Conformal Embeddings of G_2 via Graph Planar Algebra Embeddings

WRITING TASK: Finish all writing except verifications by EOD 9/22/2025
Swap "coefficients" to "coordinates" where applicable

Contents

1. Introduction

This paper gives generators-and-relations presentations for two categories

$$\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3} \quad \text{and} \quad \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$$

where A_3 and A_4 are the commutative Etale algebras corresponding, respectively, to the conformal embeddings $G_{2,3} \subseteq E_{6,1}$ and $G_{2,4} \subseteq D_{7,1}$ of affine Lie algebras. The correspondence $H_k \leftrightarrow \overline{\mathrm{Rep}(U_{q_k}(\mathfrak{h}))}$ between affine Lie algebra representations and quantum group representations [**<empty citation>**] allows one, in the course of studying conformal embeddings, to move freely from the perspective of affine Lie algebras to that of representations of quantum groups. We will initialize our statements in the language of affine Lie algebras, and then translate into quantum group representations.

By *presentation* we mean the following standard sequence of steps. Let $\mathcal{P}$ be the planar algebra generated by some object of the tensor category $\mathcal{C}$. Then we show $\mathcal{P}$ is monoidally equivalent to some skein category $\mathcal{E}$, and that $\mathrm{Ab}(\mathcal{E})$ is monoidally equivalent to $\mathcal{C}$. The presentations we give are extensions of Kuperberg's G_2 skein theory by strands of type $\mathbb{Z}_3$ (level 3) and $\mathbb{Z}_2$ (level 4).

The novel methodology offered here is the use of graph planar embeddings to deduce relations. Computing, e.g., the relation (Change of Basis) purely in the context of quantum group representations is daunting, to say the least. With a GPA embedding in hand, however, this becomes an elementary linear algebra problem.

2. Preliminaries

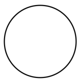
Here we define the players in this game. We begin with planar algebras, in the spirit of [13].

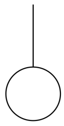
2.1. **Unoriented Planar Algebras.** We begin by recalling the theory of **rigid** monoidal categories detailed in [11]. To put it succintly, rigid monoidal categories have duals. Duals, and the associated evaluation and coevaluation maps, give us the cups and caps ubiquitous in planar algebras. A rigidity assumption gives us the ability to isotope diagrams.

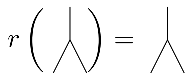
Let X be a tensor generator for the tensor category $\mathcal{C}$; that is, every object of $\mathcal{C}$ is isomorphic to a subobject of some tensor power $X^{\otimes n}$. For our pruposes it will suffice to assume X is symmetrically self-dual. Let $\mathcal{P}_{X;\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ whose objects are tensor powers $\mathbb{1} = X^{\otimes 0}, X, X^{\otimes 2}, \ldots$; we call this the (unoriented) **planar algebra** generated by X in $\mathcal{C}$. Since we assumed X tensor generates $\mathcal{C}$, it follows that $\mathcal{C} \cong \mathrm{Kar}(\mathcal{P}_{X;\mathcal{C}})$, the Cauchy completion of $\mathcal{P}_{X;\mathcal{C}}$. The universal property of $\mathrm{Kar}(\mathcal{P}_{X;\mathcal{C}})$ therefore implies that studying $\mathcal{P}_{X;\mathcal{C}}$ is sufficient to understand $\mathcal{C}$. The planar algebra $\mathcal{P}_{X;\mathcal{C}}$ is **evaluable** if $\dim \mathrm{End}_{\mathcal{P}_{X;\mathcal{C}}}(\mathbb{1}) = 1$.

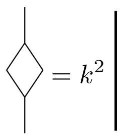
Define trivalent cat as generated by a vertex here. Do it fully diagrammatically.

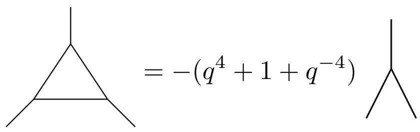
We will be presenting the our two new categories as extensions of $\mathcal{G}_2(q)$ skein theories, in the spirit of Kuperberg [12]. Up to a rescaling by a factor of $k = \sqrt{[7]-1}$ we use the same skein theory (note the sign error in the Pentagon relation of [12]). That is, we define the planar algebra $\mathcal{G}_2(q)$ generated by a trivalent vertex, with the skein relations

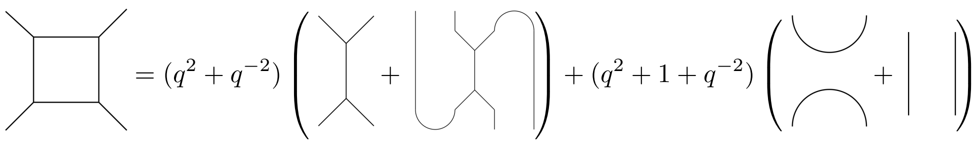
(Loop)  $= q^{10} + q^{8} + q^{2} + 1 + q^{-2} + q^{-8} + q^{-10}$

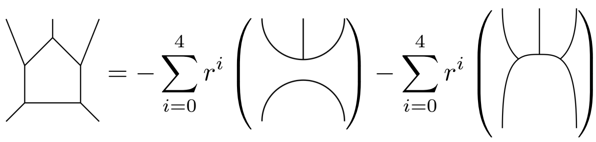
(Lollipop)  $= 0$

(Rotate) 

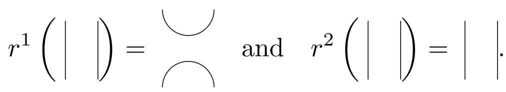
(Bigon) 

(Trigon) 

(Tetragon) 

(Pentagon) 

Here, the notation $r^i(\mathcal{E})$ means an i-click right rotation of the diagram $\mathcal{E}$. For instance,



The value of q is given in [6] and depends on the level k:

$$q = e^{\frac{2\pi i}{3(4+k)}}.$$

Thus at level 3 we have $q_3 = e^{\frac{2\pi i}{42}}$ and at level 4 we have $q_4 = e^{\frac{2\pi i}{48}}$.

2.2. **Unitarity and inner products.** Recall that a **dagger** category is one endowed with an involution, denoted by a superscript $\dagger$. The dagger operation gives us the useful ability to reverse any morphism. This will be of use when, for example, we need a way to pair parallel *or* antiparallel morphisms.

The categorical trace of a morphism $f : X \to X$ is defined by

$$tr(f) \coloneqq ev_X \circ (f \otimes id_{X^*}) \circ coev_X.$$

Informally, the trace "caps off" an endomorphism and returns the resulting scalar (since we assume $\dim \mathrm{Hom}(\mathbb{1} \to \mathbb{1}) \cong \mathbb{C}$).

