QUANTUM SUBGROUPS OF G_2 VIA GRAPH PLANAR ALGEBRA EMBEDDINGS

CALEB KENNEDY HILL

Abstract. todo

1. Introduction

Quantum subgroups are a well-known source of tensor categories. More precisely, given a conformal embedding $\mathcal{V}(\mathfrak{g}, k) \subseteq \mathcal{V}(\mathfrak{h}, 1)$ of VOAs as in [4], one obtains a corresponding Etale algebra A. This algebra then allows one to consider the category $\overline{\operatorname{Rep}(U_q(\mathfrak{g}))}_A$ of right A-modules. A half-braiding on A then gives a tensor product on $\overline{\operatorname{Rep}(U_q(\mathfrak{g}))}_A$, and one may study this new category in its own right. The free functor gives an embedding $\overline{\operatorname{Rep}(U_q(\mathfrak{g}))} \hookrightarrow \overline{\operatorname{Rep}(U_q(\mathfrak{g}))}_A$. As this embedding is, in general, not full, it remains only to find a description of the new morphisms in $\overline{\operatorname{Rep}(U_q(\mathfrak{g}))}_A$. Recent works of Edie-Michell and Snyder [6] have used this reasoning, and representation theoretic techniques to give diagrammatic descriptions of new tensor categories of modules corresponding to the family of conformal embeddings $\mathcal{V}(\mathfrak{sl}_N, N^2) \subseteq \mathcal{V}(\mathfrak{sl}_{N^2-1}, 1)$.

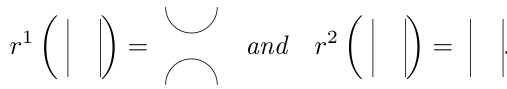
On the other hand, one may start with a known category and compute graph planar algebra (GPA) embeddings for it. This has been done for $\overline{\operatorname{Rep}(U_q(\mathfrak{sl}_N))}$ in [1] and for the extended Haagerup categories in [10]. This computation has the theoretical and practical consequences. By the GPA embedding theorem [**something**], such an embedding immediately gives a module category. It additionally gives a concrete representation of the category in which one may perform explicit computations.

The present work describes a blend of these two techniques. We begin by fidning a GPA embedding on the well-known trivalent category $\mathcal{G}_2(q)$ of [14, 15] which is a diagrammatic presentation for $\overline{\operatorname{Rep}(U_q(\mathfrak{g}_2))}$. Through the free fucntor we can view our GPA embedding as a GPA embedding for a $\otimes$-generating object's planar algebra in $\overline{\operatorname{Rep}(U_q(\mathfrak{g}_2))}_A$. This gives us a black-strand. We then search inside the GPA embedding for new morphisms. According to [4] there ought to be a projection onto a $\mathbb{Z}_k$-like simple object in $\overline{\operatorname{Rep}(U_q(\mathfrak{g}_2))}_A$, so this is what we search for inside the GPA. We view this new morphism as an orange strand. The properties of this new morphism are unknown beyond a scant basic set. Once we have or hands on the imagine in the GPA of this projection, though, we may explore its properties through explicit computations. We perform this process of extending GPA embeddings for the two conformal embeddings

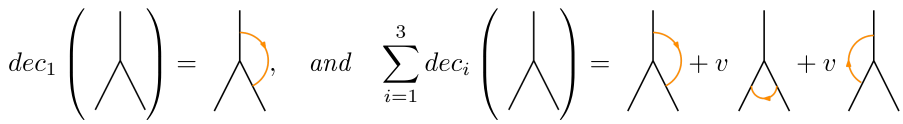
$$\mathcal{V}(\mathfrak{g}_2, 3) \subseteq \mathcal{V}(\mathfrak{e}_6, 1) \quad \text{and} \quad \mathcal{V}(\mathfrak{g}_2, 4) \subseteq \mathcal{V}(\mathfrak{d}_7, 1). \tag{1}$$

Now we begin by introducing some notation for a skein theory involving an oritned, colored strand in addition to unoriented black strands.

Definition 1. *For a diagram $\mathcal{E}$ the notation $r^i(\mathcal{E})$ means an i-click right rotation. For instance,*

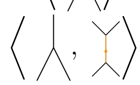


Suppose the diagram $\mathcal{E}$ has m boundary points. We define $dec_i(\mathcal{E})$ to be the i-th external single clockwise decoration of $\mathcal{E}$. For example,

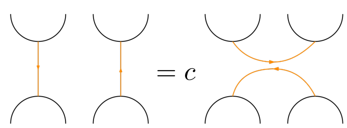


We adopt the convention that $dec_0(\mathcal{E}) = \mathcal{E}$.

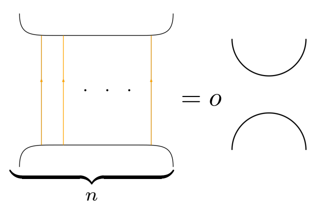
Both of the categories studied in this paper are extensions of trivalent categories by a colored, directed, $\mathbb{Z}_n$-like strand. We define the class of categories we will be working with. In Section 3 we will show that, with an assumption on the underlying skein theory, categories in this class are evaluable in general.

Definition 2. *Let $\mathcal{C} =$*  *be a trivalent category. Call $\mathcal{D}$ a* $\mathbb{Z}_n$**-like extension** *of $\mathcal{C}$ if we have $\mathcal{D} =$* *, enjoying the following relations*[1]*:*

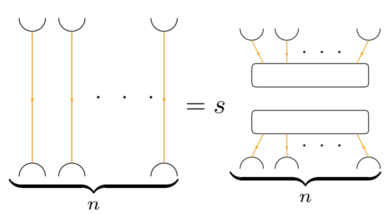
(Recouple)



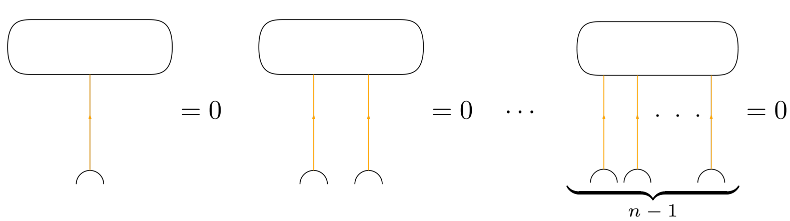
($\mathbb{Z}_n$)



(Split)

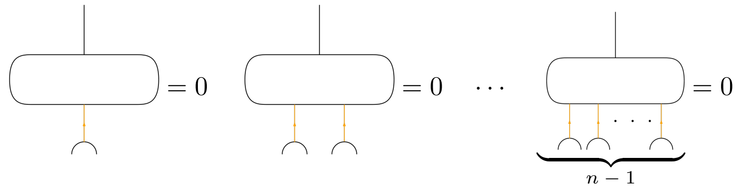


(Schur 0)

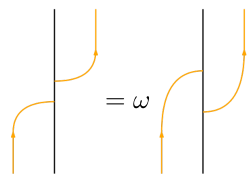


[1]Conditions from [3].

