PRESENTATIONS QUANTUM SUBGROUPS OF TYPE G_2 VIA GRAPH PLANAR ALGEBRA EMBEDDINGS

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ABSTRACT. todo

1. INTRODUCTION

Quantum subgroups are a well-known source of tensor categories. More precisely, given a conformal embedding $\mathcal{V}(\mathfrak{g}, k) \subseteq \mathcal{V}(\mathfrak{h}, 1)$ of VOAs as in [4], one obtains a corresponding Etale algebra A. This algebra then allows one to consider the category $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$ of right A-modules. A half-braiding on A then gives a tensor product on $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$, and one may study this new category in its own right. The free functor gives an embedding $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))} \hookrightarrow \overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$. As this embedding is, in general, not full, it remains only to find a description of the new morphisms in $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$ to describe this newly constructed category of modules. Recent works of Edie-Michell and Snyder [6] have used this reasoning, and representation theoretic techniques to give diagrammatic descriptions of tensor categories of modules corresponding to the family of conformal embeddings $\mathcal{V}(\mathfrak{sl}_N, N^2) \subseteq \mathcal{V}(\mathfrak{sl}_{N^2-1}, 1)$.

On the other hand, one may start with a known category and compute graph planar algebra (GPA) embeddings for it. This has been done for $\overline{\mathrm{Rep}(U_q(\mathfrak{sl}_N))}$ in [2] and for the extended Haagerup categories in [9]. This computation has the theoretical and practical consequences. By the GPA embedding theorem [**something**], such an embedding immediately gives a module category. It additionally gives a concrete representation of the category in which one may perform explicit computations.

The present work describes a blend of these two techniques. We begin by fidning a GPA embedding of the well-known trivalent category $\mathcal{G}_2(q)$ of [13, 14] which is a diagrammatic presentation for $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}$. Through the free functor we can view our GPA embedding as a GPA embedding for a $\otimes$-generating object's planar algebra in $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A$. Diagrammatically, this gives us a black-strand and a trivalent vertex:
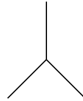
with a known skein theory. We then search inside the GPA embedding for new morphisms. According to [4] there ought to be a projection onto a $\mathbb{Z}_k$-like simple object in $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A$, so this is what we search for inside the GPA. We view this new morphism as an I with an oriented orange vertical strand:

In the case of $\mathcal{G}_2(q)$, the properties of this new morphism were unknown beyond a few basics deriving from, e.g., the fact that is is a projection onto a simple object contained in the tensor square of the $\otimes$-generating object. Once we have or hands on the image in the GPA of this projection, though, we may explore its properties through explicit computations. We perform this process of extending GPA embeddings for the two conformal embeddings

$$(1) \qquad \mathcal{V}(\mathfrak{g}_2, 3) \subseteq \mathcal{V}(\mathfrak{e}_6, 1) \quad \text{and} \quad \mathcal{V}(\mathfrak{g}_2, 4) \subseteq \mathcal{V}(\mathfrak{d}_7, 1).$$

Now we begin by introducing some notation for a skein theory involving an oriented, colored strand in addition to unoriented black strands.

Definition 1. *For a diagram $\mathcal{E}$ the notation $\rho^i(\mathcal{E})$ means an i-click right rotation. For instance,*

$$\rho^1 \left(\Big| \quad \Big) \right) = \underset{\frown}{\smile} \quad \text{and} \quad \rho^2 \left(\Big| \quad \Big) \right) = \Big| \quad \Big|.$$

Suppose the diagram $\mathcal{E}$ has m boundary points. We define $dec_i(\mathcal{E})$ to be the i-th external single clockwise decoration of $\mathcal{E}$. For example,

$$dec_1 \left(\bigwedge \right) = \bigwedge, \quad \text{and} \quad \sum_{i=1}^{3} dec_i \left(\bigwedge \right) = \bigwedge + v \bigwedge + v \bigwedge$$

We adopt the convention that $dec_0(\mathcal{E}) = \mathcal{E}$.

Both of the categories studied in this paper are extensions of trivalent categories by a colored, directed, $\mathbb{Z}_n$-like strand. We define the class of categories we will be working with. Later we will show that, with a relatively tame assumption on the underlying skein theory, categories in this class are evaluable in general.

Definition 2. *Let $\mathcal{C} = \left\langle \bigwedge \right\rangle$ be a trivalent category. Call $\mathcal{D}$ a $\mathbb{Z}_n$-**like exten-***

***sion** (or **cyclic** when n is understood) of $\mathcal{C}$ if we have $\mathcal{D} = \left\langle \bigwedge, \bigotimes \right\rangle$, enjoying the following relations*

(Recouple)

(Reverse)

(Schur 0)

$$\cdots$$

$$\underbrace{\qquad}_{n-1}$$

(Schur 1) $= 0$ $= 0$ $\cdots$ $= 0$

(Swap) $= \omega$

(decStick) $=$

(decBigon) $= b$

(Change of Basis) $= \displaystyle\sum_{i=0}^{n-1} r_i$

(decTrigon) $= \displaystyle\sum_{i=0}^{n-1} t_i$

(decTetragon)

 $= \displaystyle\sum_{i=0}^{4}\sum_{j=0}^{3} u_{i,j} dec_i\left(\rho^j\left(\text{} \right)\right) + \sum_{i=0}^{4}\sum_{j=0}^{3} v_{i,j} dec_i\left(\rho^j\left(\text{} \right)\right)$

(decPentagon)

 $= \displaystyle\sum_{i=0}^{5}\sum_{j=0}^{4} w_{i,j} dec_i\left(\rho^j\left(\text{} \right)\right) + \sum_{i=0}^{5}\sum_{j=0}^{4} x_{i,j} dec_i\left(\rho^j\left(\text{} \right)\right)$

We additionally enforce the condition that the diagrams *for* $i = 0, \ldots, n-1$
span the $2 \to 1$ *hom-space.*

Remark 1. *It is worth noting the following standard abuse of language. A diagrammatically presented category such as a cyclic extension has hom-spaces which*

are formal spans of diagrams. When applying a relation such as (decTrigon) locally, the result is a linear combination of diagrams. Usually, though, this linear combination has some desirable quality, such as a smaller number of internal faces in each summand. In this instance, we prefer to say something along the lines of, "applying (decTrigon) decreases the number of internal faces," instead of, for instance, the more wordy, "applying (decTrigon) turns this diagram into a linear combination of diagrams with fewer internal faces."

Definition 3. *Set $q_4 = e^{\frac{2\pi i}{48}}$ and define $\mathcal{D}_4$ to be the cyclic extension of $\mathcal{G}_2(q_4)$ with structure constants[1]*

One of the two primary results we give here is that $\mathcal{D}_4$ is a presentation for a category of modules corresponding to the level 4 conformal embedding of $\mathfrak{g}_2$.

Theorem 1. *There is an equivalence*

$$\mathrm{Ab}(\mathcal{D}_4) \cong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$$

where A_4 is the algebra object corresponding to the level-4 conformal embedding of 1.

Theorem 7 is an analogous theorem for level 3, where $\mathcal{D}_3$ is defined similarly to $\mathcal{D}_4$, with structure constants given in the attached Mathematica files.

It is not clear a priori that the defining relations for, say, $\mathcal{D}_4$ lead to a nontrivial tensor category. The general undecidability of the word problem for groups offers some evidence that this question is difficult for a typical presentation for a tensor category. That is, one should not expect a set of relations to yield any nontriviality.

The remainder of the paper is structured as follows. Section 2 sets up most of the theory needed, referencing that which we do not exposit here. This includes unoriented planar algebras, unoriented graph planar algebras, internal algebra and module objects, and some assorted theoretical devices and results. Section 3 then goes on to investigate some general properties of cyclic extensions. We expect

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[1]We omit the (decPentagon) equation here for brevity. It contains 44 nonzero summands, and can be found in the attached Mathematica notebooks.

this class of categories to be of use for researchers intent on conjuring examples of exotic tensor categories. In fact, in a forthcoming paper, the present author and Cain Edie-Michell diagrammatically present a number of near-group categories as cyclic extensions of $SO(3)_q$ trivalent categories. We demonstrate evaluability of this class of categories under a relatively tame assumption on the underlying trivalent skein theory. Section 4 discusses the process of arriving at GPA embeddings. We detail the techniques used to arrive at GPA embeddigns of trivalent categories, and then show how we extend these embeddings to cyclic extensions. This section also explains how we use GPA embeddings to explore relations in these extensions. This section uses examples from level 4 ($\mathcal{D}_4$) due to the fact that the numbers involved are more presentable. The process used for level 3 ($\mathcal{D}_3$) was essentially identical. Finally, in Section 5 we prove that the representations we've found are actually presentations for the respective quantum subgroups.

