QUANTUM SUBGROUPS OF G_2 VIA GRAPH PLANAR ALGEBRA EMBEDDINGS

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Abstract. todo

1. Introduction

Quantum subgroups are a well-known source of tensor categories. More precisely, given a conformal embedding $\mathcal{V}(\mathfrak{g}, k) \subseteq \mathcal{V}(\mathfrak{h}, 1)$ of VOAs as in [4], one obtains a corresponding Etale algebra A. This algebra then allows one to consider the category $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$ of right A-modules. A half-braiding on A then gives a tensor product on $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$, and one may study this new category in its own right. The free functor gives an embedding $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))} \hookrightarrow \overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$. As this embedding is, in general, not full, it remains only to find a description of the new morphisms in $\overline{\mathrm{Rep}(U_q(\mathfrak{g}))}_A$. Recent works of Edie-Michell and Snyder [6] have used this reasoning, and representation theoretic techniques to give diagrammatic descriptions of new tensor categories of modules corresponding to the family of conformal embeddings $\mathcal{V}(\mathfrak{sl}_N, N^2) \subseteq \mathcal{V}(\mathfrak{sl}_{N^2-1}, 1)$.

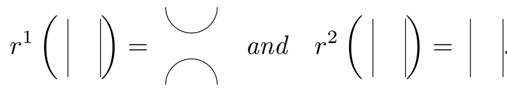
On the other hand, one may start with a known category and compute graph planar algebra (GPA) embeddings for it. This has been done for $\overline{\mathrm{Rep}(U_q(\mathfrak{sl}_N))}$ in [1] and for the extended Haagerup categories in [9]. This computation has the theoretical and practical consequences. By the GPA embedding theorem [**something**], such an embedding immediately gives a module category. It additionally gives a concrete representation of the category in which one may perform explicit computations.

The present work describes a blend of these two techniques. We begin by fidning a GPA embedding on the well-known trivalent category $\mathcal{G}_2(q)$ of [13, 14] which is a diagrammatic presentation for $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}$. Through the free fucntor we can view our GPA embedding as a GPA embedding for a $\otimes$-generating object's planar algebra in $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A$. This gives us a black-strand. We then search inside the GPA embedding for new morphisms. According to [4] there ought to be a projection onto a $\mathbb{Z}_k$-like simple object in $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}_A$, so this is what we search for inside the GPA. We view this new morphism as an orange strand. The properties of this new morphism are unknown beyond a scant basic set. Once we have or hands on the imagine in the GPA of this projection, though, we may explore its properties through explicit computations. We perform this process of extending GPA embeddings for the two conformal embeddings

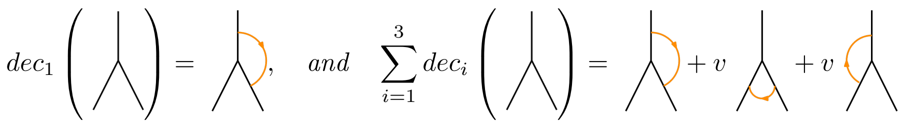
$$\mathcal{V}(\mathfrak{g}_2, 3) \subseteq \mathcal{V}(\mathfrak{e}_6, 1) \quad \text{and} \quad \mathcal{V}(\mathfrak{g}_2, 4) \subseteq \mathcal{V}(\mathfrak{d}_7, 1). \tag{1}$$

Now we begin by introducing some notation for a skein theory involving an oritned, colored strand in addition to unoriented black strands.

Definition 1. *For a diagram $\mathcal{E}$ the notation $r^i(\mathcal{E})$ means an i-click right rotation. For instance,*

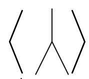
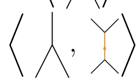


Suppose the diagram $\mathcal{E}$ has m boundary points. We define $dec_i(\mathcal{E})$ to be the i-th external single clockwise decoration of $\mathcal{E}$. For example,

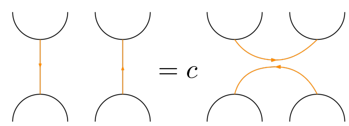


We adopt the convention that $dec_0(\mathcal{E}) = \mathcal{E}$.