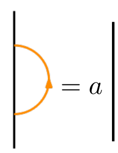
(Schur 1)



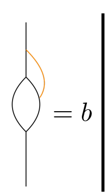
(Swap)



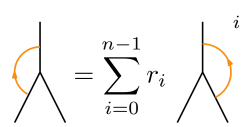
(decStick)



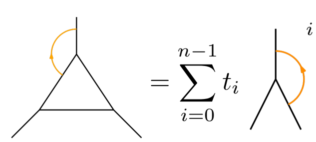
(decBigon)



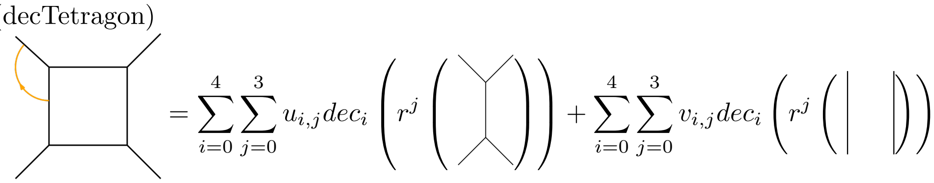
(Change of Basis)



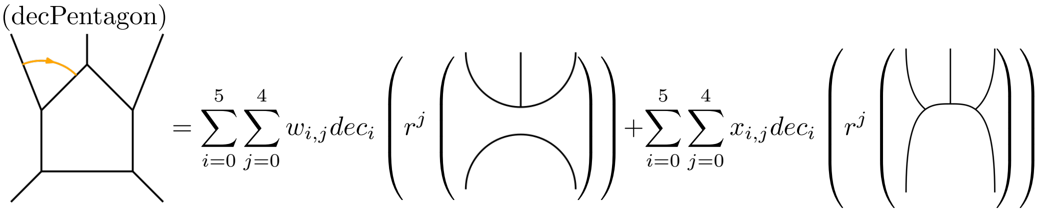
(decTrigon)



(decTetragon)



(decPentagon)



Remark 1. *A quick sketch shows that using (Order) followed by repeated applications of (Recouple) and (decStick) allows one to swap an up-oriented strand for $n-1$ down-oriented strand. This means that, upon reversing the orientations of the lefthand sides of the relations in Definition 2 will give similar relations. This fact will be used in the proof of Lemma 3.*

Remark 2. *It is worth noting the following standard abuse of language. A diagrammatically presented category such as a $\mathbb{Z}_n$-like extension has hom-spaces which are formal spans of diagrams. When applying a relation such as (decTrigon) locally, the result is clearly a linear combination of diagrams. Usually, though, this linear combination has some desirable quality, such as a smaller number of internal faces in each summand. In this instance, we prefer to say something along the lines of, "applying (decTrigon) decreases the number of internal faces," instead of, for instance, the more wordy, "applying (decTrigon) turns this diagram into a linear combination of diagrams with fewer internal faces."*

Definition 3. *Set $q_4 = e^{\frac{2\pi i}{48}}$ and deifne $\mathcal{D}_4$ to be the $\mathbb{Z}_2$-like extension of $\mathcal{G}_2(q_4)$ with structure constants*

$$\omega = -1$$

$$r_1 = e^{-\frac{\pi i}{6}}, \quad r_2 = e^{-\frac{2\pi i}{3}}$$

$$s_1 = e^{-\frac{\pi i}{6}} \quad s_2 = e^{-\frac{4\pi i}{3}}$$

$$t_1 = -1 \quad t_2 = -1$$

$$\begin{array}{cccccc} u = 1, & u = 2, & u = 3, & u = 4, & u = 5, & u = 6, \\ v = 7, & v = 8, & v = 9, & v = 10, & v = 11, & v = 12 \end{array}$$

One of the two primary results we give here is that $\mathcal{D}_4$ is a presentation for a category of modules corresponding to the level 4 conformal embedding of $\mathfrak{g}_2$.

Theorem 1. *There is an equivalence*

$$\mathrm{Ab}(\mathcal{D}_4) \cong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$$

where A_4 is the algebra object corresponding to the level-4 conformal embedding of 1.

Theorem 2 is an analogous theorem for level 3, with structure constants given in the attached Mathematica files.

It is not clear a priori that the defining relations for, say, $\mathcal{D}_4$ lead to a nontirivial tensor category. The general undecidability of the word problem for groups offers some evidence that this question is difficult for a typical presentation for a tensor category. That is, one should not expect a set of relations to yield any nontriviality. It follows that the presentations we give here are interesting and worth investigating more generally.

The remainder of the paper is structured as follows. Section 2 sets up most of the theory needed, referencing that which we do not exposit here. This includes unoriented planar algebras, unoriented graph planar algebras, internal algebra and module objects, and some assorted theoretical devices and results. Section 3 then goes on to investigate some properties of $\mathbb{Z}_n$-like extensions. We expect this class of categories to be of use for researchers intent on conjuring examples of exotic tensor categories. In fact, in a forthcoming paper, the present author and Cain Edie-Michell diagrammatically present a number of near-group categories as $\mathbb{Z}_n$-like extensions of $SO(3)_q$ trivalent categories. We demonstrate evaluability of this class of catgeories under a relatively tame assumption on the underlying trivalent skein theory. Section 4 discusses the process of arriving at GPA embeddings. Subsection 4.1 details the techniques used to arrive at GPA embeddigns of trivalent categories. Subsection 4.2 shows how we extend these embeddings of trivalent categories to embeddings of $\mathbb{Z}_n$-like extensions, and how we use GPA embeddings to explore relations in these extensions. This section uses examples from level 4 ($\mathcal{D}_4$) due to the fact that the numbers involved are more presentable. The process used for level 3 ($\mathcal{D}_3$) was essentially identical. Finally, Section 5 gives the structure constants for the newly constructed categories. This section also discusses the argument used to prove that we truly have found full presentations. The argument appears in its entirety in [6], and is adapted to the present settign without a problem.

2. Preliminaries

Here we define the players in this game. This includes planar algebras, graph planar algebras, and internal algebra and module objects. We give only a few necessary results, and refer the reader to the definitive publications. For the general theory of tensor categories, see [7].

2.1. Algebra and Module Objects.

We will ultimately show that $\mathcal{D}_3$ and $\mathcal{D}_4$ are presentations for the categories $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$ and $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$ of modules over algebra objects A_3 and A_4 coming from the conformal embeddings $\mathcal{C}(\mathfrak{g}_2, 3) \subseteq \mathcal{C}(\mathfrak{e}_6, 1)$ and $\mathcal{C}(\mathfrak{g}_2, 4) \subseteq \mathcal{C}(\mathfrak{d}_7, 1)$, respectively. In this subsection we recall basic facts about algebra and module objects, as well as conformal embeddings. See [7, 16] for more complete descriptions. The theory which will apply to our context is given in [6]. Some basic properties concerning the interaction of algebra and module objects with monoidal functors will be used in the proof of our main theorems; this material can be found in [12]. We restate a few definitions and facts here. Unless otherwise stated, we will be assuming the underlying tensor categories are braided.

Definition 4. *Let A be an algebra object of the braided tensor category $\mathcal{C}$. A is an* **Etale** *algebra if it is commutative and separable. We call A* **connected** *if it is Etale and* $\dim \mathrm{Hom}_{\mathcal{C}}(\mathbb{1} \to A) = 1$.

For an Etale algebra object A of $\mathcal{C}$, we denote by $\mathcal{C}_A$ the collection of left A-modules internal to $\mathcal{C}$. As decribed in [6], a braiding on $\mathcal{C}$ induces a tesnor product on $\mathcal{C}_A$. Separability of A implies semisimplicity of $\mathcal{C}_A$, and connectedness of A implies the unit $\mathbb{1}_{\mathcal{C}_A} = A$ is simple in $\mathcal{C}_A$ [4].

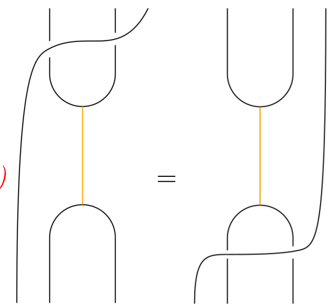
Furthermore, the free fucntor

$$\mathcal{F}_A : \mathcal{C} \xhookrightarrow{X \mapsto A \otimes X} \mathcal{C}_A$$

is a monoidal embedding which is, as we will see later, not always full. Its right adjoint is given by the forgetful functor $\mathcal{F}^\vee : \mathcal{C}_A \to \mathcal{C}$ which acts as the identity on both objects and morphisms.

One result which will help immensely in arriving at GPA embeddings is the following, which is Lemma 2.4 of [6].