2. PRELIMINARIES

Here we define the players in this game. This includes planar algebras, graph planar algebras, and internal algebra and module objects. We give only a few necessary results, and refer the reader to the definitive publications. For the general theory of tensor categories, see [7].

2.1. **Algebra and Module Objects.** We will ultimately show that $\mathcal{D}_3$ and $\mathcal{D}_4$ are presentations for the categories $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$ and $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$ of modules over algebra objects A_3 and A_4 coming from the conformal embeddings $\mathcal{C}(\mathfrak{g}_2, 3) \subseteq \mathcal{C}(\mathfrak{e}_6, 1)$ and $\mathcal{C}(\mathfrak{g}_2, 4) \subseteq \mathcal{C}(\mathfrak{d}_7, 1)$, respectively. In this subsection we recall basic facts about algebra and module objects, as well as conformal embeddings. See [7, 15] for more complete descriptions. The theory which will apply to our context is given in [6]. Some basic properties concerning the interaction of algebra and module objects with monoidal functors will be used in the proof of our main theorems; this material can be found in [11]. We restate a few definitions and facts here. Unless otherwise stated, we will be assuming the underlying tensor categories are braided.

Definition 4. *Let A be an algebra object of the braided tensor category $\mathcal{C}$. A is an **Etale** algebra if it is commutative and separable. We call A **connected** if it is Etale and $\dim \mathrm{Hom}_{\mathcal{C}}(\mathbb{1} \to A) = 1$.*

For an Etale algebra object A of $\mathcal{C}$, we denote by $\mathcal{C}_A$ the collection of left A-modules internal to $\mathcal{C}$. As described in [6], a braiding on $\mathcal{C}$ induces a tensor product on $\mathcal{C}_A$. Separability of A implies semisimplicity of $\mathcal{C}_A$, and connectedness of A implies the unit $\mathbb{1}_{\mathcal{C}_A} = A$ is simple in $\mathcal{C}_A$ [4].

The Etale conditions are precisely those required to perform the skein theory on A-modules to define the tensor product on $\mathcal{C}_A$. Etale is the same thing as multiplication having a section, which means we can pop A-bigons. Connected means theres only one A-cap and one A-cup. This lets us wiggle enough to perform the proof that $M \otimes_A N$ is well-defined.

Furthermore, the free functor

$$\mathcal{F}_A : \mathcal{C} \xrightarrow{X \mapsto A \otimes X} \mathcal{C}_A$$

is a monoidal embedding which is, as we will see later, not always full. Its right adjoint is the forgetful functor $\mathcal{F}^\vee : \mathcal{C}_A \to \mathcal{C}$ which acts as the identity on both objects and morphisms.

Beginning with the following lemma, which is recreated from [1], we now recall some facts that will help us along the way. These will consist of a few results, along with the basics of *conformal embeddings*.

Lemma 1. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a monoidal functor with faithful exact right adjoint R. If we define $A := R(\mathbb{1})$, then there is an equivalence $\mathcal{K}$ such that the diagram*

commutes up to natural isomorphism.

Lemma 2. *Suppose $\mathcal{C}$ has simple unit, $\mathcal{D}$ is unitary, and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a †-functor. Then $\overline{\mathcal{C}}$ is unitary, and F descends to a †-embedding $\overline{\mathcal{F}} : \overline{\mathcal{C}} \to \mathcal{D}$ such that*

commutes.

One result which will help immensely in arriving at GPA embeddings is the following, which is Lemma 2.4 of [6]. Since the free functor $\mathcal{F}_A$ gives an embedding $\mathcal{C} \hookrightarrow \mathcal{C}_A$, the braid in $\mathcal{C}$ is mapped to a braiding of the image of $\mathcal{C}$ in $\mathcal{C}_A$. Since the free functor is in general not full, we cannot extend this to a braiding on all of $\mathcal{C}_A$, however, there is a half-braid for arbitrary morphisms between objects in the image of $\mathcal{F}_A$.

Lemma 3 (Half-braid)**.** *Let $\mathcal{C}$ be a braided tensor category, and A an Etale algebra object. For any $f \in \mathrm{Hom}_{\mathcal{C}_A}(\mathcal{F}_A(Y_1) \to \mathcal{F}_A(Y_2))$, the following relation holds:*

(Half-braid)

Note that f need not be in the image of the free functor. We will utilize this result to obtain a rather large number (2970 at level 3 and 7776 at level 4) of linear equations constraining the GPA coordinates of the morphisms not living in the image of $\mathcal{F}_A$. Thus the half-braid relation will be key to our program, despite not being necessary to prove evaluability.

The source of our algebra objects will be conformal embeddings. We direct the reader to [4] a more complete treatment.

Definition 5. *A containment $\mathcal{V}(\mathfrak{g}, j) \subseteq \mathcal{V}(\mathfrak{h}, k)$ of affine Lie algebras is said to be* **conformal** *if the adjoint representation of $\mathcal{V}(\mathfrak{h}, j)$ restricts to a finite direct sum of simple objects in $\mathcal{C}(\mathfrak{g}, j) := \operatorname{Rep}(\mathcal{V}(\mathfrak{g}, j))$.*

Affine Lie algebras and conformal embeddings will only be used to obtain algebra objects and module fusion graphs, so we briefly recall the correspondence

$$\mathcal{C}(\mathfrak{g}_2, k) \cong \overline{\operatorname{Rep}(U_{q_k}(\mathfrak{g}_2))} \tag{2}$$

of [5], where k is the level and q_k is given by

$$q = e^{\frac{2\pi i}{3(4+k)}}.$$

At level 3 we have $q_3 = e^{\frac{2\pi i}{42}}$ and at level 4 we have $q_4 = e^{\frac{2\pi i}{48}}$. We obtain the algebra objects and fundamental graphs for GPAs from [3]:

$$A_3 = V_\emptyset \oplus V_{\Lambda_1} \quad \text{and} \quad A_4 = V_\emptyset \oplus V_{3\Lambda_1} \tag{3}$$

at levels 3 and 4, respectively.

Additionally at level 3 and 4, respectively, we have the existence of $\mathbb{Z}_3$- and $\mathbb{Z}_2$-like simple objects g_3 and g_4 [4]. From [exceptional] we see that at both levels $k = 3, 4$ we have

$$\dim \operatorname{Hom}_{\overline{\operatorname{Rep}(U_{q_k}(\mathfrak{g}_2))}_{A_k}} (\mathcal{F}_{A_k}(V_{\Lambda_1})^{\otimes 2} \to g_k) = 1.$$

Remark 2. *It follows that there are idempotents*

$$P_{g_k} : \mathcal{F}_{A_k}(V_{\Lambda_1})^{\otimes 2} \to \mathcal{F}_{A_k}(V_{\Lambda_1})^{\otimes 2}$$

projecting onto these grouplike objects. As the g_i are simple, we have $P_{g_k}^\dagger = P_{g_k}$.

The behavior of the P_{g_k} will be captured by ⌣⌢ *in $\mathcal{D}_k$. As we will see, describing the interaction of P_{g_k} with the image of the free functor will be sufficient to fully describe $\overline{\operatorname{Rep}(U_{q_k}(\mathfrak{g}_2))}_{A_k}$.*

2.2. **Unoriented Planar Algebras.** Recall the theory of **rigid** monoidal categories detailed in [12]. To put it succinctly, rigid monoidal categories have duals. Duals, and the associated evaluation and coevaluation maps, giving us the cups and caps ubiquitous in skein theory. A rigidity assumption gives us the ability to isotope diagrams. The generators we will use for our planar algebras will be symmetrically self-dual. We also assume pivotality throughout.