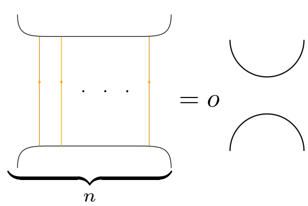
Both of the categories studied in this paper are extensions of trivalent categories by a colored, directed, $\mathbb{Z}_n$-like strand. We define the class of categories we will be working with. In Section 3 we will show that, with an assumption on the underlying skein theory, categories in this class are evaluable in general.

Definition 2. *Let $\mathcal{C} =$*  *be a trivalent category. Call $\mathcal{D}$ a* $\mathbb{Z}_n$**-like extension** *of $\mathcal{C}$ if we have $\mathcal{D} =$* *, enjoying the following relations[1]:*

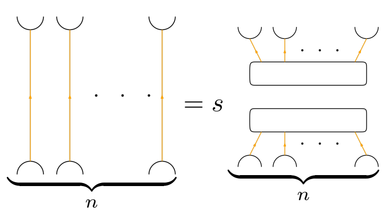
(Recouple)



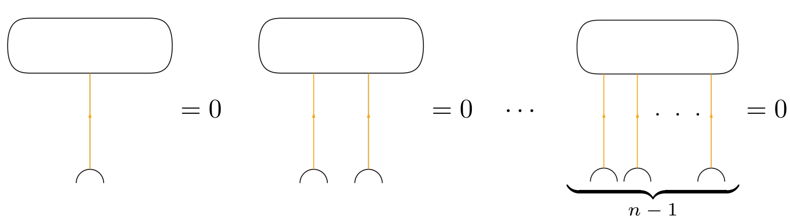
($\mathbb{Z}_n$)



(Split)

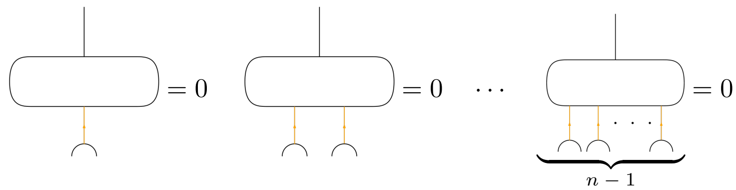


(Schur 0)

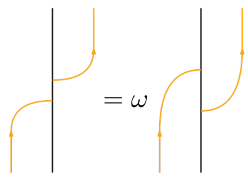


[1]Conditions from [3].

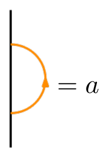
(Schur 1)



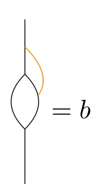
(Swap)



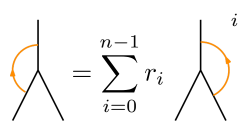
(decStick)



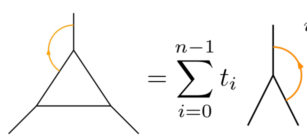
(decBigon)



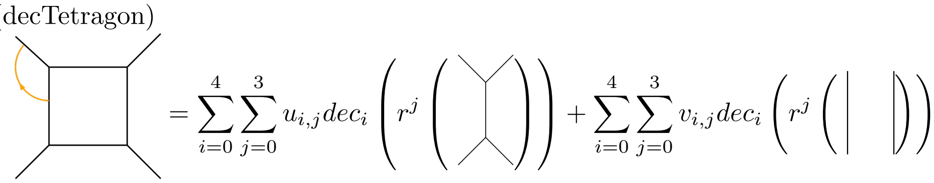
(Change of Basis)



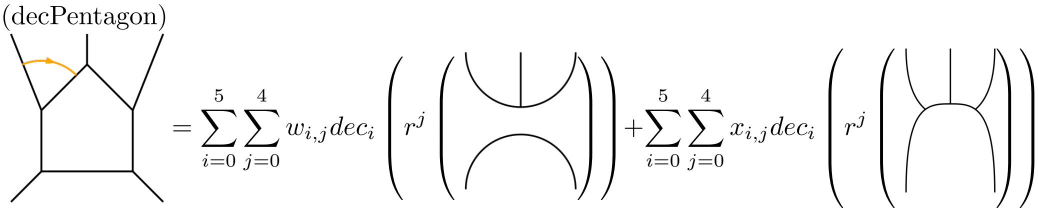
(decTrigon)