We use trace to define two different notions of pairing of morphisms:

$$\langle f, g \rangle^\dagger \coloneqq tr(g^\dagger \circ f)$$

for parallel morphisms and

$$\langle \varphi, \psi \rangle \coloneqq tr(\psi \circ \varphi)$$

for antiparallel morphisms.

Unitarity gives two tools. Computationally, it allows us to do computations by "capping" diagrams. It also has useful theoretical consequences.

Definition 1. *A category $\mathcal{C}$ is* **unitary** *if it is a dagger category and the two conditions*

- *if $\langle f, f \rangle^\dagger = 0$, then $f = 0$*
- *for every $f : X \to Y$, there is some $g : X \to X$ such that $ff^\dagger = gg^\dagger$*

are satisfied.

On a related note, we also make the following definition

Definition 2. *A morphism $f : X \to Y$ in a dagger category $\mathcal{C}$ is* **negligible** *if $\langle f, g \rangle = 0$ for all $g : Y \to X$.*

It is a fact that the negigible morphisms in a category $\mathcal{C}$ form an ideal, called the **negligible idea** and denoted by $\mathcal{N}(\mathcal{C})$. We may form the semisimple quotient $\overline{\mathcal{C}} \coloneqq \mathcal{C}/\mathcal{N}(\mathcal{C})$.

Proposition 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be pivotal, with $\mathcal{D}$ unitary, and $F : \mathcal{C} \to \mathcal{D}$ a pivotal dagger functor. Then F descends to a faithful pivotal functor $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ such that the diagram*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \\
\overline{\mathcal{C}} \arrow[ur, "\overline{F}"] &
\end{tikzcd}$$

is commutative.

Proof. Compute directly:

$$\begin{aligned}
\langle F(f), F(f) \rangle^\dagger &= tr(F(f)^\dagger \circ F(f)) \\
&= tr(F(f^\dagger) \circ F(f)) \\
&= F(tr(f^\dagger \circ f)) \\
&= F(0) = 0
\end{aligned}$$

since we assumed f was negligible. □

As one might expect, and as was alluded to above, unitarity interacts nicely with the negligible ideal.

Proposition 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be unitary, and suppose $F : \mathcal{C} \to \mathcal{D}$ is a pivotal dagger functor. Then there is a faithful $\tilde{F} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ such that*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d] \\
\overline{\mathcal{C}} \arrow[r, "\tilde{F}"] & \overline{\mathcal{D}}
\end{tikzcd}$$

If F is faithful, so is $\tilde{F}$.

Proof. Compose $\overline{F}$ of Proposition 1 with the functor $\mathcal{D} \twoheadrightarrow \overline{\mathcal{D}}$. ?? □

2.3. **The Unoriented Graph Planar Algebra.** We will study the quantum subgroups of type G_2 by embedding their skein theories into appropriate graph planar algebras. This serves two purposes:

- Giving us solid ground on which to do computations, allowing us to uncover relations by finding them in the GPA hom-spaces, and
- Implying some nice general properties for the quantum subgroups (i.e., unitarity)

In this work we have no use for generalized GPAs, such as the *oriented* [2] or *multi-color* GPA, so we consider only the unoriented case.

Definition 3. *Let $\Gamma = (V, E)$ be a finite graph. For an edge $e = (u, v) \in E$, let $\overline{e} = (v, u) \in E$. The* **graph planar algebra** *on Γ, denoted* GPA(Γ)*, is the strictly pivotal rigid monoidal category whose objects are nonnegative integers, and whose hom-spaces have basis*

$$\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(m \to n) := \mathbb{C}\left\{(p, q) \mid {}^{p \text{ an } m\text{-path}}_{q \text{ and } n\text{-path}}, {}^{s(p)=s(q)}_{t(p)=t(q)}\right\},$$

with composition law

$$(p, q) \circ (p', q') := \delta_{q=p'}(p, q'),$$

and rigidity maps

$$ev = \sum_e \sqrt{\frac{\lambda_{t(e)}}{\lambda_{s(e)}}}\langle e\overline{e}, s(e)\rangle, \quad coev = \sum_e \sqrt{\frac{\lambda_{t(e)}}{\lambda_{s(e)}}}\langle s(e)e\overline{e}\rangle.$$

Monoidal product on objects is addition, and for morphisms is defined by

$$(p, q) \otimes (p', q') := \delta_{s(p')=t(p)}(pp', qq').$$

We will be finding GPA embeddings of certain planar algebras. The following implies unitarity of these planar algebras.

Lemma 1. *For Γ a finite graph,* GPA(Γ) *is unitary.*

Proof. □

2.4. **Algebra and Module Objects.** Our ultimate goal is to find skein theoretic descriptions of two categories $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$ and $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$ of modules over algebra objects A_3 and A_4 coming from the conformal embeddings $\hat{\mathfrak{g}_2} \subseteq \hat{\mathfrak{e}_6}$ and $\hat{\mathfrak{g}_2} \subseteq \hat{\mathfrak{o}_7}$. In this subsection we define algebra and module objects. See [14] for a full description. Unless otherwise stated, we will be studying braided tensor categories.

Definition 4. *An* **algebra** *object of a braided tensor category $\mathcal{C}$ is an object A along with two maps*

$$\begin{aligned} m :& A \otimes A \to A \\ e :& \mathbb{1} \to A \end{aligned}$$

such that the following three diagrams

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "a"] \arrow[d, "m \otimes id_A"] & A \otimes (A \otimes A) \arrow[r, "id_A \otimes m"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[rr, "m"] & & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
\mathbb{1} \otimes A \arrow[r, "e"] \arrow[dr] & A \otimes A \arrow[d, "m"] & A \otimes \mathbb{1} \arrow[l, "e"'] \arrow[dl] \\
& A &
\end{tikzcd}$$

commute (where a is the associator for $\mathcal{C}$). Given that $\mathcal{C}$ has braiding c, the algebra A is **commutative** *if $m \circ c_{A,A} = m$.*

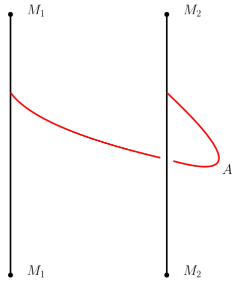
Once we have algebra objects one naturally defines modules over them.