Lemma 1 (Half-braid)**.** *Let $\mathcal{C}$ be a braided tensor category, and A an etale algebra object. For any $f \in \mathrm{Hom}_{\mathcal{C}_A}(\mathcal{F}(Y_1) \to \mathcal{F}(Y_2))$, the following relation holds:*

(stand in for general hb) 

We will utilize this result to obtain a rather large number (2970 at level 3 and 7776 at level 4) of linear equations constraining the GPA coordinates of the morphisms not living in the image of $\mathcal{F}_A$. Thus the half-braid relation will be key to our program, despite not being necessary to prove evaluability.

The source of our algebra objects will be conformal embeddings. We direct the reader to [4] a more complete treatment of conformal embeddings.

Definition 5. *A containment $\mathcal{V}(\mathfrak{g}, j) \subseteq \mathcal{V}(\mathfrak{h}, k)$ of affine Lie algebras is said to be* **conformal** *if the adjoint representation of $\mathcal{V}(\mathfrak{h}, j)$ restricts to a finite direct sum of simple objects in $\mathcal{C}(\mathfrak{g}, j) := \mathrm{Rep}(\mathcal{V}(\mathfrak{g}, j))$.*

Affine Lie algebras and conformal embeddings will only be used to obtain algebra objects and module fusion graphs, so we briefly recall the correspondence

$$\mathcal{C}(\mathfrak{g}_2, k) \cong \overline{\mathrm{Rep}(U_{q_k}(\mathfrak{g}_2))} \tag{2}$$

of [5], where k is the level and q_k is given by

$$q = e^{\frac{2\pi i}{3(4+k)}}.$$

At level 3 we have $q_3 = e^{\frac{2\pi i}{42}}$ and at level 4 we have $q_4 = e^{\frac{2\pi i}{48}}$. We obtain the algebra objects and fundamental graphs for GPAs from [2]:

$$A_3 = V_\emptyset \oplus V_{\Lambda_1} \quad \text{and} \quad A_4 = V_\emptyset \oplus V_{3\Lambda_1} \tag{3}$$

at levels 3 and 4, respectively.

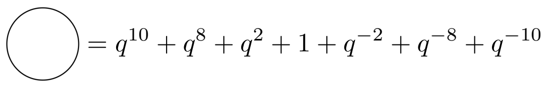
2.2. **Unoriented Planar Algebras.** Recall the theory of **rigid** monoidal categories detailed in [13]. To put it succintly, rigid monoidal categories have duals. Duals, and the associated evaluation and coevaluation maps, giving us the cups and caps ubiquitous in skein theory. A rigidity assumption gives us the ability to isotope diagrams. The generators we will use for our planar algebras will be symmetrically self-dual. We also assume pivotality throughout.

Let X be a (symmetrically self-dual) **tensor generator** for the tensor category $\mathcal{C}$; that is, every object of $\mathcal{C}$ is isomorphic to a subobject of some tensor power $X^{\otimes n}$. Let $\mathcal{P}_{X;\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ whose objects are tensor powers $\mathbb{1} = X^{\otimes 0}, X, X^{\otimes 2}, \ldots$; we call this the (unoriented) **planar algebra** generated by X in $\mathcal{C}$. The planar algebra $\mathcal{P}_{X;\mathcal{C}}$ is **evaluable** if $\dim \mathrm{End}_{\mathcal{P}_{X;\mathcal{C}}}(\mathbb{1}) = 1$.

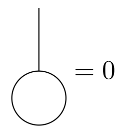
We will be presenting the our two quantum subgroups as extensions of $\mathcal{G}_2(q)$ skein theories, in the spirit of Kuperberg [14, 15]. Up to a rescaling by a factor of $\kappa = \sqrt{[7]-1}$ we use the same skein theory as [15] (note the sign error in the Pentagon relation of [14]).

Definition 6. *For q a root of unity, the $\mathcal{G}_2(q)$ skein theory is defined to be that generated by an unoriented trivalent vertex*  *satisfying the relations*

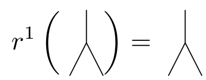
(Loop)



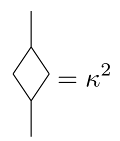
(Lollipop)



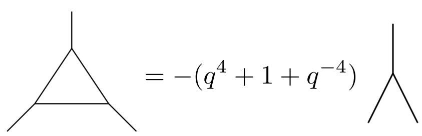
(Rotate)



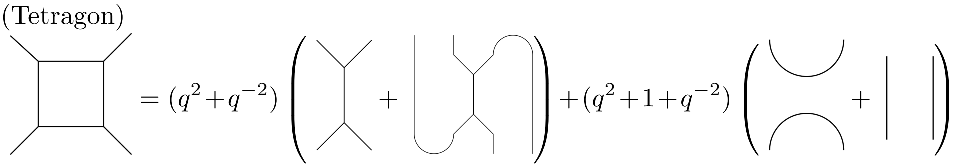
(Bigon)



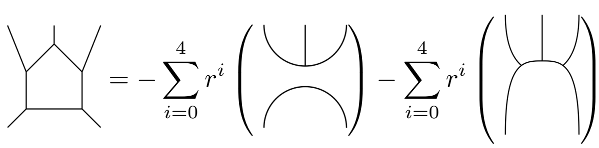
(Trigon)



(Tetragon)



(Pentagon)



Our use of planar algebras will depend entirely on the construction of the Cauchy completion, which we sketch here. See [6] for more details and [17] for a full treatment of the topic. Recall that the **idempotent competion** of a pivotal tensor category $\mathcal{C}$ consists of pairs (Z, p), where $p \in \mathrm{End}_{\mathcal{C}}(Z)$ is an idempotent. We denote the idempotent completion of $\mathcal{C}$ as $\mathrm{Idemp}(\mathcal{C})$. Furhter, we define the **additive envelope** of a pivotal, $\mathbb{C}$-linear tensor category $\mathcal{C}$ to have objects formal direct sums $\bigoplus_j Z_j$ for objects Z_j of $\mathcal{C}$. The **Cauchy completion** of $\mathcal{C}$ is defined by

$$\mathrm{Ab}(\mathcal{C}) := \mathrm{Add}(\mathrm{Idemp}(\mathcal{C})).$$

If we again assume X tensor generates $\mathcal{C}$, it follows that $\mathcal{C} \cong \mathrm{Ab}(\mathcal{P}_{X;\mathcal{C}})$ [17, Theorem3.4] The universal property of $\mathrm{Ab}(\mathcal{P}_{X;\mathcal{C}})$ therefore implies that studying $\mathcal{P}_{X;\mathcal{C}}$ is sufficient to understand $\mathcal{C}$.

The category $\mathcal{G}_2(q)$ is a **presentation** for the category $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}$ in the sense that

$$\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))} \cong \overline{\mathrm{Kar}(\mathcal{G}_2(q))}.$$

2.3. **Unoriented Graph Planar Algebras.** We will study the quantum subgroups of type G_2 by embedding their skein theories into appropriate graph planar algebras (GPAs). This serves two purposes:

- Giving us solid ground on which to do computations, allowing us to uncover relations by finding them in the GPA hom-spaces, and
- Implying some nice general properties for the quantum subgroups (i.e., unitarity)

GPAs are an invention of Vaughan Jones [11]. In this work we have no use for less specialized GPAs, such as the *oriented* [1] or *multi-color* GPA [**emily**], so we consider only the unoriented case.