(decTetragon)



(decPentagon)



Remark 1. *A quick sketch shows that using (Order) followed by repeated applications of (Recouple) and (decStick) allows one to swap an up-oriented strand for $n-1$ down-oriented strand. This means that, upon reversing the orientations of the lefthand sides of the relations in Definition 2 will give similar relations. This fact will be used in the proof of Lemma 3.*

Remark 2. *It is worth noting the following standard abuse of language. A diagrammatically presented category such as a $\mathbb{Z}_n$-like extension has hom-spaces which are formal spans of diagrams. When applying a relation such as (decTrigon) locally, the result is clearly a linear combination of diagrams. Usually, though, this linear combination has some desirable quality, such as a smaller number of internal faces in each summand. In this instance, we prefer to say something along the lines of, "applying (decTrigon) decreases the number of internal faces," instead of, for instance, the more wordy, "applying (decTrigon) turns this diagram into a linear combination of diagrams with fewer internal faces."*

Definition 3. *Set $q_4 = e^{\frac{2\pi i}{48}}$ and deifne $\mathcal{D}_4$ to be the $\mathbb{Z}_2$-like extension of $\mathcal{G}_2(q_4)$ with structure constants*

$$\omega = -1$$

$$r_1 = e^{-\frac{\pi i}{6}}, \quad r_2 = e^{-\frac{2\pi i}{3}}$$

$$s_1 = e^{-\frac{\pi i}{6}} \quad s_2 = e^{-\frac{4\pi i}{3}}$$

$$t_1 = -1 \quad t_2 = -1$$

$$\begin{array}{cccccc} u = 1, & u = 2, & u = 3, & u = 4, & u = 5, & u = 6, \\ v = 7, & v = 8, & v = 9, & v = 10, & v = 11, & v = 12 \end{array}$$

One of the two primary results we give here is that $\mathcal{D}_4$ is a presentation for a category of modules corresponding to the level 4 conformal embedding of $\mathfrak{g}_2$.

Theorem 1. *There is an equivalence*

$$\mathrm{Ab}(\mathcal{D}_4) \cong \overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$$

where A_4 is the algebra object corresponding to the level-4 conformal embedding of 1.

Theorem **??** is an analogous theorem for level 3, with structure constants given in the attached Mathematica files.

It is not clear a priori that the defining relations for, say, $\mathcal{D}_4$ lead to a nontirivial tensor category. The general undecidability of the word problem for groups offers some evidence that this question is difficult for a typical presentation for a tensor category. That is, one should not expect a set of relations to yield any nontriviality. It follows that the presentations we give here are interesting and worth investigating more generally.

The remainder of the paper is structured as follows. Section 2 sets up most of the theory needed, referencing that which we do not exposit here. This includes unoriented planar algebras, unoriented graph planar algebras, internal algebra and module objects, and some assorted theoretical devices and results. Section 3 then goes on to investigate some properties of $\mathbb{Z}_n$-like extensions. We expect this class of categories to be of use for researchers intent on conjuring examples of exotic tensor categories. In fact, in a forthcoming paper, the present author and Cain Edie-Michell diagrammatically present a number of near-group categories as $\mathbb{Z}_n$-like extensions of $SO(3)_q$ trivalent categories. We demonstrate evaluability of this class of catgeories under a relatively tame assumption on the underlying trivalent skein theory. Section **??** discusses the process of arriving at GPA embeddings. Subsection **??** details the techniques used to arrive at GPA embeddigns of trivalent categories. Subsection **??** shows how we extend these embeddings of trivalent categories to embeddings of $\mathbb{Z}_n$-like extensions, and how we use GPA embeddings to explore relations in these extensions. This section uses examples from level 4 ($\mathcal{D}_4$) due to the fact that the numbers involved are more presentable. The process used for level 3 ($\mathcal{D}_3$) was essentially identical. Finally, Section **??** gives the structure constants for the newly constructed categories. This section also discusses the argument used to prove that we truly have found full presentations. The argument appears in its entirety in [6], and is adapted to the present settign without a problem.

2. Preliminaries

Here we define the players in this game. This includes planar algebras, graph planar algebras, and internal algebra and module objects. We give only a few necessary results, and refer the reader to the definitive publications. For the general theory of tensor categories, see [7].