Definition 5. *Given an algebra object A of $\mathcal{C}$, an object M of $\mathcal{C}$ is said to be a* **right A-module** *if there is a map $s : M \otimes A \to A$ such that the diagram*

$$\begin{tikzcd}
(M \otimes A) \otimes A \arrow[r, "a"] \arrow[d, "a"] & M \otimes (A \otimes A) \arrow[r, "s"] & M \otimes A \arrow[d, "s"] \\
M \otimes A \arrow[rr, "s"] & & M
\end{tikzcd}$$

commutes. We define $\mathcal{C}_A$ to be the category of right A-module objects in $\mathcal{C}$.

The braiding on $\mathcal{C}$ allows one to give a right A-module the structure of a left A-module. Once this has been done, we may define a tensor product on $\mathcal{C}_A$ by defining $M_1 \otimes_{\mathcal{C}_A} M_2$ to be the pbject projected onto by the idempotent



Monoidal functors interact with algebra objects in the ways we would hope.

Proposition 3. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a lax-monoidal fucntor, and let A be an algebra object of $\mathcal{C}$. Then $\mathcal{F}A$ has algebra structure induced by $\mathcal{F}$.*

Proof. The approach is to use the lax-monoidal structure on $\mathcal{F}$ and its functoriality to translate diagrams in $\mathcal{C}$ to diagrams in $\mathcal{D}$. See [10] for the full argument. □

The following proposition gives us license to make similar statements about adjoints of monoidal functors.

Proposition 4. *Monoidal functors between semisimple categories have lax-monoidal right adjoints.*

The free module functor $\mathcal{F}_A : \mathcal{C} \to \mathcal{C}_A$ given by $X \mapsto X \otimes A$ has a monoidal structure induced by the braiding on $\mathcal{C}$. Its right adjoint is given by the forgetful functor $\mathcal{F}^\vee : \mathcal{C}_A \to \mathcal{C}$ which is the identity on objects. In the sequel we will be interested in using facts about the restriction of $\mathcal{F}_A$ to a planar algebra whose Karoubi completion is $\mathcal{C}$. The following fact, which is a restatement of [1][Proposition 5.1], will turn out to be key.

Proposition 5. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a monoidal functor with faithful exact right adjoint R. If we define $A := R(\mathbb{1})$, then there is an isomorphism K such that the diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[dr, "\mathcal{F}_A"'] & \mathcal{D} \arrow[d, "K"] \\ & \mathcal{C}_A \end{tikzcd}$$

commutes up to natural isomorphism.

Remark 1. *Affine Lie algebras and conformal embeddings will only be used to obtain algebra objects for the quantum subgroups, so we briefly recall the correspondence*

$$\mathcal{C}(\mathfrak{g}, k) \cong \overline{\operatorname{Rep}(U_q(\mathfrak{g}))}$$

*of [**<empty citation>**], where k is the* **level**. *For our purposes this fact translates to*

$$\mathcal{C}(\mathfrak{g}_2, k) \cong \overline{\operatorname{Rep}(U_{e^{\frac{2\pi i}{3(4+k)}}}(\mathfrak{g}_2))} \cong \overline{\operatorname{Kar}(\mathcal{G}_2(e^{\frac{2\pi i}{3(4+k)}}))} \tag{1}$$

as given in [6].

We find the algebra objects and fundamental graphs for GPAs from [3].

3. Skein Theory Generalities

The goal of this section is to develop the tools needed to prove evaluability in a certain class of extensions of trivalent categories. Namely we define the concept of a $\mathbb{Z}_n$*-like extension* of an *Euler-evaluable* trivalent category. We expect this class of extensions to be helpful in the search for novel categories. For example, there is work underway by the present author and Edie-Michell to use the techniques of this paper to construct a possibly infinitely large class of examples of *near-group* categories [8]. This work on near-group categories [**cain˙me˙near-group**] extends an underlying $SO(3)_q$ trivalent skein theory. The present author has also begun work [**unpublished work**] on a family of extensions of $SP(4)_q$, which, despite its skein theory being generated by a braid, is of the same essence.

This all begs the question of which leaves on the "tree of life" [13] may be extended in this way. We expect this tree to bear much fruit of this variety. Already we have extended both categories ($SO(3)_q$ and Fib) covered by [13, Theorem A] by group-like objects. This paper deals with all but one of the categories covered by [13, Theorem B]. The categories one might next attempt such an extension of include:

- The remaining category ABA of [13, Theorem B]
- The category H_3 of [13, Theorem C]

Now let us get to work defining the class of extensions we've been alluding to. Both of the categories studied in this paper are extensions of trivalent categories by a colored, directed, $\mathbb{Z}_n$-like strand. The trivalent categories we deal with have evaluation algorithms based on the standard Euler characteristic argument. One way to capture this evaluability is by considering dimensions of **box spaces**. In a trivalent category we let $B(k, f)$ be the span of diagrams $k \to 0$ with f internal faces. The Euler characteristic argument for evaluability may be phrased as requiring

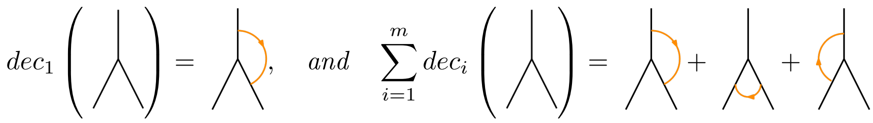
$$\dim B(k, 1) \leq \dim B(k, 0)$$

for $k = 1, \dots, 5$. We will refer to a trivalent category which exhibits such a property **Euler-evaluable**.

General methods for demonstrating evaluability of a skein theory involve identifying some measure of complexity for a closed diagram, then showing the known relations allow one to strictly decrease this measure. For our underlying trivalent categories, Euler-evaluability allows us to decrement one measure of complexity: number of internal faces. With the new strand type, we have another measure: number of colored strands. We concatenate face count with colored strand count to obtain a lexicographic measure of complexity.

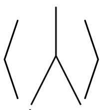
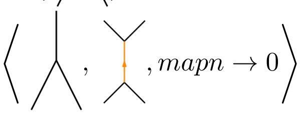
The following notation will be useful in writing down relations in $\mathbb{Z}_n$-like extensions.

Definition 6. *Suppose a diagram E has m boundary points. We define $dec_i(E)$ to be the i-th external single clockwise decorations of E. For example,*

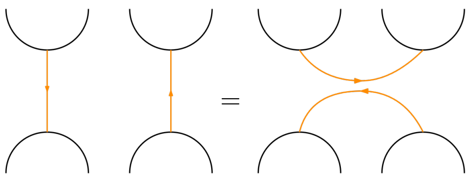


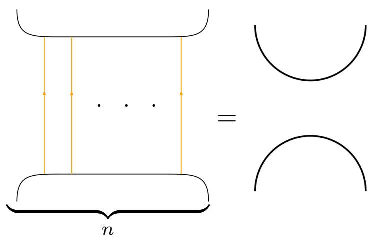
We adopt the convention that $dec_0(E) = E$.