Let X be a (symmetrically self-dual) **tensor generator** for the tensor category $\mathcal{C}$; that is, every object of $\mathcal{C}$ is isomorphic to a subobject of some tensor power $X^{\otimes n}$. Let $\mathcal{P}_{X;\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ whose objects are tensor powers $\mathbb{1} = X^{\otimes 0}, X, X^{\otimes 2}, \ldots$; we call this the (unoriented) **planar algebra** generated by X in $\mathcal{C}$. The planar algebra $\mathcal{P}_{X;\mathcal{C}}$ is **evaluable** if $\dim \operatorname{End}_{\mathcal{P}_{X;\mathcal{C}}}(\mathbb{1}) = 1$.

We will be presenting the our two quantum subgroups as extensions of $\mathcal{G}_2(q)$ skein theories, in the spirit of Kuperberg [13, 14]. Up to a rescaling by a factor of $\kappa = \sqrt{[7] - 1}$ we use the same skein theory as [14] (note the sign error in the Pentagon relation of [13]).

Definition 6. *For q a root of unity, the $\mathcal{G}_2(q)$ skein theory is defined to be that generated by an unoriented trivalent vertex* ⅄ *satisfying the relations*
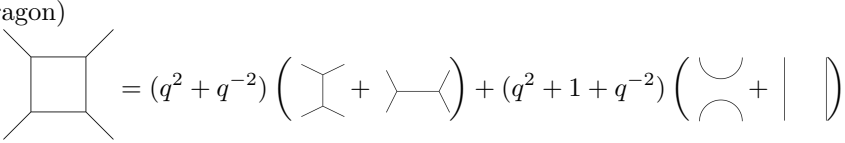
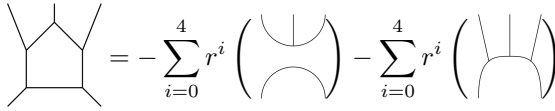$$\bigcirc = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10} \tag{Loop}$$

(Lollipop)

$$\text{(Lollipop)} \qquad \bigcirc\!\!\!\big| \; = 0$$

(Rotate)

$$r^1\left(\;\right) = \;$$

(Bigon)

$$\diamondsuit = \kappa^2 \;\Big|$$

(Trigon)

$$\triangle = -(q^4 + 1 + q^{-4}) \;$$

(Tetragon)

$$\square = (q^2 + q^{-2})\left(\;\;+\;\;\right) + (q^2 + 1 + q^{-2})\left(\;\;+\;\Big|\;\Big|\;\right)$$

(Pentagon)

$$\pentagon = -\sum_{i=0}^{4} r^i\left(\;\right) - \sum_{i=0}^{4} r^i\left(\;\right)$$

Our use of planar algebras will depend entirely on the construction of the Cauchy completion, which we sketch here. See [6] for more details and [16] for a full treatment of the topic. Recall that the **idempotent completion** of a pivotal tensor category $\mathcal{C}$ consists of pairs (Z, p), where $p \in \mathrm{End}_{\mathcal{C}}(Z)$ is an idempotent. We denote the idempotent completion of $\mathcal{C}$ as $\mathrm{Idemp}(\mathcal{C})$. Further, we define the **additive envelope** of a pivotal, $\mathbb{C}$-linear tensor category $\mathcal{C}$ to have objects formal direct sums $\bigoplus_j Z_j$ for objects Z_j of $\mathcal{C}$. The **Cauchy completion** of $\mathcal{C}$ is defined by

$$\mathrm{Ab}(\mathcal{C}) \coloneqq \mathrm{Add}(\mathrm{Idemp}(\mathcal{C})).$$

If we again assume X tensor generates $\mathcal{C}$, it follows that $\mathcal{C} \cong \mathrm{Ab}(\mathcal{P}_{X;\mathcal{C}})$ [16, Theorem 3.4]. The universal property of $\mathrm{Ab}(\mathcal{P}_{X;\mathcal{C}})$ therefore implies that studying $\mathcal{P}_{X;\mathcal{C}}$ is sufficient to understand $\mathcal{C}$.

The category $\mathcal{G}_2(q)$ is a **presentation** for the category $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}$ in the sense that

$$\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))} \cong \overline{\mathrm{Ab}(\mathcal{G}_2(q))}.$$

2.3. **Unoriented Graph Planar Algebras.** We will study the quantum subgroups of type G_2 by embedding their skein theories into appropriate graph planar algebras (GPAs). This serves two purposes:

- Giving us solid ground on which to do computations, allowing us to uncover relations by finding them in the GPA hom-spaces, and
- Implying some nice general properties for the quantum subgroups (i.e., unitarity)

GPAs are an invention of Vaughan Jones [10]. In this work we have no use for less specialized GPAs, such as the *oriented* [2] or *multi-color* GPA [**emily**], so we consider only the unoriented case.

Definition 7. *Let $\Gamma = (V, E)$ be a finite graph. For an edge $e = (u, v) \in E$, let $\overline{e} := (v, u) \in E$. The* **graph planar algebra** *on Γ, denoted $\mathrm{GPA}(\Gamma)$, is the strictly pivotal rigid monoidal category whose objects are nonnegative integers, and whose hom-spaces have basis*

$$\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(m \to n) := \mathbb{C}\left\{(p, q) \mid {}^{p \ an \ m\text{-}path}_{q \ and \ n\text{-}path}, \; {}^{s(p) \, = \, s(q)}_{t(p) \, = \, t(q)}\right\},$$

with composition law

$$(p, q) \circ (p', q') := \delta_{q = p'}(p, q'),$$

and rigidity maps

$$ev = \sum_e \sqrt{\frac{\lambda_{t(e)}}{\lambda_{s(e)}}} \langle e\overline{e}, s(e)\rangle, \quad coev = \sum_e \sqrt{\frac{\lambda_{t(e)}}{\lambda_{s(e)}}} \langle s(e) e\overline{e}\rangle.$$

Monoidal product on objects is addition, and for morphisms is defined by

$$(p, q) \otimes (p', q') := \delta_{s(p') = t(p)}(pp', qq').$$

3. Cyclic extensions

The goal of this section is to develop the tools needed to prove evaluability of general $\mathbb{Z}_n$-like extensions of trivalent categories. We expect this class of extensions to be helpful in the search for novel categories. For example, there is work underway by the present author and Edie-Michell to use the techniques of this paper to construct the largest known class of examples of *near-group* categories, as defined in [8]. This work on near-group categories extends an underlying $SO(3)_q$ trivalent skein theory. The present author has also begun work on a family of extensions of $SP(4)_q$, which, despite its skein theory being generated by a braid, is of the same essence.

This all begs the question of which leaves on the "tree of life" of [14] might bear more fruit of this variety. Already we have extended both categories ($SO(3)_q$ and *Fib*) covered by [14, Theorem A] by group-like objects. This paper deals with all but one of the categories covered by [14, Theorem B]. The categories one might next attempt such an extension of include:

- The remaining category ABA of [14, Theorem B]
- The category H_3 of [14, Theorem C]

General methods for demonstrating evaluability of a skein theory involve identifying some measure of complexity for a closed diagram, then showing the known relations allow one to strictly decrease this measure. For our underlying trivalent categories, *Euler-evaluability* allows us to decrement one measure of complexity: number of internal faces. With the new strand type, we have another measure: number of colored strands. The underlying trivalent categories we deal with have evaluation algorithms based on the standard Euler characteristic argument. One way to capture this evaluability is by considering dimensions of box spaces.

Definition 8. *In a trivalent category we define a* **box space** *$B(k, f)$ to be the span of diagrams $k \to 0$ with f internal faces. If $\mathcal{C}$ is a trivalent category such that, for*

$k = 1, \ldots, 5$, *the containment*

$$B(k, 1) \subseteq B(k, 0)$$

holds, we will refer to $\mathcal{C}$ as **Euler-evaluable**.