2.1. Algebra and Module Objects.

We will ultimately show that $\mathcal{D}_3$ and $\mathcal{D}_4$ are presentations for the categories $\overline{\mathrm{Rep}(U_{q_3}(\mathfrak{g}_2))}_{A_3}$ and $\overline{\mathrm{Rep}(U_{q_4}(\mathfrak{g}_2))}_{A_4}$ of modules over algebra objects A_3 and A_4 coming from the conformal embeddings $\mathcal{C}(\mathfrak{g}_2, 3) \subseteq \mathcal{C}(\mathfrak{e}_6, 1)$ and $\mathcal{C}(\mathfrak{g}_2, 4) \subseteq \mathcal{C}(\mathfrak{d}_7, 1)$, respectively. In this subsection we recall basic facts about algebra and module objects, as well as conformal embeddings. See [7, 15] for more complete descriptions. The theory which will apply to our context is given in [6]. Some basic properties concerning the interaction of algebra and module objects with monoidal functors will be used in the proof of our main theorems; this material can be found in [11]. We restate a few definitions and facts here. Unless otherwise stated, we will be assuming the underlying tensor categories are braided.

Definition 4. *Let A be an algebra object of the braided tensor category $\mathcal{C}$. A is an* **Etale** *algebra if it is commutative and separable. We call A* **connected** *if it is Etale and* $\dim \mathrm{Hom}_{\mathcal{C}}(\mathbb{1} \to A) = 1$.

For an Etale algebra object A of $\mathcal{C}$, we denote by $\mathcal{C}_A$ the collection of left A-modules internal to $\mathcal{C}$. As decribed in [6], a braiding on $\mathcal{C}$ induces a tesnor product on $\mathcal{C}_A$. Separability of A implies semisimplicity of $\mathcal{C}_A$, and connectedness of A implies the unit $\mathbb{1}_{\mathcal{C}_A} = A$ is simple in $\mathcal{C}_A$ [4].

Furthermore, the free fucntor

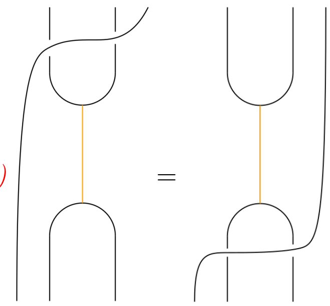
$$\mathcal{F}_A : \mathcal{C} \xhookrightarrow{X \mapsto A \otimes X} \mathcal{C}_A$$

is a monoidal embedding which is, as we will see later, not always full. Its right adjoint is given by the forgetful functor $\mathcal{F}^{\vee} : \mathcal{C}_A \to \mathcal{C}$ which acts as the identity on both objects and morphisms.

One result which will help immensely in arriving at GPA embeddings is the following, which is Lemma 2.4 of [6].

Lemma 1 (Half-braid)**.** *Let $\mathcal{C}$ be a braided tensor category, and A an etale algebra object. For any $f \in \mathrm{Hom}_{\mathcal{C}_A}(\mathcal{F}(Y_1) \to \mathcal{F}(Y_2))$, the following relation holds:*

(stand in for general hb)



We will utilize this result to obtain a rather large number (2970 at level 3 and 7776 at level 4) of linear equations constraining the GPA coordinates of the morphisms not living in the image of $\mathcal{F}_A$. Thus the half-braid relation will be key to our program, despite not being necessary to prove evaluability.

The source of our algebra objects will be conformal embeddings. We direct the reader to [4] a more complete treatment of conformal embeddings.

Definition 5. *A containment $\mathcal{V}(\mathfrak{g}, j) \subseteq \mathcal{V}(\mathfrak{h}, k)$ of affine Lie algebras is said to be* **conformal** *if the adjoint representation of $\mathcal{V}(\mathfrak{h}, j)$ restricts to a finite direct sum of simple objects in $\mathcal{C}(\mathfrak{g}, j) \coloneqq \mathrm{Rep}(\mathcal{V}(\mathfrak{g}, j))$.*