Finally we define the class of categories we will be working with, and show they are evaluable in general. I actually don't think we need the n-generator. If we have an order constraint (i.e., n neighboring strands pop), we get the generator and its absorbtion for free.

Definition 7. *Let $\mathcal{C} =$*  *be a trivalent category that is Euler-evaluable. Call $\mathcal{D}$ a $\mathbb{Z}_n$*-**like extension** *of $\mathcal{C}$ if $\mathcal{D} =$* 

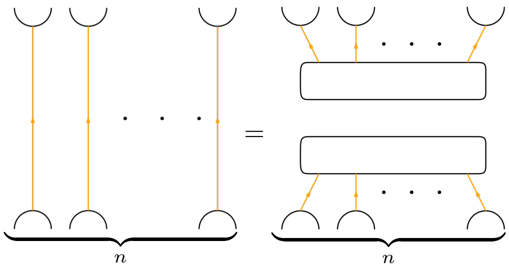
and enjoys the following relations[1]:

(Recouple) 

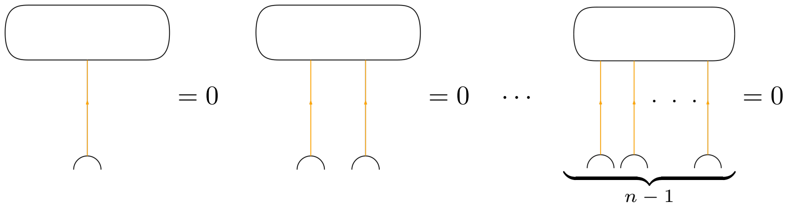
(Order) 

[1]Conditions from [4].

(Split)



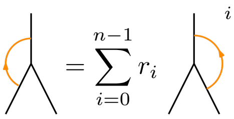
(Schur 0)



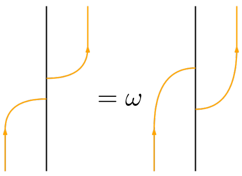
(Schur 1)

$$a = b \quad c = d \quad \cdots \quad e = f$$

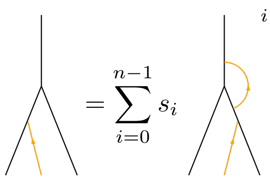
(Change of Basis)



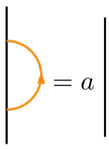
(Swap)



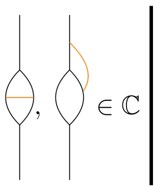
(Slide)



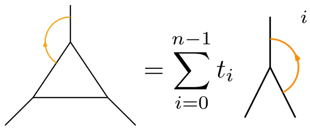
(decStick)



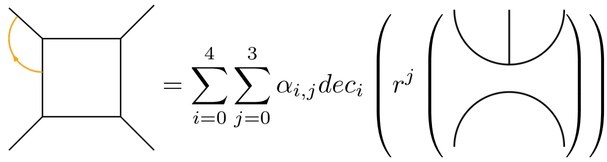
(decBigon)



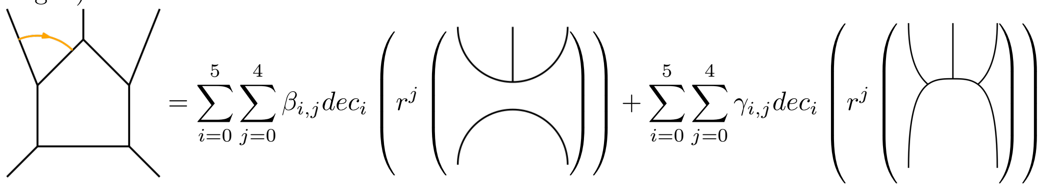
(decTrigon)



(decTetragon)



(decPentagon)



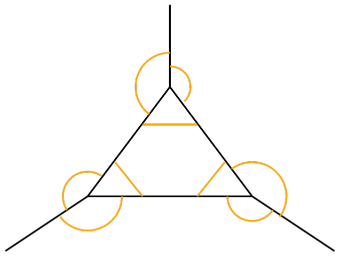
Under the above conditions we have the following fact.

Proposition 6.

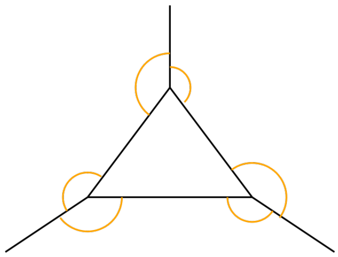
Under the conditions of 7 we obtain the following lemma.

Lemma 2. *A decorated diagram may be expressed as a combination of externally decorated diagrams*

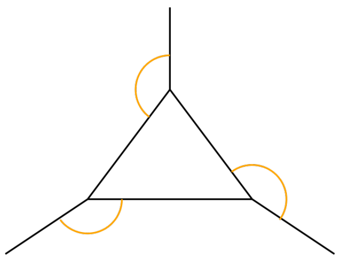
Proof. We prove the lemma for a decorated trigon, and leave the remaining cases to the reader. All less decorated cases are absorbed in this analysis. Consider a maximally-decorated trigon:



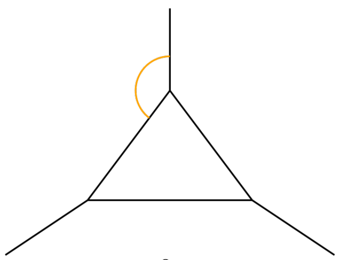
with any labeling on the colored strands. We apply the relations (Swap) and (Slide) on the internal colored strands to obtain a combination of diagrams of the form



Now apply (Change of Basis) to reduce to a combination of diagrams of the form



By another application of (Slide) and (Change of Basis) we arrive at a diagram of the form



During this last step, we pick up colored strands between the black "spokes"; one may happily move these out of the diagram. □

Proof of Proposition 6. Suppose we begin with a closed diagram consisting of a single black connected component. Begin by applying relations from the underlying trivalent category's evaluation algorithm; this decreases the number of trivalent vertices. By the standard Euler characteristic calculation, there must remain some black n-gon with $n \in \{2, \ldots, 5\}$. Choose one such face and apply Lemma 2 to reduce it to a singly-externally-decorated n-gon. Now one of the relations (decBigon), (decTrigon), (decTetragon), or (decPentagon) allows us to pop the face. This process decreases the number of faces (ignoring colored strands) in diagrams by at least 1 at every step. When only two faces remain we use the bigon relation to obtain a loop; apply (decStick) and the underlying loop relation to obtain a scalar.