Definition 7. *Let $\Gamma = (V, E)$ be a finite graph. For an edge $e = (u, v) \in E$, let $\overline{e} \coloneqq (v, u) \in E$. The* **graph planar algebra** *on Γ, denoted $\mathrm{GPA}(\Gamma)$, is the strictly pivotal rigid monoidal category whose objects are nonnegative integers, and whose hom-spaces have basis*

$$\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(m \to n) \coloneqq \mathbb{C}\left\{(p, q) \mid \begin{smallmatrix} p \text{ an } m\text{-path} \\ q \text{ and } n\text{-path} \end{smallmatrix}, \begin{smallmatrix} s(p) = s(q) \\ t(p) = t(q) \end{smallmatrix}\right\},$$

with composition law

$$(p, q) \circ (p', q') \coloneqq \delta_{q = p'}(p, q'),$$

and rigidity maps

$$ev = \sum_e \sqrt{\frac{\lambda_{t(e)}}{\lambda_{s(e)}}} \langle e\overline{e}, s(e) \rangle, \quad coev = \sum_e \sqrt{\frac{\lambda_{t(e)}}{\lambda_{s(e)}}} \langle s(e) e\overline{e} \rangle.$$

Monoidal product on objects is addition, and for morphisms is defined by

$$(p, q) \otimes (p', q') \coloneqq \delta_{s(p') = t(p)}(pp', qq').$$

We will be finding GPA embeddings of certain planar algebras. Unitarity of GPAs implies unitarity of these planar algebras.

3. $\mathbb{Z}_n$-like extensions

The goal of this section is to develop the tools needed to prove evaluability of general $\mathbb{Z}_n$-like extensions of trivalent categories. We expect this class of extensions to be helpful in the search for novel categories. For example, there is work underway by the present author and Edie-Michell to use the techniques of this paper to construct the largest known class of examples of *near-group* categories, as defined in [8]. This work on near-group categories extends an underlying $SO(3)_q$ trivalent skein theory. The present author has also begun work on a family of extensions of $SP(4)_q$, which, despite its skein theory being generated by a braid, is of the same essence.

This all begs the question of which leaves on the "tree of life" of [15] might bear more fruit of this variety. Already we have extended both categories ($SO(3)_q$ and *Fib*) covered by [15, Theorem A] by group-like objects. This paper deals with all but one of the categories covered by [15, Theorem B]. The categories one might next attempt such an extension of include:

- The remaining category ABA of [15, Theorem B]
- The category H_3 of [15, Theorem C]

General methods for demonstrating evaluability of a skein theory involve identifying some measure of complexity for a closed diagram, then showing the known relations allow one to strictly decrease this measure. For our underlying trivalent categories, Euler-evaluability allows us to decrement one measure of complexity: number of internal faces. With the new strand type, we have another measure: number of colored strands. The underlying trivalent categories we deal with have evaluation algorithms based on the standard Euler characteristic argument. One way to capture this evaluability is by considering dimensions of box spaces.

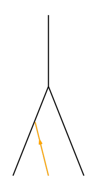
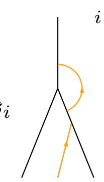
Definition 8. *In a trivalent category we define a* **box space** $B(k, f)$ *to be the span of diagrams* $k \to 0$ *with* f *internal faces. If* $\mathcal{C}$ *is a trivalent catgeory such that, for* $k = 1, \ldots, 5$*, the constraint*

$$\dim B(k, 1) \leq \dim B(k, 0)$$

holds, we will refer to $\mathcal{C}$ *as* **Euler-evaluable***.*

Diagrams inside a $\mathbb{Z}_n$-like extension exhibit the following nice properties, which will be key in proving their evaluability. Essentially, we use the following lemmas to exchange decorated faces for singly-externally-decorated faces. The defining relations for a $\mathbb{Z}_n$-like extension then pop the singly-decorated faces.

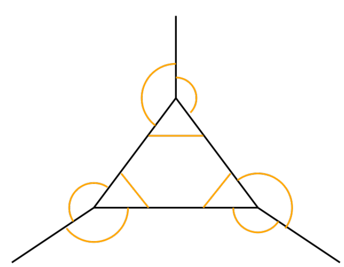
Lemma 2. *In a* $\mathbb{Z}_n$*-like extension, there exist* n *scalars* s_i *such that the following relation holds:*

(Slide)  $= \sum_{i=0}^{n-1} s_i$ 

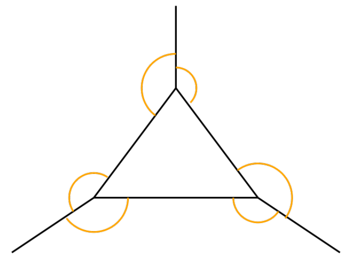
Proof. Apply (decStick), (Recouple), and (Change of Basis). □

Lemma 3. *A decorated diagram in a* $\mathbb{Z}_n$*-like extension may be expressed as a combination of singly-externally decorated diagrams*

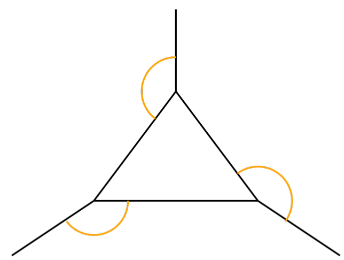
Proof. We prove the lemma for a decorated trigon, and leave the remaining cases to the reader. We begin with a maximally-decorated trigon. All less decorated cases are absorbed along the way in this analysis. Now, a maximally-decorated trigon is of the form:



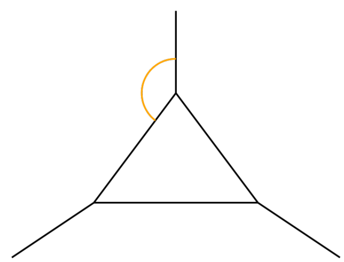
with any labeling on the colored strands. We apply the relations (Swap) and (Slide) on the internal colored strands to obtain a combination of diagrams of the form



Now apply (Change of Basis) to reduce to a combination of diagrams of the form



By another application of (Slide) and (Change of Basis) we arrive at a diagram of the form



During this last step, we pick up colored strands between the black "spokes"; one may happily move these out of the diagram. ☐

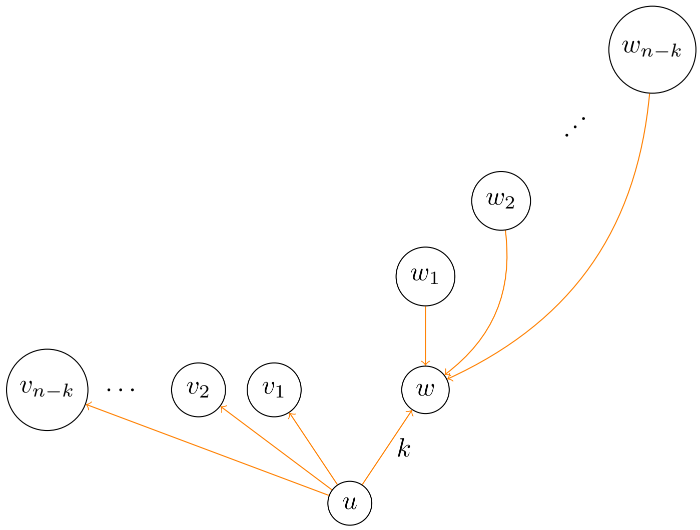
One more lemma will complete our ability to evaluate closed diagrams in $\mathbb{Z}_n$-like extensions.

Lemma 4. *Suppose a planar diagram $\mathcal{E}$ in a $\mathbb{Z}_n$-like extension consists only of black loops and colored oriented edges between them, such that the relations ($\mathbb{Z}_n$) and (Recouple) hold. Suppose furthermore that each loop of $\mathcal{E}$ has either exactly n strands or exactly n strands leaving. Then the diagram $\mathcal{E}$ evaluates to a scalar.*

Proof. Firt note that any oriented edges starting and ending from the same black loop may be removed using (Swap) and (decStick). So assume there are only oriented edges between distinct black loops. We'll use graph theoretic language, with black loops playing the role of nodes, and oriented edges playing the role of, well, orinted edges.

If a node has exactly one neighbor, use (Order n) to remove both. So assume every node has at least two neighbors. Pick one node and call it u. Choose an orintation for its neighbors. Call the rightmost neighbor by w; assume $\deg(u \to w) = k < n$. From right to left, call the remaining neighbors by $v_1, \ldots, v_{n-k}$, noting that these need not be distinct. From left to right, call the neighbors of w by $w_1, \ldots, w_{n-k}$, again noting that these need not be distinct.