Affine Lie algebras and conformal embeddings will only be used to obtain algebra objects and module fusion graphs, so we briefly recall the correspondence

$$\mathcal{C}(\mathfrak{g}_2, k) \cong \overline{\mathrm{Rep}(U_{q_k}(\mathfrak{g}_2))} \tag{2}$$

of [5], where k is the level and q_k is given by

$$q = e^{\frac{2\pi i}{3(4+k)}}.$$

At level 3 we have $q_3 = e^{\frac{2\pi i}{42}}$ and at level 4 we have $q_4 = e^{\frac{2\pi i}{48}}$. We obtain the algebra objects and fundamental graphs for GPAs from [2]:

$$A_3 = V_\emptyset \oplus V_{\Lambda_1} \quad \text{and} \quad A_4 = V_\emptyset \oplus V_{3\Lambda_1} \tag{3}$$

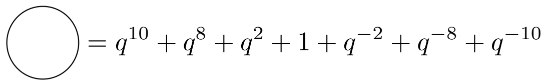
at levels 3 and 4, respectively.

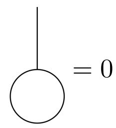
2.2. **Unoriented Planar Algebras.** Recall the theory of **rigid** monoidal categories detailed in [12]. To put it succintly, rigid monoidal categories have duals. Duals, and the associated evaluation and coevaluation maps, giving us the cups and caps ubiquitous in skein theory. A rigidity assumption gives us the ability to isotope diagrams. The generators we will use for our planar algebras will be symmetrically self-dual. We also assume pivotality throughout.

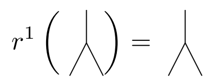
Let X be a (symmetrically self-dual) **tensor generator** for the tensor category $\mathcal{C}$; that is, every object of $\mathcal{C}$ is isomorphic to a subobject of some tensor power $X^{\otimes n}$. Let $\mathcal{P}_{X;\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ whose objects are tensor powers $\mathbb{1} = X^{\otimes 0}, X, X^{\otimes 2}, \ldots$; we call this the (unoriented) **planar algebra** generated by X in $\mathcal{C}$. The planar algebra $\mathcal{P}_{X;\mathcal{C}}$ is **evaluable** if $\dim \mathrm{End}_{\mathcal{P}_{X;\mathcal{C}}}(\mathbb{1}) = 1$.

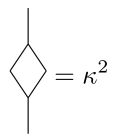
We will be presenting the our two quantum subgroups as extensions of $\mathcal{G}_2(q)$ skein theories, in the spirit of Kuperberg [13, 14]. Up to a rescaling by a factor of $\kappa = \sqrt{[7]-1}$ we use the same skein theory as [14] (note the sign error in the Pentagon relation of [13]).

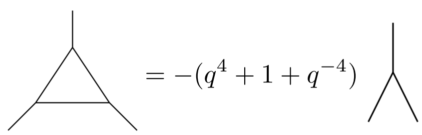
Definition 6. *For q a root of unity, the $\mathcal{G}_2(q)$ skein theory is defined to be that generated by an unoriented trivalent vertex*  *satisfying the relations*

(Loop) 

(Lollipop) 

(Rotate) 

(Bigon) 

(Trigon) 

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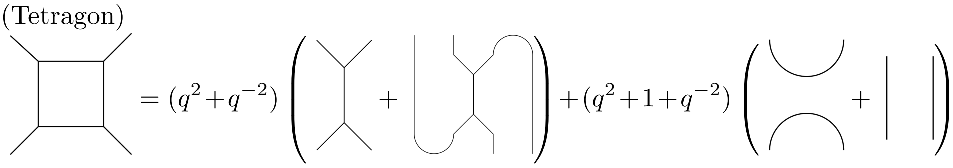
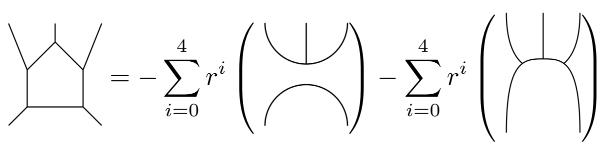


(Pentagon) 