Diagrams inside a $\mathbb{Z}_n$-like extension exhibit the following nice properties, which will be key in proving their evaluability. Essentially, we use the following lemmas to exchange decorated faces for singly-externally-decorated faces. The defining relations for a $\mathbb{Z}_n$-like extension then pop the singly-decorated faces.

Lemma 4. *(1) ($\mathbb{Z}_n$) follows from (Recouple) and (Reverse).*
(2) (Split) follows from (Recouple) and ($\mathbb{Z}_n$).

(Split)

($\mathbb{Z}_n$)
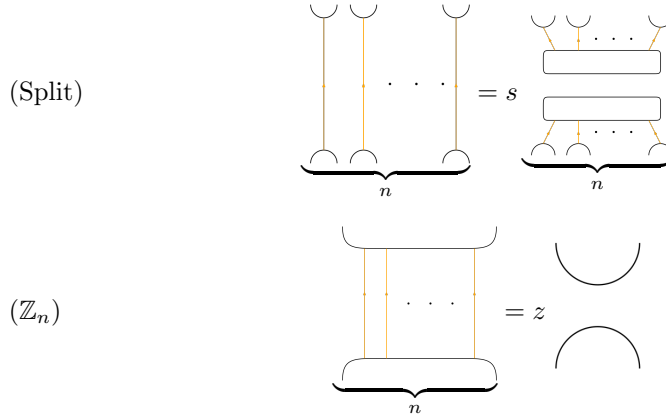
Remark 3. *The previous lemma implies that, upon reversing the orientations of the lefthand sides of the relations in Definition 2 will give similar relations. This fact will be used in the proof of Lemma 6.*

Lemma 5. *In a $\mathbb{Z}_n$-like extension, there exist n scalars s_i such that the following relation holds:*

(Slide)

Proof. Apply (decStick), (Recouple), and (Change of Basis). $\qquad\qquad\square$

**Lemma 6.** *A decorated diagram in a $\mathbb{Z}_n$-like extension may be expressed as a combination of singly-externally decorated diagrams*

Proof. We prove the lemma for a decorated trigon, and leave the remaining cases to the reader. We begin with a maximally-decorated trigon. All less decorated cases are absorbed along the way in this analysis. Now, a maximally-decorated trigon is
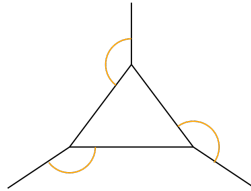
of the form:
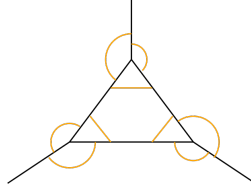
with any labeling on the colored strands. We apply the relations (Swap) and (Slide) on the internal colored strands to obtain a combination of diagrams of the form
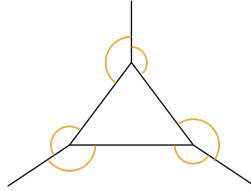
Now apply (Change of Basis) to reduce to a combination of diagrams of the form

By another application of (Slide) and (Change of Basis) we arrive at a diagram of the form
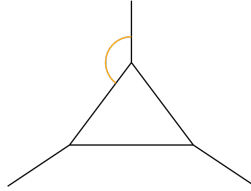
During this last step, we pick up colored strands between the black "spokes"; one may happily move these out of the diagram. □

One more lemma will complete our ability to evaluate closed diagrams in $\mathbb{Z}_n$-like extensions.

Lemma 7. *Suppose a planar diagram $\mathcal{E}$ in a $\mathbb{Z}_n$-like extension consists only of black loops and colored oriented edges between them, such that the relations ($\mathbb{Z}_n$) and (Recouple) hold. Suppose furthermore that each loop of $\mathcal{E}$ has either exactly n strands or exactly n strands leaving. Then the diagram $\mathcal{E}$ evaluates to a scalar.*

Proof. Firt note that any oriented edges starting and ending from the same black loop may be removed using (Swap) and (decStick). So assume there are only oriented edges between distinct black loops. We'll use graph theoretic language, with black loops playing the role of nodes, and oriented edges playing the role of, well, orinted edges.

If a node has exactly one neighbor, use (Order n) to remove both. So assume every node has at least two neighbors. Pick one node and call it u. Choose an orintation for its neighbors. Call the rightmost neighbor by w; assume $\deg(u \to w) = k < n$. From right to left, call the remaining neighbors by $v_1, \ldots, v_{n-k}$,

noting that these need not be distinct. From left to right, call the neighbors of w by $w_1, \ldots, w_{n-k}$, again noting that these need not be distinct.

The diagram is planar, so without loss, we may isotope it to look, locally, like
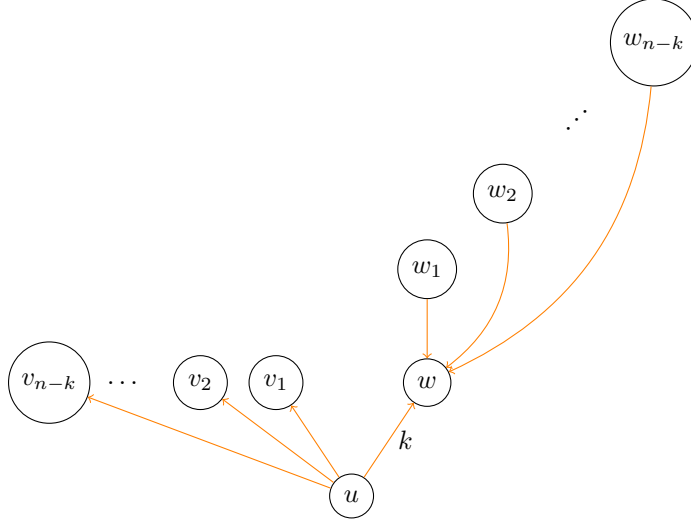
Now apply (Recouple), exchanging pairs of edges $u \to v_i$ and $w_i \to w$ for pairs of edges $u \to w$ and $w_i \to v_i$. This changes $\deg(u \to w)$ to n, allowing us, using (Order), to exchange a pair of nodes for a scalar. Continue ad nauseum.   □

Proposition 1. *A $\mathbb{Z}_n$-like extension of an Euler-evaluable trivalent category is evaluable.*

Proof. Suppose we begin with a diagram given by a closed, decorated planar trivalent graph. Begin by applying relations from the underlying trivalent category's evaluation algorithm to any undecorated faces; this decreases the number of trivalent vertices. By the standard Euler characteristic calculation, there must remain some black n-gon with $n \in \{2, \ldots, 5\}$. Choose one such face and apply Lemma 6 to reduce it to a singly-externally-decorated n-gon. Now one of the relations (decBigon), (decTrigon), (decTetragon), or (decPentagon) allows us to pop the face. This process decreases the number of faces (ignoring colored strands) in diagrams by at least 1 at every step, but also may increase the number of connected components in any summand. Continue this process until only decorated loops, or decorated loops connected by colored strands remain. If only decorated loops remain, apply (decStick).

Our diagram now consists of a number of black loops, connected by colored strands. Use (Recouple) and (Order n) to make it so every black loop has either only in-strands or only out-strands attached to it. If any black loop has more or less than n strands entering or exiting (Schur 0) implies the whole diagram is zero. So suppose each black loop has exactly n strands entering or exiting. Apply Lemma 7 to evaluate the remaining graph for a scalar.   □

As with all evaluability arguments, if we have nontriviality, we immediately deduce simplicity of the unit.

Corollary 1. *For a $\mathbb{Z}_n$-like extension $\mathcal{D}$ of an Euler-evaluable trivalent category, we have*

$$\dim \mathrm{Hom}_{\mathcal{D}}(0 \to 0) \leq 1.$$

For each quantum subgroup we construct, we will find planar algebras satisfying the conditions of Proposition 1, and thus will know the planar algebras are evaluable.

4. GPA Embeddings

This section is devoted to discussing the details of our GPA embeddings. This will include a discussion of the techniques used to solve the defining equations, along with a discussion of the coordinates these solutions define. Subsection 4.2 discusses some of the representation theory which led us to search for $\mathbb{Z}_n$-like extensions in the first place; in this respect, we discuss only the details of level 4, but the story at level 3 is essentially the same.