The case wherein the initial closed diagram has more than one black connected component requires more analysis. In this case the original diagram consists of multiple diagrams like that in the previous case, connected by colored strands. The above case tells us that each black component may be simplified to a single black loop. Our diagram now consists of a number of black loops, connected by colored strands. Use (Recouple) to make it so every black loop has either only in-strands or out-strands attached to it.

If any black loop has more or less than n strands entering or exiting (Schur 0) implies the whole diagram is zero. So suppose each black loop has exactly n strands entering or exiting. If each black loop has only one neighbor, then repeated application of (Order) pops the chain. Count the following:

- number of possible vertices mod n
- number of edges mod n
- number of nodes reachable from a given vertex

□

For each quantum subgroup we construct, we will find planar algebras satisfying the conditions of Proposition 6, and thus will know the planar algebras are evaluable.

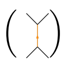
4. Methods: GPA Embeddings

The defining relations for $\mathcal{D}_3$ and $\mathcal{D}_4$ were not computed theoretically. Instead, we deduced them from embedding the planar algebras $\mathcal{P}_{Y_4;\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}$ and $\mathcal{P}_{Y_4;\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}}$ into graph planar algebras.

One may give a functor $F : \mathcal{P}_Y \to \mathrm{GPA}(\Gamma)$ by giving the image of the morphism

F  $\in \mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 1).$

This amounts to giving a list of $M := tr(\Gamma^2 \cdot \Gamma)$ complex scalars, say $a_1, \ldots, a_M$. These complex numbers satisfy equations in the a_i and $\overline{a_i}$. If we assume for now that each a_i is real, then this reduces the system to a collection of polynomials in the a_i [2]. Once we have the image of the trivalent vertex in hand, we have found an embedding of the planar algebra it generates. We can then solve for the image

F  $\in \mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 2).$

4.1. **Trivalent Vertex.** The goal of this subsection is to describe in more detail the process of finding the image of the trivalent vertex. We will walk through the details for the level 4 case. The level 3 case follows the same process, but is somewhat less instructive. See Sections **??** to find the exact graph used at level 4.

Let

$$(p_1, q_1), \ldots, (p_M, q_M)$$

be the defining basis for $\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 1)$ ($M = 88$ at level 4). Then it must be that

F  $= a_1(p_1, q_1) + \cdots + a_M(p_M, q_M).$

The Bigon relation, when sent through F, becomes the system

$$\sum_{i=1}^{M} a_i(p_i, q_i) \circ \sum_{j=1}^{M} a_j(q_j, p_j) = k^2 \sum_{e \in E(\Gamma)} (e, e).$$

[2]This assumption is useful only if it turns out to help us solve the system. In fact, any assumptions we make about this system, if they yield solutions, are in some way valid.

This system is quadratic in the a_i since it involves up to two trivalent vertices on either side. The Lollipop and Rotate relations therefore determine a system of linear equations; the others give cubic, quartic, and quintic equations. It is often useful to solve the linear subsystem first and substitute the solution into the quadratic equations. For example, at level 4, discussed in Section **??**, we solve the linear subsyetm, substitute the solution, and isolate the following resulting equations:

$$a_8^2 + a_{85}^2 = 4 - \sqrt{2} + 2\sqrt{3} - \sqrt{6}$$
$$a_{69}^2 + \left(1 + \sqrt{\frac{3}{2}}\right) a_8^2 = \frac{3 + \sqrt{3} + \sqrt{6}}{\sqrt{2}}$$
$$a_{69}^2 \left(\left(2 + \sqrt{6}\right) a_8^2 + \left(2 + \sqrt{6}\right) a_{85}^2 - 2\sqrt{2 + \sqrt{3}}\right) = 5 + \sqrt{2} + \sqrt{3} + 2\sqrt{6}$$
$$2a_{69}^4 + \left(5 + 2\sqrt{6}\right) a_{85}^4 = \left(3 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right) a_{85}^2 + 3\sqrt{6} + \sqrt{3} + 2\sqrt{2} + 7$$

Up to three choices of sign, the solution to this system is

$$a_8 = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$
$$a_{69} = \sqrt{\frac{1}{2}\left(-1 + \sqrt{2} + \sqrt{3}\right)}$$
$$a_{85} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

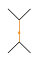
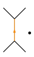
Similar equations containing a_{31}, a_{55}, and a_{63} appear as well. We may repeat this process and obtain the additional values

$$a_{31} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$
$$a_{55} = \sqrt{1 - \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}}$$
$$a_{63} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

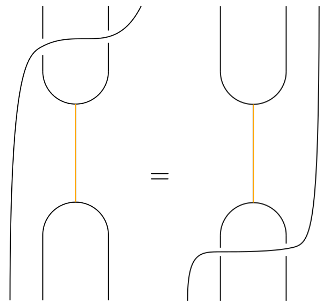
These six degree-8 algebraic numbers now begin a cascade of equation solving. They, along with the linear solution, reduce many of the original high-order equations to linear. We solve those, then repeat the process until we're forced to confront nonlinearity. The nonlinearity we encounter forces us to extract square roots, and ending up with degree-16 algebraic numbers. This lead to us concluding, for instance, that

$$a_{10} = \frac{1}{2}\left(\sqrt{1 + \sqrt{6 - 3\sqrt{3}}} + \sqrt{\sqrt{2 + \sqrt{3}} - 1}\right).$$

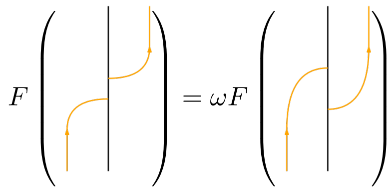
In Section **??** we will discuss these numbers further.

4.2. **Projection.** The process of finding a GPA embedding of the projection  is similar to the process of finding the trivalent vertex described above. Now, however, we may use relations involving both  and . Call the coordinates of the projection b_i. Since the coordinates of the trivalent vertex are now known, the degree of the resulting equations now depends only on the number of projection strands appearing. Moreover, we do not necessarily assume the b_i are real; hence the resulting equations are polynomial in b_i and $\overline{b_i}$. The following Proposition, Lemma 2.4 from [7], provides a system of linear equations in the projection coefficients and is therefore of great utility.