The diagram is planar, so without loss, we may isotope it to look, locally, like



Now apply (Recouple), exchanging pairs of edges $u \to v_i$ and $w_i \to w$ for pairs of edges $u \to w$ and $w_i \to v_i$. This changes $\deg(u \to w)$ to n, allowing us, using (Order), to exchange a pair of nodes for a scalar. Continue ad nauseum. □

Proposition 1. *A $\mathbb{Z}_n$-like extension of an Euler-evaluable trivalent category is evaluable.*

Proof. Suppose we begin with a diagram given by a closed, decorated planar trivalent graph. Begin by applying relations from the underlying trivalent category's evaluation algorithm to any undecorated faces; this decreases the number of trivalent vertices. By the standard Euler characteristic calculation, there must remain some black n-gon with $n \in \{2, \ldots, 5\}$. Choose one such face and apply Lemma 3 to reduce it to a singly-externally-decorated n-gon. Now one of the relations (decBigon), (decTrigon), (decTetragon), or (decPentagon) allows us to pop the face. This process decreases the number of faces (ignoring colored strands) in diagrams by at least 1 at every step, but also may increase the number of connected components in any summand. Continue this process until only decorated loops, or decorated loops connected by colored strands remain. If only decorated loops remain, apply (decStick).

Our diagram now consists of a number of black loops, connected by colored strands. Use (Recouple) and (Order n) to make it so every black loop has either only in-strands or only out-strands attached to it. If any black loop has more or less than n strands entering or exiting (Schur 0) implies the whole diagram is zero. So suppose each black loop has exactly n strands entering or exiting. Apply Lemma 4 to evaluate the remaining graph for a scalar. □

For each quantum subgroup we construct, we will find planar algebras satisfying the conditions of Proposition 1, and thus will know the planar algebras are evaluable.

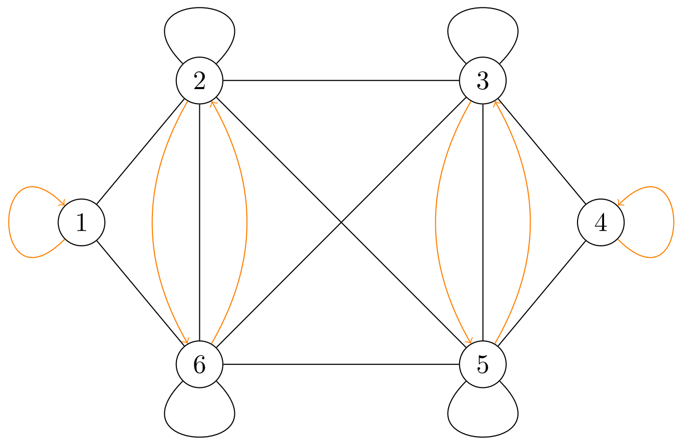


Figure 1. Fusion graphs at level 4 for Y (black) and g (orange). See [2, Figure 21b].

4. GPA Embeddings

The defining relations for $\mathcal{D}_3$ and $\mathcal{D}_4$ were not computed theoretically. Instead, we deduced them from embedding the planar algebras $\mathcal{P}_{Y_4;\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}$ and $\mathcal{P}_{Y_4;\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}}$ into graph planar algebras.

One may give a functor $F : \mathcal{G}_2(q_k) \to \mathrm{GPA}(\Gamma)$ by giving the image of the morphism

$$F\left(\quad \right) \in \mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 1).$$

This amounts to giving a list of $M := tr(\Gamma^2 \cdot \Gamma)$ complex scalars[2], say $a_1, \ldots, a_M$. These complex numbers satisfy equations in the a_i and $\overline{a_i}$. If we assume for now that each a_i is real, then this reduces the system to a collection of polynomials in the a_i [3]. Once we have the image of the trivalent vertex in hand, we have found an embedding of the planar algebra it generates. We can then solve for the image

$$F\left(\quad \right) \in \mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 2)$$

to extend the GPA embedding of $\mathcal{G}_2(q_k)$ to an embedding of $\mathcal{D}_k$.

4.1. **Trivalent Vertex.** The goal of this subsection is to describe in more detail the process of finding the image of the trivalent vertex. We will walk through the details for the level 4 case. The level 3 case follows the same process, but is less computationally instructive. However, the level 3 case does offer some insights into the relationship between embedding coordinates and symmetries of the funamental graph. In Subsection 5.1 we will explore this relationship further. Figure 1 shows the module fusion graph at level 4.

[2]We freely switch between using Γ to mean the graph itself and the graph's adjacency matrix.

[3]This assumption is useful only if it turns out to help us solve the system. In fact, any assumptions we make about this system, if they yield solutions, are in some way valid.

Let

$$(p_1, q_1), \dots, (p_M, q_M)$$[4]

be the defining basis for $\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(2 \to 1)$ ($M = 88$ at level 4). Then it must be that

$F\Bigg($  $\Bigg) = a_1(p_1, q_1) + \cdots + a_M(p_M, q_M).$

The Bigon relation, when sent through F, becomes the system

$$\sum_{i=1}^{M} a_i(p_i, q_i) \circ \sum_{j=1}^{M} a_j(q_j, p_j) = k^2 \sum_{e \in E(\Gamma)} (e, e).$$

This system is quadratic in the a_i since it involves up to two trivalent vertices on either side. The Lollipop and Rotate relations therefore determine a system of linear equations; the others give cubic, quartic, and quintic equations. It is often useful to solve the linear subsystem first and substitute the solution into the quadratic equations. For example, at level 4, discussed in Section **??**, we solve the linear subsyetm, substitute the solution, and isolate the following resulting equations:

$$a_8^2 + a_{85}^2 = 4 - \sqrt{2} + 2\sqrt{3} - \sqrt{6}$$
$$a_{69}^2 + \left(1 + \sqrt{\frac{3}{2}}\right) a_8^2 = \frac{3 + \sqrt{3} + \sqrt{6}}{\sqrt{2}}$$
$$a_{69}^2 \left(\left(2 + \sqrt{6}\right) a_8^2 + \left(2 + \sqrt{6}\right) a_{85}^2 - 2\sqrt{2 + \sqrt{3}}\right) = 5 + \sqrt{2} + \sqrt{3} + 2\sqrt{6}$$
$$2a_{69}^4 + \left(5 + 2\sqrt{6}\right) a_{85}^4 = \left(3 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right) a_{85}^2 + 3\sqrt{6} + \sqrt{3} + 2\sqrt{2} + 7$$

Up to three choices of sign, the solution to this system is

$$a_8 = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$
$$a_{69} = \sqrt{\frac{1}{2}\left(-1 + \sqrt{2} + \sqrt{3}\right)}$$
$$a_{85} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

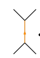
Similar equations containing a_{31}, a_{55}, and a_{63} appear as well. We may repeat this process and obtain the additional values

$$a_{31} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$
$$a_{55} = \sqrt{1 - \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}}$$
$$a_{63} = \sqrt{2 + \sqrt{3} - \sqrt{2 + \sqrt{3}}}$$

[4]See the attached Mathematica files for the specific ordering chosen.