We find our fusion graphs by orbifolding the graphs in Figures 18b and 21b of [3]. These graphs are shown in Figures 1 and 2.

One may give a monoidal functor $F : \mathcal{G}_2(q_k) \to \mathrm{GPA}(\Gamma)$ by specifying the image of the morphism

$$F\left(\begin{array}{c} \bigwedge \end{array} \right) \in \mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 1).$$

This amounts to giving a list of $M := tr(\Gamma^2 \cdot \Gamma)$ complex numbers[2], say $a_1, \ldots, a_M$. Pushing the defining relations of $\mathcal{G}_2(q_k)$ through F, we see that these complex numbers satisfy equations in the a_i and $\overline{a_i}$. If we assume for now that each a_i is real, then this reduces the system to a collection of polynomials in the a_i [3]. Once we have the image of the trivalent vertex in hand, we have found an embedding of the planar algebra it generates. We can then follow a similar approach to solve for the image

$$F\left(\begin{array}{c} \smallsmile \\ \smallfrown \end{array} \right) \in \mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 2)$$

to extend the GPA embedding of $\mathcal{G}_2(q_k)$ to an embedding of $\mathcal{D}_k$. Let us discuss our examples.

4.1. **Level 4.** We will begin with level 4. Let Γ_4 be the graph given in Figure 1. Set $q_4 := e^{2\pi i/48}$. The following result says that we hvae an embedding of $\mathcal{G}_2(q_4)$ into the GPA on Γ_4.

Theorem 2. *There is a faithful monoidal functor $F_4 : \overline{\mathcal{G}_2(q_3)} \hookrightarrow \mathrm{GPA}(\Gamma_4)$.*

Proof. We first construct a monoidal functor $\mathcal{G}_2(q_3) \hookrightarrow \mathrm{GPA}(\Gamma_4)$, and then semisimplify. See the attached Mathematica notebooks for verification of the necessary equations. □

A proof of the above result amounts to a verification of a system of linear, quadratic, cubic, quartic, and quintiuc equations. Let

$$(p_1, q_1), \ldots, (p_M, q_M)^4$$

---

[2]We freely switch between using Γ to mean the graph itself and the graph's adjacency matrix.

[3]This assumption is useful only if it turns out to help us solve the system. In fact, any assumptions we make about this system, if they yield solutions, are in some way valid.

[4]See the attached Mathematica files for the specific ordering chosen.

be the defining basis for $\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 1)$ ($M = 88$ at level 4). Then it must be that

$$F_4\left(\ \bigwedge\ \right) = a_1(p_1, q_1) + \cdots + a_M(p_M, q_M)$$

for some $a_1, \ldots, a_M \in \mathbb{C}$. The Bigon relation, when sent through F, becomes the system

$$\sum_{i=1}^{M} a_i(p_i, q_i) \circ \sum_{j=1}^{M} a_j(q_j, p_j) = k^2 \sum_{e \in E(\Gamma)} (e, e).$$

This system is quadratic in the a_i since it involves up to two trivalent vertices on either side. The Lollipop and Rotate relations therefore determine a system of linear equations; the others give cubic, quartic, and quintic equations. It is often useful to solve the linear subsystem first and substitute the solution into the quadratic equations. For example, when we substitute the linear solution into the (Bigon) and (Tetragon) equations, we are able to isolate the following resulting equations:

$$a_8^2 + a_{85}^2 = 4 - \sqrt{2} + 2\sqrt{3} - \sqrt{6}$$

$$a_{69}^2 + \left(1 + \sqrt{\frac{3}{2}}\right) a_8^2 = \frac{3 + \sqrt{3} + \sqrt{6}}{\sqrt{2}}$$

$$a_{69}^2\left(\left(2 + \sqrt{6}\right) a_8^2 + \left(2 + \sqrt{6}\right) a_{85}^2 - 2\sqrt{2 + \sqrt{3}}\right) = 5 + \sqrt{2} + \sqrt{3} + 2\sqrt{6}$$

$$2a_{69}^4 + \left(5 + 2\sqrt{6}\right) a_{85}^4 = \left(3 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right) a_{85}^2 + 3\sqrt{6} + \sqrt{3} + 2\sqrt{2} + 7$$

Up to three choices of sign, the solution to this system is

$$a_8 = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

$$a_{69} = \sqrt{\frac{1}{2}\left(-1 + \sqrt{2} + \sqrt{3}\right)}$$

$$a_{85} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

Similar equations containing a_{31}, a_{55}, and a_{63} appear as well. We may repeat this process and obtain the additional values

$$a_{31} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

$$a_{55} = \sqrt{1 - \sqrt{\frac{3}{2} + \frac{1}{\sqrt{2}}}}$$

$$a_{63} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

These six values begin a cascade of equation solving. They, along with the linear solution, reduce many of the original high-order equations to linear. We solve those,

then repeat the process until we're forced to confront nonlinearity. The nonlinearity we encounter forces us to extract square roots, and ending up with a few degree-16 algebraic numbers. For instance,

$$a_{10} = \frac{1}{2}\left(\sqrt{1 + \sqrt{6 - 3\sqrt{3}}} + \sqrt{\sqrt{2 + \sqrt{3}} - 1} \right).$$

Up to sign, the coordinates of $F_4\left(\wedge \right)$ take on the following values:

$$\alpha_1 = \sqrt{\frac{1}{2}\left(1 + 2\sqrt{2} + \sqrt{3} + \sqrt{6} \right)}$$

$$\alpha_2 = \sqrt{\frac{1}{2}\left(-1 + \sqrt{2} + \sqrt{3} \right)}$$

$$\alpha_3 = \sqrt{\frac{3}{2}\left(\sqrt{2 + \sqrt{3}} - 1 \right)}$$

$$\alpha_4 = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

$$\alpha_5 = \sqrt{\frac{1}{2}\left(\sqrt{2 + \sqrt{3}} - 1 \right)}$$

$$\alpha_6 = \sqrt{\frac{1}{2}\left(\sqrt{3} + \sqrt{2 + \sqrt{3}} \right)}$$

$$\alpha_7 = \sqrt{1 - \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}}$$

$$\alpha_8 = \frac{1}{2}\left(\sqrt{1 + \sqrt{6 - 3\sqrt{3}}} - \sqrt{\sqrt{2 + \sqrt{3}} - 1} \right)$$

$$\alpha_9 = \frac{1}{2}\left(\sqrt{1 + \sqrt{6 - 3\sqrt{3}}} + \sqrt{\sqrt{2 + \sqrt{3}} - 1} \right)$$

4.2. **Extension of level 4.** With our embedding of $\mathcal{G}_2(q_4)$, i.e. the coordinates of $F_4\left(\wedge \right)$, in hand, we now know where to find a subcategory of $\mathrm{GPA}(\Gamma_4)$ isomorphic to $\mathcal{G}_2(q_4)$. In practice, we are relying on the fact that the free functor gives an embedding

$$\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))} \hookrightarrow \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$$

which takes a simple $\otimes$-generator X_4 to a simple $\otimes$-generator Y_4.[5] Combine this with the facts that $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$ contains a subcategory equivalent to $\mathrm{Vec}\,\mathbb{Z}_2$, and that both of the grouplike simples are subobjects of the square of the $\otimes$-generator. One concludes that $\mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}}(Y^{\otimes 2} \to Y^{\otimes 2})$ contains a projection onto a $\mathbb{Z}_2$-like simple object which follows relations analogous to (Recouple) - (decBigon).

---

[5]Simplicity of Y_4 may be easily checked using the X_4 fusion graph of $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}$.

Now, to enlarge our copy of $\mathcal{G}_2(q_4)$ to a $\mathbb{Z}_2$-like extension, we must find and element of $\mathrm{Hom}_{\mathrm{GPA}(\Gamma_4)}(2 \to 2)$ which captures the behavior of the idempotent; the role will be played by ⨄.