Proposition 7. *The following relation holds:*

(Half-braid) 

4.3. **Finding Relations.** When searching for a GPA embedding of the underlying trivalent category, we have a set of known relations on the trivalent vertex; i.e., the defining relations of $\mathcal{G}_2(q)$. In order to uncover relations such as those of Proposition 6, we use a process reminiscent of the scientific method. This process begins by considering a decorated trivalent graph and searching for ways to decrease its complexity. We begin the process by assuming no more than a minimal collection of moves; i.e., (decStick), (Recouple), and (Product). Once we have applied all these minimal moves, we begin to barter. We look for the least egregious move we might make, and assume it comes at some cost. For example, being able to apply (Swap) might allow us to then apply (decStick); thus we eliminate a colored strand at the cost of a scalar ω. This is a trade we are willing to make. In order to find the exact price of this trade, we use our GPA-embedding F. We use the embedding coefficients we previously found to set up and solve the linear equation



for ω. Another example of bartering appears in the proof of Lemma 2. During the proof, we apply (Swap) and (Change of Basis) with the goal of ridding ourselves of internal strands; this comes at the cost of external strands.

5. Results: Level 3

We begin at level $k = 3$, which makes $q = q_3 = e^{\frac{2\pi i}{42}}$. From the conformal embedding $\hat{\mathfrak{g}}_2 \subseteq \hat{\mathfrak{e}}_6$ given in [5] we obtain the algebra object $A = V_\emptyset \oplus V_{\Lambda_1+\Lambda_2}$, as described in [3]. Also given in [3] is (the orbifold of) the graph Γ_3 with adjacency matrix

$$M_{\Gamma_3} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The graph Γ_3 itself is depicted on the left side of Figure 1. It is known from [5] that

$$\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A^0 \cong \mathrm{Vec}(\mathbb{Z}_3);$$

we deduce from the inclusion $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A^0 \hookrightarrow \overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A$ that $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A$ contains two $\mathbb{Z}_3$-like simple objects, denoted g and g^{-1}.

5.1. **GPA Embedding and Relations.** Here we give details of both the GPA-embedding of $\mathcal{D}(q_3)$ and its governing equations. Recall that the defining bases 3 for the spaces

$$\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(m \to n)$$

are given in terms of pairs of paths. The (undirected) graphs we are using have at most a single edge between any two vertices. Hence an edge is equivalent to a pair of vertices, and a path is equivalent to an ordered tuple of vertices. For example, the path

$$p = v_1 \longrightarrow v_2 \longrightarrow v_3$$

is equivalent to the ordered triple (v_1, v_2, v_3). Which paths q pair validly with p to form a basis element of the $2 \to 1$ hom-space of a GPA? Well, by definition, q must be parallel to p; i.e. the sources and targets of p and q must coincide. It follows that the only valid pairing for such p is

$$q = v_1 \longrightarrow v_3,$$

which may also be represented as (v_1, v_3). So the only $2 \to 1$ basis element which p appears in is

$$\big((v_1, v_2, v_3), (v_1, v_3)\big).$$

But the parallel condition defining basis elements makes including (v_1, v_3) redundant; we might just as well have called the basis element by

$$(v_1, v_2, v_3).$$

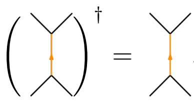
This is how we refer to $2 \to 1$ GPA basis elements. Indeed, in Table 1, the first two columns combine to specify which basis elements are being specified, and the third column gives the approximate coordinate of the trivalent embedding on that basis element. For example, the first row of Table 1 tells us that the coordinate of the $(2, 2, 2)$ basis element is approximately 1.08393; the second row tells us that the coordinate of the $(4, 2, 3)$ basis element is approximately 0.619371.

Paths of the form (i, j, i), (i, i, j), or (i, j, j) for $i, j \neq 1$ require a bit more care to describe. There is nontrivial interplay with the graph symmetry swapping vertices 2 and 4. When these two vertices are swapped, a path whose coordinate has absolute value 0.155691 is sent to one whose coordinate has absolute value 1.69414. The nine paths whose coordinates have absolute value 0.155691 are:

$$(2,3,3),\ (3,3,2),\ (3,2,3),\ (2,4,2),\ (4,3,4),\ (2,2,4),\ (3,4,4),\ (4,2,2),\ (4,4,3)$$

One may use symmetry to find the rest of the coordinates.

Table 2 holds numerical approximations to the nonzero projection coordinates. There are blocks of nonzero coordinates of length 9 and 16. These sizes, and the location of the nonzero real coordinates follow naturally when one considers three facts:

(1) The object g is $\mathbb{Z}_3$ and simple and therefore has fusion graph given in Figure **??**.
(2) There is a map $T_g \in \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(Y^{\otimes 2} \to g)$ whose outer product $T_g^\dagger \circ T_g$ equals P_g, projection onto g.
(3) The dagger of a simple projection is itself; i.e.,  .

The existence and behavior of T_g must be captured by the image of P_g in the GPA, which is all we have access to. This is indeed the case, as the only coordinates of the projection in the GPA which are nonzero are at those basis vectors

$$(i \to \_ \to j, i \to \_ \to j)$$

where $i \to j$ is a directed edge of the g-fusion graph. For $i = j = 1$ there are three possible values for $\_$; pairing them gives 9 pairs. For $i, j \neq 1$ there are four possible values for $\_$; pairing them gives 16 pairs. The columns of Table 2 give numerical approximations to the coordinates of the projection, with dictionary ordering on the pairs of $\_$ values. That is, the column labeled by $1 \to \_ \to 1$ shows the coordinates on the ordered basis

$$\begin{gathered}
(1 \to 2 \to 1, 1 \to 2 \to 1)\\
(1 \to 2 \to 1, 1 \to 3 \to 1)\\
(1 \to 2 \to 1, 1 \to 4 \to 1)\\
(1 \to 3 \to 1, 1 \to 2 \to 1)\\
(1 \to 3 \to 1, 1 \to 3 \to 1)\\
(1 \to 3 \to 1, 1 \to 4 \to 1)\\
(1 \to 4 \to 1, 1 \to 2 \to 1)\\
(1 \to 4 \to 1, 1 \to 3 \to 1)\\
(1 \to 4 \to 1, 1 \to 4 \to 1)
\end{gathered}$$

With this ordering and fact (3) above in mind, and recalling that the GPA's dagger operation swaps paths, the real coordinates appear where one would expect them.