Our use of planar algebras will depend entirely on the construction of the Cauchy completion, which we sketch here. See [6] for more details and [16] for a full treatment of the topic. Recall that the **idempotent competion** of a pivotal tensor category $\mathcal{C}$ consists of pairs (Z, p), where $p \in \mathrm{End}_{\mathcal{C}}(Z)$ is an idempotent. We denote the idempotent completion of $\mathcal{C}$ as $\mathrm{Idemp}(\mathcal{C})$. Furhter, we define the **additive envelope** of a pivotal, $\mathbb{C}$-linear tensor category $\mathcal{C}$ to have objects formal direct sums $\bigoplus_j Z_j$ for objects Z_j of $\mathcal{C}$. The **Cauchy completion** of $\mathcal{C}$ is defined by

$$\mathrm{Ab}(\mathcal{C}) := \mathrm{Add}(\mathrm{Idemp}(\mathcal{C})).$$

If we again assume X tensor generates $\mathcal{C}$, it follows that $\mathcal{C} \cong \mathrm{Ab}(\mathcal{P}_{X;\mathcal{C}})$ [16, Theorem3.4] The universal property of $\mathrm{Ab}(\mathcal{P}_{X;\mathcal{C}})$ therefore implies that studying $\mathcal{P}_{X;\mathcal{C}}$ is sufficient to understand $\mathcal{C}$.

The category $\mathcal{G}_2(q)$ is a **presentation** for the category $\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))}$ in the sense that

$$\overline{\mathrm{Rep}(U_q(\mathfrak{g}_2))} \cong \overline{\mathrm{Kar}(\mathcal{G}_2(q))}.$$

2.3. **Unoriented Graph Planar Algebras.** We will study the quantum subgroups of type G_2 by embedding their skein theories into appropriate graph planar algebras (GPAs). This serves two purposes:

- Giving us solid ground on which to do computations, allowing us to uncover relations by finding them in the GPA hom-spaces, and
- Implying some nice general properties for the quantum subgroups (i.e., unitarity)

GPAs are an invention of Vaughan Jones [10]. In this work we have no use for less specialized GPAs, such as the *oriented* [1] or *multi-color* GPA [**emily**], so we consider only the unoriented case.

Definition 7. *Let $\Gamma = (V, E)$ be a finite graph. For an edge $e = (u, v) \in E$, let $\overline{e} := (v, u) \in E$. The* **graph planar algebra** *on Γ, denoted $\mathrm{GPA}(\Gamma)$, is the strictly pivotal rigid monoidal category whose objects are nonnegative integers, and whose hom-spaces have basis*

$$\mathrm{Hom}_{\mathrm{GPA}(\Gamma)}(m \to n) := \mathbb{C}\left\{(p, q) \mid {}^{p \text{ an } m\text{-path}}_{q \text{ and } n\text{-path}}, {}^{s(p)=s(q)}_{t(p)=t(q)}\right\},$$

with composition law

$$(p, q) \circ (p', q') := \delta_{q=p'}(p, q'),$$

and rigidity maps

$$ev = \sum_e \sqrt{\frac{\lambda_{t(e)}}{\lambda_{s(e)}}}\langle e\overline{e}, s(e)\rangle, \quad coev = \sum_e \sqrt{\frac{\lambda_{t(e)}}{\lambda_{s(e)}}}\langle s(e) e\overline{e}\rangle.$$

Monoidal product on objects is addition, and for morphisms is defined by

$$(p, q) \otimes (p', q') := \delta_{s(p')=t(p)}(pp', qq').$$

We will be finding GPA embeddings of certain planar algebras. Unitarity of GPAs implies unitarity of these planar algebras.

3. $\mathbb{Z}_n$-like extensions

The goal of this section is to develop the tools needed to prove evaluability of general $\mathbb{Z}_n$-like extensions of trivalent categories. We expect this class of extensions to be helpful in the search for novel categories. For example, there is work underway by the present author and Edie-Michell to use the techniques of this paper to construct the largest known class of examples of *near-group* categories, as defined in [8]. This work on near-group categories extends an underlying $SO(3)_q$ trivalent skein theory. The present author has also begun work on a family of extensions of $SP(4)_q$, which, despite its skein theory being generated by a braid, is of the same essence.