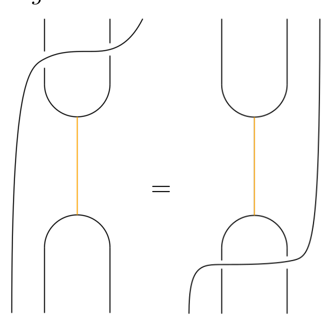
These six degree-8 algebraic numbers now begin a cascade of equation solving. They, along with the linear solution, reduce many of the original high-order equations to linear. We solve those, then repeat the process until we're forced to confront nonlinearity. The nonlinearity we encounter forces us to extract square roots, and ending up with degree-16 algebraic numbers. This lead to us concluding, for instance, that

$$a_{10} = \frac{1}{2}\left(\sqrt{1+\sqrt{6-3\sqrt{3}}} + \sqrt{\sqrt{2+\sqrt{3}}-1}\right).$$

4.2. **Projection and its relations.** The process of finding a GPA embedding of the projection  is similar to the process of finding the trivalent vertex described above. Now, however, we must use relations involving both  and . Call the coordinates of the projection b_i. Since the coordinates of the trivalent vertex are now known, the degree of the resulting equations now depends only on the number of projection strands appearing. Moreover, we do not necessarily assume the b_i are real; hence the resulting equations are polynomial in b_i and $\overline{b_i}$.

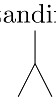
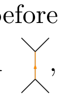
The relation (decStick) along with the one-strand relations of (Schur 0) and (Schur 1) give a system of linear equations. The following proposition, which is a specialization of Proposition **??** to the case where $f = P_g$, provides an additional set of linear equations in the projection coefficients.

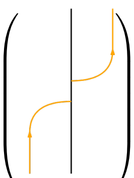
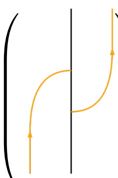
Proposition 2. *The following relation holds:*

(Half-braid) 

The relation (Half-braid) is not used in the evaluation algorithm. It is merely a vehicle for obtaining a huge set of linear equations in the b_i and $\overline{b_i}$.

In total, the linear system was enough to fully determine the projection coefficients for both levels 3 and 4. It is at this point that the present work diverges from other similar published works on the topics of GPA embeddings [1] or presentations for conformal embeddigns [6]. Previous investigations utilizing GPA embeddings have had a full set of relations on hand, and these relations were used to *find* the embeddings. On the other hand, recent progress on conformal embeddings have used theoretical means to uncover relations.

At the present step in this work, we are standing before an unexplored category, with an embedding of two of its generators,  and , in hand. We have a set of relations for  (i.e., the defining relations of $\mathcal{G}_2(q)$) sufficient to evaluate any undecorated closed diagram. However, we do not know what further relations we will find, nor exactly when we ought to stop looking. In order to uncover relations such as those of Definition 2, we use a process reminiscent of the scientific method. This process begins by considering a decorated trivalent diagram and searching for ways to decrease its complexity. We begin the process by assuming no more than a minimal collection of moves; i.e., (decStick), (Recouple), and ($\mathbb{Z}_n$). Once we have applied all these minimal moves, we begin to look for a move we might hope to make, and assume it comes at some cost. For example, being able to apply (Swap) might allow us to then apply (decStick); thus we would eliminate a colored strand at the cost of a scalar ω. This is clearly a trade we should make. In order to find the exact price of this trade, we use our GPA-embedding $F : \mathcal{D}_3 \to \mathrm{GPA}(\Gamma_3)$, and assume we know the form the cost will come in. We use the embedding coefficients we previously found to set up and solve the linear equation

F () $= \omega F$ ()

for ω.

Another example of this tradeoff appears in the proof of Lemma 3. During the proof, we apply (Swap) and (Change of Basis) with the goal of ridding ourselves of internal strands; this comes at the cost of external strands. The costs incurred in the (decTrigon), (decTetragon), and (decPentagon) relations are less straightforward to predict. The form the relations are presented in is the product of trial and error. As one will note when inspecting the coefficients of these relations in the attached Mathematica files, many of the (decTetragon) and (decPentagon) coefficients are zero. The obvious next question one asks is which of the numbers $u_{i,j}$, $v_{i,j}$, and $w_{i,j}$ one should expect to be nonzero. The examples we construct based on $\mathcal{G}_2(q)$ are too isolated to notice a pattern and draw conclusions in general. However, forthcoming work of the present author and Edie-Michell constructing many examples of near-groups as Z_n-like extensions will work to shed light on the phenomenon as a whole.

5. Results

This section is devoted to discussing the GPA embedding coordinates and the proofs of Theorem 2 and its analogue at level 4.

5.1. Level 3.

We begin at level $k = 3$, which makes $q = q_3 = e^{\frac{2\pi i}{42}}$. From the conformal embedding $\mathcal{V}(\mathfrak{g}_2, 3) \subseteq \mathcal{V}(\mathfrak{e}_6, 1)$ given in [4] we obtain the algebra object $A_3 = V_\emptyset \oplus V_{\Lambda_1+\Lambda_2}$, as described in [2]. Let $X = V_{\Lambda_1}$ be the object of $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}$ whose tensor powers constitute the objects of the planar algebra $\mathcal{P}_X \cong \mathcal{G}_2(q_3)$. Also given in [2] is (the orbifold of) the module fusion graph Γ_3 for X; its adjacency matrix is

$$M_{\Gamma_3} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The graph Γ_3 itself is depicted in Figure 2. [5]

It is known from [4] that

$$\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}^0_{A_3} \cong \mathrm{Vec}(\mathbb{Z}_3);$$

we deduce from the inclusion $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}^0_{A_3} \hookrightarrow \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$ that $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$ contains two $\mathbb{Z}_3$-like simple objects, denoted g and g^{-1}. The A_3-modules g and g^{-1} are both $V_{2\Lambda_1}$ as objects; they differ only in their A_3-multiplication maps.

Define $Y = \mathcal{F}_{A_3}(X)$ to be the image of X under the free functor. Then $\mathcal{F}_{A_3}$ restricts to an embedding

$$\mathcal{P}_{X;\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}} \hookrightarrow \mathcal{P}_{Y;\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_{A_3}}.$$

Invertibility of the objects g and g^{-1} implies $g^j \otimes Y \cong Y$, with rigidity maps for g and g^{-1} building the mutually inverse isomorphisms.

We now compute some important dimensions.

Proposition 3. *The following are true:*

(1) $\dim \mathrm{Hom}_{\mathcal{C}_{A_3}}(Y^{\otimes 2} \to Y) = 3$
(2) $\dim \mathrm{Hom}_{\mathcal{C}_{A_3}}(Y^{\otimes 2} \to g^j) = 1$
(3) $\dim \mathrm{Hom}_{\mathcal{C}_{A_3}}(Y \to Y) = 1$ *(i.e., Y is simple)*

Proof. All three are proved using fusion graph calculations, so we will only discuss (1) here. See, e.g., Figure 4 of [9] for the X fusion graph. The fusion graph tells us that

$$V_{\Lambda_1}^{\otimes 2} \cong V_{\emptyset} \oplus V_{\Lambda_1} \oplus V_{2\Lambda_1} \oplus V_{\Lambda_2}$$

and

$$A_3 \otimes V_{\Lambda_1} \cong V_{\Lambda_1} \oplus V_{2\Lambda_1} \oplus V_{3\Lambda_1} \oplus V_{\Lambda_2} \oplus V_{\Lambda_2+\Lambda_1} \oplus V_{\Lambda_2+2\Lambda_1}.$$

On the other hand, we compute

$$\begin{aligned}
\mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(Y^{\otimes 2} \to Y) &= \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(\mathcal{F}_{A_3}(V_{\Lambda_1})^{\otimes 2} \to \mathcal{F}_{A_3}(V_{\Lambda_1}))\\
&\cong \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(\mathcal{F}_{A_3}(V_{\Lambda_1}^{\otimes 2}) \to \mathcal{F}_{A_3}(V_{\Lambda_1}))\\
&= \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(\mathcal{F}_{A_3}(V_{\Lambda_1}^{\otimes 2}) \to A_3 \otimes V_{\Lambda_1})\\
&\cong \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}}(V_{\Lambda_1}^{\otimes 2} \to A_3 \otimes V_{\Lambda_1}).
\end{aligned}$$

Counting common irreducible constituents of $V_{\Lambda_1}^{\otimes 2}$ and $A_3 \otimes V_{\Lambda_1}$ gives the desired result. □

We have now set the stage for a $\mathbb{Z}_3$-like extension of $\mathcal{G}_2(q_3)$.