Finally we give explicit values for the coefficients of the equations of Proposition 6, excluding (decTetragon) and (decPentagon); the curious reader may find these in the attached Mathematica notebook. The structure constants for $\mathcal{D}_3$ are:

$$\omega = e^{2\pi i/3}$$

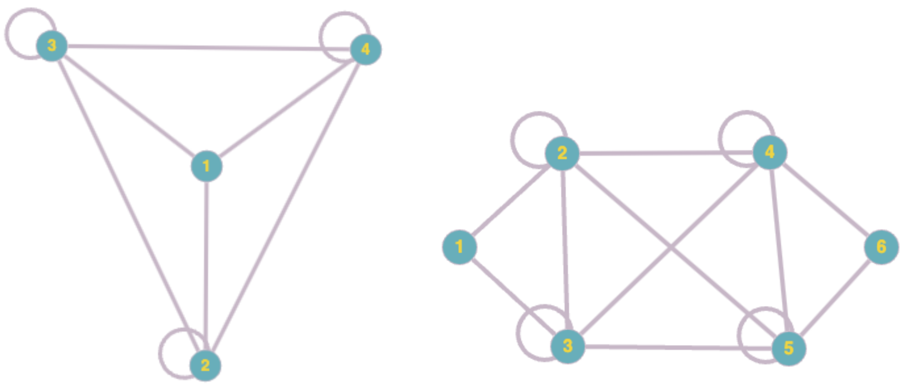


FIGURE 1. Fundamental graphs used for levels 3 (left) and 4 (right).

Vertex Path	Conditions	Coefficient
(i,i,i)	$i \neq 1$	1.08393
(i,j,k)	$\{i,j,k\} = \{2,3,4\}$	0.619371
$(i,1,k)$	$i,k \neq 1,\ i \neq k$	1.69414
$(i,1,i)$	$i \neq 1$	0.861006
$(i,i,1)$ or $(1,i,i)$	$i \neq 1$	0.967919

TABLE 1. Level 3 trivalent embedding coefficients.

$1 \to \_ \to 1$	$2 \to \_ \to 4$	$3 \to \_ \to 2$	$4 \to \_ \to 3$
1.26376	0.791288	0.791288	0.791288
$-0.631881 - 1.09445i$	$0.567622 - 0.684904i$	$0.876955 + 0.149123i$	$0.674406 + 0.580055i$
$-0.631881 + 1.09445i$	$0.674406 + 0.580055i$	$0.567622 - 0.684904i$	$0.876955 + 0.149123i$
$-0.631881 + 1.09445i$	$-0.876955 - 0.149123i$	$-0.674406 - 0.580055i$	$0.567622 - 0.684904i$
1.26376	$0.567622 + 0.684904i$	$0.876955 - 0.149123i$	$0.674406 - 0.580055i$
$-0.631881 - 1.09445i$	1	1	1
$-0.631881 - 1.09445i$	$-0.0182917 + 0.999833i$	$0.5 - 0.866025i$	$0.856735 - 0.515757i$
$-0.631881 + 1.09445i$	$-0.5 - 0.866025i$	$-0.856735 - 0.515757i$	$-0.0182917 - 0.999833i$
1.26376	$0.674406 - 0.580055i$	$0.567622 + 0.684904i$	$0.876955 - 0.149123i$
	$-0.0182917 - 0.999833i$	$0.5 + 0.866025i$	$0.856735 + 0.515757i$
	1	1	1
	$-0.856735 + 0.515757i$	$0.0182917 - 0.999833i$	$0.5 - 0.866025i$
	$-0.876955 + 0.149123i$	$-0.674406 + 0.580055i$	$0.567622 + 0.684904i$
	$-0.5 + 0.866025i$	$-0.856735 + 0.515757i$	$-0.0182917 + 0.999833i$
	$-0.856735 - 0.515757i$	$0.0182917 + 0.999833i$	$0.5 + 0.866025i$
	1	1	1

TABLE 2. Level 3 projection embedding coefficients.

$$r_1 = ...,\quad r_2 = ...,\quad r_3 = ...$$

$$s_1 = ...,\quad s_2 = ...,\quad s_3 = ...$$

$$r_1 = ...,\quad r_2 = ...,\quad r_3 = ...$$

$$t_1 = ...,\quad t_2 = ...,\quad t_3 = ...$$

5.2. **Theorem and proof.** This subsection is devoted to proving the following theorem.

Theorem 1. *There is a monoidal equivalence*

$$\mathrm{Ab}(\overline{\mathcal{D}_3}) \cong \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}.$$

Let $X = V_{\Lambda_1}$ be the object of $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}$ by which we generate the planar algebra $\mathcal{P}_X \cong \overline{\mathcal{G}_2(q)}$. Define $Y = \mathcal{F}_A(X)$ to be the image of X under the free functor. Now $\mathcal{F}_{A_3} : \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))} \to \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$ restricts to an embedding $\mathcal{P}_{X;\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}} \hookrightarrow \mathcal{P}_{Y;\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_{A_3}}$. Invertibility of the objects g and g^{-1} implies $g \otimes Y \cong Y$, with rigidity maps for g and g^{-1} building the mutually inverse isomorphisms.

We now compute some important dimensions.

Proposition 8. *The following are true:*

(1) $\dim \mathrm{Hom}_{\mathcal{C}_{A_3}}(Y^{\otimes 2} \to Y) = 3$
(2) $\dim \mathrm{Hom}_{\mathcal{C}_{A_3}}(Y^{\otimes 2} \to g^k) = 1$
(3) $\dim \mathrm{Hom}_{\mathcal{C}_{A_3}}(Y \to Y) = 1$ *(i.e., Y is simple)*

Proof. All three are proved using fusion graph calculations, so we prove only (1)[3]. The fusion graph tells us that

$$V_{\Lambda_1}^{\otimes 2} \cong V_\emptyset \oplus V_{\Lambda_1} \oplus V_{2\Lambda_1} \oplus V_{\Lambda_2}$$

and

$$V_{\Lambda_1} \otimes A \cong V_{\Lambda_1} \oplus V_{2\Lambda_1} \oplus V_{3\Lambda_1} \oplus V_{\Lambda_2} \oplus V_{\Lambda_2+\Lambda_1} \oplus V_{\Lambda_2+2\Lambda_1}.$$

On the other hand, we compute

$$\begin{aligned}
\mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(Y^{\otimes 2} \to Y) &= \mathrm{Hom}_{\mathcal{C}_{A_3}}(\mathcal{F}_{A_3}(V_7)^{\otimes 2} \to \mathcal{F}_{A_3}(V_7)) \\
&\cong \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(\mathcal{F}_{A_3}(V_7^{\otimes 2}) \to \mathcal{F}_{A_3}(V_7)) \\
&= \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(\mathcal{F}_{A_3}(V_7^{\otimes 2}) \to V_7 \otimes A_3) \\
&\cong \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}}(V_7^{\otimes 2} \to V_7 \otimes A_3).
\end{aligned}$$

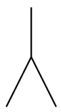
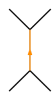
Counting common irreducible constituents of $V_{\Lambda_1}^{\otimes 2}$ and $V_{\Lambda_1} \otimes A_3$ gives the desired result. $\square$

We have an immediate consequence.