This all begs the question of which leaves on the "tree of life" of [14] might bear more fruit of this variety. Already we have extended both categories ($SO(3)_q$ and *Fib*) covered by [14, Theorem A] by group-like objects. This paper deals with all but one of the categories covered by [14, Theorem B]. The categories one might next attempt such an extension of include:

- The remaining category ABA of [14, Theorem B]
- The category H_3 of [14, Theorem C]

General methods for demonstrating evaluability of a skein theory involve identifying some measure of complexity for a closed diagram, then showing the known relations allow one to strictly decrease this measure. For our underlying trivalent categories, Euler-evaluability allows us to decrement one measure of complexity: number of internal faces. With the new strand type, we have another measure: number of colored strands. The underlying trivalent categories we deal with have evaluation algorithms based on the standard Euler characteristic argument. One way to capture this evaluability is by considering dimensions of box spaces.

Definition 8. *In a trivalent category we define a* **box space** $B(k, f)$ *to be the span of diagrams* $k \to 0$ *with* f *internal faces. If* $\mathcal{C}$ *is a trivalent catgeory such that, for* $k = 1, \ldots, 5$*, the constraint*

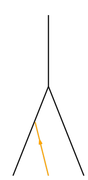
$$\dim B(k, 1) \leq \dim B(k, 0)$$

holds, we will refer to $\mathcal{C}$ *as* **Euler-evaluable**.

Diagrams inside a $\mathbb{Z}_n$-like extension exhibit the following nice properties, which will be key in proving their evaluability. Essentially, we use the following lemmas to exchange decorated faces for singly-externally-decorated faces. The defining relations for a $\mathbb{Z}_n$-like extension then pop the singly-decorated faces.

Lemma 2. *In a* $\mathbb{Z}_n$*-like extension, there exist* n *scalars* s_i *such that the following relation holds:*

(Slide)

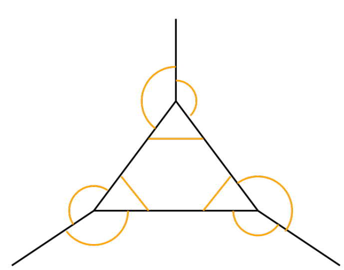


$$= \sum_{i=0}^{n-1} s_i$$

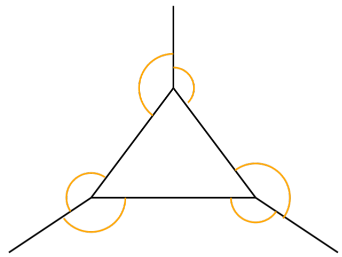
Proof. Apply (decStick), (Recouple), and (Change of Basis). □

Lemma 3. *A decorated diagram in a* $\mathbb{Z}_n$*-like extension may be expressed as a combination of singly-externally decorated diagrams*

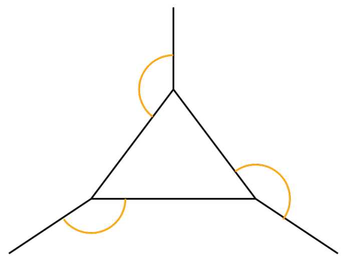
Proof. We prove the lemma for a decorated trigon, and leave the remaining cases to the reader. We begin with a maximally-decorated trigon. All less decorated cases are absorbed along the way in this analysis. Now, a maximally-decorated trigon is of the form:



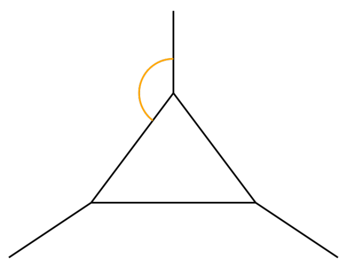
with any labeling on the colored strands. We apply the relations (Swap) and (Slide) on the internal colored strands to obtain a combination of diagrams of the form



Now apply (Change of Basis) to reduce to a combination of diagrams of the form



By another application of (Slide) and (Change of Basis) we arrive at a diagram of the form



During this last step, we pick up colored strands between the black "spokes"; one may happily move these out of the diagram. □