5.1.1. *GPA Embedding.* Here we give some details of the GPA-embedding of $\mathcal{D}_3$. Recall from Definition 7 that the defining bases for the spaces

$$\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(m \to n)$$

are given in terms of pairs of paths. The (undirected) graphs we are using have at most a single edge between any two vertices. Hence an edge is equivalent to a pair

[5] We will reference both a graph and its adjacency matrix by the same name. Whether we mean a matrix or a collection of nodes and edges should be clear from context.

of vertices, and a path is equivalent to an ordered tuple of vertices. For example, the path

$$p = v_1 \longrightarrow v_2 \longrightarrow v_3$$

is equivalent to the ordered triple (v_1, v_2, v_3). Which paths q pair validly with p to form a basis element of the $2 \to 1$ hom-space of a GPA? Well, by definition, q must be parallel to p; i.e. the sources and targets of p and q must coincide. It follows that the only valid pairing for such p is

$$q = v_1 \longrightarrow v_3,$$

which may also be represented as (v_1, v_3). So the only $2 \to 1$ basis element which p appears in is

$$\big((v_1, v_2, v_3), (v_1, v_3)\big).$$

But the parallel condition defining basis elements makes including (v_1, v_3) redundant; we might just as well have called the basis element by

$$(v_1, v_2, v_3).$$

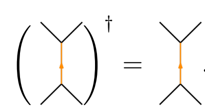
This is how we refer to $2 \to 1$ GPA basis elements. Indeed, in Table 1, the first two columns combine to specify which basis elements are being specified, and the third column gives the approximate coordinate of the trivalent embedding on that basis element. For example, the first row of Table 1 tells us that the coordinate of the $(2, 2, 2)$ basis element is approximately 1.08393; the second row tells us that the coordinate of the $(4, 2, 3)$ basis element is approximately 0.619371.

Paths of the form (i, j, i), (i, i, j), or (i, j, j) for $i, j \neq 1$ require a bit more care to describe. There is nontrivial interplay with the graph symmetry swapping vertices 2 and 4. When these two vertices are swapped, a path whose coordinate has absolute value 0.155691 is sent to one whose coordinate has absolute value 1.69414. The nine paths whose coordinates have absolute value 0.155691 are:

$$(2, 3, 3),\ (3, 3, 2),\ (3, 2, 3),\ (2, 4, 2),\ (4, 3, 4),\ (2, 2, 4),\ (3, 4, 4),\ (4, 2, 2),\ (4, 4, 3)$$

One may use symmetry to find the rest of the coordinates.

Table 2 holds numerical approximations to the nonzero projection coordinates. There are blocks of nonzero coordinates of length 9 and 16. These sizes, and the location of the nonzero real coordinates follow naturally when one considers three facts:

(1) The object g is $\mathbb{Z}_3$ and simple and therefore has fusion graph given in Figure 2.
(2) There is a map $T_g \in \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(Y^{\otimes 2} \to g)$ whose outer product $T_g^\dagger \circ T_g$ equals P_g, projection onto g.
(3) The dagger of a simple projection is itself; i.e.,  .

The existence and behavior of T_g must be captured by the image of P_g in the GPA, which is all we have access to. This is indeed the case, as the only coordinates of the projection in the GPA which are nonzero are at those basis vectors

$$(i \to \_ \to j, i \to \_ \to j)$$

where $i \to j$ is a directed edge of the g-fusion graph. For $i = j = 1$ there are three possible values for $\_$; pairing them gives 9 pairs. For $i, j \neq 1$ there are four possible

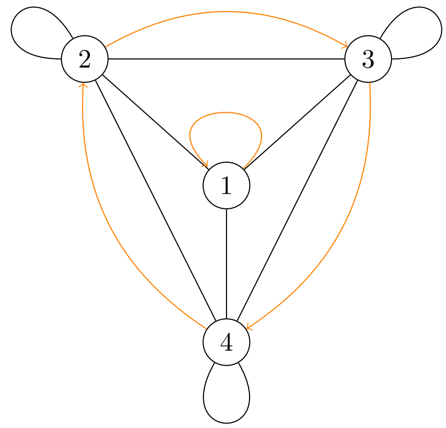


FIGURE 2. Fusion graphs at level 3 for Y (black) and g (orange). See [2, Figure 18b].

values for $\_$; pairing them gives 16 pairs. The columns of Table 2 give numerical approximations to the coordinates of the projection, with dictionary ordering on the pairs of $\_$ values. That is, the column labeled by $1 \to \_ \to 1$ shows the coordinates on the ordered basis

$$\begin{gathered}(1 \to 2 \to 1, 1 \to 2 \to 1)\\(1 \to 2 \to 1, 1 \to 3 \to 1)\\(1 \to 2 \to 1, 1 \to 4 \to 1)\\(1 \to 3 \to 1, 1 \to 2 \to 1)\\(1 \to 3 \to 1, 1 \to 3 \to 1)\\(1 \to 3 \to 1, 1 \to 4 \to 1)\\(1 \to 4 \to 1, 1 \to 2 \to 1)\\(1 \to 4 \to 1, 1 \to 3 \to 1)\\(1 \to 4 \to 1, 1 \to 4 \to 1)\end{gathered}$$

With this ordering and fact (3) above in mind, and recalling that the GPA's dagger operation swaps paths, the real coordinates appear where one would expect them.

One may find the $\mathbb{Z}_n$-like structure constants for $\mathcal{D}_3$ in the attached Mathematica files.

5.1.2. *Equivalence theorem.* This subsection discusses a proof of the following theorem. We exhibit $\overline{\mathcal{D}_3}$ as a subcategory of $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$, and then we will be in a position to apply the techniques of [6].

Theorem 2. *There is a monoidal equivalence*

$$\mathrm{Ab}(\overline{\mathcal{D}_3}) \cong \overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}.$$

That is, $\mathcal{D}_3$ gives a diagrammatic presentation for the category $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$.

Vertex Path	Conditions	Coefficient
(i,i,i)	$i \neq 1$	1.08393
(i,j,k)	$\{i,j,k\} = \{2,3,4\}$	0.619371
$(i,1,k)$	$i,k \neq 1,\ i \neq k$	1.69414
$(i,1,i)$	$i \neq 1$	0.861006
$(i,i,1)$ or $(1,i,i)$	$i \neq 1$	0.967919

TABLE 1. Level 3 trivalent embedding coefficients.

$1 \to \_ \to 1$	$2 \to \_ \to 4$	$3 \to \_ \to 2$	$4 \to \_ \to 3$
1.26376	0.791288	0.791288	0.791288
$-0.631881 - 1.09445i$	$0.567622 - 0.684904i$	$0.876955 + 0.149123i$	$0.674406 + 0.580055i$
$-0.631881 + 1.09445i$	$0.674406 + 0.580055i$	$0.567622 - 0.684904i$	$0.876955 + 0.149123i$
$-0.631881 + 1.09445i$	$-0.876955 - 0.149123i$	$-0.674406 - 0.580055i$	$0.567622 - 0.684904i$
1.26376	$0.567622 + 0.684904i$	$0.876955 - 0.149123i$	$0.674406 - 0.580055i$
$-0.631881 - 1.09445i$	1	1	1
$-0.631881 - 1.09445i$	$-0.0182917 + 0.999833i$	$0.5 - 0.866025i$	$0.856735 - 0.515757i$
$-0.631881 + 1.09445i$	$-0.5 - 0.866025i$	$-0.856735 - 0.515757i$	$-0.0182917 - 0.999833i$
1.26376	$0.674406 - 0.580055i$	$0.567622 + 0.684904i$	$0.876955 - 0.149123i$
	$-0.0182917 - 0.999833i$	$0.5 + 0.866025i$	$0.856735 + 0.515757i$
	1	1	1
	$-0.856735 + 0.515757i$	$0.0182917 - 0.999833i$	$0.5 - 0.866025i$
	$-0.876955 + 0.149123i$	$-0.674406 + 0.580055i$	$0.567622 + 0.684904i$
	$-0.5 + 0.866025i$	$-0.856735 + 0.515757i$	$-0.0182917 + 0.999833i$
	$-0.856735 - 0.515757i$	$0.0182917 + 0.999833i$	$0.5 + 0.866025i$
	1	1	1

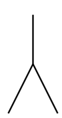
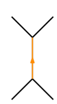
TABLE 2. Level 3 projection embedding coefficients.