Corollary 1. *There is a dominant monoidal functor*

$$\Psi_3 : \mathcal{D}_3 \to \overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_{A_3}$$

which maps

 $\mapsto \tau \in \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(Y^{\otimes 2} \to Y)$,  $\mapsto P_g \in \mathrm{End}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(Y^{\otimes 2})$

Proof. Proposition 8, along with the defining relations of $\mathcal{D}_3$ tell us this assignment is functorial. Since Y is a $\otimes$-generator and Ψ_3 surjects onto objects of $\mathcal{P}_Y$, dominance also follows. $\square$

Lemma 3. *The induced map*

$$\overline{\Psi_3} : \overline{\mathcal{D}_3} \to \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$$

is faithful.

Proof. The evaluation algorithm for $\mathcal{D}_3$ implies that $\mathcal{D}_3$ has simple unit. Therefore every ideal is contained in the ideal of negligibles, which is killed when passing to the semisimplification $\overline{\mathcal{D}}$. Hence the map $\overline{\Psi_3}$ has no kernel. $\square$

Proof of Theorem 1. We now note that since $\mathcal{G}_2(q)$ and $\mathcal{D}_3$ are unitary, we know that $\mathcal{G}_2(q) \hookrightarrow \mathcal{D}_3$ induces a $\dagger$-embedding $\overline{\mathcal{G}_2(q)} \hookrightarrow \overline{\mathcal{D}_3}$. Thus, there is a chain

$$\overline{\mathcal{G}_2(q_3)} \hookrightarrow \overline{\mathcal{D}_3} \xhookrightarrow{\overline{\Psi_3}} \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$$

[3]See, e.g., Figure 5 of [9] for the fusion graph.

of faithful dominant functors. Using the universal property of Karoubi completion, we arrive at the commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{G}_2(q_3)} \arrow[r, hook] \arrow[d, hook] & \overline{\mathcal{D}_3} \arrow[d, hook] \arrow[dr, "\overline{\Psi}"] & \\
\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))} \arrow[r, "\mathcal{F}_1"] & \mathrm{Ab}(\overline{\mathcal{D}_3}) \arrow[r, "\mathcal{F}_2"] & \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}
\end{tikzcd}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the induced functors. At this point we shift our focus to the lower layer of the diagram.

By [<**empty citation**>], $(\mathcal{F}_2 \circ \mathcal{F}_1)|_{\overline{\mathcal{G}_2(q_3)}} = \mathcal{F}_{A_3}|_{\overline{\mathcal{G}_2(q_3)}}$ implies $\mathcal{F}_2 \circ \mathcal{F}_1 = \mathcal{F}_{A_3}$. Note that both $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}$ and $\mathrm{Ab}(\overline{\mathcal{D}_3})$ are semisimple, and therefore $\mathcal{F}_1$ has a lax-monoidal right adjoint $\mathcal{F}_1^\vee$. Proposition 5 now allows us to conjure K and B such that the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd}
\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))} \arrow[r, "\mathcal{F}_1"] \arrow[dr, "\mathcal{F}_B"'] & \mathrm{Ab}(\overline{\mathcal{D}_3}) \arrow[r, "\mathcal{F}_2"] \arrow[d, "K"] & \overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A \\
& \overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_B \arrow[ur, "\mathcal{F}'"'] &
\end{tikzcd}$$

Here, $\mathcal{F}'$ is defined to complete the diagram. From here, apply $\mathcal{F}_B^\vee$ to the containment $\mathbb{1} \subseteq \mathcal{F}'^\vee(\mathbb{1})$:

$$\begin{aligned}
B = \mathcal{F}_B^\vee(\mathbb{1}) &\subseteq \mathcal{F}_B^\vee \circ \mathcal{F}'^\vee(\mathbb{1}) \\
&\cong (\mathcal{F}_2 \circ \mathcal{F}_1)^\vee(\mathbb{1}) \\
&= \mathcal{F}_{A_3}^\vee(\mathbb{1}) \\
&= \mathcal{F}_A^\vee(A_3) = A_3.
\end{aligned}$$

So $B \subseteq A_3$. Since A_3 has only two simple summands, we know A_3 has no nontrivial subalgebras: $B \cong \mathbb{1}$ or $B \cong A_3$. If $B \cong \mathbb{1}$ then

$$\mathrm{Ab}(\overline{\mathcal{D}_3}) \cong \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{\mathbb{1}} \cong \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}$$

A quick dimension count falsifies this. Hence $\mathrm{Ab}(\overline{\mathcal{D}_3}) \cong \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$. □

6. Results: Level 4

At level $k = 4$ we have $q = q_4 = e^{\frac{2\pi i}{48}}$ and $A_4 = V_\emptyset \oplus V_{3\Lambda_1}$. From [5] we have $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}^0 \cong \mathrm{Vec}(\mathbb{Z}_2)$, giving us one new $\mathbb{Z}_2$-like simple object h. The details of the GPA embedding at level 4 are less instructive than the level 3 case, so we relegate the details to the attached Mathematica files. Much of the story remains the same. The trivalent coordinates are algebraically nicer than those at level 3; up to sign, we showed 5 of 9 in Subsection 4.1. The nonzero coordinates of the projection come in blocks of 4, and 9. There is also an analogue of Theorem 1 for level 4.

Theorem 2. *There is a monoidal equivalence*

$$\mathrm{Ab}(\overline{\mathcal{D}(q_4)}) \cong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}.$$

This theorem is proved by using analogues of Proposition 8, Corollary 1, Lemma 3. The same argument used in the proof of Theorem 1 except with, e.g., every q_3 switched out for a q_4, may be applied.

Note that since $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}^0$ contributes a $\mathbb{Z}_2$-like simple object, there is only one decoration of a trivalent vertex; we represent this as an undirected colored strand. The structure constants for $\mathcal{D}_4$ are:

$$\omega = -1$$

$$r_1 = -1, \quad r_2 = -1$$

$$s_1 = ..., \quad s_2 = ...$$

$$r_1 = ..., \quad r_2 = ...$$

$$t_1 = ..., \quad t_2 = ...$$

7. Conclusion

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