To this end, we take an approach similar to the one of the previous subsection. We may identify $\mathcal{G}_2(q_4)$ with the trivalent subcategory of $\mathcal{D}_4$, and now view F_4 as a functor out of this subcategory. To finish the definition of F_4 to be a functor out of $\mathcal{D}_4$, we must find the image

$$F_4\left(\smile\atop\frown \right) \in \mathrm{Hom}_{\mathrm{GPA}(\Gamma_4)}(2 \to 2)$$

satisfying the defining relations of $\mathcal{D}_4$. We follow a process similar to solving for the image $F_4\left(\bigwedge \right)$. That is, we assume that for some $b_1, \ldots, b_N \in \mathbb{C}$ we have

$$F_4\left(\smile\atop\frown \right) = \sum_{i=1}^{N} b_i(p_i', q_i').$$

We may push the relations (Recouple) - (decBigon) through F_4, and obtain a set of equations in the b_i. Solving these new equations gives us the coordinates of the projection $F_4\left(\smile\atop\frown \right)$.

Theorem 3. *There exists an element $P_4 \in \mathrm{Hom}_{\mathrm{GPA}(\Gamma_4)}(2 \to 2)$ satisfying the $\mathbb{Z}_2$-like extension relations, with structure constants given by those in Definition 3.*

Proof. See the attahed Mathemaica notebooks for verification.   □

Table 2 holds numerical approximations of the nonzero projection coordinates There are blocks of nonzero coordinates of length 4 and 25. These sizes, and the location of the nonzero real coordinates follow naturally when one considers Remark 2. The only coordinates of the projection in the GPA which are nonzero are at those basis vectors

$$(i \to \_ \to j, i \to \_ \to j)$$

where $i \to j$ is a directed edge of the g-fusion graph. For $i = j = 1, 4$ there are two possible values for $\_$; pairing them gives 4 pairs. For $i, j \neq 1, 4$ there are five possible values for $\_$; pairing them gives 25 pairs. The columns of Table 4 give the values of the coordinates of the projection, with dictionary ordering on the pairs of $\_$ values. That is, the column labeled by $1 \to \_ \to 1$ shows the coordinates on the ordered basis

$$(1 \to 2 \to 1, 1 \to 2 \to 1)$$
$$(1 \to 2 \to 1, 1 \to 3 \to 1)$$
$$(1 \to 3 \to 1, 1 \to 2 \to 1)$$
$$(1 \to 3 \to 1, 1 \to 3 \to 1)$$

With this ordering in mind, and recalling that the GPA's dagger operation swaps paths, the conjugate pairs appear where one would expect them.

As an immediate consequence of Theorems 2 and 3 we have the following corollary.

$1 \to \_ \to 1$	$4 \to \_ \to 4$
2.22	2.22
-2.22	-2.22
-2.22	-2.22
2.22	2.22

TABLE 1. Some coefs

$2 \to \_ \to 6$	$3 \to \_ \to 5$	$5 \to \_ \to 3$	$6 \to \_ \to 2$
0.44949	1	1	0.44949
$0.474073 - 0.474073i$	$0.825482 + 0.564429i$	$0.825482 - 0.564429i$	$0.474073 + 0.474073i$
$-0.123758 - 0.658919i$	$0.123758 + 0.658919i$	$0.123758 - 0.658919i$	$-0.123758 + 0.658919i$
$-0.123758 + 0.658919i$	$-0.564429 + 0.825482i$	$-0.564429 - 0.825482i$	$-0.123758 - 0.658919i$
$0.474073 + 0.474073i$	$0.931852 - 0.362839i$	$0.931852 + 0.362839i$	$0.474073 - 0.474073i$
$0.474073 + 0.474073i$	$0.825482 - 0.564429i$	$0.825482 + 0.564429i$	$0.474073 - 0.474073i$
1	1	1	1
$0.564429 - 0.825482i$	$0.474073 + 0.474073i$	$0.474073 - 0.474073i$	$0.564429 + 0.825482i$
$-0.825482 + 0.564429i$	i	$-i$	$-0.825482 - 0.564429i$
i	$0.564429 - 0.825482i$	$0.564429 + 0.825482i$	-i
$-0.123758 + 0.658919i$	$0.123758 - 0.658919i$	$0.123758 + 0.658919i$	$-0.123758 - 0.658919i$
$0.564429 + 0.825482i$	$0.474073 - 0.474073i$	$0.474073 + 0.474073i$	$0.564429 - 0.825482i$
1	0.44949	0.44949	1
$-0.931852 - 0.362839i$	$0.474073 + 0.474073i$	$0.474073 - 0.474073i$	$-0.931852 + 0.362839i$
$-0.825482 + 0.564429i$	$-0.123758 - 0.658919i$	$-0.123758 + 0.658919i$	$-0.825482 - 0.564429i$
$-0.123758 - 0.658919i$	$-0.564429 - 0.825482i$	$-0.564429 + 0.825482i$	$-0.123758 + 0.658919i$
$-0.825482 - 0.564429i$	$-i$	i	$-0.825482 + 0.564429i$
$-0.931852 + 0.362839i$	$0.474073 - 0.474073i$	$0.474073 + 0.474073i$	$-0.931852 - 0.362839i$
1	1	1	1
$0.564429 - 0.825482i$	$-0.825482 - 0.564429i$	$-0.825482 + 0.564429i$	$0.564429 + 0.825482i$
$0.474073 - 0.474073i$	$0.931852 + 0.362839i$	$0.931852 - 0.362839i$	$0.474073 + 0.474073i$
$-i$	$0.564429 + 0.825482i$	$0.564429 - 0.825482i$	i
$-0.825482 - 0.564429i$	$-0.123758 + 0.658919i$	$-0.123758 - 0.658919i$	$-0.825482 + 0.564429i$
$0.564429 + 0.825482i$	$-0.825482 + 0.564429i$	$-0.825482 - 0.564429i$	$0.564429 - 0.825482i$
1	1	1	1

TABLE 2. Level 4 projection embedding coefficients.

Corollary 2. *The category $\mathcal{D}_4$ is a nontrivial $\mathbb{Z}_2$-like extension of $\mathcal{G}_2(q_4)$, and the semisimple quotient $\overline{\mathcal{D}_4}$ is unitary.*

Proof. We deduce $\mathcal{D}_4$ is nonzero by its embedding into a nonzero subcategory of $\mathrm{GPA}(\Gamma_4)$. Unitarity of its semisimple quotient follows from noting the unitarity of $\mathrm{GPA}(\Gamma_4)$, and applying Lemma 2. $\qquad\square$

Corollary 3. *The embedding $\mathcal{G}_2(q_4) \hookrightarrow \mathcal{D}_4$ descends to a $\dagger$-embedding $\overline{\mathcal{G}_2(q_4)} \hookrightarrow \overline{\mathcal{D}_4}$.*

Proof. Use Lemma 2 again, and compose the induced functor with the functor

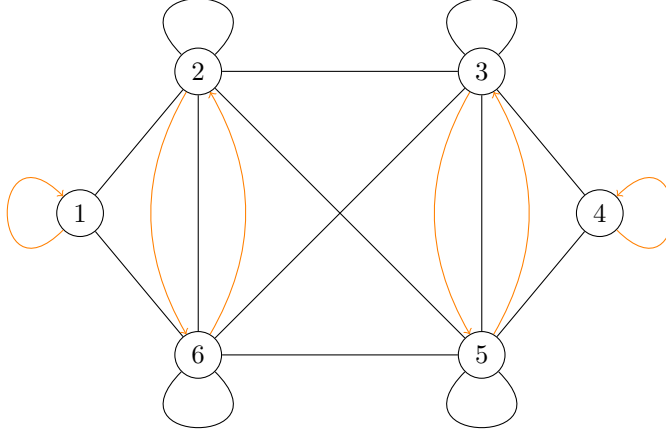$$\mathcal{D}_4 \twoheadrightarrow \overline{\mathcal{D}_4}$$

FIGURE 1. Fusion graphs at level 4 for Y (black) and g (orange).
See [3, Figure 21b].

onto the semisimple quotient.   □

4.3. **Level 3.** We now tell a similar story, but at level 3. However, for the trivalent
coefficients we use numerical approximations here, and relegate the actual numbers
to the attached Mathematica files. We were unable to find presentable representa-
tions of the GPA-embedding coordinates or structure constants. The coordinates
for the trivalent GPA embedding were algebraic numbers of degree 12 or 24. Even
worse, the structure constants for the relations (Change of Basis), (decTrigon),
(decTetragon), and (decPentagon) are all of the form

$$g_1 + g_2\alpha^2 + g_3 q_3 + g_4 \alpha^2 q_3$$

where $g_i \in \mathbb{Q}(q_3 + q_3^{-1})$, the maximal real subfield of $\mathbb{Q}(q_3)$, and α is an algebraic
number of degree 24 which appears as a coordinate of $F_3\left(\vcenter{\hbox{$\curlyvee$}} \right)$. For the relations
(Change of Basis) and (decTrigon), the power basis coordinates of the g_i are lowest
form rational numbers whose numerators have one or two digits. For (decTetrgon),
the numerators and denominators of the power basis coordinates of the g_i have
around 10 digits on average. In the (decPentagon) relation, this digit count explodes
to around 135.