We begin by giving a consequence of Proposition 3.

Corollary 1. *There is a dominant monoidal functor*

$$\Psi_3 : \mathcal{D}_3 \to \overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_{A_3}$$

which maps

 $\mapsto \tau \in \mathrm{Hom}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(Y^{\otimes 2} \to Y),$  $\mapsto P_g \in \mathrm{End}_{\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}(Y^{\otimes 2})$

Proof. Proposition 3, along with the defining relations of $\mathcal{D}_3$ tell us this assignment is functorial

$$\mathcal{D}_3 \hookrightarrow \mathcal{P}_{Y;\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}}.$$

Since Y is a $\otimes$-generator of $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$ and Ψ_3 surjects onto objects of $\mathcal{P}_Y$, dominance also follows. $\square$

Lemma 5. *The induced map*

$$\overline{\Psi_3} : \overline{\mathcal{D}_3} \to \overline{\operatorname{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$$

is faithful.

Proof. The evaluation algorithm for $\mathcal{D}_3$ implies that $\mathcal{D}_3$ has simple unit. Therefore every ideal is contained in the ideal of negligibles, which is killed when passing to the semisimplification $\overline{\mathcal{D}}$. Hence the map $\overline{\Psi_3}$ has no kernel. □

We are now in a position to apply almost the exact argument of [6]. The only difference is that in our case, it is clear the algebra object A_3 has no nontrivial subalgebras. The theorem follows.

5.2. **Level 4.** At level $k = 4$ we have $q = q_4 = e^{\frac{2\pi i}{48}}$ and $A_4 = V_\emptyset \oplus V_{3\Lambda_1}$. From [4] we have $\overline{\operatorname{Rep}(U_{q_4}(\mathfrak{g}_2))}^0_{A_4} \cong \operatorname{Vec}(\mathbb{Z}_2)$, giving us one new $\mathbb{Z}_2$-like simple object. The module fusion graph for level 4 is show in Figure 1. The details of the GPA embedding of $\mathcal{G}_2(q_4)$ are less instructive than the level 3 case, so we relegate them to the attached Mathematica files. The trivalent coordinates are algebraically nicer than those at level 3; up to sign, we showed 5 of 9 in Subsection 4.1. The nonzero coordinates of the projection come in blocks of 4, and 9. There is also an analogue of Theorem 2 for level 4.

Theorem 3. *There is a monoidal equivalence*

$$\operatorname{Ab}(\overline{\mathcal{D}(q_4)}) \cong \overline{\operatorname{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}.$$

That is, $\mathcal{D}_4$ gives a diagrammtic presentation for $\overline{\operatorname{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$.

This theorem is proved by using analogues of Proposition 3, Corollary 1, Lemma 5. The same argument used in the proof of Theorem 2 may be applied to the present context.

Note that since $\overline{\operatorname{Rep}(U_{q_4}(\mathfrak{g}_2))}^0_{A_4}$ contributes a $\mathbb{Z}_2$-like simple object, there is only one decoration of a trivalent vertex. The structure constants for $\mathcal{D}_4$ are given in Section 1, but we recreate them here:

$$\omega = -1$$

$$r_1 = e^{-\frac{\pi i}{6}}, \quad r_2 = e^{-\frac{2\pi i}{3}}$$

$$s_1 = e^{-\frac{\pi i}{6}} \quad s_2 = e^{-\frac{4\pi i}{3}}$$

$$t_1 = -1 \quad t_2 = -1$$

$$\begin{array}{cccccc} u = 1, & u = 2, & u = 3, & u = 4, & u = 5, & u = 6, \\ v = 7, & v = 8, & v = 9, & v = 10, & v = 11, & v = 12 \end{array}$$

References

[1] Daniel Copeland and Cain Edie-Michell. *Cell Systems for $\overline{\operatorname{Rep}(U_q(\mathfrak{sl}_N))}$ Module Categories.* 2023. arXiv: `2301.13172 [math.QA]`.

[2] R. Coquereaux, R. Rais, and E. H. Tahri. "Exceptional quantum subgroups for the rank two Lie algebras B2 and G2". In: *Journal of Mathematical Physics* 51.9 (Sept. 2010). ISSN: 1089-7658. DOI: 10.1063/1.3476319. URL: http://dx.doi.org/10.1063/1.3476319.

[3] Agustina Czenky. *Diagramatics for cyclic pointed fusion categories*. 2024. arXiv: 2404.08084 [math.QA]. URL: https://arxiv.org/abs/2404.08084.

[4] Alexei Davydov et al. *The Witt group of non-degenerate braided fusion categories*. 2011. arXiv: 1009.2117 [math.QA]. URL: https://arxiv.org/abs/1009.2117.

[5] Cain Edie-Michell. *Auto-equivalences of the modular tensor categories of type A, B, C and G*. 2020. arXiv: 2002.03220 [math.QA]. URL: https://arxiv.org/abs/2002.03220.

[6] Cain Edie-Michell, Noah Snyder, and Hans Wenzl. *Interpolation categories for conformal embeddings*. 2024.

[7] Pavel Etingof et al. *Tensor Categories*. en. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, Aug. 2015.

[8] David E. Evans and Terry Gannon. *Near-group fusion categories and their doubles*. 2012. arXiv: 1208.1500 [math.QA]. URL: https://arxiv.org/abs/1208.1500.

[9] David E. Evans and Mathew Pugh. "Spectral Measures for G 2". In: *Communications in Mathematical Physics* 337.3 (Feb. 2015), pp. 1161–1197. ISSN: 1432-0916. DOI: 10.1007/s00220-015-2293-0. URL: http://dx.doi.org/10.1007/s00220-015-2293-0.

[10] Pinhas Grossman et al. *The Extended Haagerup fusion categories*. 2018. arXiv: 1810.06076 [math.QA]. URL: https://arxiv.org/abs/1810.06076.

[11] Vaughan Jones. "The Planar Algebra of a bipartite graph". In: *Knots in Hellas '98* 24 (July 2000). DOI: 10.1142/9789812792679_0008.

[12] Yigal Kamel. *Strong monoidal functors and modules*. 2023. URL: https://yigalkamel.web.illinois.edu/StrongFuncModules_Mar23.pdf.

[13] David Kazhdan and Hans Wenzl. "Reconstructing monoidal categories". In: (1993). URL: https://api.semanticscholar.org/CorpusID:125218067.

[14] Greg Kuperberg. "Spiders for rank 2 Lie algebras". In: *Communications in Mathematical Physics* 180.1 (Sept. 1996), pp. 109–151. ISSN: 1432-0916. DOI: 10.1007/bf02101184. URL: http://dx.doi.org/10.1007/BF02101184.

[15] Scott Morrison, Emily Peters, and Noah Snyder. *Categories generated by a trivalent vertex*. 2016. arXiv: 1501.06869 [math.QA].

[16] Viktor Ostrik. *Module categories, weak Hopf algebras and modular invariants*. 2001. arXiv: math/0111139 [math.QA]. URL: https://arxiv.org/abs/math/0111139.

[17] Imre Tuba and Hans Wenzl. *On braided tensor categories of type BCD*. 2020. arXiv: math/0301142 [math.QA]. URL: https://arxiv.org/abs/math/0301142.