{and} \quad \pi_s(s) \coloneqq -1.$$
- Define $\rho : G \to \mathrm{GL}(\mathbb{C}^2)$ by
$$\rho(r) \coloneqq \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{bmatrix} \quad \text{and} \quad \rho(s) \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
- Define $\rho \otimes \rho : G \to \mathrm{GL}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ by
$$\rho \otimes \rho(r) \coloneqq \begin{bmatrix} e^{2\pi i/3}\rho(r) & 0 \\ 0 & e^{-2\pi i/3}\rho(r) \end{bmatrix} = \begin{bmatrix} e^{4\pi i/3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i/3} \end{bmatrix}$$
and
$$\rho \otimes \rho(s) \coloneqq \begin{bmatrix} 0 & 1\rho(s) \\ 1\rho(s) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Our goal here is to find morphisms

$$\mathbb{C}_t \to \mathbb{C}^2 \otimes \mathbb{C}^2, \quad \mathbb{C}_s \to \mathbb{C}^2 \otimes \mathbb{C}^2, \quad \text{and} \quad \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2,$$

thereby showing that $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}_t \oplus \mathbb{C}_s \oplus \mathbb{C}^2$.