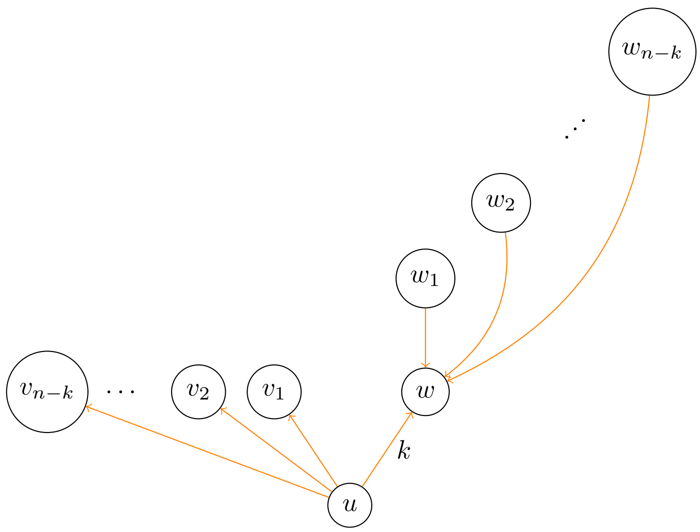
One more lemma will complete our ability to evaluate closed diagrams in $\mathbb{Z}_n$-like extensions.

Lemma 4. *Suppose a planar diagram $\mathcal{E}$ in a $\mathbb{Z}_n$-like extension consists only of black loops and colored oriented edges between them, such that the relations ($\mathbb{Z}_n$) and (Recouple) hold. Suppose furthermore that each loop of $\mathcal{E}$ has either exactly n strands or exactly n strands leaving. Then the diagram $\mathcal{E}$ evaluates to a scalar.*

Proof. Firt note that any oriented edges starting and ending from the same black loop may be removed using (Swap) and (decStick). So assume there are only oriented edges between distinct black loops. We'll use graph theoretic language, with black loops playing the role of nodes, and oriented edges playing the role of, well, orinted edges.

If a node has exactly one neighbor, use (Order n) to remove both. So assume every node has at least two neighbors. Pick one node and call it u. Choose an orintation for its neighbors. Call the rightmost neighbor by w; assume $\deg(u \to w) = k < n$. From right to left, call the remaining neighbors by $v_1, \ldots, v_{n-k}$, noting that these need not be distinct. From left to right, call the neighbors of w by $w_1, \ldots, w_{n-k}$, again noting that these need not be distinct.

The diagram is planar, so without loss, we may isotope it to look, locally, like



Now apply (Recouple), exchanging pairs of edges $u \to v_i$ and $w_i \to w$ for pairs of edges $u \to w$ and $w_i \to v_i$. This changes $\deg(u \to w)$ to n, allowing us, using (Order), to exchange a pair of nodes for a scalar. Continue ad nauseum. □

Proposition 1. *A $\mathbb{Z}_n$-like extension of an Euler-evaluable trivalent category is evaluable.*

Proof. Suppose we begin with a diagram given by a closed, decorated planar trivalent graph. Begin by applying relations from the underlying trivalent category's evaluation algorithm to any undecorated faces; this decreases the number of trivalent vertices. By the standard Euler characteristic calculation, there must remain some black n-gon with $n \in \{2, \ldots, 5\}$. Choose one such face and apply Lemma 3 to reduce it to a singly-externally-decorated n-gon. Now one of the relations (decBigon), (decTrigon), (decTetragon), or (decPentagon) allows us to pop the face. This process decreases the number of faces (ignoring colored strands) in diagrams by at least 1 at every step, but also may increase the number of connected components in any summand. Continue this process until only decorated loops, or decorated loops connected by colored strands remain. If only decorated loops remain, apply (decStick).

Our diagram now consists of a number of black loops, connected by colored strands. Use (Recouple) and (Order n) to make it so every black loop has either only in-strands or only out-strands attached to it. If any black loop has more or less than n strands entering or exiting (Schur 0) implies the whole diagram is zero. So suppose each black loop has exactly n strands entering or exiting. Apply Lemma 4 to evaluate the remaining graph for a scalar. □

For each quantum subgroup we construct, we will find planar algebras satisfying the conditions of Proposition 1, and thus will know the planar algebras are evaluable.

4. GPA Embeddings

4.1. Level 3.

Theorem 2. *There is a GPA embedding at level 3*

4.2. Extension of level 3.

Theorem 3. *There exists an element of the GPA with the following relations*

4.3. Level 4.

Theorem 4. *There is a GPA embedding at level 4*

4.4. Extension of level 4.

Theorem 5. *There exists an element of the GPA with the following relations*

5. New Relations

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