Let Γ_3 be the graph given in Figure 2. Set $q_3 := e^{2\pi i/42}$. The following result
gives us a GPA embedding of $\mathcal{G}_2(q_3)$.

Theorem 4. *There is a faithful monoidal functor $F_3 : \overline{\mathcal{G}_2(q_3)} \to \mathrm{GPA}(\Gamma_3)$.*

Proof. See the attached Mathematica notebooks for verification of the necessary
equations.   □

Despite the more difficult numbers, we are able to recover some of the strutcure
of the fusion graph in the GPA coordinates. Recall that the defining bases for the
spaces

$$\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(m \to n)$$

Vertex Path	Conditions	Coefficient
(i, i, i)	$i \neq 1$	1.08393
(i, j, k)	$\{i, j, k\} = \{2, 3, 4\}$	0.619371
$(i, 1, k)$	$i, k \neq 1, \, i \neq k$	1.69414
$(i, 1, i)$	$i \neq 1$	0.861006
$(i, i, 1)$ or $(1, i, i)$	$i \neq 1$	0.967919

TABLE 3. Level 3 trivalent embedding coefficients.

are given in terms of pairs of paths. The (undirected) graphs we are using have at most a single edge between any two vertices. Hence an edge is equivalent to a pair of vertices, and a path is equivalent to an ordered tuple of vertices. For example, the path

$$p = v_1 \longrightarrow v_2 \longrightarrow v_3$$

is equivalent to the ordered triple (v_1, v_2, v_3). Which paths q pair validly with p to form a basis element of the $2 \to 1$ hom-space of a GPA? Well, by definition, q must be parallel to p; i.e. the sources and targets of p and q must coincide. It follows that the only valid pairing for such p is

$$q = v_1 \longrightarrow v_3,$$

which may also be represented as (v_1, v_3). So the only $2 \to 1$ basis element which p appears in is

$$\big((v_1, v_2, v_3), (v_1, v_3)\big).$$

But the parallel condition defining basis elements makes including (v_1, v_3) redundant; we might just as well have called the basis element by

$$(v_1, v_2, v_3).$$

This is how we refer to $2 \to 1$ GPA basis elements. Indeed, in Table 3, the first two columns combine to specify which basis elements are being specified, and the third column gives the approximate coordinate of the trivalent embedding on that basis element. For example, the first row of Table 3 tells us that the coordinate of the $(2, 2, 2)$ basis element is approximately 1.08393; the second row tells us that the coordinate of the $(4, 2, 3)$ basis element is approximately 0.619371.

Paths of the form (i, j, i), (i, i, j), or (i, j, j) for $i, j \neq 1$ require a bit more care to describe. There is nontrivial interplay with the graph symmetry swapping vertices 2 and 4. When these two vertices are swapped, a path whose coordinate has absolute value 0.155691 is sent to one whose coordinate has absolute value 1.69414. The nine paths whose coordinates have absolute value 0.155691 are:

$$(2, 3, 3), \ (3, 3, 2), \ (3, 2, 3), \ (2, 4, 2), \ (4, 3, 4), \ (2, 2, 4), \ (3, 4, 4), \ (4, 2, 2), \ (4, 4, 3)$$

One may use symmetry to find the rest of the coordinates.
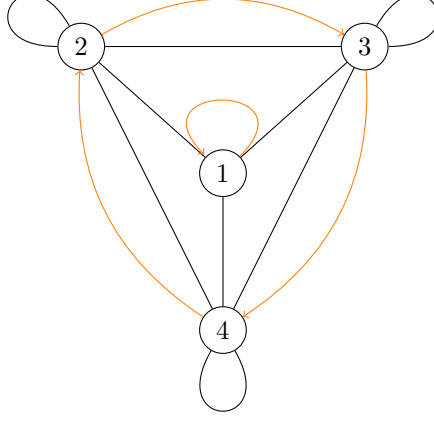
4.4. Extension of level 3.

FIGURE 2. Fusion graphs at level 3 for Y (black) and g (orange). See [3, Figure 18b].

Theorem 5. *There exists an element $P_3 \in \mathrm{Hom}_{\mathrm{GPA}(\Gamma_3)}(2 \to 2)$ satisfying the $\mathbb{Z}_2$-like extension relations, with structure constants given in the attached Mathematica notebook.*

Proof. This result is again proved by direct verification of the required equations. See the attached Mathematica notebook.   □

Similarly to the level 4 case, Theorems 4 and 5 imply nontriviality and unitarity of $\overline{\mathcal{D}_3}$.

Corollary 4. *The category $\mathcal{D}_3$ is a nontrivial $\mathbb{Z}_3$-like extension of $\mathcal{G}_2(q_3)$, and the semisimple quotient $\overline{\mathcal{D}_3}$ is unitary.*

Corollary 5. *The embedding $\mathcal{G}_2(q_3) \hookrightarrow \mathcal{D}_3$ descends to a $\dagger$-embedding $\overline{\mathcal{G}_2(q_3)} \hookrightarrow \overline{\mathcal{D}_3}$.*

4.5. **New Relations.** When finding GPA embeddings of $\mathcal{D}_3$ and $\mathcal{D}_4$, we did not use the relations (Swap), (Change of Basis), (decTrigon), (decTetragon), or (decPentagon). In fact, the structure constants in these relations were unknown before exploring our GPA embedding. Instead, we used only the relations (Schur 0), (Schur 1), (decStick), and (Half-braid). This large number of linear equations was enough to pin down the images $F_k \left(\begin{smallmatrix} \smile \\ \curlywedge \\ \frown \end{smallmatrix} \right)$.

Of primary importance were the (Half-braid) equations. These follow from Lemma 3. We translate this lemma into the hypothesis that our projection

$1 \to \_ \to 1$	$2 \to \_ \to 4$	$3 \to \_ \to 2$	$4 \to \_ \to 3$
1.26376	0.791288	0.791288	0.791288
$-0.631881 - 1.09445i$	$0.567622 - 0.684904i$	$0.876955 + 0.149123i$	$0.674406 + 0.580055i$
$-0.631881 + 1.09445i$	$0.674406 + 0.580055i$	$0.567622 - 0.684904i$	$0.876955 + 0.149123i$
$-0.631881 + 1.09445i$	$-0.876955 - 0.149123i$	$-0.674406 - 0.580055i$	$0.567622 - 0.684904i$
1.26376	$0.567622 + 0.684904i$	$0.876955 - 0.149123i$	$0.674406 - 0.580055i$
$-0.631881 - 1.09445i$	1	1	1
$-0.631881 - 1.09445i$	$-0.0182917 + 0.999833i$	$0.5 - 0.866025i$	$0.856735 - 0.515757i$
$-0.631881 + 1.09445i$	$-0.5 - 0.866025i$	$-0.856735 - 0.515757i$	$-0.0182917 - 0.999833i$
1.26376	$0.674406 - 0.580055i$	$0.567622 + 0.684904i$	$0.876955 - 0.149123i$
	$-0.0182917 - 0.999833i$	$0.5 + 0.866025i$	$0.856735 + 0.515757i$
	1	1	1
	$-0.856735 + 0.515757i$	$0.0182917 - 0.999833i$	$0.5 - 0.866025i$
	$-0.876955 + 0.149123i$	$-0.674406 + 0.580055i$	$0.567622 + 0.684904i$
	$-0.5 + 0.866025i$	$-0.856735 + 0.515757i$	$-0.0182917 + 0.999833i$
	$-0.856735 - 0.515757i$	$0.0182917 + 0.999833i$	$0.5 + 0.866025i$
	1	1	1

TABLE 4. Level 3 projection embedding coefficients.

satisfies the relation
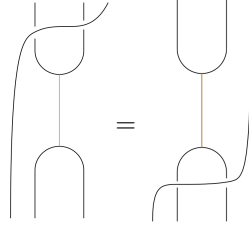
where the unoriented colored strand indicates that the equation holds whenever we pick the same orientation for the left and right sides.

Now, in order to discover the new relations (Swap), (Change of Basis), (decTrigon), (decTetragon), and (decPentagon), we hypothesize the form such a relation should take, impose that form, and solve for the structure constants. For example, we would suppose based on representation theory, that an equation of the form

$$F_k \left(\quad \right) = \omega F_k \left(\quad \right)$$

holds, for some ω.

With our explicit GPA embeddings of the projection in hand, discovering what ω is becomes a matter of solving a system of linear equations for one variable. Discovering the (decTrigon) structure constants is similarly reduced to solving a system of likear equations of the form

$$F_k \left(\quad \right) = t_1 F_k \left(\quad \right) + t_2 F_k \left(\quad \right)$$

for t_1 and t_2. In practice, the hom-spaces in the GPAs we used were sufficiently large making solving all of these equations fairly straightforward.

5. EQUIVALENCES

In this section we prove that the categories $\mathcal{D}_3$ and $\mathcal{D}_4$ are indeed presentations for quantum subgroups of $\mathcal{G}_2(q_3)$ and $\mathcal{G}_2(q_4)$, respectively. More details of the argument used may be found in [6]. It is necessary to note that from [TODO] we know that the only nontrivial Etale algebras in $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}$ and $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}$ are A_3 and A_4, respectively.

We first state and prove the theorem at level 4.

Theorem 6. *There is a monoidal equivalence*
$$\mathrm{Ab}(\overline{\mathcal{D}_4}) \cong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}.$$

Proposition 2. *There is an Etale algebra B_4 such that*
$$\mathrm{Ab}(\overline{\mathcal{D}_4}) \cong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{B_4}.$$

Proof. Recall the †-embedding $\overline{\mathcal{G}_2(q_4)} \hookrightarrow \overline{\mathcal{D}_4}$ of Corollary 3. From here we take Karoubi completion, which induces a functor
$$\mathrm{Ab}(\overline{\mathcal{G}_2(q_4)}) \hookrightarrow \mathrm{Ab}(\overline{\mathcal{D}_4})$$
which is faithful exact. By Lemma 1, we deduce that
$$\mathrm{Ab}(\overline{\mathcal{D}_4}) \cong \mathrm{Ab}(\overline{\mathcal{G}_2(q_4)})_{B_4}$$
for some Etale algebra object B_4. But it is well known that
$$\mathrm{Ab}(\overline{\mathcal{G}_2(q_4)}) \cong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}.$$
$\square$

Now we are in a position to prove Theorem 6.

Proof of Theorem 6. As we noted before, the only candidates for B_4 are A_4 and $\mathbb{1}$. Thus it will suffice to demonstrate that
$$\mathrm{Ab}(\overline{\mathcal{D}_4}) \ncong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))} \cong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{\mathbb{1}}.$$

The induced functor $\mathrm{Ab}(\overline{\mathcal{G}_2(q_4)}) \hookrightarrow \mathrm{Ab}(\overline{\mathcal{D}_4})$, whose existence we noted above, is defined to be such that the following diagram commutes:

$$
\begin{array}{ccc}
\overline{\mathcal{G}_2(q_4)} & \lhook\joinrel\longrightarrow & \overline{\mathcal{D}_4} \\
\Big\uparrow & & \Big\uparrow{\scriptstyle \mathcal{J}} \\
\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))} & \lhook\joinrel\longrightarrow & \mathrm{Ab}(\overline{\mathcal{D}_4})
\end{array}
$$

By Lemma 1 there is an equivalence $\mathcal{K}$ such that, up to isomorphism, the diagram

$$
\begin{array}{ccc}
\overline{\mathcal{G}_2(q_4)} & \lhook\joinrel\longrightarrow & \overline{\mathcal{D}_4} \\
\Big\downarrow & & \Big\downarrow {\scriptstyle \mathcal{J}} \\
\overline{\operatorname{Rep}(U_{q_4}(\mathfrak{g}_2))} & \lhook\joinrel\longrightarrow & \operatorname{Ab}(\overline{\mathcal{D}_4}) \\
& {\scriptstyle \mathcal{F}_{B_4}} \searrow & \Big\downarrow {\scriptstyle \mathcal{K}} \\
& & \overline{\operatorname{Rep}(U_{q_4}(\mathfrak{g}_2))}_{B_4}
\end{array}
$$

commutes.

Now, chasing the object $1_{\overline{\mathcal{G}_2(q_4)}}$ through the leftmost path on this diagram, we have

$$1_{\overline{\mathcal{G}_2(q_4)}} \mapsto V_{\Lambda_1} \mapsto B_4 \otimes V_{\Lambda_1}.$$

If it was true that $B_4 \cong \mathbb{1}$, we would have

$$\dim \operatorname{Hom}_{\overline{\operatorname{Rep}(U_{q_4}(\mathfrak{g}_2))}_{B_4}} ((B_4 \otimes V_{\Lambda_1})^{\otimes 2} \to B_4 \otimes V_{\Lambda_1}) = 1.$$

Now let us chase 1 along the top leftmost path to demonstrate that this is not the case. Along this second path we have

$$1_{\overline{\mathcal{G}_2(q_4)}} \mapsto 1_{\overline{\mathcal{D}_4}} \mapsto \mathcal{J}(1_{\overline{\mathcal{D}_4}}) \mapsto \mathcal{K} \circ \mathcal{J}(1_{\overline{\mathcal{D}_4}}).$$

Set $\mathcal{C}_4 := \overline{\operatorname{Rep}(U_{q_4}(\mathfrak{g}_2))}$. Now we note that

$$
\begin{aligned}
\dim \operatorname{Hom}_{(\mathcal{C}_4)_{B_4}}((B_4 \otimes V_{\Lambda_1})^{\otimes 2} \to B_4 \otimes V_{\Lambda_1}) &= \dim \operatorname{Hom}_{(\mathcal{C}_4)_{B_4}}((\mathcal{K} \circ \mathcal{J}(1_{\overline{\mathcal{D}_4}}))^{\otimes 2} \to \mathcal{K} \circ \mathcal{J}(1_{\overline{\mathcal{D}_4}})) \\
&= \dim \operatorname{Hom}_{\operatorname{Ab}(\overline{\mathcal{D}_4})}((\mathcal{J}(1_{\overline{\mathcal{D}_4}}))^{\otimes 2} \to \mathcal{J}(1_{\overline{\mathcal{D}_4}})) \\
&= \dim \operatorname{Hom}_{\overline{\mathcal{D}_4}}((1_{\overline{\mathcal{D}_4}})^{\otimes 2} \to 1_{\overline{\mathcal{D}_4}}) \\
&= \dim \operatorname{Hom}_{\overline{\mathcal{D}_4}}(2_{\overline{\mathcal{D}_4}} \to 1_{\overline{\mathcal{D}_4}}) \\
&= 2.
\end{aligned}
$$

Since $B_4 \not\cong \mathbb{1}$, we must have $B_4 \cong A_4$; the theorem is proved. $\qquad\square$

The category $\mathcal{D}_3$, whose $\mathbb{Z}_3$-like structure constants are given in the attached Mathematica notebook, admits an analogous theorem at level 3.

Theorem 7. *There is a monoidal equivalence*

$$\operatorname{Ab}(\overline{\mathcal{D}_3}) \cong \overline{\operatorname{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}.$$

The proof of Theorem 7 is analogous to the argument given above.

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